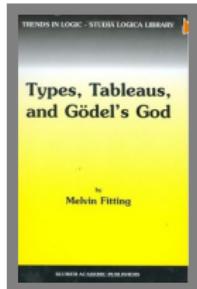


Gödel's God on the Computer

Christoph Benzmüller and Bruno Woltzenlogel Paleo

Invited Presentation, IWIL @ LPAR-2013
Stellenbosch, South Africa, December 14, 2013



$$\frac{\text{Axiom 3} \quad \frac{\neg \forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x. \varphi(x)]}{\begin{array}{c} \text{Theorem 1} \\ P(G) \end{array}}}{\begin{array}{c} \frac{P(G)}{P(G) \rightarrow \Diamond \exists x. G(x)} \\ \rightarrow_E \end{array}} \rightarrow_E \Diamond \exists x. G(x)$$

A gift to **Priest Edvaldo** in Piracicaba, Brazil

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Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Von Tobias Hüller



Kurt Gödel (um das Jahr 1935): Der Mathematiker hielt seinen Gottesbeweis Jahrzehntlang geheim
picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Ein Wesen existiert, das alle positiven Eigenschaften in sich vereint. Das bewies der legendäre Mathematiker Kurt Gödel mit einem komplizierten Formelgebiilde. Zwei Wissenschaftler haben diesen Gottesbeweis nun überprüft - und für gültig befunden.

Montag, 09.09.2013 – 12:03 Uhr

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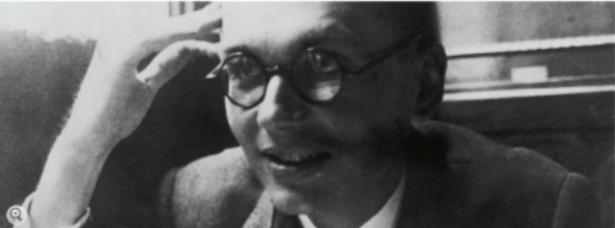
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Front Page World Europe Germany Business Zeitgeist Newsletter

English Site > Germany > Science > Scientists Use Computer to Mathematically Prove Gödel God Theorem

Holy Logic: Computer Scientists 'Prove' God Exists

By David Knight



Austrian mathematician Kurt Gödel kept his proof of God's existence a secret for decades. Now two scientists say they have proven it mathematically using a computer.

Two scientists have formalized a theorem regarding the existence of God penned by mathematician Kurt Gödel. But the God angle is somewhat of a red herring -- the real step forward is the example it sets of how computers can make scientific progress simpler.

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Researchers say they used MacBook to prove Goedel's God theorem

Oct. 23, 2013 | 8:14 PM | 1 comments

Are we in contact with Steve Jobs? No

Do you really need a MacBook to obtain the results? No

Did Apple send us some money? No
(but maybe they should)

SCIENCE NEWS

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Def: Ontological Argument

- * deductive argument
- * for the existence of God
- * starting from premises, which are justified by pure reasoning, i.e. they do not depend on observation of the world.

Existence of God: different types of arguments/proofs

- a posteriori (use experience/observation in the world)
 - teleological
 - cosmological
 - moral
 - ...
- a priori (based on pure reasoning, independent)
 - ontological argument
 - definitional
 - modal
 - ...
 - other a priori arguments

Def: Ontological Argument

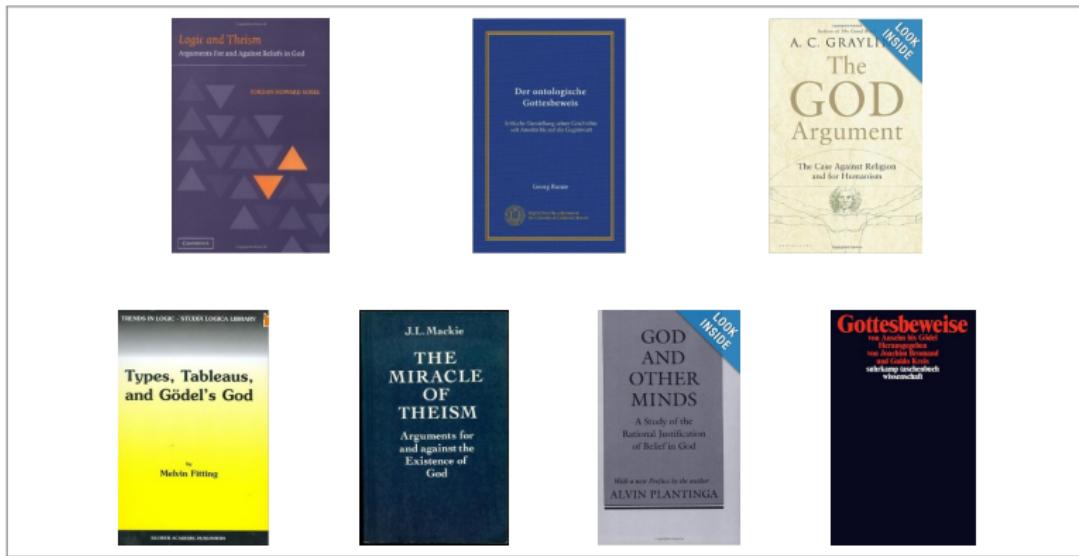
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Existence of God: different types of arguments/proofs

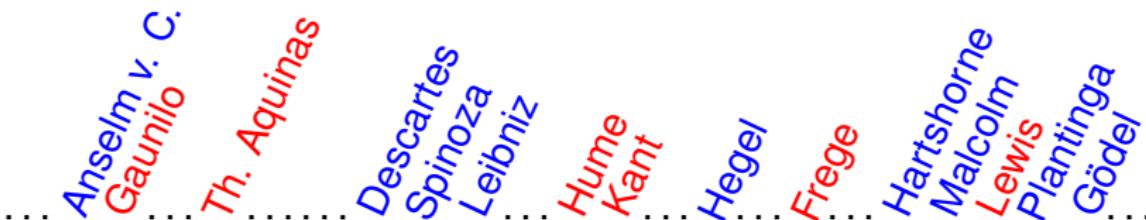
- a posteriori (use experience/observation in the world)
 - teleological
 - cosmological
 - moral
 - ...
- a priori (based on pure reasoning, independent)
 - **ontological argument**
 - definitional
 - modal
 - ...
 - other a priori arguments

Wohl eine jede Philosophie kreist um den ontologischen Gottesbeweis

(Adorno, Th. W.: Negative Dialektik. Frankfurt a. M. 1966, p.378)



Rich history on ontological arguments ([pros](#) and [cons](#))



Anselm's notion of God:

“God is that, than which nothing greater can be conceived.”

Gödel's notion of God:

“A God-like being possesses all ‘positive’ properties.”

To show by logical reasoning:

“(Necessarily) God exists.”

Rich history on ontological arguments ([pros](#) and [cons](#))



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To show by logical reasoning:

“(Necessarily) God exists.”

Different Interests in Ontological Arguments:

- **Philosophical:** Boundaries of Metaphysics & Epistemology
 - We talk about a metaphysical concept (God),
 - but we want to draw a conclusion for the real world.
- **Theistic:** Successful argument should convince atheists
- **Ours:** Can computers (theorem provers) be used ...
 - ... to formalize the definitions, axioms and theorems?
 - ... to verify the arguments step-by-step?
 - ... to fully automate (sub-)arguments?

“Computer-assisted Theoretical Philosophy”

Challenge: No provers for *Higher-order Quantified Modal Logic* (**QML**)

Our solution: Embedding in *Higher-order Classical Logic* (**HOL**)

[BenzmüllerPaulson, Logica Universalis, 2013]

What we did (**rough outline for remaining presentation!**):

- A: Pen and paper: detailed natural deduction proof
- B: Formalization: in classical higher-order logic (**HOL**)
- Automation: theorem provers LEO-II(**E**) and **SATALLAX**
- Consistency: model finder **NITPICK** (**NITROX**)
- C: Step-by-step verification: proof assistant **Coq**
- D: Automation & verification: proof assistant **ISABELLE**

Did we get any new results?

Yes — let's discuss this later!

Gödel's Manuscript: 1930's, 1941, 1946-1955, 1970

Ontologischer Begriff

FEB 10, 1970

P(φ): φ is positive ($\Leftrightarrow \varphi \in P$)

At 1: $P(\varphi) \cdot P(\psi) \supset P(\varphi \wedge \psi)$ At 2: $P(\varphi) \supset P(\neg \varphi)$

P1: $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (Ged.)

P2: $\varphi_{\text{Em},x} \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Emm. $\forall x$)

$P \supset_N = N(P \supset \varphi)$ Necessity

At 2: $P(\varphi) \supset N P(\varphi)$ } because it follows
 $\neg P(\varphi) \supset N \neg P(\varphi)$ } from the nature of the property

Th.: $G(x) \supset G_{\text{Em},x}$

Df.: $E(x) \equiv P(\varphi_{\text{Em},x} \supset N \exists x \varphi(x))$ necessary Exist.

At 3: $P(E)$

Th.: $G(x) \supset N(\exists y) G(y)$

hence $(\exists x) G(x) \supset N(\exists y) G(y)$

" $M(\exists x) G(x) \supset M N(\exists y) G(y)$ Mi = partibility
 " $\supset N(\exists y) G(y)$

any two instances of x are nec. equivalent

exclusive or * and for any number of arguments

$M(\exists x) G(x)$: means all pos. propo. w.r.t. com-
patible This is true because of:

At 4: $P(\varphi) \cdot \varphi \supset \psi \Rightarrow P(\psi)$ which impl.
~~the system of~~ { $x=x$ is positive
~~the system of~~ { $x \neq x$ is negative

But if a system S of pos. propo. were incon-
sistent it would mean that the same prop. S (which
is positive) would be $x \neq x$

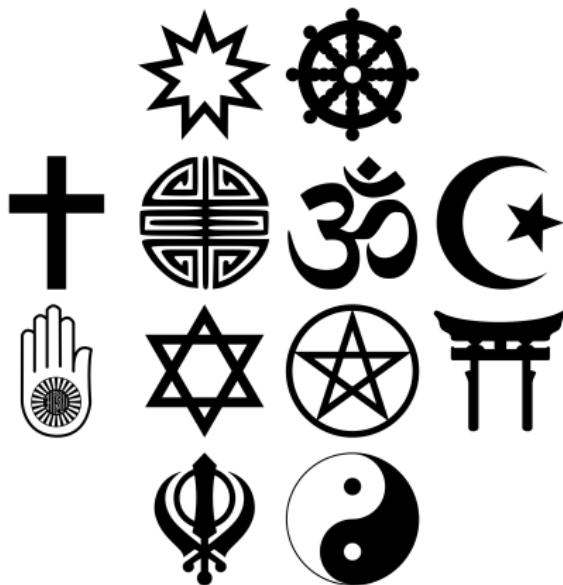
Positive means positive in the moralistic
sense (independently of the accidental structure of
the world). Only then the at time. It may
also mean "attribution" as opposed to "privation"
(or containing privation). This is Gödel's "positive proof"

$\exists x \varphi$ positive w.r.t. $(x) N \supset \varphi(x)$. Otherwise $\exists x \varphi(x) \supset_N$
 hence $x \neq x$, i.e. not $x=x$ according to At.
 or the definition of positive At.

thus
 i.e. the normal form in terms of elem. propo. contains a
 member without negation.

- A1** Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$
- A2** A property necessarily implied by a positive property is positive: $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$
- T1** Positive properties are possibly exemplified: $\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$
- D1** A *God-like* being possesses all positive properties: $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$
- A3** The property of being God-like is positive: $P(G)$
- C** Possibly, God exists: $\Diamond\exists xG(x)$
- A4** Positive properties are necessarily positive: $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$
- D2** An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi \text{ ess. } x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$
- T2** Being God-like is an essence of any God-like being: $\forall x[G(x) \rightarrow G \text{ ess. } x]$
- D3** *Necessary existence* of an individual is the necessary exemplification of all its essences: $NE(x) \leftrightarrow \forall\phi[\phi \text{ ess. } x \rightarrow \Box\exists y\phi(y)]$
- A5** Necessary existence is a positive property: $P(NE)$
- T3** Necessarily, God exists: $\Box\exists xG(x)$

— see also the Isabelle/HOL handout —



Part A: Proof Overview (ND style)

T3: $\Box \exists x.G(x)$

C1: $\Diamond \exists z.G(z)$

T3: $\Box \exists x.G(x)$

$$\frac{\mathbf{C1: } \diamond \exists z. G(z) \quad \mathbf{L2: } \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\mathbf{T3: } \Box \exists x. G(x)}$$

L2: $\diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$

C1: $\diamond \exists z. G(z)$ **L2:** $\diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$
T3: $\Box \exists x. G(x)$

S5
 $\neg \forall \xi. [\neg \diamond \square \xi \rightarrow \square \neg \xi]$

L2: $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

C1: $\diamond \exists z. G(z)$

L2: $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

T3: $\square \exists x. G(x)$

$$\frac{\diamond \exists z. G(z) \rightarrow \diamond \Box \exists x. G(x) \qquad \overline{\forall \xi. [\diamond \Box \xi \rightarrow \Box \xi]}}{\text{L2: } \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}$$

$$\frac{\text{C1: } \diamond \exists z. G(z) \qquad \text{L2: } \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\text{T3: } \Box \exists x. G(x)}$$

$$\frac{\textbf{L1: } \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)} \qquad \qquad \textbf{S5} \\ \frac{}{\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]}$$

$$\textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$$

$$\textbf{C1: } \Diamond \exists z. G(z) \qquad \textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$$

$$\textbf{T3: } \Box \exists x. G(x)$$

D1: $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

$$\frac{\frac{\textbf{L1: } \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)} \quad \textbf{S5}}{\textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}$$

$$\frac{\textbf{C1: } \Diamond \exists z. G(z) \quad \textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\textbf{T3: } \Box \exists x. G(x)}$$

D1: $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

D3*: $NE(x) \equiv \square \exists y. G(y)$

$$\frac{\frac{\frac{P(NE)}{\textbf{L1: } \exists z. G(z) \rightarrow \square \exists x. G(x)}}{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}}{\textbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)} \quad \textbf{S5: } \overline{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}$$

$$\frac{\textbf{C1: } \diamond \exists z. G(z) \quad \textbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}{\textbf{T3: } \square \exists x. G(x)}$$

D1: $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

D3*: $NE(x) \equiv \Box \exists y. G(y)$ (cheating!)

$$\frac{\frac{\frac{P(NE)}{\textbf{L1: } \exists z. G(z) \rightarrow \Box \exists x. G(x)}}{\Box \exists z. G(z) \rightarrow \Box \Box \exists x. G(x)}}{\textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)} \quad \textbf{S5: } \overline{\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]}$$

$$\frac{\textbf{C1: } \Diamond \exists z. G(z) \quad \textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\textbf{T3: } \Box \exists x. G(x)}$$

D1: $G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$

D3*: $NE(x) \equiv \square \exists y.G(y)$

D3: $NE(x) \equiv \forall \varphi.[\varphi \text{ ess } x \rightarrow \square \exists y.\varphi(y)]$

$$\frac{\begin{array}{c} \mathbf{T2:} \forall y.[G(y) \rightarrow G \text{ ess } y] \\ \hline \mathbf{L1:} \exists z.G(z) \rightarrow \square \exists x.G(x) \end{array}}{\frac{\diamond \exists z.G(z) \rightarrow \diamond \square \exists x.G(x)}{\mathbf{L2:} \diamond \exists z.G(z) \rightarrow \square \exists x.G(x)}} \quad P(NE) \quad \frac{}{\forall \xi. [\neg \square \xi \rightarrow \neg \square \xi]} \quad \mathbf{S5}$$

$$\frac{\mathbf{C1:} \diamond \exists z.G(z) \quad \mathbf{L2:} \diamond \exists z.G(z) \rightarrow \square \exists x.G(x)}{\mathbf{T3:} \square \exists x.G(x)}$$

D1: $G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$

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D3: $NE(x) \equiv \forall \varphi.[\varphi \text{ ess } x \rightarrow \square \exists y.\varphi(y)]$

$$\frac{\begin{array}{c} \textbf{T2: } \forall y.[G(y) \rightarrow G \text{ ess } y] \\ \hline \textbf{L1: } \exists z.G(z) \rightarrow \square \exists x.G(x) \end{array}}{\frac{\diamond \exists z.G(z) \rightarrow \diamond \square \exists x.G(x)}{\textbf{L2: } \diamond \exists z.G(z) \rightarrow \square \exists x.G(x)}}
 \quad
 \frac{\begin{array}{c} \textbf{A5} \\ \overline{P(NE)} \end{array}}{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}$$

$$\frac{\textbf{C1: } \diamond \exists z.G(z) \quad \textbf{L2: } \diamond \exists z.G(z) \rightarrow \square \exists x.G(x)}{\textbf{T3: } \square \exists x.G(x)}$$

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D3: $NE(x) \equiv \forall \varphi.[\varphi \text{ ess } x \rightarrow \square \exists y.\varphi(y)]$

$$\frac{\begin{array}{c} \textbf{T2: } \forall y.[G(y) \rightarrow G \text{ ess } y] \\ \hline \textbf{L1: } \exists z.G(z) \rightarrow \square \exists x.G(x) \end{array}}{\frac{\diamond \exists z.G(z) \rightarrow \diamond \square \exists x.G(x)}{\textbf{L2: } \diamond \exists z.G(z) \rightarrow \square \exists x.G(x)}}
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D1: $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

D2: $\varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \square \forall x. (\varphi(x) \rightarrow \psi(x)))$

D3*: $NE(x) \equiv \square \exists y. G(y)$

D3: $NE(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

$$\frac{\begin{array}{c} \overline{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]} \\ \textbf{A1b} \end{array} \quad \begin{array}{c} \overline{\forall \varphi. [P(\varphi) \rightarrow \square \neg P(\varphi)]} \\ \textbf{A4} \end{array}}{\textbf{T2}: \forall y. [G(y) \rightarrow G \text{ ess } y]} \quad \frac{}{P(NE)}$$

$$\frac{\textbf{L1}: \exists z. G(z) \rightarrow \square \exists x. G(x)}{\frac{\begin{array}{c} \overline{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)} \\ \textbf{L2}: \diamond \exists z. G(z) \rightarrow \square \exists x. G(x) \end{array} \quad \begin{array}{c} \overline{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]} \\ \textbf{S5} \end{array}}{\textbf{L2}: \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}}$$

$$\frac{\textbf{C1}: \diamond \exists z. G(z) \quad \textbf{L2}: \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}{\textbf{T3}: \square \exists x. G(x)}$$

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D3*: $NE(x) \equiv \square \exists y. G(y)$

D3: $NE(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

C1: $\diamond \exists z. G(z)$

$$\frac{\begin{array}{c} \overline{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]} & \overline{\forall \varphi. [P(\varphi) \rightarrow \square \neg P(\varphi)]} \\ \hline \textbf{T2: } \forall y. [G(y) \rightarrow G \text{ ess } y] & \textbf{A5} \end{array}}{\overline{P(NE)}}$$

$$\frac{\begin{array}{c} \textbf{L1: } \exists z. G(z) \rightarrow \square \exists x. G(x) \\ \hline \diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x) \end{array}}{\overline{\textbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}}$$

$$\frac{\textbf{S5}}{\overline{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}}$$

C1: $\diamond \exists z. G(z)$

L2: $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

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D3: $NE(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

$P(G)$

C1: $\diamond \exists z. G(z)$

A1b
 $\neg \forall \varphi. \neg P(\varphi) \rightarrow P(\neg \varphi)$

A4
 $\neg \forall \varphi. P(\varphi) \rightarrow \square \neg P(\varphi)$

A5
 $\neg P(NE)$

T2: $\forall y. [G(y) \rightarrow G \text{ ess } y]$

L1: $\exists z. G(z) \rightarrow \square \exists x. G(x)$
 $\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)$

S5
 $\neg \forall \xi. \neg \diamond \square \xi \rightarrow \neg \square \xi$

L2: $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

C1: $\diamond \exists z. G(z)$

L2: $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

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D3: $NE(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

A3
 $\frac{}{P(G)}$

C1: $\diamond \exists z. G(z)$

A1b
 $\frac{\neg \forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]}{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]}$

A4
 $\frac{\neg \forall \varphi. [P(\varphi) \rightarrow \square \neg P(\varphi)]}{\forall \varphi. [P(\varphi) \rightarrow \square \neg P(\varphi)]}$

A5
 $\frac{}{P(NE)}$

T2: $\forall y. [G(y) \rightarrow G \text{ ess } y]$

L1: $\exists z. G(z) \rightarrow \square \exists x. G(x)$
 $\frac{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}$

S5
 $\frac{}{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}$

L2: $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

C1: $\diamond \exists z. G(z)$

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A3
 $\frac{}{P(G)}$

T1: $\forall \varphi. [P(\varphi) \rightarrow \diamond \exists x. \varphi(x)]$

C1: $\diamond \exists z. G(z)$

A1b

$\frac{}{\neg \forall \varphi. \neg P(\varphi) \rightarrow P(\neg \varphi)}$

A4

$\frac{}{\neg \forall \varphi. P(\varphi) \rightarrow \square \neg P(\varphi)}$

T2: $\forall y. [G(y) \rightarrow G \text{ ess } y]$

A5

$\frac{}{\neg \forall \xi. \neg P(NE)}$

L1: $\exists z. G(z) \rightarrow \square \exists x. G(x)$

$\frac{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}{\quad}$

S5

$\frac{}{\neg \forall \xi. \neg \diamond \square \xi \rightarrow \square \xi}$

L2: $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

C1: $\diamond \exists z. G(z)$

L2: $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

T3: $\square \exists x. G(x)$

D1: $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

D2: $\varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \square \forall x. (\varphi(x) \rightarrow \psi(x)))$

D3*: $NE(x) \equiv \square \exists y. G(y)$

D3: $NE(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

$$\frac{\frac{\frac{\frac{\overline{A3}}{P(\bar{G})} \quad \frac{\overline{\forall \varphi. \forall \psi. [(\bar{P}(\varphi) \wedge \square \forall x. [\bar{\varphi}(x) \rightarrow \bar{\psi}(x)] \rightarrow \bar{P}(\psi)]}}{A2} \quad \frac{\overline{\forall \varphi. [\bar{P}(\neg \varphi) \rightarrow \neg \bar{P}(\varphi)]}}{A1a}}$$

T1: $\forall \varphi. [P(\varphi) \rightarrow \diamond \exists x. \varphi(x)]$

C1: $\diamond \exists z. G(z)$

$$\frac{\frac{\frac{\frac{\overline{A1b}}{\overline{\forall \varphi. [\neg \bar{P}(\varphi) \rightarrow \bar{P}(\neg \varphi)]}} \quad \frac{\overline{\forall \varphi. [\bar{P}(\varphi) \rightarrow \square \bar{P}(\varphi)]}}{A4}}{T2: \forall y. [G(y) \rightarrow G \text{ ess } y]} \quad \frac{}{A5}}{P(NE)}$$

$$\frac{\frac{\frac{\overline{L1: \exists z. G(z) \rightarrow \square \exists x. G(x)}}{\overline{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}}}{L2: \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}}{S5: \overline{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}}$$

C1: $\diamond \exists z. G(z)$

L2: $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

T3: $\square \exists x. G(x)$

$$\mathbf{D1: } G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$$

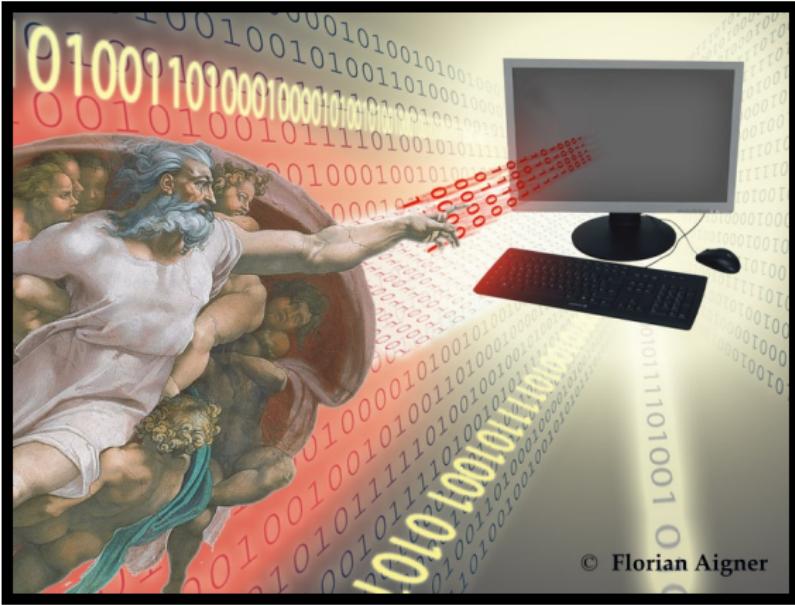
D2: $\varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \square \forall x. (\varphi(x) \rightarrow \psi(x)))$

D3: $NE(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

A3 $\frac{}{P(G)}$	$\frac{\neg \forall \varphi. \neg \forall \psi. [(\bar{P}(\varphi) \wedge \neg \square \forall x. [\varphi(x) \rightarrow \psi(x)]) \rightarrow \bar{P}(\psi)]}{\textbf{T1: } \forall \varphi. [P(\varphi) \rightarrow \diamond \exists x. \varphi(x)]}$	A2 $\frac{}{\neg \forall \varphi. [P(\neg \varphi) \rightarrow \neg P(\varphi)]}$
		C1: $\diamond \exists z. G(z)$

A1b	A4	A5
$\neg \forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]$	$\neg \forall \varphi. [P(\varphi) \rightarrow \Box P(\neg \varphi)]$	$P(NE)$
T2: $\forall y. [G(y) \rightarrow \text{G}\text{ess } y]$		
	L1: $\exists z. G(z) \rightarrow \Box \exists x. G(x)$	
	$\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)$	S5
		$\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]$
		L2: $\Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$

$$\frac{\mathbf{C1}: \Diamond \exists z.G(z) \quad \mathbf{L2}: \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\mathbf{T3}: \Box \exists x.G(x)}$$



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Part B:

Formalization:
Automation:
Consistency:

in classical higher-order logic (HOL)
theorem provers **Leo-II** and **SATALLAX**
model finder **NITPICK** (**NITROX**)

Challenge: No provers for *Higher-order Quantified Modal Logic* ([QML](#))

Our solution: Embedding in *Higher-order Classical Logic* ([HOL](#))

Then use existing [HOL](#) theorem provers for reasoning in [QML](#)

[BenzmüllerPaulson, Logica Universalis, 2013]

Previous empirical findings:

Embedding of *First-order Modal Logic* in HOL works well

[BenzmüllerOttenRaths, ECAI, 2012]

[BenzmüllerRaths, LPAR, 2013]

QML $\varphi, \psi ::= \dots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x \varphi \mid \exists x \varphi \mid \forall P \varphi$

- Kripke style semantics (possible world semantics)

HOL

$s, t ::= C \mid x \mid \lambda x s \mid s t \mid \neg s \mid s \vee t \mid \forall x t$

- meanwhile very well understood
- **Henkin semantics** vs. standard semantics
- various theorem provers do exists

interactive: Isabelle/HOL, HOL4, Hol Light, Coq/HOL, PVS, ...

automated: TPS, LEO-II, Satallax, Nitpick, Isabelle/HOL, ...

QML $\varphi, \psi ::= \dots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x \varphi \mid \exists x \varphi \mid \forall P \varphi$

HOL $s, t ::= C \mid x \mid \lambda x s \mid s t \mid \neg s \mid s \vee t \mid \forall x t$

Ax

QML $\varphi, \psi ::= \dots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x \varphi \mid \exists x \varphi \mid \forall P \varphi$

HOL $s, t ::= C \mid x \mid \lambda x s \mid s t \mid \neg s \mid s \vee t \mid \forall x t$

QML in HOL: QML formulas φ are mapped to HOL predicates $\varphi_{\iota \rightarrow o}$

\neg	$= \lambda\varphi_{t \rightarrow o} \lambda s_t \neg \varphi s$
\wedge	$= \lambda\varphi_{t \rightarrow o} \lambda\psi_{t \rightarrow o} \lambda s_t (\varphi s \wedge \psi s)$
\rightarrow	$= \lambda\varphi_{t \rightarrow o} \lambda\psi_{t \rightarrow o} \lambda s_t (\neg \varphi s \vee \psi s)$
\Box	$= \lambda\varphi_{t \rightarrow o} \lambda s_t \forall u_t (\neg rsu \vee \varphi u)$
\Diamond	$= \lambda\varphi_{t \rightarrow o} \lambda s_t \exists u_t (rsu \wedge \varphi u)$
\forall	$= \lambda h_{\mu \rightarrow (t \rightarrow o)} \lambda s_t \forall d_\mu hds$
\exists	$= \lambda h_{\mu \rightarrow (t \rightarrow o)} \lambda s_t \exists d_\mu hds$
\forall	$= \lambda H_{(\mu \rightarrow (t \rightarrow o)) \rightarrow (t \rightarrow o)} \lambda s_t \forall d_\mu Hds$
valid	$= \lambda\varphi_{t \rightarrow o} \forall w_t \varphi w$

QML $\varphi, \psi ::= \dots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x \varphi \mid \exists x \varphi \mid \forall P \varphi$

$$\text{HOL} \qquad \qquad s, t ::= C \mid x \mid \lambda x s \mid s t \mid \neg s \mid s \vee t \mid \forall x t$$

QML in HOL: QML formulas φ are mapped to HOL predicates $\varphi_{\iota \rightarrow o}$

\neg	$= \lambda\varphi_{t \rightarrow o} \lambda s_t \neg \varphi s$
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\rightarrow	$= \lambda\varphi_{t \rightarrow o} \lambda\psi_{t \rightarrow o} \lambda s_t (\neg \varphi s \vee \psi s)$
\Box	$= \lambda\varphi_{t \rightarrow o} \lambda s_t \forall u_t (\neg rsu \vee \varphi u)$
\Diamond	$= \lambda\varphi_{t \rightarrow o} \lambda s_t \exists u_t (rsu \wedge \varphi u)$
\forall	$= \lambda h_{\mu \rightarrow (t \rightarrow o)} \lambda s_t \forall d_\mu hds$
\exists	$= \lambda h_{\mu \rightarrow (t \rightarrow o)} \lambda s_t \exists d_\mu hds$
\forall	$= \lambda H_{(\mu \rightarrow (t \rightarrow o)) \rightarrow (t \rightarrow o)} \lambda s_t \forall d_\mu Hds$
valid	$= \lambda\varphi_{t \rightarrow o} \forall w_t \varphi w$

The equations in `Ax` are given as axioms to the `HOL` provers!

(Remark: Note that we are here dealing with constant domain quantification)

Example

QML formula

QML formula in HOL

expansion, $\beta\eta$ -conversion
expansion, $\beta\eta$ -conversion
expansion, $\beta\eta$ -conversion

$\diamond \exists x G(x)$

valid ($\diamond \exists x G(x))_{t \rightarrow o}$

$\forall w_t (\diamond \exists x G(x))_{t \rightarrow o} w$

$\forall w_t \exists u_t (rwu \wedge (\exists x G(x))_{t \rightarrow o})$

$\forall w_t \exists u_t (rwu \wedge \exists x Gxu)$

What are we doing?

In order to prove that φ is valid in QML,

→ we instead prove that valid $\varphi_{t \rightarrow o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Expansion:

user or prover may flexibly choose expansion depth

Example

QML formula

QML formula in HOL

expansion, $\beta\eta$ -conversion

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valid $(\diamond \exists x G(x))_{t \rightarrow o}$

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$\forall w_t \exists u_t (rwu \wedge (\exists x G(x))_{t \rightarrow o} u)$

$\forall w_t \exists u_t (rwu \wedge \exists x Gxu)$

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Example

QML formula

QML formula in HOL

expansion, $\beta\eta$ -conversionexpansion, $\beta\eta$ -conversionexpansion, $\beta\eta$ -conversion $\diamond \exists x G(x)$ valid $(\diamond \exists x G(x))_{t \rightarrow o}$ $\forall w_t (\diamond \exists x G(x))_{t \rightarrow o} w$ $\forall w_t \exists u_t (rwu \wedge (\exists x G(x))_{t \rightarrow o} u)$ $\forall w_t \exists u_t (rwu \wedge \exists x Gxu)$

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Example

QML formula**QML formula in HOL**expansion, $\beta\eta$ -conversionexpansion, $\beta\eta$ -conversionexpansion, $\beta\eta$ -conversion $\Diamond \exists x G(x)$ valid $(\Diamond \exists x G(x))_{\iota \rightarrow o}$ $\forall w_t (\Diamond \exists x G(x))_{\iota \rightarrow o} w$ $\forall w_t \exists u_t (rwu \wedge (\exists x G(x))_{\iota \rightarrow o} u)$ $\forall w_t \exists u_t (rwu \wedge \exists x Gxu)$

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QML formula**QML formula in HOL**expansion, $\beta\eta$ -conversionexpansion, $\beta\eta$ -conversionexpansion, $\beta\eta$ -conversion $\Diamond \exists x G(x)$ valid $(\Diamond \exists x G(x))_{\iota \rightarrow o}$ $\forall w_t (\Diamond \exists x G(x))_{\iota \rightarrow o} w$ $\forall w_t \exists u_t (rwu \wedge (\exists x G(x))_{\iota \rightarrow o} u)$ $\forall w_t \exists u_t (rwu \wedge \exists x Gxu)$

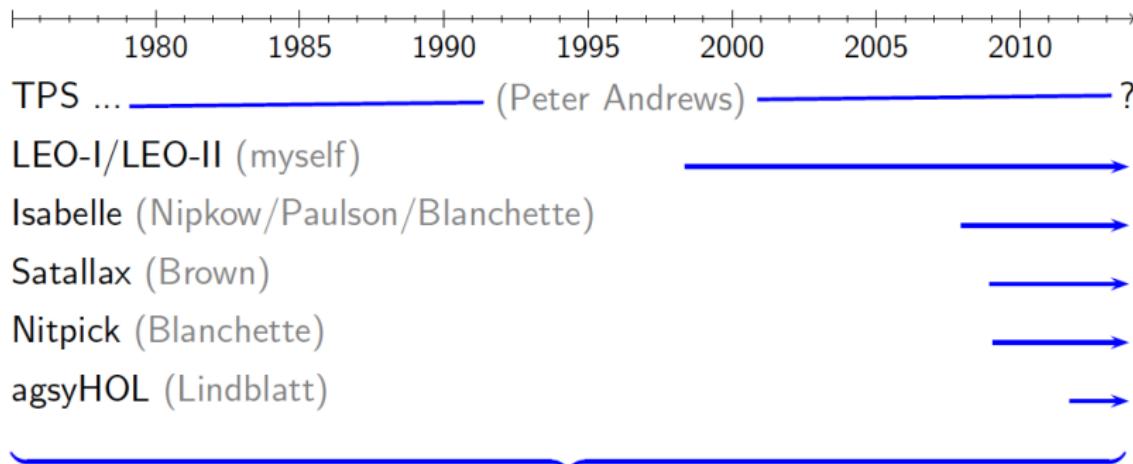
What are we doing?

In order to prove that φ is valid in QML,→ we instead prove that valid $\varphi_{\iota \rightarrow o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Expansion:

user or prover may flexibly choose expansion depth



- all accept TPTP THF Syntax [SutcliffeBenzmüller, J.Form.Reas, 2009]
 - can be called remotely via SystemOnTPTP at Miami
 - they significantly gained in strength over the last years
 - they can be bundled into a combined prover **HOL-P**

Exploit HOL with Henkin semantics as metalogic
Automate other logics (& combinations) via semantic embeddings
— HOL-P becomes a **Universal Reasoner** —

Proof Automation and Consistency Checking in THF and TPI

```
MacBook-Chris % Terminal — bash — 125x32
MacBook-Chris %
MacBook-Chris %
MacBook-Chris % ./call_tptp.sh T3.p

Asking various HOL-ATPs in Miami remotely (thanks to Geoff Sutcliffe)

MacBook-Chris % agsyHOL---1.0 : T3.p ++++++ RESULT: SOT_7L4x_Y - agsyHOL---1.0 says Unknown - CPU = 0.00 WC = 0.02
LEO-II---1.6.0 : T3.p ++++++ RESULT: SOT_E4Sch - LEO-II---1.6.0 says Theorem - CPU = 0.03 WC = 0.09
Satallax---2.7 : T3.p ++++++ RESULT: SOT_kVZ1cB - Satallax---2.7 says Theorem - CPU = 0.00 WC = 0.14
Isabelle---2013 : T3.p ++++++ RESULT: SOT_x0gEp - Isabelle---2013 says Theorem - CPU = 14.06 WC = 17.73 SolvedBy = auto
TPS---3.120601S1b : T3.p ++++++ RESULT: SOT_ROEsgs - TPS---3.120601S1b says Unknown - CPU = 33.56 WC = 41.57
Nitrox---2013 : T3.p ++++++ RESULT: SOT_WGY1Tx - Nitrox---2013 says Unknown - CPU = 75.55 WC = 49.24

MacBook-Chris %
MacBook-Chris % ./call_tptp.sh Consistency.p

Asking various HOL-ATPs in Miami remotely (thanks to Geoff Sutcliffe)

MacBook-Chris % agsyHOL---1.0 : Consistency.p ++++++ RESULT: SOT_ZtY_7o - agsyHOL---1.0 says Unknown - CPU = 0.00 WC = 0.00
Nitrox---2013 : Consistency.p ++++++ RESULT: SOT_HUz10C - Nitrox---2013 says Satisfiable - CPU = 6.56 WC = 8.50
TPS---3.120601S1b : Consistency.p ++++++ RESULT: SOT_fpJxTM - TPS---3.120601S1b says Unknown - CPU = 43.00 WC = 49.42
Isabelle---2013 : Consistency.p ++++++ RESULT: SOT_6Tpp9i - Isabelle---2013 says Unknown - CPU = 69.96 WC = 72.62
LEO-II---1.6.0 : Consistency.p ++++++ RESULT: SOT_dY10sj - LEO-II---1.6.0 says Timeout - CPU = 90 WC = 89.86
Satallax---2.7 : Consistency.p ++++++ RESULT: SOT_Q9WSLF - Satallax---2.7 says Timeout - CPU = 90 WC = 90.50

MacBook-Chris %
```

– see THF files at: <https://github.com/FormalTheology/GoedelGod/blob/master/Formalizations/THF/> –

Provers are called remotely in Miami — no local installation needed!

Cf. previous talk of Geoff Sutcliffe!

- Command and control instruction language for TPTP infrastructure
- Specify problems, avoiding repetitions
- Process the logical formulae: prove theorems, check consistency, etc.
- Report results; exploit TPTP Szs ontology
- Fully automatic
- Very useful for reproducing experiments

Productive collaboration with Geoff: new features, debugging and testing

Gödel's God as THF TPI Script

```
Defining the embedding of quantified modal logics in HOL.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
tpi(com,write,'%%%%%%%%%%%%%').
tpi(com,write,'%%% 1. Introducing the embedding, signature, definitions, axioms, and theorems.').

tpi(com,start_group,embedding).
thf(mu_type,type,(mu:$Type)).
thf(mnot_type,type,(mnot:($i>$o)>$i>$o)).
thf(mnot_definition,(mnot=(^ [Phi:$i>$o,W:$i]:~(Phi@W))).
thf(mor_type,type,(mor:($i>$o)>($i>$o)>$i>$o)).
thf(mor_definition,(mor=(^ [Phi:$i>$o,Psi:$i>$o,W:$i]:((Phi@W)|(Psi@W))))).
thf(mand_type,type,(mand:($i>$o)>($i>$o)>$i>$o)).
thf(mand_definition,(mand=(^ [Phi:$i>$o,Psi:$i>$o,W:$i]:((Phi@W)&(Psi@W))))).
thf(mimplies_type,type,(mimplies:($i>$o)>($i>$o)>$i>$o)).
thf(mimplies_definition,(mimplies=(^ [Phi:$i>$o,Psi:$i>$o,W:$i]:((Phi@W)=>(Psi@W))))).
thf(mequiv_type,type,(mequiv:($i>$o)>($i>$o)>$i>$o)).
thf(mequiv_definition,(mequiv=(^ [Phi:$i>$o,Psi:$i>$o,W:$i]:((Phi@W)<=>(Psi@W))))).
thf(mforall_ind_type,type,(mforall_ind:(mu:$i>$o)>$i>$o)).
thf(mforall_ind_definition,(mforall_ind=(^ [Phi:mu>$i>$o,W:$i]:![X:mu]:(Phi@X@W))).
thf(mforall_indset_type,type,(mforall_indset:((mu:$i>$o)>$i>$o)>$i>$o)).
thf(mforall_indset_definition,(mforall_indset=(^ [Phi:(mu:$i>$o)>$i>$o,W:$i]:![X:mu>$i>$o]:(Phi@X@W))).
thf(mexists_ind_type,type,(mexists_ind:(mu:$i>$o)>$i>$o)).
thf(mexists_ind_definition,(mexists_ind=(^ [Phi:mu>$i>$o,W:$i]:?[X:mu]:(Phi@X@W))).
thf(mequals_type,type,(mequals:mu>mu>$i>$o)).
thf(m>equals_definition,(mequals=(^ [X:mu,Y:mu,W:$i]:(X=Y))).
thf(rel_type,type,(rel:$i>$i>$o)).
thf(mbox_type,type,(mbox:($i>$o)>$i>$o)).
thf(mbox_definition,(mbox=(^ [Phi:$i>$o,W:$i]:![V:$i]:((rel@W@V)=>(Phi@V))))).
thf(mdia_type,type,(mdia:($i>$o)>$i>$o)).
thf(mdia_definition,(mdia=(^ [Phi:$i>$o,W:$i]:?[V:$i]:((rel@W@V)&(Phi@V))))).
thf(mvalid_type,type,(v:($i>$o)>$o)).
thf(mvalid_definition,(v=(^ [Phi:$i>$o]:![W:$i]:((Phi@V))))).
tpi(com,end_group,embedding).
```

Gödel's God as THF TPI Script

Defining the embedding of quantified modal logics in HOL.

```
%%%%%%
tpi(com,write,'%%%%%%%%%').
tpi(com,write,'%%% 1. Introducing the
tpi(com,start_group,embedding).
    thf(mu_type,type,(mu:$Type)).
    thf(mnot_type,type,(mnot:($i>$o)>$i>$o)).
    thf(mnot_definition,(mnot=(^[_Phi:$i>$o,W:$i]:~(_Phi@W))).
    thf(mor_type,type,(mor:($i>$o)>($i>$o)>$i>$o)).
    thf(mor_definition,(mor=(^[_Phi:$i>$o,Psi:$i>$o,W:$i]:((Phi@W)|(Psi@W)))).
    thf(mand_type,type,(mand:($i>$o)>($i>$o)>$i>$o)).
    thf(mand_definition,(mand=(^[_Phi:$i>$o,Psi:$i>$o,W:$i]:((Phi@W)&(Psi@W))).
    thf(mimplies_type,type,(mimplies:($i>$o)>($i>$o)>$i>$o)).
    thf(mimplies_definition,(mimplies=(^[_Phi:$i>$o,Psi:$i>$o,W:$i]:((Phi@W)=>(Psi@W))).
    thf(mequiv_type,type,(mequiv:($i>$o)>($i>$o)>$i>$o)).
    thf(mequiv_definition,(mequiv=(^[_Phi:$i>$o,Psi:$i>$o,W:$i]:((Phi@W)<=>(Psi@W))).
    thf(mforall_ind_type,type,(mforall_ind:(mu:$i>$o)>$i>$o)).
    thf(mforall_ind_definition,(mforall_ind=(^[_Phi:mu>$i>$o,W:$i]:![X:mu]:(_Phi@W))).
    thf(mforall_indset_type,type,(mforall_indset:(mu:$i>$o)>($i>$o)>$i>$o)).
    thf(mforall_indset_definition,(mforall_indset=(^[_Phi:(mu:$i>$o)>$i>$o,W:$i]:![X:mu>$i>$o]:(_Phi@W))).
    thf(mexists_ind_type,type,(mexists_ind:(mu:$i>$o)>$i>$o)).
    thf(mexists_ind_definition,(mexists_ind=(^[_Phi:mu>$i>$o,W:$i]:?[X:mu]:(_Phi@W))).
    thf(mequals_type,type,(mequals:mu>mu>$i>$o)).
    thf(mequals_definition,(mequals=(^[_X:mu,Y:mu,W:$i]:(X=Y))).
    thf(rel_type,type,(rel:$i>$i>$o)).
    thf(mbox_type,type,(mbox:($i>$o)>$i>$o)).
    thf(mbox_definition,(mbox=(^[_Phi:$i>$o,W:$i]:![V:$i]:((rel@W@V)=>(_Phi@V))).
    thf(mdia_type,type,(mdia:($i>$o)>$i>$o)).
    thf(mdia_definition,(mdia=(^[_Phi:$i>$o,W:$i]:?[V:$i]:((rel@W@V)&(_Phi@V))).
    thf(mvalid_type,type,(v:($i>$o)>$o)).
    thf(mvalid_definition,(v=(^[_Phi:$i>$o]:![W:$i]:(_Phi@W))).
tpi(com,end_group,embedding).
```

A (lean) QML prover in HOL

Gödel's God as THF TPI Script

```
tpi(com,start_group,symmetry).
  thf(sym,axiom,(! [S:$i,T:$i]:((rel@S@T)=>(rel@T@S)))). 
tpi(com,end_group,symmetry).

tpi(com,start_group,sig).
  thf(p_tp,type,(p:(mu>$i>$o)>$i>$o)).
  thf(g_tp,type,(g:mu>$i>$o)).
  thf(ess_tp,type,(ess:(mu>$i>$o)>mu>$i>$o)).
  thf(ne_tp,type,(ne:mu>$i>$o)).
tpi(com,end_group,sig).

tpi(com,start_group,d1).
  thf(defD1,definition,(g=(^ [X:mu]:(mforall_indset@^ [Phi:mu>$i>$o]:(mimplies@(p@Phi)@(Phi@X))))).
tpi(com,end_group,d1).

tpi(com,start_group,d2).
  thf(defD2,definition,(ess=(^ [Phi:mu>$i>$o,X:mu]:
  (mand@(Phi@X)@(mforall_indset@^ [Psi:mu>$i>$o]:
  (mimplies@(Psi@X)@(mbox@(mforall_all_ind@^ [Y:mu]:(mimplies@(Phi@Y)@(Psi@Y))))))))).
tpi(com,end_group,d2).

tpi(com,start_group,d3).
  thf(defD3,definition,(ne=(^ [X:mu]:(mforall_indset@^ [Phi:mu>$i>$o]:
  (mimplies@(ess@Phi@X)@(mbox@(mexists_ind@^ [Y:mu]:(Phi@Y))))))).
tpi(com,end_group,d3).

tpi(com,start_group,ala).
  thf(axA1a,axiom,(v@(! mforall_indset@^ [Phi:mu>$i>$o]:(mimplies@(p@^ [X:mu]:(mnot@(Phi@X)))(@mnot@(p@Phi))))).
tpi(com,end_group,ala).

tpi(com,start_group,alb).
  thf(axA1b,axiom,(v@(! mforall_indset@^ [Phi:mu>$i>$o]:(mimplies@(mnot@(p@Phi)))(@p@^ [X:mu]:(mnot@(Phi@X)))))).
tpi(com,end_group,alb).
```

Gödel's God as THF TPI Script

```
tpi(com,start_group,symmetry).
  thf(sym,axiom,(! [S:$i,T:$i]:((rel@S@T)=>(rel@T@S)))).  
tpi(com,end_group,symmetry).  
  
tpi(com,start_group,sig).
  thf(p_tp,type,(p:(mu>$i>$o)>$i>$o)).
  thf(g_tp,type,(g:mu>$i>$o)).
  thf(ess_tp,type,(ess:(mu>$i>$o)>mu>$i>$o)).
  thf(ne_tp,type,(ne:mu>$i>$o)).
tpi(com,end_group,sig).  
  
tpi(com,start_group,d1).
  thf(defD1,definition,(g=(^ [X:mu]:(mforall_indset@^[Phi:mu>$i>$o]:(mimplies@(p@Phi)@(Phi@X))))).
tpi(com,end_group,d1).  
  
tpi(com,start_group,d2).
  thf(defD2,definition,(ess=(^ [Phi:mu>$i>$o,X:mu]:
  (mand@(Phi@X)@(mforall_indset@^[Psi:mu>$i>$o]:
  (mimplies@(Psi@X)@(mbox@(mforall_all_ind@^[Y:mu]:(mimplies@(Phi@Y)@(Psi@Y))))))))).
tpi(com,end_group,d2).  
  
tpi(com,start_group,d3).
  thf(defD3,definition,(ne=(^ [X:mu]:(mforall_indset@^[Phi:mu>$i>$o]:
  (mimplies@(ess@Phi@X)@(mbox@(mexists_ind@^[Y:mu]:(Phi@Y))))))).  
tpi(com,end_group,d3).  
  
tpi(com,start_group,ala).
  thf(axA1a,axiom,(v@(!mforall_indset@^[Phi:mu>$i>$o]:(mimplies@(p@^ [X:mu]:(mnot@(Phi@X)))(@mnot@(p@Phi))))).
tpi(com,end_group,ala).  
  
tpi(com,start_group,alb).
  thf(axA1b,axiom,(v@(!mforall_indset@^[Phi:mu>$i>$o]:(mimplies@(mnot@(p@Phi)))(@p@^ [X:mu]:(mnot@(Phi@X)))))).
tpi(com,end_group,alb).
```

Axiom B (symmetry)

Gödel's God as THF TPI Script

```
tpi(com,start_group,symmetry).
  thf(sym,axiom,(! [S:$i,T:$i]:((rel@S@T)=>(rel@T@S))).
tpi(com,end_group,symmetry).

tpi(com,start_group,sig).
  thf(p_tp,type,(p:(mu>$i>$o)>$i>$o)).
  thf(g_tp,type,(g:mu>$i>$o)).
  thf(ess_tp,type,(ess:(mu>$i>$o)>mu>$i>$o)).
  thf(ne_tp,type,(ne:mu>$i>$o)).
tpi(com,end_group,sig). ↑ Signature

tpi(com,start_group,d1).
  thf(defD1,definition,(g=(^ [X:mu]:(mfa
tpi(com,end_group,d1).

tpi(com,start_group,d2).
  thf(defD2,definition,(ess=(^ [Phi:mu>$i>$o,X:mu]:
  (mand@(Phi@X)@(mforall_indset@^ [Psi:mu>$i>$o]:
  (mimplies@(Psi@X)@(mbox@(mforall_ind@^ [Y:mu]:(mimplies@(Phi@Y)@(Psi@Y))))))))))
  tpi(com,end_group,d2).

tpi(com,start_group,d3).
  thf(defD3,definition,(ne=(^ [X:mu]:(mforall_indset@^ [Phi:mu>$i>$o]:
  (mimplies@(ess@Phi@X)@(mbox@(mexists_ind@^ [Y:mu]:(Phi@Y))))))).
tpi(com,end_group,d3).

tpi(com,start_group,ala).
  thf(axA1a,axiom,(v@(mforall_indset@^ [Phi:mu>$i>$o]:(mimplies@(p@^ [X:mu]:(mnot@(Phi@X))@(mnot@(p@Phi)))))).
tpi(com,end_group,ala).

tpi(com,start_group,alb).
  thf(axA1b,axiom,(v@(mforall_indset@^ [Phi:mu>$i>$o]:(mimplies@(mnot@(p@Phi))@(p@^ [X:mu]:(mnot@(Phi@X))))))).
tpi(com,end_group,alb).
```

Gödel's God as THF TPI Script

```
tpi(com,start_group,symmetry).
  thf(sym,axiom,(! [S:$i,T:$i]:((rel@S@T)=>(rel@T@S)))).  
tpi(com,end_group,symmetry).

tpi(com,start_group,sig).
  thf(p_tp,type,(p:(mu>$i>$o)>$i>$o)).
  thf(g_tp,type,(g:mu>$i>$o).
  thf(ess_tp,type,(ess:(mu>$i>$o)>mu>$i>$o)).
  thf(ne_tp,type,(ne:mu>$i>$o)).
tpi(com,end_group,sig).

tpi(com,start_group,d1).
  thf(defD1,definition,(g=(^ [X:mu]:(mforall_indset@^ [Phi:mu>$i>$o]:(mimplies@(p@Phi)@(Phi@X))))).
tpi(com,end_group,d1).

tpi(com,start_group,d2).
  thf(defD2,definition,(ess=(^ [Phi:mu>$i>$o,X:mu]:
  (mand@(Phi@X)@(mforall_indset@^ [Psi:mu>$i>$o]:
  (mimplies@(Psi@X)@(mbox@(mforall_ind@^ [Y:mu]:(mimplies@(Phi@Y)@(Psi@Y))))))))).
tpi(com,end_group,d2).

tpi(com,start_group,d3).
  thf(defD3,definition,(ne=(^ [X:mu]:(mforall_indset@^ [Phi:mu>$i>$o]:
  (mimplies@(ess@Phi@X)@(mbox@(mexists_ind@^ [Y:mu]:(Phi@Y))))))).  
tpi(com,end_group,d3).

tpi(com,start_group,ala).
  thf(axA1a,axiom,(v@(! mforall_indset@^ [Phi:mu>$i>$o]:(mimplies@(p@^ [X:mu]:(mnot@(Phi@X)))(@(mnot@(p@Phi)))))).  
tpi(com,end_group,ala).

tpi(com,start_group,alb).
  thf(axA1b,axiom,(v@(! mforall_indset@^ [Phi:mu>$i>$o]:(mimplies@(mnot@(p@Phi)))(@(p@^ [X:mu]:(mnot@(Phi@X)))))).  
tpi(com,end_group,alb).
```

Definitions D1-D3 and Axiom A1(a/b)



Gödel's God as THF TPI Script

```
tpi(com,start_group,a2).
thf(axA2,axiom,(v@($forall $indset@^[$Phi:$mu]$i:$o] :
($forall $indset@^[$Psi:$mu]$i:$o] :
($implies@($mand@($p@Phi)@$mbox@($forall $indset@^[$X:$mu] : ($implies@($Phi@X)@$Psi@X)))) @($p@$Psi))))).
tpi(com,end_group,a2).

tpi(com,start_group,a3).
thf(axA3,axiom,(v@($p@g))). 
tpi(com,end_group,a3).

tpi(com,start_group,a4).
thf(axA4,axiom,(v@($forall $indset@^[$Phi:$mu]$i:$o] : ($implies@($p@Phi)@$mbox@($p@Phi))))).
tpi(com,end_group,a4).

tpi(com,start_group,a5).
thf(axA5,axiom,(v@($p@ne))). 
tpi(com,end_group,a5).

tpi(com,write,'%%% Done.').
tpi(com,write,'%%%').
tpi(com,write,'').

% Checking asynchronously for satisfiability of Axioms.
%%% 
tpi(com,execute_async,'SZS_AXIOM_STATUS' = 'Nitrox---2013 60 $getgroups(tpi)'). 

% Checking asynchronously for unsatisfiability of Gödel's original Axioms (modified definition D2).
%%% 
tpi(com,start_group,d2orig).
thf(defD2orig,definition,(ess=($forall $indset@^[$Phi:$mu]$i:$o,$X:$mu] : ($forall $indset@^[$Psi:$mu]$i:$o] :
($implies@($Psi@X)@$mbox@($forall $indset@^[$Y:$mu] : ($implies@($Phi@Y)@$Psi@Y)))))). 
tpi(com,end_group,d2orig).
tpi(com,execute_async,'SZS_STATUS_D2orig' =
'LEO-II---1.6.0 120 $getgroups(embedding,sig,symmetry,d2orig,ala,a2,d3,a5)').
```

Gödel's God as THF TPI Script

```
tpi(com,start_group,a2).
thf(axA2,axiom,(v@(mforall_indset@^ [Phi:mu:$i>$o] :
(mforall_indset@^ [Psi:mu:$i>$o] :
(mimplies@(mand@(p@Phi)@(mbox@(mforall_ind@^ [X:mu] : (mimplies@(Phi@X)@(Psi@X))))@(p@Psi))))))).
tpi(com,end_group,a2).

tpi(com,start_group,a3).
thf(axA3,axiom,(v@(p@g))).
tpi(com,end_group,a3).

tpi(com,start_group,a4).
thf(axA4,axiom,(v@(mforall_indset@^ [Phi:mu:$i>$o] : (mimplies@(p@Phi)@(mbox@(p@Phi)))))).
tpi(com,end_group,a4).

tpi(com,start_group,a5).
thf(axA5,axiom,(v@(p@ne))).
tpi(com,end_group,a5).

tpi(com,write,'%%% Done.').
tpi(com,write,'%%%%%%%%%%%%').
tpi(com,write,'').

% Checking asynchronously for satisfiability of Axioms.
%%%%%%%%%%%%%
tpi(com,execute_async,'Szs_AXIOM_STATUS' = 'Nitrox---2013 60 $getgroups(tpi)').  
  
% Checking asynchronously for unsatisfiability of Gödel's original Axioms (modified definition D2).
%%%%%%%%%%%%%
tpi(com,start_group,d2orig).
thf(defD2orig,definition,(ess=(^ [Phi:mu:$i>$o,X:mu] : (mforall_indset@^ [Psi:mu:$i>$o] :
(mimplies@($Psi@X)@(mbox@(mforall_ind@^ [Y:mu] : (mimplies@($Phi@Y)@$Psi@Y))))))).
tpi(com,end_group,d2orig).
tpi(com,execute_async,'Szs_Status_D2orig' =
'LEO-II---1.6.0 120 $getgroups(embedding,sig,symmetry,d2orig,ala,a2,d3,a5)').
```

Axioms A2-A5



Gödel's God as THF TPI Script

```
tpi(com,start_group,a2).
thf(axA2,axiom,(v@(mforall_indset@^ [Phi:mu:$i>$o] :
(mforall_indset@^ [Psi:mu:$i>$o] :
(mimplies@(mand@(p@Phi)@(mbox@(mforall_ind@^ [X:mu] : (mimplies@(Phi@X)@(Psi@X))))@(p@Psi))))))).
tpi(com,end_group,a2).

tpi(com,start_group,a3).
thf(axA3,axiom,(v@(p@g))).
tpi(com,end_group,a3).

tpi(com,start_group,a4).
thf(axA4,axiom,(v@(mforall_indset@^ [Phi:mu:$i>$o] : (mimplies@(p@Phi)@(mbox@(p@Phi)))))).
tpi(com,end_group,a4).

tpi(com,start_group,a5).
thf(axA5,axiom,(v@(p@ne))).
tpi(com,end_group,a5).

tpi(com,write,'%%% Done.').
tpi(com,write,'%%%%%%%%%%%%').
tpi(com,write,'').

% Checking asynchronously for satisfiability of Axioms.
%%%%%%%%%%%%%
tpi(com,execute_async,'Szs_AXIOM_STATUS' = 'Nitrox---2013 60 $getgroups(tpi)').  
  
% Checking asynchronously for unsatisfiability of Gödel's original Axioms (modified definition D2).
%%%%%%%%%%%%%
tpi(com,start_group,d2orig).
thf(defD2orig,definition,(ess=([Phi:mu:$i>$o,X:mu]:(mforall_indset@^ [Psi:mu:$i>$o] :
(mimplies@($Psi@X)@(mbox@(mforall_ind@^ [Y:mu] : (mimplies@($Phi@Y)@$Psi@Y))))))).
tpi(com,end_group,d2orig).
tpi(com,execute_async,'Szs_STATUS_D2orig' =
'LEO-II---1.6.0 120 $getgroups(embedding,sig,symmetry,d2orig,ala,a2,d3,a5)').
```

Start checking consistency (asynchronously, Nitpick)



```
tpi(com,start_group,a2).
thf(axA2,axiom,(v@(mforall_indset@^ [Phi:mu>$i>$o] :
(mforall_indset@^ [Psi:mu>$i>$o] :
(mimplies@(mand@(p@Phi)@(mbox@(mforall_ind@^ [X:mu] : (mimplies@(Phi@X)@(Psi@X))))@(p@Psi))))))).
tpi(com,end_group,a2).

tpi(com,start_group,a3).
thf(axA3,axiom,(v@(p@g))).
tpi(com,end_group,a3).

tpi(com,start_group,a4).
thf(axA4,axiom,(v@(mforall_indset@^ [Phi:mu>$i>$o] : (mimplies@(p@Phi)@(mbox@(p@Phi)))))).
tpi(com,end_group,a4).

tpi(com,start_group,a5).
thf(axA5,axiom,(v@(p@ne))).
tpi(com,end_group,a5).

tpi(com,write,'%%% Done.').
tpi(com,write,'%%%').
tpi(com,write,'').

% Checking asynchronously for satisfiability of Axioms.
%%%%%%%%%%%%%
tpi(com,execute_async,'Szs_AXIOM_STATUS')

% Checking asynchronously for unsatisfiability
%%%%%%%%%%%%%
tpi(com,start_group,d2orig). 
thf(defD2orig,definition,(ess=(^ [Phi:mu>$i>$o,X:mu] : (mforall_indset@^ [Psi:mu>$i>$o] :
(mimplies@($Psi@X)@(mbox@(mforall_ind@^ [Y:mu] : (mimplies@($Phi@Y)@$Psi@Y))))))).
tpi(com,end_group,d2orig).
tpi(com,execute_async,'Szs_STATUS_D2orig' =
'LEO-II---1.6.0 120 $getgroups(embedding,sig,symmetry,d2orig,ala,a2,d3,a5)').
```

Start checking consistency (asynchronously, LEO-II)
Gödel's original definition of D2

Gödel's God as THF TPI Script

```
% 2. Analysing theorem t1.  
%%%%%%%%%%%%%  
tpi(com,start_group,t1).  
thf(thmT1_con,conjecture,  
  (v@((mforall_indset@^ [Phi:mu>$i>$o] : (mimplies@(p@Phi)@(mdia@(mexists_ind@^ [X:mu] : (Phi@X))))))).  
tpi(com,end_group,t1).  
tpi(com,execute,'Szs_Status_t1' = 'LEO-II---1.6.0 20 $getgroups(embedding,sig,a1a,a2,t1)').  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'%%% 2. Analysing theorem t1.').  
tpi(com,write,'%%%     Checking a1a,a2 |- t1 (using LEO-II).').  
tpi(com,output,stdout = a1a).  
tpi(com,output,stdout = a2).  
tpi(com,output,stdout = t1).  
tpi(com,write,'%%%     Szs_Status for t1 is ' & '$getenv(Szs_Status_t1)').  
tpi(com,assert,$getenv('Szs_Status_t1') = 'Theorem').  
tpi(com,set_role,thmT1_con = lemma).  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'').  
  
% 3. Analysing corollary c.  
%%%%%%%%%%%%%  
tpi(com,start_group,c).  
thf(corC_con,conjecture,(v@((mdia@(mexists_ind@^ [X:mu] : (g@X)))))).  
tpi(com,end_group,c).  
tpi(com,execute,'Szs_Status_c' = 'LEO-II---1.6.0 20 $getgroups(embedding,sig,d1,a3,t1,c)').  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'%%% 3. Analysing corollary c.').  
tpi(com,write,'%%%     Checking d1,a3,t1 |- c (using LEO-II).').  
tpi(com,output,stdout = d1).  
tpi(com,output,stdout = a3).  
tpi(com,output,stdout = t1).  
tpi(com,output,stdout = c).  
tpi(com,write,'%%%     Szs_Status for c is ' & '$getenv(Szs_Status_c)').  
tpi(com,assert,$getenv('Szs_Status_c') = 'Theorem').  
tpi(com,set_role,corC_con = lemma).  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'').
```

Gödel's God as THF TPI Script

```
% 2. Analysing theorem t1.  
%%%%%%%%%%%%%  
tpi(com,start_group,t1).  
    thf(thmT1_con,conjecture,  
        (v@(mforall_all_indset@^ [Phi:mu>$i>$o] : (mimplies@(p@Phi)@(mdia@(mexists_ind@^ [X:mu] : (Phi@X)))))).  
tpi(com,end_group,t1).  
tpi(com,execute,'Szs_Status_t1' = 'LEO-II---1.6.0 20 $getgroups(embedding,sig,a1a,a2,t1)').  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'%%% 2. Analysing theorem t1.').  
tpi(com,write,'%%%     Checking a1a,a2 |- t1 (using LEO-II).').  
tpi(com,output,stdout = a1a).  
tpi(com,output,stdout = a2).  
tpi(com,output,stdout = t1).  
tpi(com,write,'%%%     Szs_Status for t1 is ' & '$getenv(Szs_Status_t1)').  
tpi(com,assert,$getenv('Szs_Status_t1') = 'Theorem').  
tpi(com,set_role,thmT1_con = lemma).  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'').
```

```
% 3. Analysing corollary c.  
%%%%%%%%%%%%%  
tpi(com,start_group,c).  
    thf(corC_con,conjecture,(v@(mdia@(me:  
tpi(com,end_group,c).  
tpi(com,execute,'Szs_Status_c' = 'LEO-II---1.6.0 20 $getgroups(embedding,sig,d1,a3,t1,c)').  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'%%% 3. Analysing corollary c.').  
tpi(com,write,'%%%     Checking d1,a3,t1 |- c (using LEO-II).').  
tpi(com,output,stdout = d1).  
tpi(com,output,stdout = a3).  
tpi(com,output,stdout = t1).  
tpi(com,output,stdout = c).  
tpi(com,write,'%%%     Szs_Status for c is ' & '$getenv(Szs_Status_c)').  
tpi(com,assert,$getenv('Szs_Status_c') = 'Theorem').  
tpi(com,set_role,corC_con = lemma).  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'').
```

Proving: T1 is a theorem (LEO-II)



Gödel's God as THF TPI Script

```
% 2. Analysing theorem t1.  
%%%%%%%%%%%%%  
tpi(com,start_group,t1).  
    thf(thmT1_con,conjecture,  
        (v@{mforall_all_indset@^ [Phi:mu>$i>$o]:(mimplies@(p@Phi)@(mdia@({mexists_ind@^ [X:mu]:(Phi@X)})))))).  
tpi(com,end_group,t1).  
tpi(com,execute,'Szs_Status_t1' = 'LEO-II---1.6.0 20 $getgroups(embedding,sig,a1a,a2,t1)').  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'%%% 2. Analysing theorem t1.').  
tpi(com,write,'%%%     Checking a1a,a2 |- t1 (using LEO-II).').  
tpi(com,output,stdout = a1a).  
tpi(com,output,stdout = a2).  
tpi(com,output,stdout = t1).  
tpi(com,write,'%%%     Szs_Status for t1 is ' & '$getenv(Szs_Status_t1)').  
tpi(com,assert,$getenv('Szs_Status_t1') = 'Theorem').  
tpi(com,set_role,thmT1_con = lemma).  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'').
```

```
% 3. Analysing corollary c.  
%%%%%%%%%%%%%  
tpi(com,start_group,c).  
    thf(corC_con,conjecture,(v@{mdia@({mexists_ind@^ [X:mu]:(g@X)}))}).
```

```
tpi(com,end_group,c).  
tpi(com,execute,'Szs_Status_c' = 'LEO-II---1.6.0 20 $getgroups(embedding,sig,d1,a3,t1,c)').  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'%%% 3. Analysing corollary c.').  
tpi(com,write,'%%%     Checking d1,a3,t1 |- c (using LEO-II).').  
tpi(com,output,stdout = d1).  
tpi(com,output,stdout = a3).  
tpi(com,output,stdout = t1).  
tpi(com,output,stdout = c).  
tpi(com,write,'%%%     Szs_Status for c is ' & '$getenv(Szs_Status_c)').  
tpi(com,assert,$getenv('Szs_Status_c') = 'Theorem').  
tpi(com,set_role,corC_con = lemma).  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'').
```

Proving: C is a theorem (LEO-II)



Gödel's God as THF TPI Script

```
% 4. Analysing theorem t2.  
%%%%%%%%%%%%%%  
tpi(com,start_group,t2).  
thf(thmT2_con,conjecture,(v@(~forall_ind@^ [X:mu]:(mimplies@(^ (g@X)@(ess@g@X)))))).  
tpi(com,end_group,t2).  
tpi(com,execute,'SZS_STATUS_t2' = 'LEO-II---1.6.0 60 $getgroups(embedding,symmetry,sig,d1,d2,a1b,a4,t2)').  
tpi(com,write,'%%%%%%%%%%%%%%').  
tpi(com,write,'%%% 4. Analysing theorem t2.').  
tpi(com,write,'%%%     Checking d1,d2,a1b,a4 |- t2 (using LEO-II)').  
tpi(com,output,stdout = d1).  
tpi(com,output,stdout = d2).  
tpi(com,output,stdout = a1b).  
tpi(com,output,stdout = a4).  
tpi(com,output,stdout = t2).  
tpi(com,write,'%%%     SZS_STATUS for t2 is ' & '$getenv(SZS_STATUS_t2)').  
tpi(com,assert,$getenv('SZS_STATUS_t2') = 'Theorem').  
tpi(com,set_role,thmT2_con = lemma).  
tpi(com,write,'%%%%%%%%%%%%%%').  
tpi(com,write,'').  
  
% 5. Analysing theorem t3.  
%%%%%%%%%%%%%%  
tpi(com,start_group,t3).  
thf(thmT3_con,conjecture,(v@(~exists_ind@^ [X:mu]:(g@X))))).  
tpi(com,end_group,t3).  
tpi(com,execute,'SZS_STATUS_t3' = 'LEO-II---1.6.0 20 $getgroups(embedding,symmetry,sig,d1,d3,c,t2,a5,t3)').  
tpi(com,write,'%%%%%%%%%%%%%%').  
tpi(com,write,'%%% 5. Analysing theorem t3.').  
tpi(com,write,'%%%     Checking sym,d1,d3,c,t2,a5 |- t3 (using LEO-II)').  
tpi(com,output,stdout = d1).  
tpi(com,output,stdout = d3).  
tpi(com,output,stdout = c).  
tpi(com,output,stdout = t2).  
tpi(com,output,stdout = a5).  
tpi(com,output,stdout = t3).  
tpi(com,write,'%%%     SZS_STATUS for t3 is ' & '$getenv(SZS_STATUS_t3)').  
tpi(com,assert,$getenv('SZS_STATUS_t3') = 'Theorem').  
tpi(com,set_role,thmT3_con = lemma).  
tpi(com,write,'%%%%%%%%%%%%%%').
```



Gödel's God as THF TPI Script

```
% 4. Analysing theorem t2.  
%%%%%%%%%%%%%  
tpi(com,start_group,t2).  
thf(thmT2_con,conjecture,(v@(mforall_ind@^X:mu):(mimplies@(g@X)@(ess@g@X)))).  
tpi(com,end_group,t2).  
tpi(com,execute,'SZS_STATUS_t2' = 'LEO-II---1.6.0 60 $getgroups(embedding,symmetry,sig,d1,d2,a1b,a4,t2)').  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'%%% 4. Analysing theorem t2.').  
tpi(com,write,'%%% Checking d1,d2,a1b,a4 |- t2 (using LEO-II)').  
tpi(com,output,stdout = d1).  
tpi(com,output,stdout = d2).  
tpi(com,output,stdout = a1b).  
tpi(com,output,stdout = a4).  
tpi(com,output,stdout = t2).  
tpi(com,write,'%%% SZS_STATUS for t2 is ' & '$getenv(SZS_STATUS_t2)').  
tpi(com,assert,$getenv('SZS_STATUS_t2') = 'Theorem').  
tpi(com,set_role,thmT2_con = lemma).  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'').  
  
% 5. Analysing theorem t3.  
%%%%%%%%%%%%%  
tpi(com,start_group,t3).  
thf(thmT3_con,conjecture,(v@(mbox@m:mu):(mforall_ind@^X:mu):(mimplies@(g@X)@(ess@g@X)))).  
tpi(com,end_group,t3).  
tpi(com,execute,'SZS_STATUS_t3' = 'LEO-II---1.6.0 20 $getgroups(embedding,symmetry,sig,d1,d3,c,t2,a5,t3)').  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'%%% 5. Analysing theorem t3.').  
tpi(com,write,'%%% Checking sym,d1,d3,c,t2,a5 |- t3 (using LEO-II)').  
tpi(com,output,stdout = d1).  
tpi(com,output,stdout = d3).  
tpi(com,output,stdout = c).  
tpi(com,output,stdout = t2).  
tpi(com,output,stdout = a5).  
tpi(com,output,stdout = t3).  
tpi(com,write,'%%% SZS_STATUS for t3 is ' & '$getenv(SZS_STATUS_t3)').  
tpi(com,assert,$getenv('SZS_STATUS_t3') = 'Theorem').  
tpi(com,set_role,thmT3_con = lemma).  
tpi(com,write,'%%%%%%%%%%%%%').
```

Proving: T2 is a theorem (LEO-II)

Gödel's God as THF TPI Script

```
% 4. Analysing theorem t2.  
%%%%%%%%%%%%%  
tpi(com,start_group,t2).  
thf(thmT2_con,conjecture,(v@(~forall_ind@^ [X:mu]:(mimplies@(^ (g@X)@(ess@g@X)))))).  
tpi(com,end_group,t2).  
tpi(com,execute,'SZS_STATUS_t2' = 'LEO-II---1.6.0 60 $getgroups(embedding,symmetry,sig,d1,d2,a1b,a4,t2)').  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'%%% 4. Analysing theorem t2.').  
tpi(com,write,'%%% Checking d1,d2,a1b,a4 |- t2 (using LEO-II)').  
tpi(com,output,stdout = d1).  
tpi(com,output,stdout = d2).  
tpi(com,output,stdout = a1b).  
tpi(com,output,stdout = a4).  
tpi(com,output,stdout = t2).  
tpi(com,write,'%%% SZS_STATUS for t2 is ' & '$getenv(SZS_STATUS_t2)').  
tpi(com,assert,$getenv('SZS_STATUS_t2') = 'Theorem').  
tpi(com,set_role,thmT2_con = lemma).  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'').  
  
% 5. Analysing theorem t3.  
%%%%%%%%%%%%%  
tpi(com,start_group,t3).  
thf(thmT3_con,conjecture,(v@(~exists_ind@^ [X:mu]:(g@X))))).  
tpi(com,end_group,t3).  
tpi(com,execute,'SZS_STATUS_t3' = 'LEO-II---1.6.0 20 $getgroups(embedding,symmetry,sig,d1,d3,c,t2,a5,t3)').  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'%%% 5. Analysing theorem t3.').  
tpi(com,write,'%%% Checking sym,d1,d3,c,t2,a5 |- t3 (using LEO-II)').  
tpi(com,output,stdout = d1).  
tpi(com,output,stdout = d3).  
tpi(com,output,stdout = c).  
tpi(com,output,stdout = t2).  
tpi(com,output,stdout = a5).  
tpi(com,output,stdout = t3).  
tpi(com,write,'%%% SZS_STATUS for t3 is ' & '$getenv(SZS_STATUS_t3)').  
tpi(com,assert,$getenv('SZS_STATUS_t3') = 'Theorem').  
tpi(com,set_role,thmT3_con = lemma).  
tpi(com,write,'%%%%%%%%%%%%%').
```

Proving: T3 is a theorem (LEO-II)



Gödel's God as THF TPI Script

```
% 6. Analysing theorem c2.  
%%%  
tpi(com,start_group,c2).  
    thf(corC2_con,conjecture,(v@({mexists_ind@^ [X:mu]:(g@X)}))).  
tpi(com,end_group,c2).  
tpi(com,execute,'Szs_Status_c2' =  
    'LEO-II---1.6.0 20 $getgroups(embedding,symmetry,sig,d1,t1,t3,c2)').  
tpi(com,write,'%%% 6. Analysing corollary c2.' ).  
tpi(com,write,'%%%     Checking sym,d1,t1,t3 |- c2 (using LEO-II).').  
tpi(com,output,stdout = d1).  
tpi(com,output,stdout = t1).  
tpi(com,output,stdout = t3).  
tpi(com,output,stdout = c2).  
tpi(com,write,'%%%     Szs_Status for c2 is ' & '$getenv(Szs_Status_c2)').  
tpi(com,assert,$getenv('Szs_Status_c2') = 'Theorem').  
tpi(com,set_role,corC2_con = lemma).  
tpi(com,write,'%%%  
tpi(com,write,'').  
  
% 7a. Report on consistency of axioms.  
%%%  
tpi(com,waitenv,'Szs_Axiom_Status').  
tpi(com,write,'%%% 7a. Checking satisfiability of all assumptions (using Nitpick).').  
tpi(com,write,'%%%     Szs_Axiom_Status for assumptions is ' & '$getenv(Szs_Axiom_Status)').  
tpi(com,assert,$getenv('Szs_Axiom_Status') = 'Satisfiable').  
tpi(com,write,'%%%  
tpi(com,write,'').
```

Gödel's God as THF TPI Script

```
% 6. Analysing theorem c2.  
%%%  
tpi(com,start_group,c2).  
    thf(corC2_con,conjecture,(v@({mexists}_ind@^*[X:mu]:(g@X)))).  
tpi(com,end_group,c2).  
tpi(com,execute,'Szs_Status_c2' =  
    'LEO-II---1.6.0 20 $getgroups(embedding,symmetry,sig,d1,t1,t3,c2)').  
tpi(com,write,'%%% 6. Analysing corollary c2.').  
tpi(com,write,'%%%     Checking sym,d1,t1,t3 |- c2 (using LEO-II).').  
tpi(com,output,stdout = d1).  
tpi(com,output,stdout = t1).  
tpi(com,output,stdout = t3).  
tpi(com,output,stdout = c2).  
tpi(com,write,'%%%     Szs_Status for c2 is ' & '$getenv(Szs_Status_c2)').  
tpi(com,assert,$getenv('Szs_Status_c2') = 'Theorem').  
tpi(com,set_role,corC2_con = lemma).  
tpi(com,write,'%%%  
tpi(com,write,'').
```

% 7a. Report on consistency of axioms.

```
%%%  
tpi(com,waitenv,'Szs_Axiom_Status').  
tpi(com,write,'%%%  
tpi(com,write,'%%% 7a. Checking satisfiability of all assumptions (using Nitpick).').  
tpi(com,write,'%%%     Szs_Axiom_Status for assumptions is ' & '$getenv(Szs_Axiom_Status)').  
tpi(com,assert,$getenv('Szs_Axiom_Status') = 'Satisfiable').  
tpi(com,write,'%%%  
tpi(com,write,'').
```

Proving: C2 is a theorem (LEO-II)

Gödel's God as THF TPI Script

```
% 6. Analysing theorem c2.  
%%%  
tpi(com,start_group,c2).  
    thf(corC2_con,conjecture,(v@({exists _ind@^[_X:mu]: (g@_X)})).  
tpi(com,end_group,c2).  
tpi(com,execute,'Szs_Status_c2' =  
    'LEO-II---1.6.0 20 $getgroups(embedding,symmetry,sig,d1,t1,t3,c2)').  
tpi(com,write,'%%% 6. Analysing corollary c2.').  
tpi(com,write,'%%%     Checking sym,d1,t1,t3 |- c2 (using LEO-II).').  
tpi(com,output,stdout = d1).  
tpi(com,output,stdout = t1).  
tpi(com,output,stdout = t3).  
tpi(com,output,stdout = c2).  
tpi(com,write,'%%%     Szs_Status for c2 is ' & '$getenv(Szs_Status_c2)').  
tpi(com,assert,$getenv('Szs_Status_c2') = 'Satisfiable').  
tpi(com,set_role,corC2_con = lemma).  
tpi(com,write,'%%%  
tpi(com,write,'').
```

Checking: Axioms are consistent (Nitpick)

% 7a. Report on consistency of axioms.

```
%%%  
tpi(com,waitenv,'Szs_Axiom_Status').  
tpi(com,write,'%%% 7a. Checking satisfiability of all assumptions (using Nitpick).').  
tpi(com,write,'%%%     Szs_Axiom_Status for assumptions is ' & '$getenv(Szs_Axiom_Status)').  
tpi(com,assert,$getenv('Szs_Axiom_Status') = 'Satisfiable').  
tpi(com,write,'%%%  
tpi(com,write,'').
```

Gödel's God as THF TPI Script

```
% 7b. Report on Inconsistency of Goedel's original axioms.  
%%%%%%%%%%%%%  
tpi(com,waitenv,'Szs_Status_D2orig').  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'%%% 7b. Proving inconsistency of Goedel's original axioms (d2orig).').  
tpi(com,write,'%%%     Checking d2orig,ala,a2,d3,a5 (using LEO-II).').  
tpi(com,output,stdout = d2orig).  
tpi(com,write,'%%%     Szs_Status_D2orig for d2orig is ' & '$getenv(Szs_Status_D2orig)').  
tpi(com,assert,$getenv('Szs_Status_D2orig') = 'Unsatisfiable').  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'').  
  
% Analysing modal collapse (asynchronously).  
%%%%%%%%%%%%%  
tpi(com,start_group,mc).  
thf(phi_type,type,(phi:$i>$o)).  
thf(thmMC_con,conjecture,(v@(mimplies@phi@(mbox@phi)))).  
tpi(com,end_group,mc).  
tpi(com,execute_async,'Szs_Status_MC' =  
'LEO-II---1.6.0 40 $getgroups(embedding,symmetry,sig,d2,t2,t3,mc)').  
  
% Analysing flawlessness (asynchronously).  
%%%%%%%%%%%%%  
tpi(com,start_group,fg).  
thf(thmFG_con,conjecture,(v@(mforall_indset@^ [Phi:mu>$i>$o]:(mforall_ind@^ [X:mu]:  
          (mimplies@(g@X)@(mimplies@(mnot@(p@Phi))@(mnot@(Phi@X))))))).  
tpi(com,end_group,fg).  
tpi(com,execute_async,'Szs_Status_FG' = 'LEO-II---1.6.0 40 $getgroups(embedding,sig,alb,d1,fg)').
```

Gödel's God as THF TPI Script

```
% 7b. Report on Inconsistency of Gödel's original axioms.  
%  
tpi(com,waitenv,'Szs_Status_D2orig').  
tpi(com,write,'%%% 7b. Proving inconsistency of Gödel's original axioms (d2orig).').  
tpi(com,write,'%%% Checking d2orig,ala,a2,d3,a5 (using LEO-II).').  
tpi(com,output,stdout = d2orig).  
tpi(com,write,'%%% Szs_Status_D2orig for d2orig is ' & '$getenv(Szs_Status_D2orig)').  
tpi(com,assert,$getenv('Szs_Status_D2orig') = 'Unsatisfiable').  
tpi(com,write,'%%%').  
tpi(com,write,'').  
  
% Analysing modal collapse (asynchronously).  
%  
tpi(com,start_group,mc).  
thf(phi_type,type,(phi:$o)).  
thf(thmMC_con,conjecture,(v@(@(implies  
tpi(com,end_group,mc).  
tpi(com,execute_async,'Szs_Status_MC' =  
'LEO-II---1.6.0 40 $getgroups(embedding,symmetry,sig,d2,t2,t3,mc)').  
  
% Analysing flawlessness (asynchronously).  
%  
tpi(com,start_group,fg).  
thf(thmFG_con,conjecture,(v@(@(mforall_indset@^ [Phi:mu>$i>$o]:(mforall_ind@^ [X:mu]:  
(@(implies@((g@X)@(implies@(@(mnot@(p@Phi))@(@(mnot@(Phi@X)))))))))).  
tpi(com,end_group,fg).  
tpi(com,execute_async,'Szs_Status_FG' = 'LEO-II---1.6.0 40 $getgroups(embedding,sig,alb,d1,fg)').
```

Checking: Gödel's original axioms are inconsistent
(LEO-II)



Gödel's God as THF TPI Script

```
% 7b. Report on Inconsistency of Goedel's original axioms.  
%  
tpi(com,waitenv,'Szs_Status_D2orig').  
tpi(com,write,'%%% 7b. Proving inconsistency of Goedel's original axioms (d2orig).').  
tpi(com,write,'%%% Checking d2orig,ala,a2,d3,a5 (using LEO-II).').  
tpi(com,output,stdout = d2orig).  
tpi(com,write,'%%% Szs_Status_D2orig for d2orig is ' & $getenv('Szs_Status_D2orig')).  
tpi(com,assert,$getenv('Szs_Status_D2orig')).  
tpi(com,write,'%%%').  
tpi(com,write,'').  
  
% Analysing modal collapse (asynchronously).  
%  
tpi(com,start_group,mc).  
thf(phi_type,type,(phi:$o)).  
thf(thmMC_con,conjecture,(v@($implies@($o)))).  
tpi(com,end_group,mc).  
tpi(com,execute_async,'Szs_Status_MC' =  
'LEO-II---1.6.0 40 $getgroups(embedding,symmetry,sig,d2,t2,t3,mc)').  
  
% Analysing flawlessness (asynchronously).  
%  
tpi(com,start_group,fg).  
thf(thmFG_con,conjecture,(v@($forall_indset@(^[$Phi:$mu:$o]):($forall_ind@(^[$X:$mu]):  
($implies@($g@$X)@($implies@($not@($p@$Phi))@($not@($Phi@$X))))))).  
tpi(com,end_group,fg).  
tpi(com,execute_async,'Szs_Status_FG' = 'LEO-II---1.6.0 40 $getgroups(embedding,sig,alb,d1,fg)').
```

Start checking modal collapse (asynchronously, LEO-II)



Gödel's God as THF TPI Script

```
% 7b. Report on Inconsistency of Gödel's original axioms.  
%  
tpi(com,waitenv,'Szs_Status_D2orig').  
tpi(com,write,'%%% 7b. Proving inconsistency of Gödel's original axioms (d2orig).').  
tpi(com,write,'%%% Checking d2orig,ala,a2,d3,a5 (using LEO-II).').  
tpi(com,output,stdout = d2orig).  
tpi(com,write,'%%% Szs_Status_D2orig for d2orig is ' & '$getenv(Szs_Status_D2orig)').  
tpi(com,assert,$getenv('Szs_Status_D2orig') = 'Unsatisfiable').  
tpi(com,write,'%%%').  
tpi(com,write,'').  
  
% Analysing modal collapse (asynchronously).  
%  
tpi(com,start_group,mc).  
thf(phi_type,type,(phi:$i>$o)).  
thf(thmMC_con,conjecture,(v@($implies@($phi@($mbox@$phi))))).  
tpi(com,end_group,mc).  
tpi(com,execute_async,'Szs_Status_MC' =  
'LEO-II---1.6.0 40 $getgroups(embedding,sig,alb,d1,fg)').  
  
% Analysing flawlessness (asynchronously).  
%  
tpi(com,start_group,fg).  
thf(thmFG_con,conjecture,(v@($forall_indset@(^[$Phi:$mu]>$i>$o):($forall_ind@(^[$X:$mu]):  
($implies@($g@$X)@($implies@($mnot@($p@$Phi))@($mnot@($Phi@$X))))))).  
tpi(com,end_group,fg).  
tpi(com,execute_async,'Szs_Status_FG' = 'LEO-II---1.6.0 40 $getgroups(embedding,sig,alb,d1,fg)').
```

Start checking 'flawlessness' of God (LEO-II)



Gödel's God as THF TPI Script

```
% 8. Report on modal collapse.  
%%%%%  
tpi(com,waitenv,'Szs_Status_mc').  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'%% 8. Analysing modal collapse.').  
tpi(com,write,'%% Checking sym,d2,t2,t3 |- mc (using LEO-II).').  
tpi(com,output,stdout = mc).  
tpi(com,write,'%% Szs_Status_mc for mc is ' & '$getenv(Szs_Status_mc)').  
tpi(com,assert,$getenv('Szs_Status_mc') = 'Theorem').  
tpi(com,set_role,thmMC_con = lemma).  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'').  
  
% 9. Report on flawlessness.  
%%%%%  
tpi(com,waitenv,'Szs_Status_fg').  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'%% 9. Analysing flawless god.').  
tpi(com,write,'%% Checking alb,d1 |- fg (using LEO-II).').  
tpi(com,output,stdout = fg).  
tpi(com,write,'%% Szs_Status_fg for fg is ' & '$getenv(Szs_Status_fg)').  
tpi(com,assert,$getenv('Szs_Status_fg') = 'Theorem').  
tpi(com,set_role,thmFG_con = lemma).  
tpi(com,write,'%%%%%%%%%%%%%').  
tpi(com,write,'').
```

```
% 8. Report on modal collapse.  
%%%%%  
tpi(com,waitenv,'Szs_Status_mc').  
tpi(com,write,'%%%%%').  
tpi(com,write,'%%% 8. Analysing modal collapse.').  
tpi(com,write,'%%% Checking sym,d2,t2,t3 |- mc (using LEO-II).').  
tpi(com,output,stdout = mc).  
tpi(com,write,'%%% Szs_Status_mc for mc is ' & '$getenv(Szs_Status_mc)').  
tpi(com,assert,$getenv('Szs_Status_mc') = 'Theorem').  
tpi(com,set_role,thmMC_con = lemma).  
tpi(com,write,'%%%%%').  
tpi(com,write,'').
```

```
% 9. Report on flawlessness.  
%%%%%  
tpi(com,waitenv,'Szs_Status_fg').  
tpi(com,write,'%%%%%').  
tpi(com,write,'%%% 9. Analysing flawless god.').  
tpi(com,write,'%%% Checking alb,d1 |- fg (using LEO-II).').  
tpi(com,output,stdout = fg).  
tpi(com,write,'%%% Szs_Status_fg for fg is ' & '$getenv(Szs_Status_fg)').  
tpi(com,assert,$getenv('Szs_Status_fg') = 'Theorem').  
tpi(com,set_role,thmFG_con = lemma).  
tpi(com,write,'%%%%%').  
tpi(com,write,'').
```

Checking: modal collapse holds (LEO-II)



```
% 8. Report on modal collapse.  
%%%%%  
tpi(com,waitenv,'Szs_Status_mc').  
tpi(com,write,'%%%%% 8. Analysing modal collapse.').  
tpi(com,write,'%%% Checking sym,d2,t2,t3 |- mc (using LEO-II).').  
tpi(com,output,stdout = mc).  
tpi(com,write,'%%% Szs_Status_mc for mc is ' & '$getenv(Szs_Status_mc)').  
tpi(com,assert,$getenv('Szs_Status_mc') = 'Theorem').  
tpi(com,set_role,thmMC_con = lemma).  
tpi(com,write,'%%%%%  
tpi(com,write,'').  
  
% 9. Report on flawlessness.  
%%%%%  
tpi(com,waitenv,'Szs_Status_fg').  
tpi(com,write,'%%%%% 9. Analysing flawless god.').  
tpi(com,write,'%%% Checking alb,d1 |- fg (using LEO-II).').  
tpi(com,output,stdout = fg).  
tpi(com,write,'%%% Szs_Status_fg for fg is ' & '$getenv(Szs_Status_fg)').  
tpi(com,assert,$getenv('Szs_Status_fg') = 'Theorem').  
tpi(com,set_role,thmFG_con = lemma).  
tpi(com,write,'%%%%%  
tpi(com,write,'').
```

Checking: 'flawlessness' of God (LEO-II)



Gödel's God as THF TPI Script

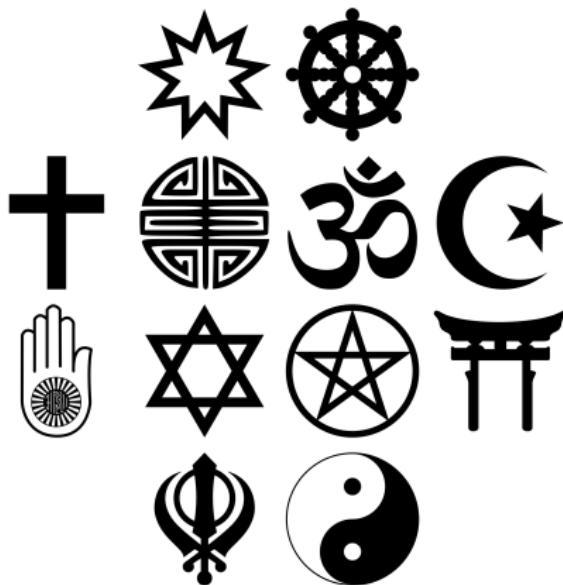
```
% 10. Analysing monotheism.  
%  
tpi(com,start_group,mt).  
thf(thmMT_con,conjecture,(v@(mforall_ind@^ [X:mu]:(mforall_ind@^ [Y:mu]:  
        (mimplies@(g@X)@(mimplies@(g@Y)@(mequals@X@Y)))))).  
tpi(com,end_group,mt).  
tpi(com,execute,'Szs_Status_MT' = 'TPS---3.120601S1b 60 $getgroups(embedding,sig,fg,d1,mt)').  
tpi(com,write,'%%% 10. Analysing monotheism.').  
tpi(com,write,'%%%     Checking fg,d1 |- mt (using TPS.)').  
%tpi(com,output,stdout = tpi_premises).  
%tpi(com,output,stdout = tpi_conjectures).  
tpi(com,output,stdout = mt).  
tpi(com,write,'%%%     Szs_Status_MT for mt is ' & '$getenv(Szs_Status_MT)').  
tpi(com,assert,$getenv('Szs_Status_MT') = 'Theorem').  
tpi(com,write,'%%%').  
tpi(com,write,'').  
  
tpi(end,exit,exit).
```

Gödel's God as THF TPI Script

```
% 10. Analysing monotheism.  
%  
tpi(com,start_group,mt).  
thf(thmMT_con,conjecture,(v@(mforall_ind@^ [X:mu]:(mforall_ind@^ [Y:mu]:  
        (mimplies@(g@X)@(mimplies@(g@Y)@(mequals@X@Y)))))).  
tpi(com,end_group,mt).  
tpi(com,execute,'Szs_Status_MT' = 'TPS---3.120601S1b 60 $getgroups(embedding,sig,fg,d1,mt)').  
tpi(com,write,'%%% 10. Analysing monotheism.').  
tpi(com,write,'%%%     Checking fg,d1 |- mt (using TPS).').  
%tpi(com,output,stdout = tpi_premises).  
%tpi(com,output,stdout = tpi_conjectures).  
tpi(com,output,stdout = mt).  
tpi(com,write,'%%%     Szs_Status_MT for mt is ' & '$getenv(Szs_Status_MT)').  
tpi(com,assert,$getenv('Szs_Status_MT') = 'Theorem').  
tpi(com,write,'%%%').  
tpi(com,write,'').  
  
tpi(end,exit,exit).
```

Proving:Monotheism (TPS)





Part C: Formalization and Verification in Coq

- Goal: verification of the natural deduction proof
 - Step-by-step formalization
 - Almost no automation (intentionally!)
- Interesting facts:
 - Embedding is transparent to the user
 - Embedding gives labeled calculus for free

The screenshot shows the CoqIDE interface with the title bar "Coqide". The menu bar includes File, Edit, Navigation, Try Tactics, Templates, Queries, Display, Compile, Windows, Help. The toolbar has icons for file operations like Open, Save, Print, and search. The left sidebar lists files: "scratch", "Modal.v", "ModalClassical.v", and "GoedelGod-Scott.v". The main window displays a Coq proof script and its state.

```

(* Constant predicate that distinguishes positive properties *)
Parameter Positive: (u -> o) -> o.

(* Axiom A1: either a property or its negation is positive, but not both *)
Axiom axiomia : V (mforall p, (Positive (fun x: u => m-(p x))) m-> (m- (Positive p))). 
Axiom axiomib : V (mforall p, (m- (Positive p)) m-> (Positive (fun x: u => m- (p x)))). 

(* Axiom A2: a property necessarily implied by a positive property is positive *)
Axiom axiom2: V (mforall p, mforall q, Positive p m/\ (box (mforall x, (p x) m-> (q x)))). 

(* Theorem T1: positive properties are possibly exemplified *)
Theorem theorem1: V (mforall p, (Positive p) m-> dia (mexists x, p x)).
Proof.
intro.
intro p.
intro H1.
proof_by_contradiction H2.
apply not_dia_box_not_in H2.
assert (H3: (box (mforall x, m- (p x))) w). (* Lemma from Scott's notes *)
  box_intro wl R1.
  intro x.
  assert (H4: ((m- (mexists x : u, p x)) wl)).
    box_elim H2 wl R1 G2.
    exact G2.

  clear H2 R1 H1 w.
  intro H5.
  apply H4.
  exists x.
  exact H5.

assert (H6: ((box (mforall x, (p x) m-> m- (x m= x))) w)). (* Lemma from Scott's notes *)
  box_intro wl R1.
  intro x.
  intro H7.
  intro H8.
  box_elim H3 wl R1 G3.
  annTu ct with / \_ := _.

```

The right pane shows the proof state:

```

2 subgoals
w : i
p : u -> o
H1 : Positive p w
H2 : box (m- (mexists x : u, p x)) w
box (mforall x : u, m- p x) w
                                         (1/2)
                                         (2/2)
False

```



Part D:

Automation and Verification in IsABELLE/HOL

Isabelle

http://isabelle.in.tum.de/index.html

Home-FU 2012-Watson Homepage 2012-FOL SPIEGEL 2012-FOL-Home GMail Google Maps M&M SigmaOnline Kita Sigma Kalender Beliebt Google Maps >>

Isabelle



Isabelle

UNIVERSITY OF CAMBRIDGE
Computer Laboratory

TUM
TECHNISCHE UNIVERSITÄT MÜNCHEN

What is Isabelle?

Isabelle is a generic proof assistant. It allows mathematical formulas to be expressed in a formal language and provides tools for proving those formulas in a logical calculus. Isabelle is developed at University of Cambridge ([Larry Paulson](#)), Technische Universität München ([Tobias Nipkow](#)) and Université Paris-Sud ([Makarius Wenzel](#)). See the [Isabelle overview](#) for a brief introduction.

Now available: Isabelle2013

Download for Mac OS X



[Download for Linux](#) - [Download for Windows](#)

Some highlights:

- Improvements of Isabelle/Scala and Isabelle/Edit Prover IDE.
- Advanced build tool based on Isabelle/Scala.
- Updated manuals: isar-ref, implementation, system.
- Pure: improved support for block-structured specification contexts.
- HOL tool enhancements: Sledgehammer, Lifting, Quickcheck.
- HOL library enhancements: HOL-Library, HOL-Probability, HOL-Cardinals.
- HOL: New BNF-based (co)datatype package.
- Improved performance thanks to PolyML 5.5.0.

See also the cumulative [NEWS](#).

Distribution & Support

Isabelle is distributed for free under the BSD license. It includes source and binary packages and documentation, see the detailed [installation instructions](#). A vast collection of Isabelle examples and applications is available from the [Archive of Formal Proofs](#).

Support is available by ample [documentation](#), the [Isabelle Community Wiki](#), and the following mailing lists:

- isabelle-users@cl.cam.ac.uk provides a forum for Isabelle users to discuss problems, exchange information, and make announcements. Users of official Isabelle releases should [subscribe](#) or see the [archive](#) (also available via [Google groups](#) and [Narkive](#)).
- isabelle-dev@in.tum.de covers the Isabelle development process, including intermediate repository versions, and administrative issues concerning the website or testing infrastructure. Early adopters of [repository versions](#) should [subscribe](#) or see the [archive](#) (also available at [mail-archive.com](#) or [gmane.org](#)).

Last updated: 2013-03-09 12:21:39

Isabelle/HOL (Cambridge University/TU Munich)

- HOL instance of the generic IsABELLE proof assistant
- User interaction and proof automation
- Automation is supported by SLEDGEHAMMER tool
- Verification of the proofs in IsABELLE/HOL's small proof kernel

What we did?

- Proof automation of Gödel's proof script (Scott's version)
- SLEDGEHAMMER makes calls to remote THF provers in Miami
- These calls suggest respective calls to the METIS prover
- METIS proofs are verified in IsABELLE/HOL's proof kernel

— see the handout (generated from the Isabelle source file) —

Automation & Verification in Proof Assistant IsABELLE/HOL

The screenshot shows the Isabelle/HOL proof assistant interface. The main window displays the theory file `GoedelGod.thy` (modified). The code includes several Sledgehammer annotations, indicated by blue underlines and yellow highlights. The annotations are:

- `sledgehammer [provers = remote_leo2] by (metis A3 T1)`
- `sledgehammer [provers = remote_leo2] by (metis A1b A4 G_def ess_def)`

The interface includes a vertical sidebar labeled "Sidekick: Theories". At the bottom, there is a status bar showing "Sledgehammering..." and other system information.

```
corollary C: "[o (E G)]"
sledgehammer [provers = remote_leo2] by (metis A3 T1)

text {* Axiom @{text "A4"} is added: $\forall \phi [P(\phi) \rightarrow \Box \; ; \; P(\phi)]$ 
(Positive properties are necessarily positive). *}

axiomatization where A4: "[!! (\lambda\Phi. P \Phi m⇒ □ (P Φ))"

text {* Symbol @{text "ess"} for 'Essence' is introduced and defined as 
$ess{\phi}{x} \; \text{biimp} \; \phi(x) \; \text{wedge} \; \forall \psi \; (\psi(x) \; \text{imp} \; \text{nec} \; \forall y \; (\phi(y) \; 
\text{imp} \; \psi(y)))$ 
(An essence of an individual is a property possessed by it 
and necessarily implying any of its properties). *}

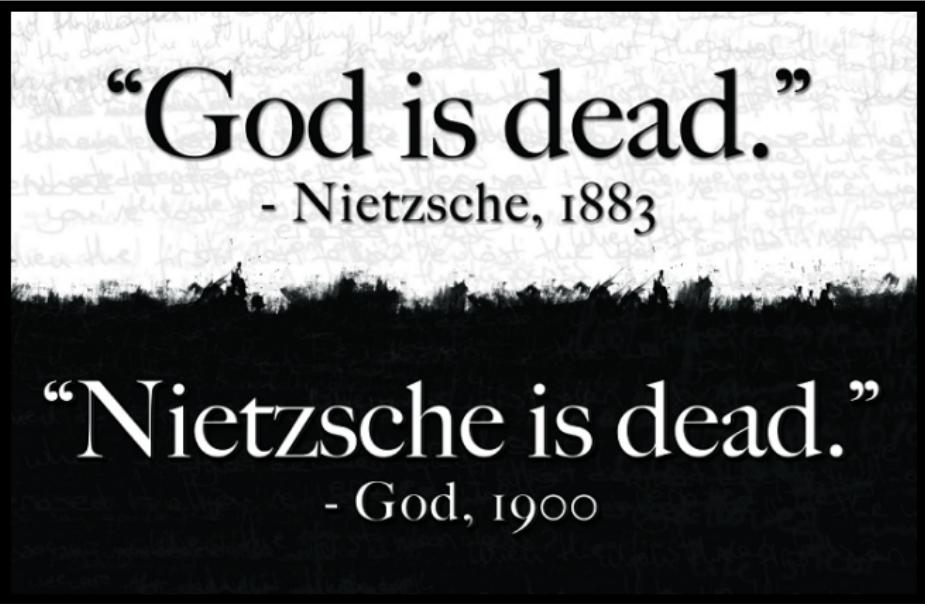
definition ess :: "(μ ⇒ σ) ⇒ μ ⇒ σ" (infixr "ess" 85) where
  "Φ ess x = Φ x m⇒ II (λψ. ψ x m⇒ □ ( ∀ (λy. Φ y m⇒ ψ y)))"

text {* Next, Sledgehammer and Metis prove theorem @{text "T2"}: $\forall x \; [G(x) \; \text{imp} \; ess(G){x}]$ 
(Being God-like is an essence of any God-like being). *}

theorem T2: "[! (λx. G x m⇒ G ess x)]"
sledgehammer [provers = remote_leo2] by (metis A1b A4 G_def ess_def)

text {* Symbol @{text "NE"}, for 'Necessary Existence', is introduced and 
defined as $NE(x) \; \text{biimp} \; \forall \phi \; (\text{ess}{\phi}{x}) \; \text{imp} \; \text{nec} \; \exists y \; (\phi(y))$ 
(Necessary 
existence of an individual is the necessary exemplification of all its essences). *}

definition NE :: "μ ⇒ σ" where "NE = (λx. II (λΦ. Φ ess x m⇒ □ ( ∃ Φ)))"
```



“God is dead.”

- Nietzsche, 1883

“Nietzsche is dead.”

- God, 1900

Part E: Criticisms

$$\forall P. [\diamond \Box P \rightarrow \Box P]$$

If something is possibly necessary, then it is necessary.

$$\diamond \Box (A \vee \neg A) \qquad \Box (A \vee \neg A)$$

S5 usually considered adequate

(But KB is sufficient! — shown by HOL ATPs)

$$\forall P. [\diamond \Box P \rightarrow \Box P]$$

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If something is possibly necessary, then it is necessary.

$$\diamond \Box (A \vee \neg A) \qquad \Box (A \vee \neg A)$$

S5 usually considered adequate

(But KB is sufficient! — shown by HOL ATPs)

$$\forall P. [\diamond \Box P \rightarrow \Box P]$$

If something is possibly necessary, then it is necessary.

$$\diamond_c \Box_c (A \vee \neg A) \quad \Box_c (A \vee \neg A)$$

S5 usually considered adequate

(But KB is sufficient! — shown by HOL ATPs)

$$\forall P.[P \rightarrow \Box P]$$

Everything that is the case is so necessarily.

Follows from T2, T3 and D2 (as shown by HOL ATPs).

There are no contingent “truths”.

Everything is determined.

There is no free will.

Many proposed solutions: Anderson, Fitting, Hájek, ...

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$$\forall \phi [P(\neg\phi) \leftrightarrow \neg P(\phi)]$$

Either a property is positive or its negation is (but never both)

Are the following properties positive or negative?

$$\lambda x.G(x) \quad \lambda x.NE(x) \quad \lambda x.x = x \quad \lambda x.\top$$

$$\lambda x.blue(x) \quad \lambda x.punishing(x) \quad \lambda x.human(x)$$

Solution:

“... positive in the moral aesthetic sense (independently of the accidental structure of the world). Only then the ax. true. ...”

- Gödel, 1970

See also my extended Isabelle formalization at:

<https://github.com/FormalTheology/GoedelGod/blob/master/Formalizations/Isabelle/DivineVersion/GoedelGodDivine.thy>

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Are the following properties positive or negative?

$$\lambda x.G(x) \quad \lambda x.NE(x) \quad \lambda x.x = x \quad \lambda x.T$$

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Part F: Conclusions

The (**new**) insights we gained from experiments include:

- Logic K sufficient for T1, C and T2
- Logic S5 not needed for T3
- Logic KB sufficient for T3 (not well known)
- We found a simpler new proof of C
- Gödel's axioms (without conjunct $\phi(x)$ in D2) are inconsistent
- Scott's axioms are consistent
- For T1, only half of A1 (A1a) is needed
- For T2, the other half (A1b) is needed

Our novel contributions to the theorem proving community include

- Powerful infrastructure for reasoning with QML
- A new natural deduction calculus for higher-order modal logic
- Difficult new benchmarks problems for HOL provers
- Huge media attention

What have we achieved

- Verification of Gödel's ontological argument with HOL provers
 - exact parameters known: constant domain quantification, Henkin Semantics
 - experiments with different parameters could be performed
- Gained some novel results and insights
- Major step towards **Computer-assisted Theoretical Philosophy**
 - see also Ed Zalta's *Computational Metaphysics* project at Stanford University
 - see also John Rushby's recent verification of Anselm's proof in PVS
 - remember Leibniz' dictum — *Calculemus!*
- Interesting bridge between CS, Philosophy and Theology

Ongoing and future work

- Formalize and verify literature on ontological arguments
 - ... in particular the criticism and improvements to Gödel
- Own contributions — supported by theorem provers

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Ongoing and future work

- Formalize and verify literature on ontological arguments
 - ... in particular the criticism and improvements to Gödel
- Own contributions — supported by theorem provers

I'm sure that God would be impressed with your proof, if only he existed :-)

Larry

Die Philosophen können so schön staunen.

Sie packen Dinge in Begriffe (gucken dabei in die Luft) werfen die Begriffe dann in ihre Philosophiekiste, schütteln ganz dolle, und freuen sich, dass ganz genau rauskommt, was sie vorher reingetan haben. Und das geht sogar, wenn eine Maschine die Kiste schüttelt.

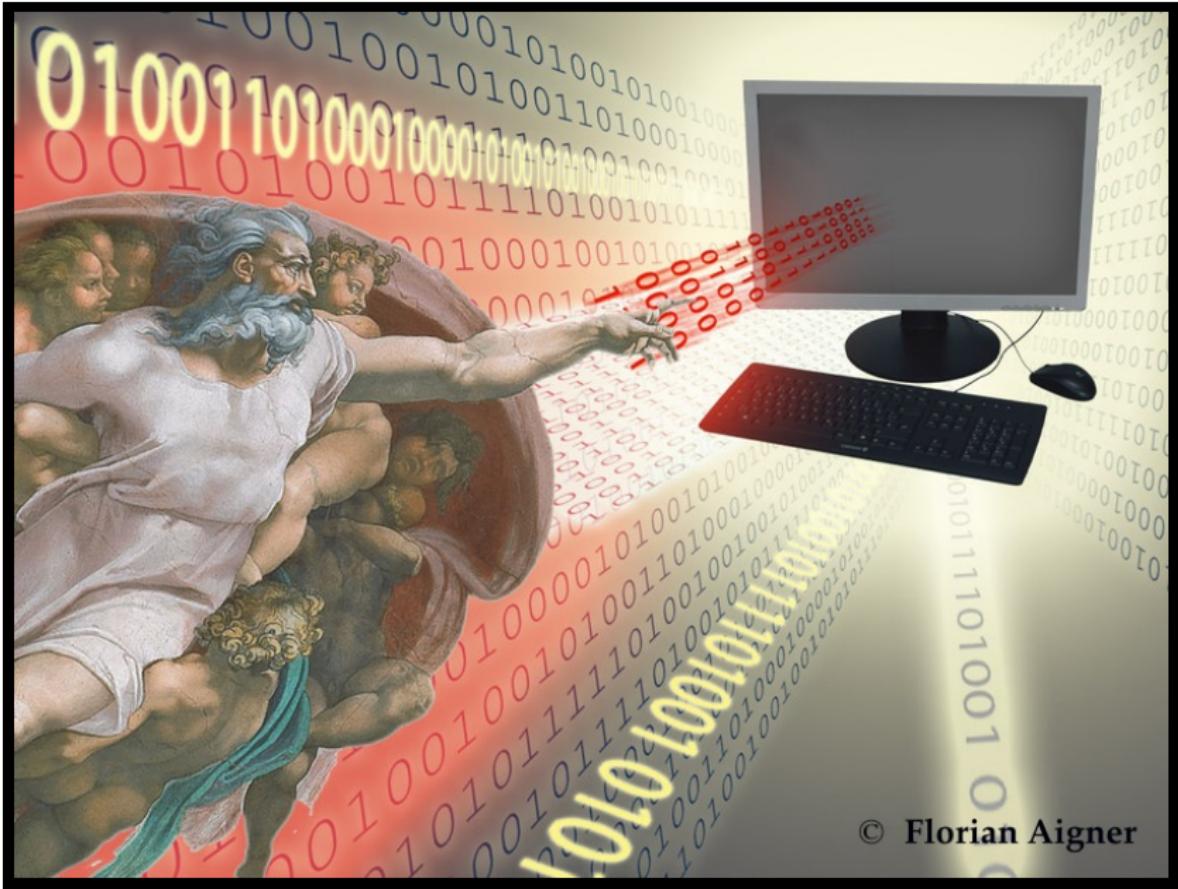
Unerstaunt
2017cp

60. Suchlauf

 souveränsatt 09.09.2013

man kann auch auf andere Weise in diesem Zusammenhang methodisch vorgehen:
bei einer längeren Autofahrt das Radio auf automatischen Suchlauf stellen. Nach zwei Tagen sieht man Gott

... find more on the internet ...



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