CSE 202: Design and Analysis of Algorithms

Lecture 7

Instructor: Kamalika Chaudhuri

Announcements

- HW2 is up! Due **Mon Apr 25** in class
- Remember: Midterm on **Wed May 4**
- Midterm is **closed book**
- **Syllabus:** Greedy, Divide and Conquer, Dynamic Programming, Flows (upto Ford-Fulkerson)

Last class: Three steps of Dynamic Programming

Main Steps:

1. Divide the problem into **subtasks**

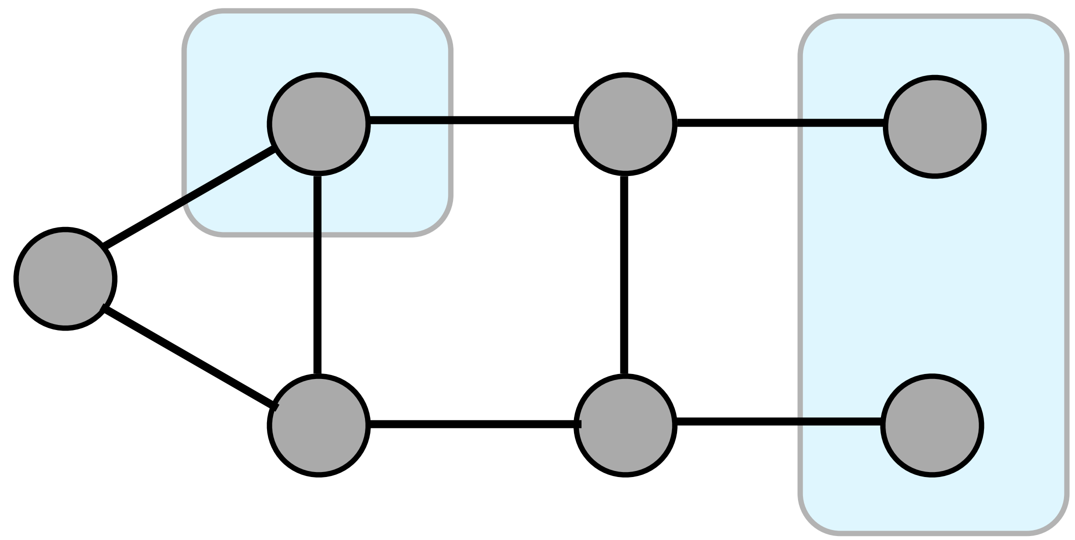
2. Define the subtasks **recursively** (express larger subtasks in terms of smaller ones)

3. Find the **right order** for solving the subtasks (but do not solve them recursively!)

Last Class: Dynamic Programming

- String Reconstruction
- Longest Common Subsequence
- Edit Distance
- Subset Sum
- Independent Set in a Tree

Independent Set



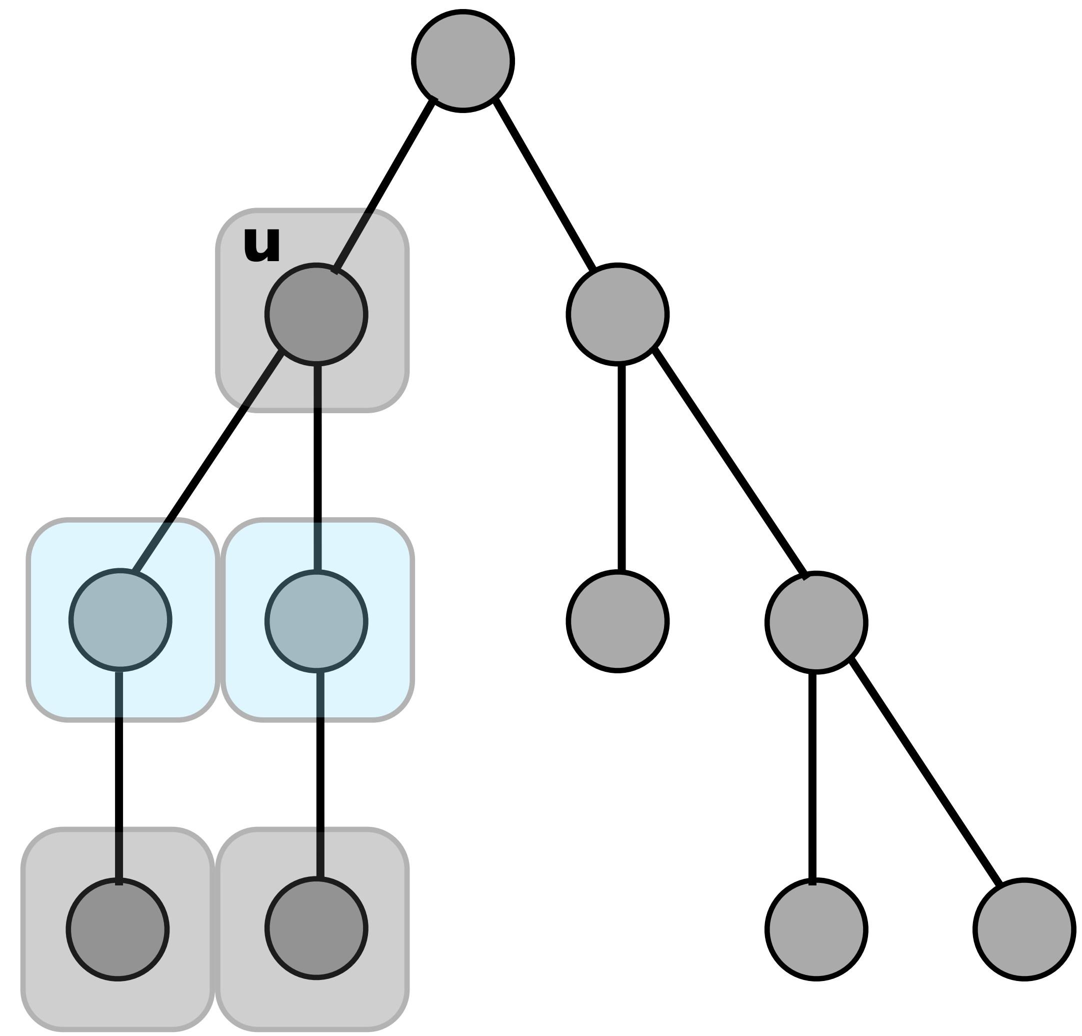
Independent Set: Given a graph G = (V, E), a subset of vertices S is an independent set if there are no edges between them

Max Independent Set Problem: Given a graph G = (V, E), find the largest independent set in G

Max Independent Set is a notoriously hard problem!
We will look at a restricted case, when G is a **tree**

Max. Independent Set in a Tree

A set of nodes is an **independent set** if there are no edges between the nodes



Two Cases at node u:

1. Don’t include u
2. Include u, and don’t include its children

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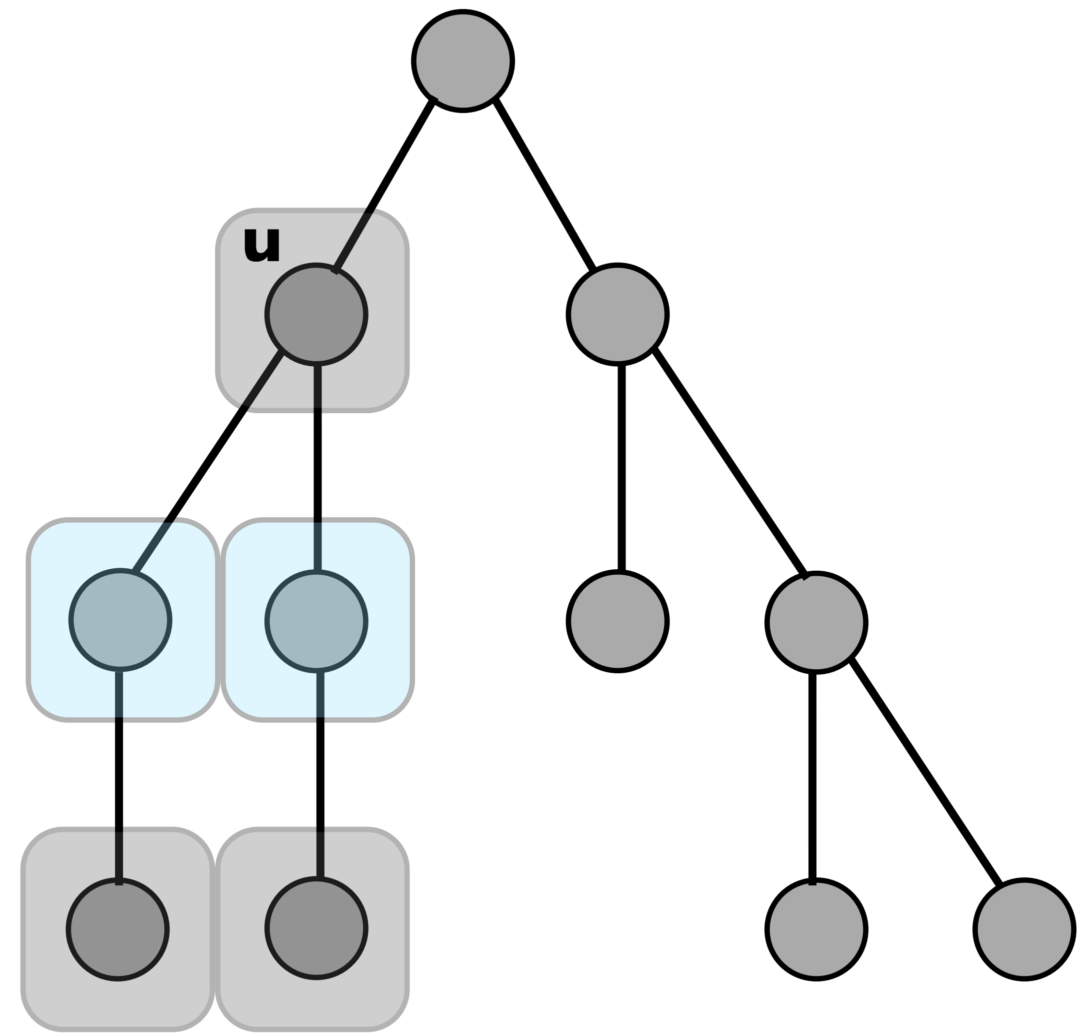
STEP 1: Define subtask

I(u) = size of largest independent set in subtree rooted at u

We want I(r), where r = root



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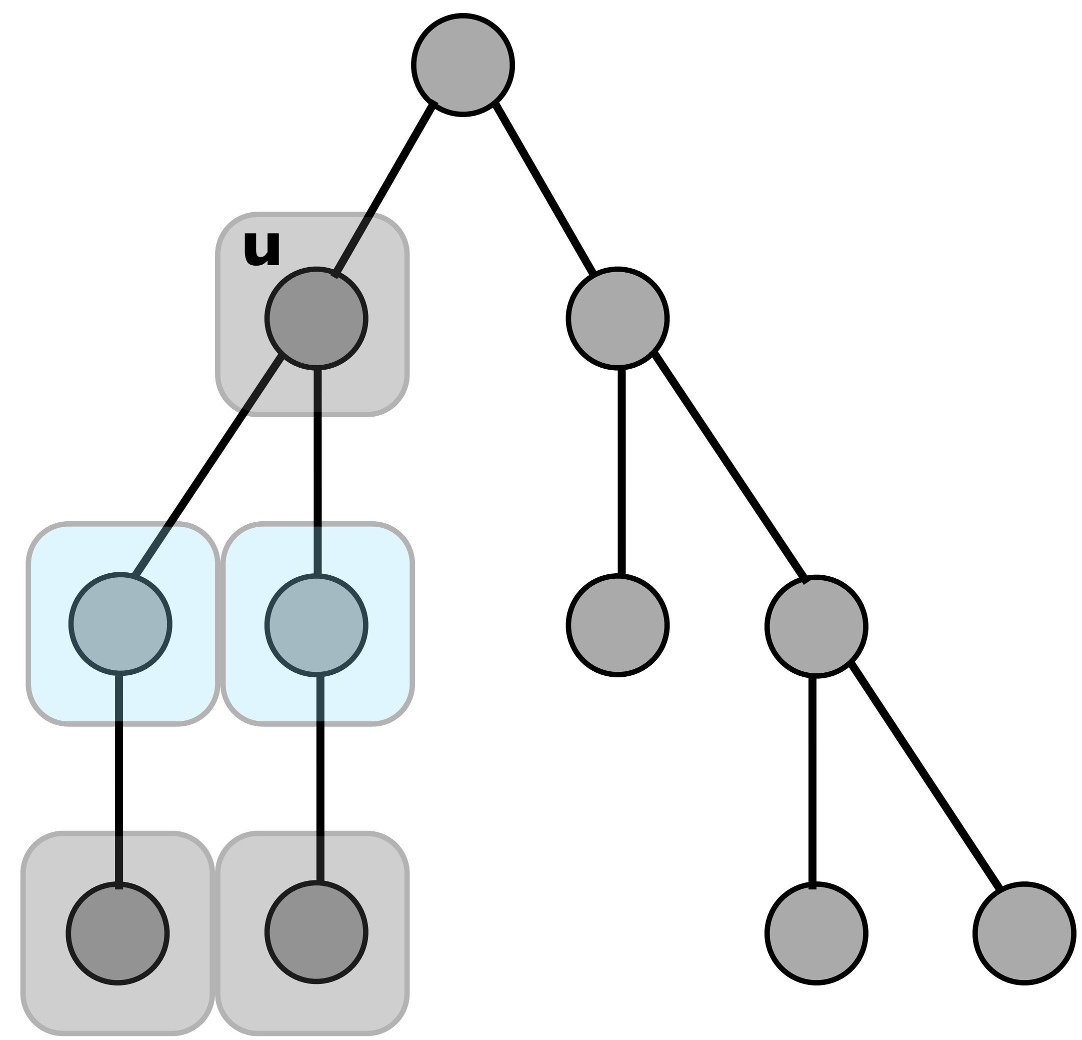

STEP 2: Express recursively

$$I(u) = \max \begin{cases} \sum_{\substack{\text{children} \\ w \text{ of } u}} I(w) \\ 1 + \sum_{\substack{\text{grandchildren} \\ w \text{ of } u}} I(w) \end{cases}$$

Base case: for leaf nodes, I(u) = 1

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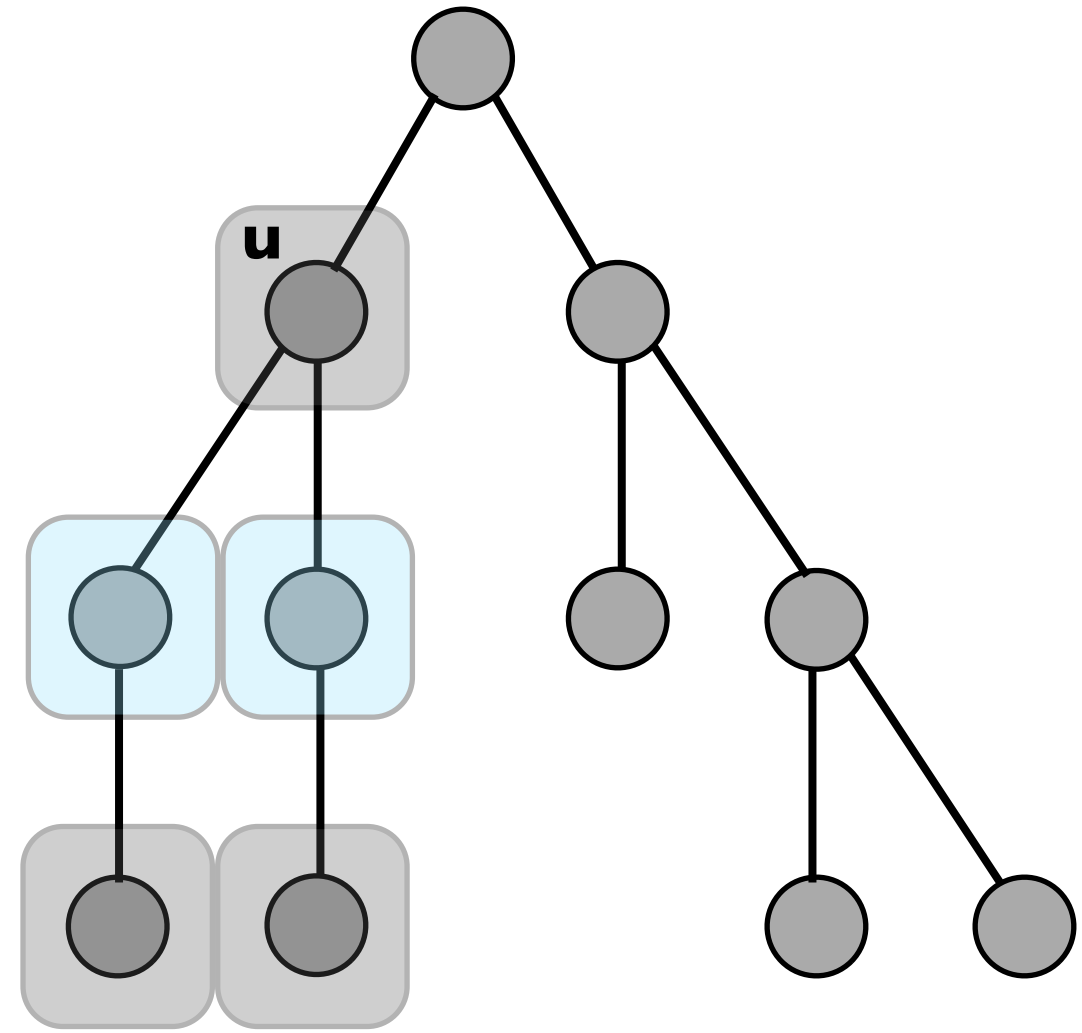
STEP 3: Order of subtasks

Reverse order of distance from root; use BFS!



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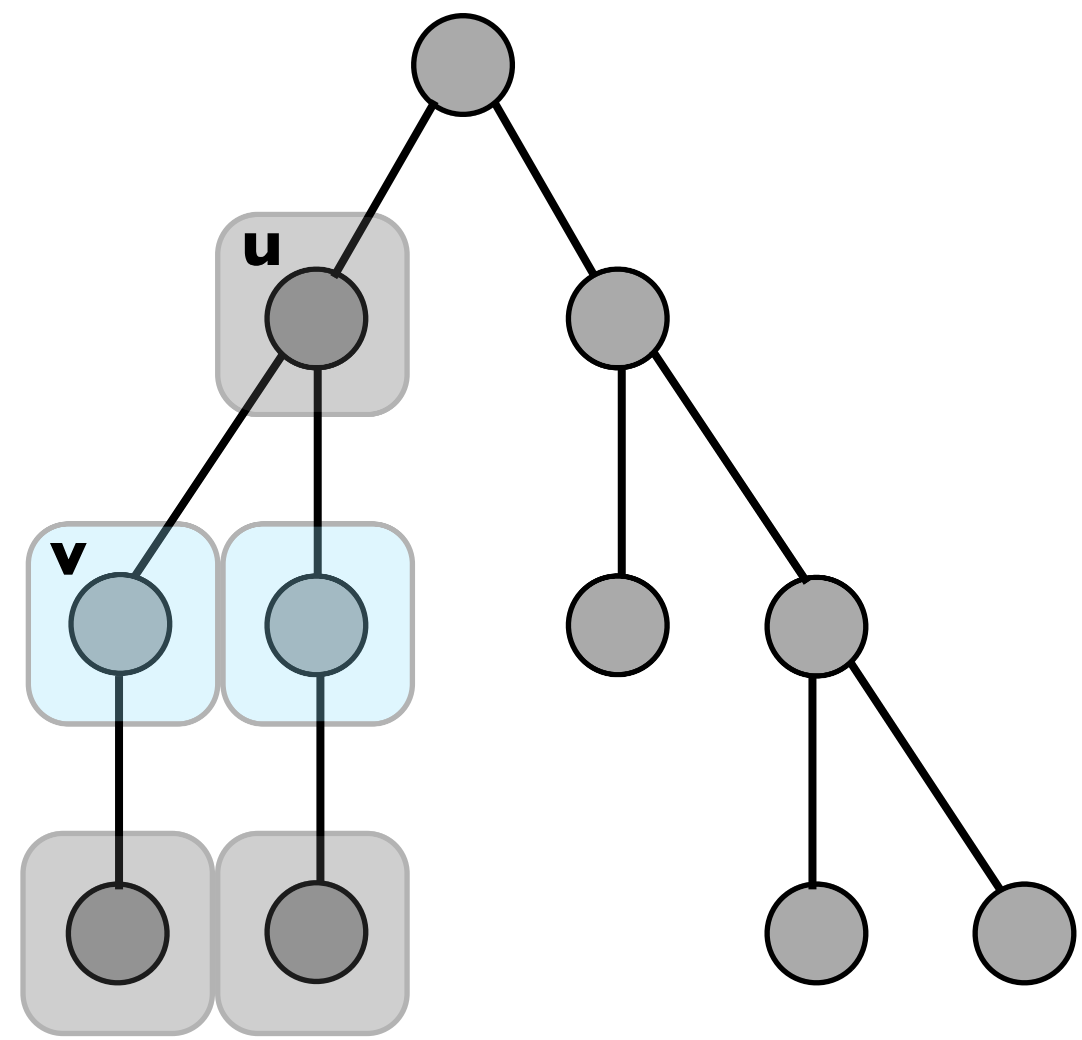


Running Time: O(n)

Edge (u, v) is examined in Step 2 at most twice:

(1) v is a child of u

(2) v is a grandchild of u's parent

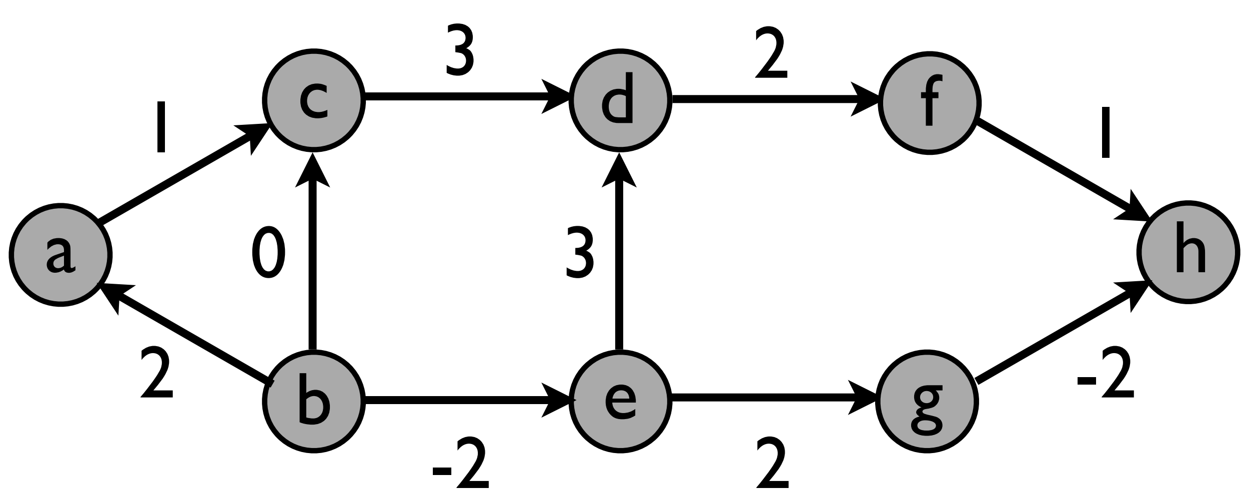
There are n-1 edges in a tree on n nodes

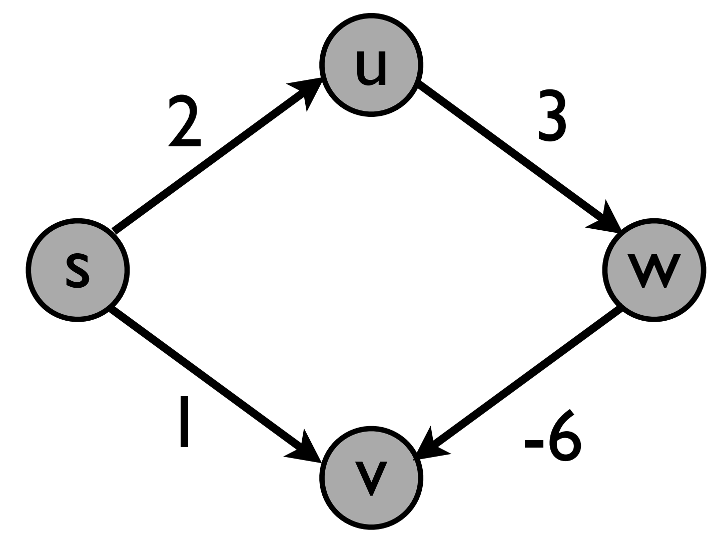
Dynamic Programming

- String Reconstruction
- Longest Common Subsequence
- Edit Distance
- Subset Sum
- Independent Set in a Tree
- All Pairs Shortest Paths

All Pairs Shortest Paths

Problem: Given n nodes and distances d_{ij} (which could be negative, or 0, or positive) on all edges, find shortest path distances between all pairs of nodes.





Does Dijkstra's algorithm work?

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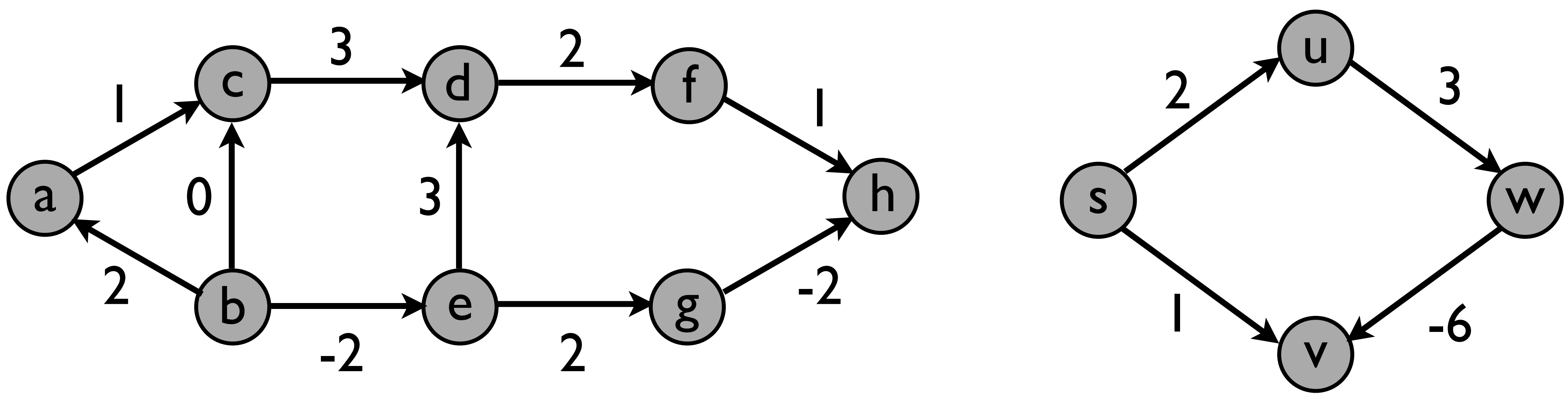


Does Dijkstra's algorithm work?

Ans: No! Example: s-v Shortest Paths

All Pairs Shortest Paths (APSP)

Problem: Given n nodes and distances d_{ij} (which could be negative, or 0, or positive) on all edges, find shortest path distances between all pairs of nodes.

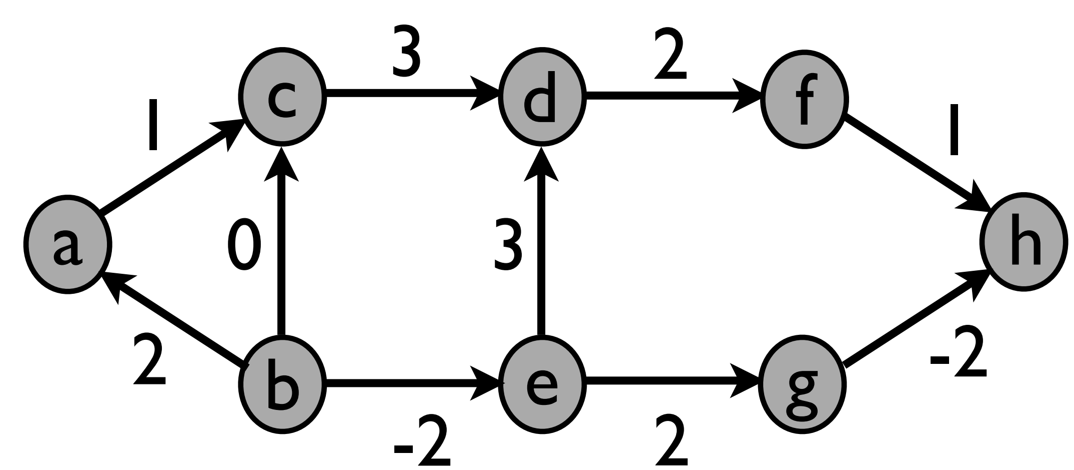
Structure:

For all x, y:

either $SP(x, y) = d_{xy}$

Or there exists some z s.t

$SP(x, y) = SP(x, z) + SP(y, z)$



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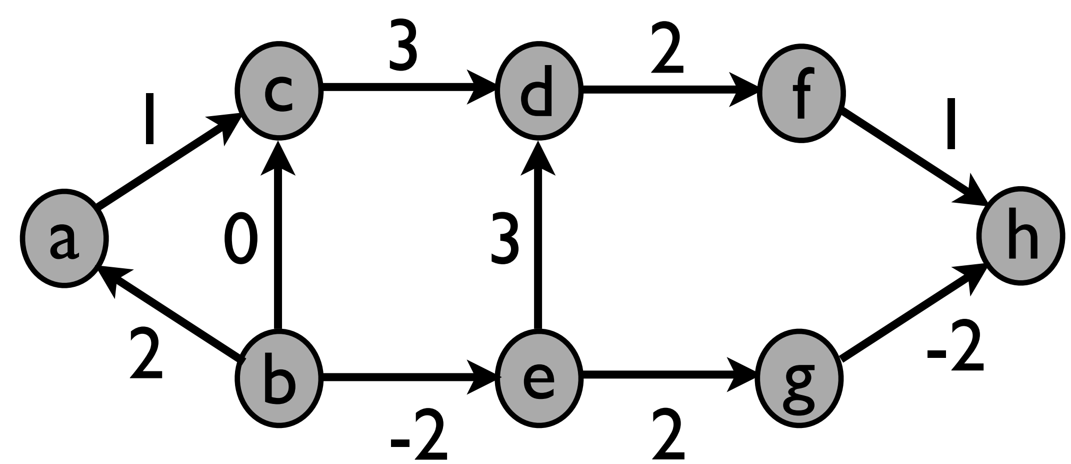
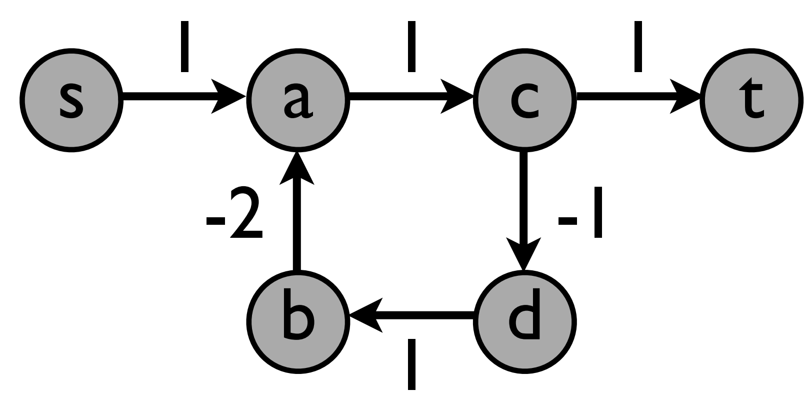


Property: If there is no negative weight cycle, then for all x, y, SP(x, y) is simple (that is, includes no cycles)



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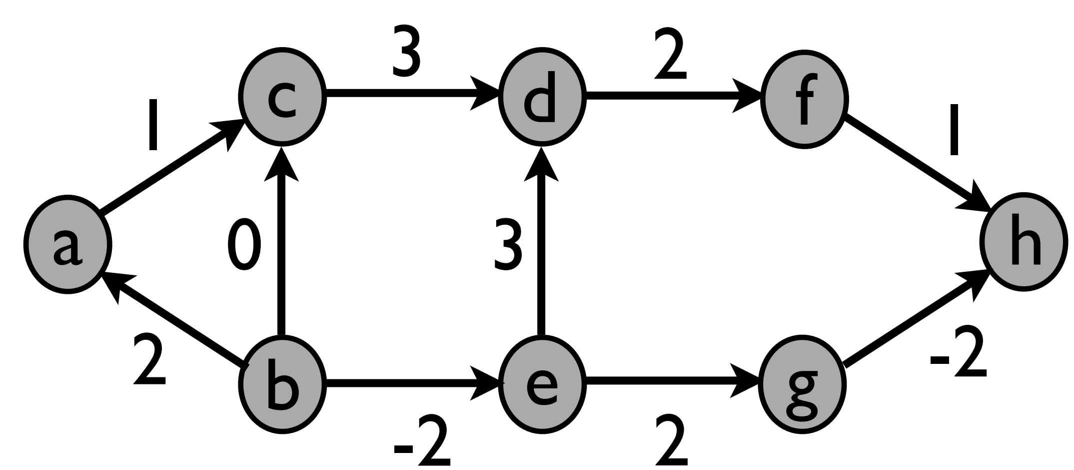

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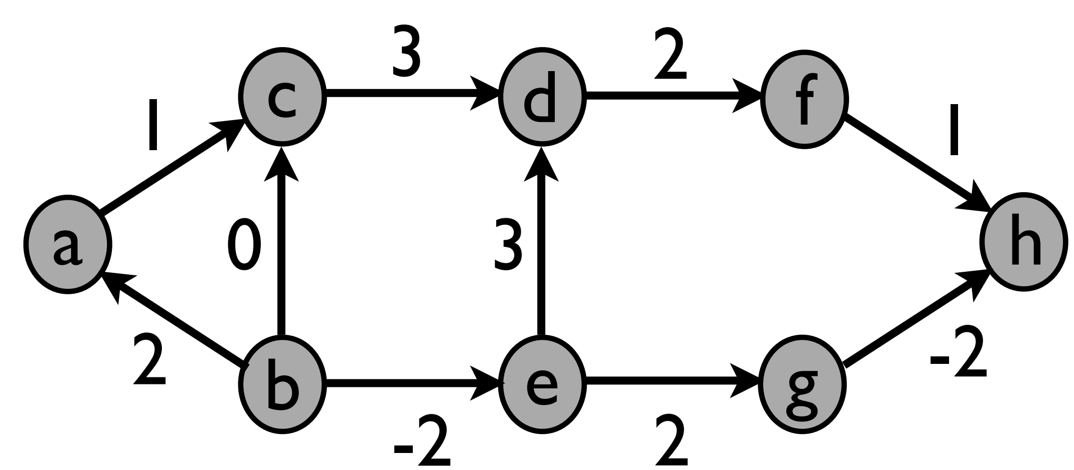

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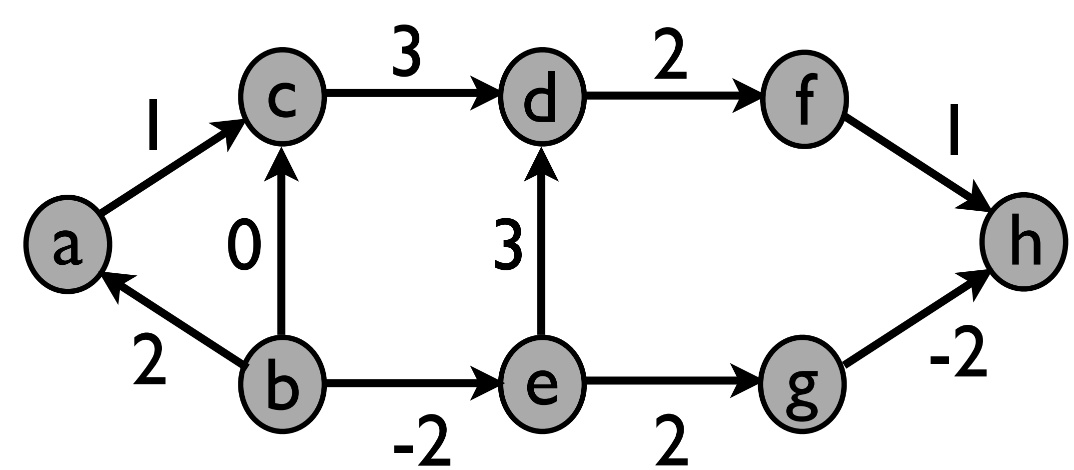

STEP 2: Express Recursively

$D(i,j,k) = \min\{D(i,j,k-1), D(i,k,k-1) + D(k,j,k-1)\}$

Base case: $D(i,j,0) = d_{ij}$

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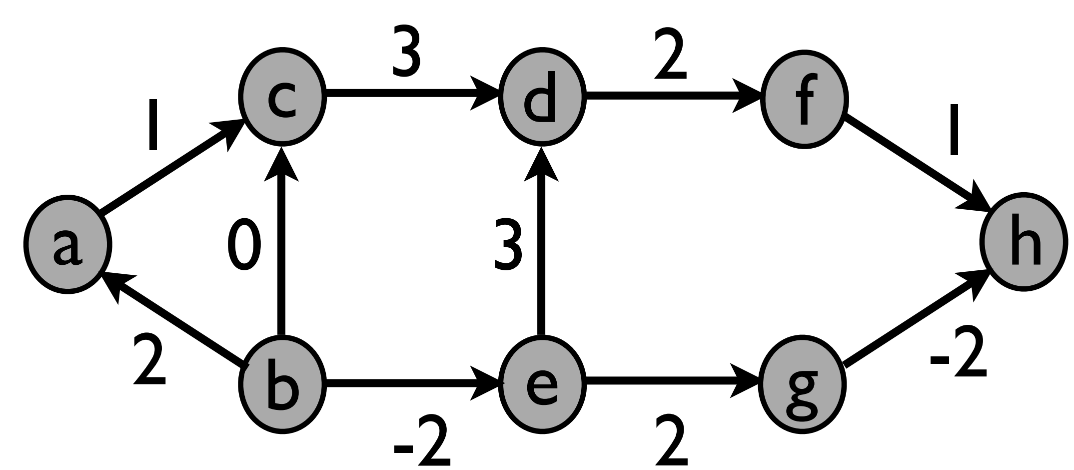

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By increasing order of k

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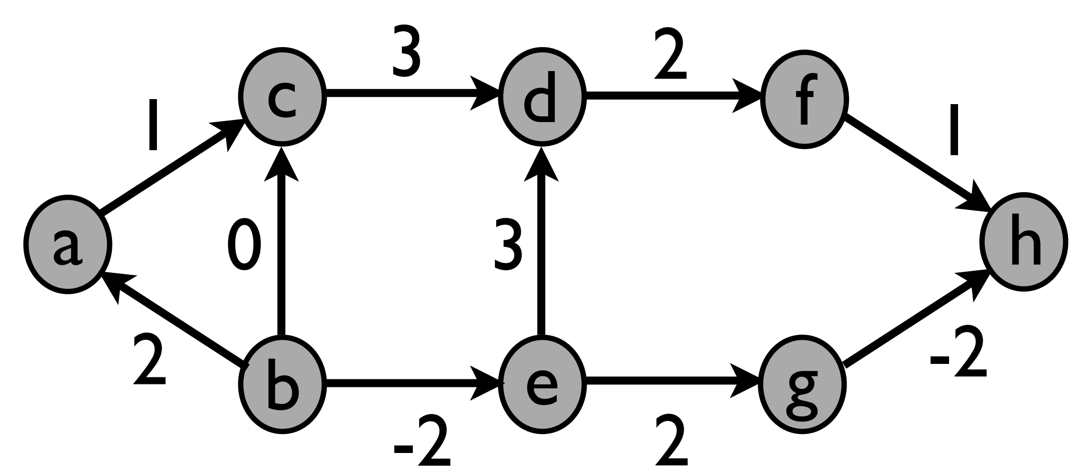

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STEP 3: Order of Subtasks

By increasing order of k

Running Time = $O(n^3)$

Exercise:

Reconstruct the shortest paths

Summary: Dynamic Programming

Main Steps:

1. Divide the problem into **subtasks**

2. Define the subtasks **recursively** (express larger subtasks in terms of smaller ones)

3. Find the **right order** for solving the subtasks (but do not solve them recursively!)

Summary: Dynamic Programming vs Divide and Conquer

Divide-and-conquer

A problem of size n is decomposed into a few subproblems which are significantly smaller (e.g. n/2, 3n/4,...)

Therefore, size of subproblems decreases geometrically.

eg. n, n/2, n/4, n/8, etc

Use a recursive algorithm.

Dynamic programming

A problem of size n is expressed in terms of subproblems that are not much smaller (e.g. n-1, n-2,...)

A recursive algorithm would take exp. time.

Saving grace: in total, there are only polynomially many subproblems.

Avoid recursion and instead solve the subproblems one-by-one, saving the answers in a table, in a clever explicit order.

Summary: Common Subtasks in DP

Case 1: Input: $x_1, x_2,..,x_n$ Subproblem: $x_1, .., x_i$.

$$\boxed{x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6} \quad x_7 \quad x_8 \quad x_9 \quad x_{10}$$

Case 2: Input: $x_1, x_2,..,x_n$ and $y_1, y_2,..,y_m$ Subproblem: $x_1, .., x_i$ and $y_1, y_2,..,y_j$

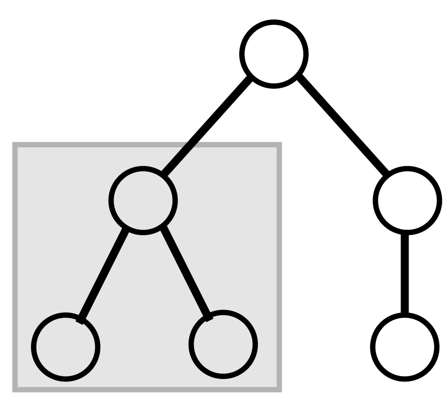
$$\boxed{x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6} \quad x_7 \quad x_8 \quad x_9 \quad x_{10}$$

$$\boxed{y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5} \quad y_6 \quad y_7 \quad y_8$$

Case 3: Input: $x_1, x_2,..,x_n$. Subproblem: $x_i, .., x_j$

$$x_1 \quad x_2 \quad \boxed{x_3 \quad x_4 \quad x_5 \quad x_6} \quad x_7 \quad x_8 \quad x_9 \quad x_{10}$$

Case 4: Input: a rooted tree. Subproblem: a subtree



Next: Network Flow

Oil Through Pipelines

Problem: Given directed graph G=(V,E), source s, sink t, edge capacities c(e), how much oil can we ship from s to t?

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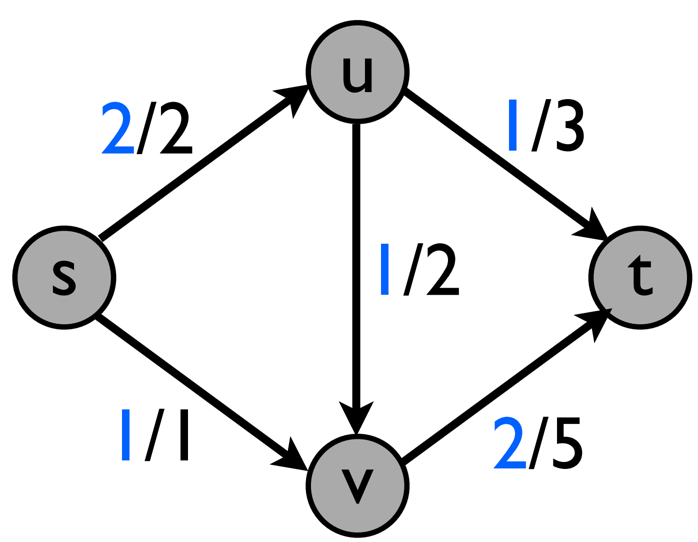
An s-t flow is a function: E $\longrightarrow$ R such that:

- 0 <= f(e) <= c(e), for all edges e
- flow into node v = flow out of node v, for all nodes v except s and t,

$$\sum_{e\ into\ v} f(e) = \sum_{e\ out\ of\ v} f(e)$$

Size of flow f = Total flow out of s = total flow into t



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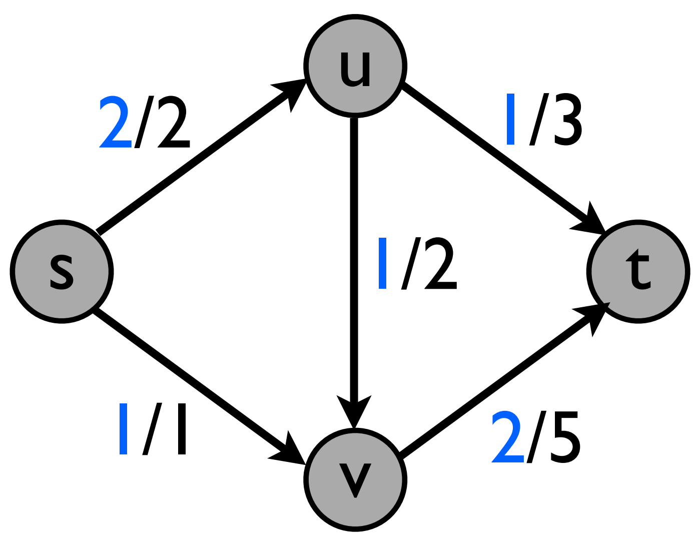

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The Max Flow Problem: Given directed graph G=(V,E), source s, sink t, edge capacities c(e), find an s-t flow of maximum size

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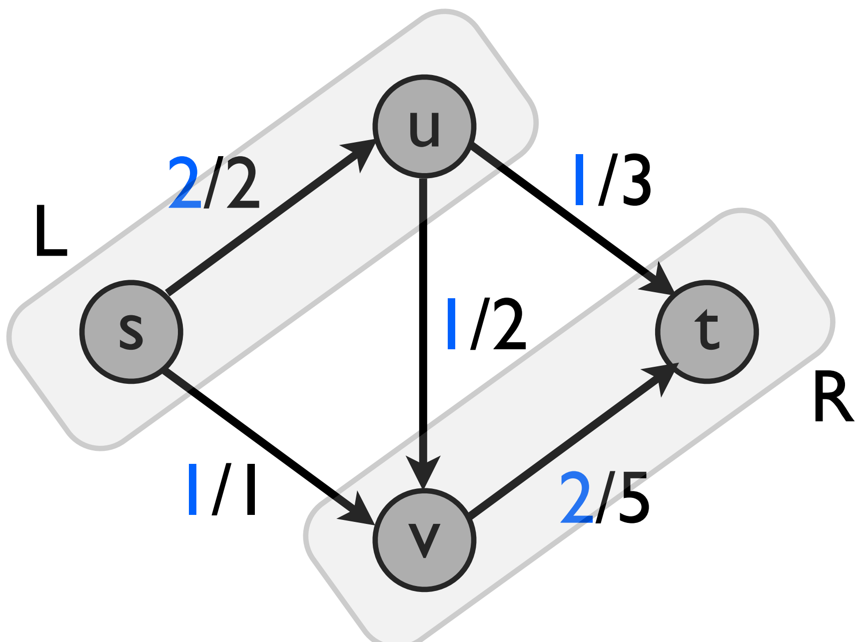

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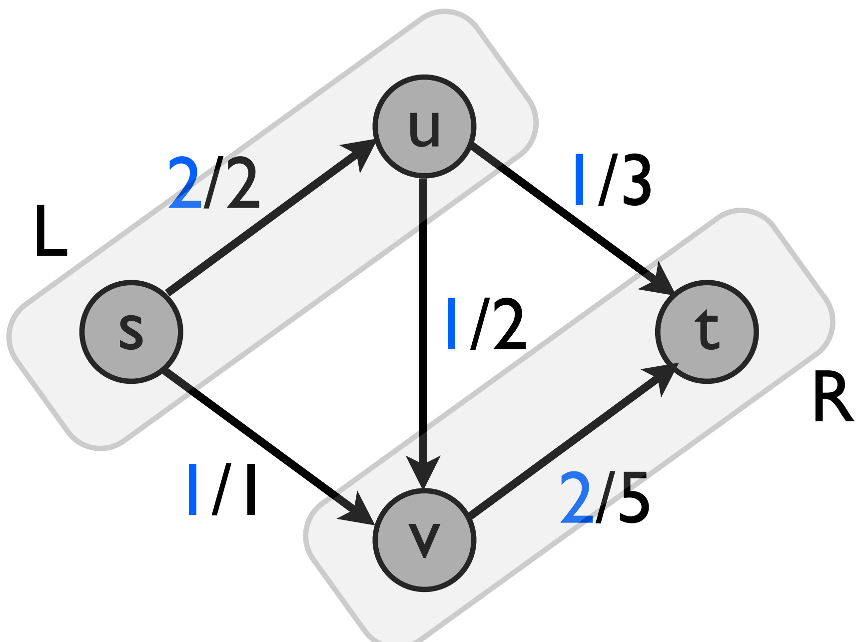

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Flow across (L,R) = $\sum_{(u,v)\in E, u\in L, v\in R} f(u,v) - \sum_{(v,u)\in E, u\in L, v\in R} f(v,u)$

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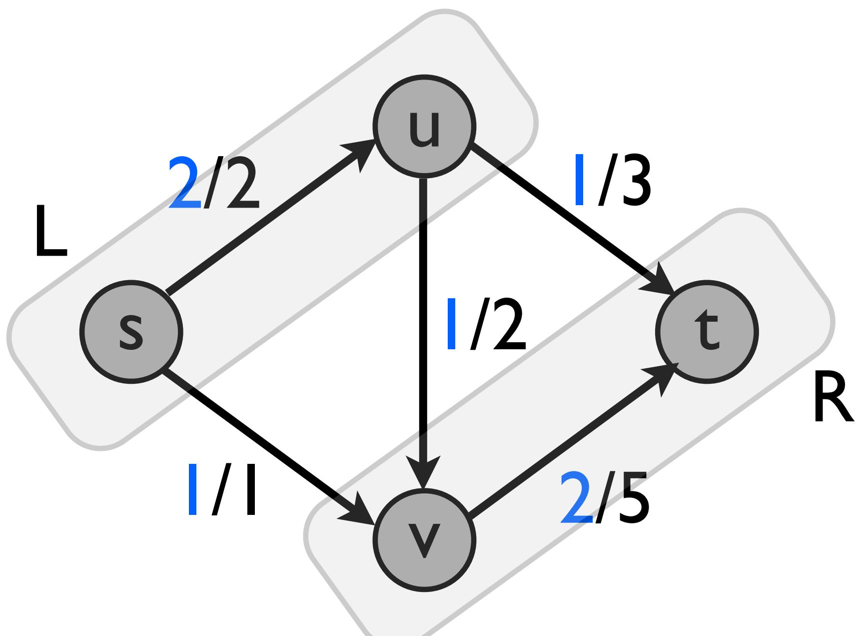

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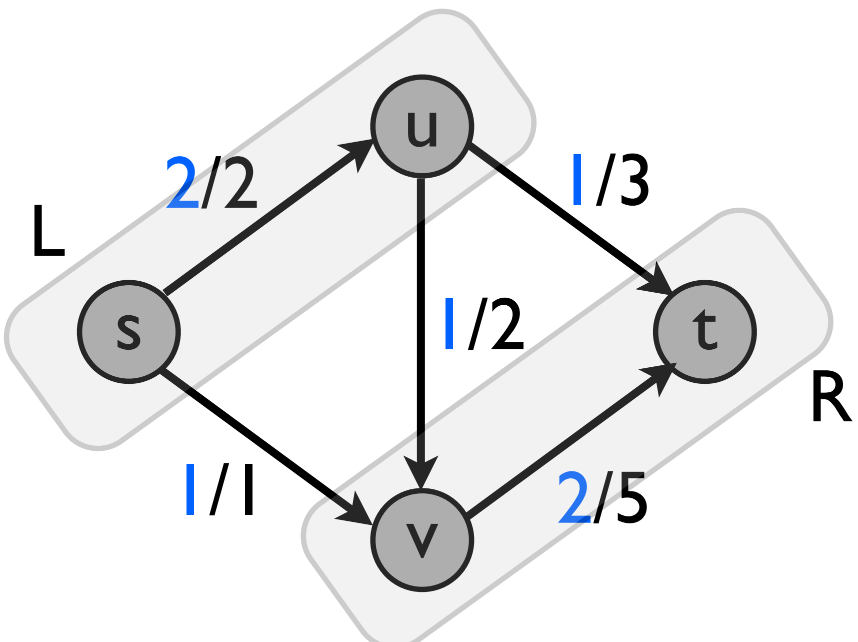

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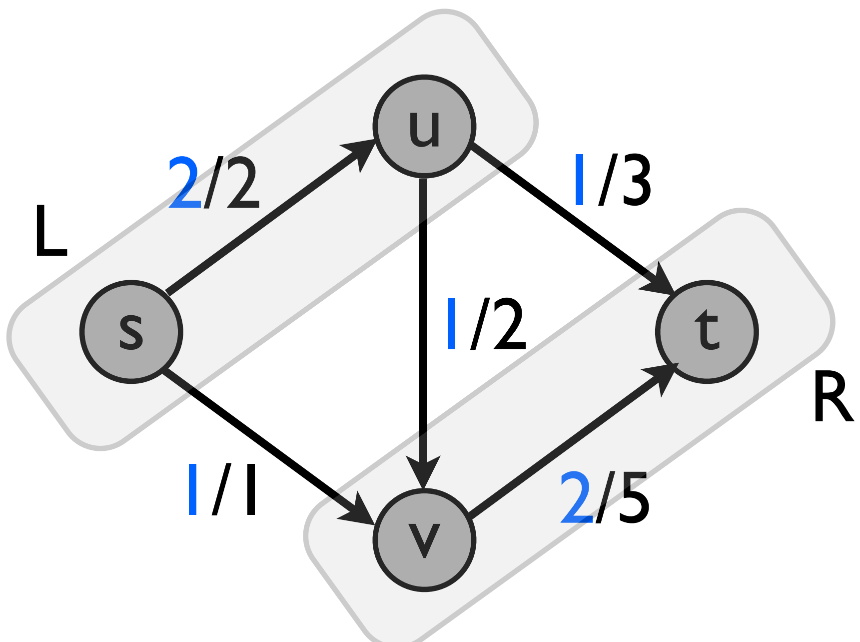

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Max-Flow <= Min-Cut

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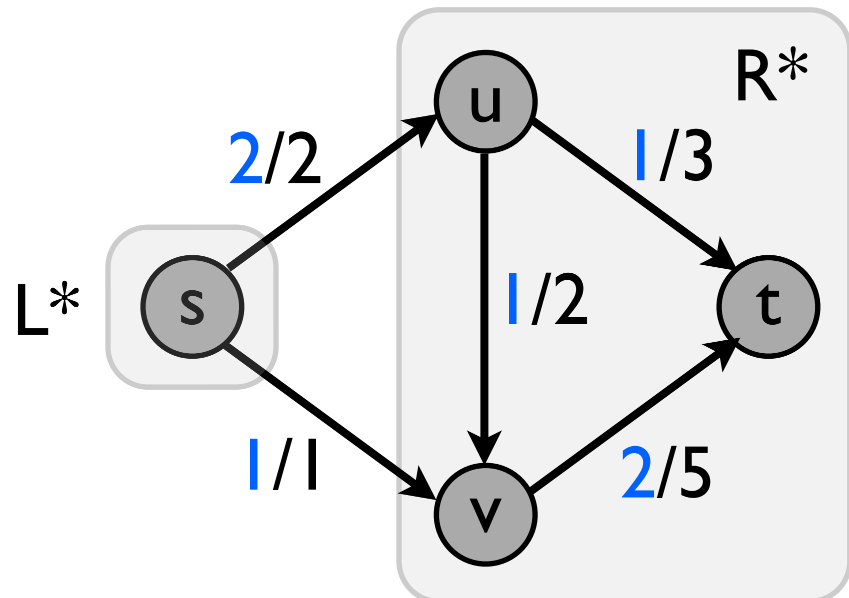

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Cuts

Flows

In our example: Size of f = 3, Capacity of Cut (s,V - s) = 3

Flows and Cuts

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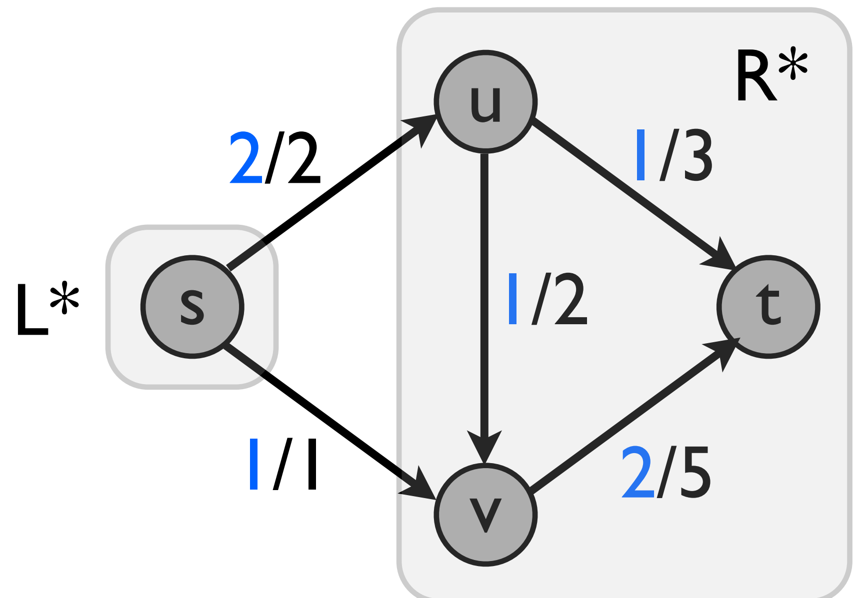

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Cuts

Flows

In our example: Size of f = 3, Capacity of Cut (s,V - s) = 3.
Thus, a Min Cut is a **certificate of optimality for a flow**

Ford-Fulkerson algorithm

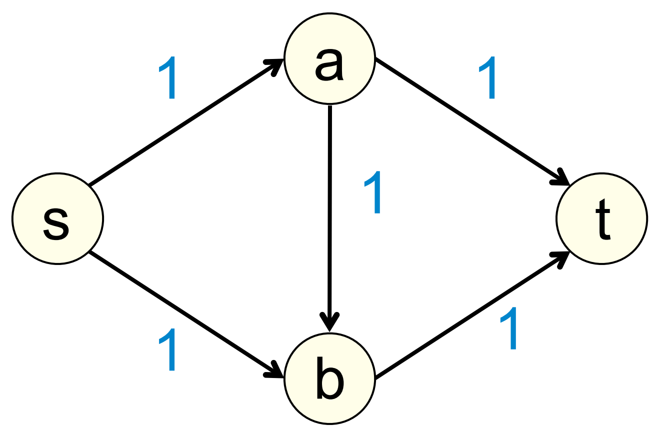
FF Algorithm: Start with zero flow

Repeat:

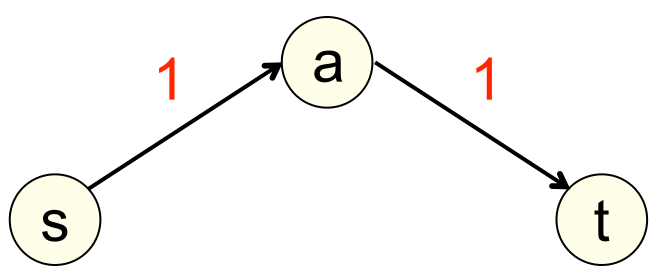
Find a path from s to t along which flow can be increased

Increase the flow along that path

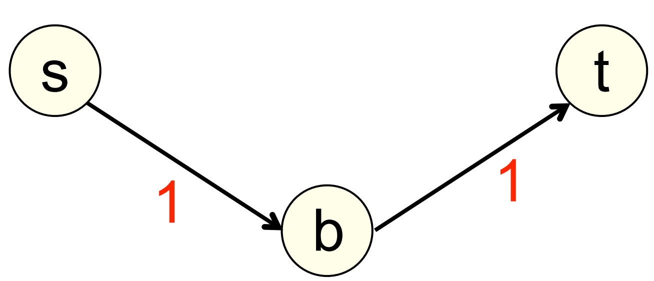
Example



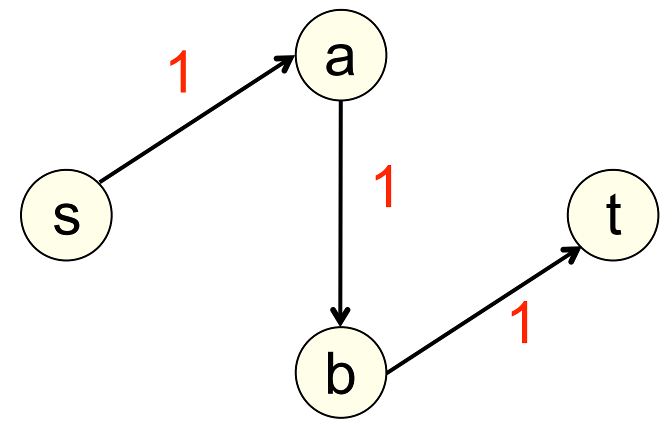
First choose:



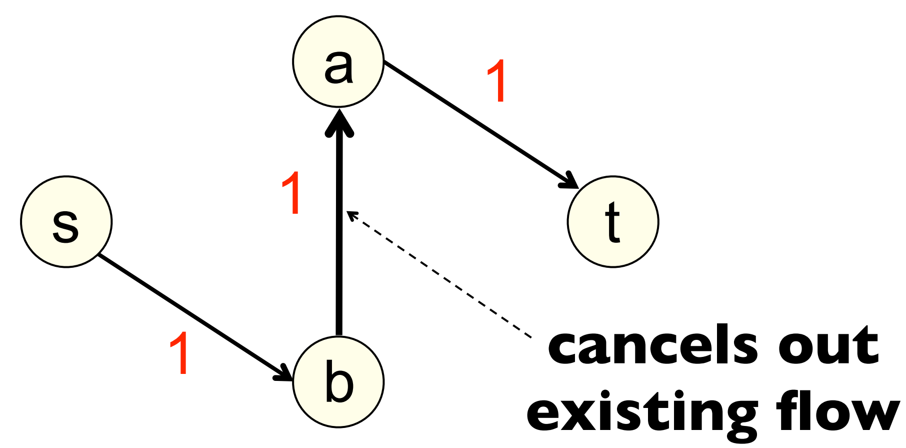
Next choose:



But what if we first chose:



Then we'd have to allow:



Ford-Fulkerson, continued

FF Algorithm: Start with zero flow

Repeat:

Find a path from s to t along which flow can be increased

Increase the flow along that path

In any iteration, we have some flow f and we are trying to improve it. How to do this?

1: Construct a residual graph G_f ("what's left to take?")

$G_f = (V, E_f)$ where $E_f \subseteq E \cup E^R$

For any (u,v) in E or E^R,

$c_f(u,v) = c(u,v) - f(u,v) + f(v,u)$

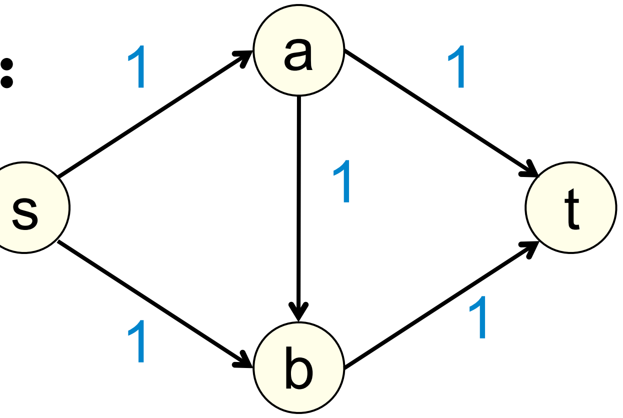
[ignore edges with zero c_f: don't put them in E_f]

2: Find a path from s to t in G_f

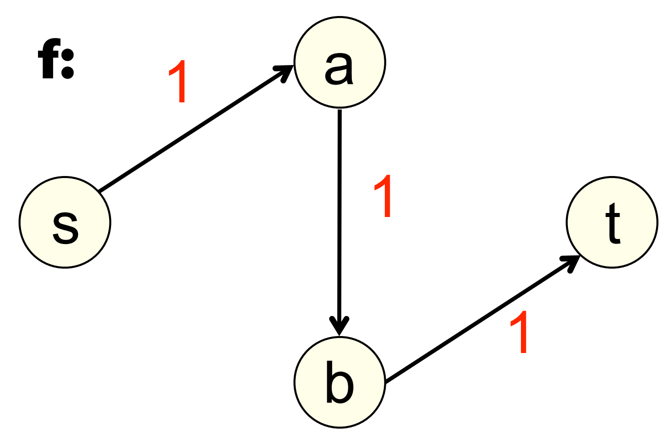
3: Increase flow along this path, as much as possible

Example

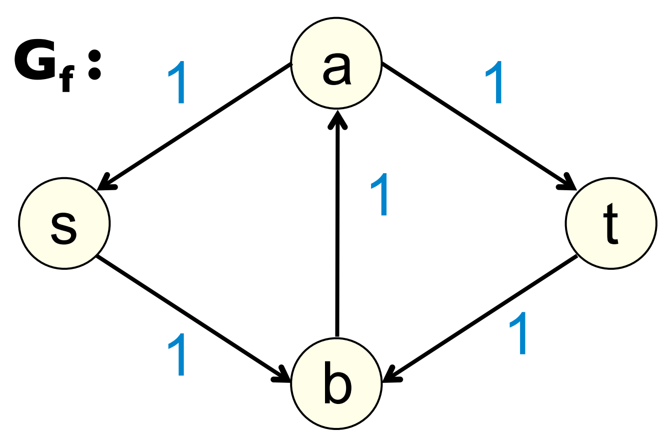
G:



f:



G_f:



Example: Round 1

Construct residual graph $G_f = (V, E_f)$

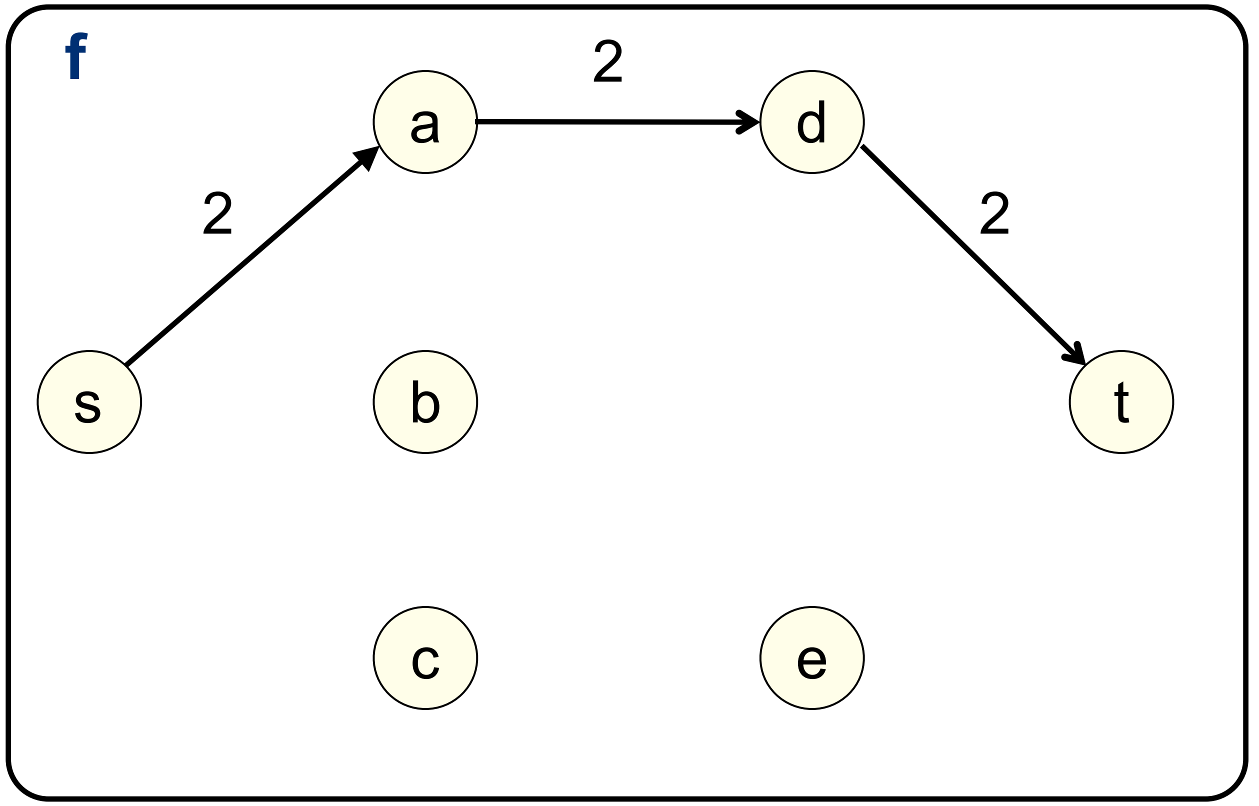
$E_f \subseteq E \cup E^R$

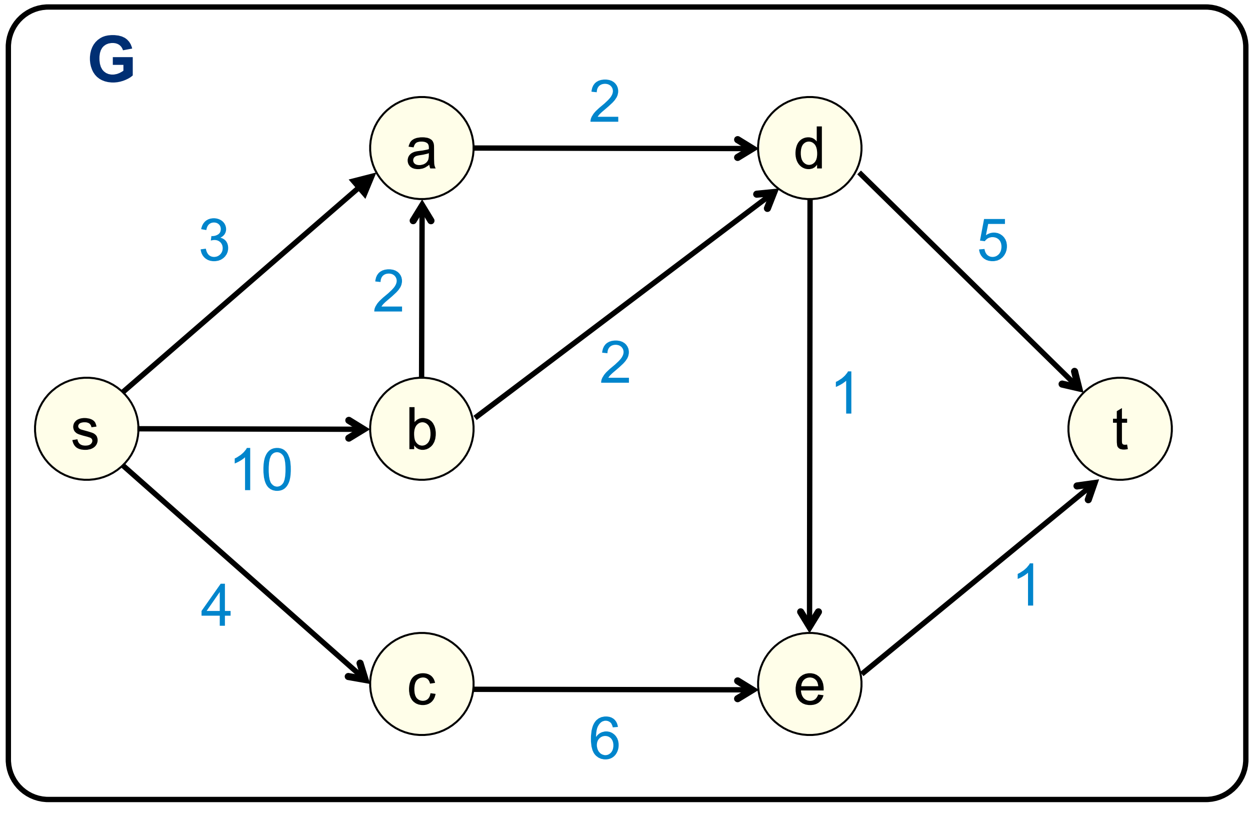
For any (u,v) in E or E^R,

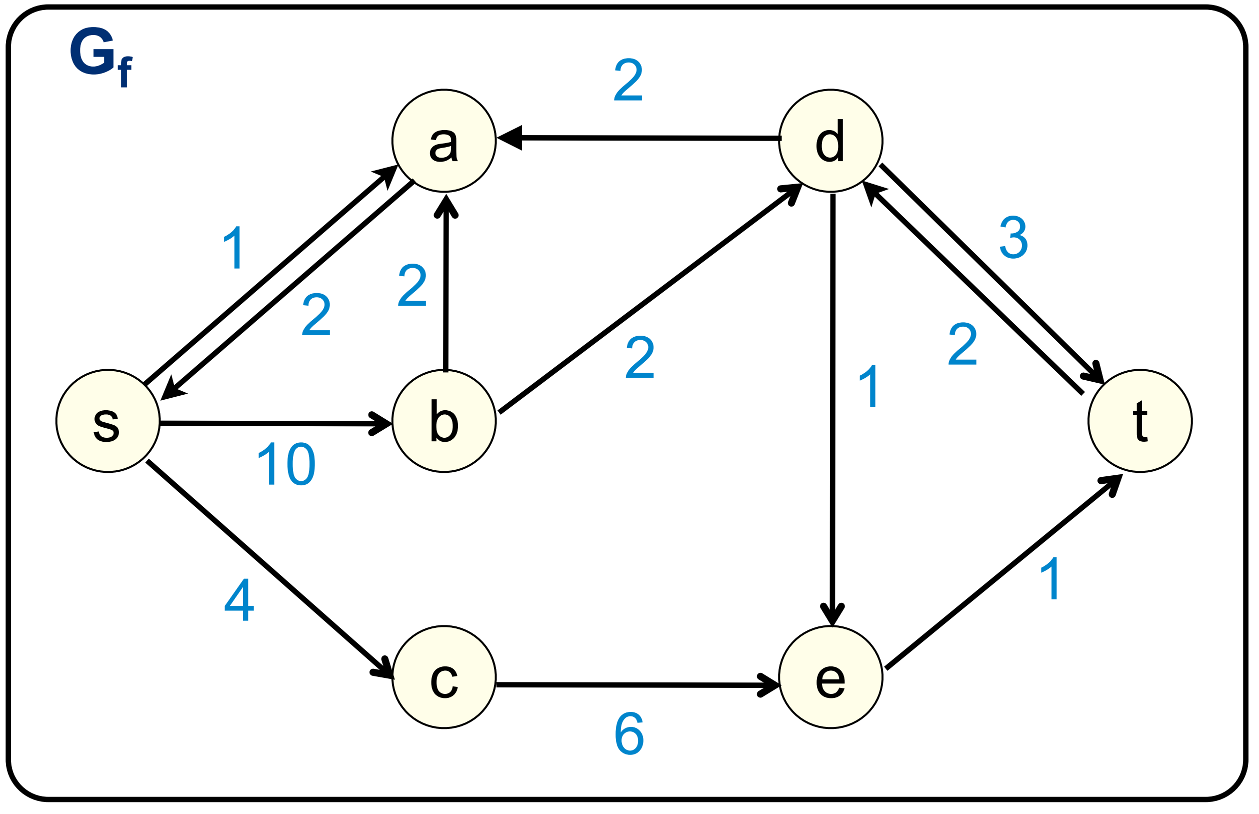
$c_f(u,v) = c(u,v) - f(u,v) + f(v,u)$

Find a path from s to t in G_f

Augment f along this path







Example: Round 2

Construct residual graph $G_f = (V, E_f)$

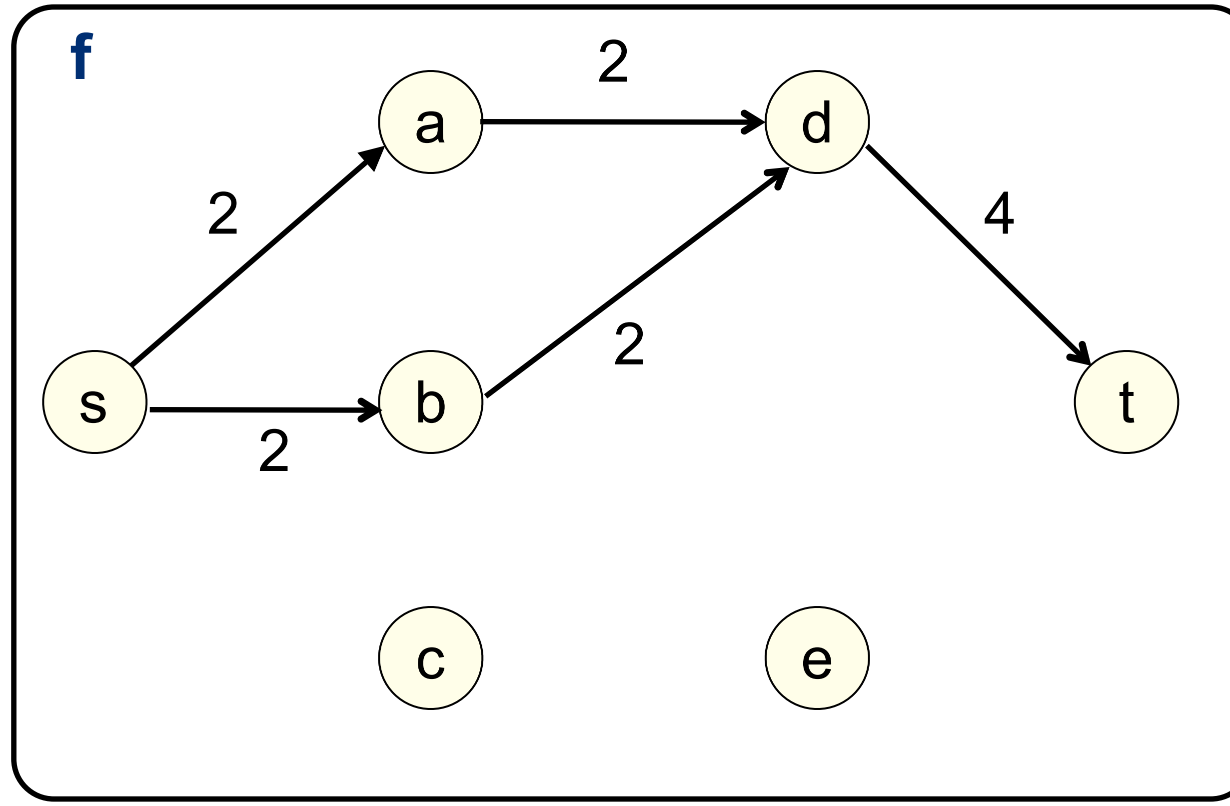
$E_f \subseteq E \cup E^R$

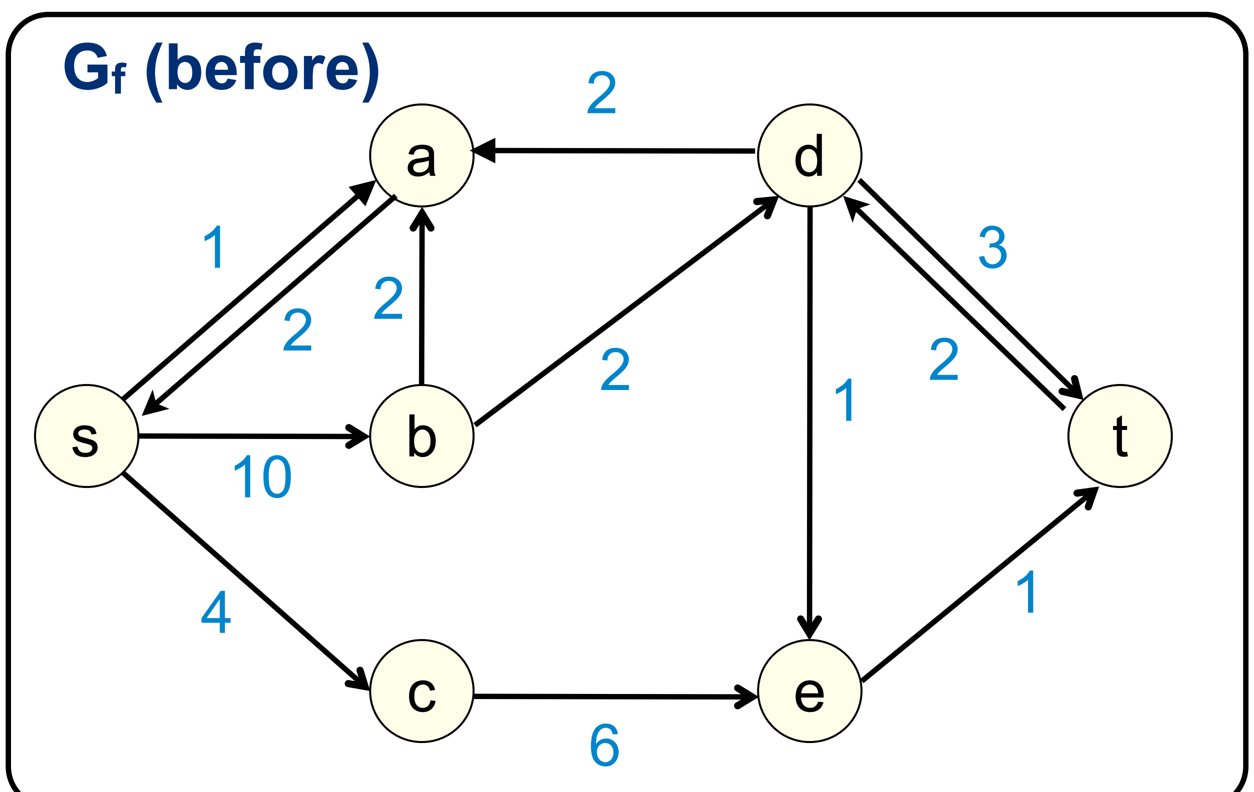
For any (u,v) in E or E^R,

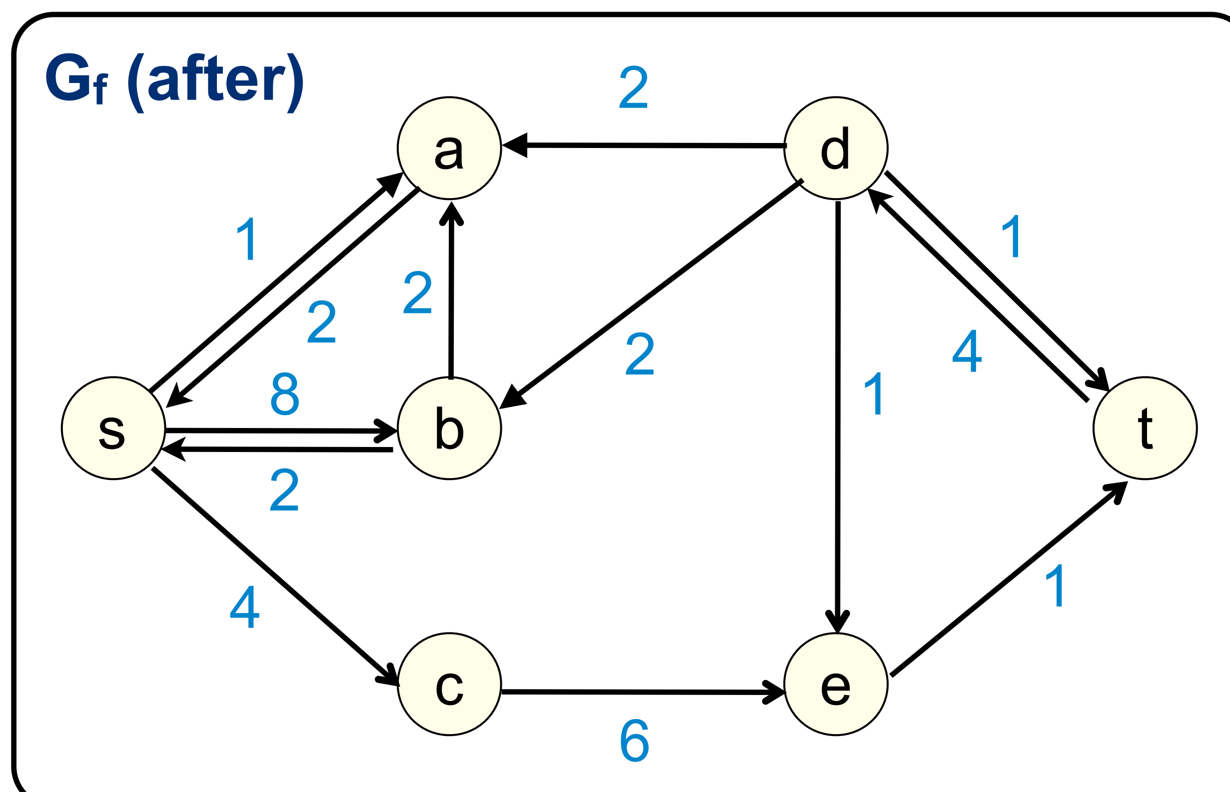
$c_f(u,v) = c(u,v) - f(u,v) + f(v,u)$

Find a path from s to t in G_f

Augment f along this path







Example: Round 3

Construct residual graph $G_f = (V, E_f)$

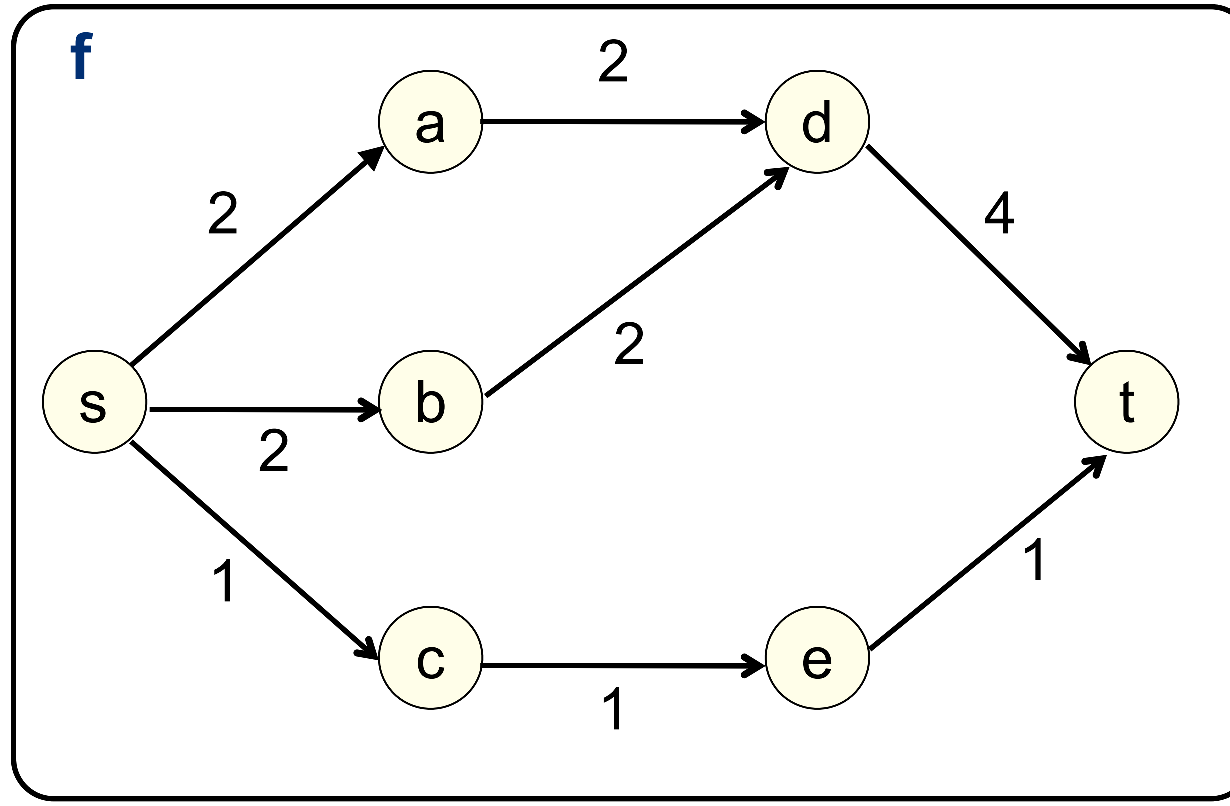
$E_f \subseteq E \cup E^R$

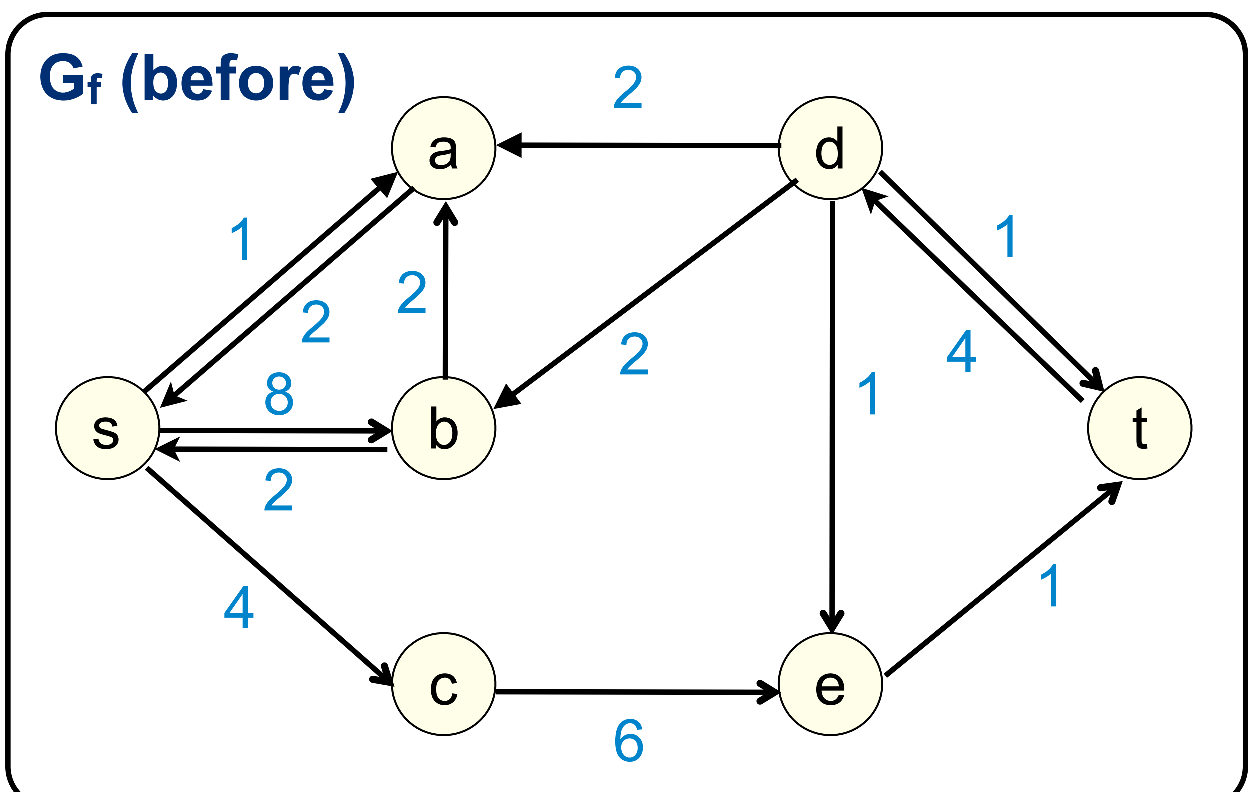
For any (u,v) in E or E^R,

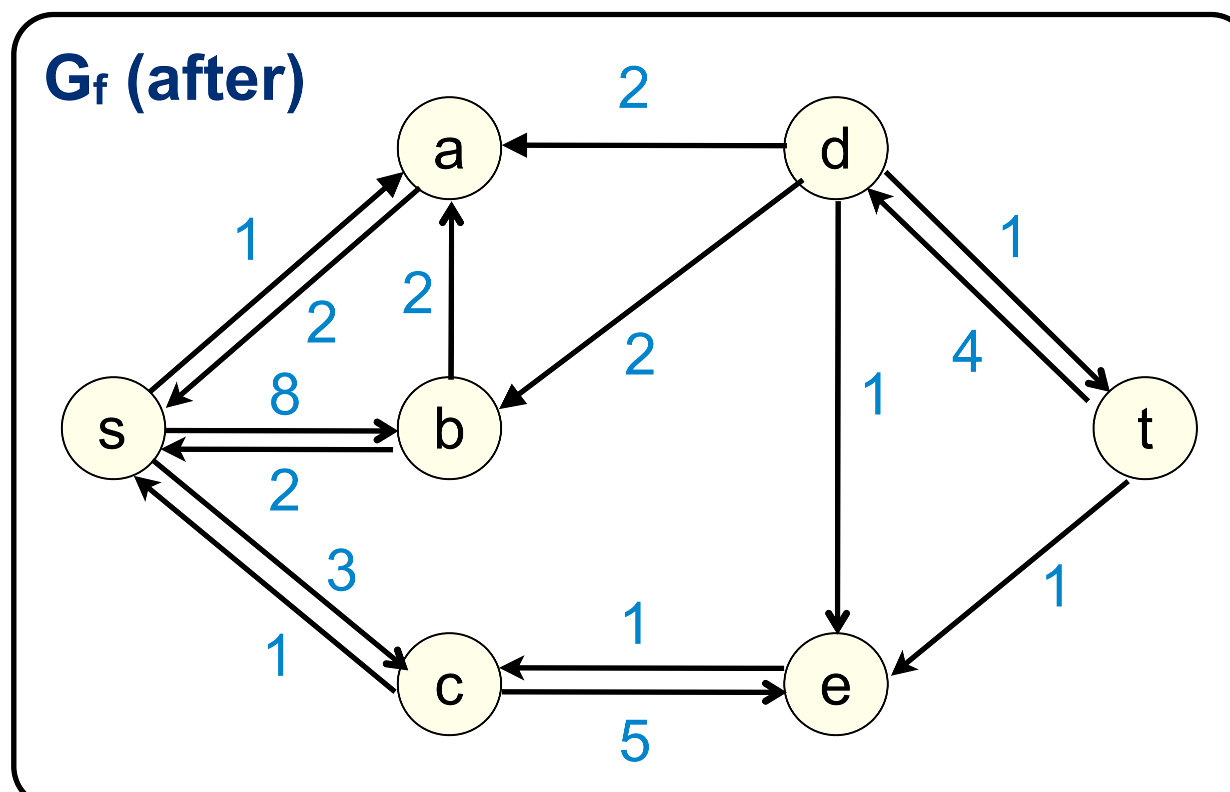
$c_f(u,v) = c(u,v) - f(u,v) + f(v,u)$

Find a path from s to t in G_f

Augment f along this path







Example: Round 3

Construct residual graph $G_f = (V, E_f)$

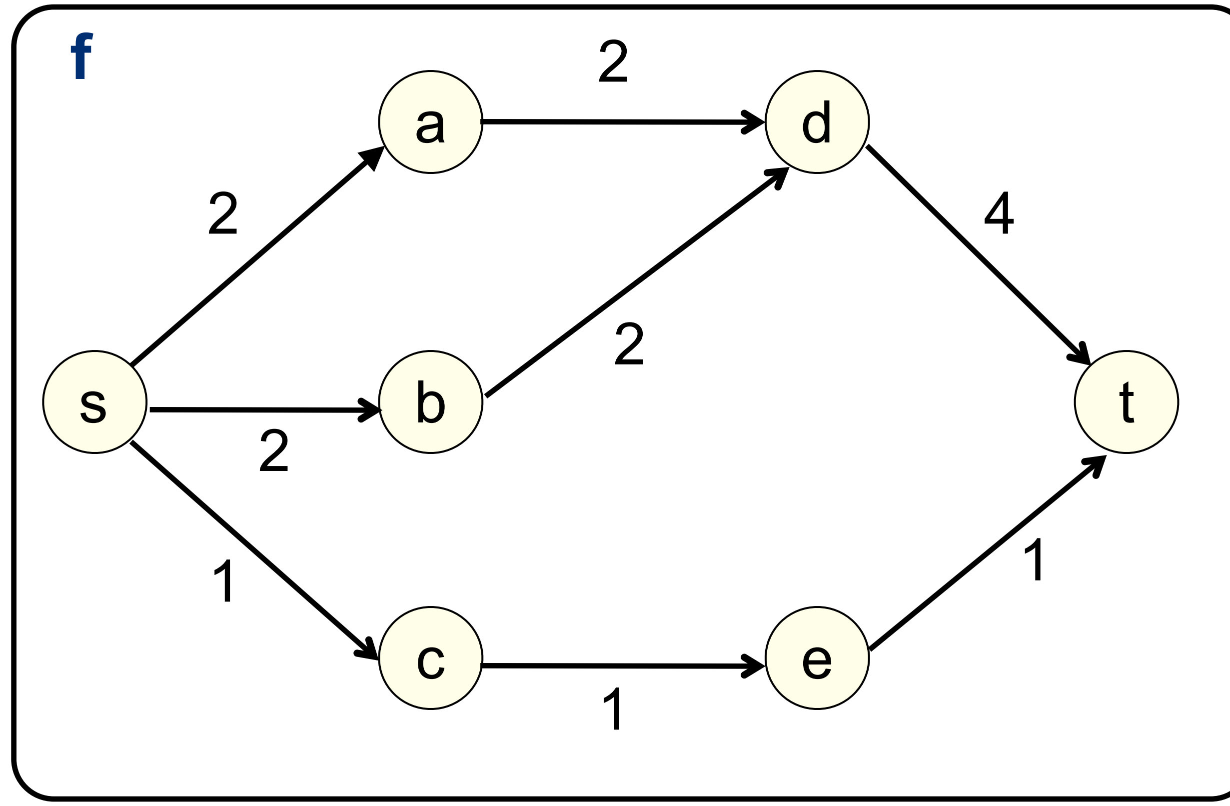
$E_f \subseteq E \cup E^R$

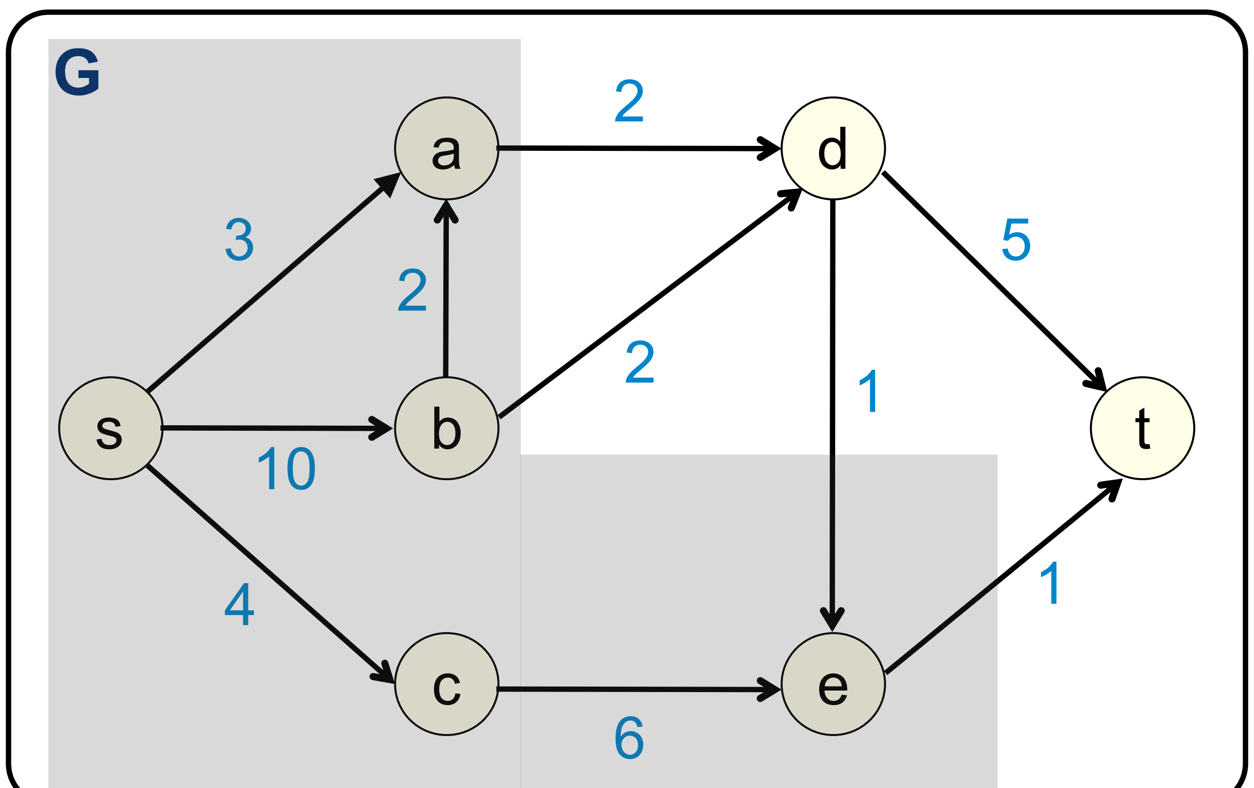
For any (u,v) in E or E^R,

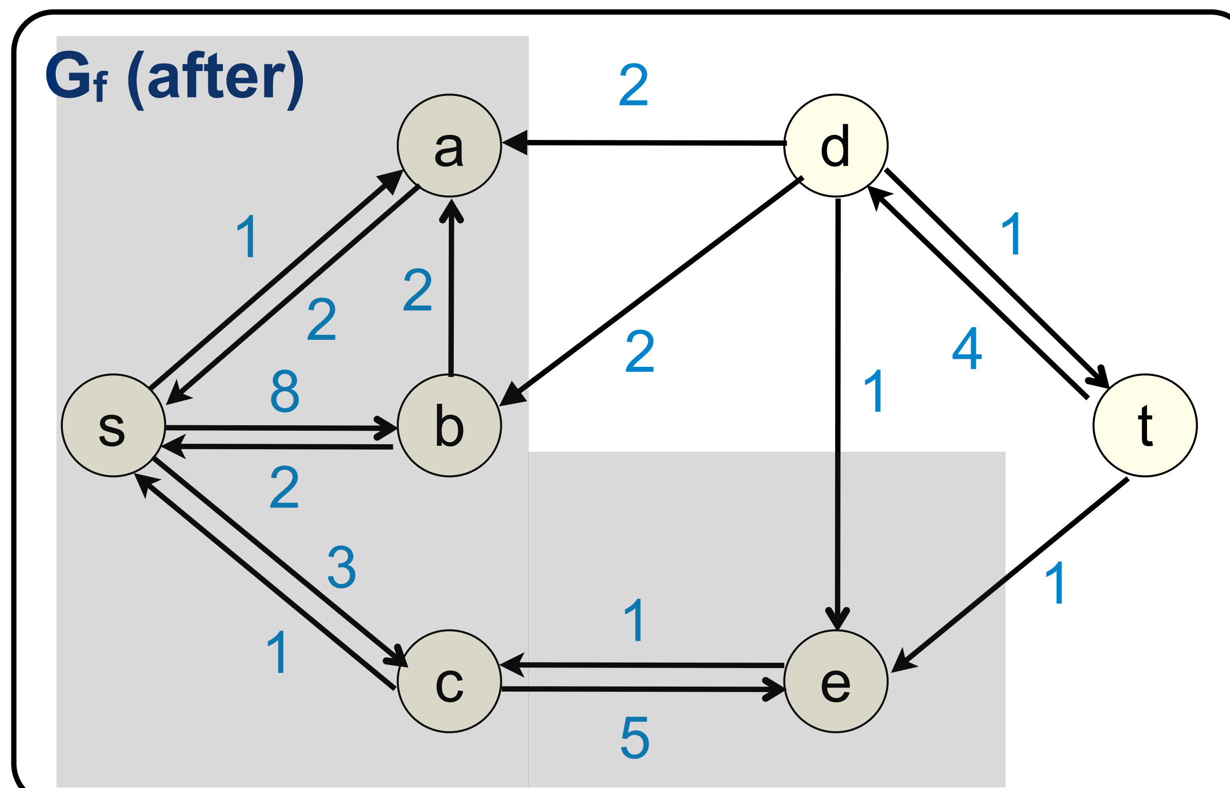
$c_f(u,v) = c(u,v) - f(u,v) + f(v,u)$

Find a path from s to t in G_f

Augment f along this path



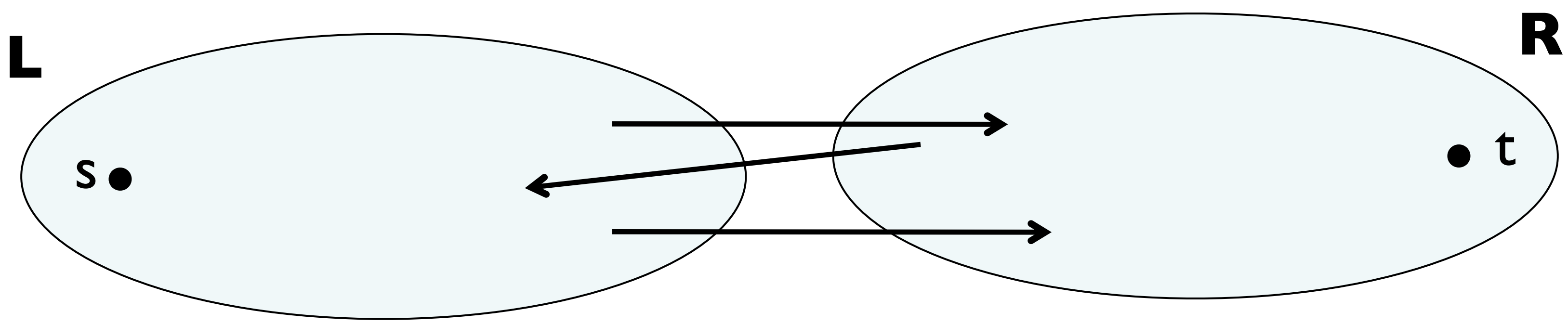




Analysis: Correctness

FF algorithm gives us a valid flow. But is it the **maximum** possible flow?

Consider final residual graph $G_f = (V, E_f)$
Let L = nodes reachable from s in G_f and let R = rest of nodes = V – L
So $s \in L$ and $t \in R$



Edges from L to R must be at full capacity
Edges from R to L must be empty
Therefore, flow across cut (L,R) is

$$\sum_{(u,v)\in E, u\in L, v\in R} c(u,v)$$

Thus, size(f) = capacity(L,R)

Recall: for any flow and any cut,
size(flow) <= capacity(cut)

Therefore f is the **max flow** and (L,R) is the **min cut**!

Thus, **Max Flow** = **Min Cut**