Deus Ex Machina
Computer-Assisted Formalization of Gödel's Proof of God's Existence

Christoph Benzmüller and **Bruno Woltzenlogel Paleo**

6th of November 2013

$$\dfrac{\dfrac{\text{Axiom 3}}{P(G)} \qquad \dfrac{\dfrac{\text{Theorem 1}}{\forall\varphi.[P(\varphi)\to\Diamond\exists x.\varphi(x)]}}{P(G)\to\Diamond\exists x.G(x)}\,\forall_E}{\Diamond\exists x.G(x)}\,\to_E$$

A gift to **Priest Edvaldo** and his church in Piracicaba, Brazil

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Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Von Tobias Hürter

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Kurt Gödel (um das Jahr 1935): Der Mathematiker hielt seinen Gottesbeweis jahrzehntelang geheim

Ein Wesen existiert, das alle positiven Eigenschaften in sich vereint. Das bewies der legendäre Mathematiker Kurt Gödel mit einem komplizierten Formelgebilde. Zwei Wissenschaftler haben diesen Gottesbeweis nun überprüft - und für gültig befunden.

Montag, 09.09.2013 – 12:03 Uhr

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Jetzt sind die letzten Zweifel ausgeräumt: Gott existiert tatsächlich. Ein Computer hat es mit kalter Logik bewiesen - das MacBook des Computerwissenschaftlers Christoph Benzmüller von der Freien Universität Berlin.

Germany
- Telepolis & Heise
- Spiegel Online
- FAZ
- Die Welt
- Berliner Morgenpost
- Hamburger Abendpost

Austria
- Die Presse
- Wiener Zeitung
- ORF

International
- Spiegel International
- Yahoo Finance
- United Press Intl.

India
- DNA India
- Delhi Daily News
- India Today

US
- ABC News

Italy
- Repubblica

Spain, Russia, Brazil, Bulgaria

. . .

… Anselm v. C. Gaunilo … Th. Aquinas …… Descartes Spinoza Leibniz … Hume Kant … Hegel … Frege … Hartshorne Malcolm Lewis Plantinga Gödel …

Anselm's notion of God:

"God is that, than which nothing greater can be conceived."

Gödel's notion of God:

"A God-like being possesses all 'positive' properties."

To show by logical reasoning:

"(Necessarily) God exists."

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Wohl eine jede Philosophie kreist um den ontologischen Gottesbeweis

(Adorno, Th. W.: Negative Dialektik. Frankfurt a. M. 1966, p.378)

Motivations

- Philosophical:
  - Limits of metaphysics & epistemology
  - *Metaphysical* versus *logical* necessary existence
- Theological:
  - Investigations of the nature of God
  - Arguments to convince atheists
- Computational: can automated reasoners be used ...
  - ...to formalize the definitions, axioms and theorems?
  - ...to verify the arguments step-by-step?
  - ...to fully automate (sub-)arguments?

"Computer-assisted Theoretical Philosophy"

Ontologischer Beweis Feb 10, 1970

$P(\varphi)$ φ is positive (or $\varphi \in P$)

At. 1 $P(\varphi).P(\psi) \supset P(\varphi.\psi)$ At 2 $P(\varphi) \vee P(\sim\varphi)$

P1 $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (God)

P2 $\varphi \text{ Ess } x \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Essence of x)

$p \supset_N q = N(p \supset q)$ Necessity

At 2 $P(\varphi) \supset NP(\varphi)$, $\sim P(\varphi) \supset N \sim P(\varphi)$ } because it follows from the nature of the property

Th. $G(x) \supset G \text{ Ess. } x$

Df. $E(x) \equiv (\varphi)[\varphi \text{ Ess } x \supset N \exists x\, \varphi(x)]$ necessary Existence

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists y)\, G(y)$

hence $(\exists x)\, G(x) \supset N(\exists y)\, G(y)$

" $M(\exists x)\, G(x) \supset MN(\exists y)\, G(y)$

" $\supset N(\exists y)\, G(y)$ M = possibility

any two essences of x are nec. equivalent

exclusive or • and for any number of summands

$M(\exists x)\, G(x)$ means the system of all pos. props. is compatible. This is true because of:

At 4: $P(\varphi).\varphi \supset_N \psi :\supset P(\psi)$ which impl

~~hence~~ { $x = x$ is positive

~~and~~ { $x \neq x$ is negative

But if a system S of pos. props. were incom

It would mean that the sum prop. s (which is positive) would be $x \neq x$

Positive means positive in the moral aesth. sense (independently of the accidental structure of the world). Only then the ax. true. It m. also mean pure "attribution"* as opposed to "privation" (or containing privation). This interpret. simpler proof

If φ pos. not: $(x)\, N \sim\varphi(x)$. Otherwise: $\varphi(x) \supset_N x \neq x$

hence $x \neq x$ positive so $x = x$ neg. contrary Ax or the exist. of pos. props.

* i.e. the disj. normal form in terms of elem. prop. contains a member without negation.

A1 Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

A2 A property necessarily implied by a positive property is positive: $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

T1 Positive properties are possibly exemplified: $\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$

D1 A *God-like* being possesses all positive properties: $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$

A3 The property of being God-like is positive: $P(G)$

C Possibly, God exists: $\Diamond\exists x G(x)$

A4 Positive properties are necessarily positive: $\forall\phi[P(\phi) \rightarrow \Box\, P(\phi)]$

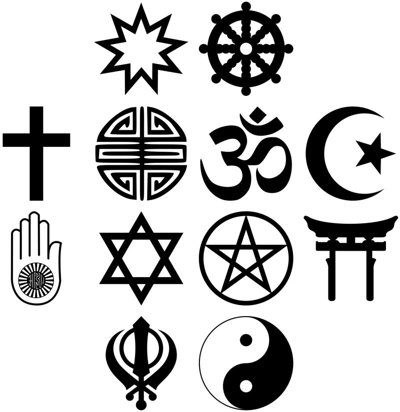
D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi\ ess.\ x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

T2 Being God-like is an essence of any God-like being: $\forall x[G(x) \rightarrow G\ ess.\ x]$

D3 *Necessary existence* of an individual is the necessary exemplification of all its essences: $NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$

A5 Necessary existence is a positive property: $P(NE)$

T3 Necessarily, God exists: $\Box\exists x G(x)$



Proof Overview

Proof Overview

T3: $\Box \exists x. G(x)$

Proof Overview

$$\frac{\textbf{C1: } \Diamond \exists z. G(z)}{\textbf{T3: } \Box \exists x. G(x)}$$

Proof Overview

$$\frac{\textbf{C1: } \Diamond \exists z. G(z) \qquad \textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\textbf{T3: } \Box \exists x. G(x)}$$

Proof Overview

L2: $\Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$

$$\frac{\textbf{C1: } \Diamond \exists z. G(z) \qquad \textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\textbf{T3: } \Box \exists x. G(x)}$$

$$\dfrac{\dfrac{\textbf{S5}}{\overline{\forall}\xi.[\overline{\Diamond}\,\overline{\Box}\xi\ \overline{\rightarrow}\ \overline{\Box}\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

$$\frac{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x) \qquad\qquad \dfrac{\mathbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\mathbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\frac{\mathbf{C1:}\ \Diamond\exists z.G(z) \qquad \mathbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\mathbf{T3:}\ \Box\exists x.G(x)}$$

$$\dfrac{\dfrac{\textbf{L1: } \exists z.G(z) \rightarrow \Box\exists x.G(x)}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1:}\ G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$$

$$\dfrac{\dfrac{\textbf{L1:}\ \exists z.G(z) \rightarrow \Box\exists x.G(x)}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall \xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1:}\ \Diamond\exists z.G(z) \qquad \textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3:}\ \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1: } G(x) \equiv \forall\varphi.[P(\varphi) \to \varphi(x)]$$

$$\textbf{D3*: } NE(x) \equiv \Box\exists y.G(y)$$

$$\dfrac{\dfrac{\dfrac{P(NE)}{\textbf{L1: } \exists z.G(z) \to \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \to \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \to \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

D1: $G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$

D3*: $NE(x) \equiv \Box \exists y.G(y)$ (cheating!)

$$\dfrac{\dfrac{\dfrac{P(NE)}{\textbf{L1: } \exists z.G(z) \rightarrow \Box \exists x.G(x)}}{\Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall \xi.[\Diamond \Box \xi \rightarrow \Box \xi]}}{\textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond \exists z.G(z) \qquad \textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\textbf{T3: } \Box \exists x.G(x)}$$

D1: $G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$

D3*: $NE(x) \equiv \Box\exists y.G(y)$

D3: $NE(x) \equiv \forall\varphi.[\varphi \; ess \; x \rightarrow \Box\exists y.\varphi(y)]$

$$
\dfrac{\dfrac{\dfrac{\textbf{T2: } \forall y.[G(y) \rightarrow G \; ess \; y] \qquad P(NE)}{\textbf{L1: } \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}
$$

$$
\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}
$$

Proof Overview

$$\mathbf{D1:}\ G(x) \equiv \forall\varphi.[P(\varphi) \to \varphi(x)]$$

$$\mathbf{D3^*:}\ NE(x) \equiv \Box\exists y.G(y) \qquad\qquad \mathbf{D3:}\ NE(x) \equiv \forall\varphi.[\varphi\ ess\ x \to \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\dfrac{\mathbf{T2:}\ \forall y.[G(y) \to G\ ess\ y] \qquad \dfrac{\mathbf{A5}}{P(NE)}}{\mathbf{L1:}\ \exists z.G(z) \to \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \to \Diamond\Box\exists x.G(x)} \qquad \dfrac{\mathbf{S5}}{\forall\xi.[\Diamond\Box\xi \to \Box\xi]}}{\mathbf{L2:}\ \Diamond\exists z.G(z) \to \Box\exists x.G(x)}$$

$$\dfrac{\mathbf{C1:}\ \Diamond\exists z.G(z) \qquad \mathbf{L2:}\ \Diamond\exists z.G(z) \to \Box\exists x.G(x)}{\mathbf{T3:}\ \Box\exists x.G(x)}$$

$$\textbf{D1:}\ G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$$

$$\textbf{D3*:}\ NE(x) \equiv \Box\exists y.G(y) \qquad \textbf{D3:}\ NE(x) \equiv \forall\varphi.[\varphi\ ess\ x \rightarrow \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\textbf{T2:}\ \forall y.[G(y) \rightarrow G\ ess\ y] \qquad \dfrac{\textbf{A5}}{P(NE)}}{\dfrac{\textbf{L1:}\ \exists z.G(z) \rightarrow \Box\exists x.G(x)}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)}} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1:}\ \Diamond\exists z.G(z) \qquad \textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3:}\ \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1:}\ G(x) \equiv \forall\varphi.[P(\varphi) \to \varphi(x)]$$

$$\textbf{D2:}\ \varphi\ ess\ x \equiv \varphi(x) \wedge \forall\psi.(\psi(x) \to \Box\forall x.(\varphi(x) \to \psi(x)))$$

$$\textbf{D3*:}\ NE(x) \equiv \Box\exists y.G(y) \qquad \textbf{D3:}\ NE(x) \equiv \forall\varphi.[\varphi\ ess\ x \to \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{A1b}}{\forall\varphi.[\neg P(\varphi) \to P(\neg\varphi)]} \quad \dfrac{\textbf{A4}}{\forall\varphi.[P(\varphi) \to \Box P(\varphi)]}}{\textbf{T2:}\ \forall y.[G(y) \to G\ ess\ y]} \quad \dfrac{\textbf{A5}}{P(NE)}}{\textbf{L1:}\ \exists z.G(z) \to \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \to \Diamond\Box\exists x.G(x)} \quad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \to \Box\xi]}}{\textbf{L2:}\ \Diamond\exists z.G(z) \to \Box\exists x.G(x)}}{}$$

$$\dfrac{\textbf{C1:}\ \Diamond\exists z.G(z) \qquad \textbf{L2:}\ \Diamond\exists z.G(z) \to \Box\exists x.G(x)}{\textbf{T3:}\ \Box\exists x.G(x)}$$

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$$\textbf{D1:}\ G(x) \equiv \forall\varphi.[P(\varphi) \to \varphi(x)]$$

$$\textbf{D2:}\ \varphi\ ess\ x \equiv \varphi(x) \wedge \forall\psi.(\psi(x) \to \Box\forall x.(\varphi(x) \to \psi(x)))$$

$$\textbf{D3*:}\ NE(x) \equiv \Box\exists y.G(y) \qquad \textbf{D3:}\ NE(x) \equiv \forall\varphi.[\varphi\ ess\ x \to \Box\exists y.\varphi(y)]$$

$$\textbf{C1:}\ \Diamond\exists z.G(z)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\textbf{A1b}}{\overline{\forall\varphi.[\neg P(\varphi) \to P(\neg\varphi)]}} \quad \overset{\textbf{A4}}{\overline{\forall\varphi.[P(\varphi) \to \Box P(\varphi)]}}}{\textbf{T2:}\ \forall y.[G(y) \to G\ ess\ y]} \quad \overset{\textbf{A5}}{\overline{P(NE)}}}{\textbf{L1:}\ \exists z.G(z) \to \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \to \Diamond\Box\exists x.G(x)} \quad \overset{\textbf{S5}}{\overline{\forall\xi.[\Diamond\Box\xi \to \Box\xi]}}}{\textbf{L2:}\ \Diamond\exists z.G(z) \to \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1:}\ \Diamond\exists z.G(z) \qquad \textbf{L2:}\ \Diamond\exists z.G(z) \to \Box\exists x.G(x)}{\textbf{T3:}\ \Box\exists x.G(x)}$$

$$\textbf{D1:}\ G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$$

$$\textbf{D2:}\ \varphi\ ess\ x \equiv \varphi(x) \wedge \forall\psi.(\psi(x) \rightarrow \Box\forall x.(\varphi(x) \rightarrow \psi(x)))$$

$$\textbf{D3*:}\ NE(x) \equiv \Box\exists y.G(y) \qquad \textbf{D3:}\ NE(x) \equiv \forall\varphi.[\varphi\ ess\ x \rightarrow \Box\exists y.\varphi(y)]$$

$$\dfrac{P(G)}{\textbf{C1:}\ \Diamond\exists z.G(z)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall\varphi.[\neg P(\varphi) \rightarrow P(\neg\varphi)]}\textbf{A1b} \quad \dfrac{}{\forall\varphi.[P(\varphi) \rightarrow \Box P(\varphi)]}\textbf{A4}}{\textbf{T2:}\ \forall y.[G(y) \rightarrow G\ ess\ y]} \quad \dfrac{}{P(NE)}\textbf{A5}}{\textbf{L1:}\ \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \quad \dfrac{}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}\textbf{S5}}{\textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1:}\ \Diamond\exists z.G(z) \qquad \textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3:}\ \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1: } G(x) \equiv \forall\varphi.[P(\varphi) \to \varphi(x)]$$

$$\textbf{D2: } \varphi\ ess\ x \equiv \varphi(x) \wedge \forall\psi.(\psi(x) \to \Box\forall x.(\varphi(x) \to \psi(x)))$$

$$\textbf{D3*: } NE(x) \equiv \Box\exists y.G(y) \qquad \textbf{D3: } NE(x) \equiv \forall\varphi.[\varphi\ ess\ x \to \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\textbf{A3}}{P(G)}}{\textbf{C1: } \Diamond\exists z.G(z)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\textbf{A1b}}{\forall\varphi.[\neg P(\varphi) \to P(\neg\varphi)]} \quad \dfrac{\textbf{A4}}{\forall\varphi.[P(\varphi) \to \Box P(\varphi)]}}{\textbf{T2: } \forall y.[G(y) \to G\ ess\ y]} \quad \dfrac{\textbf{A5}}{P(NE)}}{\dfrac{\textbf{L1: } \exists z.G(z) \to \Box\exists x.G(x)}{\Diamond\exists z.G(z) \to \Diamond\Box\exists x.G(x)}} \quad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \to \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1:}\ G(x) \equiv \forall\varphi.[P(\varphi) \to \varphi(x)]$$

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$$\textbf{D3*:}\ NE(x) \equiv \Box\exists y.G(y) \qquad \textbf{D3:}\ NE(x) \equiv \forall\varphi.[\varphi\ ess\ x \to \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\textbf{A3}}{P(G)} \qquad \textbf{T1:}\ \forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]}{\textbf{C1:}\ \Diamond\exists z.G(z)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{A1b}}{\forall\varphi.[\neg P(\varphi) \to P(\neg\varphi)]} \qquad \dfrac{\textbf{A4}}{\forall\varphi.[P(\varphi) \to \Box P(\varphi)]}}{\textbf{T2:}\ \forall y.[G(y) \to G\ ess\ y]} \qquad \dfrac{\textbf{A5}}{P(NE)}}{\textbf{L1:}\ \exists z.G(z) \to \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \to \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \to \Box\xi]}}{\textbf{L2:}\ \Diamond\exists z.G(z) \to \Box\exists x.G(x)}$$

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$$\dfrac{\dfrac{\textbf{A3}}{P(G)} \qquad \dfrac{\dfrac{\textbf{A2}}{\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]} \qquad \dfrac{\textbf{A1a}}{\forall\varphi.[P(\neg\varphi) \rightarrow \neg P(\varphi)]}}{\textbf{T1:}\ \forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]}}{\textbf{C1:}\ \Diamond\exists z.G(z)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{A1b}}{\forall\varphi.[\neg P(\varphi) \rightarrow P(\neg\varphi)]} \qquad \dfrac{\textbf{A4}}{\forall\varphi.[P(\varphi) \rightarrow \Box\, P(\varphi)]}}{\textbf{T2:}\ \forall y.[G(y) \rightarrow G\ ess\ y]} \qquad \dfrac{\textbf{A5}}{P(NE)}}{\textbf{L1:}\ \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1:}\ \Diamond\exists z.G(z) \qquad \textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3:}\ \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1: } G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$$

$$\textbf{D2: } \varphi\ ess\ x \equiv \varphi(x) \wedge \forall\psi.(\psi(x) \rightarrow \Box\forall x.(\varphi(x) \rightarrow \psi(x)))$$

$$\textbf{D3: } NE(x) \equiv \forall\varphi.[\varphi\ ess\ x \rightarrow \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\textbf{A3}}{P(G)} \qquad \dfrac{\dfrac{\textbf{A2}}{\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]} \qquad \dfrac{\textbf{A1a}}{\forall\varphi.[P(\neg\varphi) \rightarrow \neg P(\varphi)]}}{\textbf{T1: } \forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]}}{\textbf{C1: } \Diamond\exists z.G(z)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{A1b}}{\forall\varphi.[\neg P(\varphi) \rightarrow P(\neg\varphi)]} \qquad \dfrac{\textbf{A4}}{\forall\varphi.[P(\varphi) \rightarrow \Box\, P(\varphi)]}}{\textbf{T2: } \forall y.[G(y) \rightarrow G\ ess\ y]} \qquad \dfrac{\textbf{A5}}{P(NE)}}{\textbf{L1: } \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Modal Natural Deduction Proof

Natural Deduction Calculus

$$\frac{A \vee B \quad \begin{array}{c}\overline{A}\\ \vdots \\ C\end{array} \quad \begin{array}{c}\overline{B}\\ \vdots \\ C\end{array}}{C}\ \vee_E \qquad \frac{A \quad B}{A \wedge B}\ \wedge_I \qquad \frac{\begin{array}{c}\overline{A}\ n\\ \vdots \\ B\end{array}}{A \longrightarrow B}\ \longrightarrow_I^n$$

$$\frac{A}{A \vee B}\ \vee_{I_1} \qquad \frac{A \wedge B}{A}\ \wedge_{E_1} \qquad \frac{B}{A \longrightarrow B}\ \longrightarrow_I$$

$$\frac{B}{A \vee B}\ \vee_{I_2} \qquad \frac{A \wedge B}{B}\ \wedge_{E_2} \qquad \frac{A \quad A \longrightarrow B}{B}\ \longrightarrow_E$$

$$\frac{A[\alpha]}{\forall x.A[x]}\ \forall_I \qquad \frac{\forall x.A[x]}{A[t]}\ \forall_E \qquad \frac{A[t]}{\exists x.A[x]}\ \exists_I \qquad \frac{\exists x.A[x]}{A[\beta]}\ \exists_E$$

$$\neg A \equiv A \longrightarrow \bot \qquad \frac{\neg\neg A}{A}\ \neg\neg_E$$

$$\dfrac{\alpha : \boxed{\begin{array}{c}\vdots \\ A\end{array}}}{\Box A}\ \Box_I \qquad\qquad \dfrac{\Box A}{t : \boxed{\begin{array}{c}A \\ \vdots\end{array}}}\ \Box_E$$

$$\Diamond A \equiv \neg\Box\neg A$$

$$\dfrac{t : \boxed{\begin{array}{c}\vdots \\ A\end{array}}}{\Diamond A}\ \Diamond_I \qquad\qquad \dfrac{\Diamond A}{\beta : \boxed{\begin{array}{c}A \\ \vdots\end{array}}}\ \Diamond_E$$

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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{A2}}{\forall\varphi.\forall\psi.[(P(\varphi)\wedge\Box\forall x.[\varphi(x)\rightarrow\psi(x)])\rightarrow P(\psi)]}}{\forall\psi.[(P(\rho)\wedge\Box\forall x.[\rho(x)\rightarrow\psi(x)])\rightarrow P(\psi)]}\ \forall_E}{(P(\rho)\wedge\Box\forall x.[\rho(x)\rightarrow\neg\rho(x)])\rightarrow P(\neg\rho)}\ \forall_E}{(P(\rho)\wedge\Box\forall x.[\neg\rho(x)])\rightarrow P(\neg\rho)}\qquad \dfrac{\dfrac{\textbf{A1a}}{\forall\varphi.[P(\neg\varphi)\rightarrow\neg P(\varphi)]}}{P(\neg\rho)\rightarrow\neg P(\rho)}\ \forall_E}{(P(\rho)\wedge\Box\forall x.[\neg\rho(x)])\rightarrow\neg P(\rho)}}{P(\rho)\rightarrow\Diamond\exists x.\rho(x)}}{\textbf{T1: }\forall\varphi.[P(\varphi)\rightarrow\Diamond\exists x.\varphi(x)]}\ \forall_I
$$

$$
\dfrac{\dfrac{\textbf{A3}}{P(G)}\qquad \dfrac{\dfrac{\textbf{T1}}{\forall\varphi.[P(\varphi)\rightarrow\Diamond\exists x.\varphi(x)]}}{P(G)\rightarrow\Diamond\exists x.G(x)}\ \forall_E}{\Diamond\exists x.G(x)}\ \rightarrow_E
$$

$$
\dfrac{\dfrac{\dfrac{\psi(x)^6 \qquad \dfrac{\Pi_2}{\psi(x) \to \Box P(\psi)}}{\Box P(\psi)}\,{\to_E} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Box P(\psi)^7}{P(\psi)}\,{\Box_E} \qquad \dfrac{\Pi_3}{P(\psi) \to \forall x.(G(x) \to \psi(x))}}{\forall x.(G(x) \to \psi(x))}\,{\to_E}}{\Box \forall x.(G(x) \to \psi(x))}\,{\Box_I}}{\Box P(\psi) \to \Box \forall x.(G(x) \to \psi(x))}\,{\to_I^7}}{\Box \forall x.(G(x) \to \psi(x))}\,{\to_E}}{\psi(x) \to \Box \forall x.(G(x) \to \psi(x))}\,{\to_I^6}
$$

Embedding Modal Logic into Higher-Order Logic

Challenge: No provers for *Higher-order Quantified Modal Logic* (QML)

Our solution: Embedding in *Higher-order Classical Logic* (HOL)

Then use existing HOL theorem provers for reasoning in QML

[BenzmüllerPaulson, Logica Universalis, 2013]

Previous empiricial findings:

Embedding of *First-order Modal Logic* in HOL works well

[BenzmüllerOttenRaths, ECAI, 2012]

[Benzmüller, LPAR, 2013]

QML $\varphi, \psi \quad ::= \quad \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x\, \varphi \mid \exists x\, \varphi \mid \forall P\, \varphi$

- Kripke style semantics (possible world semantics)

HOL $s, t \quad ::= \quad C \mid x \mid \lambda x s \mid s\, t \mid \neg s \mid s \vee t \mid \forall x\, t$

- various theorem provers exist

  interactive: Isabelle/HOL, HOL4, Hol Light, Coq/HOL, PVS, ...

  automated: TPS, LEO-II, Satallax, Nitpick, Isabelle/HOL, ...

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HOL $s, t \quad ::= \quad C \mid x \mid \lambda x s \mid s\, t \mid \neg s \mid s \vee t \mid \forall x\, t$

QML in HOL: QML formulas φ are mapped to HOL predicates $\varphi_{\iota \to o}$

$$
\begin{aligned}
\neg &= \lambda \varphi_{\iota\to o} \lambda w_\iota \neg \varphi w \\
\wedge &= \lambda \varphi_{\iota\to o} \lambda \psi_{\iota\to o} \lambda w_\iota (\varphi w \wedge \psi w) \\
\rightarrow &= \lambda \varphi_{\iota\to o} \lambda \psi_{\iota\to o} \lambda w_\iota (\neg \varphi w \vee \psi w) \\
\forall &= \lambda h_{\gamma\to(\iota\to o)} \lambda w_\iota \forall d_\gamma\, h d w \\
\exists &= \lambda h_{\gamma\to(\iota\to o)} \lambda w_\iota \exists d_\gamma\, h d w \\
\Box &= \lambda \varphi_{\iota\to o} \lambda w_\iota \forall u_\iota\, (\neg r w u \vee \varphi u) \\
\Diamond &= \lambda \varphi_{\iota\to o} \lambda w_\iota \exists u_\iota\, (r w u \wedge \varphi u) \\
\text{valid} &= \lambda \varphi_{\iota\to o} \forall w_\iota \varphi w
\end{aligned}
$$

Ax

The equations in Ax are given as axioms to the HOL provers!

QML $\varphi, \psi \quad ::= \quad \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x\, \varphi \mid \exists x\, \varphi \mid \forall P\, \varphi$

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Ax

The equations in Ax are given as axioms to the HOL provers!

Example

QML formula	$\Diamond \exists x G(x)$
QML formula in HOL	valid $(\Diamond \exists x G(x))_{\iota \to o}$
expansion, $\beta\eta$-conversion	$\forall w_\iota (\Diamond \exists x G(x))_{\iota \to o}\, w$
expansion, $\beta\eta$-conversion	$\forall w_\iota \exists u_\iota (rwu \wedge (\exists x G(x))_{\iota \to o} u)$
expansion, $\beta\eta$-conversion	$\forall w_\iota \exists u_\iota (rwu \wedge \exists x Gxu)$

What are we doing?

In order to prove that φ is valid in QML,
–> we instead prove that valid $\varphi_{\iota \to o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Expansion: user or prover may flexibly choose expansion depth

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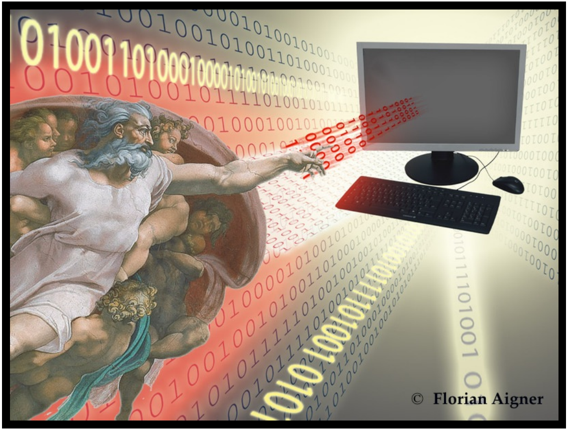
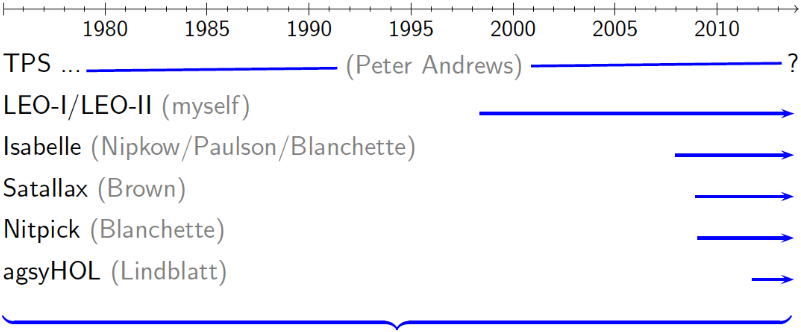


Automated Proof Search and Consistency Check



- all accept TPTP THF Syntax [SutcliffeBenzmüller, J.Form.Reas, 2009]
- can be called remotely via SystemOnTPTP at Miami
- they significantly gained in strength over the last years
- they can be bundled into a combined prover HOL-P

Exploit HOL with Henkin semantics as metalogic
Automate other logics (& combinations) via semantic embeddings
— HOL-P becomes a Universal Reasoner —

Proof Automation and Consistency Checking: Demo!

```
Terminal — bash — 125×32
MacBook-Chris %
MacBook-Chris %
MacBook-Chris % ./call_tptp.sh T3.p

Asking various HOL-ATPs in Miami remotely (thanks to Geoff Sutcliffe)

MacBook-Chris % agsyHOL---1.0 : T3.p ++++++ RESULT: SOT_7L4x_Y - agsyHOL---1.0 says Unknown - CPU = 0.00 WC = 0.02
LEO-II---1.6.0 : T3.p ++++++ RESULT: SOT_E4SCha - LEO-II---1.6.0 says Theorem - CPU = 0.03 WC = 0.09
Satallax---2.7 : T3.p ++++++ RESULT: SOT_kVZ1cB - Satallax---2.7 says Theorem - CPU = 0.00 WC = 0.14
Isabelle---2013 : T3.p ++++++ RESULT: SOT_xa0gEp - Isabelle---2013 says Theorem - CPU = 14.06 WC = 17.73 SolvedBy = auto
TPS---3.120601S1b : T3.p ++++++ RESULT: SOT_R0Egsg - TPS---3.120601S1b says Unknown - CPU = 33.56 WC = 41.57
Nitrox---2013 : T3.p ++++++ RESULT: SOT_WGY1Tx - Nitrox---2013 says Unknown - CPU = 75.55 WC = 49.24

MacBook-Chris %
MacBook-Chris % ./call_tptp.sh Consistency.p

Asking various HOL-ATPs in Miami remotely (thanks to Geoff Sutcliffe)

MacBook-Chris % agsyHOL---1.0 : Consistency.p ++++++ RESULT: SOT_ZtY_7o - agsyHOL---1.0 says Unknown - CPU = 0.00 WC = 0.00
Nitrox---2013 : Consistency.p ++++++ RESULT: SOT_HUz10C - Nitrox---2013 says Satisfiable - CPU = 6.56 WC = 8.50
TPS---3.120601S1b : Consistency.p ++++++ RESULT: SOT_fpJxTM - TPS---3.120601S1b says Unknown - CPU = 43.00 WC = 49.42
Isabelle---2013 : Consistency.p ++++++ RESULT: SOT_6Tpp9i - Isabelle---2013 says Unknown - CPU = 69.96 WC = 72.62
LEO-II---1.6.0 : Consistency.p ++++++ RESULT: SOT_dY10sj - LEO-II---1.6.0 says Timeout - CPU = 90 WC = 89.86
Satallax---2.7 : Consistency.p ++++++ RESULT: SOT_Q9WSLf - Satallax---2.7 says Timeout - CPU = 90 WC = 90.50

MacBook-Chris %
```

Provers are called remotely in Miami — no local installation needed!

Formalization and Verification in Coq

- Goal: verification of the natural deduction proof
  - Step-by-step formalization
  - Almost no automation (intentionally!)
- Interesting facts:
  - Embedding is transparent to the user
  - Embedding gives labeled calculus for free

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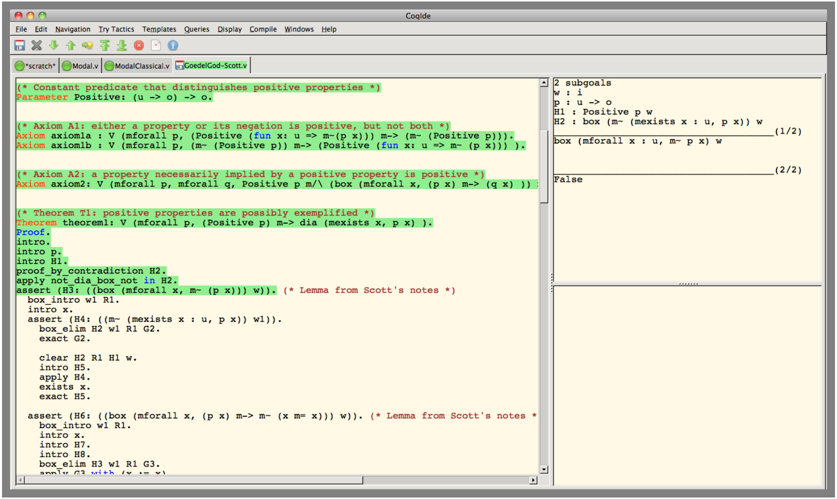
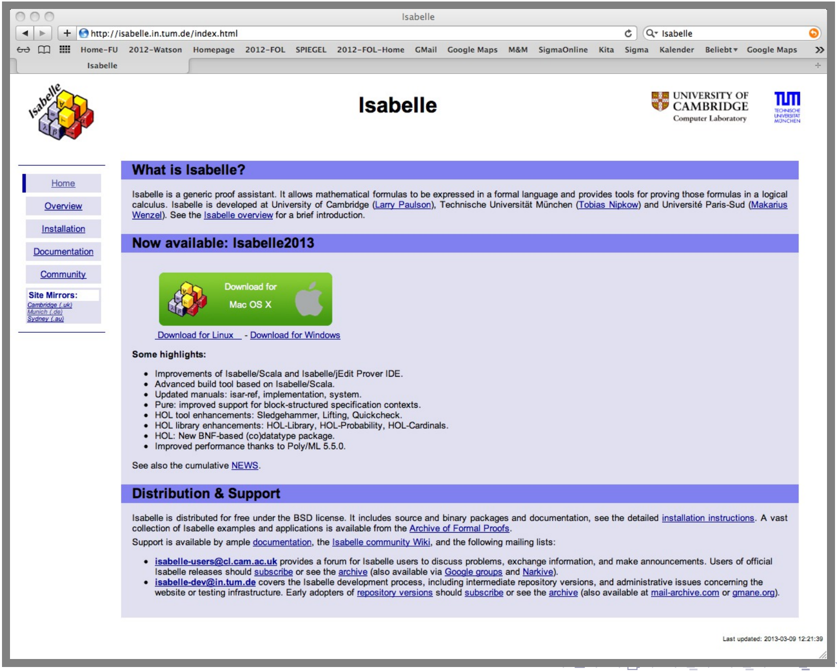
Coq Proof


CoqIde
File Edit Navigation Try Tactics Templates Queries Display Compile Windows Help
scratch Modal.v ModalClassical.v GoedelGod-Scott.v
(* Constant predicate that distinguishes positive properties *)
Parameter Positive: (u -> o) -> o.
(* Axiom A1: either a property or its negation is positive, but not both *)
Axiom axiom1a : V (mforall p, (Positive (fun x: u => m~(p x))) m-> (m~ (Positive p))).
Axiom axiom1b : V (mforall p, (m~ (Positive p)) m-> (Positive (fun x: u => m~ (p x)))).
(* Axiom A2: a property necessarily implied by a positive property is positive *)
Axiom axiom2: V (mforall p, mforall q, Positive p m/\ (box (mforall x, (p x) m-> (q x)))
(* Theorem T1: positive properties are possibly exemplified *)
Theorem theorem1: V (mforall p, (Positive p) m-> dia (mexists x, p x)).
Proof.
intro.
intro p.
intro H1.
proof_by_contradiction H2.
apply not_dia_box_not in H2.
assert (H3: ((box (mforall x, m~ (p x))) w)). (* Lemma from Scott's notes *)
box_intro w1 R1.
intro x.
assert (H4: ((m~ (mexists x : u, p x)) w1)).
box_elim H2 w1 R1 G2.
exact G2.
clear H2 R1 H1 w.
intro H5.
apply H4.
exists x.
exact H5.
assert (H6: ((box (mforall x, (p x) m-> m~ (x m= x))) w)). (* Lemma from Scott's notes *
box_intro w1 R1.
intro x.
intro H7.
intro H8.
box_elim H3 w1 R1 G3.
2 subgoals
w : i
p : u -> o
H1 : Positive p w
H2 : box (m~ (mexists x : u, p x)) w
(1/2)
box (mforall x : u, m~ p x) w
(2/2)
False

automation & verification: proof assistant Isabelle



Isabelle

http://isabelle.in.tum.de/index.html

Isabelle

UNIVERSITY OF CAMBRIDGE
Computer Laboratory

TUM TECHNISCHE UNIVERSITÄT MÜNCHEN

- Home
- Overview
- Installation
- Documentation
- Community

Site Mirrors:
Cambridge (.uk)
Munich (.de)
Sydney (.au)

What is Isabelle?

Isabelle is a generic proof assistant. It allows mathematical formulas to be expressed in a formal language and provides tools for proving those formulas in a logical calculus. Isabelle is developed at University of Cambridge (Larry Paulson), Technische Universität München (Tobias Nipkow) and Université Paris-Sud (Makarius Wenzel). See the Isabelle overview for a brief introduction.

Now available: Isabelle2013

Download for Mac OS X

Download for Linux - Download for Windows

Some highlights:

- Improvements of Isabelle/Scala and Isabelle/jEdit Prover IDE.
- Advanced build tool based on Isabelle/Scala.
- Updated manuals: isar-ref, implementation, system.
- Pure: improved support for block-structured specification contexts.
- HOL tool enhancements: Sledgehammer, Lifting, Quickcheck.
- HOL library enhancements: HOL-Library, HOL-Probability, HOL-Cardinals.
- HOL: New BNF-based (co)datatype package.
- Improved performance thanks to Poly/ML 5.5.0.

See also the cumulative NEWS.

Distribution & Support

Isabelle is distributed for free under the BSD license. It includes source and binary packages and documentation, see the detailed installation instructions. A vast collection of Isabelle examples and applications is available from the Archive of Formal Proofs.

Support is available by ample documentation, the Isabelle community Wiki, and the following mailing lists:

- **isabelle-users@cl.cam.ac.uk** provides a forum for Isabelle users to discuss problems, exchange information, and make announcements. Users of official Isabelle releases should subscribe or see the archive (also available via Google groups and Narkive).
- **isabelle-dev@in.tum.de** covers the Isabelle development process, including intermediate repository versions, and administrative issues concerning the website or testing infrastructure. Early adopters of repository versions should subscribe or see the archive (also available at mail-archive.com or gmane.org).

Last updated: 2013-03-09 12:21:39

Isabelle/HOL (Cambridge University/TU Munich)

- HOL instance of the generic ISABELLE proof assistant
- User interaction and proof automation
- Automation is supported by SLEDGEHAMMER tool
- Verification of the proofs in ISABELLE/HOL's small proof kernel

What we did?

- Proof automation of Gödel's proof script (Scott version)
- SLEDGEHAMMER makes calls to remote THF provers in Miami
- These calls the suggest respective calls to the METIS prover
- METIS proofs are verified in ISABELLE/HOL's proof kernel

See the handout (generated from the Isabelle source file).

GoedelGod.thy (modified)

GoedelGod.thy (~/GoedelGod/Formalizations/Isabelle/GoedelGodSession/)

```
  corollary C: "[◇ (∃ G)]"
  sledgehammer [provers = remote_leo2] by (metis A3 T1)

text {* Axiom @{text "A4"} is added: $\all \phi [P(\phi) \to \Box \; P(\phi)]$
(Positive properties are necessarily positive). *}

  axiomatization where A4:  "[Π (λΦ. P Φ m⇒ □ (P Φ))]"

text {* Symbol @{text "ess"} for `Essence' is introduced and defined as
$\ess{\phi}{x} \biimp \phi(x) \wedge \all \psi (\psi(x) \imp \nec \all y (\phi(y)
\imp \psi(y)))$ (An \emph{essence} of an individual is a property possessed by it
and necessarily implying any of its properties). *}

  definition ess :: "(μ ⇒ σ) ⇒ μ ⇒ σ" (infixr "ess" 85) where
    "Φ ess x = Φ x m∧ Π (λψ. ψ x m⇒ □ (∀ (λy. Φ y m⇒ ψ y)))"

text {* Next, Sledgehammer and Metis prove theorem @{text "T2"}: $\all x [G(x) \imp \ess{G}{x}]$
(Being God-like is an essence of any God-like being). *}

  theorem T2: "[∀ (λx. G x m⇒ G ess x)]"
  sledgehammer [provers = remote_leo2] by (metis A1b A4 G_def ess_def)

text {* Symbol @{text "NE"}, for `Necessary Existence', is introduced and
defined as $\NE(x) \biimp \all \phi [\ess{\phi}{x} \imp \nec \ex y \phi(y)]$ (Necessary
existence of an individual is the necessary exemplification of all its essences). *}

  definition NE :: "μ ⇒ σ" where "NE = (λx. Π (λΦ. Φ ess x m⇒ □ (∃ Φ)))"
```

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```
Sledgehammering...
```

Output README Symbols

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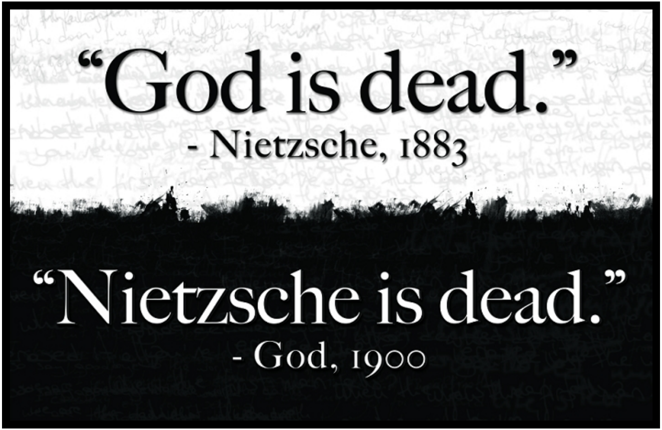

Criticisms

$$\forall P.[\Diamond\Box P \rightarrow \Box P]$$

If something is possibly necessary, then it is necessary.

$$\Diamond\Box(A \vee \neg A) \qquad \Box(A \vee \neg A)$$

logical necessity $\sim$ validity

for all $M, M \models F \quad \longrightarrow \quad \Box F$

logical possibility $\sim$ satisfiability

exists $M, M \models F \quad \longrightarrow \quad \Diamond F$

What about iterations?

$$\Diamond\Box\Diamond\Diamond F$$

weak intuitions $\Rightarrow$ dozens of modal logics

S5 is considered adequate

(But KB is sufficient!)

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Follows from T2, T3 and D2.

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There is no free will.

Many proposed solutions: Anderson, Fitting, Hájek, ...

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$$\forall \phi [P(\neg \phi) \leftrightarrow \neg P(\phi)]$$

Either a property is positive or its negation is (but never both)

Are the following properties positive or negative?

$\lambda x.G(x)$ $\lambda x.E(x)$ $\lambda x.x = x$ $\lambda x.\top$

$\lambda x.blue(x)$ $\lambda x.red(x)$ $\lambda x.human(x)$

Solution:
"…positive in the moral aesthetic sense (independently of the accidental structure of the world). Only then the ax. true. …"
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Conclusions

The (new) insights we gained from experiments include:

- Logic K sufficient for T1, C and T2
- Logic S5 *not* needed for T3
- Logic KB sufficient for T3 (not well known)
- We found a simpler new proof of C
- Gödel's axioms (without conjunct $\phi(x)$ in D2) are inconsistent
- Scott's axioms are consistent
- For T1, only half of A1 (A1a) is needed
- For T2, the other half (A1b) is needed

- Contributions to the theorem proving community include
  - Powerful infrastructure for reasoning with QML
  - A (new?) natural deduction calculus for higher-order (intuitionistic) modal logic
  - Difficult new benchmark problems for HOL provers
  - Huge media attention
- Interesting bridge between CS, Philosophy and Theology
- Major step towards Computer-assisted Theoretical Philosophy
  - see also Ed Zalta's *Computational Metaphysics* project at Stanford University and John Rushby's formalization of Anselm's proof using PVS
  - remember Leibniz' dictum — *Calculemus!*

- Formalize and verify other ontological arguments
  - . . . particularly the criticisms and improvements
- Experiment with other embeddings (e.g. varying domains)
- Eliminate and introduce cuts

I'm sure that God would be impressed with your proof, if only he existed :-)

Larry

Die Philosophen können so schön staunen.

Sie packen Dinge in Begriffe (gucken dabei in die Luft) werfen die Begriffe dann in ihre Philosophiekiste, schütteln ganz dolle, und freuen sich, dass ganz genau rauskommt, was sie vorher reingetan haben. Und das geht sogar, wenn eine Maschine die Kiste schüttelt.

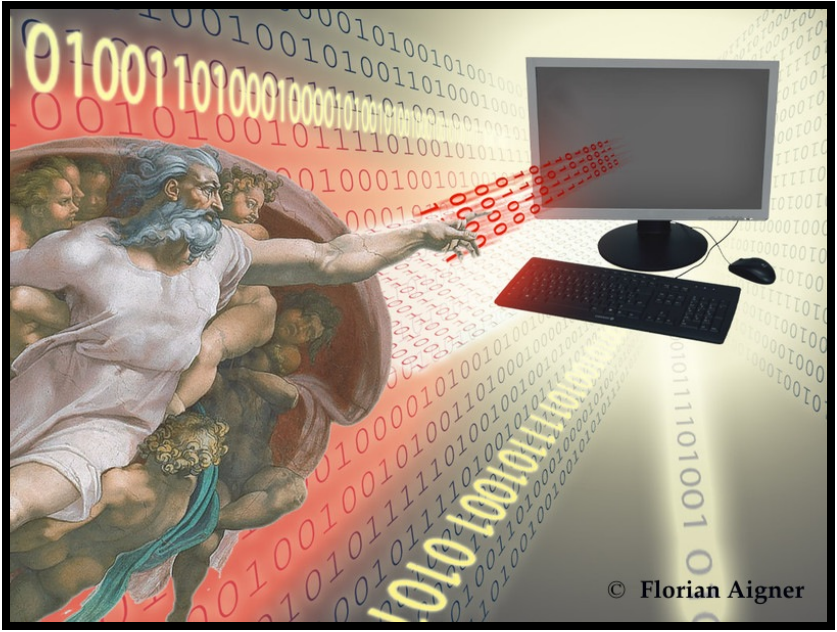
Unerstaunt
2017cp

60. Suchlauf

souveränsatt 09.09.2013

man kann auch auf andere Weise in diesem Zusammenhang methodisch vorgehen: bei einer längeren Autofahrt das Radio auf automatischen Suchlauf stellen. Nach zwei Tagen sieht man Gott

...find more on the internet ...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