INFORMATION THEORY & CODING

Week 5 : Source Coding 1

Dr. Rui Wang

Department of Electrical and Electronic Engineering
Southern Univ. of Science and Technology (SUSTech)

Email: wang.r@sustech.edu.cn

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Review Summary

Theorem (AEP)

"Almost all events are almost equally surprising." Specifically, if $X_1, X_2, \ldots$ are i.i.d. $\sim p(x)$, then

$$-\frac{1}{n}\log p(X_1, X_2, \ldots, X_n) \to H(X) \text{in probability.}$$

Definition

The *typical set* $A_\epsilon^{(n)}$ is the set of sequences $x_1, x_2, \ldots, x_n$ satisfying

$$2^{-n(H(X)+\epsilon)} \le p(x_1, x_2, \ldots, x_n) \le 2^{-n(H(X)-\epsilon)}.$$

Review Summary

Properties of the typical set

1. If $(x_1, x_2, \ldots, x_n) \in A_\epsilon^{(n)}$, then
$H(X) - \epsilon \leq -\frac{1}{n} \log p(x_1, x_2, \ldots, x_n) \leq H(X) + \epsilon$.
2. $\Pr[A_\epsilon^{(n)}] > 1 - \epsilon$ for n sufficiently large.
3. $|A_\epsilon^{(n)}| \leq 2^{n(H(X)+\epsilon)}$, where $|A|$ denotes the cardinality of the set A.
4. $|A_\epsilon^{(n)}| \geq (1-\epsilon)2^{n(H(X)-\epsilon)}$ for n sufficiently large.

Theorem

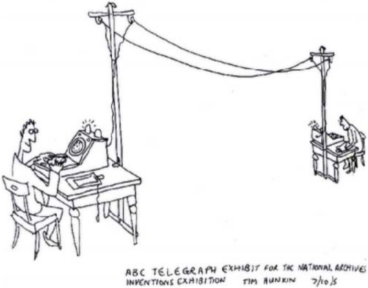
Let X^n be i.i.d. $\sim p(x)$. There exists a code that one-to-one maps sequences x^n of length n into binary strings with

$$E[\frac{1}{n}\ell(X^n)] \leq H(X) + \epsilon$$

for n sufficiently large.

Source Coding

Which horse won in the horse racing?



Source Coding

source alphabet $\mathcal{X}=\{0,1,\cdots,7\}$

Which horse won in the horse racing?

code alphabet $D=\{0,1\}$

alphabet extension

$D^*=\{0,1,00,\cdots\}$

X	Pr	Code I	Code II
0	1/2	000	0
1	1/4	001	10
2	1/8	010	110
3	1/16	011	1110
4	1/64	100	111100
5	1/64	101	111101
6	1/64	110	111110
7	1/64	111	111111

$$H(X) = -\sum p_i \log p_i = 2\text{bits}$$

Which code is better?

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Source Coding (Data Compression)

- We interpret that $H(X)$ is the best achievable data compression.

- We want to develop practical lossless coding algorithms that approach, or achieve the entropy limit $H(X)$.

Terminology

X	Pr	Code I	Code II
0	1/2	000	0
1	1/4	001	10
2	1/8	010	110
3	1/16	011	1110
4	1/64	100	111100
5	1/64	101	111101
6	1/64	110	111110
7	1/64	111	111111

- Source alphabet $\mathcal{X} = \{0, 1, 2, 3, 4, 5, 6, 7\}$.
- Code alphabet $\mathcal{D} = \{0, 1\}$.
- Codeword, e.g., 010 for $X = 2$ in Code 1.
- Codeword length, e.g., codeword length for Code 1 is 3.
- Codebook: all the codewords.

Notation (Alphabet Extension)

The set of all possible sequences based on a finite alphabet $\mathcal{D}$ is denoted by $\mathcal{D}^*$. E.g.,
$\mathcal{D} = \{0, 1\} \rightarrowtail \mathcal{D}^* = \{0, 1, 00, 01, 10, 11, 000, ...\}$.

Definition (Source Code)

Let $\mathcal{X}$ be the alphabet of a random variable X, and $\mathcal{D}$ be the alphabet of code. A *source code* C for the random variable X is a map

$$\begin{aligned} C: \quad & \mathcal{X} \to \mathcal{D}^* \\ & x \mapsto C(x) \end{aligned}$$

where $C(x)$ is the codeword associated with x. Let $\ell(x)$ denote the length of C(x).

Definition

The *expected length* $L(X)$ of a source code C for a random variable X with probability mass function $p(x)$ is

$$L(X) = E\ell(X) = \sum_{x \in \mathcal{X}} p(x)\ell(x).$$

X	Pr	Code I	Code II
0	1/2	000	0
1	1/4	001	10
2	1/8	010	110
3	1/16	011	1110
4	1/64	100	111100
5	1/64	101	111101
6	1/64	110	111110
7	1/64	111	111111

$L_1(X) = 3$

$L_2(X) = 2$

Source Coding Applications

- Magnetic recording: cassette, hard drive ...
- Speech compression
- Compact disk (CD)
- Image compression: JPEG

Source Coding: Set of codes

For $\mathcal{X} = \{1, 2, 3, 4\}$ and $\mathcal{D} = \{0, 1\}$, consider

x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	$1/2$	0	0	10	0
2	$1/4$	0	1	00	10
3	$1/8$	1	00	11	110
4	$1/8$	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

- Code efficiency $= H(X)/E[\ell(X)]$
- Which code is best? Would we prefer C_I or C_{II}?

  Consider C_I and decode string: 00001. It would come from $1, 2, 1, 2, 3$ or $2, 1, 2, 1, 3$ or $1, 1, 1, 1, 3$, or etc.

Source Coding: Set of codes

For $\mathcal{X} = \{1, 2, 3, 4\}$ and $\mathcal{D} = \{0, 1\}$, consider

x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	1/2	0	0	10	0
2	1/4	0	1	00	10
3	1/8	1	00	11	110
4	1/8	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

- Code efficiency $= H(X)/E[\ell(X)]$
- Which code is best? Would we prefer C_I or C_{II}?

  Consider C_{II} and decode string: 0011. It could be either $1, 1, 2, 2$ or $3, 4$.

Source Coding: Set of codes

For $\mathcal{X} = \{1, 2, 3, 4\}$ and $\mathcal{D} = \{0, 1\}$, consider

x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	1/2	0	0	10	0
2	1/4	0	1	00	10
3	1/8	1	00	11	110
4	1/8	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

- Consider C_{III}. Can we decode 1100000000?

  Yes. But if we only see a prefix, such as 11, we don't know until we see more bits to the end.

  $1100000000 = 3, 2, 2, 2, 2$

  $11000000000 = 4, 2, 2, 2, 2$

Source Coding: Set of codes

For $\mathcal{X} = \{1, 2, 3, 4\}$ and $\mathcal{D} = \{0, 1\}$, consider

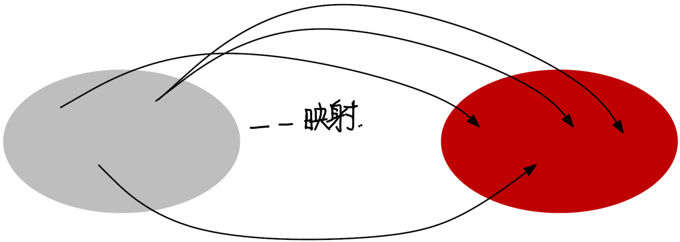
x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	1/2	0	0	10	0
2	1/4	0	1	00	10
3	1/8	1	00	11	110
4	1/8	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

- Consider C_{IV}. This code seems at least feasible (since $E[\ell] \geq H$). Decoding seems easy: (e.g., $111110100 = 111, 110, 10, 0 = 4, 3, 2, 1$).

Definition (Nonsingular Code)

A code C is called *nonsingular* if every realization of $\mathcal{X}$ maps onto a difference codeword in $\mathcal{D}^*$, i.e.,

$$x \neq x' \Rightarrow C(x) \neq C(x').$$



Definition (Nonsingular Code)

→ 不能区分编码方式的好坏.

A code C is called *nonsingular* if every element of $\mathcal{X}$ maps onto a difference string in $\mathcal{D}^*$, i.e.,

$$x \neq x' \Rightarrow C(x) \neq C(x').$$

x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	1/2	0	0	10	0
2	1/4	0	1	00	10
3	1/8	1	00	11	110
4	1/8	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

C_I is singular.

Definition (Code Extension)

The *extension* of a code $C\colon \mathcal{X} \to \mathcal{D}^*$ is defined by

$$C(x_1 x_2 \cdots x_n) = C(x_1)C(x_2)\cdots C(x_n).$$

Definition (Unique Decodable Code)

区分编码方式的好坏.

不能区分解码方式的好坏.

A code is called *uniquely decodable* if its extension is nonsingular.

$$x_1 x_2 \ldots x_m \neq x_1' x_2' \ldots x_n' \Rightarrow C\left(x_1 x_2 \ldots x_m\right) \neq C\left(x_1' x_2' \ldots x_n'\right)$$

Definition (Unique Decodable Code)

A code is called *uniquely decodable* if its extension is nonsingular.

C_{II}^* is singular. $(C(1,1) = C(3) = 00)$

x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	1/2	0	0	10	0
2	1/4	0	1	00	10
3	1/8	1	00	11	110
4	1/8	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

C_I is singular.
C_{II} is NOT u.d..

Source Coding: Code types

Definition (Unique Decodable Code)

A code is called *uniquely decodable* if its extension is nonsingular.

C_{III} is uniquely decodable.

x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	1/2	0	0	10	0
2	1/4	0	1	00	10
3	1/8	1	00	11	110
4	1/8	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

C_I is singular.
C_{II} is NOT u.d..

Source Coding: Code types

Definition (Unique Decodable Code)

A code is called *uniquely decodable* if its extension is nonsingular.

$$1100000000 = 3, 2, 2, 2, 2$$
$$11000000000 = 4, 2, 2, 2, 2$$

To know the source, we have to wait until the end!

x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	1/2	0	0	10	0
2	1/4	0	1	00	10
3	1/8	1	00	11	110
4	1/8	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

C_I is singular.
C_{II} is NOT u.d..

Source Coding: Code types

Definition (Prefix Code) →区分解码方式的好坏.

A code C is called a *prefix code* (a.k.a. *instantaneous*) iff no codeword of C is a prefix of any other codeword of C.

x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	1/2	0	0	10	0
2	1/4	0	1	00	10
3	1/8	1	00	11	110
4	1/8	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

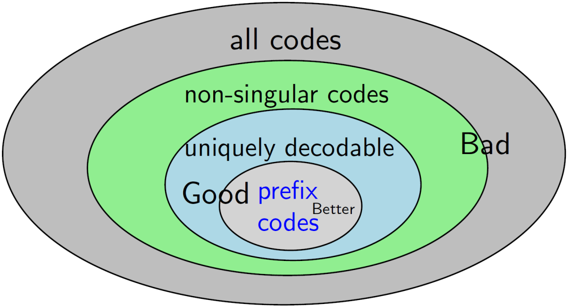
C_I is singular.
C_{II} is NOT u.d..
C_{III} is NOT prefix.
C_{IV} is prefix.

For $\mathcal{X} = \{1, 2, 3, 4\}$ and binary code, consider

x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	1/2	0	0	10	0
2	1/4	0	1	00	10
3	1/8	1	00	11	110
4	1/8	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

- C_I is singular.
- C_{II} is non-singular, but not uniquely decodable.
- C_{III} is non-singular, uniquely decodable, but NOT prefix.
- C_{IV} is non-singular, uniquely decodable, and prefix.

Source Coding: Classes of codes



- Goal: to find a prefix code with minimum expected length.

Kraft Inequality

Theorem 5.2.1 (Kraft Inequality)

For any prefix code over an alphabet of size D, the codeword lengths $\ell_1, \ell_2, \ldots, \ell_m$ must satisfy the inequality

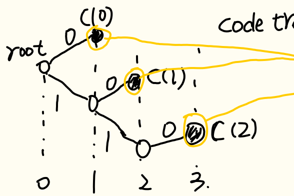
$$\sum_i D^{-\ell_i} \leq 1.$$

Conversely, given a set of codeword lengths that satisfy this inequality, there exists a prefix code with these codeword lengths.

x	$C(x)$
0	0
1	10
2	110

$\mathcal{X}=\{0,1,2\}$

$D=\{0,1\}$.



code tree.

leaf. node.

ancestor node

descendant. node.

full tree.

把所有子结点展开到最大level.

$l_{max} = \max_x l(x) = 3$.

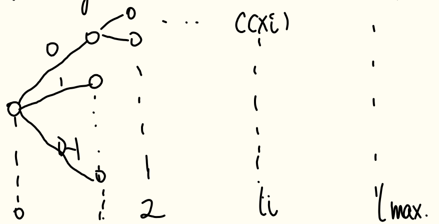
In prefix code.

codewords are leaf nodes of code tree

Proof:

Suppose there is a D-ary prefix code $\{C(x_1), C(x_2), \cdots C(x_n)\}$.

with lengths $l_1, l_2, \cdots l_m$, and code alphabet $D=\{0,1,\cdots,D-1\}$.



① expand the code tree to a full tree with l_{max} levels.

② In the full tree, codeword $C(x_i)$ has $l_{max}-l_i$ levels of descendants.

$\Rightarrow D^{l_{max}-l_i}$ leaf nodes in the full tree are the descendants of $C(x_i)$.

Proof Idea. (A small example) To prove: A prefix code with lengths $\ell_1, \ell_2, \ldots, \ell_m$, the inequality

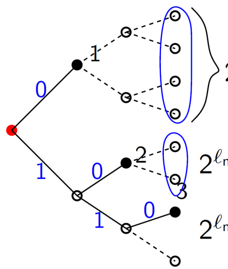
$$\sum_i D^{-\ell_i} \le 1 \qquad \text{holds.}$$

$D = \{0, 1\}$

$\mathcal{X} = \{1, 2, 3\}$

x	$c(x)$
1	0
2	10
3	110

Depth: 0 1 2 3



$2^{\ell_{\max}-\ell_i} = 2^{3-1} = 4$

$2^{\ell_{\max}-\ell_j} = 2^{3-2} = 2$

$2^{\ell_{\max}-\ell_{\max}} = 2^{3-3} = 1$

$\ell_{\max} = 3$

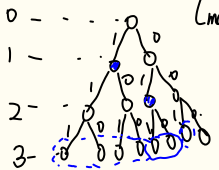
$$\sum_i 2^{-\ell_i} \le 1 \Leftarrow \sum_i 2^{\ell_{\max}-\ell_i} \le 2^{\ell_{\max}}$$

③ In full tree, leaf nodes of different codewords are not overlapped.

$\sum_i D^{l_{max}-l_i} \le D^{l_{max}} \Rightarrow \sum_i D^{-l_i} \le 1.$

Consider binary codeword lengths $\{l_1, l_2, l_3\} = \{1, 2, 3\}$

$l_{max} = 3$



$8 = 2^3$ leaf nodes.

l_1	l_2	l_3
1	2	3
2^{3-1}	2^{3-2}	2^{3-3}

$\{1, 01, 001\}$

Algorithm: $l_1, l_2, \cdots l_m$.

前提: m有限 ← suppose: $l_1 \le l_2 \le l_3 \cdots \le l_m$

推广 ⟹

① Generate a D-ary full tree with l_m levels, give each leaf node an index from left to right, $L = \{0, 1, \cdots, D^{l_m} - 1\}$.

② let j be the minimum index of leaf nodes which has not been allocated, initially $j = 0$.

③ for $i = 1$ to m.

pick up leaf nodes in $\{j, j+1, \cdots, j + D^{l_m - l_i} - 1\}$

find the node N_i, which is the closest common ancestor of above leaf nodes. $j = j + D^{l_m - l_i}$

end.

output $= \{N_1, N_2, \cdots N_m\} \Rightarrow$ prefix code.

例外:

当 $m = \infty$,

Poisson Distribution.

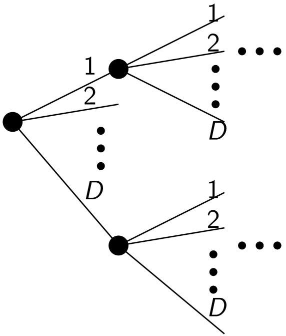
$Pr[X = k] = \frac{\lambda^k}{k!} e^{-\lambda}, \ k = 0, 1, 2, \cdots$

$\mathcal{X} = \{0, 1, 2, 3 \cdots\} \quad |\mathcal{X}| = \infty.$

Kraft Inequality

Proof. (in general)

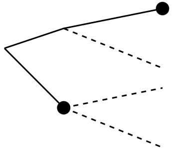
- Represent the set of prefix codes on a D-ary tree:



- Codewords correspond to leaves
- Path from root to each leaf determines a codeword
- Prefix condition: won't get to a codeword until we get to a leaf (no descendants of codewords are codewords)

Proof. (in general)

- $\ell_{\max} = \max_i(\ell_i)$ is the length of the longest codeword.
- We can expand the full-tree down to depth $\ell_{\max}$:



The nodes at the level $\ell_{\max}$ are either

1. codewords
2. descendants of codewords
3. neither

- Consider a codeword i at depth ℓ_i in tree
- There are $D^{\ell_{\max}-\ell_i}$ descendants in the tree at depth $\ell_{\max}$
- Descendants of code i are disjoint from decedents of code j (prefix free condition)

Proof. (in general)

- All the above implies:

$$\sum_i D^{\ell_{\max}-\ell_i} \leq D^{\ell_{\max}} \quad \Rightarrow \sum_i D^{-\ell_i} \leq 1$$

- Conversely: given codewords lengths $\ell_1, \ell_2, \ldots, \ell_m$ satisfying Kraft inequality, try to construct a prefix code.

Proof. (in general)

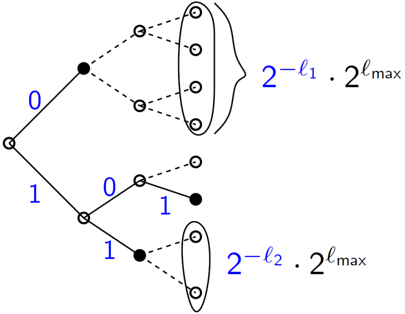
- Conversely: given codewords lengths $\ell_1, \ell_2, \ldots, \ell_m$ satisfying Kraft inequality, try to construct a prefix code.

$$\{\ell_1, \ell_2, \ell_3\} = \{1, 2, 3\}$$

$$2^{-1} + 2^{-2} + 2^{-3} \leq 1$$

x	$c(x)$
1	0
2	11
3	101

C is prefix.



Reading & Homework

Reading : 5.1, 5.2

Homework : Problems 5.1, 5.3, 5.18, 5.37