

# Deus Ex Machina

## Computer-Assisted Formalization of Gödel's Proof of God's Existence

Christoph Benzmüller and Bruno Woltzenlogel Paleo

6th of November 2013

$$\frac{\text{Axiom 3} \quad \frac{\overline{\forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x. \varphi(x)]}}{P(G) \rightarrow \Diamond \exists x. G(x)} \text{Theorem 1}}{\Diamond \exists x. G(x)} \rightarrow_E \forall_E$$

A gift to **Priest Edvaldo** and his church in Piracicaba, Brazil

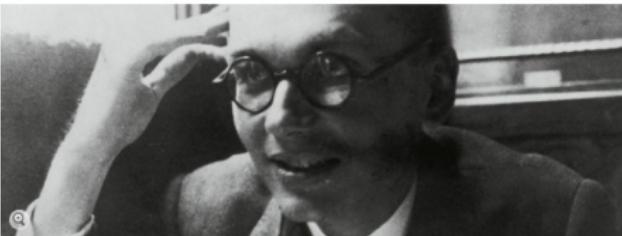
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Nachrichten > Wissenschaft > Mensch > Mathematik > Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

## Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Von Tobias Hürtler



Kurt Gödel (um das Jahr 1935): Der Mathematiker hielt seinen Gottesbeweis Jahrzehntlang geheim

picture-alliance/ Imagoeconomica/ Wiener Stadt- und Landesbibliothek

**Ein Wesen existiert, das alle positiven Eigenschaften in sich vereint. Das bewies der legendäre Mathematiker Kurt Gödel mit einem komplizierten Formelegebilde. Zwei Wissenschaftler haben diesen Gottesbeweis nun überprüft - und für gültig befunden.**

Montag, 09.09.2013 – 12:03 Uhr  
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Jetzt sind die letzten Zweifel ausgeräumt: Gott existiert tatsächlich. Ein Computer hat es mit kalter Logik bewiesen - das MacBook des Computerwissenschaftlers Christoph Benzmüller von der Freien Universität Berlin.

## Germany

- Telepolis & Heise
- Spiegel Online
- FAZ
- Die Welt
- Berliner Morgenpost
- Hamburger Abendpost

## Austria

- Die Presse
- Wiener Zeitung
- ORF

## International

- Spiegel International
- Yahoo Finance
- United Press Intl.

## India

- DNA India
- Delhi Daily News
- India Today

## US

- ABC News

## Italy

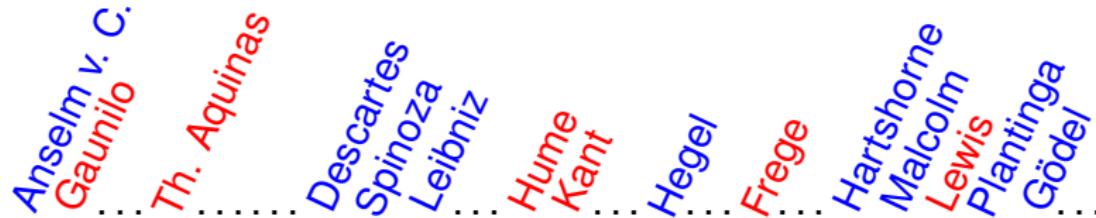
- Repubblica

Spain, Russia, Brazil, Bulgaria

...

# A Long History

pros and cons



Anselm's notion of God:

*“God is that, than which nothing greater can be conceived.”*

Gödel's notion of God:

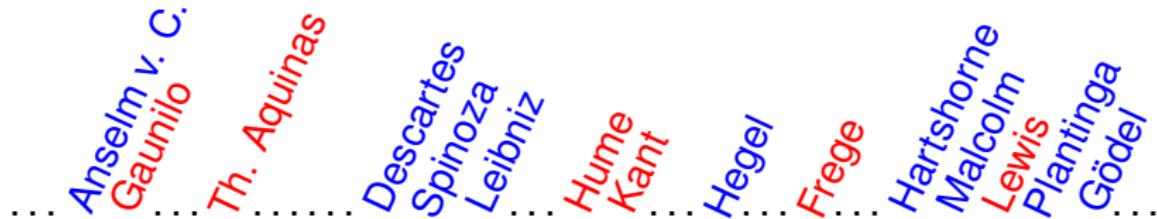
*“A God-like being possesses all ‘positive’ properties.”*

To show by logical reasoning:

*“(Necessarily) God exists.”*

# A Long History

pros and cons



Anselm's notion of God:

*"God is that, than which nothing greater can be conceived."*

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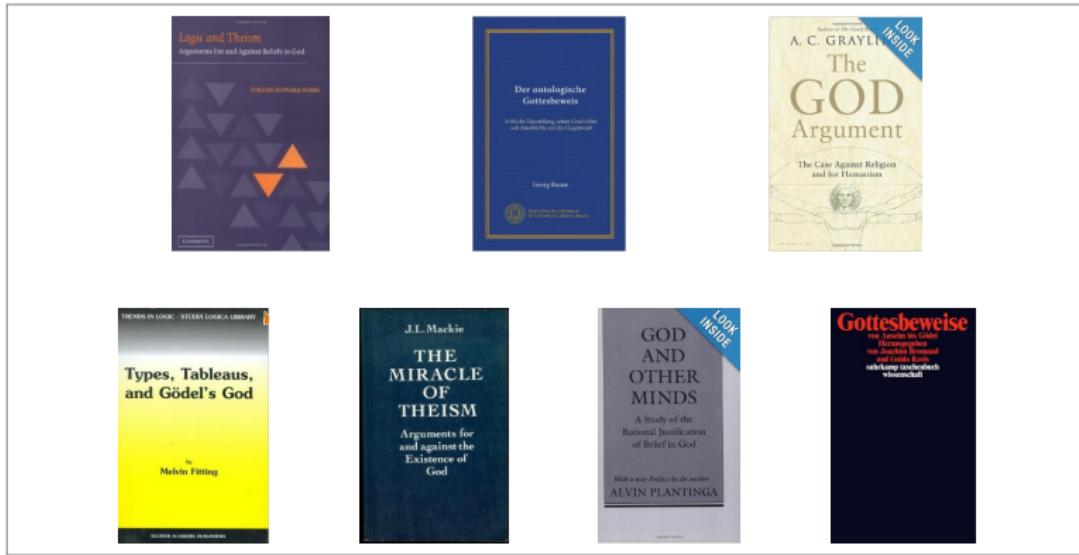
*"A God-like being possesses all 'positive' properties."*

To show by logical reasoning:

*"(Necessarily) God exists."*

## *Wohl eine jede Philosophie kreist um den ontologischen Gottesbeweis*

(Adorno, Th. W.: Negative Dialektik. Frankfurt a. M. 1966, p.378)



- Philosophical:
  - Limits of metaphysics & epistemology
  - *Metaphysical* versus *logical* necessary existence
- Theological:
  - Investigations of the nature of God
  - Arguments to convince atheists
- Computational: can automated reasoners be used ...
  - ... to formalize the definitions, axioms and theorems?
  - ... to verify the arguments step-by-step?
  - ... to fully automate (sub-)arguments?

*“Computer-assisted Theoretical Philosophy”*

# Gödel's Manuscript: 1930's, 1941, 1946-1955, 1970

Ontologischer Beweis

FEB 10, 1970

$P(\varphi)$ :  $\varphi$  is positive ( $\Leftrightarrow \varphi \in P$ )

At 1:  $P(\varphi), P(\psi) \Rightarrow P(\varphi \wedge \psi)$  • At 2:  $P(\varphi) \wedge P(\neg \varphi)$

P1:  $G(x) \equiv (\varphi)[P(\varphi) \Rightarrow \varphi(x)]$  (Ged.)

P2:  $\varphi_{\text{Em},x} \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$  (Emm.  $\forall x$ )

$P \supset_N = N(P \supset \varphi)$  Necessity

At 2:  $P(\varphi) \supset N P(\varphi)$  } because it follows  
 $\neg P(\varphi) \supset N \neg P(\varphi)$  } from the nature of the  
 property

Th.:  $G(x) \supset \varphi_{\text{Em},x}$

Df.:  $E(x) \equiv \exists \varphi[\varphi_{\text{Em},x} \supset N \exists x \varphi(x)]$  necessary Existence

At 3:  $P(E)$

Th.:  $G(x) \supset N(\exists y) G(y)$

$(\exists x) G(x) \supset N(\exists y) G(y)$

"  $M(\exists x) G(x) \supset M N(\exists y) G(y)$

MI = partibility

any two instances of  $x$  are nec. equivalent

exclusive or \* and for any number of instances

$M(\exists x) G(x)$ : means all pos. propo. w.r.t. com-  
 patible. This is true because of:

At 4:  $P(\varphi), \varphi \supset_N \psi \Rightarrow P(\psi)$  which impl.  
~~the system~~ {  $x=x$  is positive  
~~the system~~ {  $x \neq x$  is negative

But if a system S of pos. propo. were incom-  
 patible it would mean that the comp. s (which  
 is positive) would be  $x \neq x$

Positive means positive in the moralistic  
 sense (independently of the accidental structure of  
 the world). Only then the at time. It may  
 also mean "attribution" as opposed to "privation"  
 (or containing privation). This is Gödel's problem point

$\neg P(\varphi)$  positive w.r.t.  $(\exists x) N \supset \varphi(x)$ . Otherwise  $\neg P(\varphi) \supset x \neq$   
 hence  $x \neq x$ , but we have  $x=x$  contrary At.  
 or the definition of pos. At. 2

X i.e. the normal form in terms of elem. propo. contains a  
 member without negation.

**A1** Either a property or its negation is positive, but not both:  $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

**A2** A property necessarily implied

by a positive property is positive:

$$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

$$\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$$

**T1** Positive properties are possibly exemplified:

**D1** A *God-like* being possesses all positive properties:

$$G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$$

$$P(G)$$

**A3** The property of being God-like is positive:

**C** Possibly, God exists:

$$\Diamond\exists xG(x)$$

**A4** Positive properties are necessarily positive:

$$\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$$

**D2** An *essence* of an individual is

a property possessed by it and

necessarily implying any of its properties:  $\phi \text{ ess. } x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

**T2** Being God-like is an essence of any God-like being:

$$\forall x[G(x) \rightarrow G \text{ ess. } x]$$

**D3** *Necessary existence* of an individual is

the necessary exemplification of all its essences:

$$NE(x) \leftrightarrow \forall\phi[\phi \text{ ess. } x \rightarrow \Box\exists y\phi(y)]$$

$$P(NE)$$

**A5** Necessary existence is a positive property:

**T3** Necessarily, God exists:

$$\Box\exists xG(x)$$



## Proof Overview

T3:  $\Box \exists x.G(x)$

**C1:**  $\Diamond \exists z.G(z)$

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**T3:**  $\Box \exists x.G(x)$

$$\frac{\mathbf{C1: } \diamond \exists z. G(z) \quad \mathbf{L2: } \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\mathbf{T3: } \Box \exists x. G(x)}$$

**L2:**  $\diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$

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**C1:**  $\diamond \exists z. G(z)$       **L2:**  $\diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$   
**T3:**  $\Box \exists x. G(x)$

**S5**  
 $\neg \forall \xi. [\neg \diamond \square \xi \rightarrow \square \neg \xi]$

---

**L2:**  $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

**C1:**  $\diamond \exists z. G(z)$

**L2:**  $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

---

**T3:**  $\square \exists x. G(x)$

$$\frac{\begin{array}{c} \diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x) \\ \hline \textbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x) \end{array}}{\begin{array}{c} \forall \xi. [\neg \diamond \square \xi \rightarrow \neg \square \xi] \\ \hline \textbf{S5} \end{array}}$$

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$$\frac{\begin{array}{c} \textbf{C1: } \diamond \exists z. G(z) \\ \textbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x) \\ \hline \textbf{T3: } \square \exists x. G(x) \end{array}}{\quad}$$

$$\frac{\begin{array}{c} \mathbf{L1: } \exists z. G(z) \rightarrow \Box \exists x. G(x) \\ \hline \Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x) \end{array}}{\mathbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}$$
$$\frac{\mathbf{S5} \quad \forall \xi. [\neg \Diamond \Box \xi \rightarrow \Box \neg \xi]}{\neg \forall \xi. [\Diamond \Box \xi \rightarrow \Box \neg \xi]}$$

$$\frac{\mathbf{C1: } \Diamond \exists z. G(z) \quad \mathbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\mathbf{T3: } \Box \exists x. G(x)}$$

**D1:**  $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

$$\frac{\frac{\textbf{L1: } \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)} \quad \textbf{S5}}{\textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}$$

$$\frac{\textbf{C1: } \Diamond \exists z. G(z) \quad \textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\textbf{T3: } \Box \exists x. G(x)}$$

**D1:**  $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

**D3\*:**  $NE(x) \equiv \square \exists y. G(y)$

$$\frac{\frac{\frac{P(NE)}{\textbf{L1: } \exists z. G(z) \rightarrow \square \exists x. G(x)}}{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}}{\textbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)} \quad \textbf{S5: } \overline{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}$$


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$$\frac{\textbf{C1: } \diamond \exists z. G(z) \quad \textbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}{\textbf{T3: } \square \exists x. G(x)}$$

**D1:**  $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

**D3\*:**  $NE(x) \equiv \Box \exists y. G(y)$  (cheating!)

$$\frac{\frac{\frac{P(NE)}{\textbf{L1: } \exists z. G(z) \rightarrow \Box \exists x. G(x)}}{\Box \exists z. G(z) \rightarrow \Box \exists x. G(x)}}{\textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)} \quad \textbf{S5: } \overline{\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]}$$


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$$\frac{\textbf{C1: } \Diamond \exists z. G(z) \quad \textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\textbf{T3: } \Box \exists x. G(x)}$$

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**D3:**  $NE(x) \equiv \forall \varphi.[\varphi \text{ ess } x \rightarrow \square \exists y.\varphi(y)]$

**T2:**  $\forall y.[G(y) \rightarrow G \text{ ess } y]$   $P(NE)$

**L1:**  $\exists z.G(z) \rightarrow \square \exists x.G(x)$

$\frac{\diamond \exists z.G(z) \rightarrow \diamond \square \exists x.G(x)}{\quad}$

**S5**

$\neg \forall \xi. [\neg \diamond \square \xi \rightarrow \neg \square \xi]$

**L2:**  $\diamond \exists z.G(z) \rightarrow \square \exists x.G(x)$

**C1:**  $\diamond \exists z.G(z)$

**L2:**  $\diamond \exists z.G(z) \rightarrow \square \exists x.G(x)$

**T3:**  $\square \exists x.G(x)$

**D1:**  $G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$

**D3\*:**  $NE(x) \equiv \Box \exists y.G(y)$

**D3:**  $NE(x) \equiv \forall \varphi.[\varphi \text{ ess } x \rightarrow \Box \exists y.\varphi(y)]$

$$\frac{\begin{array}{c} \textbf{T2: } \forall y.[G(y) \rightarrow G \text{ ess } y] \\ \hline \textbf{L1: } \exists z.G(z) \rightarrow \Box \exists x.G(x) \end{array}}{\Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x)} \quad \frac{\textbf{A5} \quad \overline{P(NE)}}{\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]} \quad \frac{\textbf{S5} \quad \overline{\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]}}{\textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}$$

$$\frac{\textbf{C1: } \Diamond \exists z.G(z) \quad \textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\textbf{T3: } \Box \exists x.G(x)}$$

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$$\frac{\begin{array}{c} \textbf{T2: } \forall y.[G(y) \rightarrow G \text{ ess } y] \\ \hline \textbf{L1: } \exists z.G(z) \rightarrow \square \exists x.G(x) \end{array}}{\frac{\diamond \exists z.G(z) \rightarrow \diamond \square \exists x.G(x)}{\textbf{L2: } \diamond \exists z.G(z) \rightarrow \square \exists x.G(x)}} \quad \frac{\textbf{A5} \quad \overline{P(NE)}}{\forall \xi. [\overline{\diamond \square \xi} \rightarrow \overline{\square \xi}]} \quad \frac{}{\textbf{S5} \quad \overline{\forall \xi. [\overline{\diamond \square \xi} \rightarrow \overline{\square \xi}]}}$$

$$\frac{\textbf{C1: } \diamond \exists z.G(z) \quad \textbf{L2: } \diamond \exists z.G(z) \rightarrow \square \exists x.G(x)}{\textbf{T3: } \square \exists x.G(x)}$$

$$\mathbf{D1: } G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$$

$$\mathbf{D2: } \varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \square \forall x. (\varphi(x) \rightarrow \psi(x)))$$

$$\mathbf{D3^*: } NE(x) \equiv \square \exists y. G(y)$$

$$\mathbf{D3: } NE(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$$

$$\begin{array}{c}
 \dfrac{\neg \forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)] \quad \neg \forall \varphi. [P(\varphi) \rightarrow \square \neg P(\varphi)]}{\mathbf{T2: } \forall y. [G(y) \rightarrow G \text{ ess } y]} \qquad \dfrac{}{P(NE)} \\
 \hline
 \dfrac{\mathbf{L1: } \exists z. G(z) \rightarrow \square \exists x. G(x) \quad \neg \forall \xi. [\diamond \square \xi \rightarrow \neg \square \xi]}{\mathbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)} \qquad \mathbf{S5} \\
 \hline
 \mathbf{C1: } \diamond \exists z. G(z) \qquad \mathbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x) \\
 \hline
 \mathbf{T3: } \square \exists x. G(x)
 \end{array}$$

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$$\mathbf{D3: } NE(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$$

$$\mathbf{C1: } \diamond \exists z. G(z)$$

$$\frac{\begin{array}{c} \mathbf{A1b} \\ \neg \forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)] \end{array}}{\mathbf{T2: } \forall y. [G(y) \rightarrow G \text{ ess } y]} \quad 
 \frac{\begin{array}{c} \mathbf{A4} \\ \neg \forall \varphi. [P(\varphi) \rightarrow \square \neg P(\varphi)] \end{array}}{\mathbf{A5} \quad \neg P(NE)} \quad 
 \frac{\mathbf{L1: } \exists z. G(z) \rightarrow \square \exists x. G(x)}{\frac{\mathbf{L2: } \diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}{\frac{\mathbf{S5} \quad \neg \forall \xi. [\diamond \square \xi \rightarrow \neg \square \xi]}{\mathbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}}}}$$

$$\mathbf{C1: } \diamond \exists z. G(z)$$

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**D3\***:  $NE(x) \equiv \square \exists y. G(y)$

**D3:**  $NE(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

$P(G)$

---

**C1:**  $\diamond \exists z. G(z)$

**A1b**  
 $\neg \forall \varphi. \neg P(\varphi) \rightarrow P(\neg \varphi)$

**A4**  
 $\neg \forall \varphi. P(\varphi) \rightarrow \neg \square P(\varphi)$

**A5**  
 $\neg P(NE)$

---

**T2:**  $\forall y. [G(y) \rightarrow G \text{ ess } y]$

**L1:**  $\exists z. G(z) \rightarrow \neg \exists x. G(x)$   
 $\neg \exists z. G(z) \rightarrow \diamond \exists x. G(x)$

**S5**  
 $\neg \forall \xi. [\diamond \square \xi \rightarrow \square \xi]$

---

**L2:**  $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

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**C1:**  $\diamond \exists z. G(z)$

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**T3:**  $\square \exists x. G(x)$

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**D3:**  $NE(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

**A3**  
 $\frac{}{P(G)}$

---

**C1:**  $\diamond \exists z. G(z)$

**A1b**  
 $\frac{\neg \forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]}{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]}$

**A4**  
 $\frac{\neg \forall \varphi. [P(\varphi) \rightarrow \square \neg P(\varphi)]}{\forall \varphi. [P(\varphi) \rightarrow \square \neg P(\varphi)]}$

**A5**  
 $\frac{}{P(NE)}$

---

**T2:**  $\forall y. [G(y) \rightarrow G \text{ ess } y]$

**L1:**  $\exists z. G(z) \rightarrow \square \exists x. G(x)$   
 $\frac{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}$

**S5**  
 $\frac{\neg \forall \xi. [\diamond \square \xi \rightarrow \square \xi]}{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}$

---

**L2:**  $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

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**L2:**  $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

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**D1:**  $G(x) \equiv \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$

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**D3:**  $NE(x) \equiv \forall \varphi.[\varphi \text{ ess } x \rightarrow \square \exists y.\varphi(y)]$

**A3**  
 $\frac{}{P(G)}$

**T1:**  $\forall \varphi.[P(\varphi) \rightarrow \diamond \exists x.\varphi(x)]$

**C1:**  $\diamond \exists z.G(z)$

**A1b**

$\frac{}{\neg \forall \varphi. \neg P(\varphi) \rightarrow P(\neg \varphi)}$

**A4**

$\frac{}{\neg \forall \varphi. P(\varphi) \rightarrow \square \neg P(\varphi)}$

**T2:**  $\forall y.[G(y) \rightarrow G \text{ ess } y]$

**A5**

$\frac{}{P(NE)}$

**L1:**  $\exists z.G(z) \rightarrow \square \exists x.G(x)$

$\frac{\diamond \exists z.G(z) \rightarrow \diamond \square \exists x.G(x)}{\square \exists x.G(x)}$

**S5**

$\frac{}{\neg \forall \xi. \neg \diamond \square \xi \rightarrow \neg \square \xi}$

**L2:**  $\diamond \exists z.G(z) \rightarrow \square \exists x.G(x)$

**C1:**  $\diamond \exists z.G(z)$

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**D1:**  $G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$

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**D3\***:  $NE(x) \equiv \square \exists y. G(y)$

**D3:**  $NE(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

$$\frac{\frac{\frac{\frac{\overline{A3}}{P(\bar{G})} \quad \frac{\overline{A2}}{\overline{\forall \varphi. \forall \psi. [(P(\bar{\varphi}) \wedge \square \forall x. [\bar{\varphi}(x) \rightarrow \bar{\psi}(x)]] \rightarrow \bar{P}(\bar{\psi})]} \quad \frac{\overline{A1a}}{\overline{\forall \varphi. [P(\neg \varphi) \rightarrow \neg P(\bar{\varphi})]}}}{T1: \forall \varphi. [P(\varphi) \rightarrow \diamond \exists x. \varphi(x)]}}}{C1: \diamond \exists z. G(z)}$$

$$\frac{\frac{\frac{\overline{A1b}}{\overline{\forall \varphi. [\neg P(\bar{\varphi}) \rightarrow \bar{P}(\neg \bar{\varphi})]}} \quad \frac{\overline{A4}}{\overline{\forall \varphi. [P(\bar{\varphi}) \rightarrow \square \bar{P}(\bar{\varphi})]}}}{T2: \forall y. [G(y) \rightarrow G \text{ ess } y]} \quad \frac{}{P(NE)}}{\frac{\frac{\frac{L1: \exists z. G(z) \rightarrow \square \exists x. G(x)}{\frac{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}{L2: \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}}}{S5: \overline{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}}}{C1: \diamond \exists z. G(z) \quad L2: \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}}{T3: \square \exists x. G(x)}}$$

$$\mathbf{D1: } G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$$

**D2:**  $\varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \square \forall x. (\varphi(x) \rightarrow \psi(x)))$

**D3:**  $NE(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

<b>A3</b> $\frac{}{P(G)}$	$\frac{\neg \forall \varphi. \neg \forall \psi. [(\bar{P}(\varphi) \wedge \Box \forall x. [\varphi(x) \rightarrow \psi(x)]) \rightarrow \bar{P}(\psi)]}{\textbf{T1: } \forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x. \varphi(x)]}$	<b>A2</b> $\frac{}{\neg \forall \varphi. [P(\neg \varphi) \rightarrow \neg P(\varphi)]}$
		<b>C1:</b> $\Diamond \exists z. G(z)$

<b>A1b</b>	<b>A4</b>	<b>A5</b>
$\neg \forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]$	$\neg \forall \varphi. [P(\varphi) \rightarrow \Box P(\varphi)]$	$P(NE)$
<b>T2:</b> $\forall y. [G(y) \rightarrow \text{G}\text{ess } y]$		
	<b>L1:</b> $\exists z. G(z) \rightarrow \Box \exists x. G(x)$	
	$\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)$	<b>S5</b>
		$\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]$
		<b>L2:</b> $\Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$

$$\frac{\mathbf{C1}: \Diamond \exists z.G(z) \quad \mathbf{L2}: \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\mathbf{T3}: \Box \exists x.G(x)}$$



## Modal Natural Deduction Proof

$$\frac{\overline{A} \quad \overline{B} \quad \vdots \quad \vdots}{\frac{A \vee B \quad C \quad C}{C} \vee_E} \quad \frac{A \quad B}{A \wedge B} \wedge_I \quad \frac{\overline{A} \quad \overline{B} \quad \vdots \quad \vdots}{\frac{B}{A \rightarrow B} \rightarrow_I^n}$$

$$\frac{A}{A \vee B} \vee_{I_1} \quad \frac{A \wedge B}{A} \wedge_{E_1} \quad \frac{B}{A \rightarrow B} \rightarrow_I$$

$$\frac{B}{A \vee B} \vee_{I_2} \quad \frac{A \wedge B}{B} \wedge_{E_2} \quad \frac{A \quad A \rightarrow B}{B} \rightarrow_E$$

$$\frac{A[\alpha]}{\forall x.A[x]} \forall_I \quad \frac{\forall x.A[x]}{A[t]} \forall_E \quad \frac{A[t]}{\exists x.A[x]} \exists_I \quad \frac{\exists x.A[x]}{A[\beta]} \exists_E$$

$$\neg A \equiv A \rightarrow \perp \qquad \frac{\neg\neg A}{A} \neg\neg_E$$

# Natural Deduction Calculus

## Rules for Modalities

$$\alpha : \boxed{\begin{array}{c} \vdots \\ A \end{array}} \quad \frac{}{\Box A} \quad \Box_I$$

$$t : \boxed{\begin{array}{c} \Box A \\ A \\ \vdots \end{array}} \quad \frac{\Box A}{A} \quad \Box_E$$

$$\Diamond A \equiv \neg \Box \neg A$$

$$t : \boxed{\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ A \end{array}} \quad \frac{}{\Diamond A} \quad \Diamond_I$$

$$\beta : \boxed{\begin{array}{c} \Diamond A \\ A \\ \vdots \end{array}} \quad \frac{\Diamond A}{A} \quad \Diamond_E$$

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## Rules for Modalities

$$\alpha : \boxed{\begin{array}{c} \vdots \\ A \end{array}} \quad \frac{}{\Box A} \quad \Box_I$$

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$$\Diamond A \equiv \neg \Box \neg A$$

$$t : \boxed{\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ A \end{array}} \quad \frac{}{\Diamond A} \quad \Diamond_I$$

$$\beta : \boxed{\begin{array}{c} \Diamond A \\ A \\ \vdots \end{array}} \quad \frac{\Diamond A}{A} \quad \Diamond_E$$

# Natural Deduction Calculus

## Rules for Modalities

$$\alpha : \boxed{\begin{array}{c} \vdots \\ A \end{array}} \quad \frac{}{\Box A} \quad \Box_I$$

$$t : \boxed{\begin{array}{c} \Box A \\ A \\ \vdots \end{array}} \quad \frac{\Box A}{A} \quad \Box_E$$

$$\Diamond A \equiv \neg \Box \neg A$$

$$t : \boxed{\begin{array}{c} \vdots \\ \neg \\ A \end{array}} \quad \frac{}{\Diamond A} \quad \Diamond_I$$

$$\beta : \boxed{\begin{array}{c} \Diamond A \\ A \\ \vdots \end{array}} \quad \frac{\Diamond A}{A} \quad \Diamond_E$$

## Natural Deduction Proofs

## T1 and C1

$\frac{\forall \varphi. \forall \psi. [(P(\varphi) \wedge \Box \forall x. [\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]}{\forall_E}$	<b>A2</b>
$\frac{\forall \psi. [(P(\rho) \wedge \Box \forall x. [\rho(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]}{\forall_E}$	$\frac{(P(\rho) \wedge \Box \forall x. [\rho(x) \rightarrow \neg \rho(x)]) \rightarrow P(\neg \rho)}{\forall_E}$
$\frac{(P(\rho) \wedge \Box \forall x. [\neg \rho(x)]) \rightarrow P(\neg \rho)}{\forall_E}$	<b>A1a</b>
$\frac{(P(\rho) \wedge \Box \forall x. [\neg \rho(x)]) \rightarrow \neg P(\rho)}{\forall_E}$	$\frac{\forall \varphi. [P(\neg \varphi) \rightarrow \neg P(\varphi)]}{P(\neg \rho) \rightarrow \neg P(\rho)}$
$\frac{(P(\rho) \wedge \Box \forall x. [\neg \rho(x)]) \rightarrow \neg P(\rho)}{\forall_E}$	
$\frac{\frac{\frac{P(\rho) \rightarrow \Diamond \exists x. \rho(x)}{\forall_I}}{(P(\rho) \wedge \Box \forall x. [\neg \rho(x)]) \rightarrow \neg P(\rho)}}{\forall_E}$	
$\frac{\frac{\frac{P(\rho) \rightarrow \Diamond \exists x. \rho(x)}{\forall_I}}{(P(\rho) \wedge \Box \forall x. [\neg \rho(x)]) \rightarrow \neg P(\rho)}}{\forall_E}$	<b>T1</b>
$\frac{\forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x. \varphi(x)]}{\forall_E}$	
$\frac{\frac{\forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x. \varphi(x)]}{\forall_E}}{P(G) \rightarrow \Diamond \exists x. G(x)}$	<b>T1</b>
$\frac{\frac{\forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x. \varphi(x)]}{\forall_E}}{P(G) \rightarrow \Diamond \exists x. G(x)}$	<b>A3</b>
$\frac{\frac{\forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x. \varphi(x)]}{\forall_E}}{\Diamond \exists x. G(x)}$	$\frac{\frac{\forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x. \varphi(x)]}{\forall_E}}{P(G) \rightarrow \Diamond \exists x. G(x)}$
$\frac{\frac{\forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x. \varphi(x)]}{\forall_E}}{\Diamond \exists x. G(x)}$	$\rightarrow_E$

# Natural Deduction Proofs

T2 (Partial)

$$\frac{\psi(x)^6 \quad \frac{\neg \psi(x) \rightarrow \square P(\psi)}{\square P(\psi)}^{\neg\neg E}}{\psi(x) \rightarrow \square \forall x.(G(x) \rightarrow \psi(x))}^{\rightarrow_E}$$

$$\frac{\square P(\psi)^7 \quad \frac{\neg P(\psi) \rightarrow \forall x.(G(x) \rightarrow \psi(x))}{\forall x.(G(x) \rightarrow \psi(x))}^{\neg\neg E}}{\square \forall x.(G(x) \rightarrow \psi(x))}^{\square I}$$

$$\frac{\square P(\psi) \rightarrow \square \forall x.(G(x) \rightarrow \psi(x))}{\square \forall x.(G(x) \rightarrow \psi(x))}^{\rightarrow_I}$$

$$\frac{\square \forall x.(G(x) \rightarrow \psi(x))}{\psi(x) \rightarrow \square \forall x.(G(x) \rightarrow \psi(x))}^{\rightarrow_E}$$



## Embedding Modal Logic into Higher-Order Logic

Challenge: No provers for *Higher-order Quantified Modal Logic* (**QML**)

Our solution: Embedding in *Higher-order Classical Logic* (**HOL**)

Then use existing **HOL** theorem provers for reasoning in **QML**

[BenzmüllerPaulson, Logica Universalis, 2013]

Previous empirical findings:

Embedding of *First-order Modal Logic* in HOL works well

[BenzmüllerOttenRaths, ECAI, 2012]

[Benzmüller, LPAR, 2013]

**QML**     $\varphi, \psi ::= \dots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x \varphi \mid \exists x \varphi \mid \forall P \varphi$

- Kripke style semantics (possible world semantics)

**HOL**                           $s, t ::= C \mid x \mid \lambda x s \mid s t \mid \neg s \mid s \vee t \mid \forall x t$

- various theorem provers exist

interactive: Isabelle/HOL, HOL4, Hol Light, Coq/HOL, PVS, ...

automated: TPS, LEO-II, Satallax, Nitpick, Isabelle/HOL, ...

## Embedding QML in HOL

**QML**     $\varphi, \psi ::= \dots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x \varphi \mid \exists x \varphi \mid \forall P \varphi$

HOL  $s, t ::= C \mid x \mid \lambda x s \mid s t \mid \neg s \mid s \vee t \mid \forall x t$

Ax

**QML**     $\varphi, \psi ::= \dots | \neg\varphi | \varphi \wedge \psi | \varphi \rightarrow \psi | \Box\varphi | \Diamond\varphi | \forall x \varphi | \exists x \varphi | \forall P \varphi$

**HOL**                           $s, t ::= C | x | \lambda x s | s t | \neg s | s \vee t | \forall x t$

**QML in HOL:**    QML formulas  $\varphi$  are mapped to HOL predicates  $\varphi_{t \rightarrow o}$

$\neg$	$= \lambda \varphi_{t \rightarrow o} \lambda w_t \neg \varphi s$
$\wedge$	$= \lambda \varphi_{t \rightarrow o} \lambda \psi_{t \rightarrow o} \lambda w_t (\varphi s \wedge \psi s)$
$\rightarrow$	$= \lambda \varphi_{t \rightarrow o} \lambda \psi_{t \rightarrow o} \lambda w_t (\neg \varphi s \vee \psi s)$
$\Box$	$= \lambda \varphi_{t \rightarrow o} \lambda w_t \forall u_t (\neg r s u \vee \varphi u)$
$\Diamond$	$= \lambda \varphi_{t \rightarrow o} \lambda w_t \exists u_t (r s u \wedge \varphi u)$
$\forall$	$= \lambda h_{\mu \rightarrow (t \rightarrow o)} \lambda w_t \forall d_\mu h ds$
$\exists$	$= \lambda h_{\mu \rightarrow (t \rightarrow o)} \lambda w_t \exists d_\mu h ds$
$\forall$	$= \lambda H_{(\mu \rightarrow (t \rightarrow o)) \rightarrow (t \rightarrow o)} \lambda w_t \forall d_\mu H ds$
<b>valid</b>	$= \lambda \varphi_{t \rightarrow o} \forall w_t \varphi w$

Ax

The equations in Ax are given as axioms to the HOL provers!

**QML**    $\varphi, \psi ::= \dots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x \varphi \mid \exists x \varphi \mid \forall P \varphi$

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$\Diamond$	$= \lambda\varphi_{t \rightarrow o} \lambda w_t \exists u_t (rsu \wedge \varphi u)$
$\forall$	$= \lambda h_{\mu \rightarrow (t \rightarrow o)} \lambda w_t \forall d_\mu hds$
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Ax

The equations in `Ax` are given as axioms to the `HOL` provers!

## Example

QML formula

QML formula in HOL

expansion,  $\beta\eta$ -conversion

expansion,  $\beta\eta$ -conversion

expansion,  $\beta\eta$ -conversion

$\diamond \exists x G(x)$

valid  $(\diamond \exists x G(x))_{t \rightarrow o}$

$\forall w_t (\diamond \exists x G(x))_{t \rightarrow o} w$

$\forall w_t \exists u_t (rwu \wedge (\exists x G(x))_{t \rightarrow o})$

$\forall w_t \exists u_t (rwu \wedge \exists x Gxu)$

### What are we doing?

In order to prove that  $\varphi$  is valid in QML,

→ we instead prove that valid  $\varphi_{t \rightarrow o}$  can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Expansion:

user or prover may flexibly choose expansion depth

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### What are we doing?

In order to prove that  $\varphi$  is valid in QML,

→ we instead prove that valid  $\varphi_{\iota \rightarrow o}$  can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Expansion:

user or prover may flexibly choose expansion depth

## Example

**QML formula****QML formula in HOL**expansion,  $\beta\eta$ -conversionexpansion,  $\beta\eta$ -conversionexpansion,  $\beta\eta$ -conversion $\Diamond \exists x G(x)$ valid  $(\Diamond \exists x G(x))_{\iota \rightarrow o}$  $\forall w_t (\Diamond \exists x G(x))_{\iota \rightarrow o} w$  $\forall w_t \exists u_t (rwu \wedge (\exists x G(x))_{\iota \rightarrow o} u)$  $\forall w_t \exists u_t (rwu \wedge \exists x Gxu)$ 

## What are we doing?

In order to prove that  $\varphi$  is valid in **QML**,→ we instead prove that **valid**  $\varphi_{\iota \rightarrow o}$  can be derived from **Ax** in **HOL**.This can be done with interactive or automated **HOL** theorem provers.

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## Example

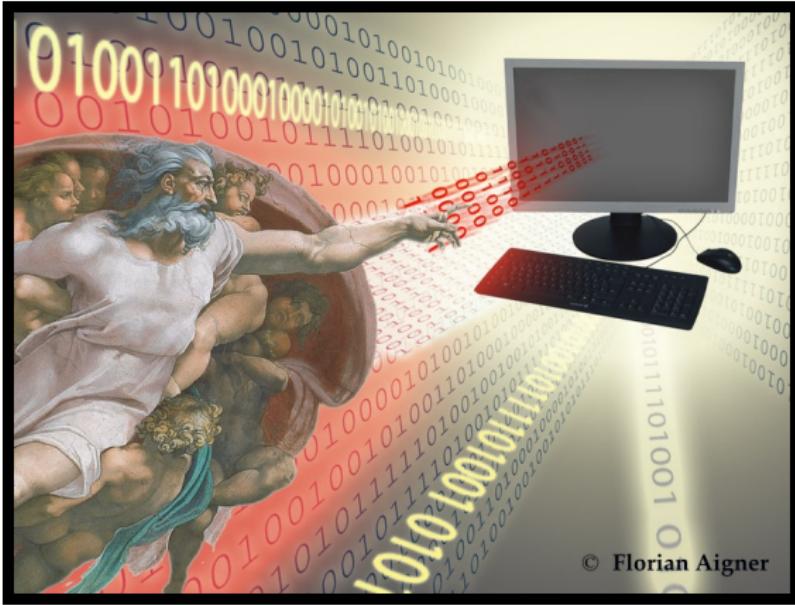
**QML formula****QML formula in HOL**expansion,  $\beta\eta$ -conversionexpansion,  $\beta\eta$ -conversionexpansion,  $\beta\eta$ -conversion $\Diamond \exists x G(x)$ valid  $(\Diamond \exists x G(x))_{\iota \rightarrow o}$  $\forall w_t (\Diamond \exists x G(x))_{\iota \rightarrow o} w$  $\forall w_t \exists u_t (rwu \wedge (\exists x G(x))_{\iota \rightarrow o} u)$  $\forall w_t \exists u_t (rwu \wedge \exists x Gxu)$ 

## What are we doing?

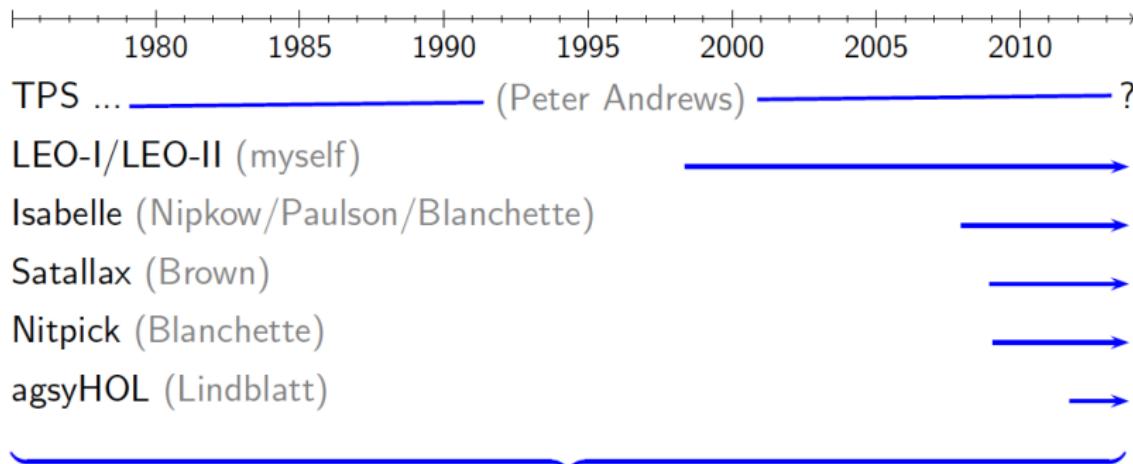
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## Automated Proof Search and Consistency Check



- all accept TPTP THF Syntax [SutcliffeBenzmüller, J.Form.Reas, 2009]
  - can be called remotely via SystemOnTPTP at Miami
  - they significantly gained in strength over the last years
  - they can be bundled into a combined prover **HOL-P**

Exploit HOL with Henkin semantics as metalogic  
Automate other logics (& combinations) via semantic embeddings  
— **HOL-P** becomes a **Universal Reasoner** —

# Proof Automation and Consistency Checking: Demo!

```
MacBook-Chris % Terminal — bash — 125x32
MacBook-Chris %
MacBook-Chris %
MacBook-Chris % ./call_tptp.sh T3.p

Asking various HOL-ATPs in Miami remotely (thanks to Geoff Sutcliffe)

MacBook-Chris % agsyHOL---1.0 : T3.p ++++++ RESULT: SOT_7L4x.Y - agsyHOL---1.0 says Unknown - CPU = 0.00 WC = 0.02
LEO-II---1.6.0 : T3.p ++++++ RESULT: SOT_E4Scha - LEO-II---1.6.0 says Theorem - CPU = 0.03 WC = 0.09
Satallax---2.7 : T3.p ++++++ RESULT: SOT_kVZ1cB - Satallax---2.7 says Theorem - CPU = 0.00 WC = 0.14
Isabelle---2013 : T3.p ++++++ RESULT: SOT_xa0gEp - Isabelle---2013 says Theorem - CPU = 14.06 WC = 17.73 SolvedBy = auto
TPS---3.120601S1b : T3.p ++++++ RESULT: SOT_ROEsgs - TPS---3.120601S1b says Unknown - CPU = 33.56 WC = 41.57
Nitrox---2013 : T3.p ++++++ RESULT: SOT_WGY1Tx - Nitrox---2013 says Unknown - CPU = 75.55 WC = 49.24

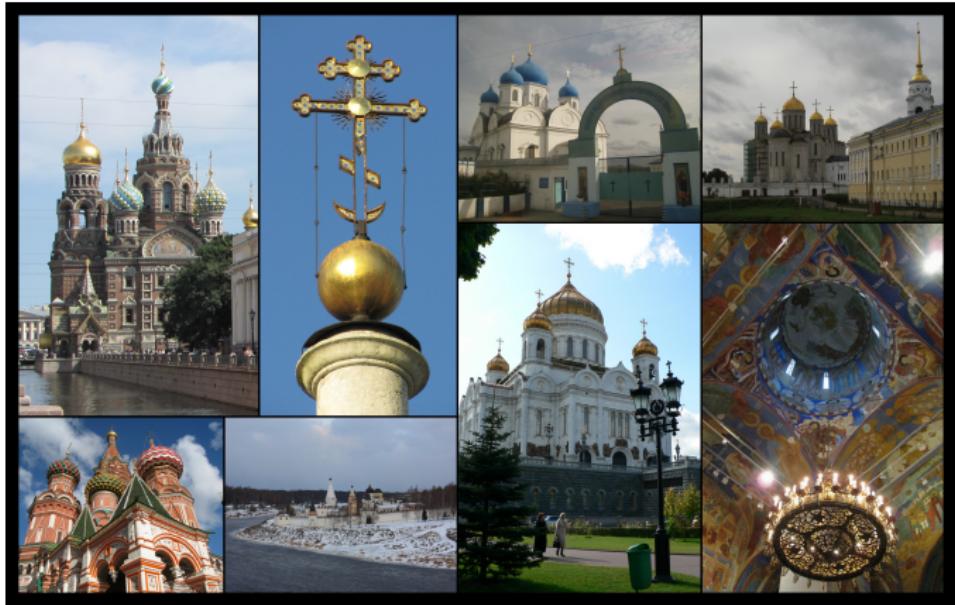
MacBook-Chris %
MacBook-Chris % ./call_tptp.sh Consistency.p

Asking various HOL-ATPs in Miami remotely (thanks to Geoff Sutcliffe)

MacBook-Chris % agsyHOL---1.0 : Consistency.p ++++++ RESULT: SOT_ZtY_7o - agsyHOL---1.0 says Unknown - CPU = 0.00 WC = 0.00
Nitrox---2013 : Consistency.p ++++++ RESULT: SOT_HUz1OC - Nitrox---2013 says Satisfiable - CPU = 6.56 WC = 8.50
TPS---3.120601S1b : Consistency.p ++++++ RESULT: SOT_fpjxTM - TPS---3.120601S1b says Unknown - CPU = 43.00 WC = 49.42
Isabelle---2013 : Consistency.p ++++++ RESULT: SOT_6Tpp9i - Isabelle---2013 says Unknown - CPU = 69.96 WC = 72.62
LEO-II---1.6.0 : Consistency.p ++++++ RESULT: SOT_dy10sj - LEO-II---1.6.0 says Timeout - CPU = 90 WC = 89.86
Satallax---2.7 : Consistency.p ++++++ RESULT: SOT_Q9WSLF - Satallax---2.7 says Timeout - CPU = 90 WC = 90.50

MacBook-Chris %
```

Provers are called remotely in Miami — no local installation needed!



## Formalization and Verification in Coq

- Goal: verification of the natural deduction proof
  - Step-by-step formalization
  - Almost no automation (intentionally!)
- Interesting facts:
  - Embedding is transparent to the user
  - Embedding gives labeled calculus for free

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The screenshot shows the CoqIDE interface with the title "Coqide". The menu bar includes File, Edit, Navigation, Try Tactics, Templates, Queries, Display, Compile, Windows, Help. The toolbar has icons for file operations like Open, Save, and Print. The tabs at the top are "scratch", "Modal.v", "ModalClassical.v", and "GoedelGod-Scott.v". The main window displays a Coq proof script and its state.

```

(* Constant predicate that distinguishes positive properties *)
Parameter Positive: (u -> o) -> o.

(* Axiom A1: either a property or its negation is positive, but not both *)
Axiom axiomia : V (mforall p, (Positive (fun x: u => m-(p x))) m-> (m- (Positive p))). 
Axiom axiomib : V (mforall p, (m- (Positive p)) m-> (Positive (fun x: u => m- (p x)))). 

(* Axiom A2: a property necessarily implied by a positive property is positive *)
Axiom axiom2: V (mforall p, mforall q, Positive p m/\ (box (mforall x, (p x) m-> (q x)))). 

(* Theorem T1: positive properties are possibly exemplified *)
Theorem theorem1: V (mforall p, (Positive p) m-> dia (mexists x, p x)).
Proof.
intro.
intro p.
intro H1.
proof_by_contradiction H2.
apply not_dia_box_not_in H2.
assert (H3: (box (mforall x, m- (p x))) w). (* Lemma from Scott's notes *)
  box_intro wl R1.
  intro x.
  assert (H4: ((m- (mexists x : u, p x)) wl)).
    box_elim H2 wl R1 G2.
    exact G2.

  clear H2 R1 H1 w.
  intro H5.
  apply H4.
  exists x.
  exact H5.

assert (H6: ((box (mforall x, (p x) m-> m- (x m= x))) w)). (* Lemma from Scott's notes *)
  box_intro wl R1.
  intro x.
  intro H7.
  intro H8.
  box_elim H3 wl R1 G3.
  annTu ct with / \_ := _

```

The right pane shows the state of the proof. It lists 2 subgoals:

- $w : i$
- $p : u \rightarrow o$
- $H1 : \text{Positive } p \ w$
- $H2 : \text{box } (\text{mexists } x : u, p \ x) \ w$  (1/2)
- $\text{box } (\text{mforall } x : u, m- \ p \ x) \ w$  (2/2)

Below the subgoals is the term `False`.



automation & verification: proof assistant **ISABELLE**

Isabelle

http://isabelle.in.tum.de/index.html

Home-FU 2012-Watson Homepage 2012-FOL SPIEGEL 2012-FOL-Home GMail Google Maps M&M SigmaOnline Kita Sigma Kalender Beliebt Google Maps >>

Isabelle



**Isabelle**

UNIVERSITY OF CAMBRIDGE Computer Laboratory

TUM TECHNISCHE UNIVERSITÄT MÜNCHEN

**What is Isabelle?**

Isabelle is a generic proof assistant. It allows mathematical formulas to be expressed in a formal language and provides tools for proving those formulas in a logical calculus. Isabelle is developed at University of Cambridge ([Larry Paulson](#)), Technische Universität München ([Tobias Nipkow](#)) and Université Paris-Sud ([Makarius Wenzel](#)). See the [Isabelle overview](#) for a brief introduction.

**Now available: Isabelle2013**

Download for Mac OS X

Download for Linux Download for Windows

**Some highlights:**

- Improvements of Isabelle/Scala and Isabelle/Edit Prover IDE.
- Advanced build tool based on Isabelle/Scala.
- Updated manuals: Isar-ref, implementation, system.
- Pure: improved support for block-structured specification contexts.
- HOL tool enhancements: Sledgehammer, Lifting, Quickcheck.
- HOL library enhancements: HOL-Library, HOL-Probability, HOL-Cardinals.
- HOL: New BNF-based (co)datatype package.
- Improved performance thanks to PolyML 5.5.0.

See also the cumulative [NEWS](#).

**Distribution & Support**

Isabelle is distributed for free under the BSD license. It includes source and binary packages and documentation, see the detailed [installation instructions](#). A vast collection of Isabelle examples and applications is available from the [Archive of Formal Proofs](#).

Support is available by ample [documentation](#), the [Isabelle Community Wiki](#), and the following mailing lists:

- [isabelle-users@cl.cam.ac.uk](mailto:isabelle-users@cl.cam.ac.uk) provides a forum for Isabelle users to discuss problems, exchange information, and make announcements. Users of official Isabelle releases should [subscribe](#) or see the [archive](#) (also available via [Google groups](#) and [Narkive](#)).
- [isabelle-dev@in.tum.de](mailto:isabelle-dev@in.tum.de) covers the Isabelle development process, including intermediate repository versions, and administrative issues concerning the website or testing infrastructure. Early adopters of [repository versions](#) should [subscribe](#) or see the [archive](#) (also available at [mail-archive.com](#) or [gmane.org](#)).

Last updated: 2013-03-09 12:21:39

## Isabelle/HOL (Cambridge University/TU Munich)

- HOL instance of the generic IsABELLE proof assistant
- User interaction and proof automation
- Automation is supported by SLEDGEHAMMER tool
- Verification of the proofs in IsABELLE/HOL's small proof kernel

## What we did?

- Proof automation of Gödel's proof script (Scott version)
- SLEDGEHAMMER makes calls to remote THF provers in Miami
- These calls suggest respective calls to the METIS prover
- METIS proofs are verified in IsABELLE/HOL's proof kernel

See the handout (generated from the Isabelle source file).

# Automation & Verification in Proof Assistant IsABELLE/HOL

The screenshot shows the Isabelle/HOL proof assistant interface. The main window displays the theory file `GoedelGod.thy` (modified). The code includes several Sledgehammer annotations, indicated by blue underlines and yellow highlights. The annotations are:

- `sledgehammer [provers = remote_leo2] by (metis A3 T1)`
- `text {* Axiom @{text "A4"} is added: $\forall \phi [P(\phi) \rightarrow \Box \; ; \; P(\phi)]$ (Positive properties are necessarily positive). *}`
- `axiomatization where A4: "[!! (\lambda\Phi. P \Phi m⇒ □ (P Φ))]"`
- `text {* Symbol @{text "ess"} for 'Essence' is introduced and defined as $ess{\phi}{x} \; \text{biimp} \; \phi(x) \wedge \forall \psi \; (\psi(x) \; \text{imp} \; \nec \; \forall y \; (\phi(y) \; \text{imp} \; \psi(y)))$ (An essence of an individual is a property possessed by it and necessarily implying any of its properties). *}`
- `definition ess :: "(μ ⇒ σ) ⇒ μ ⇒ σ" (infixr "ess" 85) where "Φ ess x = Φ x m⇒ II (λψ. ψ x m⇒ □ ( ∀ (λy. Φ y m⇒ ψ y)))"`
- `text {* Next, Sledgehammer and Metis prove theorem @{text "T2"}: $\forall x \; [G(x) \; \text{imp} \; ess(G){x}]$ (Being God-like is an essence of any God-like being). *}`
- `theorem T2: "[! (λx. G x m⇒ G ess x)]"`
- `sledgehammer [provers = remote_leo2] by (metis A1b A4 G_def ess_def)`
- `text {* Symbol @{text "NE"}, for 'Necessary Existence', is introduced and defined as $NE(x) \; \text{biimp} \; \forall \phi \; [ess{\phi}{x}] \; \text{imp} \; \nec \; \exists y \; \phi(y)$ (Necessary existence of an individual is the necessary exemplification of all its essences). *}`
- `definition NE :: "μ ⇒ σ" where "NE = (λx. II (λΦ. Φ ess x m⇒ □ ( ∃ Φ)))"`

The status bar at the bottom left shows "Sledgehammering...". The bottom right corner contains icons for zoom, auto update, and detach.

**“God is dead.”**

- Nietzsche, 1883

**“Nietzsche is dead.”**

- God, 1900

## Criticisms

$$\forall P. [\diamond \Box P \rightarrow \Box P]$$

If something is possibly necessary, then it is necessary.

$$\diamond \Box (A \vee \neg A) \quad \Box (A \vee \neg A)$$

logical necessity ~ validity

for all  $M, M \models F \rightarrow \Box F$

logical possibility ~ satisfiability

exists  $M, M \models F \rightarrow \diamond F$

What about iterations?

$$\diamond \Box \diamond \diamond F$$

weak intuitions  $\Rightarrow$  dozens of modal logics

S5 is considered adequate

(But KB is sufficient!)

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$$\diamond_c \Box_c (A \vee \neg A) \quad \Box_c (A \vee \neg A)$$

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Everything that is the case is so necessarily.

Follows from T2, T3 and D2.

There are no *contingent truths*.

Everything is determined.

There is no free will.

Many proposed solutions: Anderson, Fitting, Hájek, ...

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$$\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$$

Either a property is positive or its negation is (but never both)

Are the following properties positive or negative?

$$\lambda x.G(x) \quad \lambda x.E(x) \quad \lambda x.x = x \quad \lambda x.\top$$

$$\lambda x.blue(x) \quad \lambda x.red(x) \quad \lambda x.human(x)$$

Solution:

“...positive in the moral aesthetic sense (independently of the accidental structure of the world). Only then the ax. true. ....”

- Gödel, 1970

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- Gödel, 1970



## Conclusions

The (**new**) insights we gained from experiments include:

- Logic K sufficient for T1, C and T2
- Logic S5 *not* needed for T3
- Logic KB sufficient for T3 (not well known)
- We found a simpler new proof of C
- Gödel's axioms (without conjunct  $\phi(x)$  in D2) are inconsistent
- Scott's axioms are consistent
- For T1, only half of A1 (A1a) is needed
- For T2, the other half (A1b) is needed

- Contributions to the theorem proving community include
  - Powerful infrastructure for reasoning with QML
  - A (new?) natural deduction calculus for higher-order (intuitionistic) modal logic
  - Difficult new benchmark problems for HOL provers
  - Huge media attention
- Interesting bridge between CS, Philosophy and Theology
- Major step towards **Computer-assisted Theoretical Philosophy**
  - see also Ed Zalta's *Computational Metaphysics* project at Stanford University
  - remember Leibniz' dictum — *Calculemus!*

- Formalize and verify other ontological arguments
  - ... particularly the criticisms and improvements
- Experiment with other embeddings (e.g. varying domains)
- Eliminate and introduce cuts

I'm sure that God would be impressed with your proof, if only he existed :-)

Larry

Die Philosophen können so schön staunen.

Sie packen Dinge in Begriffe (gucken dabei in die Luft) werfen die Begriffe dann in ihre Philosophiekiste, schütteln ganz dolle, und freuen sich, dass ganz genau rauskommt, was sie vorher reingetan haben. Und das geht sogar, wenn eine Maschine die Kiste schüttelt.

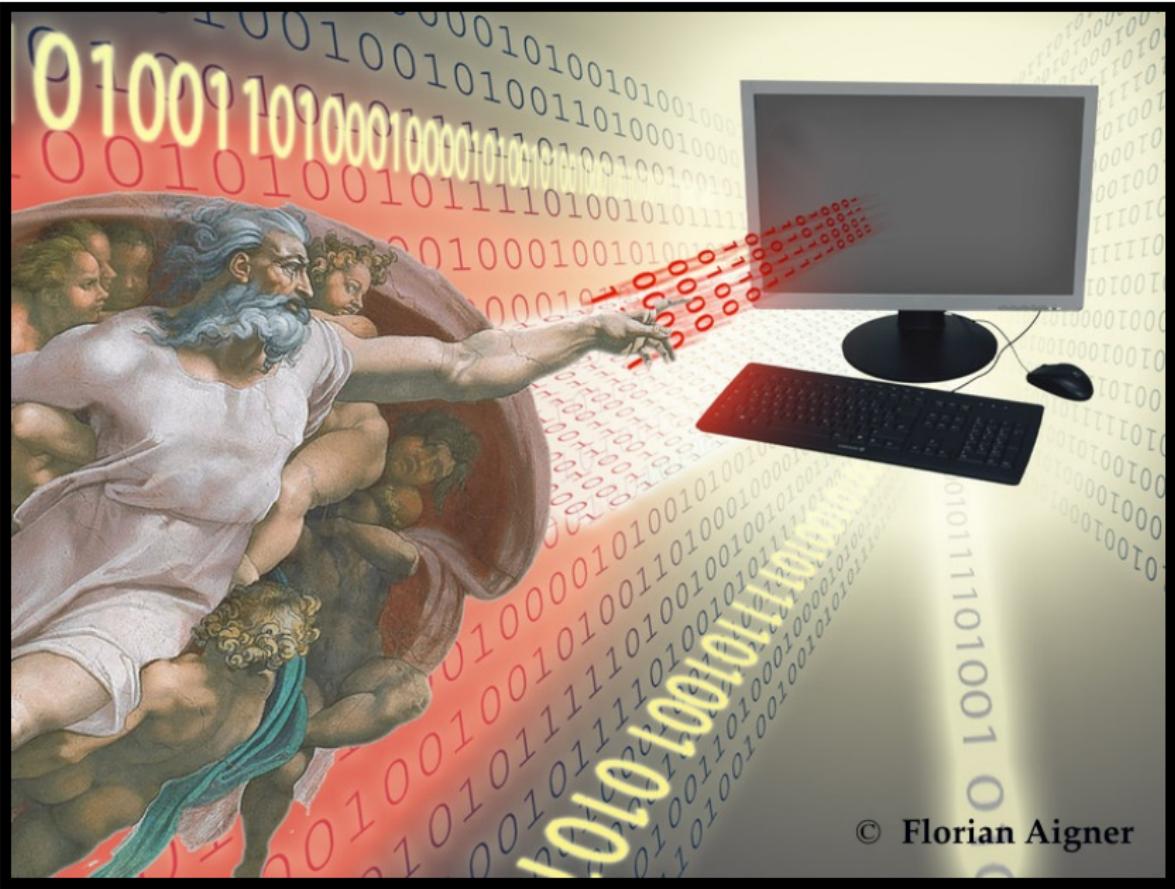
Unerstaunt  
2017cp

## 60. Suchlauf

 souveränsatt 09.09.2013

man kann auch auf andere Weise in diesem Zusammenhang methodisch vorgehen:  
bei einer längeren Autofahrt das Radio auf automatischen Suchlauf stellen. Nach zwei Tagen sieht man Gott

... find more on the internet ...



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