Interpolation Error

$$\text{Error} = |f(x) - P_n(x)|$$

From weierstrass Theorem, we know that if we increase the degree of polynomial, then the error reduces.

$$|f(x) - P_2(x)| > |f(x) - P_{200}(x)|$$

But it is not true for all functions.

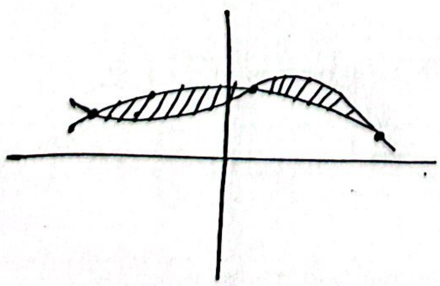
Convergence

$\uparrow$ nodes $\downarrow$ error [We know]

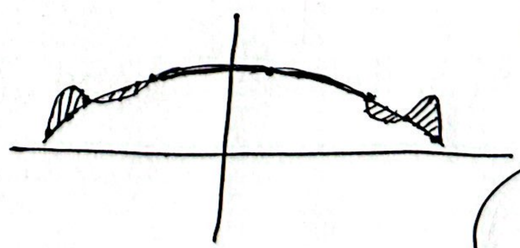
if nodes $= \infty$, error $= 0$

Let's take a function, $f(x) = \dfrac{1}{1+25x^2}$ on $[-1, 1]$

nodes $= 3$

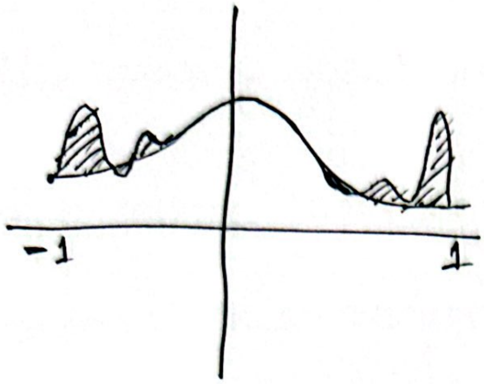


nodes $= 7$

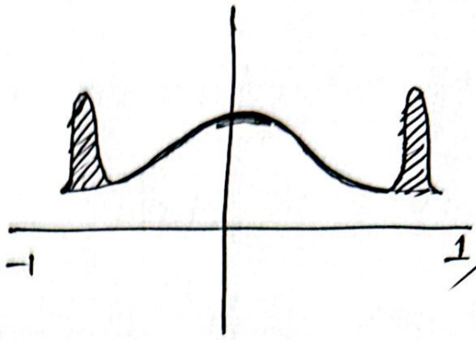


equally Spaced

nodes = 12



nodes = 18



equally spaced

From here, we can notice that the error is decreasing in the middle when we increase the equally space nodes but it diverging more at the ends → the interval. There are spikes at the end of the polynomial specially at the interval point -1 and 1. This phenomena is known as Runge phenomena.

Runge phenomena occurs:

1) Depends on the function. Specially it occurs on symmetric functions.

2) Depends on nodes. Specially it occurs because of taking equally spaced nodes.

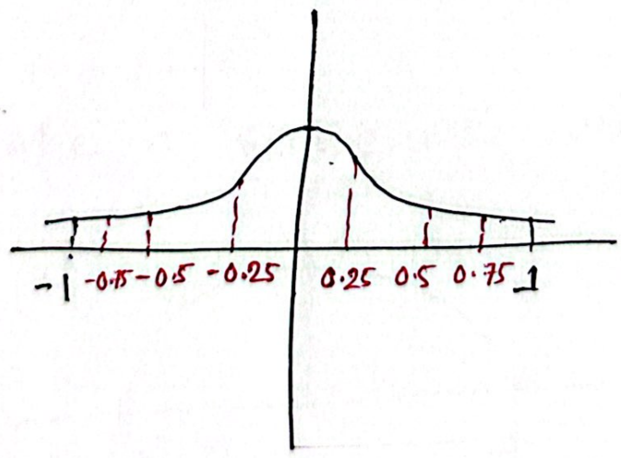
As we noticed the error is occuring at the corner point or at the interval, we can take more nodes at those points to avoid errors. This mean we cann't take equally spaced node.

Solution of these problems

1) Piece wise Interpolation:

Here we have to take small intervals rathe than taking the whole, then interpolate. Lastly add them up.

Given intervals
$[-1,1]$



The more we divide the intervals then interpolate and merge, the better the result willbe.

2) Take non equal distant nodes. This is known as (Chebyshev Nodes.)

- We will take more nodes at end points.
- Rather than taking equidistant nodes, we will take equal angled nodes.

Chebyshev nodes for Runge functions.

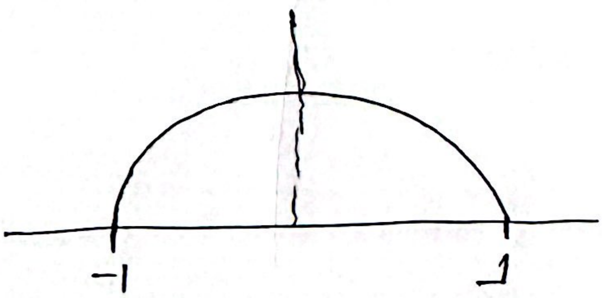
Given, $f(x) = \frac{1}{4+3x^2}$ $[-1, 1]$ interval

Step 1:

Draw a line as per the interval

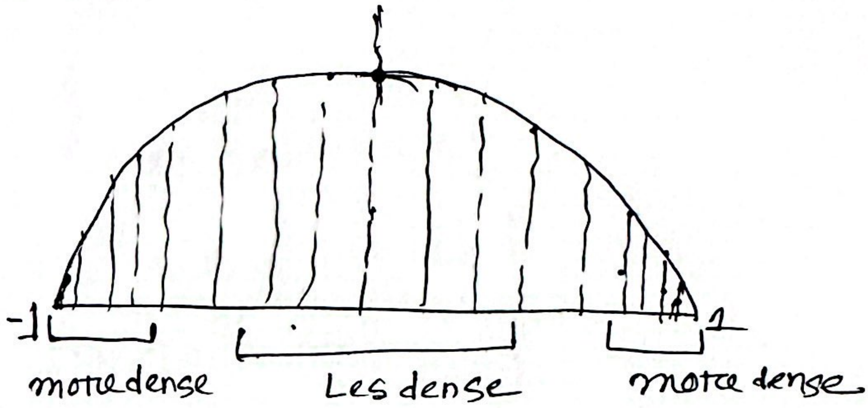
Step 2:

~~Sa~~ make a semicircle with the end points

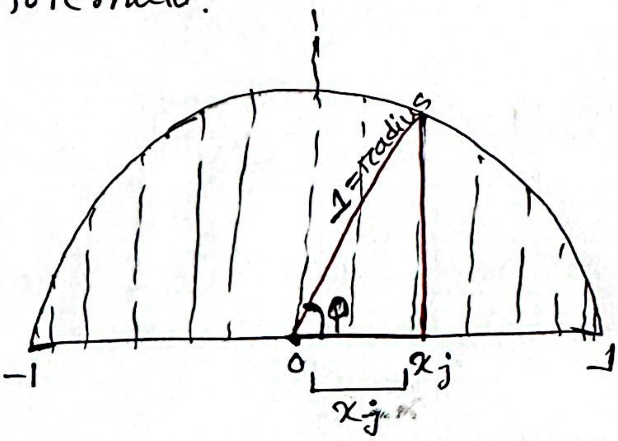


Step 3

Now equally distant nodes must be taken on the circumference. Then draw verticle lines



Now the formula:



root node

$$\varphi_j = \frac{(2j+1)\pi}{2(n+1)}$$

degree

$$\cos\varphi_j = \frac{x_j}{\text{radius}}$$

$$x_j = \cos\frac{(2j+1)\pi}{2(n+1)} \times \text{radius} + \text{center}.$$

Example:

Given, $f(x) = \frac{1}{1+25x^2}$, $\underbrace{[-1, 1]}_{\text{interval}}$, $n = 3$ ↳ degree

$n = 3$ means nodes $= 4$

So, $j = 0, 1, 2, 3$ (Total 4 nodes)
$[x_0, x_1, x_2, x_3]$

$$x_0 = 1 \times \left(\cos \frac{(2 \times 0 + 1)\pi}{2(3+1)}\right) + 0 = \cos \pi/8$$

$$x_1 = 1 \times \left(\cos \frac{(2 \times 1 + 1)\pi}{2(3+1)}\right) + 0 = \cos 3\pi/8$$

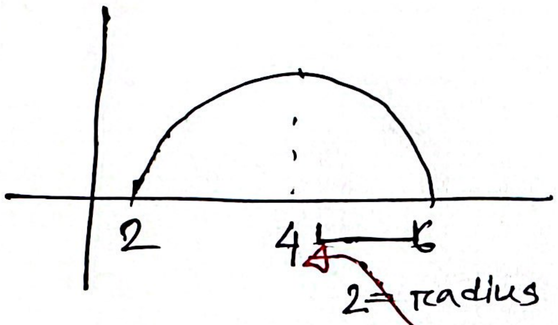
$$x_2 = 1 \times \left(\cos \frac{(2 \times 2 + 1)\pi}{2 \times 4}\right) + 0 = \cos 5\pi/8$$

$$x_3 = 1 \times \left(\cos \frac{(2 \times 3 + 1)\pi}{2 \times 4}\right) + 0 = \cos 7\pi/8$$

Now by using these nodes you can find the polynomial using any method.

Example 2

$$f(x) = \frac{1}{2+3x^2} \qquad n = 3 \qquad [2, 6]$$



$$\text{center} = \frac{6+2}{2} = 4$$

then

$$x_j = r\cos\phi_j + \text{center}$$

$$= r\cos\phi_j + \text{center}$$

$$= 2\cos\phi_j + 4$$