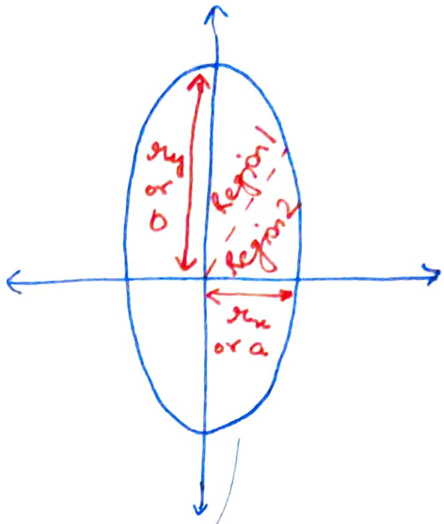
Midpoint ellipse algorithm

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$b^2x^2 + a^2y^2 - a^2b^2 = 0$$

$$y = \pm b\sqrt{1-(x^2/a^2)}$$

$$x = \pm a\sqrt{1-(y^2/b^2)}$$



The algorithm is applied through the first quadrant in two parts, i.e. region 1 and region 2.
If the slope of the curve is less than −1, then we are in region 1, otherwise in region 2.

$$\frac{dy}{dx} = -\frac{2r_y^2 x}{2r_x^2 y} = \frac{-2b^2x}{2a^2y}$$

We move out of the region when

$$2b^2x \geq 2a^2y$$

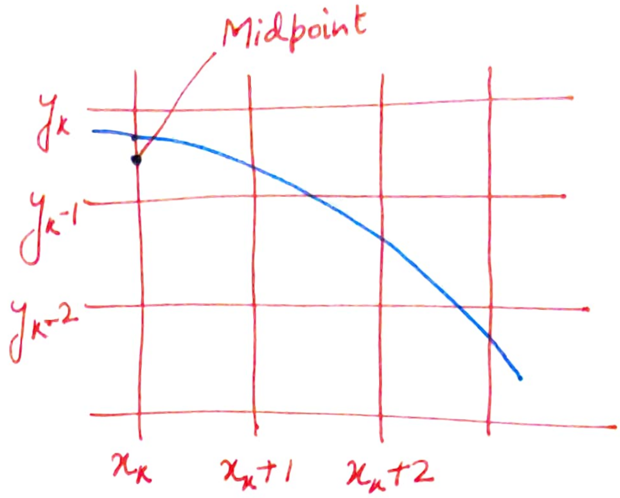
$$f(x,y) = b^2x^2 + a^2y^2 - a^2b^2 \begin{cases} <0 & \text{if } (x,y) \text{ is inside the ellipse} \\ =0 & \text{if } (x,y) \text{ is on the ellipse} \\ >0 & \text{if } (x,y) \text{ is outside the ellipse} \end{cases}$$

The next position on the ellipse can be evaluated by decision parameter at midpoint.

$$p1_k = f_{ellipse}\left(x_k+1,\ y_k - \frac{1}{2}\right)$$

$$= b^2(x_k+1)^2 + a^2\left(y_k - \frac{1}{2}\right)^2 - a^2b^2$$

If $p1_k < 0$, midpoint is inside the ellipse and pixel on y_k is closer to ellipse boundary, else midpoint is outside ~~or~~ on the ellipse boundary and $y_k - 1$ is closer to the ellipse boundary.



Next point $x_{k+1} = x_k + 1$

$$p1_{k+1} = f_{ellipse}\left(x_{k+1}+1,\ y_{k+1} - \frac{1}{2}\right)$$

$$= b^2\left((x_k+1)+1\right)^2 + a^2\left(y_{k+1} - \frac{1}{2}\right)^2 - a^2b^2$$

$$p1_{k+1} = p1_k + 2b^2(x_k+1) + b^2 + a^2\left[\left(y_{k+1} - \frac{1}{2}\right)^2 - \left(y_k - \frac{1}{2}\right)^2\right]$$

where y_{k+1} is either y_k or $y_k - 1$ depending on sign of $p1_k$.

If $p1_k < 0$, $y_{k+1} = y_k$

$$\therefore p1_{k+1} = p1_k + 2b^2x_{k+1} + b^2$$

Else if $p1_k \geq 0$, $y_{k+1} = y_k - 1$

$$p1_{k+1} = p1_k + 2b^2x_{k+1} - 2a^2y_{k+1} + b^2$$

where $2b^2x_{k+1} = 2b^2x_k + 2b^2$ and

$$2a^2y_{k+1} = 2a^2y - 2a^2$$

Start position in region 1 is $(0, b)$.

Initial decision parameter, $p1_0 = f_{ellipse}\left(1,\ b - \frac{1}{2}\right)$

$$= b^2 + a^2\left(b - \frac{1}{2}\right)^2 - a^2b^2$$

$$p1_0 = b^2 + a^2b^2 + \frac{1}{4}a^2 - a^2b - a^2b^2$$

$$= b^2 + \frac{1}{4}a^2 - a^2b$$

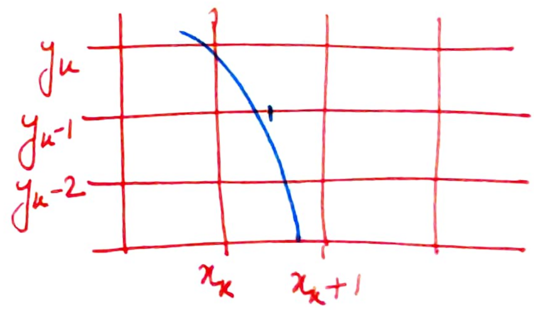
For region 2, we sample at unit steps in y direction and midpoint is now taken between horizontal pixels.

$$p2_k = f_{ellipse}\left(x_k + \frac{1}{2}, y_k - \frac{1}{2}\right)$$

$$= b^2\left(x_k + \frac{1}{2}\right)^2 + a^2(y_k - 1)^2 - a^2b^2$$

If $p2_k > 0$, midpoint is outside the ellipse boundary, so x_k is selected. i.e. $(x_k, y_k - 1)$

If $p2_k \leq 0$, midpoint is inside or on the ellipse boundary, so $x_k + 1$ is selected. i.e. $(x_k + 1, y_k - 1)$.



$$p2_{k+1} = f_{ellipse}\left(x_{k+1} + \frac{1}{2}, y_{k+1} - 1\right)$$

$$= b^2\left(x_{k+1} + \frac{1}{2}\right)^2 + a^2\left[(y_k - 1) - 1\right]^2 - a^2b^2$$

$$p2_{k+1} - p2_k = b^2\left(x_{k+1} + \frac{1}{2}\right)^2 - b^2\left(x_k + \frac{1}{2}\right)^2 + a^2(y_k - 2)^2 - a^2(y_k - 1)^2$$

$$= b^2\left(x_{k+1} + \frac{1}{2}\right)^2 - b^2\left(x_k + \frac{1}{2}\right)^2 + a^2\left[y_k^2 + 4 - 4y_k - y_k^2 - 1 + 2y_k\right]$$

$$= b^2\left(x_{k+1} + \frac{1}{2}\right)^2 - b^2\left(x_k + \frac{1}{2}\right)^2 + a^2(3 - 2y_k)$$

$$= b^2\left(x_{k+1} + \frac{1}{2}\right)^2 - b^2\left(x_k + \frac{1}{2}\right)^2 - 2a^2(y_k - 1) + a^2$$

If $p2_k > 0$, $x_{k+1} = x_k$

$$p2_{k+1} = p2_k - 2a^2(y_k - 1) + a^2$$

$$= p2_k - 2a^2 y_{k+1} + a^2$$

Else, $x_{k+1} = x_k + 1$

$$p2_{k+1} = p2_k + b^2\left(x_k + \frac{3}{2}\right)^2 - b^2\left(x_k + \frac{1}{2}\right)^2 - 2a^2 y_{k+1} + a^2$$

$$= p2_k + b^2\left[x_k^2 + \frac{9}{4} + 3x_k - x_k^2 - \frac{1}{4} - x_k\right] - 2a^2 y_{k+1} + a^2$$

$$= p2_k + b^2(2 + 2x_k) - 2a^2 y_{k+1} + a^2$$

$$= p2_k + 2b^2 x_{k+1} - 2a^2 y_{k+1} + a^2$$

$$p2_0 = f_{ellipse}\left(x_0 + \frac{1}{2}, y_0 - 1\right)$$

$$= b^2\left(x + \frac{1}{2}\right)^2 + a^2(y-1)^2 - a^2 b^2$$