Homework 3

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1 Prove Chernoff Bound (2)(3)

Let $X_1, X_2, ..., X_n$ be independent Poisson trials such that, for $1 \le i \le n, Pr[X_i = 1] = p_i$, where $0 < p_i < 1$. Then, for $X = \Sigma_{i=1}^n X_i, \mu = E[X] = \Sigma_{i=1}^n p_i$, and any $0 < \delta < 1$,

$$Pr[X < (1-\delta)\mu] < [\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}]^{\mu}$$
$$Pr[|X-\mu| > \delta\mu] < 2e^{-\frac{\delta^2}{3}\mu}$$

Prove(2)
For any positive real λ, and applying Markov inequaity, we have

$$\begin{aligned} Pr[X < (1-\delta)\mu] =& Pr[-X > -(1-\delta)\mu] \\ =& Pr[e^{-\lambda X} > e^{-\lambda(1-\delta)\mu}] \\ <& \frac{E[e^{-\lambda X}]}{e^{-\lambda(1-\delta)\mu}} \end{aligned}$$

Since X_i are independent, the variable $e^{-\lambda X_i}$ are also independent.

$$\begin{aligned} & E[e^{-\lambda X}] \\ =& \prod_{1\le i\le n} E[e^{-\lambda X_i}] \\ =& \prod_{1\le i\le n} [p_i e^{-\lambda} + (1-p_i)] \\ =& \prod_{1\le i\le n} [p_i(e^{-\lambda} - 1) + 1] \\ \le& \prod_{1\le i\le n} [e^{p_i(e^{-\lambda}-1)}] \\ =& e^{\mu(e^{-\lambda}-1)} \end{aligned}$$

so we have

$$Pr[X < (1-\delta)\mu] < \frac{e^{\mu(e^{-\lambda}-1)}}{e^{-\lambda(1-\delta)\mu}}$$

let $\lambda = -\ln(1-\delta)$, which is positive, to obtain

$$Pr[X < (1-\delta)\mu] < \frac{e^{-\mu\delta}}{(1-\delta)^{(1-\delta)\mu}}$$

This yields the desired result. $\square$

Prove(3)
Using McLaurin expansion for $\ln(1-\delta)$ and $\ln(1+\delta)$ to obtain

$$\begin{aligned}(1-\delta)^{1-\delta} &> e^{-\delta+\frac{\delta^2}{2}} \\ (1+\delta)^{1+\delta} &> e^{\delta+\frac{\delta^2}{2}-\frac{\delta^3}{6}}\end{aligned}$$

Combine these two inequalities with chernoff bound (1) and (2)

$$\begin{aligned}&Pr[|X-\mu| > \delta\mu] \\ =&Pr[X > (1+\delta)\mu] + Pr[X < (1-\delta)\mu] \\ <&e^{-\frac{\delta^2}{2}\mu} + e^{-\frac{\delta^2}{2}\mu+\frac{\delta^3}{6}\mu} \\ <&2e^{-\frac{\delta^2}{3}\mu}\end{aligned}$$

2.bounds for Binomial coefficient

for integer x range from 1 to m

$$(\frac{m}{x})^x \le \binom{m}{x} \le (\frac{em}{x})^x$$

Prove:

$$\binom{m}{x} = \frac{m}{x}\frac{m-1}{x-1}...\frac{m-x+2}{2}\frac{m-x+1}{1} \ge (\frac{m}{x})^x$$

The other side

$$\binom{m}{x} = \frac{m}{x}\frac{m-1}{x-1}...\frac{m-x+2}{2}\frac{m-x+1}{1} \le \frac{m^x}{x!}$$

Applying stirling formula $x! = \sqrt{2\pi x}(\frac{x}{e})^x$ to obtain

$$\binom{m}{x} \leq \frac{1}{\sqrt{2\pi x}}(\frac{em}{x})^x < (\frac{em}{x})^x$$

3.$Cor(Y_i, Y_j) < 0$

Prove:

$$Cor(Y_i, Y_j) = E(Y_iY_j) - E(Y_i)E(Y_j)$$

Assume the probability that $Y_i = 1$ is p, that is $E(Y_i) = E(Y_j) = p$,

$$E(Y_iY_j) = E(Y_i|Y_j)E(Y_j) = E(Y_i|Y_j)p$$

We can focus on $E(Y_i|Y_j)$, which means the expectation of Y_i with the condition that more than cx balls fall into another bin.

$$E(Y_i|Y_j) < \sum_{i=cx}^{m-cx} \binom{m-cx}{i}(\frac{1}{n})^i(\frac{n-1}{n})^{m-cx-i} < \sum_{i=cx}^{m} \binom{m}{i}(\frac{1}{n})^i(\frac{n-1}{n})^{m-i} = E(Y_j)$$

It yields the desired result that $Cor(Y_i, Y_j) < 0$