GEOMETRIC INTRODUCTION TO TRIGONAL CURVES OF GENUS FIVE

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1.

Let C be a non-hyperelliptic trigonal curve of genus five (assume for a moment that it exists; we will construct one in a few moments). Then it has a plane model C' which is a quintic with a double point:

Indeed, we can write the trigonal structure as $g^1_3 = |p+q+r|$, where the points p, q and r on C are distinct. (In other words, a trigonal structure is a map

$$\pi : C \to \mathbb{P}^1,$$

, and for some point $a \in \mathbb{P}^1$ the preimage of the point a consists of three separate points, $\pi^{-1}(a) = p+q+r$.)

Then, by the geometric version of the Riemann-Roch theorem, the points p, q and r all belong to the same line l in the canonical embedding of C.

When we vary (move) the divisor $D = p+q+r$ in the linear system g^1_3, the linear span $\langle D \rangle$, which is a line, is changing, too; forming a ruled surface in $\mathbb{P}^4$ containing the curve C. More on this later; this surface containing C is important.

Now consider the linear system of hyperplanes $|H|$ in $\mathbb{P}^4$ and the linear subsystem $|H-l|$ of hyperplanes containing the fixed line l for some divisor $D = p+q+r$ in the linear system g^1_3. This linear system of hyperplanes cuts out a linear system $|K_C - p - q - r|$ of dimension $4 - (\dim l + 1) = 2$ and degree 8 - 3 = 5 on C:

$$K_C - g^1_3 = g^2_5$$

This g^2_5 gives a plane model C' of the curve C which is a curve of degree 5; by construction, it is just the projection of the curve C with the center l.

Since $p_a(C') = 4 \cdot 3/2 = 6$, we have $\delta(C') = p_a(C') - p_g(C') = 6-5 = 1$, and thus C' is a singular curve with one singular point, which can be either a node or one cusp. (See Serre, "Algebraic groups and class fields" for an excellent introduction to the δ - invariant of singular curves.)

2.

We can now prove the existence of such a curve C. Consider a plane curve C' of degree 5 with one node p. (We omit the case of a cusp.) It is clear that such a curve exists.

The geometric genus of C' is $4 \cdot 3/2 - 1 = 5$. Let C be the smooth model of C'. It can be constructed by blowing up the point p on $\mathbb{P}^2$, and taking the proper preimage of the singular curve C'.

It is immediately clear that the curve C is trigonal: the linear system of lines on the plane through the point p, $|l-p|$, gives a pencil of (effective) divisors of degree 3 on C (and gives a projection of C to a projective line which is a degree 3 map).

3.

The linear system of conics through p on the plane, $|2l-p|$, cuts out a linear system of degree 10 on C'. The base locus of this linear system is $2p$. By the adjunction formula for nodal curves, (cf. Enrico Arbarello, Maurizio Cornalba, Phillip Griffiths, Joseph Harris, Geometry of Algebraic Curves: Volume I, Appendix A), after removing the base point $2p$, we get the *canonical* linear system on C. (For any plane nodal curve C', the canonical class of its smooth model is $(d-3)l-\{$ 0-cycle of nodes on $C'\}$.)

Note that the linear system $|2l-p|$ embeds the blown-up plane $\mathrm{Bl}_p(\mathbb{P}^2)$ as a ruled surface in $\mathbb{P}^4$.

This gives an embedding

$$i(C) \subset i(\mathrm{Bl}(\mathbb{P}^2)) \subset \mathbb{P}^4 = |K_C|^*$$

given by the linear system $|2l-p|$.

Note that any line l_0 on $\mathbb{P}^2$ via p becomes a line L_0 under the embedding i. (It follows, or example, from the computation $C \cdot l_0 = p+p'$ for any conic C in $|2l-p|$.

It follows that the image $S = \mathrm{Bl}(\mathbb{P}^2)$ in $\mathbb{P}^4$ is a ruled surface, with every line of the ruling giving a trisecant to the $i(C)$. Thus, S is the spanned by the trisecant lines to C.

4.

It is easy to see that $S = \mathrm{Bl}(\mathbb{P}^2)$ is the intersection of quadrics through S (and even through C, as we will see in a moment):

Note that $i(C)$ is not an intersection of quadrics: for a trisecant line L, L intersects any quadric Q through C at three points, $i(p1), .., i(p3)$, and this is contained in Q . Thus, S is the intersection of quadrics through C .

5.

In fact, the surface S is clearly one and the same for all the curves C' with a node at the same point p . A computation with Riemann-Roch theorem gives that every C is cut out in S by its own pencil of cubic hypersurfaces in $\mathbb{P}^4$. The attached program computes this pencil.

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