Reflection

Graphical Methods

Graphical methods provided me with an intuitive introduction to linear programming. By plotting the constraints in two dimensions, I could visually observe the formation of the feasible region and the interaction between the objective function and the constraints, but this quickly became impractical in higher-dimensional spaces.

Standard Form and Relaxed Form

Transforming a problem into standard or relaxed form highlights the importance of mathematical consistency. By introducing slack variables, inequalities become equalities, making the problem algorithmically tractable.

Gaussian Elimination

Revisiting Gaussian elimination in the context of linear programming, elimination is an abstract algebraic tool. Although Gaussian elimination can directly solve smaller systems of equations, it is not.

Linear Programming: Simplex Algorithm

Unlike graphical methods, the simplex method can be extended to higher dimensions and is guaranteed to obtain an optimal solution under feasible conditions. Stepping through the simplex table forced me to pay attention to details such as pivot elements, basis variables, and optimality tests. It systematically moves from one vertex of the feasible region to another until a maximum value is reached.

Task 3: Graphical Method for LP

Given the following LP problem, solve it using the graphical method:

Maximize $Z = 5X_1 + 3X_2$, subject to

$2X_1 + X_2 \leq 18,\ 2X_1 + 3X_2 \leq 42,\ 3X_1 + X_2 \leq 24,\ X_1, X_2 \geq 0$

Plot the feasible region, identify the corner (extreme) points, compute the objective value at each corner. Identify the optimal solution.

A :

Defining the constraints and plotting them

Finding all corner points of the feasible region by solving systems of equations

Filtering to keep only feasible points that satisfy all constraints

Calculating the objective function value at each feasible corner point

Identifying the optimal solution with the maximum objective value

The LP problem is: Maximize $Z = 5X_1 + 3X_2$

Subject to:

$2X_1 + X_2 \leq 18$

$2X_1 + 3X_2 \leq 42$

$3X_1 + X_2 \leq 24$

$X_1, X_2 \geq 0$

The graphical method involves plotting the constraints, identifying the feasible region, evaluating the objective function at each corner point, and selecting the corner with the optimal value.

Task 4: LP Application Scenario

A factory produces two products using the same machine. Product A yields a profit of $40, and Product B yields $30. Each unit of Product A takes 2 hours to produce; B takes 1 hour. The machine runs for at most 40 hours per week. Formulate and solve the LP problem.

A:

Decision variables:

X_1: Units of Product A to produce

X_2: Units of Product B to produce

Objective function: Maximize profit $Z = 40X_1 + 30X_2$

Constraints:

Machine time: $2X_1 + X_2 \leq 40$ hours

Non-negativity: $X_1 \geq 0$, $X_2 \geq 0$

The problem is solved using the linprog function from SciPy, which requires:

Converting maximization to minimization (negating the coefficients)

Specifying the constraint coefficients and bounds

Solving and interpreting the results

The solution provides the optimal production quantities that maximize profit while respecting the machine time constraint.

Task 5: Solve a Linear Program Using Simplex

Maximize $Z = 18X_1 + 12.5X_2$, subject to

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$X_1 + X_2 \le 20, X_1 \le 12, X_2 \le 16, X_1, X_2 \ge 0$

Solve the following linear programming problem by hand using the Simplex algorithm. No code is required. Convert the inequalities into equations using slack variables. Construct the Simplex tableau. Perform Simplex iterations manually. Clearly show:

- Pivot operations,
- Basic variables at each step,
- The final optimal solution.

A:

The first step in the simplex method is to convert the inequality constraints into equality form, which is achieved by introducing slack variables. We add three slack variables S_1, S_2, $S_3 \ge 0$, transforming the original problem into: $X_1 + X_2 + S_1 = 20$, $X_1 + S_2 = 12$, $X_2 + S_3 = 16$, while maintaining the objective function of maximizing $Z = 18X_1 + 12.5X_2 + 0S_1 + 0S_2 + 0S_3$.

Var	X_1	X_2	S_1	S_2	S_3	result
S_1	1	1	1	0	0	20
X_1	1	0	0	1	0	12
S_3	0	1	0	0	1	16
Z	-18	-12.5	0	0	0	0

First Iteration

Select the X_1 column as the pivot column (most negative coefficient -18)

Select the S_2 row as the pivot row (smallest ratio 12/1 = 12)

After the pivot operation, the new table is:

Var	X_1	X_2	S_1	S_2	S_3	result
S_1	0	1	1	-1	0	8
X_1	1	0	0	-1	0	12
S_3	0	1	0	0	1	16
Z	0	-12.5	0	18	0	216

Second Iteration

Select the X_2 column as the pivot column (most negative coefficient -12.5)

Select the S_1 row as the pivot row (smallest ratio 8/1 = 8)

After the pivot operation, the new table is:

Var	X_1	X_2	S_1	S_2	S_3	result
S_1	0	1	1	-1	0	8
X_1	1	0	0	1	0	12
S_3	0	0	-1	1	1	8
Z	0	0	12.5	5.5	0	316

Final result:

$X_1 = 12$

$X_2 = 8$

$S_3 = 8$

Maximum Z = 316

The simplex method iteratively improves the solution. In the first iteration, we select X_1 as the entering variable (due to its most negative coefficient in the target row, -18) and then use a ratio test to determine S_2 as the leaving variable (the smallest ratio is 12/1=12). After the pivot operation, X_1 enters the basis set and S_2 leaves, resulting in an improved solution of X_1=12, X_2=0, S_1=8, and S_3=16, which improves the objective function value to Z=216.

The optimal solution is reached when all coefficients in the objective function row are nonnegative. The final table shows that all coefficients are nonnegative, indicating that we have found the optimal solution: X_1=12, X_2=8, Z=316. The slack variable S_3=8 indicates that the third constraint ($X_2 \le 16$) has 8 units of surplus resources, while the first two constraints are tight (no surplus resources).

The economic significance of this solution is that producing 12 units of product A and 8 units of product B maximizes profit of 316, fully utilizing the first two resources (machine time and worker time), and leaving 8 units of the third resource (raw materials) surplus. The simplex method systematically moves from one vertex to the adjacent vertex to ultimately find this optimal vertex solution.