Princeton Competitive Programming

Dynamic Programming II

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Outline

Recap of Lecture I

What is DP?

Algorithm design technique based on breaking problems into simpler subproblems

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Finding the right subproblem break down is part art part science

Can only be learned by looking at lots of examples

Warm Up DP

Longest Increasing Subsequence

Problem

Given an array $a_1, \ldots a_n$ with n elements, find the length of the longest increasing subsequence.

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Given an array $a_1, \ldots a_n$ with n elements, find the length of the longest increasing subsequence.

- $a = 1, 2, 3$ then $\text{LIS}(a) = 3$, the whole sequence is increasing
- $a = 1, 2, 1, 4, 3, 4$ then $\text{LIS}(a) = 4$, take $[1, 2, 3, 4]$
- $a = 5, 4, 3, 2, 1$ then $\text{LIS}(a) = 1$, take $[1]$

Longest Increasing Subsequence

First possible state:

dp(i) is the longest increasing subsequence within $a_1, \ldots, a_i$

Longest Increasing Subsequence

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> $\mathrm{dp}(i)$ is the longest increasing subsequence within $a_1, \ldots, a_i$

It's not clear how to write a recursion on this...

Longest Increasing Subsequence

Another suggestion:

dp(i) is the longest increasing subsequence within $a_1, \ldots, a_i$ that ends at a_i

Longest Increasing Subsequence

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But now:

$$\mathrm{dp}(i) = \max(1, \max_{\substack{j<i \\ a_j<a[i]}} (1 + \mathrm{dp}(j)))$$

```java
public static int lis(int i) {
  if (dp[i] != -1) {
    return dp[i];
  }

  int res = 1;
  for (int j = 0; j < i; j++) {
    if (a[j] < a[i]) {
      res = max(res, 1 + lis(j));
    }
  }

  return dp[i] = res;
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int mx = 0;
for (int i = 0; i < n; i++)
  mx = Math.max(mx, lis(i));
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What is the time complexity?

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What is the time complexity? It's $O(n^2)$.

DP on trees

Tree Diameter

Problem

Given a tree T of n nodes, calculate longest path between any two nodes

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First things first: root the tree at an arbitrary node

Now note:

- The longest path can be entirely on one of the root's subtrees
- The longest path can start in a subtree of root, go through root and end at a different subtree

Tree Diameter

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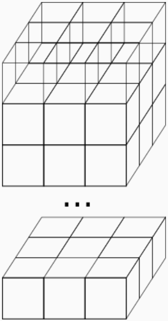
$$\begin{aligned}\text{longest}(v) = \max(&\max_{u \text{ child of } v} \text{longest}(v), \\ &1 + \max_{u \text{ child of } v} \text{height}(v) + \text{second} \max_{u \text{ child of } v} \text{height}(v))\end{aligned}$$

The Twin Tower

The Twin Tower

Problem

Consider a tower of $3 \times 3 \times n$ blocks, count how many ways there are to tile it using $1 \times 1 \times 2$ blocks.



The Twin Tower

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- Key idea: when we fill the nth layer we might fill some of the $(n-1)$th layer, so we define a state like: $\mathrm{dp}(n, S)$ is the number of ways to tile a $3 \times 3 \times n$ tower where the tiles in S are prefilled

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But wait, don't we have to handle a huge number of cases?

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- $\texttt{b}\&(\texttt{1} << \texttt{i})$ is 0 if $i \notin S$ and positive otherwise
- $\texttt{b}|(\texttt{1} << \texttt{i})$ adds i to S, so it is $S \cup \{i\}$

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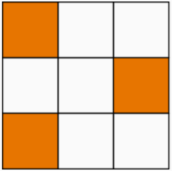
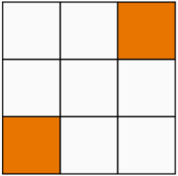


$$= \begin{matrix}100\\001\\100\end{matrix} = 100001100 = 268$$



$$= \begin{matrix}001\\000\\100\end{matrix} = 001000100 = 68$$

We represent 3×3 tiles as a bitmask like so

Useful functions to use:

```
public static int isNotPresent(int x, int y, int mask) {
  return (mask & (1 << (y * 3 + x))) == 0;
}

public static int add(int x, int y, int mask) {
  return (mask | (1 << (y * 3 + x)));
}

public static int remove(int x, int y, int mask) {
  return (mask ^ (1 << (y * 3 + x)));
}
```

```java
public static void fill(int maskN, int maskN1, int origMask) {
    if (maskN == (1 << 9) - 1) {
        transition[origMask][maskN1]++;
        return;
    }

    int x = 0, y = 0;
    // find first unused on last layer
    for (int i = 0; i < 3; i++)
        for (int j = 0; j < 3; j++)
            if (isNotPresent(i, j, maskN)) {
                x = i;
                y = j;
                break;
            }

    maskN = add(x, y, maskN);
    // horizontal
    if (x > 0 && isNotPresent(x - 1, y, maskN))
        fill(add(x - 1, y, maskN), maskN1, origMask);
    // vertical
    if (y < 2 && isNotPresent(x, y + 1, maskN))
        fill(add(x, y + 1, maskN), maskN1, origMask);
    // down
    if (isNotPresent(x, y, maskN1))
        fill(maskN, add(x, y, maskN1), origMask);
}
```

```java
public static int calc(int n, int mask) {
    //
    // Base Case
    // ?????????
    //

    if (dp[n][mask] != -1)
        return dp[n][mask];

    int res = 0;
    for (int i = 0; i < (1 << 9); i++) {
        if (transition[mask][i] == 0) continue;
        res += transition[mask][i] * calc(n - 1, i);
    }

    return dp[n][mask] = res;
}
```

```java
public static int calc(int n, int mask) {
    if (n == 0 && mask == 0)
        return 1;
    if (n == 0)
        return 0;

    if (dp[n][mask] != -1)
        return dp[n][mask];

    int res = 0;
    for (int i = 0; i < (1 << 9); i++) {
        if (transition[mask][i] == 0) continue;
        res += transition[mask][i] * calc(n - 1, i);
    }

    return dp[n][mask] = res;
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