INFORMATION THEORY & CODING

Week 3 : Inequalities

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Review Summary

- **Definitions:**

$$
\begin{aligned}
H(X) &= E_p \log \frac{1}{p(X)} \\
H(X,Y) &= E_p \log \frac{1}{p(X,Y)} \\
H(X|Y) &= E_p \log \frac{1}{p(X|Y)} \\
I(X;Y) &= E_p \log \frac{p(X,Y)}{p(X)p(Y)} \\
D(p\|q) &= E_p \log \frac{p(X)}{q(X)} \\
I(X;Y) &= H(X) - H(X|Y) \\
&= H(Y) - H(Y|X) \\
&= H(X) + H(Y) - H(X,Y).
\end{aligned}
$$

Review Summary

- **Chain rules:**

  Entropy:

  $H(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} H(X_i | X_{i-1}, \ldots, X_1)$.

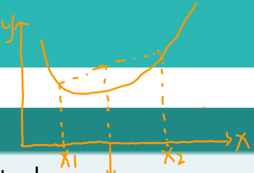
  Mutual information:

  $I(X_1, X_2, \ldots, X_n; Y) = \sum_{i=1}^{n} I(X_i; Y | X_1, X_2, \ldots, X_{i-1})$.

  Relative entropy:

  $D\big(p(x, y) \| q(x, y)\big) = D\big(p(x) \| q(x)\big) + D\big(p(y|x) \| q(y|x)\big)$.

Jensen's Inequality



$f(x_1),\ f(x_2),\ f[\lambda x_1+(1-\lambda)x_2]$

$k=\frac{f(x_2)-f(x_1)}{x_2-x_1}$

$\frac{f(x_2)-f(x_1)}{x_2-x_1}[(\lambda-1)x_1+(1-\lambda)x_2]+f(x_1)$

Definition (Convexity)

A function $f(x)$ is said to be *convex* over an interval (a, b) if $\forall x_1, x_2 \in (a, b)$ and $0 \le \lambda \le 1$,

$\lambda x_1+(1-\lambda)x_2$

$(1-\lambda)f(x_2)-(1-\lambda)f(x_1)+f(x_1)$

$$f(\lambda x_1 + (1-\lambda)x_2) \le \underline{\lambda f(x_1) + (1-\lambda) f(x_2)}.$$

A function f is called *strictly convex* if equality holds only if $\lambda = 0$ or $\lambda = 1$.

Definition (Concavity)

A function f is *concave* if $-f$ is convex.

A function is convex if it always lies below any chord. A function is concave if it always lies above any chord.

Definition (Convexity)

A function $f(x)$ is said to be *convex* over an interval (a, b) if for every $x_1 x_2 \in (a, b)$ and $0 \leq \lambda \leq 1$,

$$f\left(\lambda x_1 + (1-\lambda)x_2\right) \leq \lambda f(x_1) + (1-\lambda) f(x_2).$$

A function f is called *strictly convex* if equality holds only if $\lambda = 0$ or $\lambda = 1$.

Definition (Concavity)

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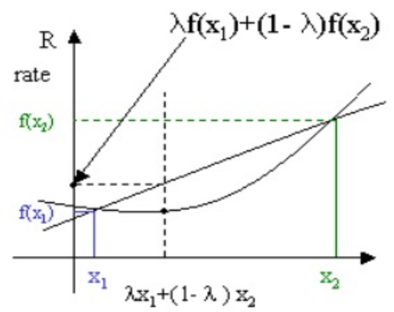


Jensen's Inequality

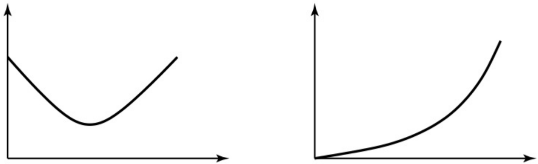
Example

convex convex convex

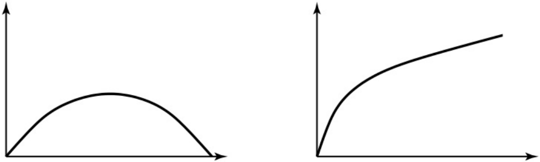
$$f(x) = x^2, \qquad |x|, \qquad e^x, \qquad x\log x \qquad (x > 0)$$

$$g(x) = \log x, \qquad \sqrt{x}, \qquad (x \geq 0)$$

concave concave



(a)



(b)

Jensen's Inequality

Theorem 2.6.2 (Jensen's Inequality)

→ 前提条件.

If f is a convex function and X is a random variable,

$E[f(X)] = \sum_{i=1}^{k} p(x_i) f(x_i)$

$$E[f(X)] \geq f(E[X]).$$

$f(E[X]) = f\left(\sum_{i=1}^{k} x_i p(x_i)\right)$

Moreover, if f is strictly convex, $E[f(X)] = f(E[X])$ implies that $X = E[X]$ with probability 1 (i.e., X is a constant).

Proof.

(数学归纳法)

By mathematical induction.

- $k = 2$: $\to p(x_1) = \lambda,\ p(x_2) = 1-\lambda$

  $p(x_1)f(x_1) + p(x_2)f(x_2) \geq f\left(p(x_1)x_1 + p(x_2)x_2\right).$
- Hypothesis: $\sum_{i=1}^{k-1} p(x_i)f(x_i) \geq f\left(\sum_{i=1}^{k-1} p(x_i)x_i\right).$
- Induction: $\sum_{i=1}^{k} p(x_i)f(x_i).$

when $k=m$

$\sum_{i=1}^{m} p(x_i)f(x_i) = \sum_{i=1}^{m-1} p_m(x_i)f(x_i) + p_m(x_m)f(x_m)$

令 $a = \sum_{i=1}^{m-1} p_m(x_i)$

$= a\sum_{i=1}^{m-1} \frac{p_m(x_i)}{a} f(x_i) + p_m(x_m)f(x_m)$

$\geq a f\left[\sum_{i=1}^{m-1} \frac{p_m(x_i)}{a} x_i\right] + p_m(x_m)f(x_m)$

$\geq f\left[\sum_{i=1}^{m-1} p_m(x_i)\cdot x_i + p_m(x_m)\cdot x_m\right]$

$= f\left[\sum_{i=1}^{m} p(x_i)\cdot x_i\right]$

□

Information Inequality

Theorem 2.6.3 (*Information Inequality*)

Let $p(x)$, $q(x)$, $x \in X$, be two probability mass functions. Then

$$D(p\|q) \geq 0$$

with equality iff $p(x) = q(x)$ for all x.

concave .

let $f(x) = \log x$, 令 $g(x) = \frac{q(x)}{p(x)}$

Proof.

Let $A = \{x : p(x) > 0\}$ be the support set of $p(x)$. Then

$$
\begin{aligned}
-D(p\|q) &= -\sum_{x \in A} p(x) \log \frac{p(x)}{q(x)} \\
&= \sum_{x \in A} p(x) \log \frac{q(x)}{p(x)} \\
&\leq \log \sum_{x \in A} p(x) \frac{q(x)}{p(x)} \quad \text{(Jensen's Inequality)} \\
&= \log \sum_{x \in A} q(x) \\
&\leq \log \sum_{x \in \mathcal{X}} q(x) = 0
\end{aligned}
$$

$= \sum_x p(x) f[g(x)] = E\{f[g(X)]\}$

$\leq f[E(g(X))]$

Corollaries

Corollary (*Nonnegativity of mutual information*)

For any two random variables, X, Y,

$$I(X;Y) \geq 0,$$

$I(X;Y) = D[p(x,y) \| p(x)p(y)]$

with equality iff X and Y are independent.

Corollary

$$D\left(p(y|x) \| q(y|x)\right) \geq 0,$$

with equality iff $p(y|x) = q(y|x)$ for all y and x such that $p(x) > 0$.

Corollary

$$I(X;Y|Z) \geq 0,$$

with equality iff X and Y are conditionally independent given Z.

The maximum entropy distribution

$p(x)=\frac{1}{|X|}$

$H(X)=\sum_x p(x)\cdot\log\frac{1}{p(x)}=\sum_x\frac{1}{|X|}\log|X|=\log|X|.$

Theorem 2.6.4

$H(X) \leq \log|\mathcal{X}|$, where $|\mathcal{X}|$ denotes the number of elements in the range of X, with equality iff X has a uniform distribution over $|\mathcal{X}|$.

Proof.

Let $u(x) = \frac{1}{|\mathcal{X}|}$ be the uniform probability mass function over $\mathcal{X}$, and let $p(x)$ be the probability mass function for X. Then

$$0 \leq D(p\|u) = \sum p(x) \log \frac{p(x)}{u(x)} = \log|\mathcal{X}| - H(X).$$

□

连续随机变量正态分布最大.

$H(X|Y) = H(X) - I(X;Y) \geq 0$

Theorem 2.6.5 (*Conditioning reduces entropy*)

$$H(X|Y) \leq H(X)$$

with equality iff X and Y are independent.

$H(X_1, \cdots X_n) = \sum_{i=1}^{n} H(X_i | X_1 \cdots X_{i-1}) \leq \sum_{i=1}^{n} H(X_i)$.

Theorem 2.6.6 (*Independence bound on entropy*)

Let $X_1, X_2, \ldots, X_n$ be drawn according to $p(x_1, x_2, \ldots, x_n)$, then

$$H(X_1, X_2, \ldots, X_n) \leq \sum_{i=1}^{n} H(X_i)$$

with equality iff the X_i's are independent.

Data-processing inequality

z: future state, y: current state x: past state

$p(z|y,x) = Pr[Z=z|Y=y, X=x] = p(z,y)$, given y, z is independent of X.

Definition (*Markov Chain*)

Random variables X, Y, Z are said to *form a Markov chain* in that order (denoted by $X \to Y \to Z$) if the conditional distribution of Z depends only on Y and is conditionally independent of X. Specifically, X, Y and Z form a Markov chain $X \to Y \to Z$ if the join probability mass function can be written as $\Leftrightarrow p(z|x,y) = p(z|y)$

$$p(x,y,z) = p(x)p(y|x)p(z|y).$$ $\Leftrightarrow X \to Y \to Z$.

$p(x|y,z) = p(x|y) \Leftrightarrow \frac{p(x,y,z)}{p(y,z)} = \frac{p(x,y)}{p(y)} \Leftrightarrow \frac{p(x,y,z)}{p(x,y)} = \frac{p(y,z)}{p(y)}$

- $X \to Y \to Z \Rightarrow p(x,z|y) = p(x|y)p(z|y)$ $= \frac{p(x,y,z)\,p(x,y)}{p(y)\,p(x,y)} = p(x|y)\,p(z|x,y) = p(x|y)\,p(z|y).$
- $X \to Y \to Z \Rightarrow Z \to Y \to X$
- If $Z = f(Y)$, then $X \to Y \to Z$.

Data-processing inequality

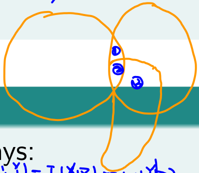
Theorem 2.8.1 (*Data-processing inequality*)

If $X \to Y \to Z$, *then* $I(X;Y) \geq I(X;Z)$.

①+② $= I(X;Y) = I(X;Y|Z)$

② $= I(X;Z)$

意义：X与Y之间传递的信息量大于等于X与Z之间传递的信息量.



Proof.

By the chain rule, we expand $I(X;Y,Z)$ in two ways:

① $= I(X;Y) - I(X;Z) = I(X;Y|Z)$

③ $= I(Y;Z|X)$.

$$
\begin{aligned}
I(X;Y,Z) &= I(X;Z) + I(X;Y|Z) \\
&= I(X;Y) + I(X;Z|Y).
\end{aligned}
$$

Since $X \to Y \to Z$, we have $I(X;Z|Y) = 0$. Since $I(X;Y|Z) \geq 0$, we have $I(X;Y) \geq I(X;Z)$. □

$X_1 \to X_2 \to X_3 \to \cdots \Rightarrow I(X_1;X_4) \leq I(X_1;X_3) \leq I(X_1;X_2)$

Corollaries

Corollary

In particular, if $Z = g(Y)$, we have $I(X;Y) \geq I(X;g(Y))$.

Corollary

If $X \to Y \to Z$, then $I(X;Y|Z) \leq I(X;Y)$.

Fano's inequality → 物理意义：$X \to Y \to \hat{X}$，判别去检测X.

① given Y, X can be uniquely determined.

② In $p(x,y)$, $\forall y$ with $p(y)>0$, $\exists$ only one x with $p(x,y)>0$

Problem 2.5 (*Zero conditional entropy*)

Show that if $H(X|Y)=0$, then X is a function of Y, i.e., for all y with $p(y)>0$, there is only one possible value of x with $p(x,y)>0$.

Proof.

Assume that there exists an y, say y_0 and two different values of x, say x_1 and x_2 such that $p(y_0,x_1)>0$ and $p(y_0,x_2)>0$. Then $p(y_0)\ge p(y_0,x_1)+p(y_0,x_2)>0$, and $p(x_1|y_0)$ and $p(x_2|y_0)$ are not equal to 0 or 1. Thus,

$$
\begin{aligned}
H(X|Y) &= -\sum_y p(y)\sum_x p(x|y)\log p(x|y) \\
&\ge p(y_0)\left(-p(x_1|y_0)\log p(x_1|y_0) - p(x_2|y_0)\log p(x_2|y_0)\right) \\
&> 0
\end{aligned}
$$

since $-t\log t\ge 0$ for $0\le t\le 1$, and is strictly positive for $t\ne 0,1$, which is a contradiction to $H(X|Y)=0$. $\square$

proof: $p(y_0)=\sum_x p(x,y_0) \to \frac{p(x_1,y_0)}{p(y_0)}<1$

$= -p(x_1,y_0)\log p(x_1|y_0) - p(x_2,y_0)\log p(x_2|y_0) = -p(x_1,y_0)\log\left[\overset{<1}{\frac{p(x_1,y_0)}{p(y_0)}}\right] - p(x_2,y_0)\log\left[\overset{<1}{\frac{p(x_2,y_0)}{p(y_0)}}\right]$

Fano's inequality

- The conditional entropy of a random variable X given another random variable Y is zero ($H(X|Y) = 0$) iff X is a function of Y. Hence we can estimate X from Y with zero probability of error iff $H(X|Y) = 0$.
- We can estimate X with a low probability of error P_e only if the conditional entropy $H(X|Y)$ is small. *Fano's inequality* quantifies this idea.

Why do we need to related P_e to entropy $H(X|Y)$? When we have a communication system, we send X, but receive a corrupted version Y. We want to infer X from Y. Our estimate is $\hat{X}$ and we will make a mistake as

$$P_e = \Pr[\hat{X} \neq X]$$

Markov chain $X \rightarrow Y \rightarrow \hat{X}$.

Fano's inequality →把P_e与$H(X|Y)$建立联系.

Problem

A random variable Y is related to another random variable X with a distribution $p(x)$. From Y, we calculate a function $g(Y) = \hat{X}$, where $\hat{X}$ is an estimate of X and takes on values in $\hat{\mathcal{X}}$. We observe that $X \to Y \to \hat{X}$ forms a Markov chain. How to bound the estimate error probability $P_e = \Pr[\hat{X} \neq X]$?

Fano's inequality

$H(P_e) = -P_e \log P_e - (1-P_e)\log(1-P_e)$

Theorem 2.11.1

For Markov chain $X \to Y \to \hat{X}$, with $P_e = \Pr\{X \neq \hat{X}\}$, we have

$$H(P_e) + P_e \log(|\mathcal{X}| - 1) \geq H(X|\hat{X}) \geq H(X|Y).$$

P_e服从(0,1)分布，其entropy ≤ 1

This inequality can be weakened to

$$1 + P_e \log(|\mathcal{X}| - 1) \geq H(X|Y)$$

or

$$P_e \geq \frac{H(X|Y) - 1}{\log(|\mathcal{X}| - 1)}.$$

Remark: $\hat{X}$ can be treated as an estimation of X based on Y.

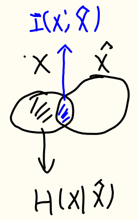
$X \to Y \to \hat{X}$

proof: $H(X|\hat{X}) = H(X) - I(X;\hat{X})$

$H(X|Y) = H(X) - I(X;Y)$

$\because I(X;Y) \geq I(X;\hat{X})$

$\therefore H(X|\hat{X}) \geq H(X|Y)$



Fano's inequality

Proof. given $X, \hat{X}$, E is certain, $H(E|X,\hat{X})=0$

Define an error random variable as

$$E = \begin{cases} 1 & \text{if } \hat{X} \neq X, \quad P_e \\ 0 & \text{if } \hat{X} = X. \quad 1-P_e \end{cases}$$

$H(E)=H(P_e)$

Using the chain rule for entropies to expand $H(E, X|\hat{X})$ in two different ways, we have

$$H(E, X|\hat{X}) = H(X|\hat{X}) + \underbrace{H(E|X, \hat{X})}_{=0} = \underbrace{H(E|\hat{X})}_{\leq H(P_e)} + \underbrace{H(X|E, \hat{X})}_{\leq P_e \log(|X|-1)} .$$

?

$H(E|\hat{X})<H(E)$

Since conditioning reduces entropy, $H(E|\hat{X}) \leq H(E) = H(P_e)$. Since E is a function of X and $\hat{X}$, the conditional entropy $H(E|X, \hat{X})$ is equal to 0. We now look at $H(X|E, \hat{X})$. By the equation $H(X|Y) = \sum_y p(y)H(X|Y = y)$, we have

$$H(X|E, \hat{X}) = \sum_{\hat{x} \in \mathcal{X}} \{\Pr[\hat{X} = \hat{x}, E = 0]H(X|\hat{X} = \hat{x}, E = 0) + \Pr[\hat{X} = \hat{x}, E = 1]H(X|\hat{X} = \hat{x}, E = 1)\}.$$

□

Fano's inequality

Proof.

$$H(E, X|\hat{X}) = H(X|\hat{X}) + \underbrace{H(E|X, \hat{X})}_{=0} = \underbrace{H(E|\hat{X})}_{\leq H(P_e)} + \underbrace{H(X|E, \hat{X})}_{\leq P_e \log(|X|-1)} .$$

$$H(X|E, \hat{X}) = \sum_{\hat{x} \in \mathcal{X}} \{\Pr[\hat{X} = \hat{x}, E = 0] H(X|\hat{X} = \hat{x}, E = 0)$$
$$+ \Pr[\hat{X} = \hat{x}, E = 1] H(X|\hat{X} = \hat{x}, E = 1)\}.$$

By definition of E, X is conditionally deterministic given $\hat{X} = \hat{x}$ and $E = 0$, then $H(X|\hat{X} = \hat{x}; E = 0) = 0$. If $\hat{X} = \hat{x}$ and $E = 1$, then X must take a value in the set $\{x \in \mathcal{X} : x \neq x\hat{x}\}$ which contains $|\mathcal{X}| - 1$ elements. Then $H(X|\hat{X} = \hat{x}, E = 1) \leq \log(|\mathcal{X}| - 1)$.均匀分布的熵,均匀分布的熵最大.

$$H(X|E, \hat{X}) \leq \sum_{\hat{x} \in \mathcal{X}} \Pr[\hat{X} = \hat{x}, E = 1] \log(|\mathcal{X}| - 1)$$
$$= \Pr[E = 1] \log(|\mathcal{X}| - 1)$$
$$= P_e \log(|\mathcal{X}| - 1)$$

□

Fano's inequality

Proof.

$$H(E, X|\hat{X}) = H(X|\hat{X}) + \underbrace{H(E|X, \hat{X})}_{=0} = \underbrace{H(E|\hat{X})}_{\leq H(P_e)} + \underbrace{H(X|E, \hat{X})}_{\leq P_e \log(|X|-1)} .$$

$$\begin{aligned} H(X|E, \hat{X}) = \sum_{\hat{x} \in \mathcal{X}} \{ &\Pr[\hat{X} = \hat{x}, E = 0] H(X|\hat{X} = \hat{x}, E = 0) \\ &+ \Pr[\hat{X} = \hat{x}, E = 1] H(X|\hat{X} = \hat{x}, E = 1) \}. \end{aligned}$$

$$\begin{aligned} H(X|E, \hat{X}) &\leq \sum_{\hat{x} \in \mathcal{X}} \Pr[\hat{X} = \hat{x}, E = 1] \log(|\mathcal{X}| - 1) \\ &= \Pr[E = 1] \log(|\mathcal{X}| - 1) \\ &= P_e \log(|\mathcal{X}| - 1) \end{aligned}$$

By the data-processing inequality, we have $I(X; \hat{X}) \leq I(X; Y)$ and therefore $H(X|\hat{X}) \geq H(X|Y)$. □

Corollary

Corollary

For any two random variables X and Y , let $p = Pr(X \neq Y)$.

$$H(p) + p\log(|\mathcal{X}| - 1) \geq H(X|Y).$$

Proof.

Let $\hat{X} = Y$ in Fano's inequality. $\square$

Applications of Fano's inequality

- Prove converse in many theorems (including channel capacity)

- Compressed sensing signal model

$$y = Ax + w$$

where $A \in \mathcal{R}^{M \times d}$: projection matrix for dimension reduction. Signal x is sparse. Want to estimate x from y.

Fano's inequality

Lemma 2.10.1

If X and X' are i.i.d. with entropy $H(X)$,

$$\Pr[X = X'] \geq 2^{-H(X)},$$

with equality iff X has a uniform distribution.

Corollary

Let X, X' be independent with $X \sim p(x)$, $X' \sim r(x)$, $x, x' \in \mathcal{X}$. Then

$$\Pr\left[X = X'\right] \geq 2^{-H(p)-D(p\|r)}$$

$$\Pr\left[X = X'\right] \geq 2^{-H(r)-D(r\|p)}$$

Reading & Homework

Reading : Whole Chapter 2

Homework : Problems 2.13, 2.15, 2.26, 2.29, 2.35