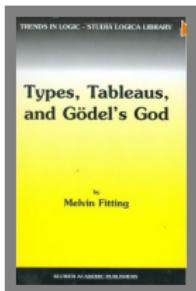


Formalization, Mechanization and Automation of Gödel's Proof of God's Existence

Christoph Benzmüller and Bruno Woltzenlogel Paleo

November 1, 2013



$$\frac{\text{Axiom 3} \quad \frac{\overline{\forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x. \varphi(x)]}}{\overline{P(G)} \rightarrow \Diamond \exists x. G(x)}}{\Diamond \exists x. G(x)} \rightarrow_E \forall_E \text{Theorem 1}$$

A gift to **Priest Edvaldo** and his church in Piracicaba, Brazil

SPIEGEL ONLINE WISSENSCHAFT

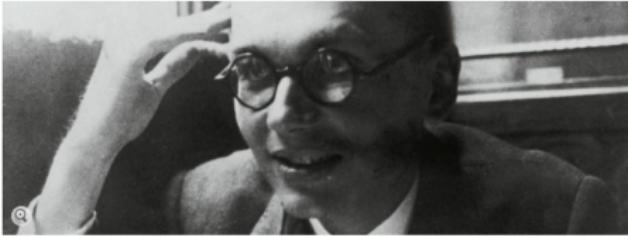
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Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Von Tobias Hürter



Kurt Gödel (um das Jahr 1935): Der Mathematiker hält seinen Gottesbeweis Jahrzehntlang geheim

Ein Wesen existiert, das alle positiven Eigenschaften in sich vereint. Das bewies der legendäre Mathematiker Kurt Gödel mit einem komplizierten Formelegebilde. Zwei Wissenschaftler haben diesen Gottesbeweis nun überprüft - und für gültig befunden.

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Montag, 09.09.2013 – 12:03 Uhr

Drucken | Versenden | Merken

Jetzt sind die letzten Zweifel ausgeräumt: Gott existiert tatsächlich. Ein Computer hat es mit kalter Logik bewiesen - das MacBook des Computerwissenschaftlers Christoph Benzmüller von der Freien Universität Berlin.

Germany

- Telepolis & Heise
- Spiegel Online
- FAZ
- Die Welt
- Berliner Morgenpost
- Hamburger Abendpost
- ...

Austria

- Die Presse
- Wiener Zeitung
- ORF
- ...

Italy

- Repubblica
- IlSussidario
- ...

India

- DNA India
- Delhi Daily News
- India Today
- ...

US

- ABC News
- ...

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- Spiegel International
- Yahoo Finance
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- ...

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Researchers say they used MacBook to prove Goedel's God theorem

Oct. 23, 2013 | 8:14 PM | 1 comments

Are we in contact with Steve Jobs? No

Do you really need a MacBook to obtain the results? No

Is Apple sending us money? No
(but maybe they should)

SCIENCE NEWS

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- | | |
|---|----|
| Are we in contact with Steve Jobs? | No |
| Do you really need a MacBook to obtain the results? | No |
| Is Apple sending us money? (but maybe they should) | No |

Def: Ontological Argument/Proof

- * deductive argument
- * for the existence of God
- * starting from premises, which are justified by pure reasoning, i.e. they do not depend on observation of the world.

Existence of God: different types of arguments/proofs

- a posteriori (use experience/observation in the world)
 - teleological
 - cosmological
 - moral
 - ...
- a priori (based on pure reasoning, independent)
 - ontological argument
 - definitional
 - modal
 - ...
 - other a priori arguments

Def: Ontological Argument/Proof

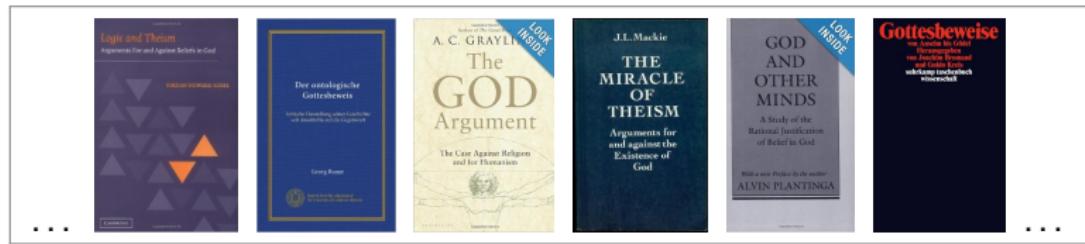
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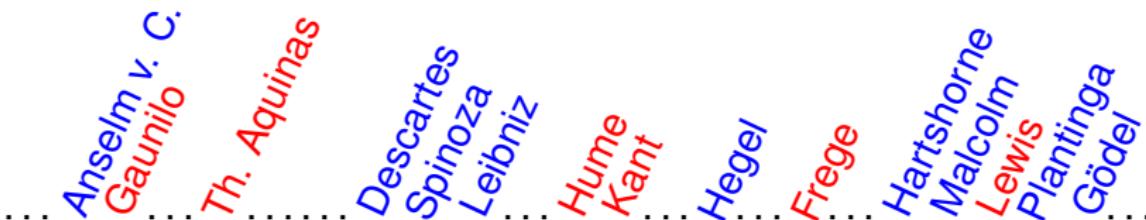
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 - ontological argument
 - definitional
 - modal
 - ...
 - other a priori arguments

Wohl eine jede Philosophie kreist um den ontologischen Gottesbeweis

(Adorno, Th. W.: Negative Dialektik. Frankfurt a. M. 1966, p.378)



Rich history on ontological arguments ([pros](#) and [cons](#))



Anselm's notion of God:

"God is that, than which nothing greater can be conceived."

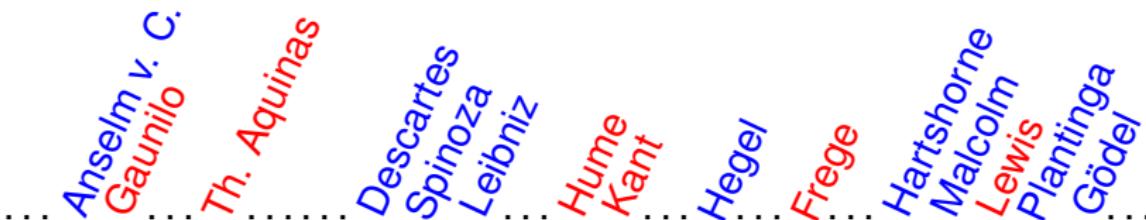
Gödel's notion of God:

"A God-like being possesses all 'positive' properties."

To show by logical reasoning:

"(Necessarily) God exists."

Rich history on ontological arguments ([pros](#) and [cons](#))



Anselm's notion of God:

“God is that, than which nothing greater can be conceived.”

Gödel's notion of God:

“A God-like being possesses all ‘positive’ properties.”

To show by logical reasoning:

“(Necessarily) God exists.”

Different Interests in Ontological Arguments:

- **Philosophical:** Boundaries of Metaphysics & Epistemology
 - We talk about a metaphysical concept (God),
 - but we want to draw a conclusion for the real world.
 - Necessary Existence (NE): metaphysical NE vs. logical NE vs. modal NE
- **Theistic:** Successful argument should convince atheists
- **Ours:** Can computers (theorem provers) be used ...
 - ... to formalize the definitions, axioms and theorems?
 - ... to verify the arguments step-by-step?
 - ... to fully automate (sub-)arguments?

“Computer-assisted Theoretical Philosophy”

Challenge: No provers for *Higher-order Quantified Modal Logic* (**QML**)

Our solution: Embedding in *Higher-order Classical Logic* (**HOL**)

[BenzmüllerPaulson, Logica Universalis, 2013]

What we did (**rough outline for remaining presentation!**):

- | | |
|-------------------------------|---|
| A: Pen and paper: | detailed natural deduction proof |
| B: Formalization: | in classical higher-order logic (HOL) |
| Automation: | theorem provers Leo-II and SATALLAX |
| Consistency: | model finder NITPICK (NITROX) |
| C: Step-by-step verification: | proof assistant Coq |
| D: Automation & verification: | proof assistant ISABELLE |

Did we get any new results?

Yes — let's discuss this later!



Part A:

Informal Proof and Natural Deduction Proof

Gödel's Manuscript: 1930's, 1941, 1946-1955, 1970

Ontologischer Beweis

FEB 10, 1970

P(φ): φ is positive ($\Leftrightarrow \varphi \in P$)

At 1: $P(\varphi) \cdot P(\psi) \supset P(\varphi \wedge \psi)$ At 2: $P(\varphi) \supset P(\neg \varphi)$

P1: $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (Ged.)

P2: $\varphi_{\text{Em},x} \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Emm. $\forall x$)

$P \supset_N = N(P \supset \varphi)$ Necessity

At 2: $P(\varphi) \supset N P(\varphi)$ } because it follows
 $\neg P(\varphi) \supset N \neg P(\varphi)$ } from the nature of the property

Th.: $G(x) \supset \varphi_{\text{Em},x}$

Df.: $E(x) \equiv \exists \varphi[\varphi_{\text{Em},x} \supset N \exists x \varphi(x)]$ necessary Existence

At 3: $P(E)$

Th.: $G(x) \supset N(\exists y) G(y)$

hence $(\exists x) G(x) \supset N(\exists y) G(y)$

" $M(\exists x) G(x) \supset M N(\exists y) G(y)$ Mi = partibility
 " $\supset N(\exists y) G(y)$

any two instances of x are nec. equivalent

exclusive or * and for any number of instances

$M(\exists x) G(x)$: means all pos. propo. w.r.t. com-
patible This is true because of:

At 4: $P(\varphi) \cdot \varphi \supset \psi \Rightarrow P(\psi)$ which impl.
~~the system of~~ { $x=x$ is positive
~~the system of~~ { $x \neq x$ is negative

But if a system S of pos. propo. were incon-
sistent it would mean that the same prop. S (which
is positive) would be $x \neq x$

Positive means positive in the moralistic sense (independently of the accidental structure of the world). Only then the at time. It also means "attribution" as opposed to "privation" (or containing privation). This is Gödel's problem proof

$\neg P(\varphi)$ positive w.r.t. $(\exists x) N \supset \varphi(x)$. Otherwise $\neg P(\varphi) \supset_N$
 hence $x \neq x$, but we have $x=x$ contrary At
 to the definition of pos. prop.

i.e. the normal form in terms of elem. propo. contains a member without negation.

A1 Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

A2 A property necessarily implied

by a positive property is positive:

$$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

$$\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$$

T1 Positive properties are possibly exemplified:

D1 A *God-like* being possesses all positive properties:

$$G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$$

$$P(G)$$

A3 The property of being God-like is positive:

C Possibly, God exists:

$$\Diamond\exists xG(x)$$

A4 Positive properties are necessarily positive:

$$\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$$

D2 An *essence* of an individual is

a property possessed by it and

necessarily implying any of its properties: $\phi \text{ ess. } x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

T2 Being God-like is an essence of any God-like being:

$$\forall x[G(x) \rightarrow G \text{ ess. } x]$$

D3 *Necessary existence* of an individual is

the necessary exemplification of all its essences:

$$NE(x) \leftrightarrow \forall\phi[\phi \text{ ess. } x \rightarrow \Box\exists y\phi(y)]$$

$$P(NE)$$

A5 Necessary existence is a positive property:

T3 Necessarily, God exists:

$$\Box\exists xG(x)$$

T3: $\Box \exists x.G(x)$

C1: $\Diamond \exists z.G(z)$

T3: $\Box \exists x.G(x)$

$$\frac{\mathbf{C1: } \diamond \exists z. G(z) \quad \mathbf{L2: } \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\mathbf{T3: } \Box \exists x. G(x)}$$

L2: $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

C1: $\diamond \exists z. G(z)$

L2: $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

T3: $\square \exists x. G(x)$

S5
 $\neg \forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]$

L2: $\Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$

C1: $\Diamond \exists z. G(z)$

L2: $\Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$

T3: $\Box \exists x. G(x)$

$$\frac{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x) \quad \forall \xi. [\overline{\diamond \square \xi} \rightarrow \overline{\square \xi}] \text{ S5}}{\mathbf{L2:} \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}$$

$$\frac{\mathbf{C1:} \diamond \exists z. G(z) \quad \mathbf{L2:} \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}{\mathbf{T3:} \square \exists x. G(x)}$$

$$\frac{\frac{\text{L1: } \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)} \quad \text{S5}}{\text{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}$$

$$\frac{\text{C1: } \Diamond \exists z. G(z) \quad \text{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\text{T3: } \Box \exists x. G(x)}$$

D1: $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

$$\frac{\frac{\text{L1: } \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)} \quad \text{S5}}{\text{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}$$

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D1: $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

D3^{*}: $E(x) \equiv \square \exists y. G(y)$

$$\frac{\frac{\frac{P(E)}{\text{L1: } \exists z. G(z) \rightarrow \square \exists x. G(x)}}{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)} \quad \overline{\forall \xi. [\overline{\diamond \square \xi} \rightarrow \overline{\square \xi}]}}{\text{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}$$

$$\frac{\text{C1: } \diamond \exists z. G(z) \quad \text{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}{\text{T3: } \square \exists x. G(x)}$$

D1: $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

D3*: $E(x) \equiv \square \exists y. G(y)$ (cheating!)

$$\frac{\frac{\frac{P(E)}{\mathbf{L1}: \exists z. G(z) \rightarrow \square \exists x. G(x)}}{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}}{\mathbf{L2}: \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}$$
$$\frac{\mathbf{S5} \quad \neg \forall \bar{\xi}. [\neg \diamond \bar{\xi} \rightarrow \neg \square \bar{\xi}]}{\mathbf{T3}: \square \exists x. G(x)}$$

$$\frac{\mathbf{C1}: \diamond \exists z. G(z) \quad \mathbf{L2}: \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}{\mathbf{T3}: \square \exists x. G(x)}$$

D1: $G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$

D3*: $E(x) \equiv \Box \exists y.G(y)$

D3: $E(x) \equiv \forall \varphi.[\varphi \text{ ess } x \rightarrow \Box \exists y.\varphi(y)]$

$$\begin{array}{c}
 \frac{\textbf{T2: } \forall y.[G(y) \rightarrow G \text{ ess } y] \qquad P(E)}{\frac{\textbf{L1: } \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\frac{\textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x)}{\frac{\textbf{S5}}{\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]}}}} \\
 \hline
 \textbf{C1: } \Diamond \exists z.G(z) \qquad \textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x) \\
 \hline
 \textbf{T3: } \Box \exists x.G(x)
 \end{array}$$

D1: $G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$

D3*: $E(x) \equiv \Box \exists y.G(y)$

D3: $E(x) \equiv \forall \varphi.[\varphi \text{ ess } x \rightarrow \Box \exists y.\varphi(y)]$

$$\frac{\begin{array}{c} \textbf{T2: } \forall y.[G(y) \rightarrow G \text{ ess } y] \\ \hline \textbf{L1: } \exists z.G(z) \rightarrow \Box \exists x.G(x) \\ \hline \textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x) \end{array}}{\textbf{C1: } \Diamond \exists z.G(z)} \quad \frac{\begin{array}{c} \textbf{A5} \\ \overline{P(E)} \\ \hline \textbf{S5} \\ \overline{\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]} \end{array}}{\textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}$$

$$\frac{\textbf{C1: } \Diamond \exists z.G(z) \quad \textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\textbf{T3: } \Box \exists x.G(x)}$$

D1: $G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$

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$$\frac{\begin{array}{c} \mathbf{T2: } \forall y.[G(y) \rightarrow G \text{ ess } y] \\ \hline \mathbf{L1: } \exists z.G(z) \rightarrow \square \exists x.G(x) \\ \hline \frac{\diamond \exists z.G(z) \rightarrow \diamond \square \exists x.G(x)}{\mathbf{L2: } \diamond \exists z.G(z) \rightarrow \square \exists x.G(x)} \end{array}}{\mathbf{A5} \quad \overline{P(E)} \quad \mathbf{S5} \quad \overline{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}}$$

$$\frac{\mathbf{C1: } \diamond \exists z.G(z) \quad \mathbf{L2: } \diamond \exists z.G(z) \rightarrow \square \exists x.G(x)}{\mathbf{T3: } \square \exists x.G(x)}$$

D1: $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

D2: $\varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \square \forall x. (\varphi(x) \rightarrow \psi(x)))$

D3^{*}: $E(x) \equiv \square \exists y. G(y)$

D3: $E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

$$\frac{\begin{array}{c} \textbf{A1b} \\ \overline{\forall \varphi. [\neg \bar{P}(\varphi) \rightarrow \bar{P}(\neg \varphi)]} \end{array}}{\textbf{T2: } \forall y. [G(y) \rightarrow G \text{ ess } y]} \quad \frac{\begin{array}{c} \textbf{A4} \\ \overline{\forall \varphi. [\bar{P}(\varphi) \rightarrow \square \bar{P}(\varphi)]} \end{array}}{\textbf{A5} \quad \overline{P(E)}}$$

$$\frac{\textbf{L1: } \exists z. G(z) \rightarrow \square \exists x. G(x)}{\frac{\textbf{L2: } \diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}{\frac{\textbf{S5} \quad \overline{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}}{\textbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}}}$$

$$\frac{\textbf{C1: } \diamond \exists z. G(z)}{\textbf{T3: } \square \exists x. G(x)} \quad \textbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$$

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D3: $E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

C1: $\diamond \exists z. G(z)$

$$\frac{\begin{array}{c} \textbf{A1b} \\ \neg \forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)] \end{array}}{\textbf{T2}: \forall y. [G(y) \rightarrow G \text{ ess } y]} \quad
 \frac{\begin{array}{c} \textbf{A4} \\ \neg \forall \varphi. [P(\varphi) \rightarrow \square \neg P(\varphi)] \end{array}}{} \quad
 \frac{\textbf{A5}}{P(E)} \quad
 \frac{\begin{array}{c} \textbf{L1: } \exists z. G(z) \rightarrow \square \exists x. G(x) \\ \hline \diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x) \end{array}}{\textbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)} \quad
 \frac{\textbf{S5}}{\neg \forall \xi. [\diamond \square \xi \rightarrow \square \xi]}$$

C1: $\diamond \exists z. G(z)$

L2: $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

T3: $\square \exists x. G(x)$

D1: $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

D2: $\varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \square \forall x. (\varphi(x) \rightarrow \psi(x)))$

D3*: $E(x) \equiv \square \exists y. G(y)$

D3: $E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

$P(G)$

C1: $\diamond \exists z. G(z)$

A1b
 $\neg \forall \varphi. \neg P(\varphi) \rightarrow P(\neg \varphi)$

A4
 $\neg \forall \varphi. P(\varphi) \rightarrow \neg \square P(\varphi)$

A5
 $\neg P(E)$

T2: $\forall y. [G(y) \rightarrow G \text{ ess } y]$

L1: $\exists z. G(z) \rightarrow \square \exists x. G(x)$
 $\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)$

S5
 $\neg \forall \xi. \neg \diamond \square \xi \rightarrow \neg \square \xi$

L2: $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

C1: $\diamond \exists z. G(z)$

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D2: $\varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \square \forall x. (\varphi(x) \rightarrow \psi(x)))$

D3*: $E(x) \equiv \square \exists y. G(y)$

D3: $E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

A3
 $\frac{}{P(G)}$

C1: $\diamond \exists z. G(z)$

A1b
 $\frac{\neg \forall \varphi. \neg P(\varphi) \rightarrow P(\neg \varphi)}{\forall \varphi. \neg P(\varphi) \rightarrow P(\neg \varphi)}$

A4
 $\frac{\neg \forall \varphi. P(\varphi) \rightarrow \square \neg P(\varphi)}{\forall \varphi. P(\varphi) \rightarrow \square P(\varphi)}$

A5
 $\frac{}{P(E)}$

T2: $\forall y. [G(y) \rightarrow G \text{ ess } y]$

L1: $\exists z. G(z) \rightarrow \square \exists x. G(x)$
 $\frac{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}{\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}$

S5
 $\frac{\forall \xi. \neg \diamond \square \xi \rightarrow \neg \square \xi}{\forall \xi. \neg \diamond \square \xi \rightarrow \square \xi}$

L2: $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

C1: $\diamond \exists z. G(z)$

L2: $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

T3: $\square \exists x. G(x)$

D1: $G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$

D2: $\varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi.(\psi(x) \rightarrow \square \forall x.(\varphi(x) \rightarrow \psi(x)))$

D3*: $E(x) \equiv \square \exists y.G(y)$

D3: $E(x) \equiv \forall \varphi.[\varphi \text{ ess } x \rightarrow \square \exists y.\varphi(y)]$

A3
—
 $P(G)$

T1: $\forall \varphi.[P(\varphi) \rightarrow \diamond \exists x.\varphi(x)]$

C1: $\diamond \exists z.G(z)$

A1b

$\neg \forall \varphi. \neg P(\varphi) \rightarrow P(\neg \varphi)$

A4

$\neg \forall \varphi. P(\varphi) \rightarrow \neg \square P(\varphi)$

A5

$\neg P(E)$

T2: $\forall y.[G(y) \rightarrow G \text{ ess } y]$

L1: $\exists z.G(z) \rightarrow \square \exists x.G(x)$

$\diamond \exists z.G(z) \rightarrow \diamond \square \exists x.G(x)$

S5

$\neg \forall \xi. \neg \diamond \square \xi \rightarrow \neg \square \xi$

L2: $\diamond \exists z.G(z) \rightarrow \square \exists x.G(x)$

C1: $\diamond \exists z.G(z)$

L2: $\diamond \exists z.G(z) \rightarrow \square \exists x.G(x)$

T3: $\square \exists x.G(x)$

D1: $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

D2: $\varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \square \forall x. (\varphi(x) \rightarrow \psi(x)))$

D3*: $E(x) \equiv \square \exists y. G(y)$

D3: $E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

$$\frac{\frac{\frac{\frac{\frac{\overline{A3}}{P(\bar{G})} \quad \frac{\overline{A2}}{\overline{\forall \varphi. \forall \psi. [(\bar{P}(\varphi) \wedge \square \forall x. [\varphi(\bar{x}) \rightarrow \psi(\bar{x})]) \rightarrow \bar{P}(\psi)]}}}{\mathbf{T1}: \forall \varphi. [P(\varphi) \rightarrow \diamond \exists x. \varphi(x)]} \quad \frac{\overline{A1a}}{\overline{\forall \varphi. [\bar{P}(\neg \varphi) \rightarrow \neg \bar{P}(\varphi)]}}}{\mathbf{C1}: \diamond \exists z. G(z)}$$

$$\frac{\frac{\frac{\frac{\frac{\overline{A1b}}{\overline{\forall \varphi. [\neg \bar{P}(\varphi) \rightarrow \bar{P}(\neg \varphi)]}} \quad \frac{\overline{A4}}{\overline{\forall \varphi. [\bar{P}(\varphi) \rightarrow \square \bar{P}(\varphi)]}}}{\mathbf{T2}: \forall y. [G(y) \rightarrow G \text{ ess } y]} \quad \overline{A5}}{P(E)} \quad \frac{\mathbf{L1}: \exists z. G(z) \rightarrow \square \exists x. G(x)}{\frac{\overline{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}}{\mathbf{L2}: \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}} \quad \frac{\mathbf{S5}}{\overline{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}}$$

$$\frac{\mathbf{C1}: \diamond \exists z. G(z) \quad \mathbf{L2}: \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}{\mathbf{T3}: \square \exists x. G(x)}$$

D1: $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

D2: $\varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \square \forall x. (\varphi(x) \rightarrow \psi(x)))$

D3: $E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

$$\frac{\frac{\frac{\frac{\frac{\overline{A3}}{P(\bar{G})} \quad \frac{\frac{\overline{A2}}{\forall \varphi. \forall \psi. [(P(\varphi) \wedge \square \forall x. [\varphi(x) \rightarrow \psi(x)]] \rightarrow P(\psi)]} \quad \frac{\overline{A1a}}{\forall \varphi. [P(\neg \varphi) \rightarrow \neg P(\varphi)]}}{\textbf{T1: } \forall \varphi. [P(\varphi) \rightarrow \diamond \exists x. \varphi(x)]}}}{\textbf{C1: } \diamond \exists z. G(z)}$$

$$\frac{\frac{\frac{\frac{\overline{A1b}}{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]} \quad \frac{\overline{A4}}{\forall \varphi. [P(\varphi) \rightarrow \square P(\varphi)]}}{\textbf{T2: } \forall y. [G(y) \rightarrow G \text{ ess } y]} \quad \overline{A5} \quad \overline{P(E)}}}{\frac{\frac{\textbf{L1: } \exists z. G(z) \rightarrow \square \exists x. G(x)}{\frac{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}{\textbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}}}{\frac{\overline{S5}}{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}}}$$

$$\frac{\textbf{C1: } \diamond \exists z. G(z) \quad \textbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}{\textbf{T3: } \square \exists x. G(x)}$$

$$\frac{\overline{A} \quad \overline{B} \quad \vdots \quad \vdots \quad A \vee B \quad C \quad C}{C} \vee_E$$

$$\frac{A \quad B}{A \wedge B} \wedge_I \quad \frac{\overline{A} \quad \overline{B} \quad \vdots \quad \vdots \quad B}{A \rightarrow B} \rightarrow_I^n$$

$$\frac{A}{A \vee B} \vee_{I_1}$$

$$\frac{A \wedge B}{A} \wedge_{E_1}$$

$$\frac{B}{A \rightarrow B} \rightarrow_I$$

$$\frac{B}{A \vee B} \vee_{I_2}$$

$$\frac{A \wedge B}{B} \wedge_{E_2}$$

$$\frac{A \quad A \rightarrow B}{B} \rightarrow_E$$

$$\frac{A[\alpha]}{\forall x.A[x]} \forall_I$$

$$\frac{\forall x.A[x]}{A[t]} \forall_E$$

$$\frac{A[t]}{\exists x.A[x]} \exists_I$$

$$\frac{\exists x.A[x]}{A[\beta]} \exists_E$$

$$\neg A \equiv A \rightarrow \perp$$

$$\frac{\neg\neg A}{A} \neg\neg_E$$

Natural Deduction Calculus

Rules for Modalities

$$\alpha : \boxed{\begin{array}{c} \vdots \\ A \end{array}} \quad \frac{}{\Box A} \quad \Box_I$$

$$t : \boxed{\begin{array}{c} \Box A \\ A \\ \vdots \end{array}} \quad \frac{}{\Box_E}$$

$$t : \boxed{\begin{array}{c} \vdots \\ A \end{array}} \quad \frac{}{\Diamond A} \quad \Diamond_I$$

$$\beta : \boxed{\begin{array}{c} \Diamond A \\ A \\ \vdots \end{array}} \quad \frac{}{\Diamond_E}$$

$$\Diamond A \equiv \neg \Box \neg A$$

Natural Deduction Proofs

T1 and C1

| | |
|--|--|
| $\frac{\forall \varphi. \forall \psi. [(P(\varphi) \wedge \Box \forall x. [\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]}{\forall \psi. [(P(\rho) \wedge \Box \forall x. [\rho(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]} \forall_E$ | $\frac{\forall \varphi. [P(\neg \varphi) \rightarrow \neg P(\varphi)]}{P(\neg \rho) \rightarrow \neg P(\rho)} \text{ A1a}$ |
| $\frac{(P(\rho) \wedge \Box \forall x. [\neg \rho(x)]) \rightarrow P(\neg \rho)}{(P(\rho) \wedge \Box \forall x. [\neg \rho(x)]) \rightarrow \neg P(\rho)}$ | $\frac{(P(\rho) \wedge \Box \forall x. [\neg \rho(x)]) \rightarrow \neg P(\rho)}{P(\rho) \rightarrow \Diamond \exists x. \rho(x)}$ |

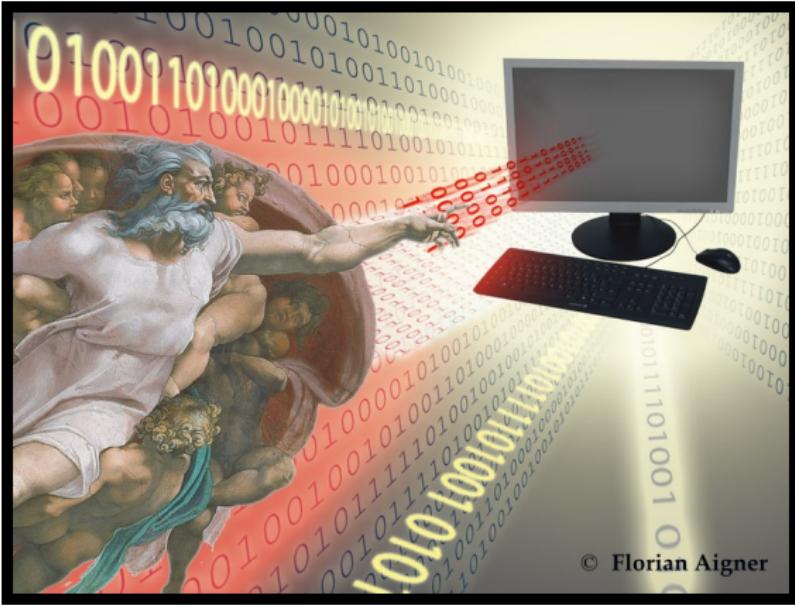
Natural Deduction Proofs

T2 (Partial)

$$\frac{\psi(x)^6 \quad \frac{\neg \psi(x) \rightarrow \square P(\psi)}{\square P(\psi)}^{\neg\neg E}}{\psi(x) \rightarrow \square \forall x.(G(x) \rightarrow \psi(x))}^{\rightarrow_E}$$

$$\frac{\square P(\psi)^7 \quad \frac{\neg \square P(\psi) \quad \frac{\neg \forall x.(G(x) \rightarrow \psi(x))}{\forall x.(G(x) \rightarrow \psi(x))}^{\neg\neg E}}{\square \forall x.(G(x) \rightarrow \psi(x))}^{\square I}}{\square P(\psi) \rightarrow \square \forall x.(G(x) \rightarrow \psi(x))}^{\rightarrow_I}$$

$$\frac{\square P(\psi) \quad \frac{\neg \square P(\psi) \quad \frac{\neg \forall x.(G(x) \rightarrow \psi(x))}{\forall x.(G(x) \rightarrow \psi(x))}^{\neg\neg E}}{\square \forall x.(G(x) \rightarrow \psi(x))}^{\square I}}{\forall x.(G(x) \rightarrow \psi(x))}^{\rightarrow_E}$$



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Part B:

Formalization:
Automation:
Consistency:

in classical higher-order logic (HOL)
theorem provers **Leo-II** and **SATALLAX**
model finder **NITPICK** (**NITROX**)

Challenge: No provers for *Higher-order Quantified Modal Logic* ([QML](#))

Our solution: Embedding in *Higher-order Classical Logic* ([HOL](#))

Then use existing [HOL](#) theorem provers for reasoning in [QML](#)

[BenzmüllerPaulson, Logica Universalis, 2013]

Previous empirical findings:

Embedding of *First-order Modal Logic* in HOL works well

[BenzmüllerOttenRaths, ECAI, 2012]

[Benzmüller, LPAR, 2013]

QML $\varphi, \psi ::= \dots | \neg\varphi | \varphi \wedge \psi | \varphi \rightarrow \psi | \Box\varphi | \Diamond\varphi | \forall x \varphi | \exists x \varphi | \forall P \varphi$

- Kripke style semantics (possible world semantics)

HOL

$s, t ::= C | x | \lambda x s | s t | \neg s | s \vee t | \forall x t$

- meanwhile very well understood
- **Henkin semantics** vs. standard semantics
- various theorem provers do exists

interactive: Isabelle/HOL, HOL4, Hol Light, Coq/HOL, PVS, ...

automated: TPS, LEO-II, Satallax, Nitpick, Isabelle/HOL, ...

QML $\varphi, \psi ::= \dots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x \varphi \mid \exists x \varphi \mid \forall P \varphi$

HOL $s, t ::= C \mid x \mid \lambda x s \mid s t \mid \neg s \mid s \vee t \mid \forall x t$

QML $\varphi, \psi ::= \dots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x \varphi \mid \exists x \varphi \mid \forall P \varphi$

$$\text{HOL} \qquad \qquad s, t ::= C \mid x \mid \lambda x s \mid s t \mid \neg s \mid s \vee t \mid \forall x t$$

QML in HOL: QML formulas φ are mapped to HOL predicates $\varphi_{\iota \rightarrow o}$

| | |
|---------------|---|
| \neg | $= \lambda\varphi_{t \rightarrow o} \lambda s_t \neg \varphi s$ |
| \wedge | $= \lambda\varphi_{t \rightarrow o} \lambda\psi_{t \rightarrow o} \lambda s_t (\varphi s \wedge \psi s)$ |
| \rightarrow | $= \lambda\varphi_{t \rightarrow o} \lambda\psi_{t \rightarrow o} \lambda s_t (\neg \varphi s \vee \psi s)$ |
| \Box | $= \lambda\varphi_{t \rightarrow o} \lambda s_t \forall u_t (\neg rsu \vee \varphi u)$ |
| \Diamond | $= \lambda\varphi_{t \rightarrow o} \lambda s_t \exists u_t (rsu \wedge \varphi u)$ |
| \forall | $= \lambda h_{\mu \rightarrow (t \rightarrow o)} \lambda s_t \forall d_\mu hds$ |
| \exists | $= \lambda h_{\mu \rightarrow (t \rightarrow o)} \lambda s_t \exists d_\mu hds$ |
| \forall | $= \lambda H_{(\mu \rightarrow (t \rightarrow o)) \rightarrow (t \rightarrow o)} \lambda s_t \forall d_\mu Hds$ |
| valid | $= \lambda\varphi_{t \rightarrow o} \forall w_t \varphi w$ |

QML $\varphi, \psi ::= \dots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x \varphi \mid \exists x \varphi \mid \forall P \varphi$

$$\text{HOL} \qquad \qquad s, t ::= C \mid x \mid \lambda x s \mid s t \mid \neg s \mid s \vee t \mid \forall x t$$

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| \rightarrow | $= \lambda\varphi_{t \rightarrow o} \lambda\psi_{t \rightarrow o} \lambda s_t (\neg \varphi s \vee \psi s)$ |
| \Box | $= \lambda\varphi_{t \rightarrow o} \lambda s_t \forall u_t (\neg rsu \vee \varphi u)$ |
| \Diamond | $= \lambda\varphi_{t \rightarrow o} \lambda s_t \exists u_t (rsu \wedge \varphi u)$ |
| \forall | $= \lambda h_{\mu \rightarrow (t \rightarrow o)} \lambda s_t \forall d_\mu hds$ |
| \exists | $= \lambda h_{\mu \rightarrow (t \rightarrow o)} \lambda s_t \exists d_\mu hds$ |
| \forall | $= \lambda H_{(\mu \rightarrow (t \rightarrow o)) \rightarrow (t \rightarrow o)} \lambda s_t \forall d_\mu Hds$ |
| valid | $= \lambda\varphi_{t \rightarrow o} \forall w_t \varphi w$ |

The equations in `Ax` are given as axioms to the `HOL` provers!

(Remark: Note that we are here dealing with constant domain quantification)

Example

QML formula

QML formula in HOL

expansion, $\beta\eta$ -conversion
expansion, $\beta\eta$ -conversion
expansion, $\beta\eta$ -conversion

$\diamond \exists x G(x)$

valid $(\diamond \exists x G(x))_{t \rightarrow o}$

$\forall w_t (\diamond \exists x G(x))_{t \rightarrow o} w$

$\forall w_t \exists u_t (rwu \wedge (\exists x G(x))_{t \rightarrow o})$

$\forall w_t \exists u_t (rwu \wedge \exists x Gxu)$

What are we doing?

In order to prove that φ is valid in QML,

→ we instead prove that valid $\varphi_{t \rightarrow o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Expansion:

user or prover may flexibly choose expansion depth

Example

QML formula

QML formula in HOL

expansion, $\beta\eta$ -conversion
 expansion, $\beta\eta$ -conversion
 expansion, $\beta\eta$ -conversion

 $\diamond \exists x G(x)$ valid $(\diamond \exists x G(x))_{t \rightarrow o}$ $\forall w_t (\diamond \exists x G(x))_{t \rightarrow o} w$ $\forall w_t \exists u_t (rwu \wedge (\exists x G(x))_{t \rightarrow o})$ $\forall w_t \exists u_t (rwu \wedge \exists x Gxu)$

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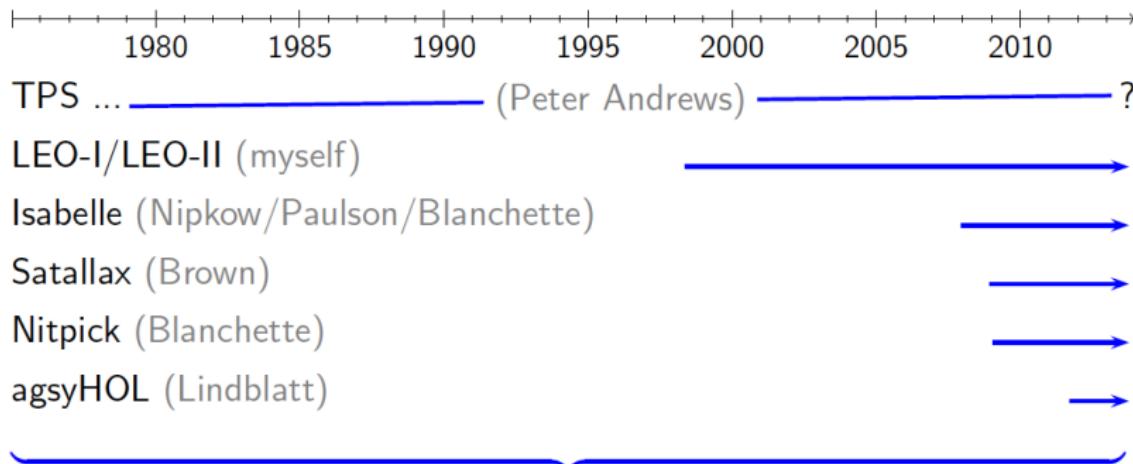
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Expansion:

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- all accept TPTP THF Syntax [SutcliffeBenzmüller, J.Form.Reas, 2009]
 - can be called remotely via SystemOnTPTP at Miami
 - they significantly gained in strength over the last years
 - they can be bundled into a combined prover **HOL-P**

Exploit HOL with Henkin semantics as metalogic
Automate other logics (& combinations) via semantic embeddings
— **HOL-P** becomes a **Universal Reasoner** —

Proof Automation and Consistency Checking: Demo!

```
MacBook-Chris % Terminal — bash — 125x32
MacBook-Chris %
MacBook-Chris %
MacBook-Chris % ./call_tptp.sh T3.p

Asking various HOL-ATPs in Miami remotely (thanks to Geoff Sutcliffe)

MacBook-Chris % agsyHOL---1.0 : T3.p ++++++ RESULT: SOT_7L4x.Y - agsyHOL---1.0 says Unknown - CPU = 0.00 WC = 0.02
LEO-II---1.6.0 : T3.p ++++++ RESULT: SOT_E4Scha - LEO-II---1.6.0 says Theorem - CPU = 0.03 WC = 0.09
Satallax---2.7 : T3.p ++++++ RESULT: SOT_kVZ1cB - Satallax---2.7 says Theorem - CPU = 0.00 WC = 0.14
Isabelle---2013 : T3.p ++++++ RESULT: SOT_xa0gEp - Isabelle---2013 says Theorem - CPU = 14.06 WC = 17.73 SolvedBy = auto
TPS---3.120601S1b : T3.p ++++++ RESULT: SOT_ROEsgs - TPS---3.120601S1b says Unknown - CPU = 33.56 WC = 41.57
Nitrox---2013 : T3.p ++++++ RESULT: SOT_WGY1Tx - Nitrox---2013 says Unknown - CPU = 75.55 WC = 49.24

MacBook-Chris %
MacBook-Chris % ./call_tptp.sh Consistency.p

Asking various HOL-ATPs in Miami remotely (thanks to Geoff Sutcliffe)

MacBook-Chris % agsyHOL---1.0 : Consistency.p ++++++ RESULT: SOT_ZtY_7o - agsyHOL---1.0 says Unknown - CPU = 0.00 WC = 0.00
Nitrox---2013 : Consistency.p ++++++ RESULT: SOT_HUz1OC - Nitrox---2013 says Satisfiable - CPU = 6.56 WC = 8.50
TPS---3.120601S1b : Consistency.p ++++++ RESULT: SOT_fpjxTM - TPS---3.120601S1b says Unknown - CPU = 43.00 WC = 49.42
Isabelle---2013 : Consistency.p ++++++ RESULT: SOT_6Tpp9i - Isabelle---2013 says Unknown - CPU = 69.96 WC = 72.62
LEO-II---1.6.0 : Consistency.p ++++++ RESULT: SOT_dy10sj - LEO-II---1.6.0 says Timeout - CPU = 90 WC = 89.86
Satallax---2.7 : Consistency.p ++++++ RESULT: SOT_Q9WSLF - Satallax---2.7 says Timeout - CPU = 90 WC = 90.50

MacBook-Chris %
```

Provers are called remotely in Miami — no local installation needed!



Part C: Formalization and Verification in Coq

- Goal: verification of the natural deduction proof
 - Step-by-step formalization
 - Almost no automation (intentionally!)
- Interesting facts:
 - Embedding is transparent to the user
 - Embedding gives labeled calculus for free

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The screenshot shows the CoqIDE interface with the title bar "Coqide". The menu bar includes File, Edit, Navigation, Try Tactics, Templates, Queries, Display, Compile, Windows, Help. The toolbar has icons for file operations like Open, Save, Print, and search. The tabs at the top are "scratch", "Modal.v", "ModalClassical.v", and "GödelGod-Scott.v". The main window displays a Coq proof script:

```

(* Constant predicate that distinguishes positive properties *)
Parameter Positive: (u -> o) -> o.

(* Axiom A1: either a property or its negation is positive, but not both *)
Axiom axiomia : V (mforall p, (Positive (fun x: u => m-(p x))) m-> (m- (Positive p))). 
Axiom axiomib : V (mforall p, (m- (Positive p)) m-> (Positive (fun x: u => m- (p x)))). 

(* Axiom A2: a property necessarily implied by a positive property is positive *)
Axiom axiom2: V (mforall p, mforall q, Positive p m/\ (box (mforall x, (p x) m-> (q x))).

(* Theorem T1: positive properties are possibly exemplified *)
Theorem theorem1: V (mforall p, (Positive p) m-> dia (mexists x, p x)).
Proof.
intro.
intro p.
intro H1.
proof_by_contradiction H2.
apply not_dia_box_not_in H2.
assert (H3: (box (mforall x, m- (p x))) w). (* Lemma from Scott's notes *)
  box_intro w1 R1.
  intro x.
  assert (H4: ((m- (mexists x : u, p x)) w1)).
    box_elim H2 w1 R1 G2.
    exact G2.

  clear H2 R1 H1 w.
  intro H5.
  apply H4.
  exists x.
  exact H5.

assert (H6: ((box (mforall x, (p x) m-> m- (x m= x))) w)). (* Lemma from Scott's notes *)
  box_intro w1 R1.
  intro x.
  intro H7.
  intro H8.
  box_elim H3 w1 R1 G3.
  annTu ct with / \_ := _

```

The right pane shows the proof state with two subgoals:

- Subgoal 1: $w : i$
H1 : Positive p w
H2 : box (m- (mexists x : u, p x)) w (1/2)
- Subgoal 2: (2/2)
False



Part D:

automation & verification: proof assistant ISABELLE

Isabelle

http://isabelle.in.tum.de/index.html

Home-FU 2012-Watson Homepage 2012-FOL SPIEGEL 2012-FOL-Home GMail Google Maps M&M SigmaOnline Kita Sigma Kalender Beliebt Google Maps >>

Isabelle



Isabelle

UNIVERSITY OF CAMBRIDGE Computer Laboratory

TUM TECHNISCHE UNIVERSITÄT MÜNCHEN

What is Isabelle?

Isabelle is a generic proof assistant. It allows mathematical formulas to be expressed in a formal language and provides tools for proving those formulas in a logical calculus. Isabelle is developed at University of Cambridge ([Larry Paulson](#)), Technische Universität München ([Tobias Nipkow](#)) and Université Paris-Sud ([Makarius Wenzel](#)). See the [Isabelle overview](#) for a brief introduction.

Now available: Isabelle2013

Download for Mac OS X



[Download for Linux](#) - [Download for Windows](#)

Some highlights:

- Improvements of Isabelle/Scala and Isabelle/Edit Prover IDE.
- Advanced build tool based on Isabelle/Scala.
- Updated manuals: Isar-ref, implementation, system.
- Pure: improved support for block-structured specification contexts.
- HOL tool enhancements: Sledgehammer, Lifting, Quickcheck.
- HOL library enhancements: HOL-Library, HOL-Probability, HOL-Cardinals.
- HOL: New BNF-based (co)datatype package.
- Improved performance thanks to PolyML 5.5.0.

See also the cumulative [NEWS](#).

Distribution & Support

Isabelle is distributed for free under the BSD license. It includes source and binary packages and documentation, see the detailed [installation instructions](#). A vast collection of Isabelle examples and applications is available from the [Archive of Formal Proofs](#).

Support is available by ample [documentation](#), the [Isabelle Community Wiki](#), and the following mailing lists:

- isabelle-users@cl.cam.ac.uk provides a forum for Isabelle users to discuss problems, exchange information, and make announcements. Users of official Isabelle releases should [subscribe](#) or see the [archive](#) (also available via [Google groups](#) and [Narkive](#)).
- isabelle-dev@in.tum.de covers the Isabelle development process, including intermediate repository versions, and administrative issues concerning the website or testing infrastructure. Early adopters of [repository versions](#) should [subscribe](#) or see the [archive](#) (also available at [mail-archive.com](#) or [gmane.org](#)).

Last updated: 2013-03-09 12:21:39

Isabelle/HOL (Cambridge University/TU Munich)

- HOL instance of the generic IsABELLE proof assistant
- User interaction and proof automation
- Automation is supported by SLEDGEHAMMER tool
- Verification of the proofs in IsABELLE/HOL's small proof kernel

What we did?

- Proof automation of Gödel's proof script (Scott version)
- SLEDGEHAMMER makes calls to remote THF provers in Miami
- These calls suggest respective calls to the METIS prover
- METIS proofs are verified in IsABELLE/HOL's proof kernel

See the handout (generated from the Isabelle source file).

Automation & Verification in Proof Assistant IsABELLE/HOL

The screenshot shows the Isabelle/HOL proof assistant interface. The main window displays the theory file `GoedelGod.thy` (modified). The code includes various text blocks, corollaries, and definitions. A specific section involves the `sledgehammer` command to prove theorem T2. The interface features a sidebar titled "Sidekick: Theories" and a status bar at the bottom.

```
corollary C: "[o (E G)]"
sledgehammer [provers = remote_leo2] by (metis A3 T1)

text {* Axiom @{text "A4"} is added: $\forall \phi [P(\phi) \rightarrow \Box \; ; \; P(\phi)]$ 
(Positive properties are necessarily positive). *}

axiomatization where A4: "[!! (\lambda\Phi. P \Phi m⇒ □ (P Φ))"

text {* Symbol @{text "ess"} for 'Essence' is introduced and defined as 
$ess{\phi}{x} \; \text{biimp} \; \phi(x) \; \text{wedge} \; \forall \psi \; (\psi(x) \; \text{imp} \; \nec \; \forall y \; (\phi(y) \; 
\text{imp} \; \psi(y)))$ 
(An essence of an individual is a property possessed by it 
and necessarily implying any of its properties). *}

definition ess :: "(μ ⇒ σ) ⇒ μ ⇒ σ" (infixr "ess" 85) where
  "Φ ess x = Φ x m ∧ II (λψ. ψ x m ⇒ □ ( ∀ (λy. Φ y m ⇒ ψ y)))"

text {* Next, Sledgehammer and Metis prove theorem @{text "T2"}: $\forall x \; [G(x) \; \text{imp} \; ess(G){x}]$ 
(Being God-like is an essence of any God-like being). *}

theorem T2: "[! (λx. G x m⇒ G ess x)]"
sledgehammer [provers = remote_leo2] by (metis A1b A4 G_def ess_def)

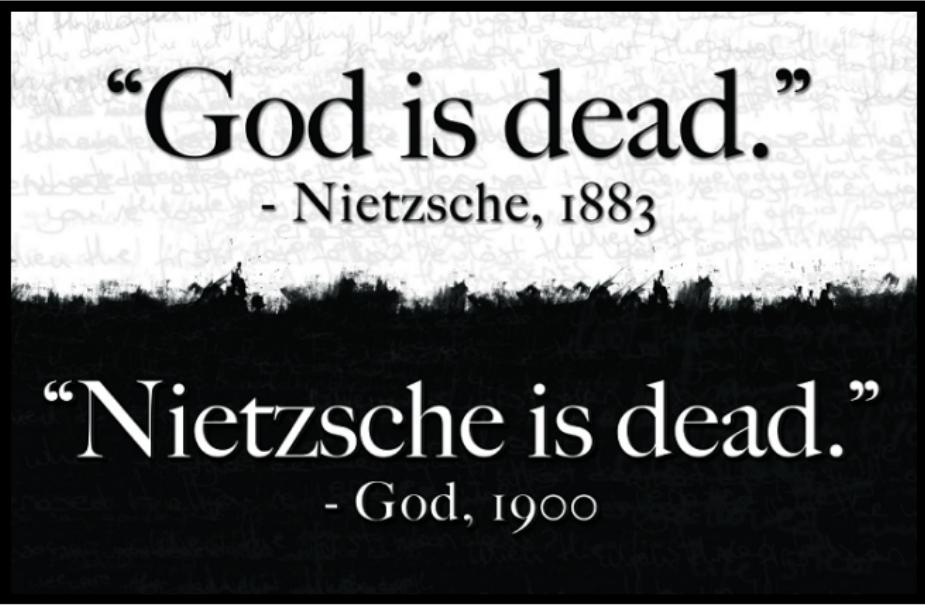
text {* Symbol @{text "NE"}, for 'Necessary Existence', is introduced and 
defined as $NE(x) \; \text{biimp} \; \forall \phi \; [ess{\phi}{x}] \; \text{imp} \; \nec \; \exists y \; \phi(y)$ 
(Necessary 
existence of an individual is the necessary exemplification of all its essences). *}

definition NE :: "μ ⇒ σ" where "NE = (λx. II (λΦ. Φ ess x m⇒ □ ( ∃ Φ)))"
```

Sledgehammering...

Output: README Symbols

139.39 (6613/8211) (isabelle,sidekick,UTF-8-isabelle) 16:02



“God is dead.”

- Nietzsche, 1883

“Nietzsche is dead.”

- God, 1900

Part E: Criticisms

$$\forall P. [\diamond \Box P \rightarrow \Box P]$$

If something is possibly necessary, then it is necessary.

$$\diamond \Box (A \vee \neg A) \quad \Box (A \vee \neg A)$$

logical necessity ~ validity

for all $M, M \models F \rightarrow \Box F$

logical possibility ~ satisfiability

exists $M, M \models F \rightarrow \diamond F$

What about iterations?

$$\diamond \Box \diamond \diamond F$$

weak intuitions \Rightarrow dozens of modal logics

S5 is considered adequate

(But KB is sufficient!)

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Everything that is the case is so necessarily.

Follows from T2, T3 and D2.

There are no contingent “truths”.

Everything is determined.

There is no free will.

Many proposed solutions: Anderson, Fitting, Hájek, ...

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$$\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$$

Either a property is positive or its negation is (but never both)

Are the following properties positive or negative?

$$\lambda x.G(x) \quad \lambda x.E(x) \quad \lambda x.x = x \quad \lambda x.\top$$

$$\lambda x.blue(x) \quad \lambda x.white(x) \quad \lambda x.human(x)$$

Solution:

“...positive in the moral aesthetic sense (independently of the accidental structure of the world). Only then the ax. true.”

- Gödel, 1970

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Part F: Conclusions

The (**new**) insights we gained from experiments include:

- Logic K sufficient for T1, C and T2
- Logic S5 not needed for T3
- Logic KB sufficient for T3 (not well known)
- We found a simpler new proof of C
- Gödel's axioms (without conjunct $\phi(x)$ in D2) are inconsistent
- Scott's axioms are consistent
- For T1, only half of A1 (A1a) is needed
- For T2, the other half (A1b) is needed

Our novel contributions to the theorem proving community include

- Powerful infrastructure for reasoning with QML
- A new natural deduction calculus for higher-order modal logic
- Difficult new benchmarks problems for HOL provers
- Huge media attention

What have we achieved

- Verification of Gödel's ontological argument with HOL provers
 - exact parameters known: constant domain quantification, Henkin Semantics
 - experiments with different parameters could be performed
- Gained some novel results and insights
- Major step towards **Computer-assisted Theoretical Philosophy**
 - see also Ed Zalta's *Computational Metaphysics* project at Stanford University
 - remember Leibniz' dictum — *Calculemus!*
- Interesting bridge between CS, Philosophy and Theology

Ongoing and future work

- Formalize and verify literature on ontological arguments
 - ... in particular the criticism and improvements to Gödel
- Own contributions — supported by theorem provers

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Ongoing and future work

- Formalize and verify literature on ontological arguments
 - ... in particular the criticism and improvements to Gödel
- Own contributions — supported by theorem provers

I'm sure that God would be impressed with your proof, if only he existed :-)

Larry

Die Philosophen können so schön staunen.

Sie packen Dinge in Begriffe (gucken dabei in die Luft) werfen die Begriffe dann in ihre Philosophiekiste, schütteln ganz dolle, und freuen sich, dass ganz genau rauskommt, was sie vorher reingetan haben. Und das geht sogar, wenn eine Maschine die Kiste schüttelt.

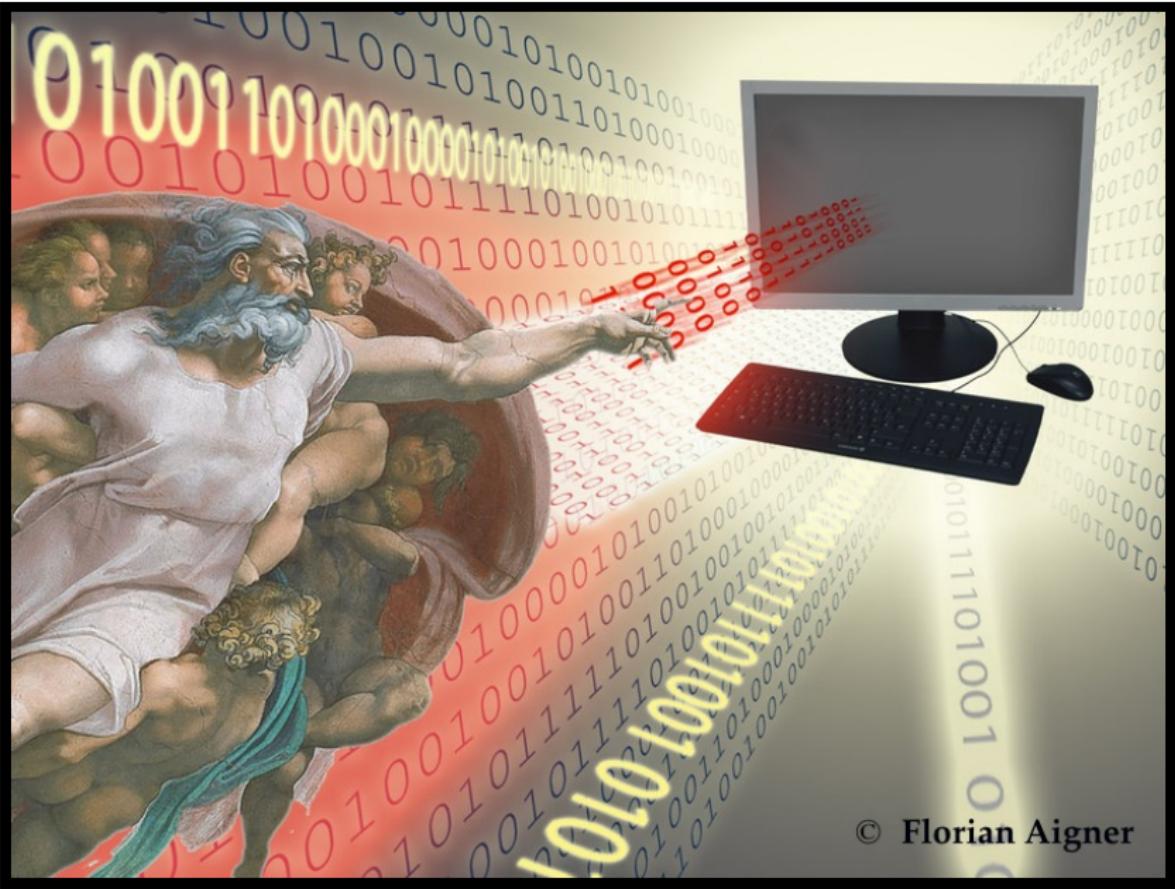
Unerstaunt
2017cp

60. Suchlauf

 souveränsatt 09.09.2013

man kann auch auf andere Weise in diesem Zusammenhang methodisch vorgehen:
bei einer längeren Autofahrt das Radio auf automatischen Suchlauf stellen. Nach zwei Tagen sieht man Gott

... find more on the internet ...



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