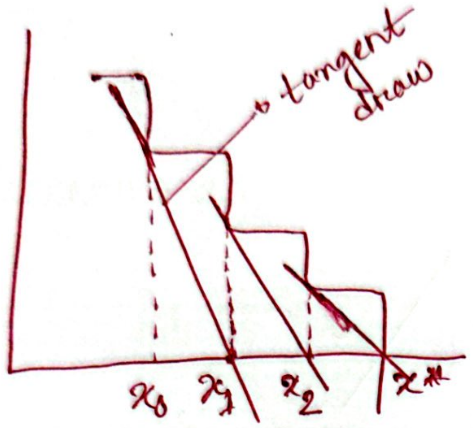
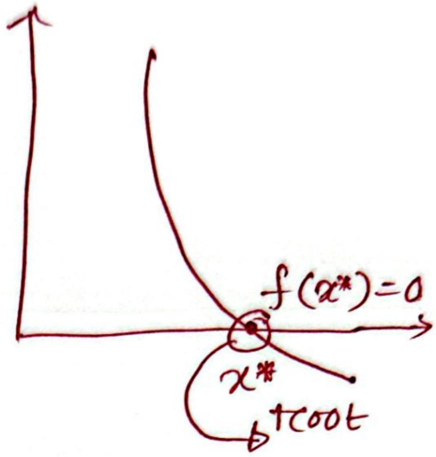
Newton's Method / Newton Raphson Method

(Super linear Convergence)

$\lambda = 0 \rightarrow$ Super linear convergent





Formula:

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

Example 1: $f(x) = x^2 - 2xe^{-x} + e^{-2x}$. Initial point $x_0 = 1$ and Error bound $= 1 \times 10^{-5} = 0.00001$.

Using this formula

$$f'(x) = 2x - \left[2e^{-x} + 2xe^{-x}(-1)\right] + e^{-2x}(-2)$$

k (item)	x_k	$f(x_k)$	$f(x_k) <$ Error bound
0	1	0.3995	No
1	0.7687	0.093	No
2	0.6648	0.0226	NO
⋮	⋮	⋮	⋮
8	0.59	0.49×10^{-5}	Yes

Solution of the function / root of the function

Example 2: $f(x) = x^3 - 0.105x^2 + 3.993 \times 10^{-4}$. Show first three iterations with relative error at each iterations. Initial point $x_0 = 0.05$

$$f'(x) = 3x^2 - 0.33x$$

Iteration 1:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.05 - \frac{f(0.05)}{f'(0.05)}$$

$$= 0.0624$$

Relative Error $= |x_{new} - x_{old}| = |0.0624 - 0.05|$
$= 0.0124$

Iteration 2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.0624 - \frac{f(0.0624)}{f'(0.0624)} = 0.0623$$

Relative Error $= |0.0624 - 0.0623| = 0.0001$

Iteration 3:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.0623 - \frac{f(0.0623)}{f'(0.0623)}$$

$$= 0.0623$$

Relative Error $= |0.0623 - 0.0623| = 0$

** Important **

4. Prove that newton raphson method is a super linear convergent.

$$\boxed{x_{k+1}} = x_k - \frac{f(x_k)}{f'(x_k)}$$

→ we want to denote it as $g(x)$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

We know, $\lambda = |g'(x)|$

→ we want to differentiate this part.

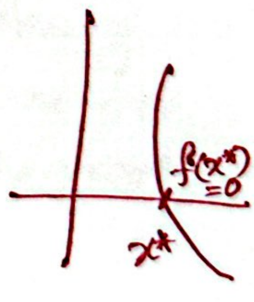
$$\lambda = |g'(x)| = \left| 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{\{f'(x)\}^2} \right|$$

→ $\boxed{\dfrac{v\frac{d}{du}u - u\frac{d}{dv}(v)}{v^2}}$

$$= \left| \frac{f(x)f''(x)}{\{f'(x)\}^2} \right|$$

→ root of a function

$$\lambda = |g'(x_*)| = \left| \frac{f(x_*)f''(x_*)}{\{f'(x_*)\}^2} \right|$$



$$= \left| \frac{0 \times f''(x_*)}{\{f'(x_*)\}^2} \right|$$

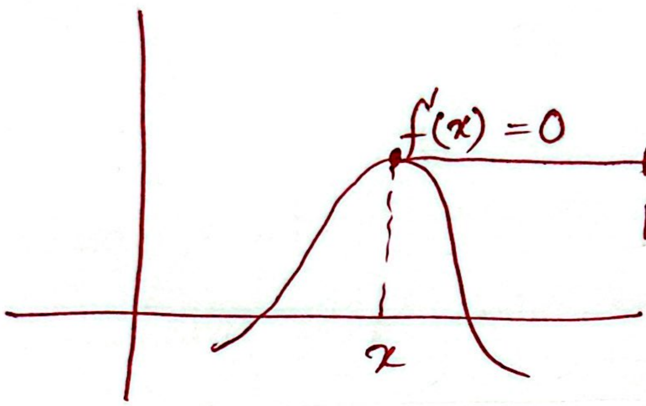
$$= 0$$

$\therefore \lambda = 0$ [Proved]

Drawback

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}} \longrightarrow \text{main formula}$$

Limitation no: 1 :

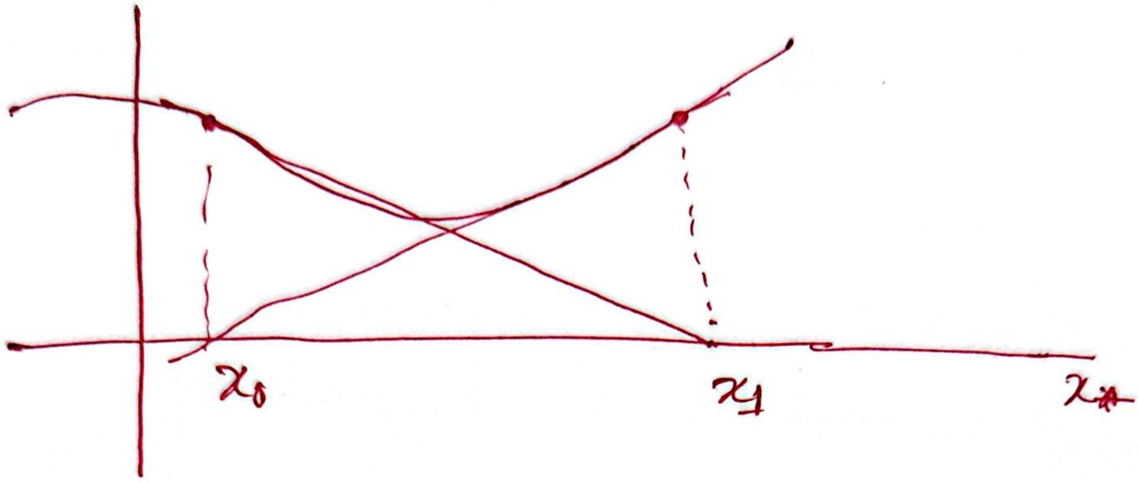


→ if the iteration leads to a turning point, then the first derivative $f'(x)$ become 0, which causes infinity

$$x_{k+1} = x_k - \frac{f(x_k)}{\boxed{0}}$$

→ This cause infinity and we will never find the root.

Limitation no: 2



If the initial point is chosen such that it lies near a turning point, the iteration may enter a loop and making it impossible to find the root.