Computational Methodologies: Distributed Computing

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Question 1

Let $G = (V_G, E_G)$ be a connected topology graph with specified root $p_r \in V_G$. Let $D = \max\{dist(p_i, p_j) \mid p_i, p_j \in V_G\}$, be the diameter of the graph.

(a)

Theorem
The time complexity of the synchronous model of the Flooding algorithm on G is $O(D)$.

Proof
In the worst case the distance from p_r to some $p_i \in V_G$ is D as this is the furthest any message can travel from p_r without reaching an already visited node.

- In this case it takes at most D rounds for $\langle M \rangle$ to reach p_i.
- This is because $\langle M \rangle$ must be passed along each of the D nodes on the path from p_r to p_i and the message can only be transferred once per round.
- As p_i is the furthest node from p_r, after D rounds every node will have received $\langle M \rangle$.

Once a node has received $\langle M \rangle$, each node responds in one round. Therefore, after $D + 1$ rounds every node will have responded with either $\langle parent \rangle$ or $\langle already \rangle$. As all nodes have responded, each node will also have terminated.
Therefore, the overall time complexity is $D + 1 = O(D)$

(b)

Theorem
The time complexity of the asynchronous model of the Flooding algorithm on G is $O(D)$

Proof
Assume that the distance from p_r to some $p_i \in V_G$ is D.
Let P be a path of length D between p_r and p_i.
Suppose there is a maximum delay on every edge in G.
Then the total max time for $\langle M \rangle$ to get from p_r to p_i is D.
As p_i is the furthest any node can be from p_r, every node in V_G will have received $\langle M \rangle$ after D maximum edge delays.
Once a node has received $\langle M \rangle$, it responds immediately. This response takes at most the maximum edge delay. Once every node has received a response from all its neighbours (excluding the parent), it terminates.
Overall, all nodes have responded and terminated in at most $D + 1$ edge delays.
Therefore the overall time complexity is $O(D)$.

Question 2

(a)

We shall prove this by contradiction. First, assume that there is some uniform synchronous algorithm ,A, that correctly computes the AND of the input bits.
Run A on a ring where all input bits are 1. In any round i, the states of all the processors are the same. Therefore, there must be some round r where all the processors terminate.
Now run A on a ring with $2(r + 1)$ processors, where processor p_0 has input bit 0 and the remaining processors have input bit 1. For A to be correct some message originating from p_0 must reach all the nodes in the ring, however, it takes $r + 1$ rounds for the message to reach processor p_{r+1}. As p_{r+1} has initial bit 1, it will terminate in r rounds with its result as 1. This is incorrect, so A cannot be correct which is a contradiction. Therefore, a uniform synchronous algorithm for this problem does not exist.

(b)

Code for each node. Initially, $asleep =$ `true` for all nodes. in is the binary input, n is the total number of nodes

```
upon receiving no message :
  if asleep then
    asleep = false
    send ⟨in,0⟩ to left

upon receiving ⟨bit,count⟩ from right :
```

```
  if count == n:   // If visited nodes == total number of nodes, end
    result = bit
    terminate
  else:
    send ⟨bit & in, count + 1⟩ to left
```

Plain English description

Each processor sends a message to the left with its input bit and a counter that starts at 0.

Upon receiving a message of this form, a processor compares the counter to n.

- If the counter is equal to n, then its original message has been passed around the ring, so it can set the result to the received bit and terminate
- If the counter is not equal to n, send a message to the left containing the received bit AND input bit, and the counter incremented by 1.

Each processor sends a message that must be forwarded to all n of the processors in the ring. This means that in total n^2 messages are sent, so the message complexity is $O(n^2)$.

Possibly more formal version

Each processor p_i sends a message $\langle b_i, c_i \rangle$ to the right with its input bit b_i and a counter c_i that starts at 0.

Upon receiving a message $\langle b_{j-1}, c_{j-1} \rangle$, a processor p_j compares the c_{j-1} to n.

- If c_{j-1} is equal to n, then its original message has been passed around the ring, so it can set the result to b_{j-1} and terminate
- If c_{j-1} is not equal to n, send a message $\langle b_{j-1}$ AND $b_j, c_{j-1} + 1 \rangle$ to the right.

Each processor sends a message that must be forwarded to all n of the processors in the ring. This means that in total n^2 messages are sent, so the message complexity is $O(n^2)$.

(c)

Code for each node. Initially, $asleep =$ `true` for all nodes. in is the binary input, n is the total number of nodes.

This algorithm has $\lfloor n/2 \rfloor$ rounds numbered $1, \ldots, \lfloor n/2 \rfloor$.

```
upon receiving no message :
  if in == 0 then
    result = 0
    send ⟨terminate⟩ to left and right
    terminate
  else if asleep then
```

```
    asleep = false

upon receiving ⟨terminate⟩ from left (resp., right):
  result = 0
  send ⟨terminate⟩ to right (resp., left)
  terminate

if no messaged received by round ⌊n/2⌋:
  result = 1
  terminate
```

Plain English description If the input bit of a processor is 0, then the processor sets its result to be 0, sends a $\langle terminate \rangle$ message in both directions and terminates.
A processor with input bit 1, waits for $\lfloor n/2 \rfloor$ rounds. If it receives a $\langle terminate \rangle$ in this time, then it sets its result to 0, forwards the $\langle terminate \rangle$ message and terminates.
If no message is received within $\lfloor n/2 \rfloor$ rounds, then it sets its result to 1 and terminates.

In the worst case, where all the processors have 0 as an input bit, they all send 2 $\langle terminate \rangle$ messages to the left and right. This means that in total $2n$ messages are sent, so overall the message complexity is $O(n)$.