Numerical Integration

$$I(f) = \int_a^b f(x)\,dx$$

now we are going to see how to approximate $f(x)$

Previously we observed that to approximate $f(x)$, we used a polynomial. We will apply the same concept here.

$f(x)$ will be approximated to $P_n(x)$

So instead of $f(x)$, we will Integrate $P_n(x)$.

Now we will use lagrange basis concept to represent the polynomial.

Let, $n = 1$ [degree, $n = 1$]

it means there will be two nodes.

So the polynomial.

$$P_1(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$$

Integration of the general form of polynomial :

$$\int_a^b P_n(x) = \int_a^b \sum_{k=0}^{n} l_k(x) f(x_k)\, dx$$

$$I_n = \int_a^b \sum_{k=0}^{n} l_k(x) f(x_k)\, dx$$

*** Weight function ***

$$\delta_k = \int_a^b l_k(x)\, dx$$

[if the nodes are of equal distant this formula is called Newton's Cotes formula]

$$I_n = \boxed{\int_a^b} \sum_{k=0}^{n \rightarrow \text{degree}} \boxed{l_k(x)} f(x_k) \boxed{dx}$$

$$\therefore I_n = \sum_{k=0}^{n} \delta_k f(x_k)$$

Closed Newton's formula

Given range $[a, b]$, $a \to x_0$, $b \to x_n$

$$\boxed{a = x_0} < x_1 < x_2 < x_3 \ldots < x_{n-1} < \boxed{x_n = b}$$

$$** \boxed{h = \frac{b-a}{n}} **$$

Open Newton's Cotes formula

Given range $\rightarrow [a, b]$

$$a < x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n < b$$

$$** \boxed{h = \frac{b-a}{n+2}} **$$

In the close Newton's formula "a" itself represent x_0 and "b" itself represent x_n. However in open Newton's formula a is less than x_0 and b is greater than x_n.

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** ** Trapezium Rule ** $\boxed{n = 1}$ ***

[Closed Newton's - Cotes Formula] $\boxed{x \in [a,b]}$

For trapezium Rule we will consider two node. So the polynomial's degree will be 1.

** $\boxed{n = 1}$ **

From here we know

$$h = \frac{b-a}{n}$$

$$= \frac{b-a}{1}$$

$$= b-a$$

$$P_1(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1}$$

[As $x \in [a,b]$
$x_0 = a$
$x_1 = b$]

$$= \frac{x-b}{a-b}$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0}$$

$$= \frac{x-a}{b-a}$$

$$\delta_0 = \int_a^b l_0(x)\, dx$$

$$= \int_a^b \frac{(x-b)}{(a-b)}\, dx$$

$$= \frac{1}{a-b} \int_a^b (x-b)\, dx$$

$$= \frac{1}{a-b} \left[\frac{x^2}{2} - bx \right]_a^b$$

$$= \frac{b-a}{2}$$

$$\delta_1 = \frac{1}{b-a} \int_a^b (x-a)\, dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2} - ax \right]_a^b = \frac{b-a}{2}$$

$$I_n = \int_a^b P_1(x)\, dx$$

$$= \int_a^b l_0(x) f(x_0) + l_1(x) f(x_1)\, dx$$

$$= \int_a^b l_0(x) f(x_0)\, dx + \int_a^b l_1(x) f(x_1)\, dx$$

$$= \delta_0 f(x_0) + \delta_1 f(x_1)$$

$$= \boxed{\frac{b-a}{2}\left(f(a) + f(b)\right)}$$

Closed Newton's - cotes Formula with $n=1$

(Trapezium Rule)

⊛ Example :

Given that, $f(x) = e^{x}$ and $[0, 2]$

1) Find numerical integration using trapezium Rule

2) Find the Actual value?

3) % Error

1

We know

$$\frac{b-a}{2}\left(f(a)+f(b)\right)$$

$$= \frac{2-0}{2}\left(e^{0}+e^{2}\right)$$

$$= 8.3891$$

(2)

$$\int_0^2 e^x \, dx$$

$$= \left[e^x\right]_0^2$$

$$= e^2 - e^0$$

$$= 6.3891$$

(3)

$$\text{Percentange error} = \left| \frac{8.3891 - 6.3891}{6.3891} \right| \times 100\%$$

$$= 31.65\%$$

Upper bound of Interpolation Error

(Cauchy's theorem)

$$\left|\frac{f^{n+1}(\xi)}{(n+1)!}\right| \Big| \underbrace{(x-x_0)(x-x_1)\cdots(x-x_n)}_{W(x)} \Big|$$

$\downarrow$ max

$W(x) \rightarrow$ Integration

$\in [a,b]$

Example: $n = 1$, $f(x) = e^x$ $[0, 2]$

$$\left|\frac{f^{(n+1)}(\xi)}{(n+1)!}\right| = \left|\frac{f^{1+1}(\xi)}{2!}\right|$$

$f(x) = e^x$

$f'(x) = e^x$

$f^2(x) = e^x$

$$= \left|\frac{e^x}{2!}\right|$$

$$= \left|\frac{e^2}{2!}\right|$$

$$= \frac{1}{2!} e^2$$

$$W(x) = (x - x_0)(x - x_1)$$
$$= (x-0)(x-2)$$
$$= x(x-2) = x^2 - 2x$$

$$\left|\int_0^2 W(x)\,dx\right| = \left|\int_0^2 (x^2 - 2x)\,dx\right|$$
$$= \left|\left[\frac{x^3}{3} - \frac{2x^2}{2}\right]_0^2\right|$$
$$= \left|-4/3\right| = 4/3$$

$\therefore$ Upper bound Error $= \frac{1}{2!} e^2 \times 4/3$

$$= \frac{1}{2} e^2 \times 4/3$$
$$= \frac{2}{3} e^2$$

Composite Newton's cotes Formula

This method improves the result without increasing the actual node numbers.

$[a,b]$ into m subintervals equal widths.

$C_{1,m}$ → Composite newton's cotes formula notation

$$C_{1,m}(f) = \sum_{i=0}^{m} l_{1i} = \frac{h}{2}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + f(x_m)\Big]$$

Example: $[0, 2]$ $f(x) = e^x$ $\boxed{m = 2}$

$a = 0$, $b = 2$

$$* \; * \; * \quad \boxed{h = \frac{b-a}{m}} \quad * \; * \; *$$

$$h = \frac{b-a}{m} = \frac{2-0}{2} = 1$$

$x_0 = 0$

$x_1 = \text{unknown}$

$x_2 = 2$

$$x_1 = x_0 + h = 0 + 1 = 1$$

$$I_n = \frac{h}{2}\left[f(x_0) + 2f(x_1) + f(x_2)\right]$$

$$= \frac{1}{2}\left[e^0 + 2e^1 + e^2\right]$$

$$= 6.9128$$

<u>Example 2:</u> $[0, 2]$ $f(x) = e^x$ $m = 4$

$$h = \frac{b-a}{m} = \frac{2-0}{4} = 0.5$$

$x_0 = 0$

$x_1 = 0 + 0.5 = 0.5$

$x_2 = 0.5 + 0.5 = 1$

$x_3 = 1 + 0.5 = 1.5$

$x_4 = 1.5 + 0.5 = 2$

$$I_n = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{0.5}{2}\left[e^0 + 2e^{0.5} + 2e^1 + 2e^{1.5} + e^2\right]$$

$$= 6.5216$$

* Note: The more subintervals there are, the smaller the error becomes.

$\therefore$ The error decreases as m increases.