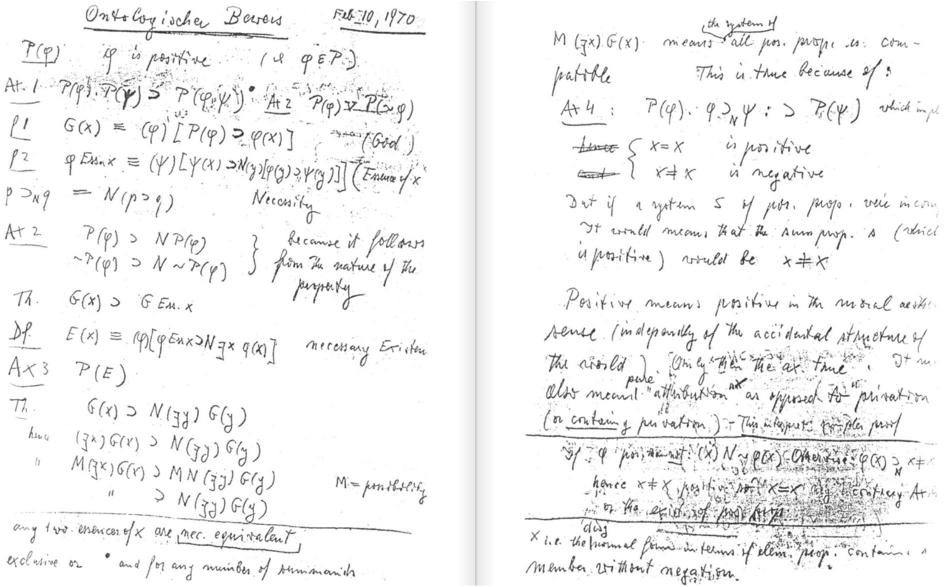
How to “lazily” prove modal logic theorems using higher-order theorem provers

(and “prove” God’s existence with the push of a button)

Christoph Benzmüller and **Bruno Woltzenlogel Paleo**

20th of July 2014



Ontologischer Beweis   Feb 10, 1970

$P(\varphi)$ φ is positive   (or $\varphi \in P$)

At. 1 $P(\varphi).P(\psi) \supset P(\varphi.\psi)$ At 2 $P(\varphi) \vee P(\sim\varphi)$

P1 $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (God)

P2 $\varphi \text{ Ess. } x \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Essence of x)

$p \supset_N q = N(p \supset q)$ Necessity

At 2 $P(\varphi) \supset NP(\varphi)$
$\sim P(\varphi) \supset N\sim P(\varphi)$ } because it follows from the nature of the property

Th. $G(x) \supset G \text{ Ess. } x$

Df. $E(x) \equiv (\varphi)[\varphi \text{ Ess } x \supset N \exists x\, \varphi(x)]$ necessary Existence

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists y)\, G(y)$

hence $(\exists x)\, G(x) \supset N(\exists y)\, G(y)$

" $M(\exists x)\, G(x) \supset MN(\exists y)\, G(y)$

" $\supset N(\exists y)\, G(y)$ M = possibility

any two essences of x are nec. equivalent

exclusive or * and for any number of summands

$M(\exists x)\, G(x)$ means the system of all pos. prop. is compatible. This is true because of:

At 4 : $P(\varphi).\ \varphi \supset_N \psi :\supset P(\psi)$ which impl

~~true~~ { $x = x$ is positive
~~and~~ { $x \neq x$ is negative

But if a system S of pos. prop. were incon it would mean that the sum prop. s (which is positive) would be $x \neq x$

Positive means positive in the moral aesth. sense (independently of the accidental structure of the world). Only then the at. true. It m also meant pure "attribution"* as opposed to privation (or containing privation). This interpret. simpler proof

If φ pos. not: $(x)\, N\sim\varphi(x)$. Otherwise: $\varphi(x) \supset_N x \neq x$ hence $x \neq x$ positive not $x = x$ contrary At. … or the exist. of pos. prop.

* i.e. the normal form in terms of elem. prop. contains a member without negation.

... Anselm v. C. Gaunilo ... Th. Aquinas ...... Descartes Spinoza Leibniz ... Hume Kant ... Hegel ... Frege ... Hartshorne Malcolm Lewis Plantinga Gödel ...

Anselm's notion of God:

> *"God is that, than which nothing greater can be conceived."*

Gödel's notion of God:

> *"A God-like being possesses all 'positive' properties."*

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The Ontological Proof Today

Proof Overview

T3: $\Box\exists x.G(x)$

Proof Overview

$$\frac{\textbf{C1: } \Diamond \exists z. G(z)}{\textbf{T3: } \Box \exists x. G(x)}$$

Proof Overview

$$\frac{\textbf{C1: } \Diamond \exists z. G(z) \qquad \textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\textbf{T3: } \Box \exists x. G(x)}$$

Proof Overview

$$\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)$$

$$\frac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

$$\dfrac{\textbf{S5} \atop \overline{\forall}\xi.[\overline{\Diamond}\overline{\Box}\xi \mathbin{\overline{\rightarrow}} \overline{\Box}\xi]}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\dfrac{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x) \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\dfrac{\dfrac{\textbf{L1: } \exists z.G(z) \to \Box\exists x.G(x)}{\Diamond\exists z.G(z) \to \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall \xi.[\Diamond\Box\xi \to \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1:}\ G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$$

$$\frac{\dfrac{\textbf{L1:}\ \exists z.G(z) \rightarrow \Box\exists x.G(x)}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall \xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

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D1: $G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$

D3*: $NE(x) \equiv \Box \exists y.G(y)$

$$\dfrac{\dfrac{\dfrac{P(NE)}{\textbf{L1: } \exists z.G(z) \rightarrow \Box \exists x.G(x)}}{\Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall \xi.[\Diamond \Box \xi \rightarrow \Box \xi]}}{\textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond \exists z.G(z) \qquad \textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\textbf{T3: } \Box \exists x.G(x)}$$

D1: $G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$

D3*: $NE(x) \equiv \Box\exists y.G(y)$ (cheating!)

$$\dfrac{\dfrac{\dfrac{P(NE)}{\textbf{L1: } \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1: } G(x) \equiv \forall\varphi.[P(\varphi) \to \varphi(x)]$$

$$\textbf{D3*: } NE(x) \equiv \Box\exists y.G(y) \qquad\qquad \textbf{D3: } NE(x) \equiv \forall\varphi.[\varphi\ ess\ x \to \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\dfrac{\textbf{T2: } \forall y.[G(y) \to G\ ess\ y] \qquad\qquad P(NE)}{\textbf{L1: } \exists z.G(z) \to \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \to \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \to \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}$$

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D1: $G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$

D3*: $NE(x) \equiv \Box \exists y.G(y)$

D3: $NE(x) \equiv \forall \varphi.[\varphi \; ess \; x \rightarrow \Box \exists y.\varphi(y)]$

$$\dfrac{\dfrac{\dfrac{\textbf{T2: } \forall y.[G(y) \rightarrow G \; ess \; y] \qquad \dfrac{\textbf{A5}}{P(NE)}}{\textbf{L1: } \exists z.G(z) \rightarrow \Box \exists x.G(x)}}{\Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall \xi.[\Diamond \Box \xi \rightarrow \Box \xi]}}{\textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}$$

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$$\textbf{D1:}\ G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$$

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$$\dfrac{\dfrac{\dfrac{\textbf{T2:}\ \forall y.[G(y) \rightarrow G\ ess\ y] \qquad \dfrac{\textbf{A5}}{P(NE)}}{\textbf{L1:}\ \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

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Proof Overview

$$\textbf{D1: } G(x) \equiv \forall\varphi.[P(\varphi) \to \varphi(x)]$$

$$\textbf{D2: } \varphi\ ess\ x \equiv \varphi(x) \land \forall\psi.(\psi(x) \to \Box\forall x.(\varphi(x) \to \psi(x)))$$

$$\textbf{D3*: } NE(x) \equiv \Box\exists y.G(y) \qquad \textbf{D3: } NE(x) \equiv \forall\varphi.[\varphi\ ess\ x \to \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\textbf{A1b}}{\forall\varphi.[\neg P(\varphi) \to P(\neg\varphi)]} \qquad \overset{\textbf{A4}}{\forall\varphi.[P(\varphi) \to \Box P(\varphi)]}}{\textbf{T2: } \forall y.[G(y) \to G\ ess\ y]} \qquad \overset{\textbf{A5}}{P(NE)}}{\textbf{L1: } \exists z.G(z) \to \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \to \Diamond\Box\exists x.G(x)} \qquad \overset{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \to \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}$$

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$$\textbf{D1:}\ G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$$

$$\textbf{D2:}\ \varphi\ ess\ x \equiv \varphi(x) \wedge \forall \psi.(\psi(x) \rightarrow \Box \forall x.(\varphi(x) \rightarrow \psi(x)))$$

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$$\textbf{C1:}\ \Diamond \exists z.G(z)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\textbf{A1b}}{\forall \varphi.[\neg P(\varphi) \rightarrow P(\neg \varphi)]} \qquad \overset{\textbf{A4}}{\forall \varphi.[P(\varphi) \rightarrow \Box P(\varphi)]}}{\textbf{T2:}\ \forall y.[G(y) \rightarrow G\ ess\ y]} \qquad \overset{\textbf{A5}}{P(NE)}}{\textbf{L1:}\ \exists z.G(z) \rightarrow \Box \exists x.G(x)}}{\Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x)} \qquad \overset{\textbf{S5}}{\forall \xi.[\Diamond \Box \xi \rightarrow \Box \xi]}}{\textbf{L2:}\ \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}}{}$$

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$$\frac{P(G)}{\textbf{C1: } \Diamond\exists z.G(z)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\textbf{A1b}}{\forall\varphi.[\neg P(\varphi) \to P(\neg\varphi)]} \quad \overset{\textbf{A4}}{\forall\varphi.[P(\varphi) \to \Box P(\varphi)]}}{\textbf{T2: } \forall y.[G(y) \to G\ ess\ y]} \quad \overset{\textbf{A5}}{P(NE)}}{\textbf{L1: } \exists z.G(z) \to \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \to \Diamond\Box\exists x.G(x)} \quad \overset{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \to \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}$$

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$$\dfrac{\dfrac{\textbf{A3}}{P(G)}}{\textbf{C1:}\ \Diamond\exists z.G(z)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{A1b}}{\forall\varphi.[\neg P(\varphi) \rightarrow P(\neg\varphi)]} \quad \dfrac{\textbf{A4}}{\forall\varphi.[P(\varphi) \rightarrow \Box P(\varphi)]}}{\textbf{T2:}\ \forall y.[G(y) \rightarrow G\ ess\ y]} \quad \dfrac{\textbf{A5}}{P(NE)}}{\textbf{L1:}\ \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \quad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

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$$\textbf{D1:}\ G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$$

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$$\dfrac{\dfrac{\textbf{A3}}{P(G)} \qquad \textbf{T1:}\ \forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]}{\textbf{C1:}\ \Diamond\exists z.G(z)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{A1b}}{\forall\varphi.[\neg P(\varphi) \rightarrow P(\neg\varphi)]} \qquad \dfrac{\textbf{A4}}{\forall\varphi.[P(\varphi) \rightarrow \Box P(\varphi)]}}{\textbf{T2:}\ \forall y.[G(y) \rightarrow G\ ess\ y]} \qquad \dfrac{\textbf{A5}}{P(NE)}}{\textbf{L1:}\ \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

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$$\dfrac{\dfrac{\textbf{A3}}{P(G)} \qquad \dfrac{\dfrac{\textbf{A2}}{\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]} \qquad \dfrac{\textbf{A1a}}{\forall\varphi.[P(\neg\varphi) \rightarrow \neg P(\varphi)]}}{\textbf{T1:}\ \forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]}}{\textbf{C1:}\ \Diamond\exists z.G(z)}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{A1b}}{\forall\varphi.[\neg P(\varphi) \rightarrow P(\neg\varphi)]} \qquad \dfrac{\textbf{A4}}{\forall\varphi.[P(\varphi) \rightarrow \Box\, P(\varphi)]}}{\textbf{T2: } \forall y.[G(y) \rightarrow G\ ess\ y]} \qquad \dfrac{\textbf{A5}}{P(NE)}}{\textbf{L1: } \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

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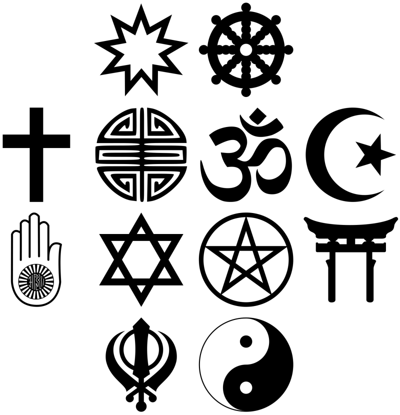

Modal Natural Deduction Proof

Natural Deduction Calculus

$$\frac{A \vee B \quad \begin{array}{c}\overline{A}\\ \vdots \\ C\end{array} \quad \begin{array}{c}\overline{B}\\ \vdots \\ C\end{array}}{C}\ \vee_E \qquad \frac{A \quad B}{A \wedge B}\ \wedge_I \qquad \frac{\begin{array}{c}\overline{A}^{\,n}\\ \vdots \\ B\end{array}}{A \rightarrow B}\ \rightarrow_I^n$$

$$\frac{A}{A \vee B}\ \vee_{I_1} \qquad \frac{A \wedge B}{A}\ \wedge_{E_1} \qquad \frac{B}{A \rightarrow B}\ \rightarrow_I$$

$$\frac{B}{A \vee B}\ \vee_{I_2} \qquad \frac{A \wedge B}{B}\ \wedge_{E_2} \qquad \frac{A \quad A \rightarrow B}{B}\ \rightarrow_E$$

$$\frac{A[\alpha]}{\forall x.A[x]}\ \forall_I \qquad \frac{\forall x.A[x]}{A[t]}\ \forall_E \qquad \frac{A[t]}{\exists x.A[x]}\ \exists_I \qquad \frac{\exists x.A[x]}{A[\beta]}\ \exists_E$$

$$\neg A \equiv A \rightarrow \bot \qquad \frac{\neg\neg A}{A}\ \neg\neg_E$$

$$\dfrac{\alpha : \boxed{\begin{array}{c}\vdots\\ A\end{array}}}{\Box A}\;\Box_I \qquad\qquad \dfrac{\Box A}{t : \boxed{\begin{array}{c} A\\ \vdots\end{array}}}\;\Box_E$$

$$\Diamond A \equiv \neg\Box\neg A$$

$$\dfrac{t : \boxed{\begin{array}{c}\vdots\\ A\end{array}}}{\Diamond A}\;\Diamond_I \qquad\qquad \dfrac{\Diamond A}{\beta : \boxed{\begin{array}{c} A\\ \vdots\end{array}}}\;\Diamond_E$$

$$\frac{\alpha : \boxed{\begin{array}{c}\vdots\\ A\end{array}}}{\Box A}\ \Box_I \qquad\qquad \frac{\Box A}{t : \boxed{\begin{array}{c}A\\ \vdots\end{array}}}\ \Box_E$$

$$\Diamond A \equiv \neg\Box\neg A$$

$$\frac{t : \boxed{\begin{array}{c}\vdots\\ A\end{array}}}{\Diamond A}\ \Diamond_I \qquad\qquad \frac{\Diamond A}{\beta : \boxed{\begin{array}{c}A\\ \vdots\end{array}}}\ \Diamond_E$$

Natural Deduction Calculus
Rules for Modalities

$$\dfrac{\alpha : \boxed{\begin{array}{c}\vdots\\ A\end{array}}}{\Box A}\ \Box_I \qquad\qquad \dfrac{\Box A}{t : \boxed{\begin{array}{c}A\\ \vdots\end{array}}}\ \Box_E$$

$$\Diamond A \equiv \neg\Box\neg A$$

$$\dfrac{t : \boxed{\begin{array}{c}\vdots\\ A\end{array}}}{\Diamond A}\ \Diamond_I \qquad\qquad \dfrac{\Diamond A}{\beta : \boxed{\begin{array}{c}A\\ \vdots\end{array}}}\ \Diamond_E$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathbf{A2}}{\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]}}{\forall\psi.[(P(\rho) \wedge \Box\forall x.[\rho(x) \to \psi(x)]) \to P(\psi)]}\,\forall_E}{(P(\rho) \wedge \Box\forall x.[\rho(x) \to \neg\rho(x)]) \to P(\neg\rho)}\,\forall_E}{(P(\rho) \wedge \Box\forall x.[\neg\rho(x)]) \to P(\neg\rho)} \qquad \dfrac{\dfrac{\mathbf{A1a}}{\forall\varphi.[P(\neg\varphi) \to \neg P(\varphi)]}}{P(\neg\rho) \to \neg P(\rho)}\,\forall_E}{(P(\rho) \wedge \Box\forall x.[\neg\rho(x)]) \to \neg P(\rho)}}{\dfrac{P(\rho) \to \Diamond\exists x.\rho(x)}{\textbf{T1: } \forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]}\,\forall_I}
$$

$$
\dfrac{\dfrac{\mathbf{A3}}{P(G)} \qquad \dfrac{\dfrac{\mathbf{T1}}{\forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]}}{P(G) \to \Diamond\exists x.G(x)}\,\forall_E}{\Diamond\exists x.G(x)}\,\to_E
$$

$$\dfrac{\dfrac{\dfrac{\psi(x)^6 \qquad \dfrac{\Pi_2}{\psi(x) \to \Box P(\psi)}}{\Box P(\psi)}\to_E \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Box P(\psi)^7}{P(\psi)}\Box_E \qquad \dfrac{\Pi_3}{P(\psi) \to \forall x.(G(x) \to \psi(x))}}{\forall x.(G(x) \to \psi(x))}\to_E}{\Box \forall x.(G(x) \to \psi(x))}\Box_I}{\Box P(\psi) \to \Box \forall x.(G(x) \to \psi(x))}\to_I^7}{\Box \forall x.(G(x) \to \psi(x))}\to_E}{\psi(x) \to \Box \forall x.(G(x) \to \psi(x))}\to_I^6$$

Embedding Modal Logic into Higher-Order Logic

Challenge: No provers for *Higher-order Quantified Modal Logic* (QML)

Our solution: Embedding in *Higher-order Classical Logic* (HOL)
Then use existing HOL theorem provers for reasoning in QML

[BenzmüllerPaulson, Logica Universalis, 2013]

Previous empirical findings:

Embedding of *First-order Modal Logic* in HOL works well

[BenzmüllerOttenRaths, ECAI, 2012]

[Benzmüller, LPAR, 2013]

QML $\quad \varphi, \psi \quad ::= \quad \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x\, \varphi \mid \exists x\, \varphi \mid \forall P\, \varphi$

- Kripke style semantics (possible world semantics)

HOL $\quad s, t \quad ::= \quad C \mid x \mid \lambda x s \mid s\, t \mid \neg s \mid s \vee t \mid \forall x\, t$

- various theorem provers exist

  interactive: Isabelle/HOL, HOL4, Hol Light, Coq/HOL, PVS, ...

  automated: TPS, LEO-II, Satallax, Nitpick, Isabelle/HOL, ...

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QML in HOL: QML formulas φ are mapped to HOL predicates $\varphi_{\iota\rightarrow o}$

$$
\begin{array}{lcl}
\neg & = & \lambda\varphi_{\iota\rightarrow o}\lambda w_\iota \neg\varphi w \\
\wedge & = & \lambda\varphi_{\iota\rightarrow o}\lambda\psi_{\iota\rightarrow o}\lambda w_\iota(\varphi w \wedge \psi w) \\
\rightarrow & = & \lambda\varphi_{\iota\rightarrow o}\lambda\psi_{\iota\rightarrow o}\lambda w_\iota(\neg\varphi w \vee \psi w) \\
\forall & = & \lambda h_{\gamma\rightarrow(\iota\rightarrow o)}\lambda w_\iota \forall d_\gamma\, h d w \\
\exists & = & \lambda h_{\gamma\rightarrow(\iota\rightarrow o)}\lambda w_\iota \exists d_\gamma\, h d w \\
 & & \\
\Box & = & \lambda\varphi_{\iota\rightarrow o}\lambda w_\iota \forall u_\iota\, (\neg r w u \vee \varphi u) \\
\Diamond & = & \lambda\varphi_{\iota\rightarrow o}\lambda w_\iota \exists u_\iota\, (r w u \wedge \varphi u) \\
 & & \\
\text{valid} & = & \lambda\varphi_{\iota\rightarrow o}\forall w_\iota \varphi w
\end{array}
\qquad \text{Ax}
$$

The equations in Ax are given as axioms to the HOL provers!

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The equations in Ax are given as axioms to the HOL provers!

Example

QML formula	$\Diamond \exists x G(x)$
QML formula in HOL	valid $(\Diamond \exists x G(x))_{\iota\to o}$
expansion, $\beta\eta$-conversion	$\forall w_\iota (\Diamond \exists x G(x))_{\iota\to o}\, w$
expansion, $\beta\eta$-conversion	$\forall w_\iota \exists u_\iota (rwu \wedge (\exists x G(x))_{\iota\to o} u)$
expansion, $\beta\eta$-conversion	$\forall w_\iota \exists u_\iota (rwu \wedge \exists x Gxu)$

What are we doing?

In order to prove that φ is valid in QML,
–> we instead prove that valid $\varphi_{\iota\to o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Expansion: user or prover may flexibly choose expansion depth

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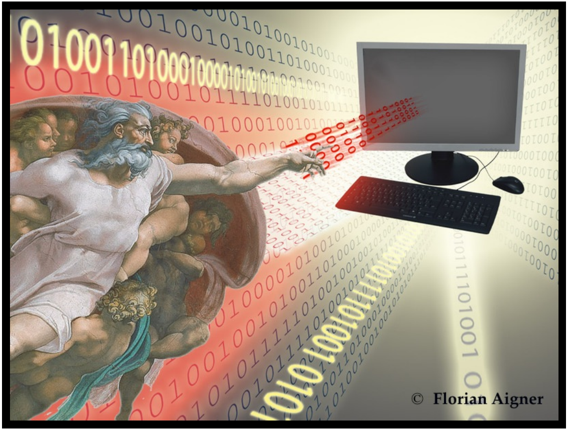
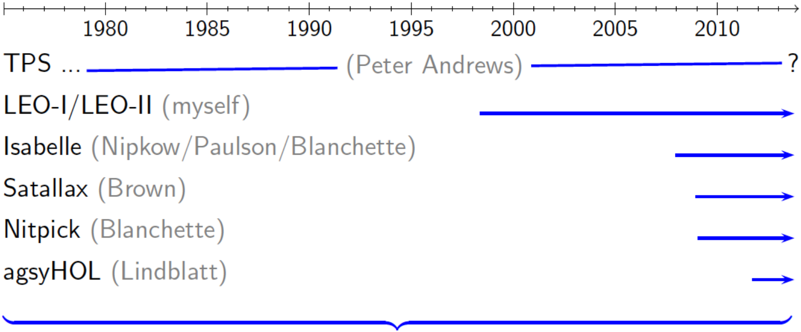


Automated Proof Search and Consistency Check

Automated Theorem Provers and Model Finders for HOL



- all accept TPTP THF Syntax [SutcliffeBenzmüller, J.Form.Reas, 2009]
- can be called remotely via SystemOnTPTP at Miami
- they significantly gained in strength over the last years
- they can be bundled into a combined prover HOL-P

Exploit HOL with Henkin semantics as metalogic
Automate other logics (& combinations) via semantic embeddings
— HOL-P becomes a Universal Reasoner —

Proof Automation and Consistency Checking

```
Terminal — bash — 125×32
MacBook-Chris %
MacBook-Chris %
MacBook-Chris % ./call_tptp.sh T3.p

Asking various HOL-ATPs in Miami remotely (thanks to Geoff Sutcliffe)

MacBook-Chris % agsyHOL---1.0 : T3.p ++++++ RESULT: SOT_7L4x_Y - agsyHOL---1.0 says Unknown - CPU = 0.00 WC = 0.02
LEO-II---1.6.0 : T3.p ++++++ RESULT: SOT_E4SCha - LEO-II---1.6.0 says Theorem - CPU = 0.03 WC = 0.09
Satallax---2.7 : T3.p ++++++ RESULT: SOT_kVZ1cB - Satallax---2.7 says Theorem - CPU = 0.00 WC = 0.14
Isabelle---2013 : T3.p ++++++ RESULT: SOT_xa0gEp - Isabelle---2013 says Theorem - CPU = 14.06 WC = 17.73 SolvedBy = auto
TPS---3.120601S1b : T3.p ++++++ RESULT: SOT_R0Egsg - TPS---3.120601S1b says Unknown - CPU = 33.56 WC = 41.57
Nitrox---2013 : T3.p ++++++ RESULT: SOT_WGY1Tx - Nitrox---2013 says Unknown - CPU = 75.55 WC = 49.24

MacBook-Chris %
MacBook-Chris % ./call_tptp.sh Consistency.p

Asking various HOL-ATPs in Miami remotely (thanks to Geoff Sutcliffe)

MacBook-Chris % agsyHOL---1.0 : Consistency.p ++++++ RESULT: SOT_ZtY_7o - agsyHOL---1.0 says Unknown - CPU = 0.00 WC = 0.00
Nitrox---2013 : Consistency.p ++++++ RESULT: SOT_HUz10C - Nitrox---2013 says Satisfiable - CPU = 6.56 WC = 8.50
TPS---3.120601S1b : Consistency.p ++++++ RESULT: SOT_fpJxTM - TPS---3.120601S1b says Unknown - CPU = 43.00 WC = 49.42
Isabelle---2013 : Consistency.p ++++++ RESULT: SOT_6Tpp9i - Isabelle---2013 says Unknown - CPU = 69.96 WC = 72.62
LEO-II---1.6.0 : Consistency.p ++++++ RESULT: SOT_dY10sj - LEO-II---1.6.0 says Timeout - CPU = 90 WC = 89.86
Satallax---2.7 : Consistency.p ++++++ RESULT: SOT_Q9WSLf - Satallax---2.7 says Timeout - CPU = 90 WC = 90.50

MacBook-Chris %
```

Provers are called remotely in Miami — no local installation needed!

GoedelGod.thy (modified)

GoedelGod.thy (~/GoedelGod/Formalizations/Isabelle/GoedelGodSession/)

```
  corollary C: "[◇ (∃ G)]"
  sledgehammer [provers = remote_leo2] by (metis A3 T1)

text {* Axiom @{text "A4"} is added: $\all \phi [P(\phi) \to \Box \; P(\phi)]$
(Positive properties are necessarily positive). *}

  axiomatization where A4:  "[Π (λΦ. P Φ m⇒ □ (P Φ))]"

text {* Symbol @{text "ess"} for `Essence' is introduced and defined as
$\ess{\phi}{x} \biimp \phi(x) \wedge \all \psi (\psi(x) \imp \nec \all y (\phi(y)
\imp \psi(y)))$ (An \emph{essence} of an individual is a property possessed by it
and necessarily implying any of its properties). *}

  definition ess :: "(μ ⇒ σ) ⇒ μ ⇒ σ" (infixr "ess" 85) where
    "Φ ess x = Φ x m∧ Π (λψ. ψ x m⇒ □ (∀ (λy. Φ y m⇒ ψ y)))"

text {* Next, Sledgehammer and Metis prove theorem @{text "T2"}: $\all x [G(x) \imp \ess{G}{x}]$
(Being God-like is an essence of any God-like being). *}

  theorem T2: "[∀ (λx. G x m⇒ G ess x)]"
  sledgehammer [provers = remote_leo2] by (metis A1b A4 G_def ess_def)

text {* Symbol @{text "NE"}, for `Necessary Existence', is introduced and
defined as $\NE(x) \biimp \all \phi [\ess{\phi}{x} \imp \nec \ex y \phi(y)]$ (Necessary
existence of an individual is the necessary exemplification of all its essences). *}

  definition NE :: "μ ⇒ σ" where "NE = (λx. Π (λΦ. Φ ess x m⇒ □ (∃ Φ)))"
```

100% Auto update Update Detach

```
Sledgehammering...
```

Output README Symbols

139,39 (6613/8211) (isabelle,sidekick,UTF-8-Isabelle) N m r o UG 157/183MB 16:02

Results

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot{\forall}\phi_{\mu\rightarrow\sigma}.\, p_{(\mu\rightarrow\sigma)\rightarrow\sigma}(\lambda X_{\mu}.\, \dot{\neg}(\phi X)) \,\dot{\equiv}\, \dot{\neg}(p\phi)]$						
A2	$[\dot{\forall}\phi_{\mu\rightarrow\sigma}.\, \dot{\forall}\psi_{\mu\rightarrow\sigma}.\, (p_{(\mu\rightarrow\sigma)\rightarrow\sigma}\phi \,\dot{\wedge}\, \dot{\Box}\dot{\forall}X_{\mu}.\, (\phi X \,\dot{\supset}\, \psi X)) \,\dot{\supset}\, p\psi]$						
T1	$[\dot{\forall}\phi_{\mu\rightarrow\sigma}.\, p_{(\mu\rightarrow\sigma)\rightarrow\sigma}\phi \,\dot{\supset}\, \dot{\Diamond}\dot{\exists}X_{\mu}.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\rightarrow\sigma} = \lambda X_{\mu}.\, \dot{\forall}\phi_{\mu\rightarrow\sigma}.\, p_{(\mu\rightarrow\sigma)\rightarrow\sigma}\phi \,\dot{\supset}\, \phi X$						
A3	$[p_{(\mu\rightarrow\sigma)\rightarrow\sigma}\, g_{\mu\rightarrow\sigma}]$						
C	$[\dot{\Diamond}\dot{\exists}X_{\mu}.\, g_{\mu\rightarrow\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot{\forall}\phi_{\mu\rightarrow\sigma}.\, p_{(\mu\rightarrow\sigma)\rightarrow\sigma}\phi \,\dot{\supset}\, \dot{\Box}p\phi]$						
D2	$\mathbf{ess}_{(\mu\rightarrow\sigma)\rightarrow\mu\rightarrow\sigma} = \lambda\phi_{\mu\rightarrow\sigma}.\, \lambda X_{\mu}.\, \phi X \,\dot{\wedge}\, \dot{\forall}\psi_{\mu\rightarrow\sigma}.\, (\psi X \,\dot{\supset}\, \dot{\Box}\dot{\forall}Y_{\mu}.\, (\phi Y \,\dot{\supset}\, \psi Y))$						
T2	$[\dot{\forall}X_{\mu}.\, g_{\mu\rightarrow\sigma}X \,\dot{\supset}\, (\mathbf{ess}_{(\mu\rightarrow\sigma)\rightarrow\mu\rightarrow\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathbf{NE}_{\mu\rightarrow\sigma} = \lambda X_{\mu}.\, \dot{\forall}\phi_{\mu\rightarrow\sigma}.\, (\mathbf{ess}\, \phi X \,\dot{\supset}\, \dot{\Box}\dot{\exists}Y_{\mu}.\, \phi Y)$						
A5	$[p_{(\mu\rightarrow\sigma)\rightarrow\sigma}\mathbf{NE}_{\mu\rightarrow\sigma}]$						
T3	$[\dot{\Box}\dot{\exists}X_{\mu}.\, g_{\mu\rightarrow\sigma}X]$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_{\sigma} \,\dot{\supset}\, \dot{\Box}s_{\sigma}]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot{\forall}\phi_{\mu\rightarrow\sigma}.\, \dot{\forall}X_{\mu}.\, (g_{\mu\rightarrow\sigma}X \,\dot{\supset}\, (\dot{\neg}(p_{(\mu\rightarrow\sigma)\rightarrow\sigma}\phi) \,\dot{\supset}\, \dot{\neg}(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot{\forall}X_{\mu}.\, \dot{\forall}Y_{\mu}.\, (g_{\mu\rightarrow\sigma}X \,\dot{\supset}\, (g_{\mu\rightarrow\sigma}Y \,\dot{\supset}\, X \doteq Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\mathbf{ess}_{(\mu\rightarrow\sigma)\rightarrow\mu\rightarrow\sigma} = \lambda\phi_{\mu\rightarrow\sigma}.\, \lambda X_{\mu}.\, \dot{\forall}\psi_{\mu\rightarrow\sigma}.\, (\psi X \,\dot{\supset}\, \dot{\Box}\dot{\forall}Y_{\mu}.\, (\phi Y \,\dot{\supset}\, \psi Y))$						
CO'	$\emptyset$ (no goal, check for consistency)	A1(⊃), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

Formalization and Verification in Coq

- Goal: verification of the natural deduction proof
  - Step-by-step formalization
  - Almost no automation (intentionally!)
- Interesting facts:
  - Embedding is transparent to the user
  - Embedding gives labeled calculus for free

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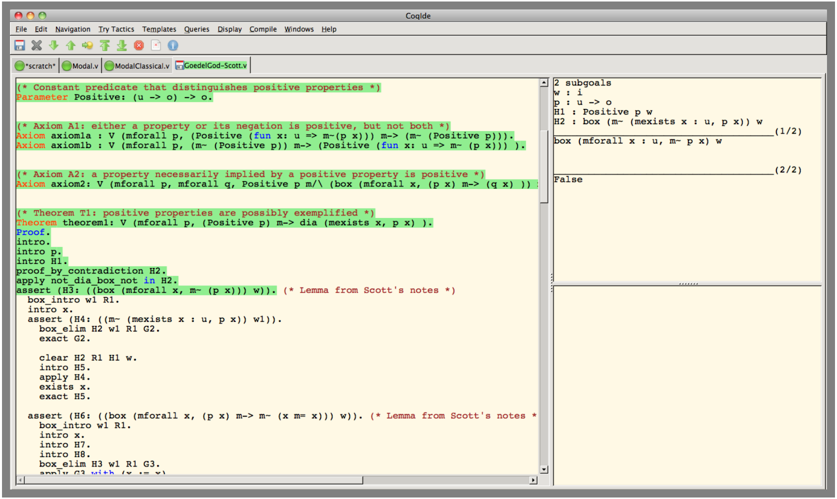
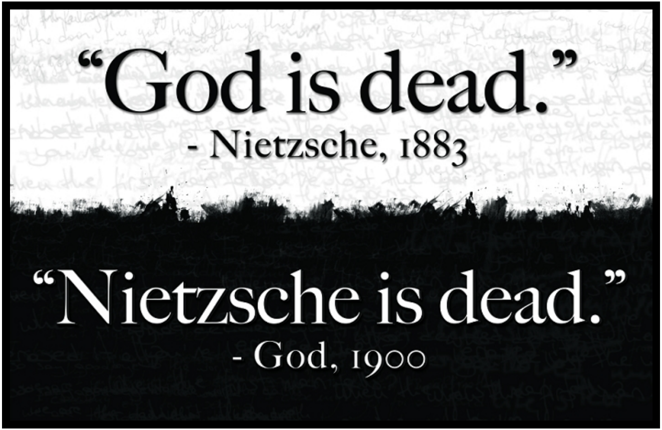

CoqIde
File Edit Navigation Try Tactics Templates Queries Display Compile Windows Help
scratch Modal.v ModalClassical.v GoedelGod-Scott.v
(* Constant predicate that distinguishes positive properties *)
Parameter Positive: (u -> o) -> o.
(* Axiom A1: either a property or its negation is positive, but not both *)
Axiom axiom1a : V (mforall p, (Positive (fun x: u => m~(p x))) m-> (m~ (Positive p))).
Axiom axiom1b : V (mforall p, (m~ (Positive p)) m-> (Positive (fun x: u => m~ (p x)))).
(* Axiom A2: a property necessarily implied by a positive property is positive *)
Axiom axiom2: V (mforall p, mforall q, Positive p m/\ (box (mforall x, (p x) m-> (q x)))
(* Theorem T1: positive properties are possibly exemplified *)
Theorem theorem1: V (mforall p, (Positive p) m-> dia (mexists x, p x)).
Proof.
intro.
intro p.
intro H1.
proof_by_contradiction H2.
apply not_dia_box_not in H2.
assert (H3: ((box (mforall x, m~ (p x))) w)). (* Lemma from Scott's notes *)
box_intro w1 R1.
intro x.
assert (H4: ((m~ (mexists x : u, p x)) w1)).
box_elim H2 w1 R1 G2.
exact G2.
clear H2 R1 H1 w.
intro H5.
apply H4.
exists x.
exact H5.
assert (H6: ((box (mforall x, (p x) m-> m~ (x m= x))) w)). (* Lemma from Scott's notes *
box_intro w1 R1.
intro x.
intro H7.
intro H8.
box_elim H3 w1 R1 G3.
2 subgoals
w : i
p : u -> o
H1 : Positive p w
H2 : box (m~ (mexists x : u, p x)) w
(1/2)
box (mforall x : u, m~ p x) w
(2/2)
False



Criticisms

$$\forall P.[\Diamond\Box P \rightarrow \Box P]$$

If something is possibly necessary, then it is necessary.

$$\forall P.[P \rightarrow \Box P]$$

Everything that is the case is so necessarily.

Follows from T2, T3 and D2.

Many proposed solutions: Anderson, Fitting, Hájek, ...

$$\forall \phi [P(\neg \phi) \leftrightarrow \neg P(\phi)]$$

Either a property is positive or its negation is (but never both)

Are the following properties positive or negative?

$$\lambda x.G(x) \qquad \lambda x.E(x) \qquad \lambda x.x = x \qquad \lambda x.\top$$

$$\lambda x.blue(x) \qquad \lambda x.red(x) \qquad \lambda x.human(x)$$

Possible Solution:
"...positive in the moral aesthetic sense (independently of the accidental structure of the world). Only then the ax. true. ..."

- Gödel, 1970

Conclusions and Future Work

Conclusions

- Contributions:
  - Powerful infrastructure for reasoning with QML
  - A (new?) natural deduction calculus for higher-order (intuitionistic) modal logic
  - Non-trivial new benchmark problems for HOL provers
  - Verification of existing results about Gödel's proof (e.g. Modal Collapse)
  - New results about Gödel's proof (e.g. Inconsistency)
- Major step towards Computer-assisted Theoretical Philosophy
  - see also Ed Zalta's *Computational Metaphysics* project at Stanford University and John Rushby's formalization of Anselm's proof using PVS

- Embedding of other non-classical logics
  - Quantified Conditional Logics,
    Modal Logics based on Neighbourhood Semantics, . . .
  - Other Suggestions?
  - Human Resources?

- New non-classical calculi via HOL embedding:
  (when does this approach work?)
  1. Embed your favorite logic L into HOL
  2. Play with Coq and invent new tactics to encapsulate the embedding and make it transparent to the user
  3. Interpret the tactics as new rules for a new natural deduction calculus C
  4. Prove soundness and completeness indirectly:

$$\models_L s_o \quad \text{iff} \quad \models \text{valid}(\lceil s_o \rceil) \qquad\qquad \vdash \text{valid}(\lceil s_o \rceil) \quad \text{iff} \quad \vdash_C s_o$$

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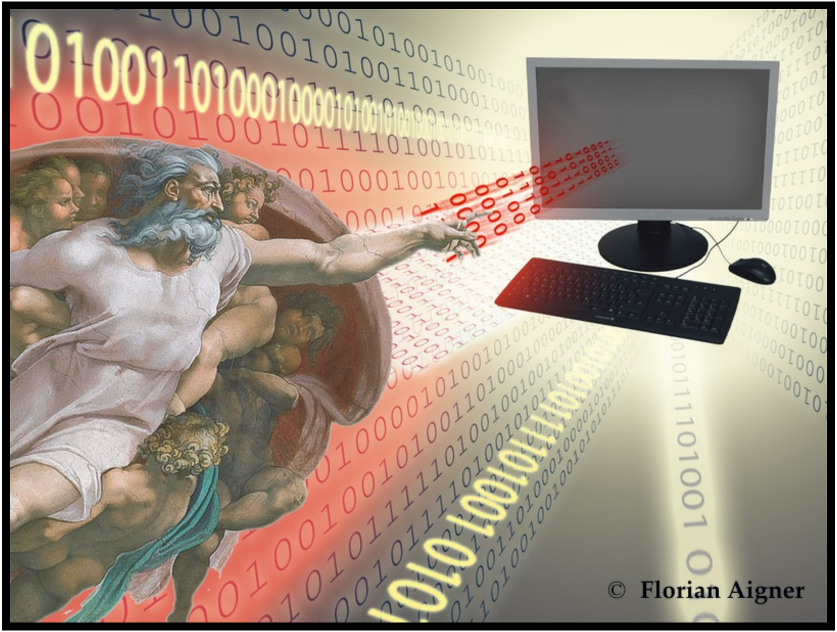
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