

Sample Space and Probability

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1 Sets (Quick Review)

Set, element, empty set \emptyset , finite set, countably finite set, uncountable set, sub set, equal, universal set Ω

1.1 Set operations:

1. Complement of a set S , with respect to the universe Ω , denoted by S^c
2. Union of two sets S, T , $S \cup T$
3. Intersection of two sets S, T , $S \cap T$
4. Union of several, $\bigcup_{n=1}^{\infty} S_n = S_1 \cup S_2 \cup \dots$
5. Intersection of several, $\bigcap_{n=1}^{\infty} S_n = S_1 \cap S_2 \cap \dots$
6. Sets are Disjoint if they share no element
7. A collection of sets is a partition of set S , if they are disjoint and the union of them are S

1.2 The Algebra of Sets:

De Morgan's laws:

$$\left(\bigcup_n S_n\right)^c = \bigcap_n S_n^c$$

$$\left(\bigcap_n S_n\right)^c = \bigcup_n S_n^c$$

2 Probabilistic Models

Elements of a Probabilistic Model

- The sample space Ω , the set of all possible outcomes
- The probability law, which assigns any event A a non-negative number $P(A)$

*

2.1 Choosing an Appropriate Sample Space

The element of the sample space should be distinct and *mutually exclusive*, and the sample space should be collectively exhaustive.

2.2 Probability Axioms

1. **(Nonnegativity)** $P(A) \geq 0$, for every event A
2. **(Additivity)** A, B are disjoint, then $P(A \cup B) = P(A) + P(B)$
3. **(Normalization)** $P(\Omega) = 1$

2.3 Discrete Models

e.g. **The toss of a coin several times** Like $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ (3 times) and the probability stuff

Discrete Probability Law

The sample space $S = \{s_1, s_2, s_3, \dots, s_n\}$ consists of finite number of elements, we have:

$$P(S) = P(\{s_1, s_2, s_3, \dots, s_n\}) = P(s_1) + P(s_2) + P(s_3) + \dots + P(s_n)$$

Discrete Uniform Probability Law

If the outcomes are equally likely, then the Probability of any single outcome A becomes:

$$P(A) = \frac{\text{number of elements of } A}{n}$$

2.4 Continuous Models

Like throwing a dart on a certain area or sth else ...

2.5 Properties of Probability Laws

1. If $A \subseteq B$, then $P(A) \leq P(B)$
2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
3. $P(A \cup B) \leq P(A) + P(B)$
4. $P(A \cup B \cup C) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$

*Insight of Probability: The term “probability” should come with an event, like the probability of event A $P(A)$, which is further a outcome of the probability law and a part of the probabilistic model. And a valid probabilistic model should contain a sample space and a probability law which agree with the probability axioms.

3 Conditional Probability

Conditional probability provides us with a way to reason about the outcome of an experiment, based on **partial information**. (The experiment is done and the only have some partial information about it.)

e.g. The experiment involving two successive rolls of a die, you are told that the sum of the two rolls are 9. What's the probability of the first roll is a 6?

In precise terms, the conditional probability is when we know the is with in a given event B , we wish to know the probability of the event A . We call this *conditional probability of A given B* , denoted by $P(A | B)$

Definition_{conditional probability}: $P(A | B) = \frac{P(A \cap B)}{P(B)}$

3.1 Verification of the Probability Laws

1. Nonnegativity is clear since the original probability is nonnegative.
2. Additivity:

$$\begin{aligned} P(A_1 \cup A_2 | B) &= \frac{P((A_1 \cup A_2) \cap B)}{P(B)} \\ &= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)} \\ &= P(A_1 | B) + P(A_2 | B) \end{aligned}$$

1. Normalization:

$$P(\Omega | B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

3.2 Properties of Conditional Probability

- Conditional probability can be viewed as a normal probability on a new universe B .
- Furthermore, if all outcomes are equally likely, then $P(A | B) = \frac{\text{num of elements of } A \cap B}{\text{num of elements of } B}$

3.3 Using Conditional Probability for Modeling

A restatement of the definition of the conditional probability is $P(A \cup B) = P(B)P(A | B)$, which can be used to calculate a non-conditional probability.

Multiplication Rule

By definition, it's easy to get

$$P\left(\bigcup_{i=1}^n A_i\right) = P(A_1)P(A_2 | A_1)P(A_3 | A_1 \cap A_2) \dots P\left(A_n | \bigcup_{i=1}^{n-1} A_i\right)$$

3.4 Total Probability Theorem and Bayes' Rule