Homework

1 successful prob of Fast-Cut

Successful probablity:
the origin size n problem is reduced to two subproblem size $\frac{n}{\sqrt{2}}$, considering the contract procedure, the probability that the result of subproblem is exactly the result of origin problem is

$$(1-\frac{2}{n})(1-\frac{2}{n-1})...(1-\frac{2}{n/\sqrt{2}}) = \frac{(\frac{n}{\sqrt{2}}-1)(\frac{n}{\sqrt{2}}-2)}{n(n-1)} \geq \frac{1}{2}$$

denote the successful probabilty of origin problem with size n by $P(k+1)$, subproblem with size $\frac{n}{\sqrt{2}}$ by $P(k)$, then we have

$$P(k+1) = 1-(1-\frac{1}{2}P(k))^2$$

so we get

$$P(k+1) = P(k) - \frac{1}{4}P(k)^2$$

perform a change of variable $P(k) = \frac{4}{q(k)+1}$, yields the following simplification:

$$q(k+1) = q(k) + 1 + \frac{1}{q(k)}$$

recursively apply the equation to the righthand side

$$q(k) = q(1) + k - 1 + \sum_{1\leq i\leq k-1} \frac{1}{q(k)} \geq k-1$$

by using $q(k) \geq k-1$,we have

$$q(k) \leq q(1) + k - 1 + \sum_{1\leq i\leq k-1} \frac{1}{i} \leq q(1) + k - 1 + \log k$$

combine the upper bound and lower bound

$$q(k) = \Omega(k)$$

$$P(k) = \Omega(\frac{1}{k})$$

observe that $k = \log n$,we can draw the conclution

$$Pr(n) = \Omega(\frac{1}{\log n})$$

2 Exercise 10.9

the time complexity of the modified algorithm is $\Theta(n^2 + t^3)$ with reducing procedure costs $\Theta(n^2)$ and cubic time algorithm for subproblem.

consider the successful probability,after reducing the problem,the probability that the subproblem result is exactly the origin problem result is

$$(1 - \frac{2}{n})(1 - \frac{2}{n-1})...(1 - \frac{2}{n/\sqrt{2}}) = \frac{t(t-1)}{n(n-1)} = \Theta(\frac{t^2}{n^2})$$

to ensure at least 1/2 successful probabilty,repeat the algorithm $\Theta(\frac{n^2}{t^2})$times

the total running time

$$\Theta(n^2 + t^3) * \Theta(\frac{n^2}{t^2}) = \Theta(\frac{n^4}{t^2} + n^2 t) = \Omega(n^{\frac{8}{3}})$$

the last step above is by using inequality of geometric and arithmetic means

$$\frac{n^4}{t^2} + \frac{n^2 t}{2} + \frac{n^2 t}{2} \geq (\frac{n^8}{4})^{\frac{1}{3}}$$

Optional(k-way cut-set)

Exercise 1.10 Page 22

暂时没搞清楚，感觉跟chernoff bound有关系，主要是不知道repeat之后到底怎么操作

Exercise 1.13 Page 27

Prove PP=coPP,
$L \in PP$,L^cdenote the complement of L. By the definition of PP,there exists an algorithm A,for x in L,Pr(A(x)accepts)>1/2,for x not in L,Pr(A(x)accepts)=0.
construct an algorithm B,B rejects when A accepts,and B accepts when A rejects.
Then

$$x \in L^c \Rightarrow Pr(B(x)accepts) = 1 - Pr(A(x)accepts) > \frac{1}{2}$$
$$x \notin L^c \Rightarrow Pr(B(x)accepts) = 1 - Pr(A(x)accepts) < \frac{1}{2}$$

which means PP=co-PP

Prove BPP=coBPP
assume $L \in BPP$ with algorithm A,construct algorithm B in the same way above.
Then

$$x \in L^c \Rightarrow Pr(B(x)accepts) = 1 - Pr(A(x)accepts) > \frac{3}{4}$$
$$x \notin L^c \Rightarrow Pr(B(x)accepts) = 1 - Pr(A(x)accepts) < \frac{1}{4}$$

implys BPP=co-BPP

Optional Problem 1.15 Page 27