Hermite Interpolation

* It's another name is derivative Conditions.

Suppose we have $[x_0, x_1, x_2 - - - -, x_n]$ nodes.

Total $(n+1)$ nodes

$[f(x_0), f(x_1), f(x_2) - - - - f(x_n)] \to (n+1)$ conditions

$[f'(x_0), f'(x_1), f'(x_2) - - - - f'(x_n)] \to (n+1)$ conditions

$2n+2$ conditions

$n+1$ nodes/condition $\Rightarrow$ degree n

$2n+2$ nodes/condition $\Rightarrow$ degree $2n+1$

$\boxed{n+n+1}+1$

Previously, only one condition used to be satisfied the condition,

$$P(x_i) = f(x_i)$$

now along with this condition $f'(x_i)$ also used.

Before, for $(n+1)$ nodes, degree $= n$

Now, for $(n+1)$ nodes, degree $= 2n+1$

Hermite Interpolation,

$$P_{2n+1}(x) = \sum_{K=0}^{n} h_K(x) \cdot f(x_K) + \hat{h}_K(x) \cdot f'(x_K)$$

- $h_K(x) = \left[1 - 2(x - x_K)\{l_K'(x_K)\}\right] \{l_K(x)\}^2$
- $\hat{h}_K(x) = (x - x_K) \{l_K(x)\}^2$

Example:

Given function, $f(x) = \sin x$ and given nodes $\{0, \pi/2\}$

Here, $n = 1$ and nodes $= 2$

x	$f(x)$	$f'(x)$
0	0	1
$\pi/2$	1	0

$f(x) = \sin x$

$f'(x) = \cos x$

$$P_{2\times 1+1}(x) = P_3(x) = h_0(x) f(x_0) + \hat{h}_0(x) f'(x_0) + h_1(x) f(x_1) + \hat{h}_1(x) f'(x_1)$$

$$= h_0(x) \times 0 + \hat{h}_0(x) \times 1 + h_1(x) \times 1 + \hat{h}_1(x) \times 0$$

$$= \hat{h}_0(x) + h_1(x)$$

$$= h_1(x) + \hat{h}_0(x)$$

$$= \left[1 - 2(x - \pi/2) \times \frac{2}{\pi}\right] \left(\frac{2x}{\pi}\right)^2 + x \left(\frac{x - \pi/2}{\pi/2}\right)^2$$

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$$h_1(x) = \left[1 - 2(x - x_1)\{l_1'(x_1)\}\right]\{l_1(x)\}^2$$

$$l_1'(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 0}{\pi/2 - 0}$$

$$= \frac{2x}{\pi}$$

$$l_1(x) = \frac{2x}{\pi}$$

$$\boxed{l_1'(x) = \frac{2}{\pi}}$$

- $l_1'(x_1) = \frac{2}{\pi}$

$$\boxed{l_1'(\pi/2) = \frac{2}{\pi}}$$

$$h_1(x) = \left[1 - 2(x - \pi/2) \times \frac{2}{\pi}\right]\left(\frac{2x}{\pi}\right)^2$$

$$\hat{h}_0(x) = (x - x_0)\, l_0^2(x)$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - \pi/2}{0 - \pi/2}$$

$$\hat{h}_0(x) = (x - 0)\left(\frac{x - \pi/2}{-\pi/2}\right)^2$$

$$= x\left(\frac{x - \pi/2}{\pi/2}\right)^2$$

Example 2

x	$f(x)$	$f'(x)$
-1	1	2
0	0	2
1	1	0

degree, $n = 2$

$$P_{2n+1}(x) = P_{2\times2+1}(x) = P_5(x) = h_0(x)f(x_0) + h_1(x)f(x_1) + h_2(x)f(x_2)$$
$$+ \hat{h}_0(x)f'(x_0) + \hat{h}_1(x)f'(x_1) + \hat{h}_2(x)f'(x_2)$$

$$= h_0(x)\times1 + h_1(x)\times0 + h_2(x)\times1 + \hat{h}_0(x)\times2$$
$$+ \hat{h}_1(x)\times2 + \hat{h}_2(x)\times0$$

$$= h_0(x) + h_2(x) + 2\hat{h}_0(x) + 2\hat{h}_1(x)$$

$$h_0(x) = \left[1-2(x-x_0)\{l_0'(x_0)\}\right]\{l_0(x)\}^2$$

$$l_0(x) = \frac{x-x_1}{x_0-x_1}\times\frac{x-x_2}{x_0-x_2}$$

$$= \frac{x-0}{-1-0}\times\frac{x-1}{-1-1}$$

$$= \frac{x(x-1)}{2}$$

$$= \frac{x^2-x}{2}$$

$$l_0'(x) = \frac{1}{2}(2x-1)$$

$$l_0'(x_0) = \frac{1}{2}(2\times(-1)-1) = \frac{1}{2}(-2-1) = -3/2$$

$$h_0(x) = \left[1-2(x+1)\left(-3/2\right)\right] \times \left(\frac{x^2-x}{2}\right)^2$$

Now, $h_2(x) = \left[1-2(x-x_2)\left(l_2'(x_2)\right)\right]\{l_2(x)\}^2$

$$l_2(x) = \frac{x-x_0}{x_2-x_0} \times \frac{x-x_1}{x-x_1}$$

$$= \frac{x+1}{1+1} \times \frac{x-0}{1-0}$$

$$= \frac{x^2+x}{2}$$

$$l_2'(x) = \frac{1}{2}(2x+1)$$

$$l_2'(x_2) = \frac{1}{2}(2x_2+1)$$

$$l_2'(1) = \frac{1}{2}(2\times 1+1) = 3/2$$

$$h_2(x) = \left[1-2(x-1)\, 3/2\right]\left(\frac{2x+1}{2}\right)^2$$

$$\hat{h}_0(x) = (x-x_0)\{l_0(x)\}^2$$

$$= (x+1)\left(\frac{x^2-x}{2}\right)^2$$

$$\hat{h}_1(x) = (x-x_1)\{l_1(x)\}^2$$

$$l_1(x) = \frac{x-x_0}{x_1-x_0} \times \frac{x-x_2}{x_1-x_2}$$

$$= \frac{x+1}{0+1} \times \frac{x-1}{0-1}$$

$$= 1-x^2$$

$$\hat{h}_1(x) = (x-0)\left(1-x^2\right)^2$$

$$= x\left(1-x^2\right)^2.$$

$$\therefore P_{2\times2+1}(x) = P_5(x) = h_0(x) + h_2(x) + 2\hat{h_0}(x) + 2\hat{h_1}(x)$$

$$= \left[1-2(x+1)\left(-3/2\right)\right] \times \left(\frac{x^2-x}{2}\right)^2 + \left[1-2(x-1)\,3/2\right] \left(\frac{2x+1}{2}\right)^2 + 2(x+1)\left(\frac{x^2-x}{2}\right)^2 + 2x\left(1-x^2\right)^2.$$

Advantage:

According to the weierstrass theorem, $|f(x)-P_n(x)|$ a centain errror is gnerated. If we increase the nodes, the error will decrease. In our first Example nodes $= 2$ so degree $= 1$. However using hermeit with same node we get degree $2n+1 = 2\times1+1 = 3$. So we can decrease the error using same number of data points.