QR decomposition

$$A = Q R$$

It is a $m \times n$ matrix

Q is an orthonormal set of vector which we can find by using $\boxed{\text{Gram-schmidt process.}}$

$$Q = (q_1 \quad q_2 \quad \cdots \quad q_n)$$

Suppose we have a coefficient matrix

$$A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\begin{array}{cc} \downarrow & \downarrow \\ u_1 & u_2 \end{array}$$

Let each column as $u_1, u_2 \cdots u_n$

to find q we need to know two formulas :

$$1.\quad P_K = U_K - \sum_{i=1}^{K-1} (U_K^T q_i)\, q_i$$

$$2.\quad q_K = \frac{P_K}{|P_K|}$$

Gram-schmidt process

Using these formulas (Gram-schmidt) process we will convert U into q.

One by One step to slove the problem

Step 1 : $K = 1$

$$P_1 = U_1 = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$$

We always start $i = 1$ and the summation loop starts from 1 to $K-1$. Here K is 1 so $K-1=0$. This not possible because we can't go from 1 to 0. So $\sum_{i=1}^{K-1} (U_K^T q_i) q_i$ won't be added here for P_1

$$q_1 = \frac{P_1}{|P_1|} = \frac{\begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}}{\sqrt{3^2+6^2+0^2}} = \frac{1}{\sqrt{45}} \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$$

Step 2: $k = 2$

$$P_2 = U_2 - \sum_{i=1}^{k-1=(2-1)} \left(U_2^T q_i\right) q_i$$

$$= U_2 - \sum_{i=1}^{1} \left(U_2^T q_i\right) q_i$$

$$= U_2 - \left(U_2^T q_1\right) q_1$$

$$= \begin{pmatrix}1\\2\\2\end{pmatrix} - \left\{ \underbrace{\begin{pmatrix}1 & 2 & 2\end{pmatrix} \frac{1}{\sqrt{45}} \begin{pmatrix}3\\6\\0\end{pmatrix}} \right\} q_1$$

$$= \begin{pmatrix}1\\2\\2\end{pmatrix} - \frac{15}{\sqrt{45}} \cdot q_1$$

$$= \begin{pmatrix}1\\2\\2\end{pmatrix} - \frac{15}{\sqrt{45}} \cdot \frac{1}{\sqrt{45}} \begin{pmatrix}3\\6\\0\end{pmatrix}$$

$$= \begin{pmatrix}1\\2\\2\end{pmatrix} - \frac{\cancel{15}^{\,1}}{\cancel{45}\,3} \begin{pmatrix}3\\6\\0\end{pmatrix}$$

$$= \begin{pmatrix}1\\2\\2\end{pmatrix} - \begin{pmatrix}\frac{3}{3}\\6/3\\0/3\end{pmatrix} = \begin{pmatrix}1\\2\\2\end{pmatrix} - \begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}0\\0\\2\end{pmatrix}$$

$$q_2 = \frac{P_2}{|P_2|}$$

$$= \frac{\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}{\sqrt{0^2+0^2+2^2}}$$

$$= \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} q_1 & q_2 \end{pmatrix}$$

$$= \begin{pmatrix} 3/\sqrt{45} & 0 \\ 6/\sqrt{45} & 0 \\ 0 & 1 \end{pmatrix}$$

$$Q^T = \begin{pmatrix} 3/\sqrt{45} & 6/\sqrt{45} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underbrace{A}_{(m\times n)}x = b$$

Previously we have used Transpose matrix to convert A into a square matrix. Again we will use it to generate the formula for QR decomposition.

$$A^T A x = A^T b$$

$$(QR)^T (QR)\, x = (QR)^T b$$

$$Q^T R^T \; Q R\, x = Q^T R^T b$$

$$\cancel{R^T} (Q^T Q) R x = Q^T \cancel{R^T} b$$

$\boxed{Q^T Q = 1}$ → According to the property of normality

We know,

$\boxed{A = QR}$ → From the first page

$$\therefore \boxed{Rx = Q^T b}$$

This is the final formula. We will use this to solve the system by QR decomposition.

Proper Example 1 :

$$f(-3) = 0$$
$$f(0) = 0$$
$$f(6) = 2$$

Here degree will be $n=2$ but in order to make this an overdetemined system we will consider P_1. which mean we will take $n = 1$.

$$P_1 = a_0 + a_1(x) = f(x)$$
$$P_1(-3) = a_0 + a_1(-3) = 0$$
$$P_1(0) = a_0 + a_1(0) = 0$$
$$P_1(6) = a_0 + a_1(6) = 2$$

$$\underset{A}{\begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix}} \underset{x}{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}$$

$$\begin{array}{cc} u_1 & u_2 \end{array}$$

In matrix A or incoefficient matix there are 2 columns now we will have two steps.

Step 1 $\boxed{K = 1}$

$$P_1 = U_1$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$q_1 = \frac{P_1}{|P_1|}$$

$$= \frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\sqrt{1^2+1^2+1^2}}$$

$$= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Step 2 $\boxed{K = 2}$

$$P_2 = U_2 - \left(U_2^T q_1\right) q_1$$

$$= \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} - \left\{ \underbrace{\begin{pmatrix} -3 & 0 & 6 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \right\} q_1$$

$$= \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} - \sqrt{3} \cdot q_1$$

$$= \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} - \sqrt{3} \cdot \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ 5 \end{pmatrix}$$

$$q_2 = \frac{P_2}{|P_2|} = \frac{\begin{pmatrix} -4 \\ -1 \\ 5 \end{pmatrix}}{\sqrt{(-4)^2 + (-1)^2 + 5^2}} = \frac{1}{\sqrt{42}} \begin{pmatrix} -4 \\ -1 \\ 5 \end{pmatrix}$$

$$Q = \begin{pmatrix} q_1 & q_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & 5/\sqrt{42} \end{pmatrix}$$

$$Q^T = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -4/\sqrt{42} & -1/\sqrt{42} & 5/\sqrt{42} \end{pmatrix}$$

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Now we have to find R

$$R = \begin{bmatrix} U_1^T q_1 & U_2^T q_1 \\ 0 & U_2^T q_2 \end{bmatrix}$$ → This is work if U_1 and U_2 is given

$$U_1^T q_1 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \sqrt{3}$$

$$U_2^T q_1 = \begin{pmatrix} -3 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \sqrt{3}$$

$$U_2^T q_2 = \begin{pmatrix} -3 & 0 & 6 \end{pmatrix} \begin{pmatrix} -4/\sqrt{42} \\ -1/\sqrt{42} \\ 5/\sqrt{42} \end{pmatrix} = \sqrt{42}$$

$$\therefore R = \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix}$$

$$Rx = Q^T b$$

$$\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -4/\sqrt{42} & -1/\sqrt{42} & 5/\sqrt{42} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{3} \\ 10/\sqrt{42} \end{bmatrix}$$

$$\sqrt{42}\, a_1 = 10/\sqrt{42}$$

$$a_1 = \frac{10}{\sqrt{42} \times \sqrt{42}}$$

$$= \frac{10}{42}$$

$$\boxed{a_1 = \frac{5}{21}}$$

$$\sqrt{3}\, a_0 + \sqrt{3}\, a_1 = \frac{2}{\sqrt{3}}$$

$$\sqrt{3}\, a_0 + \sqrt{3} \times \frac{5}{21} = \frac{2}{\sqrt{3}}$$

$$\boxed{a_0 = 3/7}$$

Example 2

$$a_0 + a_1 + a_2 = 2$$
$$a_0 + 2a_1 + 4a_2 = 3$$
$$a_0 + 3a_1 + 9a_2 = 6$$
$$a_0 + 4a_1 + 16a_2 = 4$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 2 \\ 3 \\ 6 \\ 4 \end{bmatrix}}_{b}$$

$\rightarrow$ 4×3 matrix (It is an overdetermined System)

$$A = \begin{array}{ccc} \left[\begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{matrix}\right] \\ u_1 \quad u_2 \quad u_3 \end{array}$$

There will be 3 steps.

Step 1 $\boxed{k=1}$

$$p_1 = u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$q_1 = \frac{p_1}{|p_1|} = \frac{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}{\sqrt{1^2+1^2+1^2+1^2}} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Step 2 $\boxed{k=2}$

$$p_2 = u_2 - \left(u_2^T q_1\right) q_1$$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} - \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\} \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1.5 \\ -0.5 \\ 0.5 \\ 1.5 \end{pmatrix}$$

$$q_2 = \frac{p_2}{|p_2|} = \frac{1}{\sqrt{5}} \begin{pmatrix} -1.5 \\ -0.5 \\ 0.5 \\ 1.5 \end{pmatrix}$$

Step 3: $\boxed{K = 3}$

$$P_3 = U_3 - \left[(U_3^T q_1) q_1 + (U_3^T q_2) q_2 \right]$$

$$= \begin{pmatrix} 1 \\ 4 \\ 9 \\ 16 \end{pmatrix} - \left[15 \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} + 5\sqrt{5} \begin{pmatrix} -1.5/\sqrt{5} \\ -0.5/\sqrt{5} \\ -0.5/\sqrt{5} \\ 1.5/\sqrt{5} \end{pmatrix} \right]$$

$$= \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$q_3 = \frac{P_3}{|P_3|} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$Q = \begin{array}{c} \begin{pmatrix} 0.5 & -1.5/\sqrt{5} & 0.5 \\ 0.5 & -0.5/\sqrt{5} & -0.5 \\ 0.5 & 0.5/\sqrt{5} & -0.5 \\ 0.5 & 1.5/\sqrt{5} & 0.5 \end{pmatrix} \\ \begin{array}{ccc} q_1 & \quad q_2 & \quad q_3 \end{array} \end{array}$$

$$Q^T = \begin{pmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ -1.5/\sqrt{5} & -0.5/\sqrt{5} & -0.5/\sqrt{5} & 1.5/\sqrt{5} \\ 0.5 & -0.5 & -0.5 & 0.5 \end{pmatrix}$$

Now For R matrix

$$R = \begin{bmatrix} U_1^T q_1 & U_2^T q_1 & U_3^T q_1 \\ 0 & U_2^T q_2 & U_3^T q_2 \\ 0 & 0 & U_3^T q_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & 15 \\ 0 & \sqrt{5} & 5\sqrt{5} \\ 0 & 0 & 2 \end{bmatrix}$$

$$Rx = Q^T b$$

$$\begin{bmatrix} 2 & 5 & 15 \\ 0 & \sqrt{5} & 5\sqrt{5} \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ -1.5/\sqrt{5} & 1.5/\sqrt{5} & 0.5/\sqrt{5} & 1.5/\sqrt{5} \\ -0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 6 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & 15 \\ 0 & \sqrt{5} & 5\sqrt{5} \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 5.5 \\ 27\sqrt{5}/10 \\ -3/2 \end{bmatrix}$$

$$a_2 = -3/4$$

$$\sqrt{5}\, a_1 + 5\sqrt{5} \times -3/4 = 27\sqrt{5}/10$$

$$a_1 = \frac{129}{20}$$

$$= 6.45$$

$$2a_0 + 5 \times 6.45 + 15 \times (-3/4) = 5.5$$

$$a_0 = -7.75$$

* Note: kindly check the values (calculations)