

# How to “lazily” prove modal logic theorems using higher-order theorem provers

*(and “prove” God’s existence with the push of a button)*

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# Gödel's Manuscript: 1930's, 1941, 1946-1955, 1970

Ontologischer Begriff

FEB 10, 1970

P( $\varphi$ ):  $\varphi$  is positive ( $\Leftrightarrow \varphi \in P$ )

At 1:  $P(\varphi) \cdot P(\psi) \supset P(\varphi \wedge \psi)$  At 2:  $P(\varphi) \supset P(\neg \varphi)$

P1:  $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$  (Ged.)

P2:  $\varphi_{\text{Em},x} \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$  ( $\varphi_{\text{Em},y/x}$ )

$P \supset_N = N(P \supset \varphi)$  Necessity

At 2:  $P(\varphi) \supset N P(\varphi)$  } because it follows  
 $\neg P(\varphi) \supset N \neg P(\varphi)$  } from the nature of the property

Th.:  $G(x) \supset G_{\text{Em},x}$

Df.:  $E(x) \equiv P(\varphi_{\text{Em},x} \supset N \exists x \varphi(x))$  necessary Existence

At 3:  $P(E)$

Th.:  $G(x) \supset N(\exists y) G(y)$

hence  $(\exists x) G(x) \supset N(\exists y) G(y)$

"  $M(\exists x) G(x) \supset M N(\exists y) G(y)$  Mi = partibility  
 "  $\supset N(\exists y) G(y)$

any two instances of  $x$  are nec. equivalent

exclusive or \* and for any number of arguments

$M(\exists x) G(x)$ : means all pos. propo. w.r.t. com-  
 patible This is true because of:

At 4:  $P(\varphi) \cdot \varphi \supset \psi \Rightarrow P(\psi)$  which impl.  
~~the system of~~ {  $x=x$  is positive  
~~the system of~~ {  $x \neq x$  is negative

But if a system S of pos. propo. were incon-  
 sistent it would mean that the same propo. S (which  
 is positive) would be  $x \neq x$

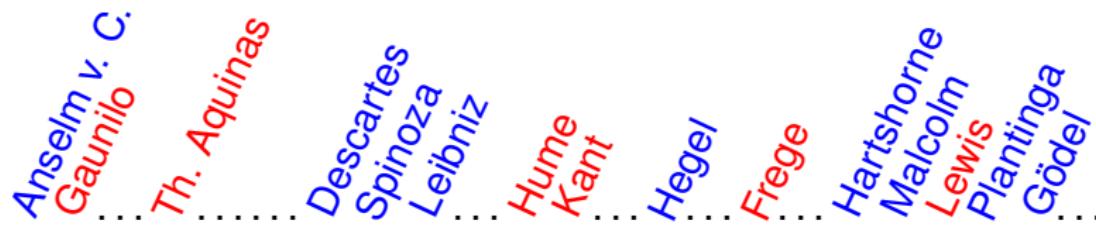
Positive means positive in the modal sense  
 sense (independently of the accidental structure of  
 the world). Only then the at time. It may  
 also mean "attribution" as opposed to "privation  
 (or containing privation). This is Gödel's problem part

$\exists x \varphi$  positive w.r.t.  $(x) N \supset \varphi(x)$ . Otherwise  $\exists x \varphi(x) \supset_N$   
 hence  $x \neq x$ , i.e. not  $x=x$  according to  
 the definition of pos. At this

i.e. the normal form in terms of elem. propo. contains a  
 member without negation.

# A Long History

pros and cons



Anselm's notion of God:

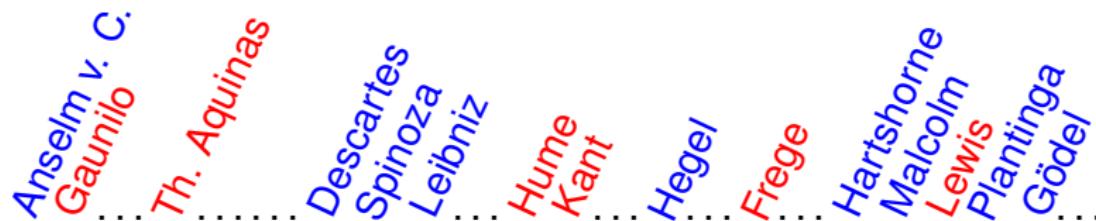
*“God is that, than which nothing greater can be conceived.”*

Gödel's notion of God:

*“A God-like being possesses all ‘positive’ properties.”*

# A Long History

pros and cons



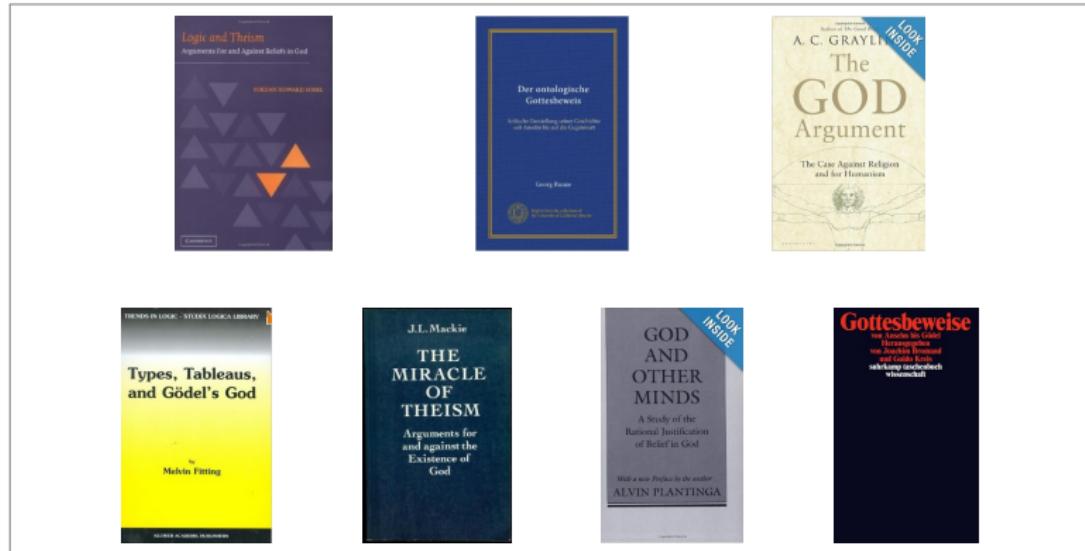
Anselm's notion of God:

*“God is that, than which nothing greater can be conceived.”*

Gödel's notion of God:

*“A God-like being possesses all ‘positive’ properties.”*

# The Ontological Proof Today





## Proof Overview

**T3:**  $\Box \exists x. G(x)$

**C1:**  $\Diamond \exists z. G(z)$

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**T3:**  $\Box \exists x. G(x)$

$$\frac{\mathbf{C1: } \diamond \exists z. G(z) \quad \mathbf{L2: } \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\mathbf{T3: } \Box \exists x. G(x)}$$

**L2:**  $\diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$

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**C1:**  $\diamond \exists z. G(z)$       **L2:**  $\diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$   
**T3:**  $\Box \exists x. G(x)$

**S5**  
 $\neg \forall \xi. [\neg \diamond \square \xi \rightarrow \square \neg \xi]$

---

**L2:**  $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

**C1:**  $\diamond \exists z. G(z)$

**L2:**  $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

---

**T3:**  $\square \exists x. G(x)$

$$\frac{\diamond \exists z. G(z) \rightarrow \diamond \Box \exists x. G(x) \qquad \overline{\forall \xi. [\diamond \Box \xi \rightarrow \Box \xi]}}{\text{L2: } \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}$$

$$\frac{\text{C1: } \diamond \exists z. G(z) \qquad \text{L2: } \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\text{T3: } \Box \exists x. G(x)}$$

$$\frac{\textbf{L1: } \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)} \qquad \qquad \textbf{S5} \\ \frac{}{\neg \forall \xi. [\neg \Diamond \neg \xi \rightarrow \neg \Box \neg \xi]}$$

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$$\textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$$

---

$$\textbf{C1: } \Diamond \exists z. G(z) \qquad \textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \\ \textbf{T3: } \Box \exists x. G(x)$$

**D1:**  $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

$$\frac{\frac{\textbf{L1: } \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)} \quad \textbf{S5}}{\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]} \quad \textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}$$

---

$$\textbf{C1: } \Diamond \exists z. G(z) \quad \textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$$

---

$$\textbf{T3: } \Box \exists x. G(x)$$

**D1:**  $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

**D3\*:**  $NE(x) \equiv \Box \exists y. G(y)$

$$\frac{\frac{P(NE)}{\frac{\frac{\textbf{L1: } \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\frac{\Box \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)}{\frac{\textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\frac{\frac{\textbf{C1: } \Diamond \exists z. G(z)}{\textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}}{\textbf{T3: } \Box \exists x. G(x)}}}}{\textbf{S5: } \forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]}}$$

**D1:**  $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

**D3\*:**  $NE(x) \equiv \Box \exists y. G(y)$  (cheating!)

$$\frac{\frac{\frac{P(NE)}{\textbf{L1: } \exists z. G(z) \rightarrow \Box \exists x. G(x)}}{\Box \exists z. G(z) \rightarrow \Box \Box \exists x. G(x)}}{\textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)} \quad \textbf{S5: } \overline{\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]}$$


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$$\frac{\textbf{C1: } \Diamond \exists z. G(z) \quad \textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\textbf{T3: } \Box \exists x. G(x)}$$

**D1:**  $G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$

**D3\*:**  $NE(x) \equiv \Box \exists y.G(y)$

**D3:**  $NE(x) \equiv \forall \varphi.[\varphi \text{ ess } x \rightarrow \Box \exists y.\varphi(y)]$

**T2:**  $\forall y.[G(y) \rightarrow G \text{ ess } y]$   $P(NE)$

**L1:**  $\exists z.G(z) \rightarrow \Box \exists x.G(x)$

$\frac{\Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x)}{\quad}$

**S5**

$\neg \forall \xi. [\neg \Box \xi \rightarrow \neg \Box \xi]$

**L2:**  $\Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)$

**C1:**  $\Diamond \exists z.G(z)$

**L2:**  $\Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)$

**T3:**  $\Box \exists x.G(x)$

**D1:**  $G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$

**D3\*:**  $NE(x) \equiv \Box \exists y.G(y)$

**D3:**  $NE(x) \equiv \forall \varphi.[\varphi \text{ ess } x \rightarrow \Box \exists y.\varphi(y)]$

$$\frac{\begin{array}{c} \textbf{T2: } \forall y.[G(y) \rightarrow G \text{ ess } y] \\ \hline \textbf{L1: } \exists z.G(z) \rightarrow \Box \exists x.G(x) \end{array}}{\Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x)} \quad \frac{\textbf{A5} \quad \overline{P(NE)}}{\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]} \quad \frac{\textbf{S5} \quad \overline{\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]}}{\textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}$$

$$\frac{\textbf{C1: } \Diamond \exists z.G(z) \quad \textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\textbf{T3: } \Box \exists x.G(x)}$$

**D1:**  $G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$

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$$\frac{\begin{array}{c} \textbf{T2: } \forall y.[G(y) \rightarrow G \text{ ess } y] \\ \hline \textbf{L1: } \exists z.G(z) \rightarrow \Box \exists x.G(x) \end{array}}{\Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x)} \quad \frac{\textbf{A5} \quad \overline{P(NE)}}{\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]} \quad \frac{\textbf{S5} \quad \overline{\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]}}{\textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}$$

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$$\mathbf{D1: } G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$$

$$\mathbf{D2: } \varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \square \forall x. (\varphi(x) \rightarrow \psi(x)))$$

$$\mathbf{D3^*: } NE(x) \equiv \square \exists y. G(y)$$

$$\mathbf{D3: } NE(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$$

$$\frac{\begin{array}{c} \mathbf{A1b} \\ \overline{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]} \end{array}}{\mathbf{T2: } \forall y. [G(y) \rightarrow G \text{ ess } y]} \quad 
 \frac{\begin{array}{c} \mathbf{A4} \\ \overline{\forall \varphi. [P(\varphi) \rightarrow \square \neg P(\varphi)]} \end{array}}{\mathbf{A5} \quad \overline{P(NE)}}$$
  

$$\frac{\mathbf{L1: } \exists z. G(z) \rightarrow \square \exists x. G(x)}{\frac{\mathbf{L2: } \diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}{\frac{\mathbf{S5} \quad \overline{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}}{\mathbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}}}$$

$$\mathbf{C1: } \diamond \exists z. G(z)$$

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$$\mathbf{C1: } \diamond \exists z. G(z)$$

$$\frac{\neg \forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]}{\mathbf{T2: } \forall y. [G(y) \rightarrow G \text{ ess } y]}$$

$$\frac{\neg \forall \varphi. [P(\varphi) \rightarrow \square \neg P(\varphi)]}{\mathbf{A4} }$$

$$\frac{}{P(NE)}$$

$$\frac{\mathbf{L1: } \exists z. G(z) \rightarrow \square \exists x. G(x)}{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}$$

$$\frac{}{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}$$

$$\mathbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$$

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$P(G)$

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**C1:**  $\diamond \exists z. G(z)$

**A1b**  
 $\neg \forall \varphi. \neg P(\varphi) \rightarrow P(\neg \varphi)$

**A4**  
 $\neg \forall \varphi. P(\varphi) \rightarrow \square \neg P(\varphi)$

**A5**  
 $\neg P(NE)$

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**T2:**  $\forall y. [G(y) \rightarrow G \text{ ess } y]$

**L1:**  $\exists z. G(z) \rightarrow \square \exists x. G(x)$   
 $\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)$

**S5**  
 $\neg \forall \xi. \neg \diamond \square \xi \rightarrow \neg \square \xi$

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**L2:**  $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

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**C1:**  $\diamond \exists z. G(z)$

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**D3:**  $NE(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

**A3**  
 $\frac{}{P(G)}$

---

**C1:**  $\diamond \exists z. G(z)$

**A1b**  
 $\frac{\neg \forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]}{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]}$

**A4**  
 $\frac{\neg \forall \varphi. [P(\varphi) \rightarrow \square \neg P(\varphi)]}{\forall \varphi. [P(\varphi) \rightarrow \square \neg P(\varphi)]}$

**A5**  
 $\frac{}{P(NE)}$

---

**T2:**  $\forall y. [G(y) \rightarrow G \text{ ess } y]$

**L1:**  $\exists z. G(z) \rightarrow \square \exists x. G(x)$   
 $\frac{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}$

**S5**  
 $\frac{}{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}$

---

**L2:**  $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

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**A3**  
 $\frac{}{P(G)}$

**T1:**  $\forall \varphi. [P(\varphi) \rightarrow \diamond \exists x. \varphi(x)]$

**C1:**  $\diamond \exists z. G(z)$

**A1b**

$\frac{}{\neg \forall \varphi. \neg P(\varphi) \rightarrow P(\neg \varphi)}$

**A4**

$\frac{}{\neg \forall \varphi. P(\varphi) \rightarrow \square \neg P(\varphi)}$

**T2:**  $\forall y. [G(y) \rightarrow G \text{ ess } y]$

**A5**

$\frac{}{\neg P(NE)}$

**L1:**  $\exists z. G(z) \rightarrow \square \exists x. G(x)$

$\frac{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}{\quad}$

**S5**

$\frac{}{\neg \forall \xi. \neg \diamond \square \xi \rightarrow \neg \square \xi}$

**L2:**  $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

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**D3\***:  $NE(x) \equiv \square \exists y. G(y)$

**D3:**  $NE(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

$$\frac{\frac{\frac{\frac{\overline{A3}}{P(\bar{G})} \quad \frac{\overline{A2}}{\overline{\forall \varphi. \forall \psi. [(\bar{P}(\varphi) \wedge \square \forall x. [\bar{\varphi}(x) \rightarrow \bar{\psi}(x)]] \rightarrow \bar{P}(\psi)]}}}{\mathbf{T1}: \forall \varphi. [P(\varphi) \rightarrow \diamond \exists x. \varphi(x)]} \quad \frac{\overline{A1a}}{\overline{\forall \varphi. [\bar{P}(\neg \varphi) \rightarrow \neg \bar{P}(\varphi)]}}}{\mathbf{C1}: \diamond \exists z. G(z)}$$

$$\frac{\frac{\frac{\frac{\overline{A1b}}{\overline{\forall \varphi. [\neg \bar{P}(\varphi) \rightarrow \bar{P}(\neg \varphi)]}} \quad \frac{\overline{A4}}{\overline{\forall \varphi. [\bar{P}(\varphi) \rightarrow \square \bar{P}(\varphi)]}}}{\mathbf{T2}: \forall y. [G(y) \rightarrow G \text{ ess } y]} \quad \frac{\overline{A5}}{\overline{P(NE)}}}{\frac{\mathbf{L1}: \exists z. G(z) \rightarrow \square \exists x. G(x)}{\frac{\mathbf{L2}: \diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}{\frac{\mathbf{S5}}{\overline{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}}}}}}{\mathbf{L2}: \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}$$

**C1:**  $\diamond \exists z. G(z)$

**L2:**  $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

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$$\mathbf{D1: } G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$$

**D2:**  $\varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \square \forall x. (\varphi(x) \rightarrow \psi(x)))$

**D3:**  $NE(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

$$\frac{\begin{array}{c} \textbf{A3} \\ \hline \neg P(G) \end{array} \quad \frac{\begin{array}{c} \textbf{A2} \\ \hline \neg \forall \varphi. \neg \psi. [(\neg P(\varphi) \wedge \neg \exists x. [\varphi(x) \rightarrow \psi(x)]) \rightarrow \neg P(\psi)] \end{array}}{\textbf{T1}: \forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x. \varphi(x)]} \quad \frac{}{\textbf{A1a}}} {\textbf{C1}: \Diamond \exists z. G(z)}$$

$\frac{\neg \forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]}{\text{T2: } \forall y. [G(y) \rightarrow G \text{ ess } y]}$	$\frac{\neg \forall \varphi. [P(\varphi) \rightarrow \Box \neg P(\varphi)]}{\text{A4}}$	$\frac{\neg P(NE)}{\text{A5}}$
$\frac{\text{L1: } \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)}$		$\frac{\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]}{\text{S5}}$

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**C1:**  $\Diamond \exists z.G(z)$       **L2:**  $\Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)$

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**T3:**  $\Box \exists x.G(x)$



## Modal Natural Deduction Proof

$$\frac{\overline{A} \quad \overline{B} \quad \vdots \quad \vdots \quad A \vee B \quad C \quad C}{C} \vee_E \quad \frac{A \quad B}{A \wedge B} \wedge_I \quad \frac{\overline{A} \quad \overline{B} \quad \vdots \quad \vdots \quad B}{A \rightarrow B} \rightarrow_I^n$$

$$\frac{A}{A \vee B} \vee_{I_1} \quad \frac{A \wedge B}{A} \wedge_{E_1} \quad \frac{B}{A \rightarrow B} \rightarrow_I$$

$$\frac{B}{A \vee B} \vee_{I_2} \quad \frac{A \wedge B}{B} \wedge_{E_2} \quad \frac{A \quad A \rightarrow B}{B} \rightarrow_E$$

$$\frac{A[\alpha]}{\forall x.A[x]} \forall_I \quad \frac{\forall x.A[x]}{A[t]} \forall_E \quad \frac{A[t]}{\exists x.A[x]} \exists_I \quad \frac{\exists x.A[x]}{A[\beta]} \exists_E$$

$$\neg A \equiv A \rightarrow \perp \qquad \frac{\neg\neg A}{A} \neg\neg_E$$

# Natural Deduction Calculus

## Rules for Modalities

$$\alpha : \boxed{\begin{array}{c} \vdots \\ A \end{array}} \quad \frac{}{\Box A} \quad \Box_I$$

$$t : \boxed{\begin{array}{c} \Box A \\ A \\ \vdots \end{array}} \quad \frac{\Box A}{\Box_E}$$

$$\Diamond A \equiv \neg \Box \neg A$$

$$t : \boxed{\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ A \end{array}} \quad \frac{}{\Diamond A} \quad \Diamond_I$$

$$\beta : \boxed{\begin{array}{c} \Diamond A \\ A \\ \vdots \end{array}} \quad \frac{\Diamond A}{\Diamond_E}$$

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$$\alpha : \boxed{\begin{array}{c} \vdots \\ A \end{array}} \quad \frac{}{\Box A} \quad \Box_I$$

$$t : \boxed{\begin{array}{c} \Box A \\ A \\ \vdots \end{array}} \quad \frac{\Box A}{\Box_E}$$

$$\Diamond A \equiv \neg \Box \neg A$$

$$t : \boxed{\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ A \end{array}} \quad \frac{}{\Diamond A} \quad \Diamond_I$$

$$\beta : \boxed{\begin{array}{c} \Diamond A \\ A \\ \vdots \end{array}} \quad \frac{\Diamond A}{\Diamond_E}$$

# Natural Deduction Calculus

## Rules for Modalities

$$\alpha : \boxed{\begin{array}{c} \vdots \\ A \end{array}} \quad \frac{}{\Box A} \quad \Box_I$$

$$t : \boxed{\begin{array}{c} \Box A \\ A \\ \vdots \end{array}} \quad \frac{\Box A}{A} \quad \Box_E$$

$$\Diamond A \equiv \neg \Box \neg A$$

$$t : \boxed{\begin{array}{c} \vdots \\ \Diamond A \end{array}} \quad \frac{}{\Diamond A} \quad \Diamond_I$$

$$\beta : \boxed{\begin{array}{c} \Diamond A \\ A \\ \vdots \end{array}} \quad \frac{\Diamond A}{A} \quad \Diamond_E$$

## Natural Deduction Proofs

T1 and C1

$\frac{\forall \varphi. \forall \psi. [(P(\varphi) \wedge \square \forall x. [\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]}{\forall \psi. [(P(\rho) \wedge \square \forall x. [\rho(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]} \forall_E$	<b>A2</b>
$\frac{(P(\rho) \wedge \square \forall x. [\rho(x) \rightarrow \neg \rho(x)]) \rightarrow P(\neg \rho)}{(P(\rho) \wedge \square \forall x. [\neg \rho(x)]) \rightarrow P(\neg \rho)}$	<b>A1a</b>
$\frac{(P(\rho) \wedge \square \forall x. [\neg \rho(x)]) \rightarrow P(\neg \rho)}{(P(\rho) \wedge \square \forall x. [\neg \rho(x)]) \rightarrow \neg P(\rho)}$	$\frac{\neg \forall \varphi. [P(\neg \varphi) \rightarrow \neg P(\varphi)]}{P(\neg \rho) \rightarrow \neg P(\rho)} \forall_E$
$\frac{\frac{\frac{(P(\rho) \wedge \square \forall x. [\neg \rho(x)]) \rightarrow \neg P(\rho)}{P(\rho) \rightarrow \diamond \exists x. \rho(x)}}{\mathbf{T1}: \forall \varphi. [P(\varphi) \rightarrow \diamond \exists x. \varphi(x)]} \forall_I}{\mathbf{T1}}$	

# Natural Deduction Proofs

T2 (Partial)

$$\frac{\psi(x)^6 \quad \frac{\overline{\psi(x)} \rightarrow \overline{\square P(\psi)}}{\square P(\psi)}^{\Pi_2}}{\psi(x) \rightarrow \square \forall x.(G(x) \rightarrow \psi(x))}^{\rightarrow_E}$$

$$\frac{\frac{\square P(\psi)^7 \quad \frac{\overline{P(\psi)} \rightarrow \overline{\forall x.(G(x) \rightarrow \psi(x))}}{\forall x.(G(x) \rightarrow \psi(x))}^{\Pi_3}}{\square \forall x.(G(x) \rightarrow \psi(x))}^{\rightarrow_I}}{\square P(\psi) \rightarrow \square \forall x.(G(x) \rightarrow \psi(x))}^{\rightarrow_I}$$

$$\frac{\square \forall x.(G(x) \rightarrow \psi(x))}{\psi(x) \rightarrow \square \forall x.(G(x) \rightarrow \psi(x))}^{\rightarrow_E}$$



## Embedding Modal Logic into Higher-Order Logic

Challenge: No provers for *Higher-order Quantified Modal Logic* (**QML**)

Our solution: Embedding in *Higher-order Classical Logic* (**HOL**)

Then use existing **HOL** theorem provers for reasoning in **QML**

[BenzmüllerPaulson, Logica Universalis, 2013]

Previous empirical findings:

Embedding of *First-order Modal Logic* in HOL works well

[BenzmüllerOttenRaths, ECAI, 2012]

[Benzmüller, LPAR, 2013]

**QML**     $\varphi, \psi ::= \dots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x \varphi \mid \exists x \varphi \mid \forall P \varphi$

- Kripke style semantics (possible world semantics)

**HOL**                           $s, t ::= C \mid x \mid \lambda x s \mid s t \mid \neg s \mid s \vee t \mid \forall x t$

- various theorem provers exist

interactive: Isabelle/HOL, HOL4, Hol Light, Coq/HOL, PVS, ...

automated: TPS, LEO-II, Satallax, Nitpick, Isabelle/HOL, ...

**QML**     $\varphi, \psi ::= \dots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x \varphi \mid \exists x \varphi \mid \forall P \varphi$

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Ax

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$$\text{HOL} \qquad \qquad s, t ::= C \mid x \mid \lambda x s \mid s t \mid \neg s \mid s \vee t \mid \forall x t$$

**QML in HOL:** QML formulas  $\varphi$  are mapped to HOL predicates  $\varphi_{\mapsto o}$

$\neg$	$= \lambda\varphi_{t \rightarrow o} \lambda w_t \neg \varphi w$
$\wedge$	$= \lambda\varphi_{t \rightarrow o} \lambda\psi_{t \rightarrow o} \lambda w_t (\varphi w \wedge \psi w)$
$\rightarrow$	$= \lambda\varphi_{t \rightarrow o} \lambda\psi_{t \rightarrow o} \lambda w_t (\neg \varphi w \vee \psi w)$
$\forall$	$= \lambda h_{\gamma \rightarrow (t \rightarrow o)} \lambda w_t \forall d_\gamma hdw$
$\exists$	$= \lambda h_{\gamma \rightarrow (t \rightarrow o)} \lambda w_t \exists d_\gamma hdw$
$\Box$	$= \lambda\varphi_{t \rightarrow o} \lambda w_t \forall u_t (\neg rwu \vee \varphi u)$
$\Diamond$	$= \lambda\varphi_{t \rightarrow o} \lambda w_t \exists u_t (rwu \wedge \varphi u)$
valid	$= \lambda\varphi_{t \rightarrow o} \forall w_t \varphi w$

Ax

**QML**     $\varphi, \psi$  ::= ... |  $\neg\varphi$  |  $\varphi \wedge \psi$  |  $\varphi \rightarrow \psi$  |  $\Box\varphi$  |  $\Diamond\varphi$  |  $\forall x \varphi$  |  $\exists x \varphi$  |  $\forall P \varphi$

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$\Diamond$	$=$	$\lambda\varphi_{t \rightarrow o}\lambda w_t\exists u_t(rwu \wedge \varphi u)$
valid	$=$	$\lambda\varphi_{t \rightarrow o}\forall w_t\varphi w$

Ax

The equations in `Ax` are given as axioms to the `HOL` provers!

## Example

QML formula

QML formula in HOL

expansion,  $\beta\eta$ -conversion

expansion,  $\beta\eta$ -conversion

expansion,  $\beta\eta$ -conversion

$\diamond \exists x G(x)$

valid  $(\diamond \exists x G(x))_{t \rightarrow o}$

$\forall w_t (\diamond \exists x G(x))_{t \rightarrow o} w$

$\forall w_t \exists u_t (rwu \wedge (\exists x G(x))_{t \rightarrow o})$

$\forall w_t \exists u_t (rwu \wedge \exists x Gxu)$

### What are we doing?

In order to prove that  $\varphi$  is valid in QML,

→ we instead prove that valid  $\varphi_{t \rightarrow o}$  can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Expansion:

user or prover may flexibly choose expansion depth

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## Example

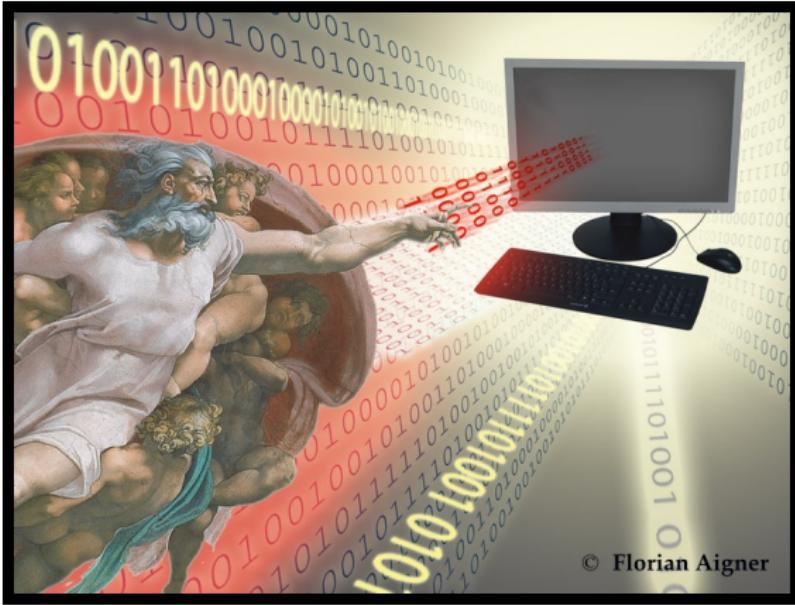
**QML formula****QML formula in HOL**expansion,  $\beta\eta$ -conversionexpansion,  $\beta\eta$ -conversionexpansion,  $\beta\eta$ -conversion $\Diamond \exists x G(x)$ valid  $(\Diamond \exists x G(x))_{\iota \rightarrow o}$  $\forall w_t (\Diamond \exists x G(x))_{\iota \rightarrow o} w$  $\forall w_t \exists u_t (rwu \wedge (\exists x G(x))_{\iota \rightarrow o} u)$  $\forall w_t \exists u_t (rwu \wedge \exists x Gxu)$ 

## What are we doing?

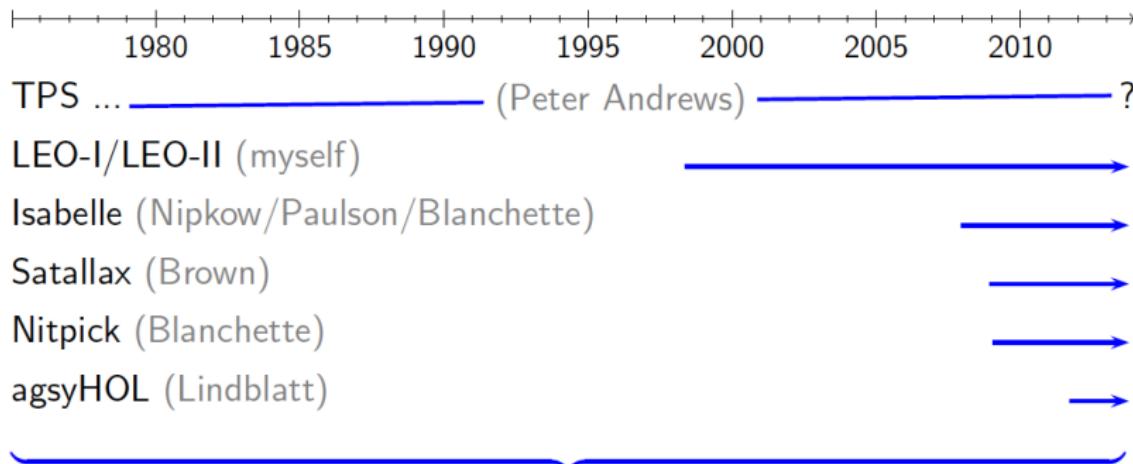
In order to prove that  $\varphi$  is valid in **QML**,→ we instead prove that **valid**  $\varphi_{\iota \rightarrow o}$  can be derived from **Ax** in **HOL**.This can be done with interactive or automated **HOL** theorem provers.

Expansion:

user or prover may flexibly choose expansion depth



## Automated Proof Search and Consistency Check



- all accept TPTP THF Syntax [SutcliffeBenzmüller, J.Form.Reas, 2009]
  - can be called remotely via SystemOnTPTP at Miami
  - they significantly gained in strength over the last years
  - they can be bundled into a combined prover **HOL-P**

Exploit HOL with Henkin semantics as metalogic  
Automate other logics (& combinations) via semantic embeddings  
— **HOL-P** becomes a **Universal Reasoner** —

# Proof Automation and Consistency Checking

```
MacBook-Chris % Terminal — bash — 125x32
MacBook-Chris %
MacBook-Chris %
MacBook-Chris % ./call_tptp.sh T3.p

Asking various HOL-ATPs in Miami remotely (thanks to Geoff Sutcliffe)

MacBook-Chris % agsyHOL---1.0 : T3.p ++++++ RESULT: SOT_7L4x.Y - agsyHOL---1.0 says Unknown - CPU = 0.00 WC = 0.02
LEO-II---1.6.0 : T3.p ++++++ RESULT: SOT_E4Scha - LEO-II---1.6.0 says Theorem - CPU = 0.03 WC = 0.09
Satallax---2.7 : T3.p ++++++ RESULT: SOT_kVZ1cB - Satallax---2.7 says Theorem - CPU = 0.00 WC = 0.14
Isabelle---2013 : T3.p ++++++ RESULT: SOT_xa0gEp - Isabelle---2013 says Theorem - CPU = 14.06 WC = 17.73 SolvedBy = auto
TPS---3.120601S1b : T3.p ++++++ RESULT: SOT_ROEsgs - TPS---3.120601S1b says Unknown - CPU = 33.56 WC = 41.57
Nitrox---2013 : T3.p ++++++ RESULT: SOT_WGY1Tx - Nitrox---2013 says Unknown - CPU = 75.55 WC = 49.24

MacBook-Chris %
MacBook-Chris % ./call_tptp.sh Consistency.p

Asking various HOL-ATPs in Miami remotely (thanks to Geoff Sutcliffe)

MacBook-Chris % agsyHOL---1.0 : Consistency.p ++++++ RESULT: SOT_ZtY_7o - agsyHOL---1.0 says Unknown - CPU = 0.00 WC = 0.00
Nitrox---2013 : Consistency.p ++++++ RESULT: SOT_HUz1OC - Nitrox---2013 says Satisfiable - CPU = 6.56 WC = 8.50
TPS---3.120601S1b : Consistency.p ++++++ RESULT: SOT_fpjxTM - TPS---3.120601S1b says Unknown - CPU = 43.00 WC = 49.42
Isabelle---2013 : Consistency.p ++++++ RESULT: SOT_6Tpp9i - Isabelle---2013 says Unknown - CPU = 69.96 WC = 72.62
LEO-II---1.6.0 : Consistency.p ++++++ RESULT: SOT_dy10sj - LEO-II---1.6.0 says Timeout - CPU = 90 WC = 89.86
Satallax---2.7 : Consistency.p ++++++ RESULT: SOT_Q9WSLF - Satallax---2.7 says Timeout - CPU = 90 WC = 90.50

MacBook-Chris %
```

Provers are called remotely in Miami — no local installation needed!

# Automation & Verification in Proof Assistant IsABELLE/HOL

The screenshot shows the Isabelle/HOL proof assistant interface. The main window displays the theory file `GoedelGod.thy` with the following content:

```
corollary C: "[o (E G)]"
sledgehammer [provers = remote_leo2] by (metis A3 T1)

text {* Axiom @{text "A4"} is added: $\forall \phi [P(\phi) \rightarrow \Box \; ; \; P(\phi)]$ 
(Positive properties are necessarily positive). *}

axiomatization where A4: "[!! (\lambda\Phi. P \Phi m⇒ □ (P Φ))]" 

text {* Symbol @{text "ess"} for 'Essence' is introduced and defined as 
$ess{\phi}{x} \; \text{biimp} \; \phi(x) \; \text{wedge} \; \forall \psi \; (\psi(x) \; \text{imp} \; \nec \; \forall y \; (\phi(y) \; 
\text{imp} \; \psi(y)))$ 
(An essence of an individual is a property possessed by it 
and necessarily implying any of its properties). *}

definition ess :: "(μ ⇒ σ) ⇒ μ ⇒ σ" (infixr "ess" 85) where
  "Φ ess x = Φ x m⇒ II (λψ. ψ x m⇒ □ (λy. Φ y m⇒ ψ y))"

text {* Next, Sledgehammer and Metis prove theorem @{text "T2"}: $\forall x \; [G(x) \; \text{imp} \; ess(G){x}]$ 
(Being God-like is an essence of any God-like being). *}

theorem T2: "[! (λx. G x m⇒ G ess x)]"
sledgehammer [provers = remote_leo2] by (metis A1b A4 G_def ess_def)

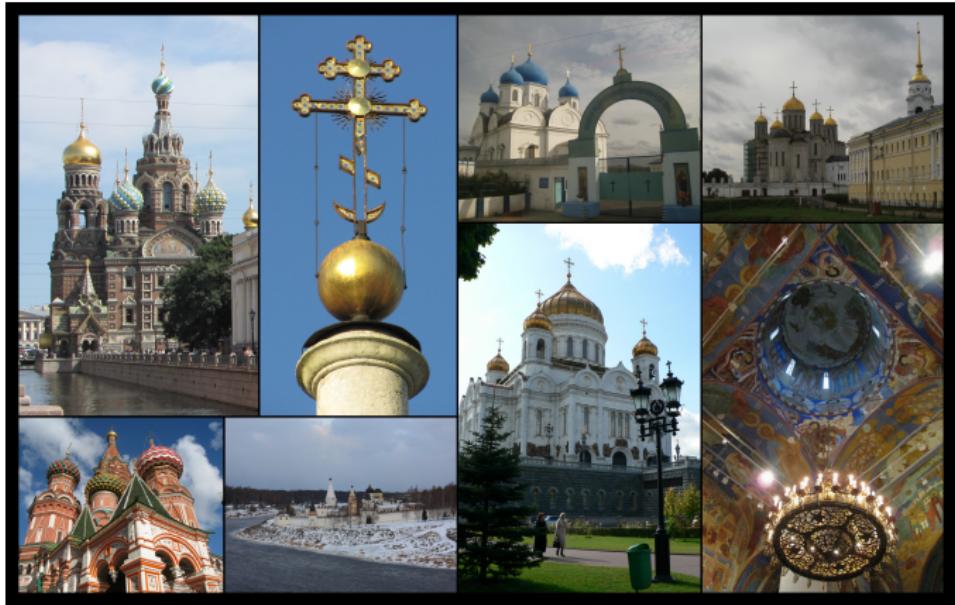
text {* Symbol @{text "NE"}, for 'Necessary Existence', is introduced and 
defined as $NE(x) \; \text{biimp} \; \forall \phi \; (\text{ess}{\phi}{x}) \; \text{imp} \; \nec \; \exists y \; \phi(y)$ 
(Necessary 
existence of an individual is the necessary exemplification of all its essences). *}

definition NE :: "μ ⇒ σ" where "NE = (λx. II (λΦ. Φ ess x m⇒ □ (E Φ)))"
```

The status bar at the bottom left shows "Sledgehammering...". The bottom right corner includes a zoom control (100%), an "Auto update" checkbox, and "Update" and "Detach" buttons.

# Results

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \dot{\forall} X_\mu. (\phi X \dot{\wedge} \psi X)) \dot{\wedge} p\psi$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Diamond} \dot{\exists} X_\mu. \phi X$	A1( $\supset$ ), A2 A1, A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \phi X$		K	THM	0.1/0.1	0.0/5.2	—/—
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$\dot{\Diamond} \dot{\exists} X_\mu. g_{\mu \rightarrow \sigma} X$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} p\phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu. \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu. (\phi Y \dot{\wedge} \psi Y))$						
T2	$\dot{\forall} X_\mu. g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)$	A1, D1, A4, D2 A1, A2, D1, A3, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess } \phi X \dot{\wedge} \dot{\Box} \dot{\exists} Y_\mu. \phi Y)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$\dot{\Box} \dot{\exists} X_\mu. g_{\mu \rightarrow \sigma} X$	D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5 D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5	K K KB KB	CSA CSA THM THM	—/— —/— 0.0/0.1 —/—	—/— —/— 0.1/5.3 —/—	3.8/6.2 8.2/7.5 —/— —/—
MC	$[s_\sigma \dot{\wedge} \dot{\Box} s_\sigma]$	D2, T2, T3 A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	17.9/—	3.3/3.2	—/—
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu. (g_{\mu \rightarrow \sigma} X \dot{\wedge} (\dot{\neg}(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \dot{\wedge} \dot{\neg}(\phi X)))$	A1, D1 A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/— 16.5/—	—/— 0.0/0.0	—/— —/—
MT	$\dot{\forall} X_\mu. \dot{\forall} Y_\mu. (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} Y \dot{\wedge} X \dot{\equiv} Y))$	D1, FG A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	12.8/15.1 —/—	0.0/5.4 0.0/3.3	—/— —/—
CO	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu. \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu. (\phi Y \dot{\wedge} \psi Y))$						
CO'	∅ (no goal, check for consistency)	A1( $\supset$ ), A2, D2', D3, A5 A1, A2, D1, A3, A4, D2', D3, A5	KB KB	UNS UNS	7.5/7.8 —/—	—/— —/—	—/— —/—



## Formalization and Verification in Coq

- Goal: verification of the natural deduction proof
  - Step-by-step formalization
  - Almost no automation (intentionally!)
- Interesting facts:
  - Embedding is transparent to the user
  - Embedding gives labeled calculus for free

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The screenshot shows the CoqIDE interface with the following details:

- File Menu:** File, Edit, Navigation, Try Tactics, Templates, Queries, Display, Compile, Windows, Help.
- Toolbar:** Standard icons for file operations like Open, Save, Print, and Undo/Redo.
- Tab Bar:** scratch, Modal.v, ModalClassical.v, GoedelGod-Scott.v
- Left Panel (Script Editor):**

```
(* Constant predicate that distinguishes positive properties *)
Parameter Positive: (u -> o) -> o.

(* Axiom A1: either a property or its negation is positive, but not both *)
Axiom axiomia : V (mforall p, (Positive (fun x: u => m-(p x))) m-> (m- (Positive p))).
Axiom axiomib : V (mforall p, (m- (Positive p)) m-> (Positive (fun x: u => m- (p x)))).

(* Axiom A2: a property necessarily implied by a positive property is positive *)
Axiom axiom2: V (mforall p, mforall q, Positive p m/\ (box (mforall x, (p x) m-> (q x)) ).

(* Theorem T1: positive properties are possibly exemplified *)
Theorem theorem1: V (mforall p, (Positive p) m-> dia (mexists x, p x) ).
```

**Proof.**

```
intro.
intro p.
intro H1.
intro H2.
proof_by_contradiction H2.
apply not_dia_box_not_in H2.
assert (H3: (box (mforall x, m- (p x))) w). (* Lemma from Scott's notes *)
  box_intro w1 R1.
  intro x.
  assert (H4: ((m- (mexists x : u, p x)) w1)).
    box_elim H2 w1 R1 G2.
    exact G2.

  clear H2 R1 H1 w.
  intro H5.
  apply H4.
  exists x.
  exact H5.

assert (H6: ((box (mforall x, (p x) m-> m- (x m= x))) w)). (* Lemma from Scott's notes *)
  box_intro w1 R1.
  intro x.
  intro H7.
  intro H8.
  box_elim H3 w1 R1 G3.
  annTu ct with / \_ := _;
```
- Right Panel (Proof View):**
  - 2 subgoals
  - w : i
  - p : u -> o
  - H1 : Positive p w
  - H2 : box (m- (mexists x : u, p x)) w (1/2)
  - box (mforall x : u, m- p x) w (2/2)
  - False

**“God is dead.”**

- Nietzsche, 1883

**“Nietzsche is dead.”**

- God, 1900

## Criticisms

$$\forall P. [\diamond \Box P \rightarrow \Box P]$$

If something is possibly necessary, then it is necessary.

$$\forall P. [P \rightarrow \Box P]$$

Everything that is the case is so necessarily.

Follows from T2, T3 and D2.

Many proposed solutions: Anderson, Fitting, Hájek, ...

$$\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$$

Either a property is positive or its negation is (but never both)

Are the following properties positive or negative?

$$\lambda x.G(x) \quad \lambda x.E(x) \quad \lambda x.x = x \quad \lambda x.\top$$

$$\lambda x.blue(x) \quad \lambda x.red(x) \quad \lambda x.human(x)$$

Possible Solution:

“...positive in the moral aesthetic sense (independently of the accidental structure of the world). Only then the ax. true. ....”

- Gödel, 1970



## Conclusions and Future Work

- Contributions:
  - Powerful infrastructure for reasoning with QML
  - A (new?) natural deduction calculus for higher-order (intuitionistic) modal logic
  - Non-trivial new benchmark problems for HOL provers
  - Verification of existing results about Gödel's proof (e.g. Modal Collapse)
  - New results about Gödel's proof (e.g. Inconsistency)
- Major step towards Computer-assisted Theoretical Philosophy
  - see also Ed Zalta's *Computational Metaphysics* project at Stanford University and John Rushby's formalization of Anselm's proof using PVS

- Embedding of other non-classical logics
  - Quantified Conditional Logics,
  - Modal Logics based on Neighbourhood Semantics, ...
  - Other Suggestions?
  - Human Resources?
- New non-classical calculi via HOL embedding:  
(when does this approach work?)
  - ① Embed your favorite logic  $L$  into HOL
  - ② Play with Coq and invent new tactics to encapsulate the embedding and make it transparent to the user
  - ③ Interpret the tactics as new rules for a new natural deduction calculus  $C$
  - ④ Prove soundness and completeness indirectly:

$$\models_L s_o \quad \text{iff} \quad \models \text{valid}(\llbracket s_o \rrbracket)$$

$$\vdash \text{valid}(\llbracket s_o \rrbracket) \quad \text{iff} \quad \vdash_C s_o$$

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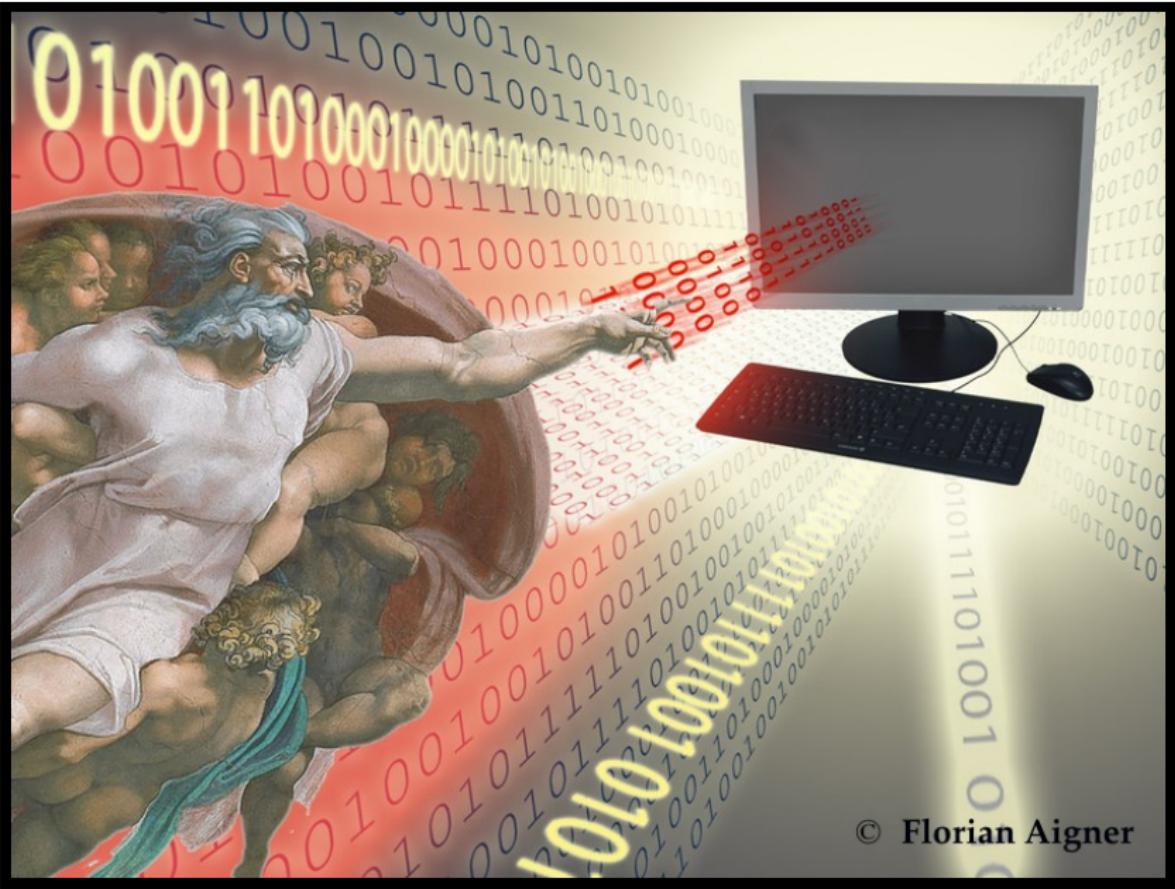
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