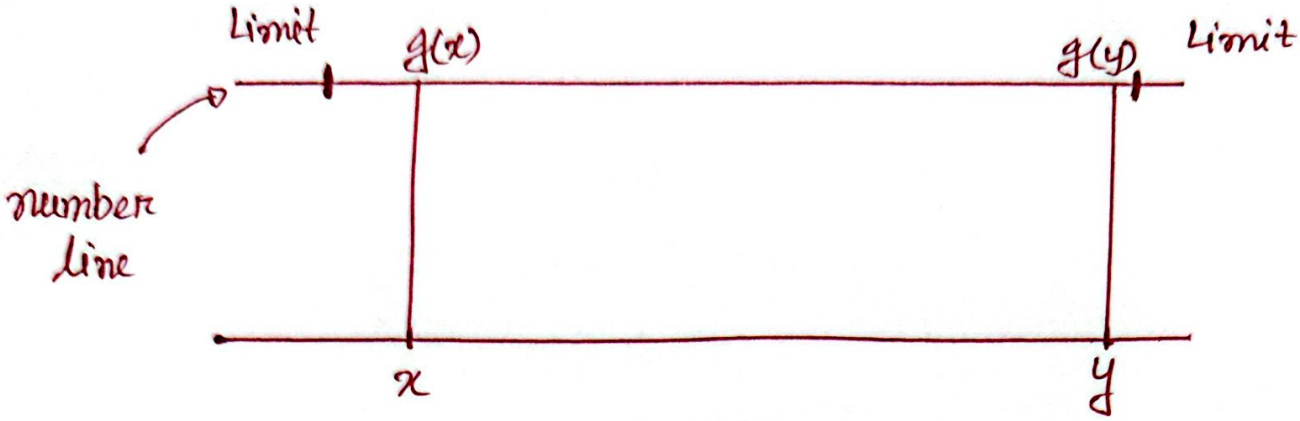
Contraction Mapping Theorem



$$\lambda = \left| \frac{g(y) - g(x)}{y - x} \right|$$

It is an ingenerical formula.
(Gradient type function)

Converging rate, $\lambda = |g'(x)|$

$\lambda = 0$ — Super linear convergent
↳ (less number of iteration) {less root step to find root}

$0 < \lambda < 1$ — linear convergent
↳ (It will find the root but needs more steps)

$\lambda \geq 1$ — divergent
↳ (will not find the root)

Example: $f(x) = x^3 - 2x^2 - x + 2$

a) Find the roots of the soln?

$$x^3 - 2x^2 - x + 2 = 0$$
$$x^2(x-2) - 1(x-2) = 0$$
$$(x^2-1)(x-2) = 0$$
$$(x+1)(x-1)(x-2) = 0$$
$$x = 1, -1, 2$$

b) Construct three $g(x)$ from $f(x)$.

$$x^3 - 2x^2 - x + 2 = 0$$
$$2x^2 = x^3 - x + 2$$
$$x^2 = \frac{1}{2}(x^3 - x + 2)$$
$$x = \sqrt{\frac{1}{2}(x^3 - x + 2)}$$
$$g_1(x) = \sqrt{\frac{1}{2}(x^3 - x + 2)}$$

$$x^3 - 2x^2 - x + 2 = 0$$
$$x = x^3 - 2x^2 + 2$$
$$g_2(x) = x^3 - 2x^2 + 2$$

$$x^3 - 2x^2 - x + 2 = 0$$
$$x(x^2 - 2x - 1) + 2 = 0$$
$$x = \frac{-2}{x^2 - 2x - 1}$$
$$g_3(x) = \frac{-2}{x^2 - 2x - 1}$$

c) Determine which $g(x)$ are convergent and which are divergent / Find the converging of $g(x)$ and which root it will converge to?

$$\lambda = |g_1'(x)| = \left| \frac{1}{\sqrt{2}} (x^3 - x + 2)^{1/2} \right|$$

$$= \left| \frac{1}{\sqrt{2}} \frac{1}{2} (x^3 - x + 2)^{-1/2} (3x^2 - 1) \right|$$

$$= \left| \frac{3x^2 - 1}{2\sqrt{2} (x^3 - x + 2)^{1/2}} \right|$$

From (a)

$x = 1$	,	$\lambda = 0.5$	linear convergent
$x = -1$	,	$\lambda = 0.6$	linear convergent
$x = 2$	,	$\lambda = 3.75$	divergent

$$\lambda = |g_2'(x)| = |x^3 - 2x^2 + 2|$$

$$= |3x^2 - 4x|$$

$x = 1$	,	$\lambda = 1$	divergent
$x = -1$	,	$\lambda = 7$	divergent
$x = 2$	,	$\lambda = 4$	divergent.

$$\lambda = |g_3'(x)| = \left| \frac{-2}{x^2-2x-1} \right|$$

$$= \left| -2 (x^2-2x-1)^{-1} \right|$$

$$= \left| +2 (x^2-2x-1)^{-2} (2x-2) \right|$$

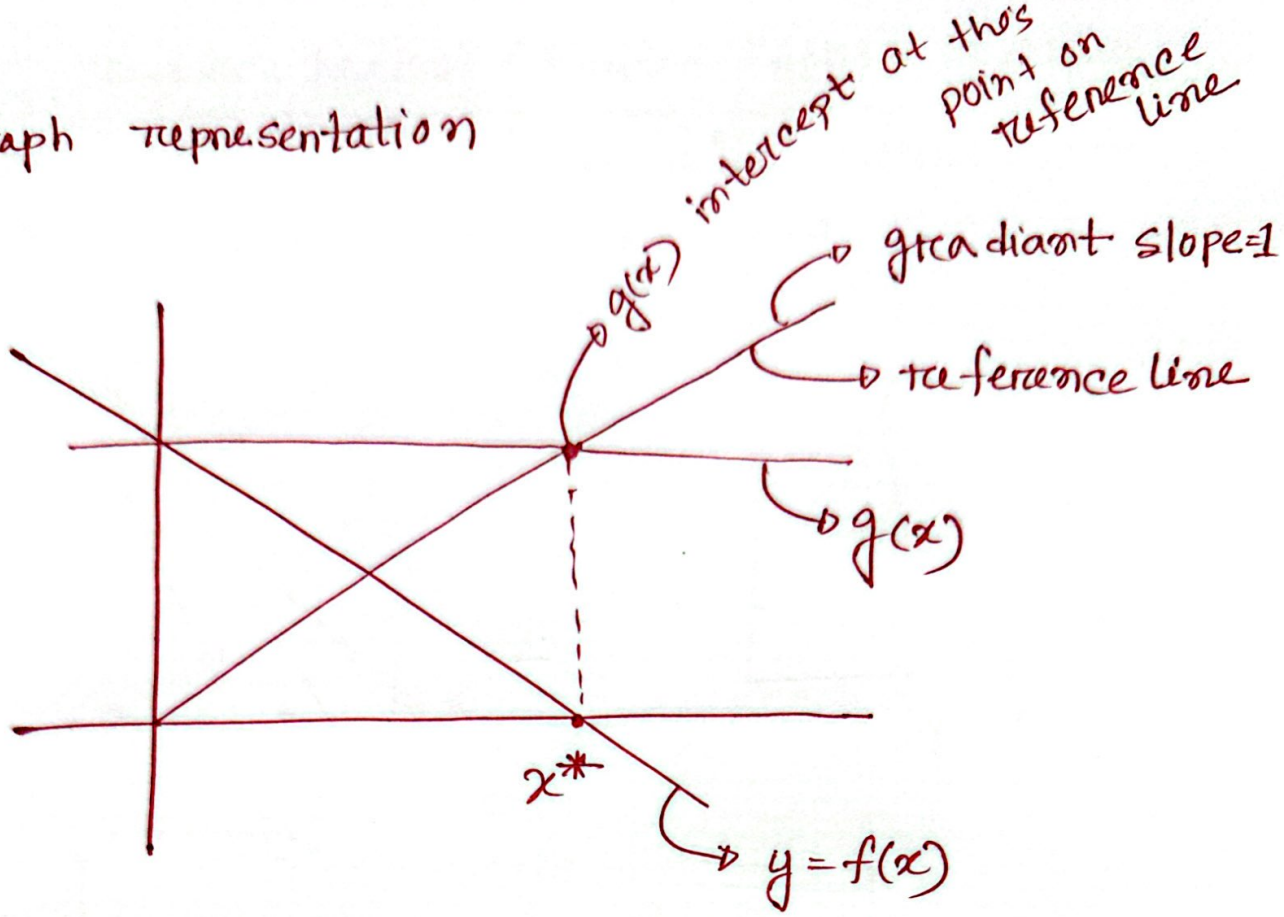
$$= \left| \frac{2(2x-2)}{(x^2-2x-1)^2} \right|$$

$x = 1$, $\lambda = 0$ Super linear Convergent

$x = -1$, $\lambda = 2$ divergent

$x = 2$, $\lambda = 4$ divergent.

Graph representation



* Important Note:

In fixed point iteration, we observe two key aspects. The root we find depends on two main factors.

1) Initial point of x_0

2) Whether the function is convergent or divergent.

Whether a function is convergent or not depends on its converging rate, λ.