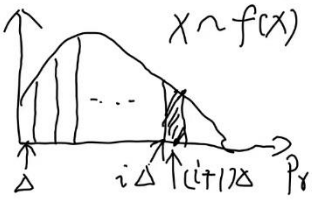
$$h(X) = h(f) = -\int_{-\infty}^{+\infty} f(x)\log f(x)\,dx$$



$\exists\ x_i \in (i\Delta, (i+1)\Delta),\ \underline{f(x_i)}\Delta = \int_{i\Delta}^{(i+1)\Delta} f(x)\,dx$

$x_0\ x_1\ \cdots\ x_i\ \cdots\ -$

$\Pr[X^\Delta = x_i] = \underline{f(x_i)\Delta} \stackrel{\Delta}{=} \underline{p_i} = \underline{\int_{i\Delta}^{(i+1)\Delta} f(x)\,dx}$

$\Delta \to 0 \qquad \underline{X^\Delta \to X}$

$\underline{H(X^\Delta)} = -\sum_i p_i \log p_i = -\sum_i f(x_i)\Delta \log[f(x_i)\Delta]$

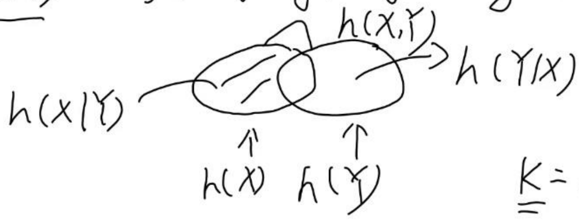
$= -\sum_i \Delta f(x_i)\log f(x_i) - \sum_i \Delta \underline{f(x_i)}\,\underline{\log\Delta} = -\sum_i \Delta f(x_i)\log f(x_i) - \underline{\log\Delta}$

$\Delta \to 0 \quad = -\int_{-\infty}^{+\infty} f(x)\log f(x)\,dx - \log\Delta = \underline{h(X)}\ \boxed{-\log\Delta}\ \infty$

$\Delta \to 0 \quad \log\Delta \to \infty$

$X_1 \cdots X_n \sim f(x_1 \cdots x_n) \quad h(X_1 \cdots X_n) = \underline{h(f)} = -\int f(x^n)\log f(x^n)\,dx^n$

$\underline{h(X|Y)} = -\int f(x,y)\log f(x|y)\,dx\,dy = h(X,Y) - h(Y)$



$\underline{K} = E\left\{\left[\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} - \mu\right]\left[(X_1 \cdots X_n) - \mu^T\right]\right\}$

$X_1 \cdots X_n \sim \underline{N(\mu, K)}$

$$\phi_K(x_1 \cdots x_n) = \frac{1}{(\sqrt{2\pi})^n |K|^{1/2}} \exp\left\{-\frac{1}{2}(\vec{x}-\mu)^T K^{-1}(\vec{x}-\mu)\right\}$$

$\vec{x} = (x_1 \cdots x_n)^T$

$$h(X_1 \cdots X_n) = -\int \phi(x^n) \log \phi(x^n)\, dx^n$$

$$= -\int \phi(x^n) \left[-\log(\sqrt{2\pi})^n |K|^{1/2} - \frac{1}{2} \log e\, (\vec{x}-\mu)^T K^{-1} (\vec{x}-\mu) \right] dx^n$$

$$= \frac{1}{2} \log(2\pi)^n |K| + \frac{\log e}{2} \int \phi(x^n) (\vec{x}-\mu)^T K^{-1} (\vec{x}-\mu)\, dx^n$$

$$= \frac{1}{2} \log(2\pi)^n |K| + \frac{\log e}{2} E (\vec{x}-\mu)^T K^{-1} (\vec{x}-\mu)$$

$$= \frac{1}{2} \log(2\pi)^n |K| + \frac{\log e}{2} E \{ tr (\vec{x}-\mu)(\vec{x}-\mu)^T K^{-1} \}$$

$$= \frac{1}{2} \log(2\pi)^n |K| + \frac{\log e}{2} tr\, E (\vec{x}-\mu)(\vec{x}-\mu)^T K^{-1}$$

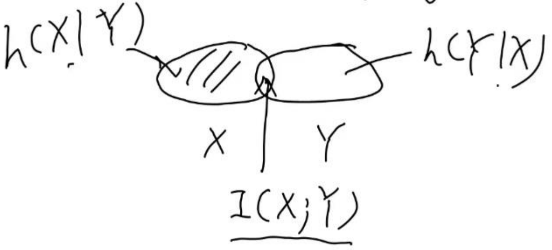
$$= \frac{1}{2} \log(2\pi)^n |K| + \frac{1}{2} \log e^n = \frac{1}{2} \log(2\pi e)^n |K|$$

$f(x^n)$ $g(x^n)$

Relative Entropy: $D(f \| g) = \int f(x^n) \log \frac{f(x^n)}{g(x^n)} dx^n \geq 0$

mutual Info: $I(X;Y) = \int f(x,y) \log \frac{f(x,y)}{f(x)f(y)} dx\,dy$

$$= D(f(x,y) \,\|\, f(x)f(y))$$



$\phi_K \sim N_n(0, K)$ $\quad g(x^n)$ $\quad 0 \quad K$ $\quad h(\phi_K) \geq h(g)$

$\int \phi(x^n) \log \phi(x^n)\, dx^n$

$0 \leq D(g \| \phi_K)$

$= \int g(x^n) \log \frac{g(x^n)}{\phi(x^n)} dx^n = -h(g) - \int g(x^n) \log \phi(x^n)\, dx^n$

$= -h(g) + h(\phi)$

$\int g(x^n) \log \phi(x^n)$

$= \int \underset{\phi(x^n)}{g(x^n)} \left[\log \frac{1}{(\sqrt{2\pi})^n |K|^{1/2}} - \frac{1}{2} \log e \cdot (\vec{x}-\mu)^T K^{-1} (\vec{x}-\mu) \right] dx^n$

$= \int g(x^n) \left[\log \frac{1}{(\sqrt{2\pi})^n |K|^{1/2}} \right] dx^n -$

$\frac{1}{2} \log e \int g(x^n) (\vec{x}-\mu)^T K^{-1} (\vec{x}-\mu)\, dx^n$

$\phi(x^n)$

$E_g \left[(\vec{x}-\mu)^T K^{-1} (\vec{x}-\mu) \right]$

$= E_g\, tr (\vec{x}-\mu)(\vec{x}-\mu)^T K^{-1}$

$= tr\, K K^{-1} = tr\, I$

$= E_\phi (\vec{x}-\mu)^T K^{-1} (\vec{x}-\mu)$

AWGN

$X \to \boxed{AWGN} \to Y = X + Z \qquad Z \sim N(0, \sigma^2)$

$\max_{f(x)} I(X;Y) = C$

$E X^2 = P$

$I(X;Y) = h(Y) - h(Y|X)$

$= h(Y) - h(Z|X) = h(Y) - h(Z)$

$h(Z) = \frac{1}{2} \log 2\pi e \cdot \sigma^2 \qquad = \frac{1}{2} \log\left(1 + \frac{P}{\sigma^2}\right)$ SNR

$E[Y^2] = E[(X+Z)^2] = E[X^2 + Z^2 + 2XZ] = E[X^2] + E[Z^2]$

$= P + \sigma^2 \qquad h(Y) = \frac{1}{2} \log 2\pi e (P + \sigma^2)$