MODULE - 01

Fundamentals of logic.

Introduction :- Logic → Method of Reasoning logic is expressed in a symbolic language is called Mathematical logic.

(*) PROPOSITIONS :-

A proposition is a statement (declaration) which in a given context, can be said to be either true (or) false, but not both.

Ex :- 1. Bangalore is in Karnataka

2. 7 is divisible by 3.

3. $xy = yx$, for this we cannot decisively say whether it is true (or) not, unless we know that what is x and y (whether it is integer (or) Real no) so it is not a proposition.

→ propositions are usually represented by small letters such as p, q, r, s,

→ The truth (or) the falsity of a proposition is called its ~~truth table~~ truth value.

→ If a proposition is true it is denoted by the truth value 1

→ If a proposition is false it is denoted by the truth value '0'.

(*) Logical connectives :-

The words like 'not', 'and', 'if then', if and only if, such words are called as Logical connectives.

(*) Compound proposition :-

The new proposition is obtained by combining the two given propositions using the logical connectives are called as compound proposition.

(*) Simple proposition :-

proposition which do not contains any logical connectives are called simple propositions.

(*) Negation (~)

A proposition is obtained by inserting the word 'not' in an appropriate place is called the negation of the given proposition.

The negation of a proposition 'p' is denoted by ~p

eg:- 1) p: 3 is a prime number – 1

~p: 3 is not a prime number – 0

Truth table:

p	~p
1	0
0	1

(2) p : 8 is divisible by 3 − 0

$\sim p$: 8 is not divisible by 3 − 1

Truth table

p	$\sim p$
0	1
1	0

(*) Conjunction ($\wedge$)

A compound proposition is obtained by inserting the word 'and' between two given propositions is called conjunction of the given proposition.

The conjunction of p and q is denoted by $p \wedge q$.

The conjunction of p and q is true only when both p and q are true, in all other cases it is false.

Truth table:

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

(*) Disjunction (∨)

A compound proposition is obtained by inserting the word 'or' between two given proposition is called disjunction of the given proposition.

The disjunction of p and q is denoted by $p \vee q$.

Truth table:

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

(*) ~~The disjunction of p and q~~

(*) The disjunction $p \vee q$ is false only when ~~only~~ both p & q are false otherwise it is true

(*) Exclusive disjunction ($\veebar$)

The exclusive disjunction of two propositions p and q is denoted by $p \veebar q$ (Read it as either p (or) q)

The exclusive disjunction is true only when either p is true (or) q is true but not both

Truth table:

p	q	$p \veebar q$
0	0	0
0	1	1
1	0	1
1	1	0

(*) conditional ($\rightarrow$)

A compound proposition is obtained by inserting the word 'if then' in an appropriate place is called the conditional.

The conditional of p and q is denoted by $p \rightarrow q$.

(read as if p then q).

The conditional $p \rightarrow q$ is false only when p is true and q is false.

Truth table:

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

(*) Bi-conditional :- ($\leftrightarrow$)

A compound proposition is obtained by inserting the word 'if and only if' in an approperiate place is called as biconditional of the given proposition.

It is denoted by $p \leftrightarrow q$.

The biconditional $p \leftrightarrow q$ is true only when both p and q have the same truth values otherwise it is false.

The biconditional of p and q is denoted by

$$p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$$

Truth table :

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

combined truth table :-

p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	1	1	0	0	0	1	1
0	1	1	0	0	1	1	1	0
1	0	0	1	0	1	1	0	0
1	1	0	0	1	1	0	1	1

$q \rightarrow p$
1
0
1
1

PROBLEMS:

1. Let p: A circle is a conic

q: $\sqrt{5}$ is a real number.

r: Exponential series is convergent.

Express the following compound propositions in words

(i) $p \wedge (\sim q)$ (ii) $(\sim p) \vee q$ (iii) $p \underline{\vee} (\sim q)$ (iv) $q \rightarrow (\sim p)$

(v) $p \rightarrow (q \underline{\vee} r)$ (vi) $\sim p \leftrightarrow q$

Soln:- (i) $p \wedge (\sim q)$:

A circle is a conic and $\sqrt{5}$ is not a real no.

(ii) $(\sim p) \vee q$:

A circle is not a conic or $\sqrt{5}$ is a real number.

(iii) $p \underline{\vee} (\sim q)$:

Either a circle is a conic or $\sqrt{5}$ is a real number. (but not both)

(iv) $q \rightarrow (\sim p)$:

If $\sqrt{5}$ is a real number, then a circle is not a conic.

(v) $p \rightarrow (q \underline{\vee} r)$:

If a circle is a conic then either $\sqrt{5}$ is a real no or the exponential series is convergent (but not both).

(vi) $\sim p \leftrightarrow q$

If a circle is not a conic then $\sqrt{5}$ is a real number and if $\sqrt{5}$ is a real number then a circle is not a conic.

(2) Construct the truth tables for the following compound propositions:

(i) $p \wedge (\sim q)$ (ii) $(\sim p) \vee q$ (iii) $p \rightarrow (\sim q)$ (iv) $(\sim p) \underline{\vee} (\sim q)$

:-

p	q	$\sim p$	$\sim q$	$p \wedge (\sim q)$	$(\sim p) \vee q$	$p \rightarrow (\sim q)$	$(\sim p) \underline{\vee} (\sim q)$
0	0	1	1	0	1	1	0
0	1	1	0	0	1	1	1
1	0	0	1	1	0	1	1
1	1	0	0	0	1	0	0

(3) Let p, q and r be the propositions having truth values 0, 0 and 1 respectively. Find the truth values of the following compound propositions.

(i) $(p \vee q) \vee r$

(ii) $(p \wedge q) \rightarrow r$

(iii) $p \rightarrow [q \rightarrow (\sim r)]$

Soln:- (i) $(p \vee q) \vee r = (0 \vee 0) \vee 1$

$= 0 \vee 1$

$= 1$

(ii) $(p \wedge q) \rightarrow r = (0 \wedge 0) \rightarrow 1$

$= 0 \rightarrow 1$

$= 1$

(iii) $p \rightarrow [q \rightarrow (\sim r)] = 0 \rightarrow [0 \rightarrow (0)]$

$= 0 \rightarrow 1$

$= 1$

(4) construct the truth table for the following compound propositions.

(i) $(p \wedge q) \rightarrow (\sim r)$

(ii) $q \wedge (\sim r \rightarrow p)$

soln:-

p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \rightarrow (\sim r)$	$\sim r \rightarrow p$	$q \wedge (\sim r \rightarrow p)$
0	0	0	0	1	1	0	0
0	0	1	0	0	1	1	0
0	1	0	0	1	1	0	0
0	1	1	0	0	1	1	1
1	0	0	0	1	1	1	0
1	0	1	0	0	1	1	0
1	1	0	1	1	1	1	1
1	1	1	1	0	0	1	1

(*) **Tautology:**

A compound proposition which is true for all possible truth values of its components is called a tautology.

(*) **Contradiction:**

A compound proposition which is false for all possible truth values of its components is called a contradiction.

(*) **contingency:**

A compound proposition that can be true or false (depending upon the truth values of its components) is called a contingency.

Note:- contingency is a compound proposition which is neither a tautology nor a contradiction.

PROBLEMS:

(1) prove that for any proposition p, $p \vee \sim p$ is tautology and $p \wedge \sim p$ is a contradiction.

:-

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
0	1	1	0
1	0	1	0

Hence $p \vee \sim p$ is a tautology and $p \wedge \sim p$ is a contradiction.

(2) Show that, for any propositions p and q the compd. proposition $p \to (p \vee q)$ is a tautology and the compd. proposition $p \wedge (\sim p \wedge q)$ is a contradiction

:-

p	q	$p \vee q$	$p \to (p \vee q)$	$\sim p$	$\sim p \wedge q$	$p \wedge (\sim p \wedge q)$
0	0	0	1	1	0	0
0	1	1	1	1	1	0
1	0	1	1	0	0	0
1	1	1	1	0	0	0

Hence, $p \to (p \vee q)$ is true for all possible truth value hence it is tautology

$p \wedge (\sim p \wedge q)$ is false for all possible truth values hence it is contradiction.

(3) prove that for any propositions p, q, r then the compound proposition $\{p \to (q \to r)\} \to \{(p \to q) \to (p \to r)\}$ is a tautology.

:-

p	q	r	$q \to r$	$p \to (q \to r)$	$p \to q$	$p \to r$	$(p \to q) \to (p \to r)$	$\{p \to (q \to r)\} \to \{(p \to q) \to (p \to r)\}$
0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	1	0	0	1	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	1	1	0	0	1	1
1	0	1	1	1	0	1	1	1
1	1	0	0	0	1	0	0	1
1	1	1	1	1	1	1	1	1

mp (4) prove that for any proposition p, q, r the compound proposition.

$[(p \vee q) \wedge \{(p \to r) \wedge (q \to r)\}] \to r$ is tautology.

:-

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$	$(p \vee q) \wedge \{(p \to r) \wedge (q \to r)\}$	$[(p \vee q) \wedge \{(p \to r) \wedge (q \to r)\}] \to r$
0	0	0	0	1	1	1	0	1
0	0	1	0	1	1	1	0	1
0	1	0	1	1	0	0	0	1
0	1	1	1	1	1	1	1	1
1	0	0	1	0	1	0	0	1
1	0	1	1	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

Hence, $[(p \vee q) \wedge \{(p \to r) \wedge (q \to r)\}] \to r$ is a tautology.

(5) prove that (i) $(p \veebar q) \vee (p \leftrightarrow q)$ is a tautology

(ii) $(p \veebar q) \wedge (p \leftrightarrow q)$ is a contradiction

(iii) $(p \veebar q) \wedge (p \to q)$ is a contingency.

:-

p	q	$p \veebar q$	$p \leftrightarrow q$	$p \to q$	(i)	(ii)	(iii)
0	0	0	1	1	1	0	0
0	1	1	0	1	1	0	1
1	0	1	0	0	1	0	0
1	1	0	1	1	1	0	0

Logical Equivalence ($\Leftrightarrow$)

Two propositions u and v are said to be logically equivalent whenever u and v have the same truth value.

We write, $u \Leftrightarrow v$, Here the symbol $\Leftrightarrow$ stands for "logically equivalent to".

Note:- 1. When the propositions u and v are not logically equivalent, we write $u \not\Leftrightarrow v$.

2. Whenever u and v are logically equivalent then $u \leftrightarrow v$ is always tautology.

problems:-

(1) prove that for any three propositions p, q, r

$[(p \vee q) \to r] \Leftrightarrow [(p \to r) \wedge (q \to r)]$.

Soln:-

p	q	r	$p \vee q$	$(p \vee q) \to r$	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	1	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	1	0	0	1	0
1	0	1	1	1	1	1	1
1	1	0	1	0	0	0	0
1	1	1	1	1	1	1	1

It is logically equivalent. //

(Q) Examine whether compound proposition $[(p \vee q) \rightarrow r] \Leftrightarrow [\sim r \rightarrow \sim(p \vee q)]$

Soln:-

p	q	r	$p \vee q$	$(p \vee q) \rightarrow r$	$\sim r$	$\sim(p \vee q)$	$\sim r \rightarrow \sim(p \vee q)$
0	0	0	0	1	1	1	1
0	0	1	0	1	0	1	1
0	1	0	1	0	1	0	0
0	1	1	1	1	0	0	1
1	0	0	1	0	1	0	0
1	0	1	1	1	0	0	1
1	1	0	1	0	1	0	0
1	1	1	1	1	0	0	1

Hence, It is logically equivalent.

(*) ~~The Laws of Logic~~

Try

1. prove that, for any three propositions p, q, r
   $[p \rightarrow (q \wedge r)] \Leftrightarrow [(p \rightarrow q) \wedge (p \rightarrow r)]$

2. show that the compound propositions $p \wedge (\sim q \vee r)$ and $p \vee (q \wedge \sim r)$ are logically equivalent.

(*) The Laws of Logic

In the following laws, T_0 denotes a tautology and F_0 denotes a contradiction.

1. Law of Double negation.

$$(\sim\sim p) \Leftrightarrow p$$

2. Idempotent law

(i) $(p \vee p) \Leftrightarrow p$ (ii) $(p \wedge p) \Leftrightarrow p$

3. Identity law

(i) $(p \vee F_0) \Leftrightarrow p$ (ii) $(p \wedge T_0) \Leftrightarrow p$

imp 4. Inverse law

(i) $(p \vee \sim p) \Leftrightarrow T_0$ (ii) $(p \wedge \sim p) \Leftrightarrow F_0$

imp 5. Domination Laws.

(i) $(p \vee T_0) \Leftrightarrow T_0$ (ii) $(p \wedge F_0) \Leftrightarrow F_0$

6. Commutative Law

(i) $(p \vee q) \Leftrightarrow (q \vee p)$ (ii) $(p \wedge q) \Leftrightarrow (q \wedge p)$

imp 7. Absorption law

(i) $[p \vee (p \wedge q)] \Leftrightarrow p$

(ii) $[p \wedge (p \vee q)] \Leftrightarrow p$

(8) De-Morgan's Law

(i) $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$

(ii) $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$

(9) Associative Law

(i) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$

(ii) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

(10) Distributive Law

(i) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

(ii) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

(11) Law for the conditional

$p \rightarrow q \Leftrightarrow \sim p \vee q$

(*) 8(ii) proof.

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
0	0	1	1	1	0	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	0	1	0

(*) 8(i) proof.

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

Hence proof

PROBLEMS:

(1) prove the following logical equivalence without using truth table.

(i) $p \vee [p \wedge (p \vee q)] \Leftrightarrow p$.

Soln: :- $p \vee [p \wedge (p \vee q)] \Leftrightarrow p \vee p$, by Absorption law

$\Leftrightarrow p$, by an idempotent law.

(ii) $[p \vee q \vee (\sim p \wedge \sim q \wedge r)] \Leftrightarrow (p \vee q \vee r)$

soln:- $[p \vee q \vee (\sim p \wedge \sim q \wedge r)]$

$\Leftrightarrow [p \vee q \vee (\sim(p \vee q) \wedge r)]$, by De-morgan's law

$\Leftrightarrow [(p \vee q) \vee (\sim(p \vee q) \wedge r)]$

$\Leftrightarrow [(p \vee q) \vee \sim(p \vee q)] \wedge [(p \vee q) \vee r]$ ($\because$ by distributive property)

$\Leftrightarrow T_0 \wedge [p \vee q \vee r]$ [by inverse law & Associativity]

$\Leftrightarrow (p \vee q \vee r)$ // (by identity)

(iii) $[(\sim p \vee \sim q) \rightarrow (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$

soln:- consider,

$[(\sim p \vee \sim q) \rightarrow (p \wedge q \wedge r)]$

$\Leftrightarrow [\sim(p \wedge q) \rightarrow (p \wedge q \wedge r)]$

$(p \wedge q) \vee (p \wedge q \wedge r)$

(WKT $p \rightarrow q \Leftrightarrow \sim p \vee q$

$\Leftrightarrow (p \wedge q) \vee [(p \wedge q) \wedge r]$

WKT Absorption Law

$\Leftrightarrow p \wedge q$ //

$p \vee (p \wedge q) \Leftrightarrow p$.

(2) prove the following logical equivalences:

(i) $[(p \vee q) \wedge (p \vee \sim q)] \vee q \Leftrightarrow p \vee q$

soln:- $[(p \vee q) \wedge (p \vee \sim q)] \vee q$

$\Leftrightarrow [p \vee (q \wedge \sim q)] \vee q$

WKT distributive law

$[\, p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

$\Leftrightarrow [p \vee F_0] \vee q$

$\Leftrightarrow p \vee q$ //

(ii) $(p \to q) \wedge [\sim q \wedge (r \vee \sim q)] \Leftrightarrow \sim (q \vee p)$

soln:- $(p \to q) \wedge [\sim q \wedge (r \vee \sim q)]$

$\Leftrightarrow (p \to q) \wedge [\sim q \wedge (\sim q \vee r)]$ by commutative

$\Leftrightarrow (p \to q) \wedge [\sim q]$ ∵ by Absorption law.

$\Leftrightarrow (\sim p \vee q) \wedge \sim q$

$\Leftrightarrow (\sim p \wedge \sim q) \vee (q \wedge \sim q)$

$\Leftrightarrow \sim (p \vee q) \vee F_0$

$\Leftrightarrow \sim (p \vee q)$

$\Leftrightarrow \sim (q \vee p)$ //

(iii) $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$

:- $p \to (q \to r) \Leftrightarrow p \to (\sim q \vee r)$

$\Leftrightarrow \sim p \vee (\sim q \vee r)$

$\Leftrightarrow (\sim p \vee \sim q) \vee r$

$\Leftrightarrow \sim (p \wedge q) \vee r$

$\Leftrightarrow (p \wedge q) \to r$ //

~~(iv)~~ $[\sim p \wedge (\sim q \wedge r)] \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$

soln:- $[\sim p \wedge (\sim q \wedge r)] \vee (q \wedge r) \vee (p \wedge r)$

$\Leftrightarrow [(\sim p \wedge \sim q) \wedge r] \vee (q \wedge \cancel{f} r) \vee (p \wedge r)$

$\Leftrightarrow [\sim (p \vee q) \wedge r] \vee (q \wedge r) \vee (p \wedge r)$

$\Leftrightarrow [\sim (p \vee q) \wedge r] \vee [(q \vee p) \wedge r]$

$\Leftrightarrow [\sim (p \vee q) \vee (q \vee p)] \wedge r$

$\Leftrightarrow [\sim (p \vee q) \vee (p \vee q)] \wedge r$

$\Leftrightarrow [T_0] \wedge r$

$\Leftrightarrow r$ //

(v). $\sim [\{(p \vee q) \wedge r\} \to \sim q] \Leftrightarrow \sim \{\sim [(p \vee q) \wedge r] \vee \sim q\}$

$\Leftrightarrow q \wedge r$

soln:- $\sim [\{(p \vee q) \wedge r\} \to \sim q]$

$\Leftrightarrow \sim \{\sim [(p \vee q) \wedge r] \vee \sim q\}$ (by demorgan's Law & by double negation)

$\Leftrightarrow [(p \vee q) \wedge r]$ ~~X~~ q

$\Leftrightarrow [p \vee (q \wedge r)] \wedge q$

$\Leftrightarrow [(p \vee q) \wedge r] \wedge q$

$\Leftrightarrow [(p \vee q) \wedge (r \wedge q)]$ (Associative law)

$\Leftrightarrow (p \vee q) \wedge (q \wedge r)$ commutative Law.

$\Leftrightarrow [(p \vee q) \wedge q] \wedge r$ Absorption law.

$\Leftrightarrow q \wedge r$ //

(3) prove that

$[(P \vee Q) \wedge \sim\{\sim P \wedge (\sim Q \vee \sim R)\}] \vee (\sim P \wedge \sim Q) \vee (\sim P \wedge \sim R)$ is a tautology

soln:- Let W denote the given proposition,

$W = [(P \vee Q) \wedge \sim\{\sim P \wedge (\sim Q \vee \sim R)\}] \vee (\sim P \wedge \sim Q) \vee (\sim P \wedge \sim R)$

$W = u \vee v$

Now $u \Leftrightarrow [(P \vee Q) \wedge \sim\{\sim P \wedge (\sim Q \vee \sim R)\}]$

$\Leftrightarrow [(P \vee Q) \wedge \sim\{\sim P \wedge \sim (Q \wedge R)\}]$ by demorgan's

$\Leftrightarrow [(P \vee Q) \wedge \sim\{\sim (P \vee (Q \wedge R))\}]$ Again demorgan's

$\Leftrightarrow [(P \vee Q) \wedge (P \vee (Q \wedge R))]$ by double negation

$\Leftrightarrow P \vee \{Q \wedge (Q \wedge R)\}$ by

$\Leftrightarrow P \vee \{(Q \wedge Q) \wedge R\}$

$\Leftrightarrow P \vee (Q \wedge R)$

$v \Leftrightarrow \sim(P \vee Q) \vee \sim(P \vee R)$

$\Leftrightarrow \sim\{(P \vee Q) \wedge (P \vee R)\}$ by demorgan's law.

$\Leftrightarrow \sim\{P \vee (Q \wedge R)\}$

Now (1) $\Rightarrow$ $w \Leftrightarrow u \vee v$

$\Leftrightarrow \{P \vee (Q \wedge R)\} \vee \sim\{P \vee (Q \wedge R)\}$

$\Leftrightarrow T_0$ //

Thus the given compound proposition is a tautology.

(*) Simplify the following compound proposition using the laws of logic.

(a) $(p \vee q) \wedge [\sim\{(\sim p \wedge q)\}]$

$\Leftrightarrow (p \vee q) \wedge [\sim\sim p \vee \sim q]$ by demorgan's law

$\Leftrightarrow (p \vee q) \wedge [p \vee \sim q]$

$\Leftrightarrow p \vee (q \wedge \sim q)$ (by distributive law)

$\Leftrightarrow p \vee (F_0)$ (by inverse law)

$\Leftrightarrow p$ // (by identity law)

(b) $\sim[\sim\{(p \vee q) \wedge r\} \vee \sim q]$

$\Leftrightarrow \sim\{\cancel{\sim\{(p \vee q) \wedge r\} \wedge}$

$\sim\sim[(p \vee q) \wedge r] \wedge \sim\sim q$ demorgan's law

$\Leftrightarrow [(p \vee q) \wedge r] \wedge q$

$\Leftrightarrow (p \vee q) \wedge (r \wedge q)$ distributive law.

$\Leftrightarrow (p \vee q) \wedge (q \wedge r)$ by commutative law

$\Leftrightarrow [(p \vee q) \wedge q] \wedge r$ Associative law

$\Leftrightarrow q \wedge r$ // by Absorption law.

(c) Let x be a specified no. write down the negation of the following conditional
"If x is an integer, then x is a rational no".

Soln:- It is of the form $p \to q$

Where, p: x is an integer

q: x is a rational number.

$\sim (p \to q) \Leftrightarrow \sim[\sim p \vee q]$

$\Leftrightarrow \sim\sim p \wedge \sim q$ demorgon's law

$\Leftrightarrow p \wedge \sim q$

i.e "x is an integer and x is not a rational no".

(d) Let x be a specified no write down the negation of the following proposition:
"If x is not a real no, then it is not a rational no and not an irrational no".

Soln:- p; x is a real number

q; x is a rational number

r: x is a irrational no

The given statement is of the form,

$$\sim p \rightarrow (\sim q \wedge \sim r)$$

Now, $\sim[\sim p \rightarrow (\sim q \wedge \sim r)]$

$\Leftrightarrow \sim[\sim p \rightarrow \{\sim (q \vee r)\}]$

$\Leftrightarrow \sim[\sim\sim p \vee \sim(q \vee r)]$

$\Leftrightarrow \sim(p \vee \sim(q \vee r))$ demorgon's law

$\Leftrightarrow \sim p \wedge \sim\sim (q \vee r)$

$\Leftrightarrow \sim p \wedge (q \vee r)$

$\therefore$ x is not a real no and it is a rational no or irrational no.

(*) Duality

Suppose 'u' is a compound proposition and its duality is obtained by replacing

(i) each $\wedge$ and $\vee$ by $\vee$ and $\wedge$ respectively

(ii) each T_0 & F_0 by F_0 & T_0 respectively

And it is denoted by u^d.

ex:- $u: p \wedge (q \vee \sim r) \vee (s \wedge T_0)$

$u^d: p \vee (q \wedge \sim r) \wedge (s \vee F_0)$

(*) Converse, Inverse and contrapositive:

consider a conditional $p \to q$ then

(1) $q \to p$ is called the converse of $p \to q$

(2) $\sim p \to \sim q$ is called the inverse of $p \to q$

(3) $\sim q \to \sim p$ is called the contrapositive of $p \to q$.

(*) RULE OF INFERENCE:

consider a set of propositions $P_1, P_2, P_3 --- P_n$ and a proposition Q. Then a compound proposition of the form $(P_1 \wedge P_2 \wedge P_3 \wedge --- \wedge P_n) \to Q$ is called a argument. Here $P_1, P_2, --- P_n$ are called the premises of the argument and Q is called a conclusion of the argument.

It is represented in the form of tabular form i.e

$$\begin{array}{l} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_n \\ \hline \therefore Q \end{array}$$

$\Rightarrow$ The preceding argument is said to be valid if whenever each of the premises $P_1, P_2, \text{---} P_n$ is true, then the conclusion Q is likewise true.

In other words, the argument

$(P_1 \wedge P_2 \wedge P_3 \wedge \text{---} \wedge P_n) \rightarrow Q$ is valid when

$(P_1 \wedge P_2 \wedge P_3 \wedge \text{---} \wedge P_n) \Rightarrow Q$

For finding the validity of Argument use the Rules of Logic and these rules are called the Rules of Inference.

(1) Rule of conjunctive simplication.

For any two propositions p and q, if $p \wedge q$ is true then p is true

i.e $\quad p \wedge q \Rightarrow p$

2) Rule of Disjunctive Amplification

for any two propositions p and q, if p is true then $p \vee q$ is true i.e

$$p \Rightarrow p \vee q$$

3.) Rule of Syllogism:

for any three propositions, p, q, r if $p \rightarrow q$ is true and $q \rightarrow r$ is true then $p \rightarrow r$ is true

i.e $\{(p \rightarrow q) \wedge (q \rightarrow r)\} \Rightarrow p \rightarrow r$

In tabular form

$$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore \; p \rightarrow r \end{array}$$

4) Modus pones

This rule states that if p is true and $p \rightarrow q$ is true, then q is true

i.e $\{p \wedge (p \rightarrow q)\} \Rightarrow q$

In tabular form

$$\begin{array}{c} p \\ p \rightarrow q \\ \hline \therefore \; q \end{array}$$

5) Modus Tollens :

This rule states that if $p \to q$ is true and q is false, then p is false

i.e $\{(p \to q) \wedge \sim q\} \Rightarrow \sim p$

In tabular form,

$$\begin{array}{l} p \to q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

6) Rule of Disjunctive Syllogism :

This rule states that if $p \vee q$ is true and p is false, then q is true

i.e $[(p \vee q) \wedge \sim p] \Rightarrow q$

In tabular form,

$$\begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore q \end{array}$$

PROBLEMS: Test whether the following are valid argument.

(1) If Sachin hits a century, then he gets a free car.

Sachin hits a century

$\therefore$ Sachin gets a free car.

Soln:- Let p: Sachin hits a century

q: Sachin gets a free car.

The given argument is of the form

$$\begin{array}{l} p \to q \\ \quad p \\ \hline \therefore \quad q \end{array}$$

$$[(p \to q) \wedge p] \Rightarrow q$$

$$\Leftrightarrow [p \wedge (p \to q)] \Rightarrow q \quad \text{(By Modus pones)}$$

This is a valid argument.

(2) If Sachin hits a century, he gets a free car

Sachin does not get a free car

$\therefore$ Sachin has not hit a century

Soln:- Let p: Sachin hits a century

q: Sachin gets a free car

The given argument is of the form

$$\begin{array}{l} p \to q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

$[(p \to q) \wedge \sim q] \Rightarrow \sim p$ (by Modus Tolens)

This is a valid argument.

(3) If Sachin hits a century, he gets a free car.

Sachin gets a free car

$\therefore$ Sachin has hit a century.

Soln:- Let p: Sachin hits a century

q: Sachin gets a free car.

The given argument is

$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array}$$

$[(p \to q) \wedge q] \Rightarrow p$

$\Leftrightarrow [(\sim p \vee q) \wedge q]$

$\Leftrightarrow q$ (absorption law)

$\therefore$ The given argument is not a valid.

(4) If I drive to work, then I will arrive tired
I am not tired (when I arrive at work)

∴ I do not drive to work.

Soln:- Let p: I drive to work
q: I arrive tried.

The given argument is of the form

$$\begin{array}{ll} & p \to q \\ & \sim q \\ \hline \therefore & \sim p \end{array}$$

$[(p \to q) \wedge \sim q] \Rightarrow \sim p$ [by Modus Tones]

It is a valid argument.

(5) I will become famous (or) I will not become a Musician

I will become a musician

∴ I will become famous.

Soln:- Let p: I will become famous
q: I will become a musician

The given argument is

$$\begin{array}{ll} & p \vee \sim q \\ & q \\ \hline \therefore & p \end{array}$$

$$[(p \vee \sim q) \wedge q] \Rightarrow p$$

$\Leftrightarrow (q \rightarrow p) \wedge q$

$\Leftrightarrow q \wedge (q \rightarrow p)$ ~~$\Rightarrow q$~~

$\Rightarrow p$

WKT by Modus pones rule $p \wedge (p \rightarrow q) \Rightarrow q$

Hence it is a valid argument.

(6) If I study, then I do not fail in the examination
If I do not fail in the examination, my father gifts a two-wheeler to me

---

$\therefore$ if I study then my father gifts a two-wheeler to me.

Soln:- Let p: I study

q: I do not fail in the examination

r: My father gifts a two-wheeler to me.

The given argument reads

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

$[(p \rightarrow q) \wedge (q \rightarrow r)] \Rightarrow p \rightarrow r$ (by Rule of syllogism)

It is a valid argument.

(7) If Ravi goes out with friends, he will not study

If Ravi does not study, his father becomes angry

His father is not angry

$\therefore$ Ravi has not gone out with friends.

Soln:- Let p: Ravi goes out with friends

q: Ravi does not study

r: His father becomes angry

The given argument is

$$\begin{array}{c} p \to q \\ q \to r \\ \sim r \\ \hline \therefore \sim p \end{array}$$

$$[(p\to q)\wedge(q\to r)\wedge\sim r] \Rightarrow \sim p$$

Now, $[\underline{(p\to q)\wedge(q\to r)}\wedge\sim r]$ (Rule of syllogism)

$\Rightarrow [(p\to r)\wedge\sim r]$ (by Rule of Modus tollens)

$\Rightarrow \sim p$

(8) If I study, I will not fail in the examination

If I do not watch TV in the evenings, I will study

I failed in the examination

$\therefore$ I must have watched TV in the evenings.

Soln:- Let p: I study

q: I fail in the examination

r: I watch TV in the evenings.

The given argument is

$$p \to \sim q$$
$$\sim r \to p$$
$$q$$
$$\therefore \; r$$

$$[(p \to \sim q) \wedge (\sim r \to p) \wedge q] \Rightarrow r$$

$$\Rightarrow [(\sim r \to p) \wedge (p \to \sim q) \wedge q]$$

Rule of syllogism

$$\Rightarrow [(\sim r \to \sim q) \wedge q]$$

Modus tollens
$[(p \to q) \wedge \sim q \Rightarrow \sim p]$

$$\Rightarrow \sim(\sim r)$$

$$\Rightarrow r$$

Its a valid argument.

(9) I will get grade A in this course or I will not graduate

If do not graduate, I will join the army

$\therefore$ I will not join the army.

Soln:- Let p: I get grade A in this course

q: I do not graduate

r: I join the army

Then the given argument is

$$\begin{array}{l} p \vee q \\ q \rightarrow r \\ p \\ \hline \therefore \sim r \end{array}$$

$$[(p \vee q) \wedge (q \rightarrow r) \wedge p]$$

$\Leftrightarrow [(\sim\sim p \vee q) \wedge (q \rightarrow r) \wedge p]$ Law for conditional

$\Leftrightarrow [(\sim p \rightarrow q) \wedge (q \rightarrow r) \wedge p]$ Rule of syllogism.

$\Leftrightarrow [(\sim p \rightarrow r) \wedge p]$ There is no rule

$\Leftrightarrow [(\sim\sim p \vee r) \wedge p]$

$\Leftrightarrow [(p \vee r) \wedge p]$ by Absorption law

$\Rightarrow r$ //

It is not a valid.

(10) If I have talent and work hard, then I will become successfull in life

If I become successful in life, then I will be happy

$\therefore$ If I will not be happy, then I did not work hard (or) I do not have talent.

Soln:- Let p: I have talent

q: I will work hard

r: I will become successfull in life

s: I will be happy

The given argument is $(p \wedge q) \to r$
$r \to s$
$\therefore \sim s \to (\sim q \vee \sim p)$

$$[((p \wedge q) \to r) \wedge (r \to s)] \Rightarrow \sim s \to (\sim q \vee \sim p)$$

Now, $((p \wedge q) \to r) \wedge (r \to s)$ Rule of syllogism

$\Rightarrow (p \wedge q) \to s$

$\Rightarrow \sim s \to \sim (p \wedge q)$ (demorgan's)

$\Rightarrow \sim s \to \sim p \vee \sim q$

$\Rightarrow \sim s \to \sim q \vee \sim p$

It is valid argument.

(11) If Ravi studies, then he will pass in Discrete Maths paper

If Ravi does not play cricket, then he will study

Ravi failed in Discrete Maths paper

$\therefore$ Ravi played cricket

soln:- p: Ravi studies

q: Ravi will pass in Discrete Maths paper

r: Ravi play cricket.

given argument, $p \to q$
$\sim r \to p$
$\sim q$
$\therefore r$

i.e $[(p \rightarrow q) \wedge (\sim r \rightarrow p) \wedge \sim q] \Rightarrow r$

Now, $[(p \rightarrow q) \wedge (\sim r \rightarrow p) \wedge \sim q]$

$\Leftrightarrow [(\sim r \rightarrow p) \wedge (p \rightarrow q) \wedge \sim q]$ commutative law

$\Rightarrow [(\sim r \rightarrow q) \wedge \sim q]$ (syllogism)

$\Rightarrow \sim(\sim r)$ (by Modus tollens)

$\Rightarrow r$

It is a valid argument.

(2) p

$p \to \sim q$

$\sim q \to \sim r$

$\therefore \sim r$

soln: $p \wedge \underline{(p \to \sim q) \wedge (\sim q \to \sim r)}$

$\Rightarrow p \wedge (p \to \sim r)$ (syllogism)

$\Rightarrow \sim r$ // by Modus pones

(3) $p \to r$

$q \to r$

$(p \vee q) \to r$

soln:- $(p \to r) \wedge (q \to r) \Rightarrow (p \vee q) \to r$

$\Leftrightarrow (\sim p \vee r) \wedge (\sim q \vee r)$

distributive law

$\Leftrightarrow (\sim p \wedge \sim q) \vee r$

demorgan's law

$\Leftrightarrow \sim (p \vee q) \vee r$

conditional law.

$\Rightarrow (p \vee q) \to r$ //

4) $p \to q$

$r \to s$

$p \vee r$

$\therefore q \vee s$

soln:- $(p \to q) \wedge (r \to s) \wedge (p \vee r)$

$\Leftrightarrow (p \to q) \wedge (r \to s) \wedge (\sim\sim p \vee r)$

$\Leftrightarrow (p \to q) \wedge (r \to s) \wedge (\sim p \to r)$

$\Leftrightarrow (p \to q) \wedge (\sim p \to r) \wedge (r \to s)$

$\Leftrightarrow (p \to q) \wedge (\sim p \to s)$ (using contrapositive $p \to q \Leftrightarrow \sim q \to \sim p$)

$\Leftrightarrow (\sim q \to \sim p) \wedge (\sim p \to s)$ (syllogism)

$\Leftrightarrow (\sim q \to s)$

$\Rightarrow q \vee s$ //

It is valid

(5) $p \to q$

$r \to s$

$\sim q \vee \sim s$

$\therefore \sim (p \wedge r)$

Soln:- $(p \to q) \wedge (r \to s) \wedge \underline{(\sim q \vee \sim s)}$ Law for condn.

$\Leftrightarrow (p \to q) \wedge (r \to s) \wedge (q \to \sim s)$ by commutative law

$\Leftrightarrow \underbrace{(p \to q) \wedge (q \to \sim s)}_{\text{(syllogism)}} \wedge (r \to s)$

$\Rightarrow (p \to \sim s) \wedge \underline{(r \to s)}$ by contrapositive

$\Rightarrow (p \to \sim s) \wedge (\sim s \to \sim r)$ (by syllogism)

$\Rightarrow (p \to \sim r)$

$\Rightarrow \sim p \vee \sim r$

$\Rightarrow \sim (p \wedge r)$ //

It is a valid argument.

(6) $p \to r$

$\sim p \to q$

$q \to s$

$\therefore \sim r \to s$

Soln:- $(p \to r) \wedge \underline{(\sim p \to q) \wedge (q \to s)}$ Syllogism

$\Rightarrow \underline{(p \to r)} \wedge (\sim p \to s)$ by contrapositive

$\Rightarrow (\sim r \to \sim p) \wedge (\sim p \to s)$

by Syllogism

$\Rightarrow \sim r \to s$ //

It is valid.

(7) $(\sim p \vee \sim q) \to (r \wedge s)$

$r \to t$

$\sim t$

$\therefore p$

Soln:- $[(\sim p \vee \sim q) \to (r \wedge s)] \wedge \underline{(r \to t) \wedge \sim t}$ by MT

$\Rightarrow [(\sim p \vee \sim q) \to (r \wedge s)] \wedge \sim r$

$\Rightarrow [(\sim p \vee \sim q) \to (r \wedge s)] \wedge (\sim r \vee \sim s)$ (by disjunction)

$\Leftrightarrow [(\sim p \vee \sim q) \to (r \wedge s)] \wedge \sim (r \wedge s)$ demorgan

$\Rightarrow \sim (\sim p \vee \sim q)$ by MT

$\Rightarrow \sim (\sim (p \wedge q))$

$\Rightarrow p \wedge q \Rightarrow p$ (by conjunction)

(8) $p \to (q \to r)$

$\sim q \to \sim p$

p

$\therefore\ r$

Soln:- $\underline{(p \to (q \to r))} \wedge (\sim q \to \sim p) \wedge \underline{p}$ by commutative

$\Rightarrow \underline{(p \to (q \to r)) \wedge p} \wedge (\sim q \to \sim p)$ by Modus pones.

$\Rightarrow (q \to r) \wedge \underline{(\sim q \to \sim p)}$ by contrapositive

$\Rightarrow (q \to r) \wedge (p \to q)$ by commutative law

$\Rightarrow (p \to q) \wedge (q \to r)$ by syllogism

$\Rightarrow p \to r$

$\Rightarrow r$ ($\because$ p is true then r is also true)

(9) $\sim p \leftrightarrow q$

$q \to r$

$\sim r$

$\therefore\ p$

(WKT: $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$)

Soln:- $\underline{(\sim p \leftrightarrow q)} \wedge (q \to r) \wedge \sim r$

$\Rightarrow (\sim p \to q) \wedge (q \to \sim p) \wedge \underline{(q \to r) \wedge \sim r}$ by MT

$\Rightarrow \underline{(\sim p \to q)} \wedge (q \to \sim p) \wedge \underline{\sim q}$ commutative law

$\Rightarrow \underline{[(\sim p \to q) \wedge \sim q]} \wedge (q \to \sim p)$ by M.T

$\Rightarrow \sim(\sim p) \wedge (q \to \sim p)$

$\Rightarrow p \wedge (q \to \sim p) \Rightarrow (q \to \sim p) \wedge p \Rightarrow p$

It is valid.

(10) $p \to (q \to r)$

$p \vee \sim s$

q

$\therefore s \to r$

Soln:- $(p \to (q \to r)) \wedge (p \vee \sim s) \wedge q$ commutative law

$\Rightarrow (p \to (q \to r)) \wedge (\sim s \vee p) \wedge q$ conditional Law

$\Rightarrow (p \to (q \to r)) \wedge (s \to p) \wedge q$ commutative law

$\Rightarrow (s \to p) \wedge (p \to (q \to r)) \wedge q$ syllogism

$\Rightarrow (s \to (q \to r)) \wedge q$ law for condn

$\Leftrightarrow [\sim s \vee (q \to r)] \wedge q$ distributive law

$\Leftrightarrow (\sim s \wedge q) \vee [(q \to r) \wedge q]$ by M.P

$\Leftrightarrow (\sim s \wedge q) \vee r$ by conjunctive simplification

$\Leftrightarrow \sim s \vee r$

$\Leftrightarrow s \to r$ //

It is a valid.

imp (11) $p \to (q \wedge r)$

$r \to s$

$\sim(q \wedge s)$

$\therefore \sim p$

Soln:- $[p \to (q \wedge r)] \wedge (r \to s) \wedge \sim(q \wedge s)$ demorgan's law

$\Leftrightarrow [p \to (q \wedge r)] \wedge (r \to s) \wedge (\sim q \vee \sim s)$ by conditional law

$\Leftrightarrow [p \to (q \wedge r)] \wedge (r \to s) \wedge (q \to \sim s)$ by contrapositive

$\Leftrightarrow [p \to (q \wedge r)] \wedge (r \to s) \wedge (s \to \sim q)$ syllogism

$\Leftrightarrow [p \to (q \wedge r)] \wedge (r \to \sim q)$ by conditional law

$\Leftrightarrow [p \to (q \wedge r)] \wedge (\sim r \vee \sim q)$ demorgan's law

$\Leftrightarrow [p \to (q \wedge r)] \wedge \sim(r \wedge q)$ commutative law

$\Leftrightarrow [p \to (q \wedge r)] \wedge \sim(q \wedge r)$ by M.T

$\Rightarrow \sim p$ //

It is valid

imp (12) $\sim(p \vee q) \to r$

$r \to (s \vee t)$

$\sim s \wedge \sim u$

$\sim u \to \sim t$

$\therefore p$

Soln:- $\underbrace{[(\sim p \vee q) \to r] \wedge [r \to (s \vee t)]}_{\text{Syllogism}} \wedge [\sim s \wedge \sim u] \wedge \underbrace{[\sim u \to \sim t]}_{\text{Associative}}$

$\Rightarrow [(\sim p \vee q) \to (s \vee t)] \wedge \sim s \wedge \underline{[\sim u \wedge (\sim u \to \sim t)]}$ by M.P

$\Rightarrow [(\sim p \vee q) \to (s \vee t)] \wedge \underline{(\sim s \wedge \sim t)}$ demorgan's law

$\Rightarrow \underline{[(\sim p \vee q) \to (s \vee t)] \wedge \sim (s \vee t)}$ by M.T

$\Rightarrow \sim (\sim p \vee q)$ demorgan's

$\Rightarrow \sim\sim p \wedge \sim q$

$\Rightarrow p \wedge \sim q$ by conjunctive simplification

$\Rightarrow p$ //

It is valid.

imp (13)

$p \vee q$

$\sim p \vee r$

$\sim r$

$\therefore q$

Soln:- $(p \vee q) \wedge \underline{(\sim p \vee r)} \wedge (\sim r)$ conditional law

$\Rightarrow (p \vee q) \wedge \underline{(p \to r) \wedge \sim r}$ by M.T

$\Rightarrow \underline{(\sim\sim p \vee q)} \wedge \sim p$

$\Rightarrow (\sim p \to q) \wedge \sim p$ by M.P

$\Rightarrow q$ // valid argument.

(14) $p \to r$

$r \to s$

$t \vee \sim s$

$\sim t \vee u$

$\sim u$

$\therefore \sim p$

Soln:- $\underbrace{(p \to r) \wedge (r \to s)}_{\text{syllogism}} \wedge \underbrace{(t \vee \sim s)}_{\text{commutative and conditional}} \wedge \underbrace{(\sim t \vee u)}_{\text{conditional}} \wedge \sim u$

$\Rightarrow (p \to s) \wedge \underbrace{(s \to t) \wedge (t \to u)}_{\text{syllogism}} \wedge \sim u$

$\Rightarrow \underbrace{(p \to s) \wedge (s \to u)}_{\text{syllogism}} \wedge \sim u$

$\Rightarrow (p \to u) \wedge \sim u$ by MT

$\Rightarrow \sim p$ // It is valid

(15) $p \to q$

$q \to (r \wedge s)$

$\sim r \vee (\sim t \vee u)$

$p \wedge t$

$\therefore u$

Soln:- $\underbrace{(p \to q) \wedge (q \to (r \wedge s))}_{\text{syllogism}} \wedge (\sim r \vee (\sim t \vee u)) \wedge (p \wedge t)$

$\Rightarrow [p \to (r \wedge s)] \wedge [\sim r \vee (\sim t \vee u)] \wedge (p \wedge t)$

$$\Rightarrow [p \to (r \wedge s)] \wedge [\underbrace{(\sim r \vee \sim t)}_{\text{demorgan's}} \vee u] \wedge (p \wedge t)$$

commutative

$$\Rightarrow [p \to (r \wedge s)] \wedge (p \wedge t) \wedge [\sim (r \wedge t) \vee u]$$

conditional

$$\Rightarrow [p \to (r \wedge s)] \wedge (p \wedge t) \wedge [(r \wedge t) \to u]$$

$$\Rightarrow \underbrace{(p \to (r \wedge s)) \wedge p}_{\text{by M.p}} \wedge t \wedge [(r \wedge t) \to u]$$

$$\Rightarrow (r \wedge s) \wedge t \wedge [(r \wedge t) \to u]$$ commutative

$$\Rightarrow (r \wedge t) \wedge s \wedge ((r \wedge t) \to u)$$ commutative

$$\Rightarrow \underbrace{(r \wedge t) \wedge [(r \wedge t) \to u]}_{\text{by M.p}} \wedge s$$

$\Rightarrow u \wedge s$ by conjunctive simplic.

$\Rightarrow u$

Try

(1) p

$p \to q$

$s \vee r$

$r \to \sim q$

$\therefore s$

(2) $C \vee D$

$(C \vee D) \to \sim H$

$\sim H \to (A \wedge \sim B)$

$(A \wedge \sim B) \to R \vee S$

$\therefore R \vee S.$

OPEN STATEMENTS ; QUANTIFIERS STATEMENTS

(*) Open Statements

The declarative sentences like $x+2=3$, $x=\sqrt{2}$, $x>0$, x divisible by 2, in these sentences 'x' is not defined, such statements are called as open statement and 'x' is called the free variables.

→ This open statement can be converted into propositions by giving the values to 'x' from universal set (U)

→ Open statements are denoted by $p(x)$, $q(x)$, $r(x)$ and so on.

consider $x+3=6$ and the set of real number R.
Now, This sentences becomes a proposition ~~and~~ if x is replaced by any element of R.

For example :-
if x is replaced by 3 → true proposition
and if x is replaced by 5, it becomes a false proposition, Here we say R is a universe (or universe of discourse).

Example (1): Suppose the universal set consists of all integers, consider the following open statements
$p(x): x \leq 3$; $q(x): x+1$ is odd , $r(x): x>0$
Write down the truth values of the following,

(i) $p(2)$

:- We've $p(x) : x \leq 3$

$x = 2, \Rightarrow p(2) : 2 \leq 3$ It is true $\rightarrow 1$

(ii) $\sim q(4)$

:- We've $q(x) : x + 1$ is odd

$\Rightarrow q(4) : 4 + 1$ is odd

$\sim q(4)$ is false $\rightarrow 0$

(iii) $p(-1) \wedge q(1)$

:- $p(-1) : -1 \leq 3 \rightarrow$ true $\rightarrow 1$

$q(1) : 1 + 1 = 2$ is not odd $\rightarrow$ false $- 0$.

$\therefore$ $p(-1) \wedge q(1)$ $[1 \wedge 0]$ is false.

(iv) $\sim p(3) \vee r(0)$

:- $p(3) : 3 \leq 3$ is true $- 1$

$\Rightarrow \sim p(3)$ is false $- 0$

$r(0) : 0 > 0$ is false $- 0$

$\therefore$ $\sim p(3) \wedge r(0) = 0 \wedge 0 = 0$

$\Rightarrow$ false

(v) $p(2) \wedge [q(0) \vee \sim r(2)]$

:- $p(2) : 2 \leq 3$ true $- 1$

$q(0) : 0 + 1 = 1$ is odd, true $- 1$

$r(2) : 2 > 0$

$\Rightarrow \sim r(2) : 2 < 0$ - false $- 0$

$\therefore\ p(2) \wedge [q(0) \vee \sim r(2)]$

$1 \wedge [1 \vee 0] \Rightarrow 1 \wedge 1$

$= 1$

$\Rightarrow p(2) \wedge [q(0) \vee \sim r(2)]$ – True – 1//

Quantifiers

consider the statements

(i) All integers are divisible by 2

(ii) Some integers are divisible by 3

(iii) Every integer is real

(iv) There exists an integer which is prime

In these propositions, the words 'all', "every", "some", "there exists" are associated with the idea of a quantity. Such words are called Quantifiers.

→ The symbol $\forall$ has been used to denote the phrases "for all", "for every", "for any", "for each" and these are called universal quantifiers.

→ The symbol $\exists$ has been used to denote the phrases 'there exists', for some, for atleast, these are called as existential quantifiers.

→ A proposition involving universal (or) existen-tial quantifier is called a <u>quantifier statement</u>

$\Rightarrow$ Thus a quantified statement is a proposition of the form "$\forall x \in S, p(x)$" (or) "$\exists x \in S, p(x)$" where $p(x)$ is an open statement and S is the universe for x in $p(x)$.

"The variable present in a quantified statement is called a bound variable."

<u>Example:-</u> For the universe of all integers, let

$p(x)$: $x > 0$

$q(x)$: x is even

$r(x)$: x is a perfect square (4, 9, 16, ---)

$s(x)$: x is divisible by 3.

$t(x)$: x is divisible by 7.

Write down the following quantified statements in symbolic form.

(i) At least one integer is even.

Soln:- $\exists x \in S, q(x)$

(ii) There exists a positive integer that is even

Soln:- $\exists, x \in S, (p(x) \wedge q(x))$

(iii) ~~Some even integers is either even (or) odd.~~

Soln:- $\exists$

(iii) Some even integers are divisible by 3.

Soln:- $\exists\, x \in S,\ [q(x) \wedge s(x)]$

(iv) Every integers is either even (or) odd

:- $\forall\, x \in S,\ [q(x) \vee \sim q(x)]$

(v) If x is even and a perfect square, then x is not divisible by 3.

:- $\forall\, x \in S,\ [(q(x) \wedge r(x)) \rightarrow \sim s(x)]$

(vi) If x is odd (or) is not divisible by 7, then x is divisible by 3.

:- $\forall\, x \in S, \{[\sim q(x) \vee \sim t(x)] \rightarrow s(x)\}$.

(*) Truth value of a Quantifier statement

Rule 1:- $\forall\, x \in S$, $p(x)$ is true, then it should be true for all values of x, suppose if it is false for atleast one value, i.e $p(a)$ is false where $a \in S$ this element 'a' is counter example, then $\forall\, x \in S$, $p(x)$ is false.

Rule 2:- For proving $\exists\, x \in S$, $p(x)$ is true, it is enough to show that $p(a)$ is true where $a \in S$.

(*) Truth value of a Quantified statement

Rule 1 : The statement "$\forall x \in S$, $p(x)$" is true only when $p(x)$ is true for each $x \in S$.

Rule 2 : The statement "$\exists x \in S$, $p(x)$" is false for every $x \in S$. only when $p(x)$ is false

Accordingly, a proposition of the form "$\forall x \in S$, $p(x)$", is false, it is enough to exhibit one element 'a' of S such that p(a) is false. This element 'a' is called counter example

IIIly a proposition of the form "$\exists x \in S$, $p(x)$" is true, it is enough to exhibit one element 'a' of S s/t p(a) is true.

(*) Two Rules of Inference :

(*) If an open statement $p(x)$ is known to be true for all x in a universe S and if $a \in S$, then p(a) is true (Rule of universal specification)

(*) If an open statement $p(x)$ is proved to be true for any x chosen from a set S, then the quantified statement $\forall x \in S$, $p(x)$ is true
(Rule of universal Generalization).

problems

(1) consider the open statement $p(x): x > 0$;

$q(x)$: x is even ; $r(x)$: x is a perfect square

$s(x)$: x is divisible by 3 ; $t(x)$: x is divisible by 7

Express each of the following symbolic statements in words. (i) $\forall x, [r(x) \rightarrow p(x)]$ (ii) $\exists x, [s(x) \wedge \sim q(x)]$

(iii) $\forall x, [\sim r(x)]$ (iv) $\forall x, [r(x) \vee t(x)]$

Soln:- (i) For all integer x, if x is a perfect square then $x > 0$

(ii) For some integer x, x is divisible by 3 and x is odd.

(iii) For all integer x, x is not a perfect square

(iv) For all integer x, x is a perfect square (or) x is divisible by 7 → False (Eg:- $x = 8$)

(2) consider the following open statements with the set of all real no as the universe.

$p(x)$: $|x| > 3$, $q(x)$: $x > 3$, Find the truth values of the statement $\forall x, [p(x) \rightarrow q(x)]$

Soln:- Given $p(x)$: $|x| > 3$

for $x = -4$, i.e $p(-4)$: $|-4| > 3$

$4 > 3$ is true - 1

And $q(x)$: $x > 3$, for $x = -4$, $q(-4)$: $-4 > 3$ - false - 0

For $x = -4$, $p(x) : 1$ and $q(x) : 0$

$$\therefore p(x) \to q(x) = 1 \to 0$$
$$= 0$$
$$\Rightarrow \text{false}$$

[We've got this result by taking $x = -4$ counter example)

(3) consider the following open statements with the set of all real numbers as the universe

$p(x) : x \geq 0$; $q(x) : x^2 \geq 0$; $r(x) : x^2 - 3x - 4 = 0$

$s(x) : x^2 - 3 > 0$

Determine the truth values of the following statements.

(i) $\exists\, x, p(x) \wedge q(x)$ (ii) $\forall\, x, p(x) \to q(x)$

(iii) $\forall\, x, q(x) \to s(x)$ (iv) $\forall\, x, r(x) \vee s(x)$

(v) $\exists\, x, p(x) \wedge r(x)$ (vi) $\forall\, x, r(x) \to p(x)$.

Soln:-

(i) there exist a real number 'x' for which both $p(x)$ and $q(x)$ are true ; for instance $x = 1$

$$\therefore p(x) \wedge q(x) = 1 \wedge 1 = 1 \to \text{true}$$

(ii) For all x, $q(x)$ is true hence $p(x) \to q(x)$ is true.

(iii) For every real no x, $s(x)$ is false for $x = 1$ and $q(x)$ is true

Thus. $q(x) \to s(x)$ is false for $x = 1$

i.e The statement $q(x) \to s(x)$ is not always true

$\therefore\ \forall x,\ q(x) \to s(x)$ is false

(iv) $r(x): x^2 - 3x - 4 = 0$

$(x-4)(x+1)$

i.e $r(x)$ is true only for $x = 4$ and $x = -1$

but for all real no x, $r(x)$ is not true i.e it is false and $s(x)$ is false for $x = 1$

Hence $r(x) \not\to s(x)$ is false (not always true)

(v) $\exists x,\ p(x) \wedge r(x)$

for $x = 4$, both $p(x)$ and $r(x)$ is true

hence, $p(x) \wedge r(x) = 1 \wedge 1 = 1$ is true.

(vi) $\forall x,\ r(x) \to p(x)$

$r(x)$ is true for $x = -1$ but $p(x)$ is false for $x = -1$

Thus. $r(x) \to p(x)$ is not always true

$\therefore \forall x,\ r(x) \to p(x)$ is false

(4). Let $p(x) = x^2 - 7x + 10 = 0$; $q(x): x^2 - 2x - 3 = 0$, $r(x): x < 0$. Determine the truth (or) falsity of the following statements where the universe U contains only the integers 2 and 5. If a statement is false, provide a counter example (or) explanation.

(i) $\forall x\ p(x) \to \sim r(x)$ (ii) $\forall x,\ q(x) \to r(x)$

(iii) $\exists x, q(x) \to r(x)$ (iv) $\exists x, p(x) \to r(x)$

Soln: Let $U = \{2, 5\}$

$p(x) : x^2 - 7x + 10 = 0$

$p(x) : (x-5)(x-2) = 0$

$p(x) : x = 5, x = 2$

$q(x) : x^2 - 2x - 3 = 0$

$q(x) : (x-3)(x+1) = 0$

$q(x) : x = 3, x = -1$

(i) $\forall x \; p(x) \to \sim r(x)$

$p(x)$ is true for all $x \in U$ and $\sim r(x)$ is true for all $x \in U$

$\therefore \forall x, p(x) \to \sim r(x)$ is true.

(ii) $\forall x \; q(x) \to r(x)$

$q(x)$ is false for all $x \in U$ and $r(x)$ is false for all $x \in U$

$\therefore \forall x \; q(x) \to r(x)$ is true.

(iii) $\exists x, q(x) \to r(x)$

$q(x)$ and $r(x)$ is false at $x = 2$

$\therefore \exists \; q(x) \to r(x)$ is true

(iv) $\exists x \, p(x) \to r(x)$

$p(x)$ is true for $x = 2$ but $r(x)$ is false for $x = 2$

$\therefore \exists x, p(x) \to r(x)$ is false.

(*) Logical Equivalence :

Two quantified statements are said to be logically equivalent whenever they have the same truth values in all possible situations.

(i) $\forall x, [p(x) \wedge q(x)] \Leftrightarrow (\forall x, p(x)) \wedge (\forall x, q(x))$

(ii) $\exists x, [p(x) \vee q(x)] \Leftrightarrow (\exists x, p(x)) \vee (\exists x, q(x))$

(iii) $\exists x, [p(x) \rightarrow q(x)] \Leftrightarrow \exists x, [\sim p(x) \vee q(x)]$

(*) Rule for Negation of a Quantified statement

(1) $\sim\{\forall x, p(x)\} \Leftrightarrow \exists x, \{\sim p(x)\}$

(2) $\sim\{\exists x, p(x)\} \Leftrightarrow \forall x, \{\sim p(x)\}$

PROBLEMS

(1) consider the open statements $p(x)$; $x > 0$, $q(x)$: x is even, $r(x)$: x is perfect square, $s(x)$: x is divisible by 3, $t(x)$: x is divisible by 7. Express each of the following symbolic statements in words and indicate its truth table,

(i) $\forall x, [r(x) \rightarrow p(x)]$

Soln.- For any integer x, if x is perfect square then $x > 0$ — false (take $x = 0$)

(2) Negate and simplify each of the following

(i) $\exists x, [p(x) \vee q(x)]$

soln:- $\sim\{\exists x, [p(x) \vee q(x)]\}$

$\Leftrightarrow \cancel{\sim\exists x, p}\ \forall x, [\sim(p(x) \vee q(x))]$

$\Leftrightarrow \forall x, [\sim p(x) \wedge \sim q(x)]$ //

(ii) $\forall x, [p(x) \wedge \sim q(x)]$

soln:- $\sim\{\forall x, [p(x) \wedge \sim q(x)]$

$\Leftrightarrow \exists x, [\sim(p(x) \wedge \sim q(x))]$

$\Leftrightarrow \exists x, [\sim p(x) \vee q(x)]$ //

(iii) $\forall x, [p(x) \rightarrow q(x)]$

soln:- $\sim\{\forall x, [p(x) \rightarrow q(x)]\}$

$\Leftrightarrow \exists x, [\sim(p(x) \rightarrow q(x))]$ Law for cond n.

$\Leftrightarrow \exists x, [\sim(\sim p(x) \vee q(x))]$ demorgan's

$\Leftrightarrow \exists x, [p(x) \wedge \sim q(x)]$

(iv) $\exists x, [\{p(x) \vee q(x)\} \rightarrow r(x)]$

soln:- $\sim\{\exists x, [\{p(x) \vee q(x)\} \rightarrow r(x)]\}$

$\Leftrightarrow \forall x, \{\sim[(p(x) \vee q(x)) \rightarrow r(x)]\}$

$\Leftrightarrow \forall x, \{\sim[\sim(p(x) \vee q(x)) \vee r(x)]\}$

$\Leftrightarrow \forall x, [(p(x) \vee q(x)) \wedge \sim r(x)], x \in A$ //

(3) Write down the following proposition in symbolic form and find its negation

" All integers are rational numbers and some rational numbers are not integers".

:- $p(x)$: x is a rational number

$q(x)$: x is an integers

Z = Set of all integers , Q = set of all rational no

Symbolic form,

$$[\forall x \in Z, p(x)] \wedge [\exists x \in Q, \sim q(x)]$$

$\Leftrightarrow \sim \{[\forall x \in Z, p(x)] \wedge [\exists x \in Q, \sim q(x)]\}$

$\Leftrightarrow \sim [\forall x \in Z, p(x)] \vee \sim [\exists x \in Q, \sim q(x)]$

$\Leftrightarrow [\exists x \in Z, \sim p(x)] \vee [\forall x \in Q, q(x)]$ //

Some integers are not rational no (or) all rational numbers are integers.

(4) Let the set Z of all integers be the universe, consider the statements

$p(x) : 3x + 1 = 5$ and $q(x) : x^2 = 9$

obtain the negation of the quantified statement $\exists x \in Z, [p(x) \wedge q(x)]$ and express it in words.

Soln:- The negation of the given statement is

$$\sim\{\exists x \in Z, [p(x) \wedge q(x)]\}$$

$$\Leftrightarrow \forall x \in Z, \sim[p(x) \wedge q(x)]$$

$$\Leftrightarrow \forall x \in Z, \sim p(x) \vee \sim q(x).$$

In words, For all integers x, $2x+1 \neq 5$ (or) $x^2 \neq 9$.

(5) Write down the following proposition in symbolic form and find its negation.

"If all triangles are right angled, then no triangle is equiangular".

Soln:- Let S be the set of all triangle

$p(x)$: x is right angled

$q(x)$: x is equiangular.

In symbolic form,

$$[\forall x \in S, p(x)] \rightarrow [\forall x \in S, \sim q(x)]$$

Now, its negation,

$$\sim\{[\forall x \in S, p(x)] \rightarrow [\forall x \in S, \sim q(x)]\}$$

$$\Leftrightarrow \sim\{\sim[\forall x \in S, p(x)] \vee [\forall x \in S, \sim q(x)]\}$$

$$\Leftrightarrow [\forall x \in S, p(x)] \wedge \sim[\forall x \in S, \sim q(x)]$$

$$\Leftrightarrow [\forall x \in S, p(x)] \wedge [\exists x \in S, q(x)]$$

In words, For All triangle are right angled and Some triangles are equiangular.

(6) Write down the negation of each of the following statements.

(i) For all integers 'n', if n is not divisible by 2 then n is odd

(ii) If k, m, n are any integers where $(k-m)$ and $(m-n)$ are odd, then $(k-n)$ is even.

(iii) For all real numbers, x, if $|x-3| < 7$, then $-4 < x < 10$.

(iv) If x is a real number where $x^2 > 16$, then $x < -4$ or $x > 4$.

Soln:- 'Z' be the set of all integers and 'R' be the set of all real no.

(i) $p(x)$: n is divisible by 2

$q(x)$: n is odd

Symbolic form, $\forall\, n \in Z,\ \sim p(x) \rightarrow q(x)$

Its negation, $\sim[\forall\, n \in Z,\ \sim p(x) \rightarrow q(x)]$

$\Leftrightarrow \exists\, n \in Z,\ \sim(\sim p(x) \rightarrow q(x))$

$\Leftrightarrow \exists\, n \in Z,\ \sim[\sim\sim p(x) \vee q(x)]$

$\Leftrightarrow \exists\, n \in Z,\ \sim p(x) \wedge \sim q(x)$

" For some integers n, n is not divisible by 2 and n is not odd".

(ii) $p(x): (k-m)$ is odd ; $r(x): (k-n)$ is even

$q(x): (m-n)$ is odd

Symbolic form; $\forall\, k, m, n \in \mathbb{Z},\ (p(x) \wedge q(x)) \rightarrow r(x)$

$\therefore$ The negation of the statement is,

$$\sim\{\forall\, k, m, n \in \mathbb{Z},\ (p(x) \wedge q(x)) \rightarrow r(x)\}$$

$$\Leftrightarrow \exists\, k, m, n \in \mathbb{Z},\ \sim[(p(x) \wedge q(x)) \rightarrow r(x)]$$

$$\Leftrightarrow \exists\, k, m, n \in \mathbb{Z},\ \sim[\sim(p(x) \wedge q(x)) \vee r(x)]$$

$$\Leftrightarrow \exists\, k, m, n \in \mathbb{Z},\ (p(x) \wedge q(x)) \wedge \sim r(x)$$

In words,

" For some integers k, m, n , $(k-m)$ and $(m-n)$ are odd and $(k-n)$ is not even".

(iii) Here, $x \in R$

$p(x) : |x-3| < 7$ & $q(x) : -4 < x < 10$ (i.e $x \in (-4, 10)$)

Symbolic form, $\forall\, x \in R,\ p(x) \rightarrow q(x)$

The negation is

$$\sim\{\forall\, x \in R,\ p(x) \rightarrow q(x)\}$$

$$\Leftrightarrow \exists\, x \in R,\ \sim(p(x) \rightarrow q(x))$$

$$\Leftrightarrow \exists\, x \in R,\ \sim(\sim p(x) \vee q(x))$$

$$\Leftrightarrow \exists\, x \in R,\ p(x) \wedge \sim q(x)$$

In words, For some real no, $|x-3| < 7$ and $x \notin (-4, 10)$.

(iv) $p(x): x^2 > 16$, $q(x): x < -4$, $r(x): x > 4$

Symbolic form, $\forall x \in R,\ p(x) \rightarrow (q(x) \vee r(x))$

Its negation is

$$\sim\{\forall x \in R,\ [p(x) \rightarrow (q(x) \vee r(x))]\}$$

$$\Leftrightarrow \exists x \in R, \sim[p(x) \rightarrow (q(x) \vee r(x))]$$

$$\Leftrightarrow \exists x \in R, \sim[\sim p(x) \vee (q(x) \vee r(x))]$$

$$\Leftrightarrow \exists x \in R,\ p(x) \wedge \sim(q(x) \vee r(x))$$

$$\Leftrightarrow \exists x \in R,\ p(x) \wedge \sim q(x) \wedge \sim r(x)$$

In words,

For some real no, $x^2 > 16$ and $x \geq -4$ and $x \leq 4$.

(*) Logical Implication Involving Quantifiers

A quantified statement P is said to logically imply a quantified statement Q if Q is true whenever P is true

then we write $P \Rightarrow Q$

An argument $P_1 \wedge P_2 \wedge P_3 \wedge --- \wedge P_n \rightarrow Q$ is valid argument if Q is true whenever each of $P_1, P_2, --- P_n$ is true (or)

$$P_1 \wedge P_2 \wedge P_3 \wedge --- \wedge P_n \Rightarrow Q$$

PROBLEMS

(1) P.T the following argument is valid

All men are mortal

Sachin is a man

$\therefore$ Sachin is mortal

:- Let S denote the set of all men

$p(x)$: x is mortal

a : Sachin

The given argument, $\forall x \in S, p(x)$

$$\begin{array}{l} \forall x \in S,\ p(x) \\ a \in S \\ \hline \therefore\ p(a) \end{array}$$

$p(x)$ is true and $a \in S$ means $p(a)$ is true

$\therefore$ This is valid by universal specification.

(2) Find whether the following is a valid argument for which the universe is the set of all students

No Engineering student is bad in studies

Anil is not bad in studies

$\therefore$ Anil is an engineering student.

Soln:- S : set of all students.

$p(x)$: x is an engineering student

$q(x)$: x is bad in studies

a : Anil.

Argument,

$$\forall x, [p(x) \rightarrow \sim q(x)]$$
$$\sim q(a)$$
$$\therefore \quad p(a)$$

$[p(x) \rightarrow \sim q(x)] \wedge \sim q(a)$

$\Leftrightarrow [p(a) \rightarrow \sim q(a)] \wedge \sim q(a)$ (by universal specification)

$\not\Rightarrow p(a)$ ($\because$ It is not M.T ~~or~~ M.P).

Because no rule is applicable

$\therefore$ It is an invalid statement.

Imp (3) Find whether the following variable is valid.

No engineering students of I^{st} or II^{nd} sem studies logic

Anil is an engineering student who studies logic

$\therefore$ Anil is not in second semester.

Soln:- S : set of all engineering students.

$p(x)$: x is in I^{st} sem

$q(x)$: x is in II^{nd} sem

$r(x)$: x studies logic

a : Anil

$\therefore \forall x \in S, \ (p(x) \vee q(x)) \rightarrow \sim r(x)$

$r(a)$

$\therefore \sim q(a)$

$\{[p(x) \vee q(x)] \rightarrow \sim r(x)\} \wedge r(a)$

$\Rightarrow \{[p(a) \vee q(a)] \rightarrow \sim r(a)\} \wedge r(a)$ [WKT by MT $(p \rightarrow \sim q) \wedge q \Leftrightarrow \sim p$

$\Rightarrow \sim [p(a) \vee q(a)]$ demorgan's law

$\Rightarrow \sim p(a) \wedge \sim q(a)$

$\Rightarrow \sim q(a) \wedge \sim p(a)$ commutative

$\Rightarrow \sim q(a)$ (by conjunction Simplification)

It is valid.

imp

(4) P.T the following argument is not valid

All squares have four sides

The Quadrilateral ABCD has four sides

$\therefore$ ABCD is a square

Soln:- S : set of all quadrilateral

$p(x)$: x is a square

$q(x)$: x has 4 sides.

a : ABCD

$$\forall x \in S, \quad \begin{array}{l} p(x) \to q(x) \\ q(a) \\ \hline \therefore\ p(a) \end{array}$$

$[p(x) \to q(x)] \wedge q(a)$

$\Rightarrow [p(a) \to q(a)] \wedge q(a)$ [neither M.P nor M.T]

$\not\Rightarrow p(a)$

$\therefore$ It is valid.

(5) Over the universe of all quadrilateral in plane geometry. Verify the validity of the argument "Since every square is a rectangle and every rectangle is a parallelogram, it follows that every square is a parallelogram."

Soln:- S: set of all ~~parall~~ quadrilateral

$p(x)$: x is a square

$q(x)$: x is a rectangle

$r(x)$: x is a parallelogram.

$$\forall x \in S, \quad \begin{array}{l} p(x) \to q(x) \\ q(x) \to r(x) \\ \hline \therefore\ p(x) \to r(x) \end{array}$$

Now, $[p(x) \to q(x)] \wedge [q(x) \to r(x)]$

$\Rightarrow [p(a) \rightarrow q(a)] \wedge [q(a) \rightarrow r(a)]$

$\Rightarrow \quad p(a) \rightarrow r(a) \quad$ [$\because$ syllogism]

$\therefore$ It is valid.

mp
(6) prove the following argument is valid

$\forall x, [p(x) \rightarrow \{q(x) \wedge r(x)\}]$

$\forall x, [p(x) \wedge s(x)]$

$\therefore \forall x, [r(x) \wedge s(x)]$

soln:- $[p(x) \rightarrow \{q(x) \wedge r(x)\}] \wedge [p(x) \wedge s(x)]$

$\Rightarrow \underline{[p(a) \rightarrow \{q(a) \wedge r(a)\}] \wedge [p(a)} \wedge s(a)]$
$\hookrightarrow$ M.P

$\Rightarrow \{q(a) \wedge r(a)\} \wedge s(a)$

$\Rightarrow \quad r(a) \wedge s(a) \quad$ (by conjunctive simplification)

$\Rightarrow \quad r(x) \wedge s(x)$

It is valid.

mp
(7) prove the following argument is valid

$\forall x, [p(x) \vee q(x)]$

$\forall x, [\{\sim p(x) \wedge q(x)\} \rightarrow r(x)]$

$\therefore \forall x, [\sim r(x) \rightarrow p(x)]$

soln:- $[p(x) \vee q(x)] \wedge [\{\sim p(x) \wedge q(x)\} \rightarrow r(x)]$

$\Rightarrow [p(a) \vee q(a)] \wedge [\underline{\{\sim p(a) \wedge q(a)\} \rightarrow r(a)}]$ Law for condn.

$\Rightarrow [p(a) \vee q(a)] \wedge [\sim(\sim p(a) \wedge q(a)) \vee r(a)]$
demorgan's

$\Rightarrow \underline{[p(a) \vee q(a)] \wedge [p(a) \vee \sim q(a)]} \vee r(a)$
distributive

$\Rightarrow [p(a) \vee (q(a) \wedge \sim q(a))] \vee r(a)$

$\Rightarrow p(a) \vee F_0 \vee r(a)$

$\Rightarrow p(a) \vee r(a)$

$\Rightarrow r(a) \vee p(a)$

$\Rightarrow \sim\sim r(a) \vee p(a)$

$\Rightarrow \sim r(a) \rightarrow p(a)$

$\Rightarrow \sim r(x) \rightarrow p(x)$ //

It is valid.

(8) prove that the following argument is valid

$\forall x, [p(x) \vee q(x)]$

$\exists x, \sim p(x)$

$\forall x, [\sim q(x) \vee r(x)]$

$\forall x, [s(x) \rightarrow \sim r(x)]$

$\therefore \exists x, \sim s(x)$

Soln:- $[p(x) \vee q(x)] \wedge \sim p(x) \wedge [\sim q(x) \vee r(x)] \wedge [s(x) \rightarrow \sim r(x)]$

$\Rightarrow \underline{[p(a) \vee q(a)] \wedge \sim p(a)} \wedge [\sim q(a) \vee r(a)] \wedge [s(a) \rightarrow \sim r(a)]$
by disjuctive syllogism

$\Rightarrow \underline{q(a) \wedge [\sim q(a) \vee r(a)]} \wedge [s(a) \rightarrow \sim r(a)]$ (by disjuctive syllogism) $(p \vee q) \wedge \sim q \Rightarrow q$

$\Rightarrow r(a) \wedge [s(a) \rightarrow \sim r(a)]$

$\Rightarrow r(a) \wedge [\sim s(a) \vee \sim r(a)]$ by disjuctive syllogism

$\Rightarrow \sim s(a)$

$\Rightarrow \sim s(x)$

$\therefore$ It is valid.

(9) Find whether the following argument is valid.

If a triangle has 2 equal sides, then it is isoceles

If a triangle is isoceles, then it has 2 equal angles

The triangle ABC does not have 2 equal angles.

$\therefore$ ABC does not have 2 equal sides.

Soln:- S : set of all triangles.

$p(x)$: x has 2 equal sides

$q(x)$: x is isoceles.

$r(x)$: x has 2 equal angles.

a : Δ^{le} ABC (counter example)

Now, $\forall x \in S$, $p(x) \rightarrow q(x)$

$q(x) \rightarrow r(x)$

$\sim r(a)$

$\sim p(a)$

$\therefore\ [p(x) \to q(x)] \wedge [q(x) \to r(x)] \wedge \sim r(a)$

$\Rightarrow\ \underline{[p(a) \to q(a)] \wedge [q(a) \to r(a)]} \wedge \sim r(a)$

Syllogism

$\Rightarrow\ [p(a) \to r(a)] \wedge \sim r(a)$ Modus pones

$\Rightarrow\ \sim p(a)$

$\therefore$ It is valid //.

(10) Determine if the argument is valid (or) not.

All people concerned about the environment, recycle their plastic containers. B is not concerned about the environment. Therefore, B does not recycle his plastic containers.

Soln:-

S : Set of all people.

$p(x)$: x is concerned about the environment

$q(x)$: x is recycle their containers.

a : B

$$\forall x \in S,\quad p(x) \to q(x)$$
$$\frac{\sim p(a)}{\therefore \sim q(a)}$$

$\therefore\ [p(x) \to q(x)] \wedge \sim p(a) \Rightarrow [p(a) \to q(a)] \wedge \sim p(a)$

$\not\Rightarrow \sim q(a)$ [both MP and MT are invalid.

$\therefore$ Argument is invalid

(*) Open statements with more than one variable

Consider the following statements $x-2y \geq 0$ and $x-y+z=0$, these are the open statements which contains more than one free variables.

These becomes propositions if each variables is replaced by an element in a universal set.

Open statements containing two variables are denoted by $p(x,y)$, $q(x,y)$ -- so on

Open statements containing 3 variables is denoted by $p(x,y,z)$, $q(x,y,z)$ --- so on.

ex:(1) Let $p(x,y): x^2 \geq y$, $q(x,y): (x+2) < y$, where x and y are the set of all real numbers. Determine the truth values of the following statements.

Soln:- (i) $p(2,4)$ (ii) $p(-3,8)$ (iii) $p(-3,8) \wedge q(1,3)$

(iv) $p\left(\frac{1}{2},\frac{1}{3}\right) \vee \sim q(-2,-3)$

Soln:- (i) $p(\underset{x}{2},\underset{y}{4}) : (2)^2 \geq 4$

$4 \geq 4 \rightarrow$ true $- 1$

(ii) $p(-3,8) : (-3)^2 \geq 8$

$\Rightarrow 9 \geq 8 \rightarrow$ true $- 1$

(iii) $p(-3,8) : -3^2 \geq 8 = 9 \geq 8$ - true $- 1$

$q(1,3) : (1+2) < 3$

$3 < 3$ — false — 0

$\therefore p(-3,8) \wedge q(1,3) = 1 \wedge 0 = 0 \rightarrow$ False

(iv) $p\left(\frac{1}{2}, \frac{1}{3}\right) : \left(\frac{1}{2}\right)^2 \geq \frac{1}{3}$

$\Rightarrow \frac{1}{4} \geq \frac{1}{3} \rightarrow$ False — 0

$q(-2,-3) : (-2+2) < -3$

$\Rightarrow 0 < -3 \Rightarrow$ false — 0

$\therefore p\left(\frac{1}{2}, \frac{1}{2}\right) \vee \sim q(-2,-3)$

$\Rightarrow 0 \vee 1$

$\Rightarrow 1$ - true.

(*) **Quantified statement with more than one variable**

When an open statement contains more than one free variable, quantification may be applied to each of the variable.

Properties

(i) $\forall x, \forall y \; p(x,y) \Leftrightarrow \forall y, \forall x \; p(x,y) \Leftrightarrow \forall x,y \; p(x,y)$

(ii) $\exists x, \exists y \; p(x,y) \Leftrightarrow \exists y, \exists x \; p(x,y) \Leftrightarrow \exists x,y \; p(x,y)$

(iii) $\forall x, \exists y \; p(x,y) \not\Leftrightarrow \exists y, \forall x, p(x,y)$

(iv) $\forall y, \exists x \; p(x,y) \not\Leftrightarrow \exists x, \forall y \; p(x,y)$

Similarly for 3 variables.

problems

(1) Let x and y denote the integers. consider the statement

$p(x,y)$: $x+y$ is even

write down the following statements in words

(i) $\forall x, \exists y\ p(x,y)$ (ii) $\exists x, \forall y, p(x,y)$

(iii) $\forall x, \forall y\ p(x,y)$

soln:- (i) For All integers x, there exist an integer y, $x+y$ is even

(ii) There exists an integers x such that $x+y$ is even for every integer y.

(iii) For all integers x and y, $x+y$ is even.

(2) Find the Negation for the following quantified statements.

(i) $\forall x, \exists y, [\{p(x,y) \wedge q(x,y)\} \rightarrow r(x,y)]$

soln:- $\sim\{\forall x, \exists y, [\{p(x,y) \wedge q(x,y)\} \rightarrow r(x,y)]\}$

$\Rightarrow \exists x, \sim\{\exists y, [(p(x,y) \wedge q(x,y))] \rightarrow r(x,y)]\}$

$\Rightarrow \exists x, \forall y, \sim\{(p(x,y) \wedge q(x,y)) \rightarrow r(x,y)\}$

$\Rightarrow \exists x, \forall y, \sim[\sim(p(x,y) \wedge q(x,y)) \vee r(x,y)]$

$\Rightarrow \exists x, \forall y, [p(x,y) \wedge q(x,y)] \wedge \sim r(x,y)$ //

(ii) $\forall x, \forall y \left[(x < y) \rightarrow \exists z, (x < z < y) \right]$

soln:- $\sim \{ \forall x, \forall y \left[(x < y) \rightarrow \exists z, (x < z < y) \right] \}$

$\Rightarrow \exists x, \exists y \sim \left[(x < y) \rightarrow \exists z, (x < z < y) \right]$

demorgan's.

$\Rightarrow \exists x, \exists y, (x < y) \wedge \sim \exists z, (x < z, < y)$

$\Rightarrow \exists x, \exists y, (x < y) \wedge \forall z \; (x > z > y)$

(iii) $\forall x, \forall y \left[|x| = |y| \rightarrow (y = \pm x) \right]$

soln:- $\sim \{ \forall x, \forall y \left[|x| = |y| \rightarrow (y = \pm x) \right] \}$

$\Rightarrow \exists x, \exists y \sim \left[|x| = |y| \rightarrow (y = \pm x) \right]$

$\Rightarrow \exists x, \exists y \sim \left[\sim (|x| = |y|) \vee (y = \pm x) \right]$

$\Rightarrow \exists x, \exists y \; (|x| = |y|) \wedge (y \neq \pm x)$ //

(iv) $\left[\forall x, \forall y, ((x < 0) \wedge (y > 0)) \right] \rightarrow \left[\exists z, (xz > y) \right]$

soln:-

$\sim \{ \left[\forall x, \forall y, ((x < 0) \wedge (y > 0)) \right] \rightarrow \left[\exists z, (xz > y) \right] \}$

$\Rightarrow \sim \{ \sim (\forall x, \forall y, \left[(x < 0) \wedge (y > 0) \right]) \vee \exists z, (xz > y) \}$

$\Rightarrow \forall x \; \forall y, \left[(x < 0) \wedge (y > 0) \right] \wedge \sim \exists z, (xz > y)$

$\Rightarrow \forall x, \forall y, \left[(x < 0) \wedge (y > 0) \right] \wedge \forall z, (xz \leq y)$ //

Methods of proof and Disproof

The propositions that commonly appear in Mathematical discussions are conditionals of the form $p \to q$, where p & q are simple (or) compd proposition & which may involves quantifiers also.

The process of establishing that the conditional is true by using the rules (or) laws of logic and other known facts constitutes a proof of the conditional.

The process of establishing that a proposition is false constitutes a disproof.

(*) Direct proof

The direct proof of a conditional $p \to q$ has the following steps.

(1) Hypothesis : First assume that p is true

(2) Analysis : Starting with the hypothesis and employing the rules (or) laws of logic and other known facts, infer that q is true

(3) Conclusion : $p \to q$ is true.

problems

(1) Give a direct proof of the statement:

"The square of an odd integer is an odd integer".

Soln:- Is of the form,

"If n is an odd integer, then n^2 is an odd integer"

Now, p : n is an odd integer

q : n^2 is an odd integer.

Hypothesis:- Assume that n is an odd integer

i.e $n = 2K+1$ for some integer K

Analysis: consequently $n^2 = (2K+1)^2$

$n^2 = 4K^2 + 1 + 4K$, which is not divisible by 2.

$\therefore$ n^2 is an odd integer.

conclusion: $p \to q$ is true.

(2) P.T, for all integers K and l, if K & l are both odd, then $K+l$ is even and Kl is odd.

Soln:- p : K & l are both odd

q : $K+l$ is even and Kl is odd

Hypo: Assume that K & l are odd

i.e $K = 2m+1$, $l = 2n+1$, $m, n \in Z$

Analy : consequently,

$K+l = (2m+1) + (2n+1)$

$K+l = 2m + 2n + 2$, which is divisible by 2

$\therefore$ $K+l$ is even

$Kl = (2m+1)(2n+1)$

$= 4mn + 2m + 2n + 1$, which is not divisil by 2

$\therefore$ Kl is odd

conclu : $p \to q$ is true.

(*) Indirect proof

WKT $p \to q \Leftrightarrow \sim q \to \sim p$ [contrapositive statement]

First show that $\sim q \to \sim p$ is true by direct proof, This shows that $p \to q$ is true indirectly, This method of proving a conditional is called indirect proof.

problems

(1) Let n be an integer, P.T "if n^2 is odd then n is odd".

soln:-

p : n^2 is odd

q : n is odd

S.T : $\sim q \to \sim p$ is true

Hypothesis : Assume that $\sim q$ is true

i.e n is even

$n = 2K$.

Analysis : $n^2 = (2K)^2$

$\Rightarrow n^2 = 4K^2$, divisible by 2

$\therefore n^2$ is even

Hence $\sim p$ is true

conclusion: $\sim q \rightarrow \sim p$ is true

Hence $p \rightarrow q$ is true.

(2) " The product of two even integers is an even integer".

Soln: If m and n are even integers then mn is an even integer.

p : m & n are even integers

q : mn is an even integer.

S.T : $\sim q \rightarrow \sim p$ is true.

Hypo : Assume that $\sim q$ is true

i.e mn is an odd integer

Analy:- mn is not divisible by 2

$\Rightarrow m$ is not divisible by 2

n is not divisible by 2

$\therefore m$ is odd & n is odd $\Rightarrow m$ & n are odd integers

$\Rightarrow \sim p$ is true

Conclu:- $\sim q \rightarrow \sim p$ is true,

Hence $p \rightarrow q$ is true

(3) For all integers k and l, if $k+l$ is even then k & l are both even (or) both odd

Soln:- p: $k+l$ is even

q: k & l are both even (or) both odd.

S.T: $\sim q \rightarrow \sim p$ is true

Hypoth: Assume that $\sim q$ is true

i.e k & l are both not even (or) both not odd (that means one is even & other one is odd)

i.e $k = 2m$, $l = 2n+1$

Analysis:- $k+l = 2m+2n+1$, is not divisible by 2

$\therefore k+l$ is odd

$\Rightarrow \sim p$ is true

Conclu: $\sim q \rightarrow \sim p$ is true

Hence $p \rightarrow q$ is true.

(4) provide an indirect proof of the following statement. "For all positive real numbers x & y If the product xy exceeds 25 then $x > 5$ or $y > 5$".

Soln:- $p : xy > 25$ S.T : $\sim q \rightarrow \sim p$ is true

$q : x > 5$ or $y > 5$

Hypo : Assume $\sim q$ is true

i.e $x \leq 5$ and $y \leq 5$

Analysis: $xy \leq 5 \times 5$

$xy \leq 25$

$\Rightarrow \sim p$ is true

conclusion: $\sim q \rightarrow \sim p$ is true

Hence $p \rightarrow q$ is true.

(*) proof by contradiction

Hypothesis : Assume that $p \rightarrow q$ is false

i.e p is true and q is false.

Analysis : Take q is false, use any rules or Laws (or) known facts to S.T p is false.

Hence our assumption is wrong.

conclusion : Hence $p \rightarrow q$ is true.

problems :

(1) provide a proof by contradiction of the following statement.

"For every integer n, if n^2 is odd, then n is odd".

<u>Sol:</u>- p : n^2 is odd

q : n is odd

S.T : $p \to q$ is true by contradiction

<u>Hypothesis</u> : Assume that $p \to q$ is false

i.e p is true and q is false

<u>Analysis</u> : take q is false

i.e n is even

$n = 2K$

consequently, $n^2 = 4K^2$, divisible by 2

$\Rightarrow n^2$ is even

Hence p is false

Hence our assumption is wrong

<u>conclu</u> : $p \to q$ is true.

(2) "If n^2 is an even integer then 'n' is an even integer".

<u>Soln</u>:- p : n^2 is an even integer

q : n is an even integer.

S.T : $p \to q$ is true by contradiction

Hypothesis : Assume that $p \to q$ is false

i.e p is true and q is false

Analysis: take q is false, means n is an odd integer

i.e $n = 2K+1$

consequently, $n^2 = (2K+1)^2$

$n^2 = 4K^2 + 1 + 4K$, not divisible by 2

$\therefore$ n^2 is an odd integer

$\Rightarrow$ p is false

Hence our assumption is wrong

conclu:- $p \rightarrow q$ is true.

(3) P.T for all real numbers x and y,
" if $x+y \geq 100$, then $x \geq 50$ or $y \geq 50$

soln:- $p: x+y \geq 100$

$q: x \geq 50$ or $y \geq 50$

Hypo: Assume that $p \rightarrow q$ is false

i.e p is true and q is false.

Analysis:- take q is false, means $x < 50$ and $y < 50$

consequently, $x+y < 50+50$

$x+y < 100$

$\Rightarrow$ p is true, hence our assumption is wrong.

conclusion:- $p \rightarrow q$ is true.

(4) P.T, "if 'm' is an even integer then 'm+7' is an odd integer".

soln:- p : m is an even integer

q : m+7 is an odd integer

Hypo: Assume that $p \to q$ is false

i.e p is true and q is false.

Analy: take q is false

m+7 is an even integer

$\Rightarrow m+7 = 2k$

$m = 2k+7$, which is not divisible by 2

$\therefore$ m is an odd integer

$\Rightarrow$ p is false, our assumption is wrong

conclusion: $p \to q$ is false.

(5) Give a (i) direct proof (ii) indirect proof (iii) proof by contradiction, for the following statement "If n is an odd integer, then n+9 is an even integer".

soln:- p : n is an odd integer

q : n+9 is an even integer

(i) direct proof :

S.T $p \to q$ is true

Hypothesis: Assume that p is true, i.e n is odd

$n = 2k + 1$

Analysis: $n + 9 = 2k + 1 + 9$

$n + 9 = 2k + 10$, divisible by 2

$\therefore$ $n + 9$ is even integer

$\therefore$ q is true

conclu :- $p \rightarrow q$ is true.

(ii) indirect proof :

S.T ; $\sim q \rightarrow \sim p$ is true

Hypo : Assume that $\sim q$ is true

i.e $n + 9$ is an odd integer

Analysis : $n + 9 = 2k + 1$

$n = 2k + 1 - 9 = 2k - 8$, is divisible by 2.

$\therefore$ n is even integer

i.e $\sim p$ is true

conclu :- $\sim q \rightarrow \sim p$ is true

Hence $p \rightarrow q$ is true

(iii) proof by contradiction :

Hypothesis : Assume $p \rightarrow q$ is false

i.e p is true and q is false

Analysis : Take q is false

$n + 9$ is an odd integers

$\therefore n+9 = 2k+1$

$n = 2k+1-9$

$n = 2k-8$, is divisible by 2

$\therefore n$ is an even integers

i.e p is false

$\therefore$ Hence our assumption is wrong.

conclu:- $p \rightarrow q$ is true.

(*) Disproof by contradiction

Hypothesis: Assume that $p \rightarrow q$ is true

i.e p is true and q is true

Analysis: Take p is true by using rules, laws or any other known facts to S.T q is false.

Hence our assumption is wrong.

conclusion: $p \rightarrow q$ is false.

Disproof the Statement:

(1) "If m is an even integer then m+7 is an even integer".

soln:- p: m is an even integer

q: m+7 is an even integer.

Hypothesis: Assume that $p \rightarrow q$ is true

i.e p is true and q is true

Analysis: Take p is true .i.e m is an even integer

$m = 2K$

consq, $m+7 = 2K+7$, is not divisible by 2

$\therefore$ $m+7$ is an odd integers

$\therefore$ q is false

Hence our assumption is wrong

conclus: $p \to q$ is false.

(*) Disproof by counter example:

Disprove the proposition.

(1) The product of any two odd integers is a perfect square.

soln:- take $m = 3$ and $n = 5$

$mn = (3)(5) = 15$

is not a perfect square

Hence the proposition is disproved.