

# Findings from a Computer-supported Analysis of Variants of Gödel's Ontological Proof

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```
>
>
> Beweise-mit-Leo2 Notwendigerweise-existiert-Gott.p
Leo-II tries to prove
=====
Goedel's Theorem T3: "Necessarily, God exists"
thf(thmT3,conjecture,
  ( v
    @ ( mbox
      @ ( mexists_ind
        @ ^ [X: mu] :
          ( g @ X ) ) ) )).
Assumptions: D1, C, T2, D3, A5
. searching for proof ..
*****
* Proof found *
*****
% S2S status Theorem for Notwendigerweise-existiert-Gott.p
. generating proof object □
```

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<sup>1</sup>Supported by DFG Heisenberg Fellowship BE 2501/9-1/2



If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other . . . : Let us calculate.

(Translation by Russell)

Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo . . . dicere: calculemus.  
(Leibniz, 1684)



Required:  
**characteristica universalis** and **calculus ratiocinator**



## Ontological argument for the existence of God

### Focus on Gödel's modern version in higher-order modal logic

### Automation with provers for higher-order classical logic (HOL)

- ▶ verification (or falsification) of known results
- ▶ some novel results (by HOL-ATPs)
- ▶ HOL as a universal metalogic via logic embeddings  
(characteristica universalis?)

- ▶ Philosophical: Boundaries of Metaphysics & Epistemology
- ▶ Theistic: Successful argument could convince atheists?
- ▶ **Ours:** Can computers (theorem provers) be used ...
  - ... to formalize the definitions, axioms and theorems?
  - ... to verify/falsify the arguments step-by-step?
  - ... to automate (sub-)arguments?
  - ... to generate new knowledge?



Anselm's notion of God (Proslogion, 1078):

**“God is that, than which nothing greater can be conceived.”**

Gödel's notion of God:

**“A God-like being possesses all ‘positive’ properties.”<sup>33</sup>**

To show by logical, deductive reasoning:

**“God exists.”**

$$\exists x G(x)$$



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## Ontologischer Beweis

Feb. 10, 1970

$P(\varphi)$   $\varphi$  is positive ( $\Leftrightarrow \varphi \in P$ )

Ax 1:  $P(p), P(\varphi) \supset P(\varphi \wedge p)$  At 2:  $P(p) \supset P(\neg p)$

$P_1$   $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$  (God)

$P_2$   $\varphi_{\text{Exn } x} \equiv (\psi)[\psi(x) \supset N(y)[P(y) \supset \psi(y)]]$  (Existence)

$P \supset_N = N(p \supset q)$  Necessity

Ax 2:  $P(\varphi) \supset N P(\varphi)$  } because it follows  
 $\neg P(\varphi) \supset N \neg P(\varphi)$  } from the nature of the  
 property

Th:  $G(x) \supset G_{\text{Exn } x}$

Df:  $E(x) \equiv P[\varphi_{\text{Exn } x} \supset N \neg x \supset \varphi(x)]$  necessary Existence

Ax 3:  $P(E)$

Th:  $G(x) \supset N(\exists y) G(y)$

hence  $(\exists x) G(x) \supset N(\exists y) G(y)$

"  $M(x) G(x) \supset M N(\exists y) G(y)$  M = possibility  
 "  $\supset N(\exists y) G(y)$

any two elements of X are nec. equivalent

exclusive or \* and for any number of them

$M(x) G(x)$  means all pos. prop. w.r.t. com-  
 patible. This is true because of:

Ax 4:  $P(\varphi) \cdot q \supset_N \varphi \supset P(\varphi)$  which impl.

$\begin{cases} x=x & \text{is positive} \\ x \neq x & \text{is negative} \end{cases}$

But if a system S of pos. prop. were incons.  
 It would mean, that the non-prop. S (which  
 is positive) would be  $x \neq x$

Positive means positive in the moral aest.  
 sense (independently of the accidental structure of  
 the world). Only in the ex. True. It is  
 also meant "Attribution" as opposed to "negation"  
 (or containing negation). This interprets the word

" $\supset$ " (positive)  $(x) N \neg P(x)$  Otherwise  $P(x) \supset x \neq x$

hence  $x \neq x$  positive not  $x=x$  negative. At  
 the end of proof Ax 4

X i.e. the normal form in terms of elem. prop. contains  
 Member without negation.

## Scott's Version of Gödel's Axioms, Definitions and Theorems

**Axiom A1** Either a property or its negation is positive, but not both:  $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

**Axiom A2** A property necessarily implied by a positive property is positive:

$$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

**Thm. T1** Positive properties are possibly exemplified:

$$\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$$

**Def. D1** A God-like being possesses all positive properties:  $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$

**Axiom A3** The property of being God-like is positive:  $P(G)$

**Cor. C** Possibly, God exists:

$$\Diamond\exists xG(x)$$

**Axiom A4** Positive properties are necessarily positive:  $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$

**Def. D2** An essence of an individual is a property possessed by it and necessarily implying any of its properties:  $\phi \text{ ess } x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

**Thm. T2** Being God-like is an essence of any God-like being:

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**Def. D3** Necessary existence of an individual is the necessary exemplification of all its essences:  $NE(x) \leftrightarrow \forall\phi[\phi \text{ ess } x \rightarrow \Box\exists y\phi(y)]$

**Axiom A5** Necessary existence is a positive property:  $P(NE)$

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Difference to Gödel (who omits this conjunct)

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Modal operators are used

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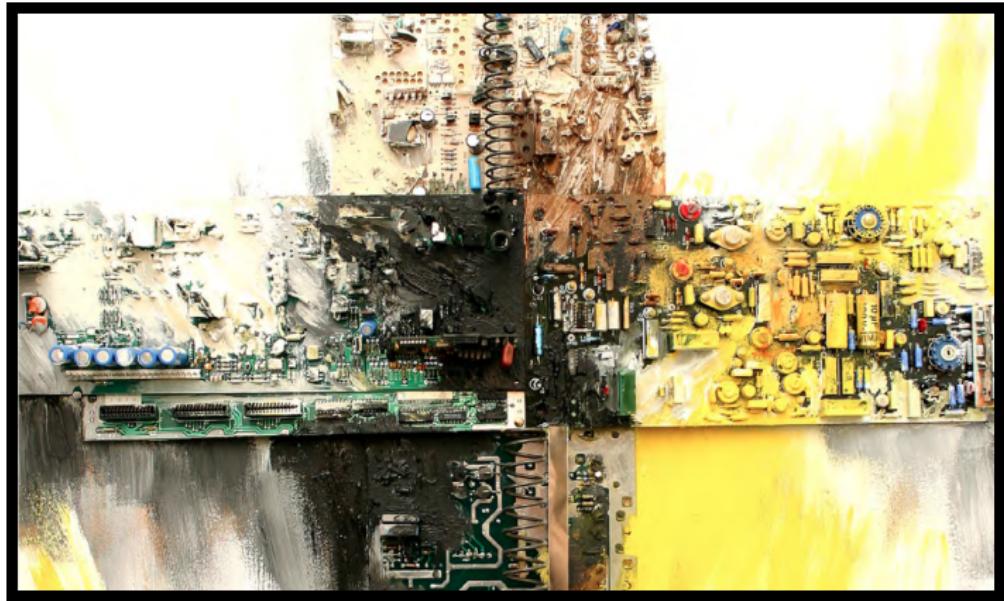
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second-order quantifiers



## How to automate Higher-Order Modal Logic?

Challenge: No provers for *Higher-order Modal Logic* (HOML)

Church's Simple Type Theory

Our solution: Embedding in **Higher-order Classical Logic** (HOL)

Then use existing **HOL** theorem provers for reasoning in **HOML**

[BenzmüllerPaulson, Logica Universalis, 2013]

Assumption: Henkin semantics for both **HOML** and **HOL**

Previous empirical findings:

Embedding of *First-order Modal Logic* in **HOL** works well

[BenzmüllerOttenRaths, ECAI, 2012]

[Benzmüller, LPAR, 2013]

**HOML**       $\varphi, \psi ::= \dots | \neg\varphi | \varphi \wedge \psi | \varphi \rightarrow \psi | \Box\varphi | \Diamond\varphi | \forall x_\gamma \varphi | \exists x_\gamma \varphi$

**HOL**       $s, t ::= C_\alpha | x_\alpha | (\lambda x_\alpha s_\beta)_{\alpha \rightarrow \beta} | (s_{\alpha \rightarrow \beta} t_\alpha)_\beta | \neg s_o | s_o \vee t_o | \forall x_\alpha t_o$

**HOML in HOL:** HOML formulas  $\varphi$  are mapped to HOL predicates  $\varphi_{\mu \rightarrow o}$   
 (explicit representation of labelled formulas)

$\neg$	$= \lambda \varphi_{\mu \rightarrow o} \lambda w_\mu \neg \varphi w$
$\wedge$	$= \lambda \varphi_{\mu \rightarrow o} \lambda \psi_{\mu \rightarrow o} \lambda w_\mu (\varphi w \wedge \psi w)$
$\rightarrow$	$= \lambda \varphi_{\mu \rightarrow o} \lambda \psi_{\mu \rightarrow o} \lambda w_\mu (\neg \varphi w \vee \psi w)$
$\forall$	$= \lambda h_{\gamma \rightarrow (\mu \rightarrow o)} \lambda w_\mu \forall d_\gamma h d w$
$\exists$	$= \lambda h_{\gamma \rightarrow (\mu \rightarrow o)} \lambda w_\mu \exists d_\gamma h d w$
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**Ax** (polymorphic over  $\gamma$ )

The equations in Ax are given as axioms to the HOL provers!

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## Example

HOML formula

HOML formula in HOL

expansion,  $\beta\eta$ -conversion

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expansion,  $\beta\eta$ -conversion

$\diamond \exists x G(x)$

valid ( $\diamond \exists x G(x)$ ) $_{\mu \rightarrow o}$

$\forall w_\mu (\diamond \exists x G(x))_{\mu \rightarrow o} w$

$\forall w_\mu \exists u_\mu (rwu \wedge (\exists x G(x))_{\mu \rightarrow o} u)$

$\forall w_\mu \exists u_\mu (rwu \wedge \exists x Gxu)$

Expansion: user or prover may flexibly choose expansion depth

## What are we doing?

In order to prove that  $\varphi$  is valid in HOML,

→ we instead prove that valid  $\varphi_{\mu \rightarrow o}$  can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

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## Modal logic axioms

- M: valid  $\forall\varphi(\Box\varphi \rightarrow \varphi)$
- B: valid  $\forall\varphi(\varphi \rightarrow \Box\Diamond\varphi)$
- D: valid  $\forall\varphi(\Box\varphi \rightarrow \Diamond\varphi)$
- 4: valid  $\forall\varphi(\Box\varphi \rightarrow \Box\Box\varphi)$
- 5: valid  $\forall\varphi(\Diamond\varphi \rightarrow \Box\Diamond\varphi)$

## Semantical conditions

- $\forall x(rxy)$
- $\forall x\forall y(rxy \rightarrow ryx)$
- $\forall x\exists y(rxy)$
- $\forall x\forall y\forall z(rxy \wedge ryz \rightarrow rxz)$
- $\forall x\forall y\forall z(rxy \wedge rxz \rightarrow ryz)$

$$\begin{array}{ll} \forall = \lambda h_{\gamma \rightarrow (\mu \rightarrow o)} \lambda w_\mu \forall d_\gamma h d w & \text{(possibilist / constant dom.)} \\ \text{becomes} & \\ \forall^{va} = \lambda h_{\gamma \rightarrow (\mu \rightarrow o)} \lambda w_\mu \forall d_\gamma (\mathbf{ExInW} d w \rightarrow h d w) & \text{(actualist / varying dom.)} \end{array}$$

where  $\mathbf{ExInW}$  is an existence predicate.

Additional axioms:

- ▶ domains are non-empty  $\forall w_\mu \exists x_\mu \mathbf{exInW} x w$
- ▶ denotation (constants & functions)  $\forall w_\mu (\mathbf{exInW} t^1 w \wedge \dots \wedge \mathbf{exInW} t^n w \supset \mathbf{exInW} (f t^1 \dots t^n) w)$

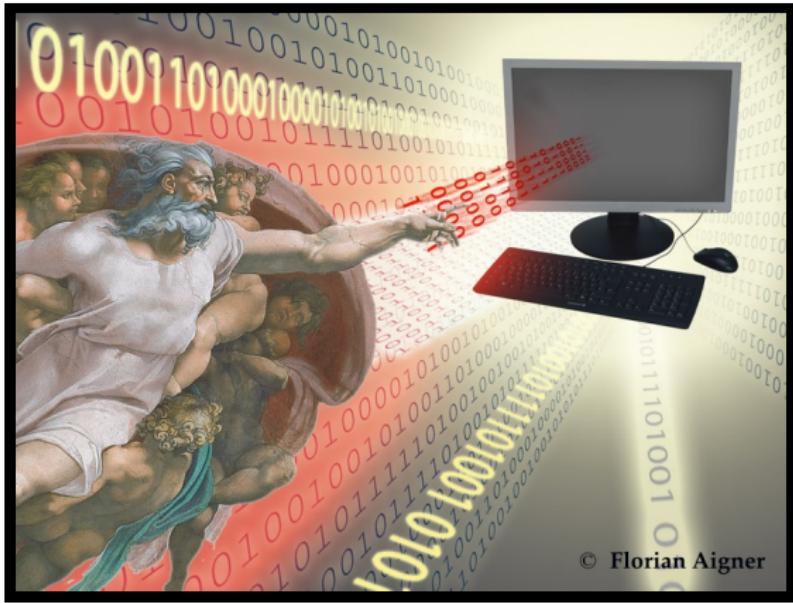
Cumulative domains:  $\forall x \forall v \forall w (\mathbf{exInW} x v \wedge r v w \supset \mathbf{exInW} x w)$

## Soundness and Completeness (and Cut-elimination)

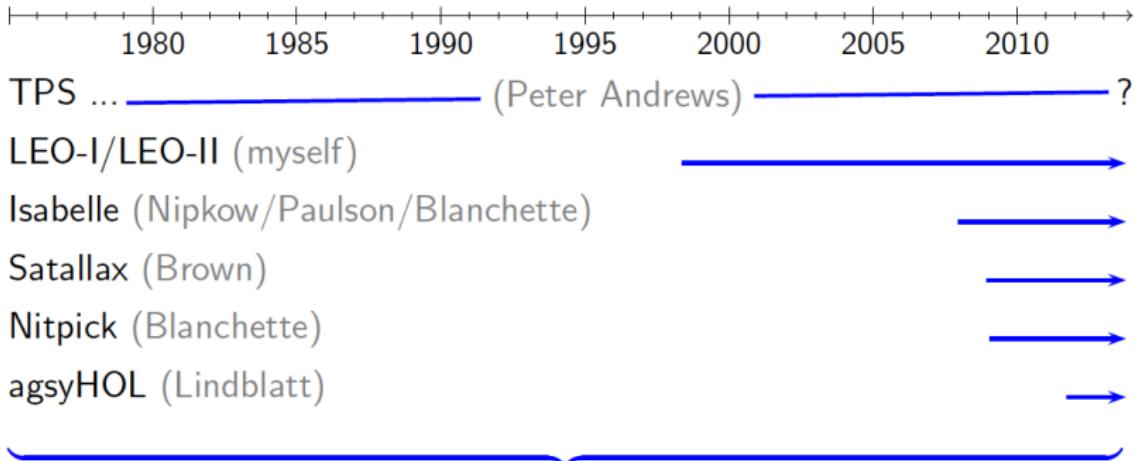
$$\models^L \varphi \text{ iff } \text{Ax} \models_{\text{Henkin}}^{\text{HOL}} \text{valid } \varphi_{\mu \rightarrow o} \text{ (iff } \text{Ax} \vdash_{\text{cut-free}}^{\text{HOL}} \text{valid } \varphi_{\mu \rightarrow o})$$

Logic L:

- ▶ Higher-order Modal Logics
- ▶ First-order Multimodal Logics
- ▶ Propositional Multimodal Logics
- ▶ Quantified Conditional Logics
- ▶ Propositional Conditional Logics
- ▶ Intuitionistic Logics
- ▶ Access Control Logics
- ▶ Logic Combinations
- ▶ ...more is on the way ... including:
  - ▶ Description Logics
  - ▶ Nominal Logics
  - ▶ Multivalued Logics (SIXTEEN)
  - ▶ Logics based on Neighborhood Semantics
  - ▶ (Mathematical) Fuzzy Logics
  - ▶ Paraconsistent Logics



## Automated and Interactive Theorem Provers for HOL



- all accept TPTP THF Syntax [SutcliffeBenzmüller, J.Form.Reas, 2009]
  - can be called remotely via SystemOnTPTP at Miami
  - they significantly gained in strength over the last years
  - they can be bundled into a combined prover **HOL-P**

Exploit HOL with Henkin semantics as metalogic  
Automate other logics (& combinations) via semantic embeddings  
— **HOL-P** becomes a **Universal Reasoner** —

```
>  
>  
> Beweise-mit-Leo2 Notwendigerweise-existiert-Gott.p
```

Leo-II tries to prove

=====

Goedel's Theorem T3: "Necessarily, God exists"

```
thf(thmT3,conjecture,  
     ( v  
       @ ( mbox  
           @ ( mexists_ind  
               @ ^ [X: mu] :  
                 ( g @ X ) ) ) ) ).
```

Assumptions: D1, C, T2, D3, A5

. searching for proof ..

\*\*\*\*\*

\* Proof found \*

\*\*\*\*\*

% Szs status Theorem for Notwendigerweise-existiert-Gott.p

. generating proof object □

Provers can be called remotely in Miami — no local installation needed!

Download our experiments from

<https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/THF>



The screenshot shows the official Isabelle website at <http://isabelle.in.tum.de/index.html>. The page features a header with the Isabelle logo, search bar, and navigation links. A sidebar on the left includes links for Home, Overview, Installation, Documentation, Community, and Site Mirrors. The main content area highlights "Isabelle 2013" and "Now available: Isabelle2013". It provides download links for Mac OS X and Linux/Windows, along with some highlights of the new features. The distribution & support section details the BSD license, documentation, and various mailing lists for users and developers.

Isabelle is a generic proof assistant. It allows mathematical formulas to be expressed in a formal language and provides tools for proving those formulas in a logical calculus. Isabelle is developed at University of Cambridge (Larry Paulson), Technische Universität München (Tobias Nipkow) and Université Paris-Sud (Makarius Wenzel). See the [Isabelle overview](#) for a brief introduction.

**Now available: Isabelle2013**

Download for Mac OS X

Download for Linux - Download for Windows

**Some highlights:**

- Improvements of Isabelle/Scala and Isabelle/jEdit Prover IDE.
- Advanced build tool based on Isabelle/Scala.
- Updated manuals: isar-ref, implementation, system.
- PolyML support for block-structured specification contexts.
- HOL tool enhancements: Sledgehammer, Lifting, Quickcheck.
- HOL library enhancements: HOL-Library, HOL-Probability, HOL-Cardinals.
- HOL: New BNF-based (co)datatype package.
- Improved performance thanks to Poly/ML 5.5.0.

See also the cumulative [NEWS](#).

**Distribution & Support**

Isabelle is distributed for free under the BSD license. It includes source and binary packages and documentation, see the detailed [Installation instructions](#). A vast collection of Isabelle examples and applications is available from the [Archive of Formal Proofs](#).

Support is available by [documentasjon](#), the [Isabelle community Wiki](#), and the following mailing lists:

- [isabelle-users@cl.cam.ac.uk](mailto:isabelle-users@cl.cam.ac.uk) provides a forum for Isabelle users to discuss problems, exchange information, and make announcements. Users of official Isabelle releases should [subscribe](#) or see the [archive](#) (also available via [Google groups](#) and [Narrows](#)).
- [isabelle-dev@in.tum.de](mailto:isabelle-dev@in.tum.de) covers the Isabelle development process, including intermediate repository versions, and administrative issues concerning the website or testing infrastructure. Early adopters of [repository versions](#) should [subscribe](#) or see the [archive](#) (also available at [mail-archive.com](#) or [gmane.org](#)).

Last updated: 2013-03-09 12:21:39

See verifiable Isabelle/HOL document (Archive of Formal Proofs) at:  
<http://afp.sourceforge.net/entries/GoedelGod.shtml>

```

(* Constant predicate that distinguishes positive properties *)
Parameter Positive : (u -> o) -> o.

(* Axiom A1: either a property or its negation is positive, but not both *)
Axiom axiomia : V (mforall p, (Positive (fun x: u => m-(p x))) m-> (m- (Positive p))). 
Axiom axiomib : V (mforall p, (m- (Positive p)) m-> (Positive (fun x: u => m- (p x)))). 

(* Axiom A2: a property necessarily implied by a positive property is positive *)
Axiom axiom2 : V (mforall p, mforall q, Positive p m\> (box (mforall x, (p x) m-> (q x)))). 

(* Theorem T1: positive properties are possibly exemplified *)
Theorem theorem1 : V (mforall p, (Positive p) m-> dia (mexists x, p x)).
Proof.
intro.
intro p.
intro H1.
proof_by_contradiction H2.
apply not_dia_box_not_in H2.
assert (H3: (box (mforall x, m- (p x))) w). (* Lemma from Scott's notes *)
box_intro w1 R1.
intro x.
assert (H4: ((m- (mexists x : u, p x)) w1)).
box_elim H2 w1 R1 G2.
exact G2.

clear H2 R1 H1 w.
intro H5.
apply H4.
exists x.
exact H5.

assert (H6: ((box (mforall x, (p x) m-> m- (x m= x))) w)). (* Lemma from Scott's notes *)
box_intro w1 R1.
intro x.
intro H7.
intro H8.
box_elim H3 w1 R1 G3.
annTu G3 with /w .:= w1

```

See verifiable Coq document at:

<https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/Coq>

“God is dead.”

- Nietzsche, 1883

“Nietzsche is dead.”

- God, 1900

## Findings from our study

[BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

# Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p \phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \dot{\forall} X_\mu^* (\phi X \dot{\wedge} \psi X) \dot{\wedge} p \psi)$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Diamond} \exists X_\mu^* \phi X$	A1(○), A2 A1, A2	K	THM	0.1/0.1 0.1/0.1	0.0/0.0 0.0/5.2	—/— —/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu^*. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \phi X$						
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\Diamond} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0 0.0/0.0	0.0/0.0 5.2/31.3	—/— —/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} p \phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu^*. \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu^* (\phi Y \dot{\wedge} \psi Y))$						
T2	$\dot{\forall} X_\mu^* g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)$	A1, D1, A4, D2 A1, A2, D1, A3, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
			K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu^*. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess } \phi X \dot{\wedge} \dot{\Box} \exists Y_\mu^* \phi Y)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\Box} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$	D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5 D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5	K K KB KB	CSA CSA THM THM	—/— —/— 0.0/0.1 —/—	—/— —/— 0.1/5.3 —/—	3.8/6.2 8.2/7.5 —/— —/—
MC	$[s_\sigma \dot{\wedge} \dot{\Box} s_\sigma]$	D2, T2, T3 A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	17.9/—	3.3/3.2	—/—
			KB	THM	—/—	—/—	—/—
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (\dot{\neg}(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \dot{\wedge} \dot{\neg}(\phi X)))$	A1, D1 A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	16.5/— 12.8/15.1	0.0/0.0 0.0/5.4	—/— —/—
MT	$\dot{\forall} X_\mu^* \dot{\forall} Y_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} Y \dot{\wedge} X \dot{\equiv} Y))$	D1, FG A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	—/— —/—	0.0/3.3 —/—	—/— —/—
CO	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu^*. \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu^* (\phi Y \dot{\wedge} \psi Y))$						
CO'	∅ (no goal, check for consistency)	A1(○), A2, D2', D3, A5 A1, A2, D1, A3, A4, D2', D3, A5	KB KB	UNS UNS	7.5/7.8 —/—	—/— —/—	—/— —/—

# Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/varv	Satallax const/varv	Nitpick const/varv
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{=} \dot{\neg}(p \phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} X_\mu^* (\phi X \dot{\wedge} \psi X)) \dot{\wedge} p \psi$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Diamond} \exists X_\mu^* \phi X$	A1(○), A2 A1, A2	K	THM	0.1/0.1 0.1/0.1	0.0/0.0 0.0/5.2	—/— —/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu^*. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \phi X$						
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\Diamond} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0 0.0/0.0	0.0/0.0 5.2/31.3	—/— —/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} p \phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu^* \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} Y_\mu^* (\phi Y \dot{\wedge} \psi Y))$						
T2	$\dot{\forall} X_\mu^* g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)$	A1, D1, A4, D2 A1, A2, D1, A3, A4, D2	K	THM	19.1/18.3 12.9/14.0	0.0/0.0 0.0/0.0	—/— —/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu^* \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess } \phi X \dot{\wedge} \dot{\Box} \exists Y_\mu^* \phi Y)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\Box} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$	D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5 D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5	K K KB KB	CSA CSA THM THM	—/— —/— 0.0/0.1 —/—	—/— —/— 0.1/5.3 —/—	3.8/6.2 8.2/7.5 —/— —/—
MC	$[s_\sigma \dot{\wedge} \dot{\Box} s_\sigma]$	D2, T2, T3 A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	17.9/— —/—	3.3/3.2 —/—	—/— —/—
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (\dot{\neg}(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \dot{\wedge} \dot{\neg}(\phi X)))$	A1, D1 A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	16.5/— 12.8/15.1	0.0/0.0 0.0/5.4	—/— —/—
MT	$\dot{\forall} X_\mu^* \dot{\forall} Y_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} Y \dot{\wedge} X \dot{=} Y))$	D1, FG A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/— —/—	0.0/3.3 —/—	—/— —/—
CO	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu^* \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} Y_\mu^* (\phi Y \dot{\wedge} \psi Y))$						
CO'	∅ (no goal, check for consistency)	A1(○), A2, D2', D3, A5 A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	7.5/7.8 —/—	—/— —/—	—/— —/—

# Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{=} \dot{\neg}(p \phi)$						
A2	$\dot{\forall} \psi_{\mu \rightarrow \sigma^*} q_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda Y_\mu. \dot{\neg} \dot{\forall} Y_\mu. (\dot{\forall} Y \dot{\neg} \dot{\forall} Y) \dot{=} p Y)$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{=} \dot{\diamond} \exists X_\mu. \phi X$	A1( $\supset$ ), A2 A1, A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \supset \phi X$						
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\diamond} \exists X_\mu. g_{\mu \rightarrow \sigma} X]$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{=} \dot{\diamond} p \phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu. \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{=} \dot{\forall} Y_\mu. (\phi Y \dot{=} \psi Y))$						
T2	$\dot{\forall} X_\mu. g_{\mu \rightarrow \sigma} X \dot{=} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)$	A1, D1, A4, D2 A1, A2, D1, A3, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\dots)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\diamond} \exists X_\mu. g_{\mu \rightarrow \sigma} X]$						
MC	$[s_\sigma \dot{=} \dot{\exists} s_\sigma]$						
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu. (s_\sigma \rightarrow X) \dot{=} (s_\sigma \rightarrow X)$						
MT	$\dot{\forall} X_\mu. \dot{\forall} Y_\mu. (g_{\mu \rightarrow \sigma} X \dot{=} (g_{\mu \rightarrow \sigma} Y \dot{=} X))$						
CO	0 (no goal, check for const)						
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda$						
CO'	0 (no goal, check for const)						

## Automating Scott's proof script

T1: "Positive properties are possibly exemplified" proved by LEO-II and Satallax

- ▶ in logic: K
- ▶ from assumptions:
  - ▶ A1 and A2
  - ▶ A1( $\supset$ ) and A2
- ▶ notion of quantification
  - ▶ possibilist quantifiers (constant dom.)
  - ▶ actualist quantifiers for individuals (varying dom.)

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p \phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \dot{\forall} X_\mu^* (\phi X \dot{\wedge} \psi X)) \dot{\wedge} p \psi$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Diamond} \exists X_\mu^* \phi X$	A1(○), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2					
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu^*. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \phi X$		K	THM	0.1/0.1	0.0/5.2	—/—
A3	$[g_{\mu \rightarrow \sigma}, g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\Diamond} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
			K	THM	0.0/0.0	5.2/31.3	—/—
A4	$\dot{\forall} \psi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \psi \dot{\wedge} \dot{\Box} p \psi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu^* \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu^* (\phi Y \dot{\wedge} \psi Y))$						
T2	$\dot{\forall} X_\mu^* g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)$	A1, D1, A4, D2 A1, A2, D1, A3, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
			K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu^* \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} \phi X \dot{\wedge} \phi X)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\Box} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$						
MC	$[s_\sigma \dot{\wedge} \dot{\Box} s_\sigma]$						
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} X \dot{\wedge} \phi X))$						
MT	$\dot{\forall} X_\mu^* \dot{\forall} Y_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} X \dot{\wedge} g_{\mu \rightarrow \sigma} Y))$						
CO	0 (no goal, check for const.)						
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu^* \phi X \dot{\wedge} \dot{\forall} Y_\mu^* (\phi X \dot{\wedge} \psi Y)$						
CO'	0 (no goal, check for const.)						

## Automating Scott's proof script

**C: "Possibly, God exists"**  
proved by LEO-II and Satallax

- ▶ in logic: K
- ▶ from assumptions:
  - ▶ T1, D1, A3
- ▶ for domain conditions:
  - ▶ possibilist quantifiers (constant dom.)
  - ▶ actualist quantifiers for individuals (varying dom.)

# Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p \phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \dot{\forall} X_\mu^* (\phi X \dot{\wedge} \psi X)) \dot{\wedge} p \psi$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \exists X_\mu^* \phi X$	A1(○), A2 A1, A2	K	THM	0.1/0.1 0.1/0.1	0.0/0.0 0.0/5.2	—/— —/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu^*. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \phi X$						
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\Box} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0 0.0/0.0	0.0/0.0 5.2/31.3	—/— —/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} p \phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu^* \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\phi X \dot{\wedge} \dot{\Box} \psi X \dot{\wedge} (\psi X \dot{\wedge} \dot{\Box} \psi Y))$						
T2	$\dot{\forall} X_\mu^* g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)$	A1, D1, A4, D2 A1, A2, D1, A3, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu^* \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} \phi X \dot{\wedge} \phi Y)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\Box} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$						

MC	$\dot{\exists} s_\sigma \dot{\wedge} \dot{\Box} s_\sigma$
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X))$
MT	$\dot{\forall} X_\mu^* \dot{\forall} Y_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} Y \dot{\wedge} \dot{\Box} X \dot{\wedge} Y))$
CO	0 (no goal, check for const.)
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu^* \phi X$
CO'	0 (no goal, check for const.)

## Automating Scott's proof script

**T2: "Being God-like is an ess. of any God-like being"** proved by LEO-II and Satallax

- ▶ in logic: K
- ▶ from assumptions:
  - ▶ A1, D1, A4, D2
- ▶ for domain conditions:
  - ▶ possibilist quantifiers (constant dom.)
  - ▶ actualist quantifiers for individuals (varying dom.)

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p \phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\forall} X_\mu^* (\phi X \dot{\wedge} \psi X)) \dot{\wedge} p \psi$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\exists} X_\mu^* \phi X$	A1(○), A2 A1, A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu^*. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \phi X$		K	THM	0.1/0.1	0.0/5.2	—/—
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\forall} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\neg} p \phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu^* \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\forall} Y_\mu^* (\phi Y \dot{\wedge} \psi Y))$						
T2	$\dot{\forall} X_\mu^* g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)$	A1, D1, A4, D2 A1, A2, D1, A3, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu^*. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (e$						
A5	$[n_\mu \rightarrow \text{NE}_\mu] \quad 1$						
T3	$[\dot{\forall} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$						

## Automating Scott's proof script

T3: "Necessarily, God exists"  
proved by LEO-II and Satallax

- ▶ in logic: KB
- ▶ from assumptions:
  - ▶ D1, C, T2, D3, A5
- ▶ for domain conditions:
  - ▶ possibilist quantifiers (constant dom.)
  - ▶ actualist quantifiers for individuals (varying dom.)

For logic K we got a countermodel by Nitpick

	HOL encoding	dependencies	logic	status	LEO-II const/varv	Satallax const/varv	Nitpick const/varv
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{=} \dot{\neg}(p \phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \dot{\forall} X_\mu. (\phi X \dot{\wedge} \psi X)) \dot{\wedge} p \psi$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Diamond} \exists X_\mu. \phi X$	A1(○), A2 A1, A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \phi X$		K	THM	0.1/0.1	0.0/5.2	—/—
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\Diamond} \exists X_\mu. g_{\mu \rightarrow \sigma} X]$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} p \phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma}. \lambda X_\mu. \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu. (\phi Y \dot{\wedge} \psi Y))$						
T2	$\dot{\forall} X_\mu. g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)$	A1, D1, A4, D2 A1, A2, D1, A3, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (e^{-\lambda V \dot{\wedge} \dot{\Box} V} - \lambda V)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\Box} \exists X_\mu. g_{\mu \rightarrow \sigma} X]$		K	THM	12.9/14.0	0.0/0.0	—/—
MC	$[s_\sigma \dot{\wedge} \dot{\Box} s_\sigma]$						
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu. (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} \phi X))$						
MT	$\dot{\forall} X_\mu. \dot{\forall} Y_\mu. (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} Y \dot{\wedge} (g_{\mu \rightarrow \sigma} X)))$						
CO	0 (no goal, check for const)						
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma}. \lambda X_\mu. \phi X \dot{\wedge} \dot{\forall} Y_\mu. (\phi Y \dot{\wedge} \psi Y)$						
CO'	0 (no goal, check for const)						

## Automating Scott's proof script

### Summary

- ▶ proof verified and automated
- ▶ KB is sufficient (criticized logic S5 not needed!)
- ▶ possibilist and actualist quantifiers (individuals)
- ▶ exact dependencies determined experimentally
- ▶ ATPs have found alternative proofs  
e.g. self-identity  $\lambda x(x = x)$  is not needed

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p \phi)]$						
A2	$[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \wedge \dot{\square} \dot{\forall} X_\mu^* (\phi X \dot{\wedge} \psi X) \dot{\wedge} p \psi)]$						
T1	$[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\square} \exists X_\mu^* \phi X]$	A1(?) A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu^*. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma}$						
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\square} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$						
A4	$[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\square} p \text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma} \lambda$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma} \lambda$						
T2	$[\dot{\forall} X_\mu^* g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma} \lambda)]$						
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu^*. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma} \lambda)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\square} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$						
<b>Consistency check: Gödel vs. Scott</b>							
<ul style="list-style-type: none"> <li>▶ Scott's assumptions are consistent; shown by Nitpick</li> <li>▶ Gödel's assumptions are inconsistent; shown by LEO-II (new philosophical result?)</li> </ul>							
A1, A2, D1, A3, A4, D2, D3, A5      K      CSA      /      /      o.z./r.s.							
D1, C, T2, D3, A5      KB      THM      0.0/0.1      0.1/5.3      —/—							
A1, A2, D1, A3, A4, D2, D3, A5      KB      THM      —/—      —/—      —/—							
MC      [s_\sigma \dot{\wedge} \dot{\square} s_\sigma]      D2, T2, T3      KB      THM      17.9/—      3.3/3.2      —/—							
FG $[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (\dot{\neg}(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \dot{\wedge} \dot{\neg}(\phi X)))]$ A1, D1      KB      THM      16.5/—      0.0/0.0      —/—							
FG $[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (\dot{\neg}(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \dot{\wedge} \dot{\neg}(\phi X)))]$ A1, A2, D1, A3, A4, D2, D3, A5      KB      THM      12.8/15.1      0.0/5.4      —/—							
MT $[\dot{\forall} X_\mu^* \dot{\forall} Y_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} Y \dot{\wedge} X \dot{\equiv} Y))]$ D1, FG      KB      THM      —/—      0.0/3.3      —/—							
MT $[\dot{\forall} X_\mu^* \dot{\forall} Y_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} Y \dot{\wedge} X \dot{\equiv} Y))]$ A1, A2, D1, A3, A4, D2, D3, A5      KB      THM      —/—      —/—      —/—							
CO $\emptyset$ (no goal, check for consistency)      A1, A2, D1, A3, A4, D2, D3, A5      KB      SAT      —/—      —/—      7.3/7.4							
D2' $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma} \lambda X_\mu^* \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\square} \dot{\forall} Y_\mu^* (\phi Y \dot{\wedge} \psi Y))$ A1(?) A2, D2', D3, A5      KB      UNS      7.5/7.8      —/—      —/—							
CO' $\emptyset$ (no goal, check for consistency)      A1, A2, D1, A3, A4, D2', D3, A5      KB      UNS      —/—      —/—      —/—							

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p \phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \wedge \dot{\square} \dot{\forall} X_\mu^* (\phi X \dot{\neg} \psi X) \dot{\neg} p \psi)$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\neg} \dot{\exists} X_\mu^* \phi X$	A1(2), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2					
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\neg} \dot{\exists} X_\mu^* \phi X$		K	THM	0.1/0.1	0.0/5.2	—/—
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\square} \dot{\exists} X_\mu^* g_{\mu \rightarrow \sigma} X]$						
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\neg} \dot{\square} p$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma} \lambda$						
T2	$\dot{\forall} X_\mu^* g_{\mu \rightarrow \sigma} X \dot{\neg} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} X)$						
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\phi$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$	A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
T3	$[\dot{\square} \dot{\exists} X_\mu^* g_{\mu \rightarrow \sigma} X]$	D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \dot{\neg} \dot{\square} s_\sigma]$	D2, T2, T3 A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	17.9/— /	3.3/3.2 /	—/— —/—
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\neg} (\dot{\neg}(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \dot{\neg} \dot{\neg}(\phi X)))$	A1, D1 A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	16.5/— 12.8/15.1	0.0/0.0 0.0/5.4	—/— —/—
MT	$\dot{\forall} X_\mu^* \dot{\forall} Y_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\neg} (g_{\mu \rightarrow \sigma} Y \dot{\neg} X \dot{\equiv} Y))$	D1, FG A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	—/— —/—	0.0/3.3 —/—	—/— —/—
CO	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma} \lambda X_\mu^* \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\neg} \dot{\square} \dot{\forall} Y_\mu^* (\phi Y \dot{\neg} \psi Y))$	A1(2), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
CO'	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

## Further Results

- ▶ Monotheism holds
- ▶ God is flawless

# Main Findings

	HOL encoding
A1	$[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu \dot{\vdash}$
A2	$[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \dot{\wedge}$
T1	$[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\exists}$
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi$
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$
C	$[\dot{\forall} \exists X_\mu. g_{\mu \rightarrow \sigma} X]$
A4	$[\dot{\forall} \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} p]$
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma}. \lambda$
T2	$[\dot{\forall} X_\mu. g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma})]$
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (e$
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$
T3	$[\dot{\Box} \exists X_\mu. g_{\mu \rightarrow \sigma} X]$

## Modal Collapse (Sobel)

$$\forall \varphi (\varphi \supset \Box \varphi)$$

- ▶ proved by LEO-II and Satallax
- ▶ for possibilist and actualist quantification (ind.)

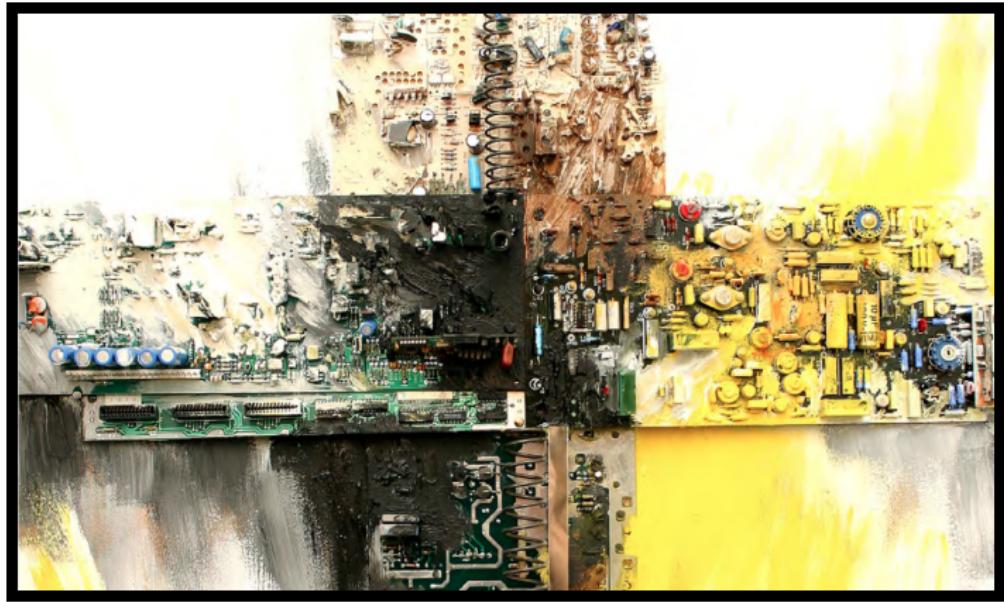
## Main critique on Gödel's ontological proof:

- ▶ there are no contingent truths
- ▶ everything is determined / no free will

MC	$[s_\sigma \dot{\wedge} \dot{\Box} s_\sigma]$	D2, T2, T3 A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	17.9/— —/—	3.3/3.2 —/—	—/— —/—
FG	$[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu. (g_{\mu \rightarrow \sigma} X \supset (\neg(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \supset \neg(\phi X)))]$	A1, D1 A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	16.5/— 12.8/15.1	0.0/0.0 0.0/5.4	—/— —/—
MT	$[\dot{\forall} X_\mu. \dot{\forall} Y_\mu. (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} Y \dot{\wedge} X \dot{\equiv} Y))]$	D1, FG A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	—/— —/—	0.0/3.3 —/—	—/— —/—
CO	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma}. \lambda X_\mu. \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu. (\phi Y \dot{\wedge} \psi Y))$	A1, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
CO'	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

## Observation

- ▶ good performance of ATPs
  - ▶ excellent match between argumentation granularity in papers and the reasoning strength of the ATPs



## Reconstruction of the Inconsistency of Gödel's Axioms

## Inconsistency (Gödel): Proof by LEO-II in KB

\*\*\*\*\* no. of clauses in derivation: 97 \*\*\*\*\*  
\*\*\*\*\* clause counter: 113 \*\*\*\*\*

```
...  
+ SZS status Unsatisfiable for ConsistencyWithoutFirstConjunction02.p : (r:0,axioms:16,ps:3,u:6,ude:false,rleibEq:true,rAndEq:true,use_choice:true,use_extnEq:true,use_extn_combined:true,expand_extnEq:false,foato:e,atp_timeout:25,atp_calls_frequency:10,ordering:none,proof_output:1,clause_count:113,loop_count:8,foato_calls:2,transl_atonleto:foat_full)  
ontoleo:DemandMaterial cbenzmuellers
```

## Inconsistency (Gödel): Reconstruction of Informal Argument (KB)

(special thanks to Chad Brown for a fruitful discussion)

**Axiom A1( $\supset$ )**

$$\forall\phi[P(\neg\phi) \rightarrow \neg P(\phi)]$$

**Axiom A2**

$$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

(special thanks to Chad Brown for a fruitful discussion)

**Axiom A1( $\supset$ )**

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**Theorem 1** Positive Properties are possibly exemplified.  $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$   
by A1( $\supset$ ), A2

# Inconsistency (Gödel): Reconstruction of Informal Argument (KB)

(special thanks to Chad Brown for a fruitful discussion)

**Axiom A1( $\supset$ )**

$$\forall\phi[P(\neg\phi) \rightarrow \neg P(\phi)]$$

**Axiom A2**

$$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

**Theorem 1** Positive Properties are possibly exemplified.  $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$   
by A1( $\supset$ ), A2

**Def. D2\***

$$\phi \text{ ess } x \leftrightarrow \cancel{\Diamond(x)} \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$$

# Inconsistency (Gödel): Reconstruction of Informal Argument (KB)

(special thanks to Chad Brown for a fruitful discussion)

**Axiom A1( $\supset$ )**

$$\forall\phi[P(\neg\phi) \rightarrow \neg P(\phi)]$$

**Axiom A2**

$$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

**Theorem 1** Positive Properties are possibly exemplified.  $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$   
by A1( $\supset$ ), A2

**Def. D2\***

$$\phi \text{ ess } x \leftrightarrow \cancel{\Diamond(x)} \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$$

**Lemma 1** The empty property is an essence of every entity.

$$\forall x(\emptyset \text{ ess } x)$$

by D2\*

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$$NE(x) \leftrightarrow \forall\phi[\phi \text{ ess } x \rightarrow \Box\exists y\phi(y)]$$

**Axiom B**

$$\forall\varphi(\varphi \rightarrow \Box\Diamond\varphi) \quad (\text{resp. } \forall x\forall y(rxy \rightarrow ryx))$$

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by B, D3, Lemma1

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**Axiom A5**

$$P(NE)$$

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**Axiom A5**

$$P(NE)$$

Inconsistency

$$\perp$$

by A5, T1, Lemma2

# Inconsistency (Gödel): Verification in Isabelle/HOL (KB)

The screenshot shows the Isabelle/HOL interface with a theory file named "GoedelGodWithoutConjunctInEss\_KB.thy". The code defines a theory "GoedelGodWithoutConjunctInEss\_KB" that imports "QML". It includes axioms A1a and A2, a theorem T1, a definition of "ess", a lemma Lemmal, a definition of "NE", and an axiomatization A5. The proof for T1 uses metis A1a A2. The proof for Lemma Lemmal uses metis ess\_def. The proof for NE\_def uses metis sym Lemmal NE\_def. The proof for A5 uses metis A5 T1 Lemma2. The interface also shows tabs for Output, Query, Sledgehammer, and Symbols, and a status bar at the bottom.

```
theory GoedelGodWithoutConjunctInEss_KB imports QML
begin
  consts P :: "(μ ⇒ σ) ⇒ σ"
  axiomatization where A1a: "[∀(λΦ. P (λx. m¬ (Φ x)) m→ m¬ (P Φ))]"
    and A2: "[∀(λΦ. ∀(λΨ. (P Φ m∧ □ (∀(λx. Φ x m→ Ψ x))) m→ P Ψ))]"
  -- {* Positive properties are possibly exemplified. *}
  theorem T1: "[∀(λΦ. P Φ m→ ◇ (exists Φ))]" by (metis A1a A2)
  definition ess (infixr "ess" 85) where "Φ ess x = ∀(λΨ. Ψ x m→ □ (forall(λy. Φ y m→ Ψ y)))"
  -- {* The empty property is an essence of every individual. *}
  lemma Lemmal: "[(forall(λx.( λy.λw. False) ess x))]" by (metis ess_def)
  definition NE where "NE x = ∀(λΦ. Φ ess x m→ □ (exists Φ))"
  axiomatization where sym: "x r y —> y r x"
  -- {* Exemplification of necessary existence is not possible. *}
  lemma Lemma2: "[m¬ (◇ (exists NE))]" by (metis sym Lemmal NE_def)
  axiomatization where A5: "[P NE]"
  -- {* Now the inconsistency follows from A5, T1 and Lemma2 *}
  lemma False by (metis A5 T1 Lemma2)
end
```

Output Query Sledgehammer Symbols

11,1 (477/1095)

(isabelle,sidekick,UTF-8-Isabelle) Nro UG 263/347 MB 17:18

# Inconsistency (Gödel): Verification in Isabelle/HOL (K)

The screenshot shows the Isabelle/HOL interface with a proof script for Gödel's Ontological Proof. The script includes definitions of 'ess' and 'NE', an axiomatization with rules A1a and A2, and theorems T1 and Lemmal. It ends with a lemma False and a note about inconsistency. The interface features a toolbar at the top, a file menu, and a sidebar on the right.

```
theory GoedelGodWithoutConjunctInEss_K imports QML
begin
  consts P :: " $(\mu \Rightarrow \sigma) \Rightarrow \sigma$ "
  definition ess (infixr "ess" 85) where " $\Phi \text{ ess } x = \forall(\lambda\Psi. \Psi x \rightarrow \square (\forall(\lambda y. \Phi y \rightarrow \Psi y)))$ "
  definition NE where " $\text{NE } x = \forall(\lambda\Phi. \Phi \text{ ess } x \rightarrow \square (\exists \Phi))$ "
  axiomatization where A1a: "[ $\forall(\lambda\Phi. P (\lambda x. m \rightarrow (\Phi x)) \rightarrow m \rightarrow (P \Phi))$ ]"
    and A2: "[ $\forall(\lambda\Phi. \forall(\lambda\Psi. (P \Phi \wedge \square (\forall(\lambda x. \Phi x \rightarrow \Psi x)) \rightarrow m \rightarrow P \Psi))$ ]"

  -- {* Positive properties are possibly exemplified. *}
  theorem T1: "[ $\forall(\lambda\Phi. P \Phi \rightarrow \diamond (\exists \Phi))$ ]" by (metis A1a A2)

  -- {* The empty property is an essence of every individual. *}
  lemma Lemmal: "[ $(\forall(\lambda x. (\lambda y. \lambda w. \text{False}) \text{ ess } x))$ ]" by (metis ess_def)

  axiomatization where A5: "[P NE]"

  -- {* Now the inconsistency follows from A5, Lemmal, NE_def and T1 *}
  lemma False
    -- {* sledgehammer [remote_leo2] *}
    by (metis A5 Lemmal NE_def T1)
end
```

Output Query Sledgehammer Symbols

21,7 (980/982)

(isabelle,sidekick,UTF-8-Isabelle)Nr o UG 302/343MB 11:37

## Ontologischer Beweis

Feb. 10, 1970

$P(\varphi)$   $\varphi$  is positive ( $\Leftrightarrow \varphi \in P$ )

Ax 1:  $P(p), P(\varphi) \supset P(\varphi \wedge p)$  • Ax 2:  $P(p) \supset P(\neg p)$

P1  $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$  (God)

P2  $\varphi_{\text{Ex} \exists x} \equiv (\psi)[\psi(x) \supset N(y)[P(y) \supset \psi(y)]]$  (Existence)

$P \supset_N q = N(p \supset q)$  Necessity

Ax 2  $P(\varphi) \supset N P(\varphi)$  } because it follows  
 $\neg P(\varphi) \supset N \neg P(\varphi)$  } from the nature of the property

Th.  $G(x) \supset G_{\text{Ex} \exists x}$

Df.  $E(x) \equiv P[\varphi_{\text{Ex} \exists x} \supset N \neg x \varphi(x)]$  necessary Existence

Ax 3  $P(E)$

Th.  $G(x) \supset N(\exists y) G(y)$

hence  $(\exists x) G(x) \supset N(\exists y) G(y)$

"  $M(x) G(x) \supset M N(\exists y) G(y)$  M = possibility  
 $\Rightarrow N(\exists y) G(y)$

any two elements of X are nec. equivalent

exclusive or \* and for any number of them

$M(x) G(x)$  means all pos. prop. w.r.t. com-  
patible. This is true because of:

Ax 4:  $P(\varphi) \cdot q \supset_N \psi \supset P(\psi)$  which impl.

$\begin{cases} x=x & \text{is positive} \\ x \neq x & \text{is negative} \end{cases}$

But if a system S of pos. prop. were incons.  
 $\neg$  it would mean that the non-prop. S (which  
is positive) would be  $x \neq x$

Positive means positive in the moral aest.  
sense (independently of the accidental structure of  
the world). Only  $\neg$  in the ex. True. It is  
also meant "Attribution" as opposed to "negation"  
(or containing negation). This interprets the word

$\neg$  as "non-existent":  $(x) \neg P(x)$  Otherwise:  $P(x) \supset x \neq x$

hence  $x \neq x$  positive not  $x=x$  negative. At  
least the ex. of prop. At

ax. i.e. the normal form in terms of elem. prop. contains  
members without negation.

## Ontologischer Beweis

Feb. 10, 1970

$P(\phi)$   $\phi$  is positive ( $\Leftrightarrow \phi \in P$ )

At 1:  $P(\phi), P(\psi) \supset P(\phi \wedge \psi)$  • At 2:  $P(\phi) \supset P(\neg \phi)$

P1  $G(x) \equiv (\phi) [P(\phi) \supset \phi(x)]$  (God)

P2  $\phi \text{ Emph.} x \equiv (\psi) [\psi(x) \supset N(y)[P(y) \supset \psi(y)]]$  (Emph. of  $x$ )

P =  $N$   $N(P \circ g)$  Necessity

At 2  $P(\phi) \supset N P(\phi)$  } because it follows  
 $\neg P(\phi) \supset N \neg P(\phi)$  } from the nature of the  
 property

Th.  $G(x) \supset G \text{ Emph.} x$

Def.  $E(x) \equiv P[\phi \text{ Emph.} x \supset N \neg x \phi(x)]$  necessary Existence

At 3  $P(E)$

Th.  $G(x) \supset N(\exists y) G(y)$

then  $(\exists x) G(x) \supset N(\exists y) G(y)$

"  $M(x) G(x) \supset M N(\exists y) G(y)$

"  $\supset N(\exists y) G(y)$

any two instances of  $x$  are nec. equivalent

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At 4:  $P(\phi), \phi \supset \psi \supset P(\psi)$  which impl.

$\begin{cases} x=x & \text{is positive} \\ x \neq x & \text{is negative} \end{cases}$

But if a system S of pos. prop. were incon-  
 sistent it would mean that the non-prop. S (which  
 is positive) would be  $x \neq x$

Positive means positive in the moral aesthe-  
 tic sense (independently of the accidental structure of  
 the world). Only  $\neg$  in the at. sense. It is  
 also pure.

## Inconsistency

Scott

$\forall \phi [P(\neg \phi) \rightarrow \neg P(\phi)]$

A1(▷)

$\forall \phi \forall \psi [(P(\phi) \wedge \square \forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

A2

$\phi \text{ ess } x \leftrightarrow \forall \psi (\psi(x) \rightarrow \square \forall y (\phi(y) \rightarrow \psi(y)))$

D2\*

$NE(x) \leftrightarrow \forall \phi [\phi \text{ ess } x \rightarrow \square \exists y \phi(y)]$

D3

$P(NE)$

A5

Gödel's version	K		KB		S5	
	constant	varying	constant	varying	constant	varying
Consistency	✗	✗	✗	✗	✗	✗
T1	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
C	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
T2	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
T3	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
Flawless God	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
Monotheism	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
Modal Collapse	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete

## Further logic details

- ▶ Henkin semantics
- ▶ full comprehension
- ▶ rigid constant symbols

Question:

Has this inconsistency be reported before?

If not, then LEO-II deserves (part of) the credit!

Scott's version	K		KB		S5	
	constant	varying	constant	varying	constant	varying
Consistency	✓	✓	✓	✓	✓	✓
T1	✓	✓	✓	✓	✓	✓
C	✓	✓	✓	✓	✓	✓
T2	✓	✓	✓	✓	✓	✓
T3	✗	✗	✓	✓	✓	✓
Flawless God	✓	✓	✓	✓	✓	✓
Monotheism	✓	✓	✓	✓	✓	✓
Modal Collapse	✓	✓	✓	✓	✓	✓

## Further logic details

- ▶ Henkin semantics
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- ▶ rigid constant symbols

## SOME EMENDATIONS OF GÖDEL'S ONTOLOGICAL PROOF

C. Anthony Anderson

Kurt Gödel's version of the ontological argument was shown by J. Howard Sobel to be defective, but some plausible modifications in the argument result in a version which is immune to Sobel's objection. A definition is suggested which permits the proof of some of Gödel's axioms.

## Der Mathematiker und die Frage der Existenz Gottes (betreffend Gödels ontologischen Beweis)

Es ist gut, daß wir nicht wissen,  
andern glauben, daß ein Gott sei.  
(Kant, Nachleid)

### 1. Einführung

Gödels zu Lebzeiten unveröffentlichter Beweis für die notwendige Existenz eines Gott-ähnlichen Wesens hat sowohl philosophisches als auch mathematisches Interesse geweckt. Zweck der vorliegenden Arbeit ist es, zu einer Deutung des Gödelischen Textes beizutragen, durch eine Emendierung des einschlägigen Literatur und 2. durch Beiträge von verschiedenen Modelltheoreten. Die Arbeit enthält keinen philosophischen Beitrag. Anlässlich der letzten Jahre habe ich etliche Male über Gödels Ontologie vorgetragen, insbesondere auf dem Symposium zur Freiheit von Professor Gerl Müller (Heidelberg, Januar 1997), doch habe ich niemals beobachtigt, eine Voröffentlichung über die Theorie zu machen. Da ich wiederholt um eine schriftliche Version gebeten wurde, entschloß ich mich, schnell eine „geweihte Kurzfassung“<sup>1</sup> zu schreiben, ohne aus ihr einen

### Gödel's Ontological Proof Revisited \*

C. Anthony Anderson and Michael Gettings  
University of California, Santa Barbara  
Department of Philosophy

Gödel's version of the modal ontological argument for the existence of God has been criticized by J. Howard Sobel [5] and modified by C. Anthony Anderson [1]. In the present paper we consider the extent to which Anderson's emendation is defeated by the type of objection first offered by the Monk Gaumilo to St. Anselm's original Ontological Argument. And we try to push the analysis of this Gödelian argument a bit further to bring it into closer agreement with the details of Gödel's own formulation. Finally, we indicate what seems to be the main weakness of this emendation of Gödel's attempted proof.

PETR HÁJEK

## A New Small Emendation of Gödel's Ontological Proof

Keywords: Ontological proof, Gödel, modal logic, comprehension, positive properties.

### 1. Introduction

Gödel's ontological proof of necessary existence of a godlike being was finally published in the third volume of Gödel's collected works [7]; but it became known in 1970 when Gödel showed the proof to Dana Scott and Scott presented it (in fact a variant of it) at a seminar at Princeton. Detailed history is found in Adams' introductory remarks to the ontological proof in [7]. The proof uses modal logic and its analysis is an exciting exercise in systems of formal modal logic. Needless to say, formal modal logic has found several

## Magari and others on Gödel's ontological proof

Petr Hájek

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### 1 Introduction

This paper is a continuation of my paper [H] and concentrates almost exclusively to mathematical properties of logical systems underlying Gödel's ontological proof [G] and its variants by Anderson [A], with special care paid to Magari's criticism [M]. Since [H] is written in German, we shall try to summarize its content in such a way that knowledge of [H] will be not obligatory for reading the present paper (even it remains advantageous). Here we describe

## Understanding Gödel's Ontological Argument

FRODE BJØRDAL

In 1970 Kurt Gödel, in a hand-written note entitled "Ontologischer Beweis", put forward an ontological argument for the existence of God, making use of second-order modal logical principles. Let the second-order formula  $P(F)$  stand for "the property F is positive", and let "God" signify the property of being God-like. Gödel presupposes the following definitions:

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# **Der Mathematiker und die Frage der Existenz Gottes**

(betreffend Gödels ontologischen Beweis)

sondern glauben, daß ein Gott sei.  
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Gödel zu Lebzeiten eindeutiger Beweis für die notwendige Existenz eines Gödel-Beweises? Wieso hat solches philosophisches als auch mathematisches Interesse geweckt und warum? Von den liegenden Möglichkeiten ist die von Deutsch (1999) vorgeschlagene Lösung durch den Kontakt mit der englischen Philosophie und Literatur zu unterscheiden. Eine andere Begründung besteht in der Darstellung von etwas Modelltheorie. Die Arbeit enthält keinen philosophischen Beitrag. Während der letzten Jahre habe ich etliche Male über Gödel-Gottesbeweise vorgetragen, insbesondere auf dem Symposium zur Feier von Professor Gerhard Heinzberg (Heidelberg, Januar 1999), doch habe ich daraus beziehungsweise eine Veröffentlichung das Thema zu machen. Da ich wiederkelte um eine schriftliche Veröffentlichung zu erhalten, entschuldigt mich, schnell eine „erweiternte Kurzfassung“<sup>1</sup> zu schreiben, über was ich gehalten wurde.

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## Magari and others on Gödel's ontological proof

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## Computer-supported Clarification of Controversy see next talk by Bruno Wolzenlogel-Paleo

## Overall Achievements

- ▶ significant contribution towards a **Computational Metaphysics**
- ▶ **novel results** contributed by **HOL-ATPs**
- ▶ infrastructure can be adapted for **other logics and logic combinations**
- ▶ **our technology is sufficiently mature** for use by philosophers

## Relevance (wrt foundations and applications)

- ▶ Philosophy, AI, Computer Science, Computational Linguistics, Maths

## Little related work: only for Anselm's simpler argument

- ▶ first-order ATP PROVER9 [OppenheimerZalta, 2011]
- ▶ interactive proof assistant PVS [Rushby, 2013]

## Ongoing/Future work

see next talk by Bruno Woltzenlogel-P.

- ▶ Landscape of verified/falsified ontological arguments
- ▶ You may consider to contribute:  
<https://github.com/FormalTheology/GoedelGod.git>

### (Interim) Culmination of two decades of related own research

- ▶ Theory of classical higher-order logic (HOL) (since 1995)
- ▶ Automation of HOL / own LEO provers (since 1998)
- ▶ Integration of interactive and automated theorem proving (since 1999)
- ▶ International TPTP infrastructure for HOL (since 2006)
- ▶ HOL as a universal logic via semantic embeddings (since 2008)
- ▶ jww Bruno Woltzenlogel-Paleo:  
**Application in Metaphysics: Ontological Argument** (since 2013)

... success story (despite strong criticism/opposition on the way!) ...  
... huge media attention ...

### (Interim) Own standpoint

- ▶ I am not fully convinced (yet) by the ontological argument.
- ▶ However, it seems to me that **the belief in a (God-like) supreme being is at least not necessarily irrational/inconsistent.**

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## Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Von Tobias Hüller



Kurt Gödel (um das Jahr 1935): Der Mathematiker hält seinen Gottesbeweis Jahrzehntlang geheim

Ein Wesen existiert, das alle positiven Eigenschaften in sich vereint. Das bewies der legendäre Mathematiker Kurt Gödel mit einem komplizierten Formelgebilde. Zwei Wissenschaftler haben diesen Gottesbeweis nun überprüft - und für gültig befunden.

Montag, 09.09.2013 – 12:03 Uhr  
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English Site | Germany > Science > Scientists Use Computer to Mathematically Prove Gödel God Theorem

**Holy Logic: Computer Scientists 'Prove' God Exists**

By David Knight



Austrian mathematician Kurt Gödel kept his proof of God's existence a secret for decades. Now two scientists say they have proven it mathematically using a computer.

Two scientists have formalized a theorem regarding the existence of God penned by mathematician Kurt Gödel. But the God angle is somewhat of a red herring -- the real step forward is the example it sets of how computers can make scientific progress simpler.

## Germany

- Telepolis & Heise
- Spiegel Online
- FAZ
- Die Welt
- Berliner Morgenpost
- Hamburger Abendpost
- ...

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The image shows the front page of the German newspaper 'DIE WELT' from Friday, October 18, 2013. The masthead 'DIE WELT' is prominently displayed at the top left, with a small globe icon between 'DIE' and 'WELT'. Below the masthead is a large photograph of three people: a man in a suit, a man in a blue shirt, and a woman in a dark jacket, all smiling and taking a selfie with a smartphone. To the left of the photo is a column titled 'Zippert zappt' with several short articles. To the right is a column titled 'Teuer erkaufte Harmonie' with an article about political negotiations. The main headline 'Mindestlohn lockt SPD in dritte Ehe mit der Union' is in the center. Below it, a sub-headline reads 'Mittwoch soll über eine große Koalition verhandelt werden. Votum des SPD-Konvents steht aus'. The page also features sections for 'Sport', 'Wirtschaft', 'Karriere', 'Aus aller Welt', 'Das Innere', and 'Wetter'. At the bottom right is an advertisement for 'UNION GLASHÜTTE/SA' featuring a wristwatch.

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# Researchers say they used MacBook to prove Goedel's God theorem

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