Chapter 2: Discrete Random Variables

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1 Basic Comcepts

In many probability models, the outcome are numerical. Then the outcome can be discribed using a *random variable.*

> ***Definition* 1: Random Variable**
> A random variable is a real-valued function of the experimental outcome.

A random variable is called discrete if its range is either finit or countably infinite. So furthermore,

> ***Definition* 2: Discrete Random Variable**
> A discrete random variable is a real-valued funciton of th e outcome of the expression that can take a finit or countably infinite number of values.

2 Probability Mass Function

The most impartant way to characterize a random variable is through the probabilities of the values that it can take. For a *discrete random variable*, these are captured by the *probability mass function.*

> ***Definition* 3: Probability Mass Function**
> A discrete random variable has an associated probability mass function (PMF), which gives the probability of each numerical value that the random variable can take.

For a discrete random variable X, if x is any value that X can take, then the *probability mass* of x, denoted $p_X(x)$ is the probability of the event $\{X = x\}$. So, the PMF

$$p_X(x) = P(\{X = x\})$$

Note that $\sum_x p_X(x) = 1$

2.1 The Bernoulli Random Variable

A very simple variable that has only two possible outcomes: 1 and 0, which is

$$X = \begin{cases} 1, \text{when } ... \\ 0, \text{when } ... \end{cases}$$

Its PMF is

$$p_X(x) = \begin{cases} p, & \text{if } k = 1 \\ 1 - p, & \text{if } k = 0 \end{cases}$$

2.2 The Binomial Random Variable

Think of tossing a coin n times.

$$p_X(k) = P(X = k) = \binom{n}{k} p^{k(1-p)^{n-k}}, \quad k = 0, 1, ..., n$$

2.3 The Geometric Random Variable

The geometric random variable is the number X of trails needed for the first success.

2.4 The Poisson Random Variable

A Poisson random variable has a PMF given by $p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}$, where λ is a positive parameter characterizing the PMF.

When the parameters of binomial PMF n is very big and p is very small, the Poisson PMF is a good approximation for a binomial PMF.

3 Function of Random Variables

From a random variable, one can generate other random variables by doing some transformations.Like $Y = aX + b$. More generally, we can derive a random variable $Y = g(X)$ from the random variable X. And the PMF of Y $p_Y(y)$ can be calculated using the PMF of X:

$$p_Y(y) = \sum_{\{x \mid g(x)=y\}} p_X(x)$$

4 Expectation, Mean, And Variance

***Definition* 4: Expectation**

We define the **expected** value (also called the **expectation** or the **mean**) of a random variable X, with PMF p_X, by

$$E[X] = \sum_x x p_X(x)$$

4.1 Variance, Moments, and Expected Value Rule

Besides the mean, there are several other quantities that we can associate with a random variable and its PMF.

***Definition* 5: Moment**

The n^{th} moment of the random variable X is $E[X^n]$, the expected value of the variable X^n.

The most important quantity associated with a random variable X, other than the mean, is the variance, which is denoted by $\text{var}(X)$.

***Definition* 6: Variance**

The variance of a random variable X is $\text{var}(X) = E\left[(X - E[X])^2\right]$

The variance provides a measure of dispersion of X around its mean. Another measurement is through the **standard deviation** of X, which is the square root of the variance, denoted by $\sigma_X = \sqrt{\text{var}(X)}$

***Theorem* 1: Expceted Value Rule for Function of Random Variable**

Let X be a random variable with PMF p_X, and let $g(X)$ be a function of X. Then the expected value of the random variable $g(X)$ in given by

$$E[g(X)] = \sum_x g(x) p_X(x).$$

So the $\text{var}(X)$ can be calculated by $\sum_x (X - E[X])^2 p_X(x) = E[X^2] - (E[X])^2$

4.1.1 Properties of Mean and Variance

Using the expected value rule, we can derive some important properties of the mean and the variance. Starting with a random variable X, we define a random variable Y of the form $Y = aX + b$.

So the mean of the random variable is:

$$E[Y] = \sum_x (ax + b)p_X(x) = \sum_x axp_X(x) + \sum_x bp_X(x) = aE[X] + b$$

And the variance:

$$\begin{aligned}
\operatorname{var}(Y) &= \sum_x (Y - E[Y])^2 p_X(x) \\
&= \sum_x (ax + b - (aE[X] + b))^2 p_X(x) \\
&= \sum_x (ax - aE[x])^2 p_X(x) \\
&= a^2 \operatorname{var}(X)
\end{aligned}$$

***Theorem* 2: Mean and Variance of a Linear Function of a Random Variable**
Let X be a random variable and let $Y = aX + b$.
Then $E[Y] = aE[X] + b$, $\operatorname{var}(Y) = a^2 \operatorname{var}(X)$.

4.1.2 Mean and variance of Some Common Random Variables

Mean of poisson: $E[X] = \lambda$

5 Joint PMFs of Multiple Random Variables

***Definition* 7: Joint PMF**

The *joint PMF* of X and Y is $p_{X,Y}(x, y) = P(\{X = x\} \cup \{Y = y\})$.

The joint PMF determines the porbability of any event that can be specified in term of the random variables X and Y. For example if A is the set of all paris (x, y) that have a certain property, then $P((X, Y) \in A) = \sum_{(x,y)\in A} p_{X,Y}(x, y)$, and $p_X = \sum_y p_{X,Y}(x, y)$, $p_Y = \sum_x p_{X,Y}(x, y)$. Sometimes we refer to p_X, p_Y as *marginal PMFs*

5.1 Function of Multiple Random Variables

Let $g(X, Y)$ of the random variables X and Y defines another random variable $Z = g(X, Y)$, so we have

$$p_Z(z) = \sum_{\{(x,y) \mid g(x,y)=z\}} p_{X,Y}(x, y)$$

$$E[g(X, Y)] = \sum_x \sum_y g(x, y)p_{X,Y}(x, y)$$

5.2 More Than Two Random Variables

Similiar to section before, skipping…

Mean of the Binomial We know the binomial random variable X is the number of "success" in a repeated independent trails, calculate the mean of the variable.

Method 1: By definition

We know the PMF of the Binomial random variable, $p_X(x) = \binom{n}{x}p^x(1-p)^{n-x}$, so the mean should be $E[X] = \sum_x x p_X(x)$ which is a huge amount of calculation (there is a tricky way through, not covered here. Hint: using the sum of PMF = 1).

Method 2: Using combination of Bernoulli random variable

The Binomial random variable can be seen as the combination of Bernoulli random variables: $X = X_1 + X_2 + X_3 + \ldots + X_n$, so it's obvious that $E[X] = E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i] = np$

6 Conditioning

6.1 Conditioning a Random Variable on an Event

Definition **8: Conditional PMF on an event**

The conditional PMF of a random variable X, conditioned on a particular event A with $P(A) > 0$, is defined by

$$p_{X \mid A}(X) = P(X = x \mid A) = \frac{P(\{X = x\} \mid A)}{P(A)}$$

6.2 Conditioning one Random Variable on Another

Definition **9: Conditional PMF on a ramdom variable**

Let X and Y be two random variables associated with the same experiment. The conditional PMF $p_{X \mid Y}(x \mid y) = \frac{P(X=x,Y=y)}{P(Y=y)} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$

It is just an new notation which con be considered as $p_{X \mid A}(x)$ where A is the event $Y = y$.

6.3 Conditional Expcetation

Similiar to normal expcetion…

6.4 Independence

Definition **10: Independence from an Event**

We say that a random variable X is independent of the event A if

$$P(X = x \text{ and } A) = P(X = x)P(A) = p_X(x)P(A), \quad \text{for all } x$$

Definition **11: Independence of Random Variables**

We say two random variables X and Y are independent if

$$p_{X,Y} = p_X(x)p_Y(y), \quad \text{for all } x, y$$