Homework 5

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Randomized Bit Fixing Algo Analysis

Question: In randomized bit fixing algorithm, consider the permutation (x,y)⟶(y,x) where x,y are $\frac{n}{2}$-bit string. Prove its running time is $2^\Omega(n)$ with high probability $1-o(1)$

Prove:
Let's consider t he permutation $(\vec{x},1,\vec{y},0)\rightarrow(\vec{y},0,\vec{x},1)$ rather than $(\vec{x},\vec{y})\rightarrow(\vec{y},\vec{x})$, notice that $|\vec{x}|=|\vec{y}|=\frac{d}{2}-1$, for simplicity, we will just use x and y.
Now, we try to compute the expectation of routes from (x,1,y,0) to (y,0,x,1) that pass the edge e : $(y,1,y,0)-(y,0,y,0)$
Assume that x and y differ in c bits, if we fix y, there are $\binom{\frac{d}{2}-1}{c}$ different choices of x. A packet will not pass edge e unless it is transported like this: (x,1,y,0)->(y,1,y,0)->(y,0,y,0)->(y,0,x,1). We can compute the expectation of the number of packets passing edge e in such form.

$$
\begin{aligned}
E=\sum_{c=0}^{\frac{d}{2}-1}\frac{c!(c+1)!}{(2c+2)!}\binom{\frac{d}{2}-1}{c} &> \frac{(\frac{d}{6}-1)!(\frac{d}{6})!}{(\frac{d}{3})!}\frac{(\frac{d}{2}-1)!}{(\frac{d}{6}-1)!(\frac{d}{3})!}\\
&\approx\frac{(\frac{d}{6e})^{\frac{d}{6}}(\frac{d}{2e})^{\frac{d}{2}}}{(\frac{d}{3e})^{\frac{d}{3}}(\frac{d}{3e})^{\frac{d}{3}}}\\
&=(\frac{27}{16})^{\frac{d}{6}}
\end{aligned}
$$

Let X denote the number of packets that pass edge e, by applying Chernoff Bound, we have

$$Pr[X<\frac{1}{2}E]<[\frac{e^{-\frac{1}{2}}}{\frac{1}{2}^{\frac{1}{2}}}]^{2^{\Omega(d)}}=o(1)$$

$$Pr[X>2E]<[\frac{e}{4}]^{2^{\Omega(d)}}=o(1)$$

This yields the result that the running time is $2^\Omega(n)$ with high probability.

Valiant two-phase algorithm

Prove: $Pr(\exists x, delay(x) > cn) = o(1)$

In Valiant algorithm, i's packet is sent to $\sigma(i)$ first, then it travels from $\sigma(i)$ to destination $d(i)$. Now, we only focus on phase one: i to $\sigma(i)$
Denote by p_i the path from i to $\sigma(i)$, $p_i = (e_1, e_2, ..., e_k), k \leq n$

Lemma: For any path p_i and p_j, once they separate, they do not rejoin.
Prove for lemma:
If $p_j = (e_1, e_2, e'_3, e'_4, ..., e'_l, e_l, ...)$, that is p_j and p_i separate at e_2 and then rejoin at e_l. In that case, there exist e_{l1} and e_{l2}, $l_1, l_2 \leq l$, and e_{l1} is parallel with e_{l2}, which is a contradiction.

Let the random variable $H_{ij} = 1$ if path p_i and p_j share at least one edge, and 0 otherwise. The total delay of packet i has an upperbound.

$$delay(i) \leq n + \sum_{j=1}^{N} H_{ij}$$

In order to estimate the upper bound of $\sum_{j=1}^{N} H_{ij}$, we introduce another random variable $T(e)$. Let $T(e)$ denote the number of paths that pass through edge e. For fixed path $p_i = (e_1, e_2, ..., e_k)$ with $k \leq n$.Then,

$$\sum_{j=1}^{N} H_{ij} \leq \sum_{l=1}^{k} T(e_l)$$

In class, we have already proved that $T(e_l) = \frac{1}{2}$, which indicates

$$E(\sum_{j=1}^{N} H_{ij}) \leq \frac{n}{2}$$

By applying Chernoff Bound, let $H = \sum_{j=1}^{N} H_{ij}$, we have

$$Pr[H > n] \leq Pr[H > 2E(H)] \leq [\frac{e^1}{2^2}]^{\frac{n}{2}} = (\frac{e}{4})^{\frac{n}{2}}$$

And therefore

$$Pr(delay(x) > 2n) \leq (\frac{e}{4})^{\frac{n}{2}}$$

Independent Set

(1)

For each vertex, it remains after deletion with probability $\frac{1}{d}$.

For each edge, it remains only if its two endpoints are not deleted.

$$E(\#vertices) = \frac{n}{d}$$
$$E(\#edges) = (\frac{1}{d})^2 \frac{nd}{2} = \frac{n}{2d}$$

(2)

For all subgraph with $\frac{n}{d}$ vertices, there exists one graph with edges less than or equal with $\frac{n}{2d}$.

For each edge that remains, delete it and delete one of its endpoints.

(3)

Don't know solution yet.