Linear System of equations

↳ highest power of x will be 1.

(*) (System of linear equations (exponent of all variables must be 1))

General form of linear System of equation

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad\qquad\qquad \vdots \quad & \quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

To solve this we will learn three methods

1) Inverse Matrix
2) Gaussian Elimination
3) LU Decomposition

1) Inverse Matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Augment matrix/ coefficient Matrix, A

$n \times n$

X $n \times 1$

b $n \times 1$

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 2x_2 + 2x_3 = 4$$
$$2x_1 + 12x_2 - 2x_3 = 4$$

first we need to create Augment matrix, A

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}}_{b}$$

$$Ax = b$$

$$x = \boxed{A^{-1}} b$$

$\rightarrow$ inverse.

There are two rules :

1) A coefficient matrix should be a square matrix . $(n \times n)$

2) $\det(A) \neq 0$, [non singular Matrix]

[linear System of equation must follow these rules otherwise we can't solve the problem]

To find inverse matrix computational cost is high. So we want to avoid this and introduce new method called Gaussian elimination.

Gaussian Elimination

Example:

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 2x_2 + 2x_3 = 4$$
$$2x_1 + 12x_2 - 2x_3 = 4$$

First find augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 2 & 12 & -2 & 4 \end{array}\right]$$

Now we have to convert it into upper triangular matrix.

Example of upper triangular matix

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

we need to do it using row operation.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 2 & 12 & -2 & 4 \end{array}\right]$$

$$R_2 = R_2 - \left(\frac{1}{1}\right) R_1 \quad \text{multiplier}$$
$$= R_2 - 1 \times R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 2 & 12 & -2 & 4 \end{array}\right]$$

$$R_3 = R_3 - \left(\frac{2}{1}\right) R_1 \quad \text{multiplier}$$
$$= R_3 - 2R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & 8 & -4 & 4 \end{array}\right]$$

$$R_3 = R_3 - \left(\frac{8}{-4}\right) R_2$$
$$= R_3 + 2R_2$$

$$= \begin{array}{c} \begin{array}{ccc} x_1 \downarrow & x_2 \downarrow & x_3 \downarrow \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & 0 & -2 & 12 \end{array}\right] \end{array}$$

$$-2x_3 = 12$$
$$\boxed{x_3 = 12/-2 = -6}$$

$$-4x_2 + x_3 = 4$$
$$\boxed{x_2 = \frac{4 - x_3}{-4} = \frac{4+6}{-4} = -2.5}$$

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 + 2 \times (-2.5) - 6 = 0$$
$$\boxed{x_1 = 11}$$

$x_1 = 11$

$x_2 = -2.5$

$x_3 = -6$

Ans

Example 2:

The upward velocity of a rocket is given at three different time in Table 1.

Time (s)	Velocity (ms^{-1})
5	106.8
8	177.2
12	279.2

* The velocity data is approximated by a polynomial as

$$v(t) = b_1 t^2 + b_2 t + b_3 \qquad 5 \leq t \leq 12$$

a) Find the values of b_1, b_2 and b_3 using the Gaussian elimination method.

b) Find the velocity at $t = 7$ seconds.

a) $V(5) = 25b_1 + 5b_2 + 1 = 106.8$

$V(8) = 64b_1 + 8b_2 + 1 = 177.2$

$V(12) = 144b_1 + 12b_2 + 1 = 279.2$

$$\left[\begin{array}{ccc|c} 25 & 5 & 1 & 106.8 \\ 64 & 8 & 1 & 177.2 \\ 144 & 12 & 1 & 279.2 \end{array}\right] \quad \begin{array}{l} R_2 = R_2 - \left(\frac{64}{25}\right)R_1 \\ R_3 = R_3 - \left(\frac{144}{25}\right)R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 25 & 5 & 1 & 106.8 \\ 0 & -4.8 & -1.56 & -96.208 \\ 0 & -16.8 & -4.76 & -335.968 \end{array}\right] \quad R_3 = R_3 - \left(\frac{-16.8}{-4.8}\right)R_2$$

$$= \left[\begin{array}{ccc|c} 25 & 5 & 1 & 106.8 \\ 0 & -4.8 & -1.56 & -96.208 \\ 0 & 0 & 0.7 & 0.76 \end{array}\right]$$

$0.7b_3 = 0.76$

$\boxed{b_3 = 1.085}$

$-4.8b_2 - 1.56\, b_3 = -96.208$

$-4.8\, b_2 - 1.56 \times 1.085 = -96.208$

$\boxed{b_2 = 19.69}$

$25b_1 + 5b_2 + b_3 = 106.8$

$25b_1 + 5 \times 19.69 + 1.085 = 106.8$

$\boxed{b_1 = 0.25976}$

$\therefore v(t) = 0.25976\,t^2 + 19.69t + 1.085$

(Ans)

b) velocity at $t = 7$

$v(7) = 0.25976\,(7)^2 + 19.69 \times 7 + 1.085$

$\simeq 151.64324$

(Ans)