Final Homework

1. Entropy and Relative Entropy

(a). A fair coin is flipped until the first head occurs. Let X denote the number of flips required. Find the entropy $H(X)$ in bits. The following expressions may be useful:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad \sum_{n=0}^{\infty} nr^n = \frac{r}{(1-r)^2}.$$

(b). Let the random variable Y has four possible outcomes $\{\mathsf{AA}, \mathsf{AG}, \mathsf{GG}, \mathsf{GA}\}$. Consider two distributions on this random variable

Symbol \ Probability	$p(y)$	$q(y)$
AA	1/4	1/2
AG	1/4	1/6
GG	1/3	1/6
GA	1/6	1/6

Calculate $H(p)$,$H(q)$, $D(p \parallel q)$ and $D(q \parallel p)$. Whether $D(p \parallel q) = D(q \parallel p)$ or not?

2. Differential Entropy

(a). Please find a continuous random variable X such that $H(X) < 0$.

(b). Suppose X and Y are *independently identically distribution* (i.i.d.) exponentially distributed random variable with mean $\frac{1}{\lambda}$. Let $Z = X - Y$. Please evaluate the dfferential entropy of Z.

(c). Suppose V_1 and V_2 are independent normal random variables with means μ_i and and variances σ_i^2, $i = 1, 2$. Let $V = V_1 + V_2$. Please evaluate the dfferential entropy of V.

3. Entropy Rates of Markov Chains.

(a). Find the entropy rate of the two-state Markov chain with transition matrix

$$\mathbf{P} = [\mathbf{P}_{ij}] = \begin{bmatrix} 1 - p_{01} & p_{01} \\ p_{10} & 1 - p_{10} \end{bmatrix},$$

where (i, j)-th element denotes the transition probability from the i-th state to the j-th state.

(b). What values of p_{01}, p_{10} maximize the entropy rate?

(c). Find the entropy rate of the two-state Markov chain with transition matrix

$$\mathbf{P} = [\mathbf{P}_{ij}] = \begin{bmatrix} 1 - p & p \\ 1 & 0 \end{bmatrix}.$$

4. Capacity.

(a). Consider the channel with $x, y \in \{0, 1, 2, 3\}$ and transition probabilities $p(y|x)$ given by the following matrix:

$$p(y|x) = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0 & 0.5 \end{bmatrix}, \quad x, y \in \{0, 1, 2, 3\},$$

where the (i, j)-th element denotes the transition probability $\Pr[y = j|x = i]$. Find the capacity of this channel.

(b). Find the capacity of ternary channel.

$$p(y|x) = \begin{bmatrix} 1/4 & 1/6 & 7/12 \\ 1/6 & 7/12 & 1/4 \\ 7/12 & 1/4 & 1/6 \end{bmatrix}.$$

5. Shannon Codes and Huffman Codes.

Consider a random variable X which takes on four values with probabilities $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{12}\right)$.

(a). Given three different sets of binary codeword $\{1, 00, 01, 110\}$, $\{00, 01, 10, 11\}$, $\{0, 11, 100, 101\}$, which are uniquely decodable?

(b). For the uniquely decodable codeword sets in (a), are they optimal (best)?

Hint: Check the expected length and Kraft inequality.

6. Huffman Coding.

Consider the random variable $X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.10 & 0.48 & 0.26 & 0.05 & 0.05 & 0.02 & 0.04 \end{pmatrix}$.

(a). Find a binary Huffman code for X.

(b). Find the expected code length for this encoding.

(c). Find a ternary Huffman code for X.