ECN 280 | Fall 2022

Cost Minimization and Cost Functions

Professor McGavock *Module 5.4*

Contents

1 Learning Outcomes for Module 5.4

(back to Table of Contents)

After completing this module, you will be able to:

- *Use the Lagrangian* to solve the cost minimization problem, obtaining the **conditional input demand functions**.
- *Derive* the **cost functions (total costs, average costs, marginal costs) using the conditional input demand functions.**

2 Assignments for Module 5.4

(back to Table of Contents)

- Watch:
  - No video
- Read:
  - Read this handout carefully and solve the problem along the way
- Turn in: [Turn in problems here.]
  - 5.4.1
  - 5.4.2

3 Lecture Notes

(back to Table of Contents)

3.1 Method 2: Lagrangian

Recall the basic steps of the Lagrangian, modified here for the cost minimization problem.

Step 1: Setup the Lagrangian.

Step 2: Take first order conditions.

Step 3: Simplify the first order conditions to get an equation relating K and L.

Step 4: Plug that relationship into the production function and solve for K^* and then L^*.

3.2 Example

This example shows all of the algebra to solve the cost minimization solution we covered, using the Lagrangian method. Let's solve now for a general production function and general input costs. Let $y = L^\alpha K^\beta$ and $C = wL + rK$.

Step 1: Setup the Lagrangian:

$$\mathcal{L}(L, K, \lambda) = wL + rK + \lambda[\bar{y} - L^\alpha K^\beta]$$

Step 2: Take first order conditions and set them equal to zero.

$$\begin{aligned}
\text{FOC L:} \quad & \frac{\partial \mathcal{L}}{\partial L} = w + \lambda(-1)(\alpha L^{\alpha-1} K^\beta) = 0 \\
\text{FOC K:} \quad & \frac{\partial \mathcal{L}}{\partial K} = r + \lambda(-1)(\beta L^\alpha K^{\beta-1}) = 0 \\
\text{FOC } \lambda\text{:} \quad & \frac{\partial \mathcal{L}}{\partial \lambda} = \bar{y} - L^\alpha K^\beta = 0
\end{aligned}$$

Step 3: Simplify the first order conditions (FOC L and FOC K) by solving each for λ and setting $\lambda = \lambda$ to get an equation relating K and L.

Simplifying FOC L gives us:

$$w = \lambda(\alpha L^{\alpha-1} K^{\beta})$$
$$\rightarrow \lambda = \frac{w}{\alpha L^{\alpha-1} K^{\beta}}$$

Simplifying FOC K gives us:

$$r = \lambda(\beta L^{\alpha} K^{\beta-1})$$
$$\rightarrow \lambda = \frac{r}{\beta L^{\alpha} K^{\beta-1}}$$

Setting the two λs equal to each other gives us:

$$\frac{w}{\alpha L^{\alpha-1} K^{\beta}} = \frac{r}{\beta L^{\alpha} K^{\beta-1}} \tag{1}$$

This is what we discussed in class that amounts to:

$$\frac{w}{MP_L} = \frac{r}{MP_K}$$

At the **optimal** combination of K and L that minimizes costs for any level of production, the cost per unit produced with capital and the cost per unit produced with labor have to be equal. Otherwise, the firm could produce at a lower cost by switching around its mix of inputs.

Our goal here is to find how K^* relates to L^* (and vice versa), so we need to further rearrange the tangency condition to get something we can work with more easily. Starting from (1) above, we can rearrange as follows:

$$\frac{\beta w}{\alpha r} = \frac{L^{\alpha-1} K^{\beta}}{L^{\alpha} K^{\beta-1}}$$
$$\rightarrow \frac{\beta w}{\alpha r} = L^{\alpha-1-\alpha} K^{\beta-\beta+1} = L^{-1} K^{1} = \frac{K}{L}$$

This gives us a nice simple version of the tangency condition:

$$K^* = \frac{\beta w}{\alpha r} L^*$$

Step 4: Plug that relationship into the production function and solve for K^* and L^*.

Remember, we want to know how much L and K will be used to produce a given level of output. The relationship we just found just tells us how K and L relate to each other *wherever* costs are minimized. To ground ourselves to a sepcific output level, plug that relationship into the production function:

$$\begin{aligned}
\bar{y} &= L^{\alpha} K^{\beta} = L^{\alpha} \left(\frac{\beta w}{\alpha r} L \right)^{\beta} \\
\bar{y} &= \left(\frac{\beta w}{\alpha r} \right)^{\beta} L^{\alpha+\beta} \\
L^{\alpha+\beta} &= \bar{y} \left(\frac{\alpha r}{\beta w} \right)^{\beta} \\
\rightarrow L^*(w, r, \bar{y}) &= \bar{y}^{\frac{1}{\alpha+\beta}} \left(\frac{\alpha r}{\beta w} \right)^{\frac{\beta}{\alpha+\beta}}
\end{aligned}$$

That's L^*! It tells us **given the cost of inputs and the desired level of output**, how much L to use. To find K^* remember that we already have an equation relating the two:

$$\begin{aligned}
K^* &= \frac{\beta w}{\alpha r} L^* \\
&= \frac{\beta w}{\alpha r} \bar{y}^{\frac{1}{\alpha+\beta}} \left(\frac{\alpha r}{\beta w} \right)^{\frac{\beta}{\alpha+\beta}} \\
&= \left(\frac{\beta w}{\alpha r} \right)^{\frac{\alpha+\beta}{\alpha+\beta}} \bar{y}^{\frac{1}{\alpha+\beta}} \left(\frac{\alpha r}{\beta w} \right)^{\frac{\beta}{\alpha+\beta}} \\
&= \bar{y}^{\frac{1}{\alpha+\beta}} \left(\frac{\beta w}{\alpha r} \right)^{\frac{\alpha+\beta}{\alpha+\beta}} \left(\frac{\beta w}{\alpha r} \right)^{\frac{-\beta}{\alpha+\beta}} \\
\rightarrow K^*(w, r, \bar{y}) &= \bar{y}^{\frac{1}{\alpha+\beta}} \left(\frac{\beta w}{\alpha r} \right)^{\frac{\alpha}{\alpha+\beta}}
\end{aligned}$$

This procedure gives you the conditional input demand functions: the expressions for K^* and L^* that we found above. Let's make our lives easier for the sake of this example and assume that $\alpha + \beta = 1$. This means that the conditional input demand functions simplify to:

$$\begin{aligned} L^*(w, r, \bar{y}) &= \bar{y} \left(\frac{\alpha r}{\beta w} \right)^{\beta} \\ K^*(w, r, \bar{y}) &= \bar{y} \left(\frac{\beta w}{\alpha r} \right)^{\alpha} \end{aligned}$$

- Finding Cost Functions:
  - The Cost Function is:
  - To find Average Cost?

$$\begin{aligned} AVC &= \frac{C^*}{y} \\ &= \frac{\left[\left(\frac{\alpha}{\beta} \right)^{\beta} + \left(\frac{\beta}{\alpha} \right)^{\alpha} \right] w^{\alpha} r^{\beta} \bar{y}}{\bar{y}} \qquad = \left[\left(\frac{\alpha}{\beta} \right)^{\beta} + \left(\frac{\beta}{\alpha} \right)^{\alpha} \right] w^{\alpha} r^{\beta} \end{aligned}$$

  - To find Marginal Cost?

$$\begin{aligned} MC &= \frac{\partial C^*}{\partial y} \\ &= \frac{\partial \left[\left(\frac{\alpha}{\beta} \right)^{\beta} + \left(\frac{\beta}{\alpha} \right)^{\alpha} \right] w^{\alpha} r^{\beta} \bar{y}}{\partial \bar{y}} \qquad = \left[\left(\frac{\alpha}{\beta} \right)^{\beta} + \left(\frac{\beta}{\alpha} \right)^{\alpha} \right] w^{\alpha} r^{\beta} \end{aligned}$$

Why does it make sense to us given this context that the AVC = MC? [Hint: look back at the production function... does the production process exhibit increasing, constant, or decreasing returns to scale?]

4 Problems

(back to Table of Contents)

** 5.4.1 *Cost Minimization using the Lagrangian*

This problem is a copy of 5.3.1 with two additional subparts (f and g) for you to solve. Start from your answer to 5.3.1! Suppose a firm's production function is $y = 10K^{\frac{1}{2}}L^{\frac{1}{2}}$. The cost of a unit of labor is \$20 and the cost of a unit of capital is \$80. They have no other inputs or costs.

(a) The firm is currently producing 100 units of output and has determined that the cost-minimizing quantities of labor and capital are 20 and 5, respectively. Graphically illustrate this using isocost lines and isoquants and write a description of what your graph is showing. Show at least two isocost lines as you explain what the firm has chosen.

(b) The firm now wants to increase production to 140 units. If capital is fixed in the short run, how much labor will the firm require? On a graph showing labor and ouptut (y), show this as movement along the firm's production function (remember capital is fixed, so the production function is only in terms of labor here).

(c) What is the change in the firm's total cost?

(d) On your graph from part a), graphically show the optimal level of capital and labor in the *long run* if the firm wants to produce 140 units of output.

(e) Use Method 1 (the tangency condition) to solve for numerical values of the optimal quantities of capital and labor for this level of production. the below will be for Friday

(f) Now use Method 2 (the Lagrangian) to solve for the optimal input demand functions for a general level of production $\bar{y}$. Confirm that these give you the same answer you got above for $y = 140$.

(g) Write down and graph on the same set of axes the total cost function, the average cost function, and the marginal cost function for this firm and assuming the prices given above.

** 5.4.2 *Cost Min Part 2*

Consider a firm that uses capital K and labor L to produce its output y, with the following production function:

$$y = f(K, L) = K^{\frac{1}{2}}L^{\frac{1}{4}}$$

(a) Use the production function to determine algebraically whether the function exhibits increasing, constant, or decreasing returns to scale. Show your work and explain.

(b) Does the firm experience diminishing marginal returns to labor and to capital? (Hint: find the marginal products first).

(c) Assume that w and r, the costs of the the inputs, are given. Write down the Lagrangian for the firm's cost minimization problem.

(d) Take first order conditions and solve for the conditional input demand functions.

(e) Using your input demand functions, find the firm's total cost function.

(f) Use your answer above to find the firm's average total cost and marginal cost functions.

(g) Now assume that $w = \$1$ and $r = \$1$. Write down and graph the three cost functions on the same set of axes.

(h) Why are the average cost and marginal cost functions *different* for this problem than for problem 5.4.1, even though the production functions are both Cobb-Douglas?