Numerical methods - Mandatory 2

Mathias Neerup

2016 April 17

Contents

1 Excercise A

Use the trapezoidal method to write an approximation to eq. 3, 4. Then write the discrete problem corresponding to eq. 1, 2 as a system of linear equations in the variables $(u_i)_i^N = 0$ and $(v_i)_i^N = 0$

Equation 1 and 2 can be seen below:

$$u(x) = \epsilon_1 \sigma T_1^4 + (1 - \epsilon_1) \int_{-\frac{1}{2}w}^{\frac{1}{2}w} F(x, y, d) v(y) dy \tag{1}$$

$$v(y) = \epsilon_2 \sigma T_2^4 + (1 - \epsilon_1) \int_{-\frac{1}{2}w}^{\frac{1}{2}w} F(x, y, d) u(x) dx \tag{2}$$

Equation 3 and 4 can be seen below:

$$I_1(x) = \int_{-\frac{1}{2}w}^{\frac{1}{2}w} F(x, y, d) v(y) dy \tag{3}$$

$$I_2(y) = \int_{-\frac{1}{2}w}^{\frac{1}{2}w} F(x, y, d) u(x) dx \tag{4}$$

The generel equation of the trapezoidal method:

$$\int_a^b f(x) dx \approx \frac{h}{2} \sum_{k=1}^{N} (f(x_k + f(x_{k+1}))) \tag{5}$$

Where h is equal to:

$$h = \frac{b - a}{N} \tag{6}$$

When the trapezoidal method is applied to eq. 1 and 2:

$$I_1 = \frac{h}{2}\sum_{k=1}^{N}(F(x,y_k,d)v(y_k) + F(x,y_{k+1},d)v(y_{k+1})) \tag{7}$$

$$I_2 = \frac{h}{2}\sum_{k=1}^{N}(F(x_k,y,d)u(x_k) + F(x_{k+1},y,d)u(x_{k+1})) \tag{8}$$

Introduce $\beta_{1,2}$ and $\beta_{1,2}$ as constants.

$$\alpha_1 = \epsilon_1 \sigma T_1^4 \qquad \alpha_2 = 1 - \epsilon_1$$

$$\beta_1 = \epsilon_2 \sigma T_2^4 \qquad \beta_2 = 1 - \epsilon_2$$

Rearrangen eq. 1, 2 to fit into $A\overrightarrow{x} = \overrightarrow{b}$

$$u(x) - \alpha_2 \int_{-\frac{1}{2}w}^{\frac{1}{2}w} F(x,y,d)v(y)dy = \alpha_1 \tag{9}$$

$$v(y) - \beta_2 \int_{-\frac{1}{2}w}^{\frac{1}{2}w} F(x,y,d)u(x)dx = \beta_1 \tag{10}$$

Setting eq. 1, 2 together with eq. 3, 4

$$u(x) - \alpha_2\frac{h}{2}\sum_{k=1}^{N}(F(x,y_k,d)v(y_k) + F(x,y_{k+1},d)v(y_{k+1})) = \alpha_1 \tag{11}$$

$$v(y) - \beta_2\frac{h}{2}\sum_{k=1}^{N}(F(x_k,y,d)u(x_k) + F(x_{k+1},y,d)u(x_{k+1})) = \beta_1 \tag{12}$$

If $N = 1$

$$u(x) - \alpha_2\frac{h}{2}F(x,a,d)v(a) + \alpha_2\frac{h}{2}F(x,b,d)v(y_b) = \alpha_1 \tag{13}$$

$$v(y) - \beta_2\frac{h}{2}F(a,y,d)u(a) + \beta_2\frac{h}{2}F(b,y,d)u(b) = \beta_1 \tag{14}$$

Expressed as matrices for $N = 2$

$$A = \begin{matrix} 1 & 0 & 0 & -\alpha_2\frac{h}{2}F(x_1,y_1,d) & -\alpha_2 hF(x_1,y_2,d) & -\alpha_2\frac{h}{2}F(x_1,y_3,d) \\ 0 & 1 & 0 & -\alpha_2\frac{h}{2}F(x_2,y_1,d) & -\alpha 2hF(x_2,y_2,d) & -\alpha 2\frac{h}{2}F(x_2,y_3,d) \\ 0 & 0 & 1 & -\alpha_2\frac{h}{2}F(x_3,y_1,d) & -\alpha 2hF(x_3,y_2,d) & -\alpha 2\frac{h}{2}F(x_3,y_3,d) \\ -\beta_2\frac{h}{2}F(x_1,y_1,d) & -\beta_2 hF(x_2,y_1,d) & -\beta_2\frac{h}{2}F(x_3,y_1,d) & 1 & 0 & 0 \\ -\beta_2\frac{h}{2}F(x_1,y_2,d) & -\beta_2 hF(x_2,y_2,d) & -\beta_2\frac{h}{2}F(x_3,y_2,d) & 0 & 1 & 0 \\ -\beta_2\frac{h}{2}F(x_1,y_3,d) & -\beta_2 hF(x_2,y_3,d) & -\beta_2\frac{h}{2}F(x_3,y_3,d) & 0 & 0 & 1 \end{matrix} \tag{15}$$

$$\overrightarrow{b} = \begin{bmatrix} \alpha_1 \\ \alpha_1 \\ \alpha_1 \\ \beta_1 \\ \beta_1 \\ \beta_1 \end{bmatrix} \qquad \overrightarrow{x} = \begin{bmatrix} u(x_1) \\ u(x_2) \\ u(x_3) \\ v(y_1) \\ v(y_2) \\ v(y_3) \end{bmatrix}$$

The A-matrix will always be diagonally 1 since it's selecting one element in vector X to be substracted from. Due to the layout of vector X, all steps from equation 11 will be in the upper right quadrant while all steps from equation 12 will be in the lower left quadrant.

The size of matrix A is $n = m = 2(N + 1)$. This is due to the trapezoidal method since when the sum runs, it creates N+1 columns which happens for both equations there by multiply by two.

$$A = \begin{bmatrix} 1 & 0 & -\alpha_2 \frac{h}{2} F(x_1, y_1, d) & -\alpha_2 \frac{h}{2} F(x_1, y_2, d) \\ 0 & 1 & -\alpha_2 \frac{h}{2} F(x_2, y_1, d) & -\alpha 2 \frac{h}{2} F(x_2, y_2, d) \\ -\beta_2 \frac{h}{2} F(x_1, y_1, d) & -\beta_2 \frac{h}{2} F(x_2, y_1, d) & 1 & 0 \\ -\beta_2 \frac{h}{2} F(x_1, y_2, d) & -\beta_2 \frac{h}{2} F(x_2, y_2, d) & 0 & 1 \end{bmatrix} \tag{16}$$

$$\overrightarrow{b} = \begin{bmatrix} \alpha_1 \\ \alpha_1 \\ \beta_1 \\ \beta_1 \end{bmatrix} \qquad \overrightarrow{x} = \begin{bmatrix} u(a) \\ u(b) \\ v(a) \\ v(b) \end{bmatrix}$$

2 Excercise B

3 Excercise C