Least Square Method

We use this approach to solve overdetermined system. In overdetermined system A is not a square matrix instead it is a $m \times n$ matrix but to solve this we need to obtain a square matrix.

coefficient matrix

$$\underset{m \times n}{A} x = b$$

$$\underset{n \times m}{A^T} \; \underset{m \times n}{A} \; x = A^T b$$

[By multiplying with transpose matrix on both side]

$$n \times m \quad m \times n$$

$$\boxed{n \times n}$$

square matrix

In this way we can solve the problem of overdetermined system and obtain a square matrix.

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Example

$$\left.\begin{aligned} f(-3) &= 0 \\ f(0) &= 0 \\ f(6) &= 2 \end{aligned}\right\}$$

$P_2(x) = a_0 + a_1 x + a_2 x^2$

$\hookrightarrow$ degree is 2

In order to make this an overdetermined system

Let degree $= 1$

then

$P_1(x) = a_0 + a_1 x$

$P_1(-3) = a_0 + a_1(-3) = 0$

$P_1(0) = a_0 + a_1 \times (0) = 0$

$P_2(6) = a_0 + a_1 \times (6) = 2$

coefficient matrix, $A = \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix}$

$$A^T A x = A^T b$$

$$\begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 45 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

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$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 45 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

$a_0 = 3/7$

$a_1 = 5/21$

[We have done it using inverse matrix]

$$P_1(x) = 3/7 + 5/21\, x$$

* Note : if you want to solve it using guassian elimination or LU Decomposition, you can. Also, in the exam follow the instruction.

Example 2

$$a_0 + 2a_1 + 4a_2 = 3$$
$$a_0 + 3a_1 + 9a_2 = 5$$
$$a_0 + 5a_1 + 25a_2 = 12$$
$$a_0 + 6a_1 + 36a_2 = 15$$

num of equations $\neq$ num of variables

4 $\neq$ 3

Using least square method:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix}$$

$$A^T A x = A^T b$$

$$\underset{3\times 4}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 6 \\ 4 & 9 & 25 & 36 \end{bmatrix}} \underset{4\times 3}{\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix}} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 6 \\ 4 & 9 & 25 & 36 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 12 \\ 15 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 4 & 16 & 74 \\ 16 & 74 & 376 \\ 74 & 376 & 2018 \end{bmatrix}}_{3\times 3 \;\to A} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 35 \\ 171 \\ 879 \end{bmatrix}}_{b}$$

Now we will solve this using guassian elimination

$$\left[\begin{array}{ccc|c} 4 & 16 & 74 & 35 \\ 16 & 74 & 376 & 171 \\ 74 & 376 & 2018 & 879 \end{array}\right] \quad \begin{array}{l} R_2 = R_2 - \left(\frac{16}{4}\right) R_1 \\ R_3 = R_3 - \left(\frac{74}{4}\right) R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 4 & 16 & 74 & 35 \\ 0 & 10 & 80 & 31 \\ 0 & 80 & 649 & 231.5 \end{array}\right] \quad R_3 = R_3 - \left(\frac{80}{10}\right) R_2$$

$$= \begin{array}{c} \begin{array}{cccc} a_0 & a_1 & a_2 & \end{array} \\ \left[\begin{array}{ccc|c} 4 & 16 & 74 & 35 \\ 0 & 10 & 80 & 31 \\ 0 & 0 & 9 & -16.5 \end{array}\right] \end{array}$$

$$9a_2 = -16.5$$

$$\therefore a_2 = -1.83$$

$$10a_1 + 80a_2 = 31$$

$$a_1 = \frac{31 - (80 \times -1.83)}{10}$$

$$= 17.74$$

$$4a_0 + 16a_1 + 74a_2 = 35$$

$$a_0 = -28.355$$

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