Subject: Lecture-1 Date:

Fixed Point Representation:

$X = 1 \, (d_1 d_2 \cdots\cdots d_{k-1} d_k \cdots d_n)_\beta$

Where $d_1, d_2, d_2 \in \{0, 1, 2, 3 \cdots\cdots \beta-1\}$

β = Base

$(12345)_{10} \Rightarrow \beta = 10$ and $\eta = 5$ digits

Evaluating fixed point numbers in Base 10

$$(101.01)_2$$

$$\begin{array}{ccc c cc} 1 & 0 & 1 & \cdot & 0 & 1 \\ \hline 2 & 1 & 0 & & -1 & -2 \end{array}$$

$= 1\times 2^2 + 0\times 2^1 + 1\times 2^0 + 0\times 2^{-1} + 1\times 2^{-2}$

$= 4+0+1+0+\frac{1}{4}$

$= 5+\frac{1}{4}$

$= \frac{21}{4}$.

Evaluating fixed point numbers in Base 9

$$(34.7)_9$$

$$\begin{array}{cc c c} 3 & 4 & \cdot & 7 \\ \hline 1 & 0 & & -1 \end{array}$$

$= 3\times 9^1 + 4\times 9^0 + 7\times 9^{-1}$

$= 27+4+\frac{7}{9}$

$= \frac{286}{9}$

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Floating Point Representation:

$$F = \left\{ \pm \left(0.d_1 d_2 \cdots d_n\right)_\beta \, \beta^e \right\} \quad \text{where } d_1 \neq 0$$

Here, β = Base

e = exponential

$0 \leq d_i \leq \beta - 1$

$\beta, d_i, e \in \mathbb{Z}$ (integer)

$e_{min} \leq 0 \leq e_{max}$

Example:

$(123.45)_{10}$

$= (12.345 \times 10)$

$= 1.2345 \times 10^2$

$= 0.12345 \times 10^{3} \longrightarrow e$

$\quad\quad\quad\;\; \hookrightarrow \beta$

Example:

$(56.2)_7$

$= 5.62 \times 7^1$

$= 0.562 \times 7^2$

Highest Possible Floating Point:

When, $\beta = 2$ $\quad e_{min} = -1$

$\quad\quad\; n = 3$ $\quad e_{max} = 2$

$(0.111)_2 \times 2^2$

Subject: Date:

Denormalization form

$$\pm (0.1\ d_1\ d_2 \ldots d_n)_\beta\ \beta^e$$

$\beta = 2$

$n = 2$

$e_{min} = -1$

$e_{max} = 2$

When $e = -1$,

$(0.100) \times 2^{-1}$

$(0\times2^0 + 1\times2^{-1} + 0\times2^{-2} + 0\times2^{-3}) \times \frac{1}{2}$

$= \frac{1}{2}\times\frac{1}{2}$

$= \frac{1}{4}$

$(0.101)\times2^{-1}$

$= (0\times2^0 + 1\times2^{-1} + 0\times2^{-2} + 1\times2^{-3}) \times \frac{1}{2}$

$= \left(\frac{1}{2} + \frac{1}{8}\right)\times\frac{1}{2}$

$= \frac{5}{16}$

$\underbrace{0.1}_{\text{fixed}}$ d_2 d_3

d_2	d_3
0	0
0	1
1	0
1	1

total 4 possible cases here.

$(0.110)\times2^{-1}$

$= (0\times2^0 + 1\times2^{-1} + 1\times2^{-2} + 0\times2^{-3})\times2^{-1}$

$= \left(\frac{1}{2}+\frac{1}{4}\right)\times\frac{1}{2}$

$= \frac{3}{8}$

e can be from

-1

0

1

2

4 posible cases

$(0.111)\times2^{-1}$

$= (0\times2^0 + 1\times2^{-1} + 1\times2^{-2} + 1\times2^{-3})\times\frac{1}{2}$

$= \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)\times\frac{1}{2}$

$= \frac{7}{16}$

total $= 4\times4$

$= 16$ cases.

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Subject: Date:

Normalized form:

$$\pm(1.d_1 d_2 d_3 d_4 \ldots d_n)_\beta \, \beta^e$$

Here, d_1 can be 1 or not 1.

for, $\beta = 2$, $e_{min} = -1$
$n = 3$, $e_{max} = 2$

Heighest possible Floating Point $= (1.111)_2 \times 2^2$

Smallest:

$(0.100)_2 \times 2^{-1}$

$= (0\times2^0 + 1\times2^{-1} + 0\times2^{-2} + 0\times2^{-3}) \times \frac{1}{2}$

$= \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{4}$

Largest:

$(0.111)_2 \times 2^2$

$= (0\times2^0 + 1\times2^{-1} + 1\times2^{-2} + 1\times2^{-3}) \times 4$

$= (\frac{1}{2} + \frac{1}{4} + \frac{1}{8}) \times 4$

$= \frac{7}{2}$

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Example:

$e_{min} = -1$

$e_{max} = 2$

$\beta = 2$

$n = 2$

For $e = -1$,

1st smallest non-negative number $= (0.100) \times 2^{-1}$

$$= (0 \times 2^{0} + 1 \times 2^{-1}) \times \frac{1}{2}$$

$$= \frac{1}{4}$$

2nd smallest non-negative number $= (0.101) \times 2^{-1}$

$$= (0 \times 2^{0} + 1 \times 2^{-1} + 1 \times 2^{-3}) \times \frac{1}{2}$$

$$= \left(\frac{1}{2} + \frac{1}{8}\right) \times \frac{1}{2}$$

$$= \frac{5}{16}$$

3rd smallest non-negative number $= (0.110) \times 2^{-1}$

$$= (0 \times 2^{0} + 1 \times 2^{-1} + 1 \times 2^{-2}) \times \frac{1}{2}$$

$$= \left(\frac{1}{2} + \frac{1}{4}\right) \times \frac{1}{2}$$

$$= \frac{3}{8}$$

4th smallest non-negative number $= (0.111) \times 2^{-1}$

$$= (0 \times 2^{0} + 1 \times 2^{-1} + 1 \times 2^{-2} + 1(2^{-3})) \frac{1}{2}$$

$$= \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) \times \frac{1}{2}$$

$$= \frac{7}{16}$$

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Subject: | Date:

When $e=0$

1st smallest non-negative number $= (0.100)\times 2^0$
$= \frac{1}{2}$

2nd smallest non-negative number $= (0.101)\times 2^0$
$= \frac{5}{8}$

3rd smallest non-negative number $= (0.110)\times 2^0$
$= \frac{3}{4}$

4th smallest non-negative number $= (0.111)\times 2^0$
$= \frac{7}{8}$

When $e=1$

1st smallest non-negative number $= (0.100)\times 2^1$
$= 1$

2nd smallest non-negative number $= (0.101)\times 2^1$
$= \frac{5}{4}$

3rd smallest non-negative number $= (0.110)\times 2^1$
$= \frac{3}{2}$

4th smallest non-negative number $= (0.111)\times 2^1$
$= \frac{7}{4}$

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Subject: Date:

When $e=0$

1st smallest non-negative number $= (0.100)\times 2^0$

$= \frac{1}{2}$

2nd smallest non-negative number $= (0.101)\times 2^0$

$= \frac{5}{8}$

3rd smallest non-negative number $= (0.110)\times 2^0$

$= \frac{3}{4}$

4th smallest non-negative number $= (0.111)\times 2^0$

$= \frac{7}{8}$

When $e=1$

1st smallest non-negative number $= (0.100)\times 2^1$

$= 1$

2nd smallest non-negative number $= (0.101)\times 2^1$

$= \frac{5}{4}$

3rd smallest non-negative number $= (0.110)\times 2^1$

$= \frac{3}{2}$

4th smallest non-negative number $= (0.111)\times 2^1$

$= \frac{7}{4}$

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When, $e = 2$

1st smallest non-negative number $= (0.100) \times 2^2$

$= 2$

2nd smallest non-negative number $= (0.101) \times 2^2$

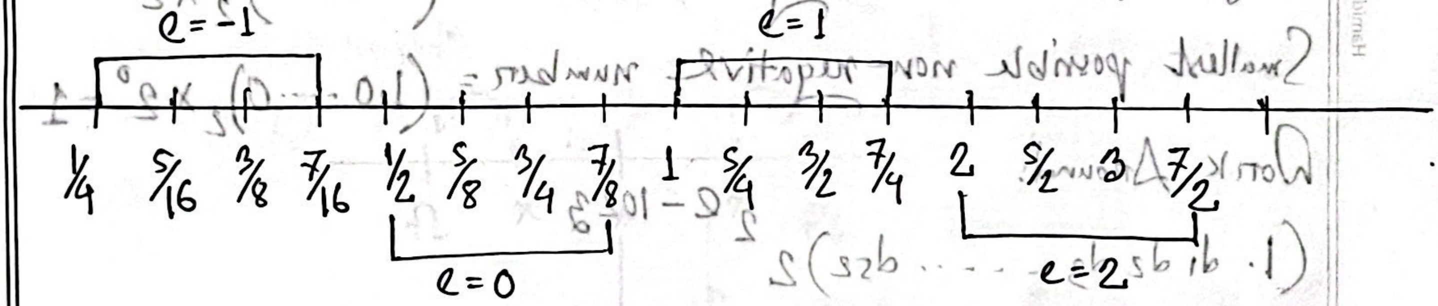
$= 5/2$

3rd smallest non-negative number $= (0.110) \times 2^2$

$= 3$

4th smallest non-negative number $= (0.111) \times 2^2$

$= 7/2$



Subject: Date:

IEEE standard for double precision:

$\beta = 2$ — 52- bits for fraction

11- bits for exponent

1- bit for sign

Normalized form:

$$(1.d_1 d_2 d_3 \cdots d_{52}) \times 2^e$$

min value of $e = 0$ [Where all 11 bits are 0]

max value of $e = 2^{11} - 1 = 2047$ [Where all 11 bits are 1]

$e_{min} = 0$

$e_{max} = 2047$

Largest possible non-negative number $= (1.11\cdots1)_2 \times 2^{2047}$

Smallest possible non-negative number $= (1.0\cdots0)_2 \times 2^0 = 1$

Work Around:

$$(1.d_1 d_2 d_3 \cdots d_{52})_2 \, 2^{e-1023}$$

$$(0.1 d_1 d_2 d_3 \cdots d_{52})_2 \, 2^{e-1022}$$

Now,

The highest possible number $= (0.111\cdots1) \times 2^{1025} \approx \infty$

The smallest possible number $= (0.100\cdots0) \times 2^{-1022} \approx 0$

highest possible exponent $= 2^{1025}$

lowest possible exponent $= 2^{-1022}$

Subject : Date :

Now,

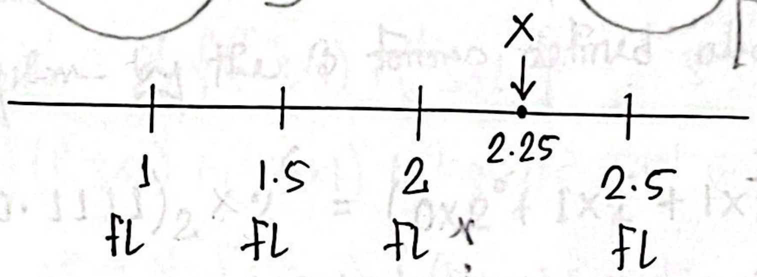
The highest possible number $= (0.111\ldots1) \times 2^{1024}$

$\approx 1.798 \times 10^{308}$

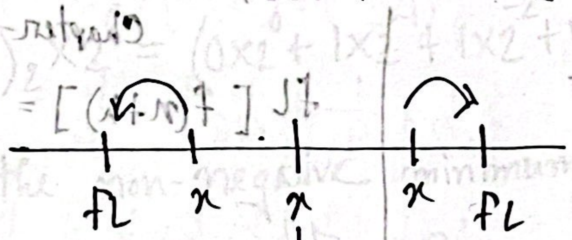
The lowest possible number $= (0.1)_2 \times 2^{-1021}$

$= 2.225 \times 10^{-308}$

Rounding:



if x is given, x should always be converted to $fl(x)$

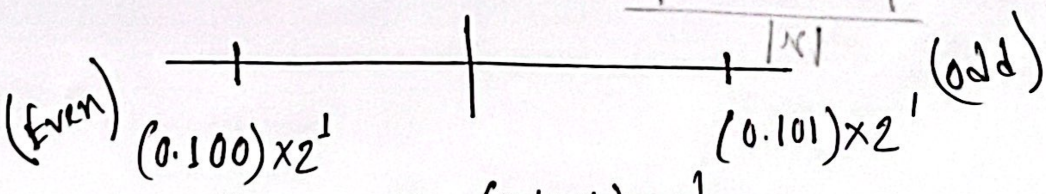


→ if perfectly in the middle, then round it to the nearest even fl

* For fl Binary,

if the number ends with 0 → Even

if the number ends with 1 → Odd



$x = (0.1001) \times 2^1$

Example:

$\beta = 2$

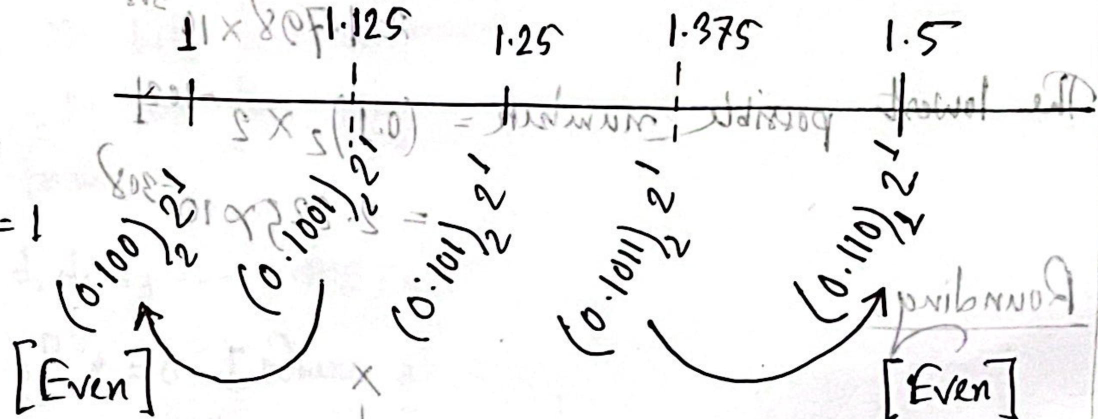
$n = 3$

$e_{min} = -1$

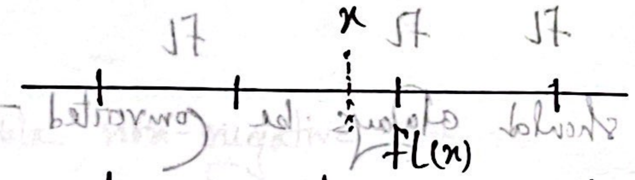
$e_{max} = 2$

Conversion = 1

When $e = 1$,



Rounding Error:



$$\delta = \frac{|fl(x) - x|}{|x|}$$

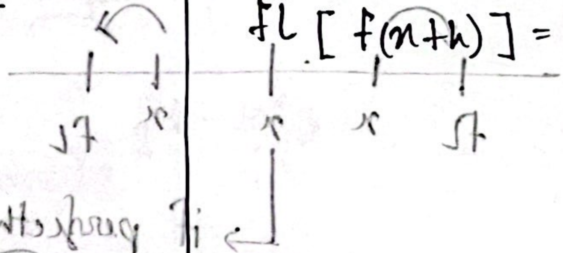
$$\delta \cdot x = fl(x) - x$$

$$fl(x) = \delta x + x$$

$$fl(x) = x(\delta + 1)$$

Chapter-3

$$fl[f(x+h)] = (1+\delta_1) f(x+h)$$



Machine Epsilon:

Maximum possible scale invariant error, δ

$$\delta = \frac{|fl(x) - x|}{|x|}$$

Subject:

Date:

1. Lecture form: $f = \pm (0.d_1 d_2 d_3 \ldots\ldots d_m)_\beta \, \beta^e$

Normalized form: $f = \pm (1.d_1 d_2 d_3 \ldots\ldots d_m)_\beta \, \beta^e$

Denormalized form: $f = \pm (0.1\, d_1 d_2 d_3 \ldots\ldots d_m)_\beta \, \beta^e$

$\beta = 2,\ m = 4,\ e_{max} = 6,\ e_{min} = -3$

(a) What are the maximum numbers that can be stored in the system by the 3 forms defined above?

1st] $(0.1111)_2 \times 2^6 = (0\times2^0 + 1\times2^{-1} + 1\times2^{-2} + 1\times2^{-3} + 1\times2^{-4})\times 2^6$

$(1.1111)_2 \times 2^6 = (1\times2^0 + 1\times2^{-1} + 1\times2^{-2} + 1\times2^{-3} + 1\times2^{-4})\times 2^6$

$(0.11111)_2 \times 2^6 = (0\times2^0 + 1\times2^{-1} + 1\times2^{-2} + 1\times2^{-3} + 1\times2^{-4} + 2\times2^{-5})\times 2^6$

(b) What are the non-negative minimum numbers that can be stored in the system by 3 forms?

① $(0.1000)_2 \times 2^{-3} = (1\times2^{-1})\times 2^{-3} = \frac{1}{16}$

② $(0.10000)_2 \times 2^{-3} = (1\times2^{-1})\times 2^{-3} = \frac{1}{16}$

③ $(1.0000)_2 \times 2^{-3} = (1\times2^{0})\times 2^{-3} = 1/8$

Subject : | Date :

ⓒ Using eq (1), find all the decimal numbers for $e=-1$ plot them in a real line.

given, $\beta = 2 \quad m=4 \quad e=-1$

① $(0.1\,000)_2 \times 2^{-1} = (1\times 2^{-1})\times 2^{-1} = 1/4$

② $(0.1\,001)_2 \times 2^{-1} = \{(1\times 2^{-1}) + (1\times 2^{-4})\}\,1/2 = 9/32$

③ $(0.1\,010)_2 \times 2^{-1} = \{(1\times 2^{-1}) + (1\times 2^{-3})\}\,1/2 = 5/16$

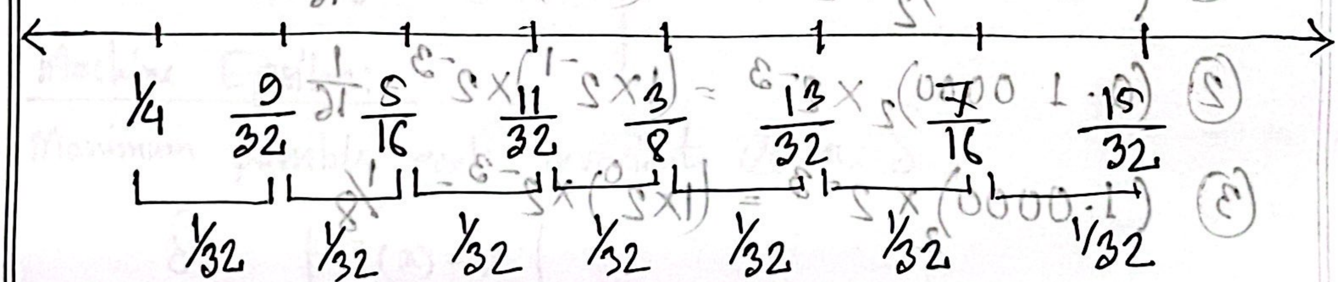
④ $(0.1\,011)_2 \times 2^{-1} = \{(1\times 2^{-1}) + (1\times 2^{-3}) + (1\times 2^{-4})\} \times 1/2 = \frac{11}{32}$

⑤ $(0.1\,100)_2 \times 2^{-1} = \{(1\times 2^{-1}) + (1\times 2^{-2})\}\,1/2 = 3/8$

⑥ $(0.1\,101)_2 \times 2^{-1} = \{(1\times 2^{-1}) + (1\times 2^{-2}) + (1\times 2^{-4})\} \times 1/2 = \frac{13}{32}$

⑦ $(0.1\,110)_2 \times 2^{-1} = \{1/2 + \frac{1}{4} + 1/18\} \times 1/2 = 7/16$

⑧ $(0.1\,111)_2 \times 2^{-1} = \{1/2 + 1/4 + 1/8 + 1/16\} \times 1/2 = \frac{15}{32}$



Subject: Date:

2. $\beta = 2, m = 4, e_{min} = -1, e_{max} = 2$

ⓐ Compute minimum of $|x|$ for normalization form

$$(1.0000)_2 \times 2^{-1} = (1 \times 2^0) \times \frac{1}{2} = \frac{1}{2}$$

ⓑ Compute minimum of $|x|$ for denormalized form

$$(0.10000)_2 \times 2^{-1} = (1 \times 2^{-1}) \times 2^{-1} = \frac{1}{4}$$

ⓒ Compute machine epsilon for normalized form

$$\varepsilon_m = \frac{1}{2}\beta^{-m} = \frac{1}{2} \times 2^{-4} = \frac{1}{32}$$

ⓓ Compute machine epsilon for denormalized form

$$\varepsilon_m = \frac{1}{2}\beta^{-m} = \frac{1}{2} \times 2^{-4} = \frac{1}{32}$$

ⓔ Compute maximum delta value for general form/convention which is discovered in lecture note.

$$\max \delta = \varepsilon_m = \frac{1}{2} \cdot \beta^{1-m}$$

$$= \frac{1}{2} 2^{(1-4)}$$

$$= \frac{1}{2} \times \frac{1}{8}$$

$$= \frac{1}{16}$$

Subject : Date :

3. Lecture form: $f = \pm(0.d_1 d_2 d_3 d_4 \cdots d_m)_\beta \beta^e$

Normalized form: $f = \pm(1.d_1 d_2 d_3 d_4 \cdots d_m)_\beta \beta^e$

Denormalized form: $f = \pm(0.1 d_1 d_2 d_3 \cdots d_m)_\beta \beta^e$

$\beta = 2,\ m = 4,\ e \in \{-2, -1, 0, 1, 2\}$

(a) How many numbers in total can be represented by this system? find for each and abrogate negative numbers.

Lecture form: $f = \pm(0.d_1 d_2 d_3 \cdots d_m)_\beta \beta^e$

$(0.1000)_2 2^e$ $(0.1001)_2 2^e$

$(0.1010)_2 2^e$ $(0.1011)_2 2^e$

$(0.1100)_2 2^e$ $(0.1101)_2 2^e$

$(0.1110)_2 2^e$ $(0.1111)_2 2^e$

Normalized form: $f = \pm(1.d_1 d_2 d_3 d_4 \cdots d_m)_\beta \beta^e$

$(1.0000)_2 2^e$ $(1.0100)_2 2^e$ $(1.1000)_2 2^e$ $(1.1100)_2 2^e$

$(1.0001)_2 2^e$ $(1.0101)_2 2^e$ $(1.1001)_2 2^e$ $(1.1101)_2 2^e$

$(1.0010)_2 2^e$ $(1.0110)_2 2^e$ $(1.1010)_2 2^e$ $(1.1110)_2 2^e$

$(1.0011)_2 2^e$ $(1.0111)_2 2^e$ $(1.1011)_2 2^e$ $(1.1111)_2 2^e$

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⑥ For each of the forms, find the smallest positive number and the largest number representable by the system.

Lecture form:

$$f = \pm (0.d_1 d_2 d_3 d_4 \dots d_m)_\beta \, \beta^e$$

Smallest positive number: $(0.1000)_2 \, 2^{-2} = (1 \times \frac{1}{2}) \times \frac{1}{4} = \frac{1}{8}$

Largest positive number: $(0.1111)_2 \, 2^{2} = (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}) \times 4 = \frac{15}{4}$

Normalized form:

$$f = \pm (1.\, d_1 d_2 d_3 \dots d_m)_\beta$$

Smallest positive number: $(1.0000)_2 \times 2^{-2} = (2^0) \times \frac{1}{4} = \frac{1}{4}$

Largest positive number: $(1.1111)_2 \times 2^{2} = 31/4$

Denormalized form:

$$f = \pm (0.1\, d_1 d_2 d_3 \dots d_m) \times_\beta \beta^e$$

Smallest positive number: $(0.10000)_2 \, 2^{-2} = 1/8$

Largest positive number: $(0.11111)_2 \, 2^{2} = \frac{31}{8}$

(c) For the IEEE standard (1985) for double-precision (64-bit) arithmetic, find the smallest, positive number and the largest number representable by a system that follow this standard. Do not find their decimal values; instead, follow the below format:

$$\pm (0.1 d_1 d_2 d_3 \ldots d_m)_\beta \beta^{e-\text{exponent Bias}}$$

Smallest positive number:

$$(0.10\ldots\ldots0)_2 2^{1-1022}$$

$$=(0.1)_2 2^{-1021}$$

Largest positive number:

$$(0.11\ldots1)_2 2^{2046-1022}$$

$$= (0.11\ldots1)_2 2^{1024}$$

(d) If the bias is 500 what is the smallest and largest positive number?

Smallest positive number:

$$(0.10\ldots0)_2 2^{1-\cancel{2022}\,500}$$

$$= (0.1)_2 2^{-499}$$

Largest positive number: $(0.11\ldots1)_2 2^{2046-500}$

$$= (0.11\ldots1)_2 2^{1546}$$

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Subject: Date:

4. $\beta = 2;\ m = 3$

$e_{min} = -1$

$e_{max} = 2$

(a) Find the floating-point representation of the number $(6.25)_{10}$ and $(6.875)_{10}$ in the normalized form. That is, find $fl[6.25]$ and $fl[6.875]$.

$$\begin{array}{r|l} 2 & 6 \\ 2 & 3-0 \\ 2 & 1-1 \\ & 0-1 \end{array}$$

$0.25 \times 2 = 0.5$

$0.5 \times 2 = 1.0$

$(6.25)_{10} = (110.01)_2 \times 2^0$

$= (1.1001)_2 \times 2^2$

$= (1.100)_2 \times 2^2$ [As $m = 3$]

$\therefore fl[6.25] = (1.100)_2 \times 2^2$

1.100 1.1001 1.101

$$\begin{array}{r|l} 2 & 6 \\ 2 & 3-0 \\ 2 & 1-1 \\ & 0-1 \end{array}$$

$0.875 \times 2 = 1.75$

$0.75 \times 2 = 1.5$

$0.5 \times 2 = 1.0$

$(6.875)_{10} = (110.111)_2 \times 2^0$

$= (1.10111)_2 \times 2^2$

$= (1.101)_2 \times 2^2$

$= (1.110)_2 \times 2^2$

1.1001 1.10111 1.110

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Subject:

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ⓑ What are the rounding errors δ_1, δ_2 in part ⓐ?

$$\delta_1 = \left| \frac{fl(x) - x}{x} \right| = \left| \frac{fl[6.25] - 6.25}{6.25} \right|$$

$$= \left| \frac{6.00 - 6.25}{6.25} \right|$$

$$= \frac{1}{25}$$

$$\delta_2 = \left| \frac{fl(x) - x}{x} \right| = \left| \frac{fl[6.875] - 6.875}{6.875} \right|$$

$$= \left| \frac{7 - 6.875}{6.875} \right|$$

$$= \frac{1}{55}$$

ⓒ Can the value represented in the Denormalized form?

$$(6.25)_{10} = (110.01)_2 \times 2^0 = (0.11001)_2 \times 2^3$$

$$(6.875)_{10} = (110.111)_2 \times 2^0 = (0.110111)_2 \times 2^3$$

Here $e = 3$ but the $e_{max} = 2$

$\therefore$ Cannot be representable in denormalized form.

Subject : Date :

(d) find the upper-bound of the rounding error for Lecture Note, Normalized and Denormalized form.

Given that,

$\beta = 2$

$m = 3$

Lecture Note : $\frac{1}{2}\beta^{1-m} = \frac{1}{2}\beta^{1-3} = \frac{1}{2}\times(2)^{-2} = \frac{1}{8}$

Normalized : $\frac{1}{2}\beta^{-m} = \frac{1}{2}(2)^{-3} = \frac{1}{16}$

Denormalized : $\frac{1}{2}\beta^{-m} = \frac{1}{2}(2)^{-3} = \frac{1}{16}$

5. Given, $\beta = 2$, $m = 4$, $e_{min} = -4$, $e_{max} = 2$

Lecture Note form : $f = \pm(0.d_1 d_2 d_3 d_4 \dots d_m)_\beta \beta^e$ — (a)

Normalized form: $f = \pm(1.d_1 d_2 d_3 \dots d_m)_\beta \beta^e$ — (b)

Denormalized form: $f = \pm(0.d_1 d_2 d_3 \dots d_m)_\beta \beta^e$ — (c)

(a) From the eq (a) find all the decimal numbers for $e = -3$ and plot it into the real line

From eq (a) : $f = \pm(0.d_1 d_2 d_3 \dots d_m)_\beta \beta^e$

Here, $\beta = 2$ and $m = 4$, $e = -3$

Subject : Date :

1. $(0.1000)_2 \times 2^{-3}$

$= (0 \times 2^0 + 1 \times 2^{-1}) \times \frac{1}{8}$

$= \frac{1}{2} \times \frac{1}{8} = \frac{1}{16}$

2. $(0.1001)_2 \times 2^{-3} = \{(0 \times 2^0) + (1 \times 2^{-1}) + (1 \times 2^{-4})\} \times \frac{1}{8}$

$= \frac{9}{128}$

3. $(0.1010)_2 \times 2^{-3} = \{(0 \times 2^0) + (1 \times \frac{1}{2}) + (1 \times \frac{1}{8})\} \times \frac{1}{8}$

$= \frac{5}{64}$

4. $(0.1011)_2 \times 2^{-3} = \{(0 \times 2^0) + (\frac{1}{2}) + (\frac{1}{8}) + (\frac{1}{16})\} \times \frac{1}{8}$

$= \frac{11}{128}$

5. $(0.1100)_2 \times 2^{-3} = \{\frac{1}{2} + \frac{1}{4}\} \times \frac{1}{8} = \frac{3}{32}$

6. $(0.1101)_2 \times 2^{-3} = \{\frac{1}{2} + \frac{1}{4} + \frac{1}{16}\} \times \frac{1}{8} = \frac{13}{128}$

7. $(0.1110)_2 \times 2^{-3} = \{\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\} \times \frac{1}{8} = \frac{7}{64}$

8. $(0.1111)_2 \times 2^{-3} = \{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\} \times \frac{1}{8} = \frac{15}{128}$

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Loss of Significance

$ax^2+bx+c=0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \quad \begin{cases} \rightarrow \dfrac{-b+\sqrt{b^2-4ac}}{2a} \\[2ex] \rightarrow \dfrac{-b-\sqrt{b^2-4ac}}{2a} \end{cases}$$

Again,

$ax^2+bx+c=0$

or, $x^2+\frac{b}{a}x+\frac{c}{a}=0$

α and β roots

$x^2+(\alpha+\beta)x+\alpha\beta$

Here, $\alpha+\beta = -\frac{b}{a}$ —— (i)

$\alpha\beta = \frac{c}{a}$ —— (ii)

Problem:

$x^2-56x+1=0$, for 4 significant figure

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-(-56) \pm \sqrt{(-56)^2-4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$x_1 = \frac{56+\sqrt{3132}}{2} \qquad x_2 = \frac{56-\sqrt{3132}}{2}$$

$= 28+3\sqrt{87}$ $\qquad$ $= 28-3\sqrt{87}$

$= 55.98213716$ $\qquad$ $= 0.01786284073$

$= 55.98$ $\qquad$ $= 0.01786$

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Subject :   Date :

Computation of Computer:

$\alpha = 28 + \sqrt{783}$

$= 28 + 27.98$

$= 55.98$

$\beta = 28 - \sqrt{783}$

$= 28 - 27.98$

$= 0.02$

$28 - 27.98213716$

$= 0.01786284073$

Relative error $= \left| \frac{fl(x) - x}{x} \right| = \left| \frac{0.01786284073 - 0.02}{0.01786284073} \right|$

$= 0.1196427432$

$\alpha\beta = c/a$

$\beta = \frac{c}{\alpha \cdot a} = \frac{1}{55.98 \times 1} = 0.01786352269$

$= 0.01786$ for 4 significant figure

$\therefore \alpha = 55.98$

$\beta = 0.01786$

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Subject: Assignment - 1 Date:

5. (a) Standard form: $f = \pm (0.d_1 d_2 d_3 \ldots d_m)_\beta \beta^e$

Normalized form: $f = \pm (1.d_1 d_2 d_3 \ldots d_m)_\beta \beta^e$

Denormalized form: $f = \pm (0.1 d_1 d_2 d_3 \ldots d_m)_\beta \beta^e$

(a) What are the maximum numbers in decimal values?

$\beta = 2$, $m = 4$, $e_{max} = 2$, $e_{min} = -3$

Standard form: $(0.1111)_2 \times 2^2 = \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\right) \times \frac{1}{4}$

$= \frac{15}{64}$

Normalized form: $(1.1111)_2 \times 2^2 = \left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\right) \times \frac{1}{4}$

$= 31/64$

Denormalized form: $(0.11111)_2 \times 2^2 = \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}\right) \times 1/4$

$= 31/128$

(b) What are the non-negative minimum numbers?

Standard form: $(0.1000)_2 \times 2^{-3} = (1 \times 1/2) \times 1/8 = 1/16$

Normalized form: $(1.0000)_2 \times 2^{-3} = (1 \times 2^0) \times 1/8 = 1/8$

Denormalized form: $(0.10000)_2 \times 2^{-3} = (1 \times 1/2) \times 1/8 = \frac{1}{16}$

Subject: Assignment-1 | Date:

ⓒ What are the maximum and minimum numbers that can be stored in the system by the three forms defined above if the system has negative support?

Standard form: $f = \pm(0.d_1 d_2 d_3 \ldots\ldots d_m)_\beta \beta^e$

maximum: $(0.1111)_2 2^2$

non-negative minimum: $(0.1000)_2 \times 2^{-3}$

negative minimum: $-$ maximum

$= -(0.1111)_2 \times 2^2$

Normalized form: $f = \pm(1.d_1 d_2 d_3 \ldots\ldots d_m)_\beta \beta^e$

maximum: $(1.1111)_2 \times 2^2$

non-negative minimum: $(1.0000)_2 \times 2^{-3}$

negative minimum: $-$ maximum

$= -(1.1111)_2 \times 2^2$

Denormalized form: $f = \pm(0.1 d_1 d_2 d_3 \ldots\ldots d_m)_\beta \beta^e$

maximum: $(0.11111)_2 \times 2^2$

non-negative minimum: $(0.10000)_2 \times 2^{-3}$

negative minimum: $-(0.11111)_2 \times 2^2$

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Subject: Date:

2. Consider the real number $x = (3.1416)_{10}$

(a) First convert the decimal number x in binary format at least up to 9 decimal/binary places

$x = (3.1416)_{10}$

Integer part:

$$\begin{array}{r|l|l} 2 & 3 & 1 \\ & 2 & \\ \hline & 1 & \end{array}$$

$$\begin{array}{r|l} 2 & 3 \\ \hline 2 & 1 - 1 \\ \hline & \end{array}$$

$(11)_2$

Fractional Part:

$0.1416 \times 2 = 0.2832$

$0.2832 \times 2 = 0.5664$

$0.5664 \times 2 = 1.1328$

$0.1328 \times 2 = 0.2656$

$0.2656 \times 2 = 0.5312$

$0.5312 \times 2 = 1.0624$

$0.0624 \times 2 = 0.1248$

$0.1248 \times 2 = 0.2496$

$0.2496 \times 2 = 0.4992$ — 9th

$0.4992 \times 2 = 0.9984$

$0.9984 \times 2 = 1.9968$

$0.9968 \times 2 = 1.9936$

$(3.1416) = (11.001001000)_2$

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Subject : Date :

(6) What will be the binary value of x [Find $fl(x)$] if you store it in a system with $m=4$ and $m=5$ using standard form of floating point representation?

For $m=4$;

$$(3.1416)_{10} \rightarrow (11.001001000)_2 \times 2^0$$

$$= (1.1001001000)_2 \times 2^1$$

$$= (0.11001001000)_2 \times 2^2$$

$$= (0.1100)_2 \times 2^2$$

0.1100 0.1101

For $m=5$;

$$(3.1416)_{10} \rightarrow (11.001001000)_2 \times 2^0$$

$$= (1.1001001000)_2 \times 2^1$$

$$= (0.11001001000)_2 \times 2^2$$

$$= (0.11001)_2 \times 2^2$$

0.11000 0.11001 0.1101

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Subject : Date :

(c) Error, $\delta = \left| \frac{fl(x) - x}{x} \right|$

When $m=4$,

$$(0.1100)_2 = (0\times2^0 + 1\times2^{-1} + 1\times2^{-2} + 0\times2^{-3} + 0\times2^{-4}) \times 4$$

$$= (0 + 1/2 + 1/4 + 0 + 0) \times 4$$

$$fl(x)_4 = 3$$

When $m=5$,

$$(0.11001)_2 \times 2^2 = (0\times2^0 + 1\times 1/2 + 1\times 1/4 + 1\times 1/32) \times 4$$

$$fl(x)_5 = 3.125$$

$$\delta_4 = \left| \frac{fl(x) - x}{x} \right| = \left| \frac{3 - 3.1416}{3.1416} \right| = 0.0450725$$

$$\delta_5 = \left| \frac{fl(x) - x}{x} \right| = \left| \frac{3.125 - 3.1416}{3.1416} \right| = 0.00528393175$$

Subject : Date :

2. (a) $x^2-70x+9=0$. Below Calculate up to 6 significant figure.

(a) Find out where the loss of significance occurs when you calculate the roots?

$x^2-70x+9=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \qquad x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-70)+\sqrt{(-70)^2-4\times1\cdot9}}{2\times1} \qquad =\frac{-(-70)-\sqrt{(-70)^2-4\cdot1\cdot9}}{2\cdot1}$$

$$=35+8\sqrt{19} \qquad =35-8\sqrt{19}$$

$$=69.87119155 \qquad =0.1288084517$$

$$=69.8711 \qquad =0.128808$$

Compalation of Computer

$$x_1=35+8\sqrt{19} \qquad x_2=35-8\sqrt{19}$$

$$=35+34.8711 \qquad =35-34.8711$$

$$=69.8711 \qquad =0.1289$$

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$$\text{Error} = \left|\frac{fl(x)-x}{x}\right| = \left|\frac{0.1288084517 - 0.1289}{0.1288084517}\right|$$

$$= 7.107320893 \times 10^{-4}$$

$$= 0.0007107320893$$

(c) $x^2 - 70x + 9 = 0$

$x^2 + (\alpha+\beta)x + \alpha\beta = 0$ $\qquad ax^2 + bx + c = 0$

$\alpha + \beta = -70$ $\qquad a = 1$

$\alpha\beta = 9$ $\qquad b = -70$

$c = 9$

$$\alpha\beta = \frac{c}{a}$$

$$\beta = \frac{c}{\alpha a}$$

$$= \frac{9}{69.8711 \times 1}$$

$$= 0.1288086204 = 0.128808$$

$\alpha = 69.8711$

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(b) we know,

$ax^2+bx+c=0$ where $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x^2+(\alpha+\beta)x+\alpha\beta=0$

where,

$\alpha+\beta=-b/a$

$\alpha\beta=c/a$

for, $x^2-70x+9=0$

$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}$

$=\frac{-(-70)+\sqrt{(-70)^2-4.1.9}}{2.1}$

$=69.87119155$

$=69.8711$

$\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$

$=0.128808$

$\alpha+\beta=-b/a$

$69.8711+0.128808=-\frac{(-70)}{1}$

or, $69.999908\neq 70$

$\alpha\beta=c/a$

$8.999956649\neq 9$

$\left|\frac{69.999908-70}{\cancel{69.999908}\ 70}\right|$

$=0.0000013142857 14$

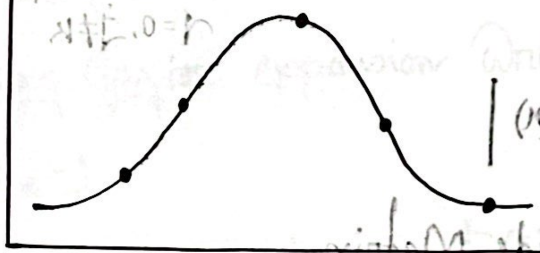
$\left|\frac{8.999956649-9}{9}\right|$

$=0.00000481 6777$

Subject: Date:

Polynomial Interpolation

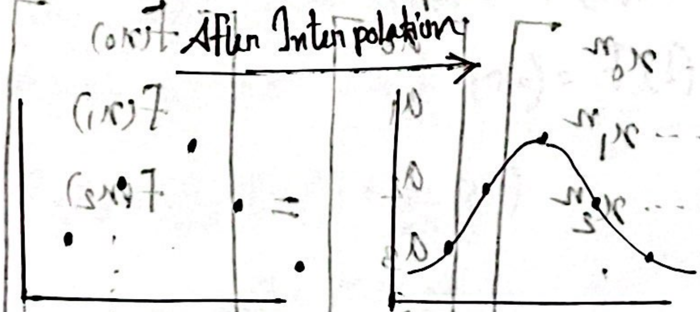
$P_n(x) = a_0x^0 + a_1x^1 + a_2x^2 + \dots + a_nx^n$



Here, n = degree

a_0, a_1, a_2 = constant/coefficients

$P_n(x)$ has $n+1$ coefficients



Basis:

$P_2(x) = a_0x^0 + a_1x^1 + a_2x^2$

$= a_0 1 + a_1x^1 + a_2x^2$

Basis $= \{1, x^1, x^2\}$

if degree $= n$ then; basis = degree+1

Taylor Series:

$$f(x) = f(x_0) + f^{(1)}(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \dots\dots$$

If we take upto the third term, it becomes a polynomial of degree 2

This terms are part of the error

Taylor's Theorem:

$$f(x) = \underbrace{\sum_{K=0}^{n} \frac{f^{(K)}(x_0)}{K!}(x-x_0)^K}_{\text{Taylor's polynomial's of degree } n} + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}}_{\text{Lagrange form of the remainder}}$$

$\boxed{\text{N.B}}$ We dont know the actual value of ξ

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Rules of Polynomial:

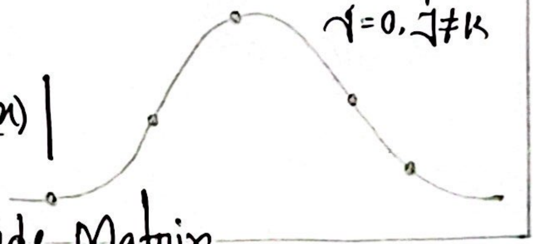
ⓐ Exponent must be non-negative.

ⓑ Must be an integer.

Transition Error: $|f(x) - P_n(x)|$

Lagrange polynomial

$$L_K(x) = \prod_{j=0,\, j\neq K}^{n} \frac{x - x_j}{x_K - x_j}$$



Vandermonde Matrix

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}$$

Vandermonde Matrix, V   A   F

$V \cdot A = F$

or $A = V^{-1} F$ [V matrix must be invertable]

Lagrange

Natural Basis: $\{1, x, x^2, \ldots\}$ | Lagrange: $\{L_0(x), L_1(x), L_2(x) \ldots\}$

Basis: $\sum_{K=0}^{n} a_K x^K$ Basis: $\sum_{K=0}^{n} f(x_K) L_K(x)$

Basis Calculation of Lagrange:

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\cdots(x_0-x_n)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)}$$

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Subject: Date:

1. $f(x) = xe^x$; An interpolating polynomial, $P_3(x)$ is computed by using Taylor expansion.

(a) Using Taylor expansion write $f(x)$ as an infinite series, centered at $x_0 = 1$

$f(x) = xe^x \longrightarrow f(x_0) = f(1) = 1e^1 = e$

$f^1(x) = xe^x + e^x \longrightarrow f^1(x_0) = f(1) = 1e^1 + e^1 = 2e$

$f^2(x) = \frac{d}{dx}(xe^x) + \frac{d}{dx}(e^x)$

$= xe^x + e^x + e^x$

$= xe^x + 2e^x \longrightarrow f^2(x_0) = f^2(1) = 1e^1 + 2e^1 = 3e$

$f^3(x) = \frac{d}{dx}(xe^x + 2e^x)$

$= xe^x + e^x + 2e^x$

$= 3e^x + xe^x \longrightarrow f^3(x_0) = f^3(1) = 3e^1 + 1e^1 = 4e$

$f(x) = f(x_0) + f^1(x_0)(x - x_0) + \frac{f^2(x_0)}{2!}(x - x_0)^2 + \frac{f^3(x_0)}{3!}(x - x_0)^3 + \cdots$

at $x_0 = 1$

$f(x) = e + 2e(x-1) + \frac{3e}{2 \times 1}(x-1)^2 + \frac{4e}{3 \times 2 \times 1}(x-1)^3 + \cdots$

$= e + 2e(x-1) + \frac{3e}{2}(x-1)^2 + \frac{4e}{6}(x-1)^3 + \cdots \infty$

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Subject : Date :

(b) Find the values of a_0, a_1, a_2 and a_3 if the function is interpolating by a degree 3 polynomial $P_3(x)$:

$a_0 = f(x_0) = e$

$a_1 = \frac{f'(x_0)}{1!} = 2e$

$a_2 = \frac{f''(x_0)}{2!} = 3e/2$

$a_3 = \frac{f^3(x_0)}{3!} = \frac{4e}{6}$

(c) Compute $f(0.1)$ and $P_3(0.1)$ up to seven significant figures.

$f(x) = xe^x$

$f(0.1) = (0.1)e^{0.1} = 0.1105170$

$P_3(x) = e + 2e(x-1) + \frac{3}{2}e(x-1)^2 + 4/6\, e(x-1)^3$

$P_3(0.1) = e + 2e(0.1-1) + 3/2\, e(0.1-1)^2 + 4/6\, e(0.1-1)^3$

$= -0.1929980098$

$= -0.1929980$

Subject : Date :

(d) Find the percent error for interpolating $f(0.1)$ by $P_3(0.1)$

$$\text{Percent error} = \frac{|f(0.1) - P_3(0.1)|}{f(0.1)} \times 100\%$$

$$= \frac{0.1105170 + 0.1929980}{0.1105170} \times 0.01$$

(e) Find the maximum error using 4th degree $(P_4(x))$ Taylor's Polynomial while approximating $f(0.2)$ on interval $[0, 0.2]$ [given $x_0 = 0$]

$f(x) = xe^x$

$$f^4(x) = \frac{d}{dx}\left(3e^x + xe^x\right)$$

$$= 3e^x + xe^x + e^x$$

$$= 4e^x + xe^x$$

$$f^5(x) = \frac{d}{dx}\left(4e^x + xe^x\right)$$

$$= 4e^x + xe^x + e^x$$

$$= 5e^x + xe^x$$

$$f^5(0.2) = 5e^{0.2} + 0.2(e)^{0.2}$$

$= 6.3513$ (max value on interval $[0, 0.2]$)

$$f(x) = P_4(x) + \frac{f^5(\xi)}{5!}(x-x_0)^5$$

$$|f(x) - P_4(x)| = \left|\cancel{P_4(x)} + \frac{f^5(\xi)}{5\times4\times3\times2}(x-0)^5 - \cancel{P_4(x)}\right|$$

$$= \frac{(5e^{\xi} + \xi e^{\xi})}{120} x^5$$

at $x = 0.2$,

$$|f(0.2) - P_4(0.2)| \le \frac{\cancel{6.5}\ 6.3513}{120} \times (0.2)^5$$

$$\le 1.6937 \times 10^{-5}$$

2. $f(x) = \sin x$. An interpolating polynomial $P_3(x)$ is computed by taylor expension

(a) Using Taylor expansion Write $f(x)$ as an infinite series. Centered at $x_0 = 1$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f^2(x_0)}{2!}(x-x_0)^2 + \frac{f^3(x_0)}{3!}(x-x_0)^3 + \dots$$

$$f(x) = \sin x = \sin(x_0) = \sin(1)$$

$$f'(x) = \cos x = \cos(x_0) = \cos(1)$$

$$f''(x) = -\sin x = -\sin(x_0) = -\sin(1)$$

$$f'''(x) = -\cos x = -\cos(x_0) = -\cos(1)$$

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Now,

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f^2(x_0)}{2!}(x-x_0)^2 + \frac{f^3(x_0)}{3!}(x-x_0)^3 + \cdots$$

$$f(x) = \sin(1) + \cos(1)(x-1) + \frac{-\sin(1)}{2\times 1}(x-1)^2 + \frac{-\cos(1)}{3\times 2\times 1}(x-1)^3 + \cdots$$

$$= \sin(1) + \cos(1)(x-1) - \frac{\sin(1)}{2}(x-1)^2 - \frac{\cos(1)}{6}(x-1)^3 + \cdots$$

(b) find the values of a_0, a_1, a_2, a_3 if the function is interpolating by a degree 3 polynomial $P_3(x)$.

$$P_3(x) = a + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3$$

$a_0 = f(x_0) = \sin(1)$

$a_1 = \frac{f'(x_0)}{1!} = \cos(1)$

$a_2 = \frac{f^2(x_0)}{2!} = \frac{-\sin(1)}{2}$

$a_3 = \frac{f^3(x_0)}{3!} = \frac{-\cos(1)}{6}$

Subject: | Date:

(c) Compute $F(0.1)$ and $P_3(0.1)$ up to 7 significant figure

$f(x) = \sin x$

$f(0.1) = \sin(0.1) = 0.09983341$ [Ans]

$P_3(x) = \sin(1) + \cos(1)(x-1) - \frac{\sin(1)}{2}(x-1)^2 - \frac{\cos(1)}{6}(x-1)^3$

$P_3(0.1) = \sin(1) + \cos 1\,(0.1-1) - \frac{\sin(1)}{2}(0.1-1)^2 - \frac{\cos(1)}{6}(0.1-1)^3$

$= 0.08004989$ [Ans]

(d) Find the percentage error for interpolating $f(0.1)$ by $P_3(0.1)$.

$$\text{Percentage Error} = \left| \frac{F(0.1) - P_3(0.1)}{F(0.1)} \right| \times 100\%$$

$$= \left| \frac{0.09983341 - 0.08004989}{0.09983341} \right|$$

$= \cancel{0.019}\ 0.1981653236 \times 100\%$

$= 19.81\%$

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(e) Find the maximum error using 4th degree Taylor's Polynomial while approximating $f(0.2)$ on interval $[0, 0.2]$. give that $x_0 = 0$

$$f^4(x) = \frac{f^5(x_0)}{5!}(x-x_0)^5$$

$$f^4(0.2) = \frac{f^5(\xi)}{5!}(x-0)^5 = \frac{f^{(5)}(\xi)}{5!}x^5 \quad \text{(a)}$$

From (a):

$$f^3(x) = -\cos(x)$$

$$f^4(x) = \frac{d}{dx}(-\cos(x)) = \sin x$$

$$f^5(x) = \frac{d}{dx}(\sin x) = \cos x$$

Now,

$$f^5(0.2) = \cos(0.2)$$
$$= 0.98000665$$

$$|f(x) - P_4(x)| = \frac{\cos(0.2)}{5!}x^5$$

at $x = 0.2$

$$|f(0.2) - P_4(0.2)| \le \frac{0.98000665}{120} \times (0.2)^5$$

$$\le 2.61335 \times 10^{-6}$$

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Subject: | Date:

Vandermonde Matrix

$f(x) = xe^x$. We are going to find interpolating polynomial by using the Vandermonde matrix.

ⓐ Construct the Vandermonde matrix √ If $f(x)$ passes through the nodes $\{-1, 0, 1\}$

Here basis = 3

Basis = degree + 1

∴ degree = 3 − 1
= 2

$x_0 = -1$; $f(x_0) = -1e^{-1} = -0.3678$

$x_1 = 0$; $f(x_1) = 0$

$x_2 = 1$; $f(x_2) = 1e^1 = 2.71828$

$$V = \begin{bmatrix} 1 & x_0^1 & x_0^2 \\ 1 & x_1^1 & x_1^2 \\ 1 & x_2^1 & x_2^2 \end{bmatrix} = \begin{bmatrix} 1 & (-1)^1 & (-1)^2 \\ 1 & (0)^1 & (0)^2 \\ 1 & (1)^1 & (1)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Now,

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{bmatrix}$$

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(b) Compute det (V)

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = 1(0-0) + 1(1-0) + 1(1-0)$$

$$= 0 + 1 + 1$$

$$= 2$$

(c) Write expression of $P_2(x)$

$$A = V^{-1} F$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -0.5 & 0 & 0.5 \\ 0.5 & -1 & 0.5 \end{bmatrix} \begin{bmatrix} -0.3678 \\ 0 \\ 2.71828 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1.575 \\ 1.1752 \end{bmatrix}$$

$$\therefore P_2(x) = a_0 + a_1 x + a_2 x^2$$

$$= 0 + 1.575x + 1.1752x^2$$

$$= 1.575x + 1.1752x^2$$

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(d) Find the Percentage Error for 0.25

$$F(0.25) = 0.25e^{0.25} = 0.3210063542$$

$$P_2(0.25) = 1.575 \times (0.25) + 1.1752(0.25)^2$$

$$= 0.4672$$

$$\text{Error} = \left| \frac{F(0.25) - P_2(0.25)}{F(0.25)} \right| \times 100\%$$

$$= \left| \frac{0.3210063542 - 0.4672}{0.3210063542} \right| \times 100\%$$

$$= 0.4554229033 \times 100\%$$

$$= 45.54\%$$

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4. Construct an appropriate polynomial for the following data using Hermite basis by following the question.

x	$f(x)$	$f'(x)$
$x_0 \to -1$	0	1
$x_1 \to 0$	1	0
$x_2 \to 1$	0	1

ⓐ Find the lagrange basis from the given data.

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{x(x-1)}{2}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+1)(x-1)}{(0+1)(0-1)} = \frac{x^2-1}{-1}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+1)(x-0)}{(1+1)(1-0)} = \frac{x(x+1)}{2}$$

ⓑ Compute Hermite basis

Total nodes = 3

$\therefore$ ~~degree~~ = ~~2~~ $n = 3-1 = 2$

We know $(2n+1)$ degree should be made.

for $n=2$ degree $= (2\times2+1) = 5$

Lagrange Polynomial

$$f(x_0)L_0 + f(x_1)L_1 + f(x_2)L_2 = \frac{x^2-1}{-1}$$

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$$P_5(x) = f(x_0)h_0(x) + f(x_1)h_1(x) + f(x_2)h_2(x) + f'(x_0)\hat{h}_0(x) + f'(x_1)\hat{h}_1(x) + f'(x_2)\hat{h}_2(x)$$

$$= 0h_0(x) + 1h_1(x) + 0h_2(x) + 1\hat{h}_0(x) + 0\hat{h}_1(x) + 1\hat{h}_2(x)$$

$$= h_1(x) + \hat{h}_0(x) + \hat{h}_2(x)$$

We know,

$$h_k(x) = L_k^2(x)\left[1 - 2(x-x_k)L_k'(x)\right]$$

$$\hat{h}_k(x) = L_k^2(x)(x-x_k)$$

$$h_1(x) = L_1^2(x)\left[1 - 2(x-x_1)L_1'(x)\right]$$

$$= \left(\frac{x^2-1}{-1}\right)^2\left[1 - 2x(-2x)\right]$$

$$= (1-x^2)^2\left[1-4x^2\right] \quad [Ans]$$

$$\hat{h}_0(x) = L_k^2(x)(x-x_k)$$

$$= L_0^2(x)(x-x_0)$$

$$= \left[\frac{x(x-1)}{2}\right]^2(x+1)$$

$$= \frac{x^2(x-1)^2}{4}(x+1) \quad [Ans]$$

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$$\hat{h}_2(x) = L_2^2(x)\,(x-x_2)$$

$$= \left[\frac{x(x+1)}{2}\right]^2 (x-1)$$

$$= \frac{x^2(x+1)^2}{4}(x-1) \quad [\text{Ans}]$$

ⓒ Find expression of interpolating Hermite polynomial

$$P_5(x) = h_1(x) + \hat{h}_0(x) + \hat{h}_2(x)$$

$$= (1-x^2)^2\left[1-4x^2\right] + \frac{x^2(x-1)^2}{4}(x+1) + \frac{x^2(x+1)^2}{4}(x-1)$$

ⓓ Use expression to determine the approximate value at $x=0.5$

$$P_5(x) = (1-x^2)^2\left[1-4x^2\right] + \frac{x^2(x-1)^2}{4}(x+1) + \frac{x^2(x+1)^2}{4}(x-1)$$

$$P_5(0.5) = (1-0.25)^2[1-1] + \frac{0.25\,(0.25)}{4}(-0.5) + \frac{0.25\,(1.5)^2}{4}(-0.5)$$

$$= -\frac{1}{128} - \frac{9}{128}$$

$$= -0.078125$$

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5. Construct an appropiate polynomial for the data using Newton's devided differance method :-

ⓐ $f(x) = \sin x$ and nodes are $= \{0, \pi/2, \pi\}$. Find the values of a_0, a_1, a_2.

$f(x) = \sin x$

$x_0 = f(x_0) = \sin x_0 = \sin 0 = 0$

$x_1 = f(x_1) = \sin(x_1) = \sin \pi/2 = 1$

$x_2 = f(x_2) = \sin(x_2) = \sin \pi = 0$

$P_2(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$

$= \cancel{f[x_0] + f_1[x_0,x_1] + f[x_0,x_1,x_2]}$

$= f[x_0] + f_1[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$

$x_0 = 0 \quad f[x_0] = 0$

$x_1 = \pi/2 \quad f[x_1] = 1$

$x_2 = \pi \quad f[x_2] = 0$

$$f[x_0,x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{1-0}{\pi/2 - 0} = 2/\pi$$

$$f[x_0,x_1,x_2] = \frac{f[x_2] - f[x_0]}{x_2 - x_0} = 0$$

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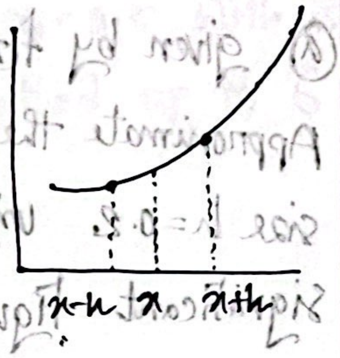
forword $= \dfrac{f(x+h) - f(x)}{h} + \dfrac{f''(\xi)}{2!}(-h)$

central $= \dfrac{f(x+h) - f(x-h)}{2h} + \dfrac{f'''(\xi)}{3!}(h^2)$



forword $= \dfrac{f(x+h) - f(x)}{h} = \dfrac{\ln(3) - \ln(2)}{1}$

Rounding Error $= \dfrac{f(x+h) - f(x-h)}{2h} = f'(x)$

$h \propto$ error
higher value of h is the cause of higher error. (for forword)

Error:

$|$Actual value of diff − value of diff by numerical$|$

$h^2 \propto$ error for central

$$= \left| \frac{f(x_1+h) - f(x_1-h)}{2h} - \frac{f^3(\xi)h^2}{3!} - \frac{fl[f(x_1+h)] - fl[f(x_1-h)]}{2h} \right|$$

$$= \left| \frac{f(x_1+h) - f(x_1-h)}{2h} - \frac{f^3(\xi)h^2}{3!} - \frac{(1+\delta_1)f(x_1+h) - (1+\delta_2)f(x_1-h)}{2h} \right|$$

$$= \left| -\frac{f^3(\xi)h^2}{3!} - \frac{\delta_1 f(x_1+h) - \delta_2 f(x_1-h)}{2h} \right|$$

$|a+b| \le |a| + |b|$

$|\delta_1|, |\delta_2| \le \varepsilon_H$

$$\le \left| \frac{f^3(\xi)h^2}{3!} \right| + \left| \frac{\delta_1 f(x_1+h) - \delta_2 f(x_1-h)}{2h} \right|$$

$$\le \underbrace{\left| \frac{f^3(\xi)}{6} \right| h^2}_{h \propto \text{rounding error}} + \underbrace{\varepsilon_H \frac{|f(x_1+h)| + |f(x_1-h)|}{2h}}_{h \propto \frac{1}{\text{rounding error}}}$$

$h \propto \dfrac{1}{\text{rounding error}}$

Subject :

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1. (a) given by $f(x) = 2x - e^{-6x}$

Approximate the derivative of $f(x)$ at $x_0 = 0.5$ With step size $h = 0.2$ using forword difference method up to 5 significant figure.

$$f(x) = 2x - e^{-6x}$$

$$x_0 = 0.5$$

$$h = 0.2$$

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \frac{f(0.5+0.2) - f(0.5)}{0.2}$$

$$= \frac{f(0.7) - f(0.5)}{0.2}$$

$$= \frac{\left[2(0.7) - e^{-6(0.7)}\right] - \left[2(0.5) - e^{-6(0.5)}\right]}{0.2}$$

$$= \frac{\left[1.4 - e^{-4.2}\right] - \left[1 - e^{-3}\right]}{0.2}$$

$$= \frac{1.385004423 - 0.9502129316}{0.2}$$

$$= 2.173957457$$

$$= 2.1739$$

Subject : Date :

⑥ Approximate the derivative of $f(x)$ at $x_0 = 0.5$ with step size $h = 0.2$ using central difference method up to 6 significant figure.

$$f(x) = 2x - e^{-6x}$$

$$x_0 = 0.5$$

$$h = 0.2$$

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

$$= \frac{f(0.5+0.2) - f(0.5-0.2)}{2 \times (0.2)}$$

$$= \frac{f(0.7) - f(0.3)}{0.4}$$

$$= \frac{\left[2 \times (0.7) - e^{-6(0.7)}\right] - \left[2 \times (0.3) - e^{-6(0.3)}\right]}{0.4}$$

$$= \frac{\left[1.4 - e^{-4.2}\right] - \left[0.6 - e^{-1.8}\right]}{0.4}$$

$$= \frac{1.385004423 - 0.4347011118}{0.4}$$

$$= 2.3757582278$$

$$= 2.37575$$

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ⓒ Calculate the upper bound of truncation error of $f(x)$ at $x_0 = 2$ using $h = 0.1$ in both of the above mentioned methods for the interval $[2.4, 2.7]$

$$f(x) = 2x - e^{-6x}$$

$$f'(x) = \frac{d}{dx}\left[2x - e^{-6x}\right]$$

$$= 2 - (-6)e^{-6x}$$

$$f'(x) = 2 + 6e^{-6x}$$

For interval $[2.4, 2.7]$ with $h = 0.1$ we will get

$x = 2.4, 2.5, 2.6, 2.7$

question ⓐ Forward method

for $x = 2.4$,

Actual value:

$$f'(2.4) = 2 + 6e^{-6\times(2.4)}$$

$$= 2 + 6e^{-14.4}$$

$$= \cancel{2.000000557}\ 2.00000334$$

Forward value:

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$$f'(2.4) = \frac{f(2.4+0.1) - f(2.4)}{0.1}$$

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$$= \frac{f(2.5) - f(2.4)}{0.1}$$

$$= \frac{\left[2(2.5) - e^{-6(2.5)}\right] - \left[2(2.4) - e^{-6(2.4)}\right]}{0.1}$$

$$= \frac{\left[5 - e^{-15}\right] - \left[4.8 - e^{-14.4}\right]}{0.1}$$

$$= \frac{4.999999694 - 4.799999443}{0.1}$$

$$= 2.00000251$$

Since, $(2.4-h)$ is out of limit there is no backward value.

Truncation Error:

$$\text{Error} = \left|\text{Actual} - \text{Forward}\right|$$

$$= \left|\cancel{2.000000557 - 2.00000251}\right|$$

$$= \left|2.000003344 - 2.00000251\right|$$

$$= 8.3434221 \times 10^{-7}$$

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$$= \frac{f(2.5) - f(2.4)}{0.1}$$

$$= \frac{\left[2(2.5) - e^{-6(2.5)}\right] - \left[2(2.4) - e^{-6(2.4)}\right]}{0.1}$$

$$= \frac{\left[5 - e^{-15}\right] - \left[4.8 - e^{-14.4}\right]}{0.1}$$

$$= \frac{4.999999694 - 4.799999443}{0.1}$$

$$= 2.00000251$$

Since. (2.4-h) is out-of limit there is no backward value.

Truncation Error:

$$\text{Error} = \left|\text{Actual} - \text{Forward}\right|$$

$$= \left|\cancel{2.000000557 - 2.00000251}\right|$$

$$= \left|2.000003344 - 2.00000251\right|$$

$$= 8.3434221 \times 10^{-7}$$

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For $x = 2.5$

Actual value:

$$f'(2.5) = 2 + 6e^{-6\times 2.5}$$

$$= 2.000001835$$

Forward Value:

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$$f'(2.5) = \frac{f(2.5+0.1) - f(2.5)}{0.1}$$

$$= \frac{f(2.6) - f(2.5)}{0.1}$$

$$= \frac{\left[2(2.6) - e^{-6\times 2.6}\right] - \left[2(2.5) - e^{-6(2.5)}\right]}{0.1}$$

$$= \frac{\left[5.2 - e^{-15.6}\right] - \left[5 - e^{-15}\right]}{0.1}$$

$$= \frac{5.199999832 - 4.999999694}{0.1}$$

$$= 2.00000138$$

Error:

$$|2.000001835 - 2.00000138| = 4.55 \times 10^{-7}$$

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Backward Value =

$$f'(x) = \frac{f(x) - f(x-h)}{h}$$

$$f'(2.5) = \frac{f(2.5) - f(2.5-0.1)}{0.1}$$

$$= \frac{\left[2(2.5) - e^{-6(2.5)}\right] - \left[2(2.4) - e^{-6(2.4)}\right]}{0.1}$$

$$= \frac{\left[5 - e^{-15}\right] - \left[4.8 - e^{-14.4}\right]}{0.1}$$

$$= \frac{4.999999694 - 4.799999443}{0.1}$$

$$= 2.00000251$$

Error:

$$\left| 2.000001835 - 2.00000251 \right|$$

$$= 6.75 \times 10^{-7}$$

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For $x = 2.6$;

Actual Value:

$$f'(2.6) = 2 + 6e^{-6(2.6)}$$

$$= 2.000001007$$

Forward Value:

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$$f'(2.6) = \frac{f(2.6+0.1) - f(2.6)}{0.1}$$

$$= \frac{\left[2(2.7) - e^{-6(2.7)}\right] - \left[2(2.6) - e^{-6(2.6)}\right]}{0.1}$$

$$= \frac{\left[5.4 - e^{-16.2}\right] - \left[5.2 - e^{-15.6}\right]}{0.1}$$

$$= \frac{5.399999908 - 5.199999832}{0.1}$$

$$= 2.000000759$$

Error:

$$\left|2.000001007 - 2.000000759\right|$$

$$= 2.48 \times 10^{-7}$$

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Backward Value:

$$f'(x) = \frac{f(x) - f(x-h)}{h}$$

$$f'(2.6) = \frac{f(2.6) - f(2.6-0.1)}{0.1}$$

$$= \frac{f(2.6) - f(2.5)}{0.1}$$

$$= \frac{\left[2(2.6) - e^{-6(2.6)}\right] - \left[2(2.5) - e^{-6(2.5)}\right]}{0.1}$$

$$= \frac{\left[5.2 - e^{-15.6}\right] - \left[5 - e^{-15}\right]}{0.1}$$

$$= \frac{5.199999832 - 4.999999694}{0.1}$$

$$= 2.00000138$$

Error:

$$\left| 2.000001007 - 2.00000138 \right|$$

$$= 3.73 \times 10^{-7}$$

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For $x = 2.7$;

Actual value:

$$f'(2.7) = 2 + 6e^{-6(2.7)}$$

$$= 2.000000553$$

Since $(2.7+h)$ is out of limit, there is no forward value

Backward Value:

$$f'(x) = \frac{f(x) - f(x-h)}{h}$$

$$= \frac{f(2.7) - f(2.6)}{h}$$

$$= \frac{[5.4 - e^{-16.2}] - [5.2 - e^{-15.6}]}{0.1}$$

$$= \frac{5.399999908 - 5.199999832}{0.1}$$

$$= 2.00000076$$

Error:

$$|2.000000553 - 2.00000076|$$

$$= 2.07 \times 10^{-7}$$

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(d) given that,

$x_0 = 0.2$

We know that,

$$D_h = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

$$D_{h/2} = \frac{f(x_0+h/2) - f(x_0-h/2)}{2(h/2)}$$

Now,

$$D_{0.5} = \frac{f(0.2+0.5) - f(0.2-0.5)}{2(0.5)}$$

$$= \frac{f(0.7) - f(-0.3)}{1}$$

$$= \cancel{2+6e} \left[2(0.7) - e^{-6(0.7)}\right] - \left[2(-0.3) - e^{-6(-0.3)}\right]$$

$$= \left[1.4 - e^{-4.2}\right] - \left[-0.6 - e^{1.8}\right]$$

$$= 1.385004423 + 6.649647464$$

$$= 8.0346518871$$

$$= 8.03465$$

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$$D_{0.25} = \frac{f(x_0+h/2) - f(x_0-h/2)}{2(h/2)}$$

$$= \frac{f(0.2+0.25) - f(0.2-0.25)}{2(0.25)}$$

$$= \frac{f(0.45) - f(-0.05)}{0.5}$$

$$= \frac{\left[2(0.45) - e^{-6(0.45)}\right] - \left[2(-0.05) - e^{-6(-0.05)}\right]}{0.5}$$

$$= \frac{\left[0.9 - e^{-2.7}\right] - \left[-0.1 - e^{0.3}\right]}{0.5}$$

$$= \frac{0.832794487 + 1.449858808}{0.5}$$

$$= 4.565306591$$

$$D_h^1 = \frac{(2)^2 D_{h/2} - D_h}{(2)^2 - 1} \quad \text{—— ②}$$

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OR, $$D_{0.5}^{(1)} = \frac{(2)^2 D_{0.25} - D_{1/2}}{(2)^2 - 1}$$

$$= \frac{4(4.565306591) - (8.03465)}{4-1}$$

$$= \frac{10.22657636}{3}$$

$$= 3.408858787$$

Now,

$$f(x) = 2x - e^{-6x}$$

$$f'(x) = \frac{d}{dx}\left[2x - e^{-6x}\right]$$

$$= 2 + 6e^{-6x}$$

$$f'(0.2) = 2 + 6e^{-6(0.2)}$$

$$= 3.807165271$$

$$\text{Error} = 3.807165271 - 3.408858787$$

$$= 0.398306484$$

$$= 0.398306$$

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$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f^2(x_0)}{2!}(x-x_0)^2 + \cdots$$

$$f(x_1) = f(x_1) + f'(x_1)(x-x_1) + \frac{f^2(x_1)}{2!}(x-x_1)^2 + \cdots\cdots$$

$$f(x_1+h) = f(x_1) + f'(x_1)(x_1+h-x_1) + \frac{f^2(x_1)}{2!}(x_1+h-x_1)^2 + \cdots$$

$$= f(x_1) + f'(x_1)(h) + \frac{f^2(x_1)}{2!}(h)^2 + \frac{f^3(x_1)}{3!}(h)^3 + O(h^4)$$

$$f(x_1-h) = f(x_1) + f'(x_1)(x_1-h-x_1) + \frac{f^2(x_2)}{2!}(x_1-h-x_1)^2 + \cdots$$

$$D_h = \frac{f(x_1+h) - f(x_1-h)}{2h}$$

$$= \frac{1}{2h} f(x_1+h) - f(x_1-h)$$

$$= \underbrace{f'(x_1)}_{\text{Value}} + \underbrace{\frac{f^3(x_1)h^2}{3!} + \frac{f^5(x_1)h^4}{5!} + O(h^6)}_{\text{Error}}$$

or, $$D_h = f'(x_1) + \frac{f^3(x_1)}{3!}h^2 + \frac{f^5(x_1)h^4}{5!} + O(h^6)$$

or, $$D_{h/2} = f'(x_1) + \frac{f^3(x_1)}{3!}\left(\frac{h}{2}\right)^2 + \frac{f^5(x_1)}{5!}\left(\frac{h}{2}\right)^4 + O(h^6)$$

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$$D_h = \frac{f(x+h) - f(x-h)}{2h}$$

$$f(x+h) = f(x) + f'(x)h + \frac{f^2(x)}{2!}h^2 + \frac{f^3(x)}{3!}h^3 + \frac{f^4(x)}{4!}h^4 + \frac{f^5(x)}{5!}h^5 + O(h^6)$$

$$f(x-h) = f(x) - f'(x)h + \frac{f^2(x)}{2!}h^2 - \frac{f^3(x)}{3!}h^3 + \frac{f^4(x)}{4!}h^4 - \frac{f^5(x)}{5!}h^5 + O(h^6)$$

Now,

$$D_h = \frac{f(x+h) - f(x-h)}{2h}$$

$$= \frac{1}{2h}\left[2f'(x)h + \frac{2f^3(x)h^3}{3!} + \frac{2f^5(x)}{5!}h^5 + O(h^7)\right]$$

$$D_h = f'(x) + \boxed{\frac{f^3(x)h^2}{3!}} + \frac{f^5(x)h^4}{5!} + O(h^6)$$

$$D_{h/2} = f'(x) + \frac{f^3(x)}{3!}\left(\frac{h}{2}\right)^2 + \frac{f^5(x)}{5!}\cancel{h^4}\left(\frac{h}{2}\right)^4 + O(h^6)$$

$$= f'(x) + \boxed{\frac{f^3(x)}{3!}\frac{h^2}{4}} + \frac{f^5(x)}{5!}\frac{h^4}{16} + O(h^6)$$

$$4D_{h/2} - \cancel{4}D_h = 3f'(x) + 3\frac{f^5(x)}{5!}\frac{h^4}{\cancel{16}4} + O(h^6)$$

$$\frac{4D_{h/2} - D_h}{3} = f'(x) + \frac{f^5(x)}{5!}\frac{h^4}{\cancel{1}4} + O(h^6) \longrightarrow D_h^{(1)}$$

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2. (a) given that

$$D_h^{(1)} = f'(x_0) - \frac{h^4}{480} f^{(5)}(x_0) + O(h^6)$$

$$D_{h/2}^{(1)} = f'(x_0) - \frac{(h/2)^4}{480} f^5(x_0) + O(h^6)$$

$$= f'(x_0) - \frac{f^5(x_0)}{480}\frac{h^4}{16} + O(h^6)$$

Now,

$$16 D_{h/2}^{(1)} - D_h^{(1)} = 16 f'(x_0) - \frac{f^5(x_0)\, h^4}{480} + O(h^6) - f'(x_0) + \frac{h^4 f^5(x_0)}{480} + O(h^6)$$

$$16 D_{h/2}^{(1)} - D_h^{(1)} = 15 f'(x_0) + O(h^6)$$

$$\frac{16 D_{h/2}^{(1)} - D_h^{(1)}}{15} = f'(x_0) + O(h^6) = D_h^{(2)}$$

(b) $D_h = \dfrac{f(x+h) - f(x-h)}{2h}$

$$f(x+h) = f(x) + f'(x)h + \frac{f^2(x)}{2!}h^2 + \frac{f^3(x)}{3!}h^3 + \frac{f^4(x)}{4!}h^4 + \frac{f^5(x)}{5!}h^5 + \cdots$$

$$f(x-h) = f(x) - f'(x)h + \frac{f^2(x)}{2!}h^2 - \frac{f^3(x)}{3!}h^3 + \frac{f^4(x)}{4!}h^4 - \frac{f^5(x)}{5!}h^5 + \cdots$$

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$$D_h = \frac{1}{2h}\left[2f'(x)h + \frac{2f^{3}(x)}{3!}h^3 + \frac{2f^{5}(x)}{5!}h^5 + O(x^7)\right]$$

$$D_h = f'(x) + \frac{f^{(3)}(x)}{3!}h^2 + \frac{f^{(5)}(x)}{5!}h^4 + O(h^6)$$

$$D_{h/3} = f'(x) + \frac{f^{(3)}(x)}{3!}\left(\frac{h}{3}\right)^2 + \frac{f^{(5)}(x)}{5!}\left(\frac{h}{3}\right)^4 + O(h^6)$$

$$D_{h/3} = f'(x) + \frac{f^{(3)}(x)}{3!}\frac{h^2}{9} + \frac{f^{(5)}(x)}{5!}\frac{h^4}{81} + O(h^6)$$

$$9D_{h/3} = 9f'(x) + \frac{f^{(3)}(x)}{3!}h^2 + \frac{f^{(5)}(x)}{5!}\frac{h^4}{9} + O(h^6)$$

Now,

$$9D_{h/3} - D_h = 9f'(x) - f'(x) + \frac{f^3(x)h^2}{3!} - \frac{f^3(x)h^2}{3!} + \frac{f^5(x)}{5!}\frac{h^4}{9} - \frac{f^5(x)h^4}{5!} + O(h^6)$$

$$9D_{h/3} - D_h = 8f'(x) - \frac{8}{9}\frac{f^5(x)h^4}{5!} + O(h^6)$$

$$9D_{h/3} - D_h = 8\left[f'(x) - \frac{1}{9}\frac{f^5(x)h^4}{5!} + O(h^6)\right]$$

$$\frac{9D_{h/3} - D_h}{8} = f'(x) - \frac{1}{9}\frac{f^5(x)h^4}{5!} + O(h^6)$$

$$D_h^{(1)} = f'(x) - \frac{1}{9}\frac{f^5(x)h^4}{5!} + O(h^6)$$

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ⓒ from question ⓑ;

$$D_h^{(1)} = f'(x) - \frac{1}{9}\frac{f^{(5)}(x)\,h^4}{5!} + O(h^6)$$

Here error term is: $-\frac{1}{9}\frac{f^{(5)}(x)h^4}{5!} + O(h^6)$ where the leading error is $-\frac{1}{9}\frac{f^{(5)}(x)h^4}{5!}$

Error bound is the absolute value of the error term

$$\left|-\frac{1}{9}\frac{f^{(5)}(x)h^4}{5!}\right| + O(h^6)$$

$$\text{or, } \frac{1}{9}\left|\frac{f^{(5)}(x)h^4}{5!}\right| + O(h^6)$$

ⓓ given that,

$f(x) = \ln(x)$

$x_0 = 1$

$h = 0.1$

Error term from the question ⓒ: $\frac{1}{9}\left|\frac{f^{(5)}(x)\,h^4}{5!}\right| + O(h^6)$

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$f(x) = \ln(x)$

$f'(x) = 1/x$

$f^{(2)}(x) = \frac{d}{dx}(1/x) = -1/x^2$

$f^3(x) = \frac{d}{dx}(-1/x^2) = \frac{d}{dx}(-x^{-2}) = -[-2x^{-3}] = 2x^{-3} = \frac{2}{x^3}$

$f^4(x) = \frac{d}{dx}(2/x^3) = \frac{d}{dx}(2x^{-3}) = 2(-3)x^{-4} = -6x^{-4} = -\frac{6}{x^4}$

$f^5(x) = \frac{d}{dx}(-6/x^4) = \frac{d}{dx}(-6x^{-4}) = -6 \times -4\, x^{-5} = 24x^{-5} = \frac{24}{x^5}$

$f^5(x) = \frac{24}{x^5}$

or, $f^5(1) = \frac{24}{(1)^5} = 24$

Now,

$$\frac{1}{9}\left|\frac{f^5(x)h^4}{5\times4\times3\times2\times1}\right|$$

$$= \frac{1}{9}\left|\frac{24(0.1)^4}{120}\right|$$

$$= \frac{1}{9}\left|\frac{1}{50000}\right|$$

$$= \frac{1}{450000} \text{ or } 2.2223\times10^{-6}$$

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All Equation

Machine Epsilon :

Normalized $\rightarrow \frac{1}{2} \beta^{-m}$

Denormalized $\rightarrow \frac{1}{2} \beta^{-m}$

Maximum Delta value : $\frac{1}{2} \beta^{1-m}$

forward $= \dfrac{f(x_0+h) - f(x_0)}{h}$

Central $= \dfrac{f(x_0+h) - f(x_0-h)}{2h}$

backward $= \dfrac{f(x) - f(x-h)}{h}$

Lagrang Basis : $L_0 = \dfrac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$ | $L_2 = \dfrac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$

Lagrange Polynomial : $f(x_0)L_0 + f(x_1)L_1 + f(x_2)L_2$

Relative Error (%) $= \dfrac{|f(x) - P(x)|}{|f(x)|} \times 100$ for $x = x$

Truncation Error :

given, $h = 1, 0.1, 0.001, 0.01$

$f(x) = 6e^{-5x}$, $x_0 = 2$

Value h	forward ; $x_0 = 2$	error
0.1 0.01 0.001	$f'(x_0) = \dfrac{f(x_0+h) - f(x)}{h}$ where $x_0 = 2$	$f(x) = 6e^{-5x}$ $f'(2) = -30e^{-5x}$ $f'(2) = -30e^{-10}$ error $= [f'(2) - \text{forward}]$
Value h	Central ; $x_0 = 2$	error
0.1 0.01 0.001	$f'(x_0) = \dfrac{f(x_0+h) - f(x_0-h)}{2h}$ where $x_0 = 2$	$f(x) = 6e^{-5x}$ $f'(x) = -30e^{-5x}$ $f'(2) = -30e^{-10}$ error $= [f'(2) - \text{central}]$

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Newton's divided difference method:

$$P_2(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) \cdots$$

$$= f[x_0] + f[x_0, x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) \cdots$$

$f(x) = \sin x \rightarrow$ nodes $\{0, \pi/2, \pi\}$

$x_0 = 0;\ f[x_0] = \sin 0 = 0$

$$f[x_0,x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0} = \frac{1-0}{\pi/2 - 0} = 2/\pi$$

$x_1 = \pi/2;\ f[x_1] = \sin \pi/2 = 1$

$$f[x_1,x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1} = \frac{0-1}{\pi-\pi/2} = -2/\pi$$

$x_2 = \pi;\ f[\pi] = \sin \pi = 0$

$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2] - f[x_0,x_1]}{x_2 - x_0}$$

$$= \frac{-2/\pi - 2/\pi}{\pi - 0}$$

$$= \frac{-4}{\pi^2}$$

$a_0 = f[x_0] = 0$

$a_1 = f[x_0,x_1] = 2/\pi$

$a_2 = f[x_0,x_1,x_2] = -4/\pi^2$

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Polynomial Formula:

$a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$ ......

$0 + 2/\pi\,[x-0] + -\frac{4}{\pi^2}[x-0][x-\pi/2]$

$= \frac{2x}{\pi} - \frac{4}{\pi^2}\left[x^2 - \frac{x\pi}{2}\right]$ (—)

Adding a new node: $\{0, \pi/2, \pi, \frac{3\pi}{2}\}$ Fetch the previous:

$x_0 = 0$

$[x_0, x_1] = 2/\pi$

$[x_0, x_1, x_2] = \frac{-2/\pi - 2/\pi}{\pi - 0} = \frac{4}{\pi^2}$

$x_1 = \pi$

$[x_1, x_2] = -2/\pi$

$[x_1, x_2, x_3] = \frac{-2/\pi - 2/\pi}{3\pi/2 - \pi/2} = -\frac{4}{\pi^2}$

$x_2 = 0$

$[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2} = \frac{-1-0}{\frac{3\pi}{2} - \pi} = -2/\pi$

$x_3 = \sin\frac{3\pi}{2} = -1$

$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$

$= \frac{-4/\pi^2 - 4/\pi^2}{\frac{3\pi}{2} - 0} = \frac{-8}{\pi^2} \times \frac{2}{3\pi}$

$= \frac{-16}{3\pi^3}$

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Richardson Extrapolation

$$f(x+h) = \cancel{f(x)} + f'(x).h + \cancel{\frac{f^2(x)}{2!}h^2} + \frac{f^3(x)}{3!}h^3 + \cancel{\frac{f^4(x)}{4!}h^4} + \frac{f^5(x)}{5!}h^5 + \cdots$$

$$f(x-h) = \cancel{f(x)} - f'(x)h + \cancel{\frac{f^2(x)}{2!}h^2} - \frac{f^3(x)}{3!}h^3 + \cancel{\frac{f^4(x)}{4!}h^4} - \frac{f^5(x)}{5!}h^5 + \cdots$$

$(-)$

$$2f'(x)h + 2\frac{f^3(x)}{3!}h^3 + 2\frac{f^5(x)}{5!}h^5$$

$$f(x+h) - f(x-h) = 2f'(x)h + \frac{2f^3(x)}{3!}h^3 + 2\frac{f^5(x)}{5!}h^5$$

Now,

$$D_h = \frac{f(x+h) - f(x-h)}{2h}$$

$$= \frac{1}{2h}\left[2f'(x)h + \frac{2f^3(x)}{3!}h^3 + \frac{2f^5(x)}{5!}h^5\right]$$

$$D_h = f'(x) + \frac{f^3(x)}{3!}h^2 + \frac{f^5(x)}{5!}h^4 + O(h^6) \quad \text{—— a}$$

@ $$D_{h/2} = f'(x) + \frac{f^3(x)}{3!}(h/2)^2 + \frac{f^5(x)}{5!}(h/2)^4 + O(h^6)$$

$$= f'(x) + \frac{f^3(x)}{3!}\frac{h^2}{4} + \frac{f^5(x)}{5!}\frac{h^4}{16} + O(h^6)$$

$$4D_{h/2} = 4f'(x) + \frac{f^3(x)}{3!}h^2 + \frac{f^5(x)}{5!}\frac{h^4}{4} + O(h^6)$$

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$$4D_{h/2} - D_h = 3f'(x) - \frac{3}{4}\frac{f^5(x)}{5!}h^4 + O(h^6)$$

$$\frac{4D_{h/2} - D_h}{3} = f'(x) - \frac{1}{4}\frac{f^5(x)}{5!}h^4 + O(h^6)$$

$$\frac{4D_{h/2} - D_h}{(2)^2 - 1} = f'(x) - \frac{1}{4}\frac{f^5(x)}{5!}h^4 + O(h^6)$$

(b)

$$D_{h/3} = f'(x) + \frac{f^3(x)}{3!}\left(\frac{h}{3}\right)^2 + \frac{f^5(x)}{5!}\left(\frac{h}{3}\right)^4 + O(h^6)$$

$$= f'(x) + \frac{f^3(x)}{3!}\frac{h^2}{9} + \frac{f^5(x)}{5!}\frac{h^4}{81} + O(h^6)$$

$$9D_{h/3} = 9f'(x) + \frac{f^3(x)}{3!}h^2 + \frac{9}{81}\frac{f^5(x)}{5!}h^4 + O(h^6)$$

$$9D_{h_3} - D_h = 8f'(x) - \frac{8}{9}\frac{f^5(x)}{5!}h^4 + O(h^6)$$

$$\frac{9D_{h_3} - D_h}{8} = f'(x) - \frac{1}{9}\frac{f^5(x)}{5!}h^4 + O(h^6)$$

$$\frac{9D_{h/3} - D_h}{(3)^2 - 1} = f'(x) - \frac{1}{9}\frac{f^5(x)}{5!}h^4 + O(h^6)$$

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$$4D_{h/2} - D_h = 3f'(x) - \frac{3}{4}\frac{f^5(x)}{5!}h^4 + O(h^6)$$

$$\frac{4D_{h/2} - D_h}{3} = f'(x) - \frac{1}{4}\frac{f^5(x)}{5!}h^4 + O(h^6)$$

$$\frac{4D_{h/2} - D_h}{(2)^2 - 1} = f'(x) - \frac{1}{4}\frac{f^5(x)}{5!}h^4 + O(h^6)$$

(b)

$$D_{h/3} = f'(x) + \frac{f^3(x)}{3!}\left(\frac{h}{3}\right)^2 + \frac{f^5(x)}{5!}\left(\frac{h}{3}\right)^4 + O(h^6)$$

$$= f'(x) + \frac{f^3(x)}{3!}\frac{h^2}{9} + \frac{f^5(x)}{5!}\frac{h^4}{81} + O(h^6)$$

$$9D_{h/3} = 9f'(x) + \frac{f^3(x)}{3!}h^2 + \frac{9}{81}\frac{f^5(x)}{5!}h^4 + O(h^6)$$

$$9D_{h/3} - D_h = 8f'(x) - \frac{8}{9}\frac{f^5(x)}{5!}h^4 + O(h^6)$$

$$\frac{9D_{h/3} - D_h}{8} = f'(x) - \frac{1}{9}\frac{f^5(x)}{5!}h^4 + O(h^6)$$

$$\frac{9D_{h/3} - D_h}{(3)^2 - 1} = f'(x) - \frac{1}{9}\frac{f^5(x)}{5!}h^4 + O(h^6)$$

Subject: Date:

Upper Bound problem:

$$f(x) = e^{x} + e^{-x} \mid \text{nodes } \{-2, 0, 2\}$$

Evaluate the upper bound of interpolation error for the given function; interval $[-1, 1]$

$f(x) = e^{x} + e^{-x}$

$f'(x) = e^{x} - e^{-x}$

$f^{2}(x) = e^{x} + e^{-x}$

$f^{3}(x) = e^{x} - e^{-x}$

for upper bound:

$f^{3}(x) = e^{x} - e^{-x}$

$f^{3}(1) = e^{1} - e^{-1}$

$$\frac{f^{3}(x)}{3!}(x - x_0)^3 = \frac{f^{3}(\xi)}{3!}(x - x_0)^3$$

$$= \frac{e^{1} - e^{-1}}{3 \times 2 \times 1}(x - (-2))(x - 0)(x - 2)$$

$$= 1.1752$$

IEEE Maths:

$\beta = 2$

52 bits for fraction

11 bits for exponent

1 bit for sign

$e_{min} = 0$

$e_{max} = 2^{11} - 1 = 2047$

max non negative number - $(1.11 \cdots 1) \times 2^{2047}$

min non-negative number $(1.00 \cdots 0) \times 2^{0}$

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$f(x) = x\ln(x)$

$x_0 = 1.0$, $h = 0.1$

upper bound for the central difference.

Truncation Error $= \frac{h^2}{6} f^3(c)$

$f(x) = x\ln x$

$f'(x) = x \frac{1}{x} + \ln x$

$= 1 + \ln x$

$f^2(x) = \frac{1}{x}$

$f^3(x) = -\frac{1}{x^2}$

$c \in [x_0 - h, x_0 + h] = [0.9, 1.1]$

$|f^{(3)}c| = \left|-\frac{1}{(0.9)^2}\right| = 1.234567$

$|f^{(3)}c| = \left|-\frac{1}{(1.1)^2}\right| = 0.82644$

Truncation Error $= \frac{h^2}{6} \max|f^{(3)}c|$

$= \frac{(0.1)^2}{6} \max|f^{(3)}c|$

$= \frac{0.01}{6} \times 1.23456$

$= 2.0575 \times 10^{-3}$

If node exist:

$E(x) \leq \frac{|f^{(3)}c|}{3!} \max|(x-x_0)(x-x_1)(x-x_2)|$, here x = intern

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ⓐ $x^3 - 2x^2 - x + 2$

$x = -1, \quad x = 2$

ⓑ given that,

$$x^3 - 2x^2 - x + 2 = 0$$

$$x(x^2 - 2x - 1) + 2 = 0$$

$$x = \left(\frac{-2}{x^2 - 2x - 1}\right) = g(x) \quad \text{— ⓐ}$$

$$x^3 - 2x^2 - x + 2 = 0$$

$$x = (x^3 - 2x^2 + 2) \quad \text{— ⓑ}$$

$$x^3 - 2x^2 - x + 2 = 0$$

$$x^3 - x + 2 = 2x^2$$

$$\text{or, } x = \sqrt{\frac{x^3 - x + 2}{2}} \quad \text{— ⓒ}$$

ⓒ $x = \sqrt{\dfrac{x^3 - x + 2}{2}}$

$$\frac{d}{dx}\frac{-2}{(x^2 - 2x - 1)} = -2\frac{d}{dx}(x^2 - 2x - 1)^{-1}$$

$$= 2(x^2 - 2x - 1)^{-2} \cdot (2x - 2)$$

$$= \frac{(4x - 4)}{(x^2 - 2x - 1)^2} \quad \text{— ⓘ}$$

$x_0 = 0$

$x_1 = f(0) = 0$

$x = f$

$$g'(x) = \frac{(4x - 4)}{(x^2 - 2x - 1)^2}$$

$g'(4) = 0.24489$

$g'(0.24489) = -1.47745$

$g(-1.47745) = -0.57880$

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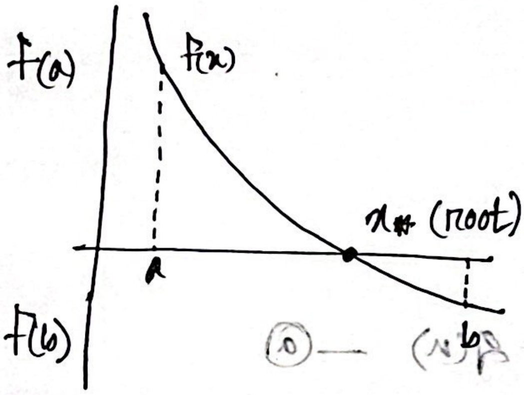
Root Finding

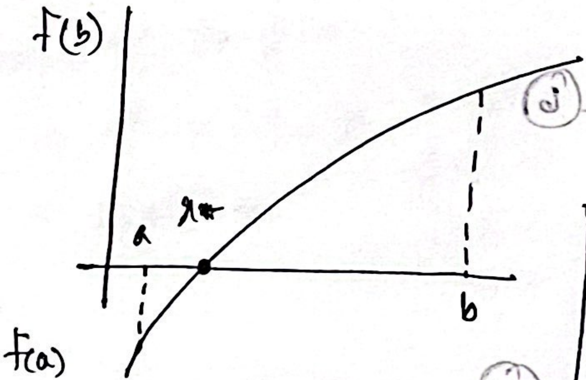
Root finding of non-linear equations



$f(a) = +ve$

$f(b) = -ve$

$$m = \left(\frac{a+b}{2}\right) \begin{cases} f(a) \times f(m) < 0 \rightarrow & [a \leftrightarrow m] \\ f(a) \times f(m) > 0 \rightarrow & [b \leftrightarrow m] \end{cases}$$



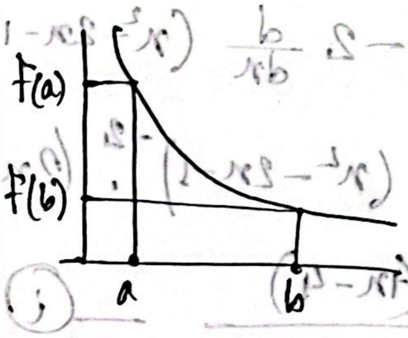
$f(a) = -ve$

$f(b) = +ve$

$\boxed{\text{MENU}} + \boxed{8}$ = spread sheet

$\boxed{\text{OPTN}} + \boxed{1}$ = set formula

Existance Check for Root $[f(a) * f(b) < 0] \rightarrow$ Exist

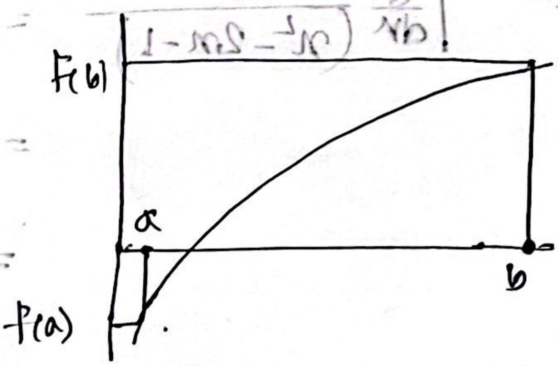


$f(a) * f(b)$

$= (+ve) * (+ve)$

$= (+ve)$

$\therefore$ Root doesnot exist



$f(a) * f(b)$

$= (-ve) * (+ve)$

$= (-ve)$

Exist Root

Subject :	

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1. $f(x) = 1/x - 0.5$ Interval $= [1.5, 3]$

2. $f(x) = x^3 - 7x^2 + 14x - 6 = 0$ Interval $[1, 3.2]$

K	a_K	m_K	b_K	$f(a_K)$	$f(b_K)$	$f(m_K)$	$x \in [\ ,\ ]$	$\lvert f(m) \rvert \leq 10^{-3}$
0	1	2.1	3.2	2	-0.112	1.791	[2.1, 3.2]	No
1	2.1	2.65	3.2	1.791	-0.112	0.552	[2.65, 3.2]	No
2.	2.65	2.925	3.2	0.5521	-0.112	0.0858	[2.925, 3.2]	No
3	2.925	3.062	3.2	0.0858	-0.112	-0.054	[2.925, 3.062]	No
9	2.998	3.000	3.002	1.96×10^{-3}	-2.3×10^{-3}	-1.95×10^{-4}		Yes

Subject :

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Finding number of iterations to find root:

Formula:

$$n \geq \frac{\log(|b-a|) - \log(\varepsilon)}{\log(2)} - 1$$

lets assume that,

Interval = $[1.5, 3]$

ε (machine epsilon) = 1.1×10^{-16}

$$n \geq \frac{\log(|3-1.5|) - \log(1.1 \times 10^{-16})}{\log(2)} - 1$$

$n \geq 52.59$ [minimum number of iteration]

$n \geq 53$ iterations

Fixed Point Iterations,

Convergence Rate, $\lambda = |g'(\text{root})|$

Here, if $\lambda < 1$; Converge

if $\lambda \geq 1$; Diverge

$\lambda = 0 \rightarrow$ super Linear Convergence

$0 < \lambda < 1 \rightarrow$ Linear Convergence

$\lambda \geq 1 \rightarrow$ Diverge

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Subject : Date :

1. Consider the function; $f(x) = x^3 - x^2 - 9x + 9$.

(a) State the exact roots of $f(x)$:

$$x^3 - x^2 - 9x + 9 = 0$$

$$\Rightarrow x^2(x-1) - 9(x-1) = 0$$

$$\Rightarrow (x^2-9)(x-1) = 0$$

$$x_k = -3, 1, 3$$

(b) Construct three different fixed point functions $g(x)$ such that $f(x)=0$; Make sure that one of the $g(x)$'s that you constructed converges to at least a root.

1st choice:

$$x^3 - x^2 - 9x + 9$$

$$9x = x^3 - x^2 + 9$$

$$x = \frac{x^3}{9} - \frac{x^2}{9} + 1$$

$$x = g(x)$$

2nd choice:

$$x(x^2 - x - 9) = -9$$

$$x = \frac{-9}{x^2 - x - 9}$$

$$x = g(x)$$

3rd choice:

$$x = x + x^3 - x^2 - 9x + 9$$

$$x = x^3 - x^2 - 8x + 9$$

$$x = g(x)$$

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(c) Find the $\boxed{\text{Convergence rate/ratio}}$ for $g(x)$ constructed in previous part and ~~ratio~~ also find which root is it converging to?

Convergence rate: $\lambda = \left| g'(x_*) \right|$

$$= \left| \frac{dg}{dx} \Big|_{x=x_*} \right|$$

For 1st case:

$$g(x) = \frac{1}{9}(x^3 - x^2 + 9)$$

$$g'(x) = \frac{1}{9}(3x^2 - 2x)$$

$$\lambda = \left| g'(x_*) \right| = \begin{cases} \frac{1}{9} < 1 \text{ for } x_* = 1 \text{ (Linear Convergence)} \\ \frac{11}{9} > 1 \text{ for } x_* = -3 \text{ (Divergence)} \\ 7/3 > 1 \text{ for } x_* = 3 \text{ (Divergence)} \end{cases}$$

$\therefore$ $g(x)$ is Converging to $x_* = 1$ for 1st case

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For 2nd case:

$$-9\,(\,(2\times A1-1)\,)\,/\,(A1\wedge(2)-A1-9)\wedge 2$$

$$g(x) = \frac{-9}{x^2-x-9}$$

$$g'(x) = -9\,\frac{-1}{(x^2-x-9)^2}(2x-1)$$

$$= \frac{-9(2x-1)}{(x^2-x-9)^2}$$

$$\lambda = \left| g'(x_*) \right| = \begin{cases} \frac{1}{9} < 1 \text{ for } x_* = 1 \text{ (Linear Convergence)} \\ 7 > 1 \text{ for } x_* = -3 \text{ (Divergence)} \\ 5 > 1 \text{ for } x_* = 3 \text{ (Divergence)} \end{cases}$$

$\therefore g(x)$ is Converging to $x_* = 1$ for second case.

For 3rd case:

$$g(x) = x^3 - x^2 - 8x + 9$$

$$g'(x) = 3x^2 - 2x - 8$$

$$\lambda = \left| g'(x_*) \right| = \begin{cases} 7 > 1 \text{ for } x_* = 1 \text{ (Divergence)} \\ 13 > 1 \text{ for } x_* = 3 \text{ (Divergence)} \\ 25 > 1 \text{ for } x_* = -3 \text{ (Divergence)} \end{cases}$$

not convergence for 3rd case.

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(d) find the approximate root, x_*, of the above function using $\boxed{\text{fixed point iterations}}$ up to 4 significant figures within the error bound of 1×10^{-3} using $x_0=0$ and any fixed point function $g(x)$ from part (b) that converges to root.

Error Bound, $\varepsilon_0 = 1\times10^{-3}$

$x_0 = 0$

$g(x) = \frac{1}{9}(x^3 - x^2 + 9)$

$g(0) = 1.000$

$g(1.000) = 1.000$

$\therefore$ 1.000 is the fixed point of $g(x)$ and it is also the root of $f(x)$.

2\. consider the function $f(x) = x^3 - 2x^2 - 11x + 12$ on the interval $[1.78, 8.25]$

(a) find the $\boxed{\text{maximum number of iterations}}$ required to find the root of the given equation within the error bound of 1×10^{-3}.

$$n \geq \frac{\log(|b-a|) - \log(\varepsilon_0)}{\log 2} - 1$$

$$n \geq \frac{\log(|8.25 - 1.78|) - \log(1\times10^{-3})}{\log 2} - 1$$

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$n \geq 11.65955$

$n \geq 12$ iterations

(b) Show 5 iterations using the $\boxed{\text{Bisection Method}}$ to find the root of the above function within the interval $[1.78, 8.25]$

K	a_K	b_K	m_K	$f(a_K)$	$f(b_K)$	$f(m_K)$	$\mathcal{E}_0 \in [\ ,\ ]$	$\lvert f(m) \rvert \leq 10^{-3}$
0	1.78	8.25	5.015	-8.27	346.64	32.66	[1.78, 5.015]	No
1	1.78	5.015	3.397	-8.27	32.66	-9.24	[3.397, 5.015]	No
2.	3.397	5.015	4.206	-9.24	32.66	4.759	[3.397, 4.20]	No
3.	3.397	4.20	3.801	-9.24	4.765	-3.78	[3.801, 4.20]	No
4.	3.801	4.20	4.003	-3.79	4.765	0.076	[3.801, 4.003]	No

3. Using $\boxed{\text{Newtons method}}$ to find the root x_* of the equation $f(x)$. $f(x) = x^2e^{-x} - 0.6$, up to machine epsilon of 1×10^{-4} starting with $x_0 = 0.2$.

$$f(x) = x^2e^{-x} - 0.6$$

$$f'(x) = -x^2e^{-x} + e^{-x}2x$$

$$= 2xe^{-x} - x^2e^{-x}$$

$$x_0 = 0.2$$

$$\mathcal{E}_0 = 1\times10^{-4}$$

$$x_{K+1} = x_K - \frac{f(x_K)}{f'(x_K)}$$

Calculator

A1 = (A1^(2) e^(-A1) } - 0.6)

÷ (2A1 e^(-A1) - A1^(2) e^(-A1))

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K	x_K	$f(x_K)$	$\lvert f(x_K) \rvert < \delta$
0	0.2	-0.56725	No
1	2.1245	-0.060	No
2	0.2049	-0.565	No
3	2.0929	----	----
4	----	----	----

4. given that $f(x) = x^3 - 2x^2 - x + 2$. This function has three roots and one root is $x_* = 1$.

(a) find the remaining two exact roots of the function $f(x)$ algebrically.

$$x^3 - 2x^2 - x + 2 = 0$$

$$\Rightarrow x^2(x-2) - 1(x-2) = 0$$

$$\Rightarrow (x-2)(x^2-1) = 0$$

Here, $x - 2 = 0$ | $x^2 - 1 = 0$

$x = 2$ | $x = \pm 1$

$\therefore x = -1, 1, 2$

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(b) Construct two different fixed point functions $g(x)$ such that $f(x) = 0$ given that,

$$f(x) = x^3 - 2x^2 - x + 2$$

Here,

$$x^3 - 2x^2 - x + 2 = 0$$

$$x(x^2 - 2x - 1) = -2$$

$$x = \left(\frac{-2}{x^2 - 2x - 1}\right) \quad \text{--- (a)}$$

Again,

$$x^3 - 2x^2 - x + 2 = 0$$

$$\Rightarrow \frac{x^3 - 2x^2 - x + 2}{x} = 0$$

$$\Rightarrow x^2 - 2x - 1 = -2/x$$

$$\Rightarrow x(x-2) = 1 - 2/x$$

$$\Rightarrow x(x-2) = \frac{x-2}{x}$$

$$\Rightarrow x = \frac{x-2}{x(x-2)}$$

$$\Rightarrow x = 1/x \quad \text{--- (b)}$$

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(8) Compute the Convergence rate, λ, for each fixed point function $g(x)$ obtained in the previous part, and state which root it is converging to or diverging.

We know that,

$$\lambda = \left| g'(x_*) \right|$$

For case 1:

$$g(x) = \left(\frac{-2}{x^2 - 2x - 1} \right)$$

$$g'(x) = \frac{d}{dx} \left[\frac{-2}{x^2 - 2x - 1} \right]$$

$$= -2 \frac{d}{dx} \left[\frac{1}{x^2 - 2x - 1} \right]$$

$$= +2 \frac{1}{(x^2 - 2x - 1)^2} (2x - 2)$$

$$= \frac{2(2x-2)}{(x^2 - 2x - 1)^2}$$

$$\lambda = \left| g'(x_*) \right| = \begin{cases} \text{when } x_* = 1;\ \lambda = 0 < 1 \text{ [Converge]} \\ \text{when } x_* = -1;\ \lambda = 2 > 1 \text{ [Diverge]} \\ \text{when } x_* = 2;\ \lambda = 4 > 1 \text{ [Diverge]} \end{cases}$$

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For case 2:

$$g(x) = \frac{1}{x}$$

$$g'(x) = \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$g'(x) = -\frac{1}{x^2}$$

$$\lambda = |g'(x_*)| = \begin{cases} \text{When } x_* = -1; \; \lambda = 1 \not< 1 \; [\text{Diverge}] \\ \text{When } x_* = 1; \; \lambda = 1 = 1 \; [\text{Diverge}] \\ \text{When } x_* = 2; \; \lambda = \frac{1}{4} < 1 \; [\text{Converge}] \end{cases}$$

4. Compute the fixed point function $g(x)$ for the Newton's method; and find the root within 10^{-6} starting with $x_0 = -3.5$

k	x_k	$f(x_k)$		$\lvert f(x_k) \rvert < 10^{-6}$
0	-3.5	-61.87		
1	-2.256	-17.4		
2	-2.752	-31.237		
3	-2.5455	-24.907		
4	-2.629	-27.3649		
5	-2.595	-26.3478		
6	-2.6093	-26.7728		
7	-2.6034			

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given that,

$$f(x) = x^3 - 2x^2 - x + 2$$

$$f'(x) = 3x^2 - 4x - 1$$

$$x_0 = -3.5$$

C_0 = Not given

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= -3.5 - \frac{(x_k)^3 - 2(x_k)^2 - (x_k) + 2}{3(x_k)^2 - 4(x_k) - 1}$$

-3.5 - (A1^(3) - 2(A1)^(2) - A1 + 2) ÷ (3A1^(2) - 4A1 - 1)

K	x_k	$f(x_k)$	$\lvert f(x_k) \rvert < 10^{-6}$
0	-3.5	-61.875	No
1	-2.2562	-17.4	No
2.	-1.508	-4.469	No
3.	-1.13	-0.866	No
4.	-1.012	-0.072	No
5.	-1	0	Yes
6.	-1		

∴ root = -1

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1. $2x_1 + x_2 - x_3 + 2x_4 = 5$

$4x_1 + 5x_2 - 3x_3 + 6x_4 = 9$

$-2x_1 + 5x_2 - 2x_3 + 6x_4 = 4$

$4x_1 + 11x_2 - 4x_3 + 8x_4 = 2$

(a) Explain how to find this system has unique solution or not

$$A = \begin{pmatrix} 2 & 1 & -1 & 2 \\ 4 & 5 & -3 & 6 \\ -2 & 5 & -2 & 6 \\ 4 & 11 & -4 & 8 \end{pmatrix}$$

This is a square matrix. $\det(A) = -12 \neq 0$. Hence A is non-singular matrix. This system has unique solution.

(b) Write down the augmented matrix.

$$\left(\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 4 & 5 & -3 & 6 & 9 \\ -2 & 5 & -2 & 6 & 4 \\ 4 & 11 & -4 & 8 & 2 \end{array}\right)$$

(c) Find the upper triangular matrix U

$$\left(\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 4 & 5 & -3 & 6 & 9 \\ -2 & 5 & -2 & 6 & 4 \\ 4 & 11 & -4 & 8 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 0 & 3 & -1 & 2 & -1 \\ 0 & 0 & -1 & 4 & 11 \\ 0 & 0 & 0 & 2 & 6 \end{array}\right)$$

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(d)

2.

$$x_1 + x_2 + x_3 = 6$$

$$2x_1 + 3x_2 + 4x_3 = 20$$

$$3x_1 + 4x_2 + 2x_3 = 17$$

(a) Construct the Frobenius matrix $F^{(1)}$ and $F^{(2)}$ from system.

$Ax = b$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 17 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$F^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$F^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

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(b) Compute the unit lower triangular matrix L.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 3 & 3/4 & 1 \end{bmatrix}$$

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1.
$$2x - 2y + z = -3$$
$$x + 3y - 2z = 1$$
$$3x - y - z = 2$$

Augmented matrix :
$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 1 & 3 & -2 & 1 \\ 3 & -1 & -1 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 1 & 3 & -2 & 1 \\ 3 & -1 & -1 & 2 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 0 & 4 & -5/2 & 5/2 \\ 0 & 2 & -5/2 & 13/2 \end{array}\right]$$

$[R_2' = R_2 - \frac{1}{2} R_1]$

$[R_3' = R_3 - \frac{3}{2} R_1]$

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 0 & 4 & -5/2 & 5/2 \\ 0 & 0 & -5/4 & 21/4 \end{array}\right]$$

$[R_3' = R_3 - \frac{1}{2} R_2]$ [Ans]

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2. previously;

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 0 & 4 & -5/2 & 5/2 \\ 0 & 0 & -5/4 & 21/4 \end{array}\right]$$

$2x_1 - 2x_2 + x_3 = -3$

$4x_2 - 5/2\, x_3 = 5/2$

$-5/4\, x_3 = 21/4$

or, $x_3 = -\frac{21}{5}$ — ⓐ

$4x_2 - 5/2\left(-\frac{21}{5}\right) = 5/2$

or, $x_2 = \cancel{-8/4} = -2$

$\cancel{2x_1 - (2\times 8/4) - \frac{21}{5} = -3}$

$\cancel{\text{or, } x_1 =}$

$2x_1 - (2\times -2) + \left(-\frac{21}{5}\right) = -3$

$2x_1 + 4 - 21/5 = -3$

$x_1 = -7/5$

$x_1 = -7/5$

$x_2 = -2$

$x_3 = -21/5$

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3.
$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 1 & 3 & -2 & 1 \\ 3 & -1 & -1 & 2 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 0 & 4 & -5/2 & 5/2 \\ 0 & 2 & -5/2 & 13/2 \end{array}\right] \qquad F_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 0 & 4 & -5/2 & 5/2 \\ 0 & 0 & -5/4 & 21/4 \end{array}\right] \qquad \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -3/2 & -1/2 & 1 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix}$$

~~Total $F = F_2 F_1$~~

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4. From (3) we get,

$$F_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix}$$

$$L = (F_1^{-1})(F_2^{-1})$$

$$F_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & 0 & 1 \end{bmatrix} \qquad F_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & 1/2 & 1 \end{bmatrix}$$

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(5) We know,

$A = LU$

$Lc = d$

$Ux = c$

U = Upper Triangular Matrix

A = Augmented Matrix

$L = (F_1^{-1})(F_2^{-1})$

d = constant of Augmented Matrix

c = from $Lc = d$ part

$Lc = d$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

$c_1 = -3$

$\frac{1}{2}c_1 + c_2 = 1$

$c_2 = 1 - (\frac{1}{2} \times -3)$

$= \frac{5}{2}$

$c_1 = -3$

$c_2 = \frac{5}{2}$

$c_3 = \frac{21}{4}$

$\frac{3}{2}c_1 + \frac{1}{2}c_2 + c_3 = 2$

or, $\frac{3}{2}(-3) + \frac{1}{2}(\frac{5}{2}) + c_3 = 2$

or $-\frac{9}{2} + \frac{5}{4} + c_3 = 2$

or, $c_3 = \frac{21}{4}$

$Ux = c$

$$\begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & -5/2 \\ 0 & 0 & -5/4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 5/2 \\ 21/4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & -5/2 \\ 0 & 0 & -5/4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 5/2 \\ 21/4 \end{bmatrix}$$

$2x_1 - 2x_2 + x_3 = 3$

$4x_2 - 5/2\, x_3 = 5/2$

$-5/4\, x_3 = 21/4$

or, $x_3 = -21/5$

$4x_2 - 5/2(-21/5) = 5/2$

$4x_2 = -2$

$2x_1 - 2(-2) + (-21/5) = -3$

or $x_1 = -7/5$

$\therefore x_1 = -7/5$

$x_2 = -2$

$x_3 = -21/5$

1. ⓐ Prove that,

$$S = \left\{ \frac{1}{\sqrt{5}} (2,-1,0)^T, \frac{1}{\sqrt{30}} (1,2,-5)^T, \frac{1}{\sqrt{24}} (2,4,2)^T \right\}$$

is orthonormal

$$S = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \frac{1}{\sqrt{24}} \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ -5/\sqrt{30} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{24} \\ 4/\sqrt{24} \\ 2/\sqrt{24} \end{bmatrix}$$

$$x = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix} \quad y = \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ -5/\sqrt{30} \end{bmatrix} \quad z = \begin{bmatrix} 2/\sqrt{24} \\ 4/\sqrt{24} \\ 2/\sqrt{24} \end{bmatrix}$$

$$x^T y = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ -5/\sqrt{30} \end{bmatrix} = 0$$

$$y^T z = \begin{bmatrix} 1/\sqrt{30} & 2/\sqrt{30} & -5/\sqrt{30} \end{bmatrix} \begin{bmatrix} 2/\sqrt{24} \\ 4/\sqrt{24} \\ 2/\sqrt{24} \end{bmatrix} = 0$$

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$$\frac{x^T z}{~~z^T x~~} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{24} \\ 4/\sqrt{24} \\ 2/\sqrt{24} \end{bmatrix} = 0$$

Now,

$$|x| = \sqrt{(2/\sqrt{5})^2 + (-1/\sqrt{5})^2 + 0} = 1$$

$$|y| = \sqrt{(1/\sqrt{30})^2 + (2/\sqrt{30})^2 + (-5/\sqrt{30})^2} = 1$$

$$|z| = \sqrt{(2/\sqrt{24})^2 + (4/\sqrt{24})^2 + (2/\sqrt{24})^2} = 1$$

Hence S is orthonormal. [proved]

(6) $f(x) = \sin x$ at the points $x_0 = 4$, $x_1 = 9$, $x_2 = -6$. Evaluate the best fit straight line using the discrete Square Approximation.

$x_0 = 4$; $f(x_0) = \sin(4) = -0.756$

$x_1 = 9$; $f(x_1) = \sin(9) = 0.412$

$x_2 = -6$; $f(x_2) = \sin(-6) = 0.279$

$$P_1(x) = a_0 + a_1 x$$

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$$\begin{bmatrix} 4^0 & 4^1 \\ 9^0 & 9^1 \\ -6^0 & -6^1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} -0.756 \\ 0.412 \\ 0.279 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 1 & 9 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} -0.756 \\ 0.412 \\ 0.279 \end{bmatrix}$$

Now,

$$A^T A x = A^T b$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 9 & -6 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 9 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 9 & -6 \end{bmatrix} \begin{bmatrix} -0.756 \\ 0.412 \\ 0.279 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 7 \\ 7 & 133 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} -0.065 \\ -0.99 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 7 & 133 \end{bmatrix}^{-1} \begin{bmatrix} -0.065 \\ -0.99 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 19/50 & -1/50 \\ -1/50 & 3/350 \end{bmatrix} \begin{bmatrix} -0.065 \\ -0.99 \end{bmatrix}$$

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$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} -4.9\times10^{-3} \\ -7.18\times10^{-3} \end{bmatrix}$$

$a_0 = -4.9\times10^{-3}$ and $a_1 = -7.18\times10^{-3}$

$$P_1(x) = a_0 + a_1 x$$

$$= -4.9\times10^{-3} - 7.18\times10^{-3}x$$

$$= -0.0049 - 0.00718x$$

2.

$f(0)=3,\ f(4)=-2,\ f(-1)=2,\ f(1)=1$

(a) From the given data find out the linearly independent column vectors u_1, u_2 and u_3.

$x_0 = 0,\ x_1 = 4,\ x_2 = -1$ and $x_3 = 1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 16 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Here $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $u_2 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 1 \end{bmatrix}$

$u_3 = \begin{bmatrix} 0 \\ 16 \\ 1 \\ 1 \end{bmatrix}$

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(b) Using Gram-Schmidt process construct the orthonormal column matrices q_1, q_2 and q_3 and then write down the Q matrix.

From question (a) we get;

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad u_2 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 1 \end{bmatrix} \quad u_3 = \begin{bmatrix} 0 \\ 16 \\ 1 \\ 1 \end{bmatrix}$$

$$P_1 = u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$P_2 = u_2 - \frac{u_2^T \cdot P_1}{P_1^T \cdot P_1} \times P_1$$

$$= \underset{A}{\begin{bmatrix} 0 \\ 4 \\ -1 \\ 1 \end{bmatrix}} - \frac{\overset{B}{\begin{bmatrix} 0 & 4 & -1 & 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\underset{C}{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \overset{D}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 \\ 4 \\ -1 \\ 1 \end{bmatrix} - \frac{4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

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$$= \begin{bmatrix} 0 \\ 4 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -2 \\ 0 \end{bmatrix}$$

$$P_3 = u_3 - \frac{u_3 \cdot P_1}{P_1 \cdot P_1} P_1 - \frac{u_3 \cdot P_2}{P_2 \cdot P_2} P_2$$

$$= \begin{bmatrix} 0 \\ 16 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 & 16 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 16 \\ 1 \\ 1 \end{bmatrix} - \frac{18}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 16 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 9/2 \\ 9/2 \\ 9/2 \\ 9/2 \end{bmatrix} = \begin{bmatrix} -9/2 \\ 23/2 \\ -7/2 \\ -7/2 \end{bmatrix}$$

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$$A\quad \frac{\begin{bmatrix} 0 & 16 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ -2 \\ 0 \end{bmatrix}}{B\quad \begin{bmatrix} -1 & 3 & -2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ -2 \\ 0 \end{bmatrix}} \quad C\quad \begin{bmatrix} -1 \\ 3 \\ -2 \\ 0 \end{bmatrix}$$

$$= \frac{46}{14} \begin{bmatrix} -1 \\ 3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -23/7 \\ 3 \times 23/7 \\ -2 \times 23/7 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{23}{7} \\ 69/7 \\ -46/7 \\ 0 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} -9/2 \\ 23/2 \\ -7/2 \\ -7/2 \end{bmatrix} - \begin{bmatrix} -23/7 \\ 69/7 \\ -46/7 \\ 0 \end{bmatrix} = \begin{bmatrix} -17/14 \\ 23/14 \\ 43/14 \\ -7/2 \end{bmatrix}$$

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Now, -

$$q_1 = \frac{P_1}{|P_1|} = \frac{1}{\sqrt{1+1+1+1}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \quad [Ans]$$

$$q_2 = \frac{P_2}{|P_2|} = \frac{1}{\sqrt{(-1)^2+(3)^2+(-2)^2}} \begin{bmatrix} -1 \\ 3 \\ -2 \\ 0 \end{bmatrix}$$

$$= \frac{\sqrt{14}}{14} \begin{bmatrix} -1 \\ 3 \\ -2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\sqrt{14}}{14} \\ \frac{3\sqrt{14}}{14} \\ -\frac{2\sqrt{14}}{14} \\ 0 \end{bmatrix} \quad [Ans]$$

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$$q_3 = \frac{P_3}{|P_3|} = \frac{1}{\sqrt{\left(\frac{-17}{14}\right)^2 + \left(\frac{23}{14}\right)^2 + \left(\frac{43}{14}\right)^2 + \left(-\frac{7}{2}\right)^2}} \begin{bmatrix} -17/14 \\ 23/14 \\ 43/14 \\ -7/2 \end{bmatrix}$$

$$= 0.196657 \begin{bmatrix} -17/14 \\ 23/14 \\ 43/14 \\ -7/2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.239 \\ 0.323 \\ 0.604 \\ -0.688 \end{bmatrix}$$

[Ans]

$$Q = \begin{bmatrix} 1/2 & -\frac{\sqrt{14}}{14} & -0.239 \\ 1/2 & \frac{3\sqrt{14}}{14} & 0.323 \\ 1/2 & \frac{-2\sqrt{14}}{14} & 0.604 \\ 1/2 & 0 & -0.688 \end{bmatrix}$$

[Ans]

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(c) Calculate the matrix elements of R. and write down the matrix R

$$R = Q^T A$$

From the question (b) we get,

$$Q = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{14}}{14} & -0.239 \\ \frac{1}{2} & \frac{3\sqrt{14}}{14} & 0.323 \\ \frac{1}{2} & \frac{-2\sqrt{14}}{14} & 0.604 \\ \frac{1}{2} & 0 & -0.688 \end{bmatrix}$$

$$Q^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{\sqrt{14}}{14} & \frac{3\sqrt{14}}{14} & \frac{-2\sqrt{14}}{14} & 0 \\ -0.239 & 0.323 & 0.604 & -0.688 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 16 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

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$$R = Q^T A$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{-\sqrt{14}}{14} & \frac{3\sqrt{14}}{14} & \frac{-2\sqrt{14}}{14} & 0 \\ -0.239 & 0.323 & 0.604 & -0.688 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 16 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 9 \\ 0 & 3.7416 & 12.2940 \\ 0 & 0 & 5.084 \end{bmatrix}$$

(d) Compute Rx and $Q^T b$.

$$R = \begin{bmatrix} 2 & 2 & 9 \\ 0 & 3.7416 & 12.2940 \\ 0 & 0 & 5.084 \end{bmatrix}$$

$$x = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$Rx = \begin{bmatrix} 2a_0 + 2a_1 + 9a_2 \\ 3.7416a_1 + 12.2940a_2 \\ 5.084a_2 \end{bmatrix}$$

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$$Q^T b = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{\sqrt{14}}{14} & \frac{3\sqrt{14}}{14} & -\frac{2\sqrt{14}}{14} & 0 \\ -0.239 & 0.323 & 0.604 & -0.688 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -3.4743 \\ -0.848 \end{bmatrix}$$

(2) let $x = (a_0\ a_1\ a_2)^T$ being the coefficients of the polynomial $P_2(x)$. Evaluate theses cofficients and write down the polynomial $P_2(x)$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 16 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 4 & -1 & 1 \\ 0 & 16 & 1 & 1 \end{bmatrix}$$

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$$A^T A x = A^T b$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 4 & -1 & 1 \\ 0 & 16 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 16 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 4 & -1 & 1 \\ 0 & 16 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 4 & 18 \\ 4 & 18 & 64 \\ 18 & 64 & 258 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -9 \\ -29 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 18 \\ 4 & 18 & 64 \\ 18 & 64 & 258 \end{bmatrix} \begin{bmatrix} 4 \\ -9 \\ -29 \end{bmatrix}$$

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$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -542 \\ -2002 \\ -7986 \end{bmatrix}$$

$a_0 = -542$, $a_1 = -2002$ and $a_3 = -7986$

$$P_2(x) = a_0 + a_1 x + a_2 x^2$$

$$= -542 - 2002x - 7986x^2$$

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Integration

(a) Exact Integration:

for interval [0,2]

$$\frac{d}{dx}(a^x) = a^x \ln(a)$$

$$\int (a^x) = \frac{a^x}{\ln(a)}$$

$f(x) = e^{2x}$

$$I(F) = \int_0^2 e^{2x}\,dx \qquad \int \ln(x) = x\ln x - x$$

$$= \left.\frac{e^{2x}}{2}\right|_0^2$$

$$= \boxed{26.80} \text{ (actual)}$$

(b) Closed Newton cotes formulae with degree $(n) = 1$

(Trapezium Rule)

For degree $(n) = 1$; nodes $= 2$ $\{x_0, x_1\}$

Formula:

$$I_1(f) = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

interval $[0,2]$ $f(x) = e^{2x}$

$$I_1(f) = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$$= \frac{2-0}{2}\left[e^{2(0)} + e^4\right]$$

$$= 55.598. \text{ (Approximate)}$$

Absolute Error $= \left|\text{Actual} - \text{Approximate}\right|$

$= 28.798$

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$$\text{Relative Error} = \frac{|\text{Actual} - \text{Approximate}|}{|\text{Actual}|}$$

$$= 1.074$$

(3) Closed Newton cotes formula With degree $(n) = 2$

(Simpson Rule)

degree $(n) = 2$; node $= 3$ $\{x_0, x_1, x_2\}$

$$I_2(f) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$f(x) = e^{2x}$ [interval $= [0,2]$]

$$I_2(f) = \frac{2-0}{6}\left[e^{2(0)} + 4\left(e^{2\left(\frac{2+0}{2}\right)}\right) + e^{2(2)}\right]$$

$$= \left[e^0 + 4e^2 + e^4\right] 2/6$$

$$= \frac{(85.15437)2}{6}$$

$$= 28.38$$

$$\text{Absolute Error} = |\text{Actual} - \text{Approximate}|$$

$$= |26.80 - 28.38|$$

$$= 1.584$$

Relative Error $= 0.0591$

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④ Composite Newton Cotes

[a,b] interval is splitted into m sub-intervals

$$h = \frac{b-a}{m}$$

sub-intervals : m

nodes : m+1

$$C_{1,m} = \frac{h}{2}\Big[1f(x_0) + \underbrace{2f(x_1) + 2f(x_2) + \dots}_{\text{Middle Node}} + 1f(x_m) \Big]$$

First Node   Last Node

Sub-interval = 2 (m=2)

$$h = \left(\frac{b-a}{m}\right) = \left(\frac{2-0}{2}\right) = 1$$

node = m+1 = 3 nodes

$x_0 = a = 0 \leftarrow$ Lower bound

$x_1 = x_0 + h = 0+1 = 1$

$x_2 = x_1 + h = 1+1 = 2 \leftarrow$ Upper bound

$$C_{1,2} = \frac{h}{2}\left[f(x_0) + 2f(x_1) + f(x_2) \right]$$

$$= \frac{1}{2}\left[e^{2(0)} + 2e^{2(1)} + e^{2\cdot(2)} \right]$$

$$= \frac{1}{2}\left[1 + 2e^2 + e^4 \right]$$

$$= 35.18813$$

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