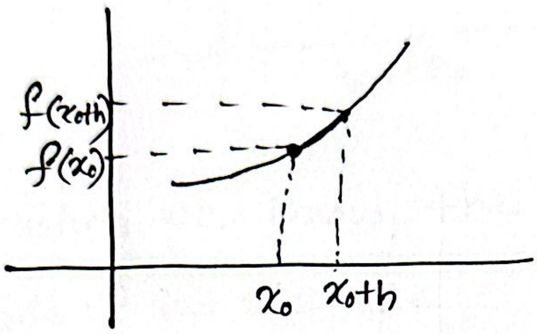
Numerical differentiation

Forward difference (F D):



We use it when we know the current node and future node.

h = step size

F.D, $$f'(x) = \frac{f(x_0+h) - f(x_0)}{h}$$

Example: The given function $f(x) = x^2 + 10x$. Find $f'(x)$ using forward difference at $x=2$, $h=0.1$.

Actual $= f'(x) = 2x + 10$

$f'(2) = 2\times 2 + 10 = 14$

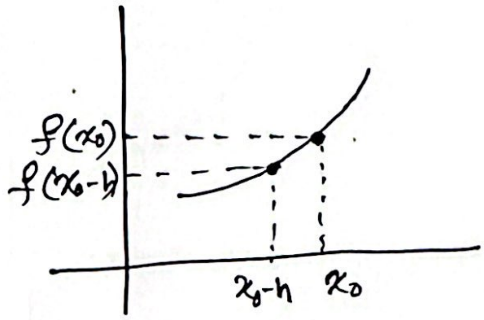
Using formula $$f'(2) = \frac{f(2+0.1) - f(2)}{0.1}$$

$$= \frac{f(2.1) - f(2)}{0.1}$$

$$= \frac{(2.1)^2 + 10\times 2.1 - 4 - 10\times 2}{0.1}$$

$$= 14.1$$

Backward difference (B.D):



We use it when we know the current node and previous node. $h =$ step size

$$B.D, \quad f'(x) = \frac{f(x_0) - f(x_0-h)}{h}$$

Example: $f(x) = x^2 + 10x$. Find $f'(x)$ using backward differance at $x = 2$, $h = 0.1$

$$\text{Actual} = f'(x) = 2x + 10$$

$$f'(2) = 2 \times 2 + 10 = 14$$

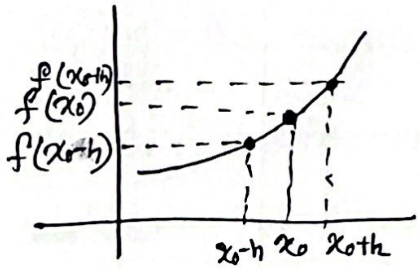
Using formula, $f'(2) = \dfrac{f(2) - f(1.9)}{0.1}$

$$= \frac{2^2 + 10 \times 2 - (1.9)^2 - 10 \times 1.9}{0.1}$$

$$= \frac{4 + 20 - 3.61 - 19}{0.1}$$

$$= 13.9$$

Central difference (C.D):



We use it when we know the current node and previous and future node. CD, $f'(x) = \dfrac{f(x_0+h) - f(x_0-h)}{2h}$

Example: $f(x) = x^2 + 10x$. Find $f'(x)$ using central difference at $x = 2$, $h = 0.1$

$$\text{Actual} = f'(x) = 2x + 10 = 2\times 2 + 10 = 14$$

Using formula, $f'(2) = \dfrac{f(2.1) - f(1.9)}{2\times 0.1}$

$$= \frac{(2.1)^2 + 10\times 2.1 - (1.9)^2 - 10\times 1.9}{2\times 0.1}$$

$$= \frac{4.41 + 21 - 3.61 - 19}{0.2}$$

$$= 14$$

Comparing all three values, we can see that central difference gives best approximation than forward and backward difference.

F.D = 14.1 B.D = 13.9 $\boxed{CD = 14}$

Actual = 14

From our observation we can see that the derivate calculated using the numerical diffenence methods are just an approximation. There are some errors to this value.

Forward difference, $$f'(x) = \underbrace{\frac{f(x+h)-f(x)}{h}}_{\text{Appoximate value.}} - \underbrace{\frac{f''(\xi)(h)}{2!}}_{\text{truncation Error/ upper bound error}}$$

Backward difference, $$f'(x) = \frac{f(x)-f(x-h)}{h} - \frac{f''(\xi)h}{2!}$$

Central difference, $$f'(x) = \frac{f(x+h)-f(x-h)}{2h} - \frac{f'''(\xi)h^2}{3!}$$

From here we get

For forward & backward diffenence

Error $\propto h$ Error $\uparrow$ $h\uparrow$

$h\downarrow$ Error$\downarrow$

For central diffenence

Error $\propto h^2$

Example 1:

1. Given, $f(x) = \ln(x)$

$x = 2$

$h = 1, 0.1, 0.01, 0.001$

find $f'(2)$ and the truncation error using forward difference.

$f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$

$f'(2) = \frac{1}{2}$

$= 0.5$

F.D

$f'(2) = \frac{f(2+h) - f(2)}{h}$

$= \frac{\ln(2+h) - \ln(2)}{h}$

h	$f'(2)$ [F.D]	Truncation Error
1	0.405465	0.5 − 0.405465 = 0.09453
0.1	0.487902	0.5 − 0.487902 = 0.012098
0.01	0.498754	0.5 − 0.498754 = 0.001246
0.001	0.499875	0.5 − 0.499875 = 0.00012

Here h is decreasing and also error is decreasing.

$\text{error} \propto h$

Example 2

Given function $f(x) = 2x^2 - e^x$. find $f'(2)$ using central difference. Also, find the truncation error.

$[h = 0.1, 0.01, 0.001]$

$$f(x) = 2x^2 - e^x$$

$$f'(x) = 4x - e^x$$

$$f'(2) = 4 \times 2 - e^2$$

$$= 0.61094$$

$$f'(2) = \frac{f(2+h) - f(2-h)}{2h}$$

h	$f'(2)$ using C.D	Truncation Error
0.1	0.598626	0.0123179
0.01	0.610820	0.00012
0.001	0.6109439	-0.000003

Example 3

x_0	4.0	4.1	4.2	4.3	4.4
$f(x_0)$	16	18	20	21	22

Now, using Forward difference and backward difference calculate $f'(4.2)$

Here we notic that h is not given.

h mainly indicates the difference between the nodes

Here, $h = 4.1 - 4.0 = 0.1$

For forward differance

$$f'(4.2) = \frac{f(4.2+0.1) - f(4.2)}{0.1}$$

$$= \frac{f(4.3) - f(4.2)}{0.1}$$

$$= \frac{21 - 20}{0.1} = \frac{1}{0.1} = 10$$

For backward difference

$$f'(4.2) = \frac{f(4.2) - f(4.2-0.1)}{0.1}$$

$$= \frac{f(4.2) - f(4.1)}{0.1} = \frac{20-18}{0.1} = 20.$$

Example

$$V(t) = 2000 \ln\left[\frac{14\times 10^4}{14\times 10^4 - 2100t}\right] - 9.8t$$. Calculate the value of $v'(16)$ using forward, backward, central difference. Also given, $\boxed{\Delta t = 2s \text{ [given]}}$

step size

Using forward difference,

$$F.D = \frac{V(16+2) - v(16)}{2} = \frac{453.0214897 - 392.073}{2}$$

$$= 30.474$$

Using backward difference,

$$B.D = \frac{V(16) - V(16-2)}{2} = \frac{392.073 - 334.244}{2}$$

$$= 28.9145$$

Using central difference.

$$C.D = \frac{V(16+2) - V(16-2)}{2\times 2} = \frac{453.0214897 - 334.244}{2\times 2}$$

$$= 29.69437$$

Upper bound of Truncation Error

Given function, $f(x) = x\sin x + x^2 \cos x$

x	1.0	1.2	1.4
$f(x)$	1.2717	1.3642	1.5496

step size, $h = 0.2$

1) Calculate $f'(1.2)$ using Central Difference ?

2) Using the above mentioned function, Compute the truncation error or Upperbound error bound using $\xi \rightarrow [1.0, 1.4]$

$\rightarrow$ [If this is not given, the range of x should be taken]

1) $$f'(1.2) = \frac{f(1.2+0.2) - f(1.2-0.2)}{2\times 0.2}$$

$$= \frac{f(1.4) - f(1)}{2\times 0.2}$$

$$= \frac{1.5496 - 1.2717}{2\times 0.2}$$

$$= 0.69475$$

2) Error $= \left| \dfrac{f'''(\xi)}{3!} (h)^2 \right|$

$$f(x) = x \sin x + x^2 \cos x$$

$$f'(x) = x\cos x + \sin x + 2x\cos x - x^2 \sin x$$

$$= 3x\cos x + \sin x - x^2 \sin x$$

$$f''(x) = 3\cos x - 3x\sin x + \cos x - 2x\sin x - x^2\cos x$$

$$= 4\cos x - 5x\sin x - x^2\cos x$$

$$f'''(x) = -4\sin x - 5\sin x - 5x\cos x - 2x\cos x + x^2\sin x$$

$$= -9\sin x - 7x\cos x + x^2\sin x$$

Error $= \left| \dfrac{f'''(\xi)}{3!} h^2 \right|$

$$= \left| \frac{9\sin(\xi) + 7(\xi)\cos(\xi) + (\xi)^2 \sin x(\xi)}{6} \times (0.2)^2 \right|$$

$$= \left| \frac{9\sin(1.4) + 7(1.4)\cos(1) + (1.4)^2 \sin(1.4)}{6} \times 0.2^2 \right|$$

Proof of the formula of forward difference

We will apply forward difference of this function, $f(x)$.

nodes

$x_0 \quad x \Rightarrow x_0$

$x_1 \quad x+h \Rightarrow x_0+h$

Using lagrange:

$$f(x) = P_1(x) + \text{Error}$$

$$P_1(x) = f(x_0)\, l_0(x) + f(x_1)\, l_1(x)$$

$$f(x) = f(x_0)\frac{x-x_1}{x_0-x_1} + f(x_1)\frac{x-x_0}{x_1-x_0} + \frac{f''(\xi)}{2}(x-x_0)(x-x_1)$$

$$f'(x) = f(x_0)\frac{1}{x_0-x_1} + f(x_1)\frac{1}{x_1-x_0} + \frac{f'''(\xi)}{2}\frac{d}{dx}(\xi)(x-x_0)(x-x_1)$$

$$+ \frac{f''(\xi)}{2}(2x-x_0-x_1)$$

$$f'(x_0) = \frac{f(x_0)}{x_0-x_0-h} + \frac{f(x_1)}{x_0+h-x_0} + \frac{f'''(\xi)}{2}\frac{d}{dx}(\xi)(x_0-x_0)(x_0-x_1)$$

$$+ \frac{f''(\xi)}{2}(2x_0-x_0-x_0-h)$$

$$= -\frac{f(x_0)}{h} + \frac{f(x_0+h)}{h} + 0 - \frac{f''(\xi)}{2}\times h$$

$$= \frac{f(x_0+h)-f(x_0)}{h} - \frac{f''(\xi)}{2}\times h$$

(Proved)

Till now, we have seen that if step size decreases, truncation error also decreases. What about rounding error?

Especially for central difference,

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

→ smaller h, better result

→ if h is very small, $f(x+h)$ and $f(x-h)$ will have similar values.

→ Subtracting 2 similar values/close values. gives "loss of significance" → chapter 1

→ Therefore rounding error increases.

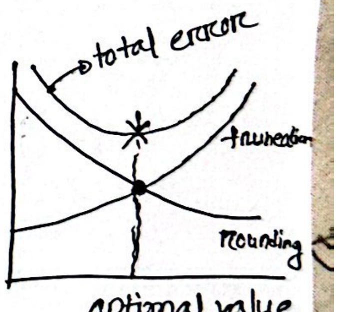
From chapter 1:

$$\delta = \frac{|fl(x) - x|}{|x|}$$

$$fl(x) = (1+\delta)x$$

$$fl[f(x_1+h)] = (1+\delta_1) f(x_1+h)$$

$$fl[f(x_1-h)] = (1+\delta_2) f(x_1-h)$$



optimal value of h

which value of h to choose?

Error = | actual value of differentiation − value of differentiation by numerical approach |

$$\text{Error} \leq \underbrace{\left|\frac{f'''(\xi)\, h^2}{6}\right|}_{\text{truncation}} + \underbrace{\varepsilon_M \frac{|f(x_1+h) - f(x_1-h)|}{2h}}_{\text{rounding error}}$$