Richardson Extrapolation Derivation

$$D_h = f'(x_1) = \frac{f(x_1+h) - f(x_1-h)}{2h}$$

$$f(x_1+h) = f(x_1) + f'(x_1)h + \frac{f^2(x_1)h^2}{2!} + \frac{f^3(x_1)h^3}{3!} + \frac{f^4(x_1)h^4}{4!} + \frac{f^5(x_1)h^5}{5!} + \frac{f^6(x_1)h^6}{6!} + O(h^7)$$

$$f(x_1-h) = f(x_1) - f'(x_1)h + \frac{f^2(x_1)h^2}{2!} - \frac{f^3(x_1)h^3}{3!} + \frac{f^4(x_1)h^4}{4!} - \frac{f^5(x_1)h^5}{5!} + \frac{f^6(x_1)h^6}{6!} + O(h^7)$$

$$D_h = \frac{1}{2h}\left\{f(x_1+h) - f(x_1-h)\right\}$$

$$= \frac{1}{2h}\left\{2f'(x_1)h + \frac{2f^3(x_1)h^3}{3!} + \frac{2f^5(x_1)h^5}{5!} + O(h^7)\right\}$$

$$= \underbrace{f'(x_1)}_{\text{Actual derivative}} + \underbrace{\frac{f^3(x_1)h^2}{3!} + \frac{f^5(x_1)h^4}{5!} + O(h^6)}_{\text{error we need to remove this}}$$

dominating factor of error (We need to remove this)

$$D_{h/2} = f'(x_1) + \frac{f^3(x_1)(h/2)^2}{3!} + \frac{f^5(x_1)(h/2)^4}{5!} + O(h^6)$$

$$2^2 D_{h/2} = 2^2 f'(x_1) + 2^2\frac{f^3(x_1)(h/2)^2}{3!} + 2^2\frac{f^5(x_1)(h/2)^4}{5!} + O(h^6)$$

$$= 2^2 f'(x_1) + \frac{f^3(x_1)h^2}{3!} + \frac{f^5(x_1)h^4/2^2}{5!} + O(h^6)$$

$$2^2 D_{h/2} - D_h = (2^2-1) f'(x_1) + \left(\frac{1}{2^2} - 1\right) \frac{f^5(x_1) h^4}{5!} + O(h^6)$$

$$\frac{2^2 D_{h/2} - D_h}{2^2-1} = f'(x_1) + \frac{\left(\frac{1}{2^2}-1\right)}{2^2-1} \frac{f^5(x_1) h^4}{5!} + O(h^6)$$

$$= f'(x_1) - \frac{1}{2^2} \frac{f^5(x_1) h^4}{5!} + O(h^6)$$

$$\therefore D_h^{(1)} = \frac{2^2 D(h/2) - D(h)}{2^2-1} = \frac{2^n D(h/2) - D(h)}{2^n - 1}$$

$$= \frac{4 D(h/2) - D(h)}{3}$$

Formula: $D_h^{(2)} = \dfrac{16 D^{(1)}(h/2) - D^{(1)}(h)}{15}$

$\boxed{D'(h) = D_h^{(1)} \text{ same}}$

Example 1:

$$f(x) = e^x \sin x$$

Question: Find $D^1_{(h)}$ using Richardson extrapolation at $x = 1$, for:

$$h = 0.5$$
$$h = 0.25$$

We know Richardson extrapolation is only for central difference:

$f'(1)$; $h = 0.5$

$$f'(1) = \frac{f(1+0.5) - f(1-0.5)}{2 \times 0.5}$$

$$= \frac{e^{1.5} \sin(1.5) - e^{0.5} \sin(0.5)}{1}$$

$$= 3.68$$

$f'(1)$; $h = 0.25$

$$f'(1) = \frac{f(1.25) - f(0.75)}{2 \times 0.25}$$

$$= 3.7385$$

Using Richardson to find more accurate value:

$h = 0.5$ $\quad D_h = 3.68$

$h = 0.25$ $\quad D_{h/2} = 3.7385$

$$D_h^{(1)} = \frac{2^2 D(h/2) - D(h)}{2^2 - 1}$$

$$= \frac{2^2 \times (3.7385) - 3.68}{3}$$

$$= 3.757$$

Example 2: $h = 0.1 \quad f'(1) = 0.7$

$h = 0.2 \quad f'(1) = 0.5$

Using Richardson Extrapolation, find $f'(1)$

$$D_h^{(1)} = \frac{4\,D(h/2) - D(h)}{3}$$

$$= \frac{4(0.7) - (0.5)}{3}$$

$$= 0.77$$

Example 3: $f(x) = e^{2x} + 3x$, find $f'(2)$ using RE.

$h = 1.2$

$h = 0.6$

$$f'(2) = \frac{f(2+1.2) - f(2-1.2)}{2 \times 1.2}$$

$$= \frac{f(3.2) - f(0.8)}{2 \times 1.2}$$

$$= \frac{e^{2 \times 3.2} + 3 \times 3.2 - e^{2 \times 0.8} - 3 \times 0.8}{2 \times 1.2}$$

$$= 251.705$$

$$f'(2) = \frac{f(2+0.6) - f(2-0.6)}{2 \times 0.6}$$

$$= \frac{f(2.6) - f(1.4)}{2 \times 0.6}$$

$$= \frac{e^{2 \times 2.6} + 3 \times 2.6 - e^{2 \times 1.4} - 3 \times 1.4}{2 \times 0.6}$$

$$= 140.356$$

$$D_n^{(1)} = \frac{2^2(140.356) - 251.705}{2^2 - 1}$$

$$= 103.23$$

Example 4: $f'(1) = ?$

$h = 0.4$

$h = 0.2$

$h = 0.1$

x	$f(x)$
0.6	0.707178
0.8	0.8559892
0.9	0.925863
1.0	0.984007
1.1	1.033743
1.2	1.074575
1.4	1.127986

h	D_n
0.4	0.52601
0.2	0.5464
0.1	0.5394

$$D_n^{(1)} = \frac{4D_{n/2} - D_n}{3} = 0.553$$

$$D_n^{(1)} = \frac{4D_{n/2} - D_n}{3} = 0.537$$

$$D_n^{(2)} = \frac{2^4 D_{n/2} - D_n}{2^4 - 1} = \frac{16 \times 0.537 - 0.553}{2^4 - 1} = 0.535933$$

$h = 0.4$

$$D_n = f'(1) = \frac{f(1+0.4) - f(1-0.4)}{2 \times 0.4}$$

$$= (1.127986 - 0.707178)/2 \times 0.4$$

$$= 0.52601$$

$h = 0.2$

$$f'(1) = \frac{f(1+0.2) - f(1-0.2)}{2 \times 0.2} = 0.5464$$

$h = 0.1$

$$f'(1) = \frac{f(1+0.1) - f(1-0.1)}{2 \times 0.1} = 0.5394$$

Example 5: $f(x) = x^2 + e^x$

Compute $D^{(1)}_{0.2}$ and $D^{(2)}_{0.2}$ at $x=1$ using Richardson Extrapolation.

$h = 0.2$

another $h = h/2 = 0.2/2 = 0.1$

$f'(1)$; $h = 0.2$

$$D_h = f'(1) = \frac{f(1.2) - f(0.8)}{2 \times 0.2}$$

$$= 4.7364$$

$f'(1)$; $h = 0.1$

$$D_h = f'(1) = \frac{f(1.1) - f(0.9)}{2 \times 0.1}$$

$$= 4.7228$$

$$D^{(1)}_{0.2} = \frac{4D(h/2) - D(h)}{3}$$

$$= \frac{4(4.7228) - 4.7364}{3} = 4.7183$$

$$D^{(2)}_{0.2} = \frac{16\,D'(h/2) - D'(h)}{15}$$

$$D'(h) = D^{(1)}_{0.2} = 4.7183$$

Calc. $D'(h/2) = D^{(1)}_{0.1}$ using the same method and just plug in the value.

Example 6:

G Deduce an expression for D_h^1 from D_h by replacing h with $(4h/3)$ using the Richardson Extrapolation method.

$$D_h = \frac{f(x+h) - f(x-h)}{2h}$$

$$f(x+h) = f(x) + f'(x)h + \frac{f^2(x)h^2}{2!} + \frac{f^3(x)h^3}{3!} + \frac{f(x)h^4}{4!} + \frac{f^5(x)h^5}{5!} + \frac{f^6(x)h^6}{6!} + O(h^7)$$

$$f(x-h) = f(x) - f'(x)h + \frac{f^2(x)h^2}{2!} - \frac{f^3(x)}{3!}h^3 + \frac{f^4(x)h^4}{4!} - \frac{f^5(x)\ ^5}{5!} + \frac{f^6(x)h^6}{6!} + O(h^7)$$

$$D_h = \frac{1}{2}\left[2f'(x)h + \frac{2f^3(x)}{3!}h^3 + 2\frac{f^5(x)}{5!}h^5 + O(h^7)\right]$$

$$= f'(x) + \frac{f^3(x)}{3!}h^2 + \frac{f^5(x)h^4}{5!} + O(h^6)$$

Replacing h by $4h/3$

$$D_{4h/3} = f'(x) + \frac{f^3(x)}{3!}\left(\frac{4h}{3}\right)^2 + \frac{f^5(x)}{5!}\left(\frac{4h}{3}\right)^4 + O(h^6)$$

$$\left(\frac{3}{4}\right)^2 D_{4h/3} = \left(\frac{3}{4}\right)^2 f'(x) + \frac{f^3(x)h^2}{3!} + \frac{f^5(x)}{5!}\left(\frac{4h}{3}\right)^4\left(\frac{3}{4}\right)^2 + O(h^6)$$

Now,

$$\left(\frac{3}{4}\right)^2 D_{4h/3} - D_h = \left(\frac{3^2}{4^2} - 1\right) f'(x) + \left(\frac{4^2}{3^2} - 1\right) \frac{f^5(x)\, h^4}{5!} + O(h^6)$$

$$\frac{\left(3/4\right)^2 D_{4h/3} - D_h}{\left(3/4\right)^2 - 1} = f'(x) + \frac{\left(4^2/3^2 - 1\right)}{\left(3/4\right)^2 - 1} \frac{f^5(x)\, h^4}{5!} + O(h^6)$$

$$D_h^{(1)} = f'(x) - \frac{16}{9} \frac{f^5(x)}{5!} h^4 + O(h^6)$$

Example 7:

Consider the function $f(x) = 4x^3 - 9e^{7x}$. Now answer the following:

a) Compute $D_{0.2}^{(1)}$ at $x = 2.7$ using Richardson extrapolation method upto 4 significant figures.

b) Compute $D_{0.2}^{(2)}$ at $x = 2.7$ using Richardson extrapolation method upto 4 significant figures.

(a)

$$D_h = D_{0.2} = \frac{f(x+h) - f(x-h)}{2h} = \frac{f(2.7+0.2) - f(2.7-0.2)}{2 \times 0.2}$$

$$= \frac{f(2.9) - f(2.5)}{0.4}$$

$$= -1.384 \times 10^{10} \text{ (4 s.f)}$$

$$D_{h/2} = D_{0.2/2} = D_{0.1} = \frac{f(x+h) - f(x-h)}{2h} = \frac{f(2.7+0.1) - f(2.7-0.1)}{2 \times 0.1}$$

$$= \frac{f(2.8) - f(2.6)}{0.2}$$

$$= -1.103 \times 10^{10} \text{ (4 s.f)}$$

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$$\therefore\ D_{0.2}^{(1)} = \frac{2^2 D_{0.1} - D_{0.2}}{2^2-1}$$

$$= \frac{2^2(-1.103\times10^{10}) - (-1.384\times10^{10})}{2^2-1}$$

$$= -1.009\times10^{10} \quad (4 \text{ s.f})$$

(b)

To find

$$D_{0.2}^{(2)} = \frac{2^4 D_{0.1}^{(1)} - D_{0.2}^{(1)}}{2^4-1} \quad \text{- - - (1)}$$

$$D_{0.1}^{(1)} = \frac{2^2 D_{0.05} - D_{0.1}}{2^2-1} \quad \left[\frac{2^n D(h/2) - D(h)}{2^n-1}\right]$$

$$= \frac{4(-1.038\times10^{10}) - (-1.103\times10^{10})}{2^2-1}$$

$$= -1.016\times10^{10} \quad (4 \text{ s.f})$$

$$D_{h/4} = D_{0.05} = \frac{f(x+h)-f(x-h)}{2h} = \frac{f(2.7+0.05)-f(2.7-0.05)}{2\times0.05}$$

$$= \frac{f(2.75)-f(2.65)}{0.1}$$

$$\boxed{= -1.038\times10^{10}} \quad (4 \text{ s.f})$$

Now plug all the values in equation (i)

$$D_{0.2}^{(2)} = \frac{2^4 D_{0.1}^{(1)} - D_{0.2}^{(1)}}{2^4 - 1}$$

$$= \frac{2^4 \times (-1.016 \times 10^{10}) - (-1.009 \times 10^{10})}{2^4 - 1}$$

$$= -1.016 \times 10^{10} \qquad (4 \cdot s \cdot f)$$