HIGHER TRANSCENDENTAL FUNCTIONS

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For Singularity.

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (n^2 - x^2)y = 0$$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(\frac{n^2-x^2}{x^2}\right)y = 0$$

Infinite Product

$$P_n = \prod_{k=1}^{n}(1+a_k)$$

$$= (1+a_1)(1+a_2)(1+a_3)\cdots(1+a_n)$$

If $n \to \infty$, Infinite Product.

Any term $= 0$

Product $= 0 \Rightarrow$ Divi. (Divergence)

Product $= \infty \Rightarrow$ Div.

Converging to P.

$$P = P_n = \prod_{k=1}^{n}(1+a_k)$$

$\lim n \to \infty$

$$\rightarrow \frac{\lim_{n\to\infty}\prod_{k=1}^{n}(1+a_k)}{\lim_{n\to\infty}\prod_{k=1}^{n}(1+a_{k-1})} = \frac{\cancel{P}}{\cancel{P}} = 1$$

$$\Rightarrow \lim_{n\to\infty}(1+a_n) = 1 \Rightarrow \lim_{n\to\infty} a_n = 0.$$

$\prod_{k=1}^{n} (1+a_k) =$

$\prod_{k=1}^{n-1} (1+a_k) =$

$\rightarrow \dfrac{(1+\cancel{a_1})(1+\cancel{a_2})(1+\cancel{a_3}) \cdots (1+\cancel{a_{n-1}})(1+a_n)}{(1+a_0)(1+\cancel{a_1})(1+\cancel{a_2}) \cdots (1+\cancel{a_{n-1}})}$

$= \lim_{n\to\infty} \left(\dfrac{1+a_n}{1+a_0}\right)$

$= \lim_{n\to\infty} \cancel{1}+a_n = \cancel{1}+a_0$

$= \lim a_n = a_0$

$\Rightarrow 0 = a_0$

$\Rightarrow \boxed{a_0 = 0}$

$\rightarrow \dfrac{(1+\cancel{a_1})(1+\cancel{a_2}) \cdots (1+\cancel{a_{n-1}})(1+a_n)}{(1+\cancel{a_1})(1+\cancel{a_2}) \cdots (1+\cancel{a_{n-1}})}$

7/1 Let $P_n = \prod_{k=1}^{n} (1+a_k)$

$S_n = \sum_{k=1}^{n} \log(1+a_k)$

Now, $\exp S_n = P_n$

$\lim_{n\to\infty} \exp S_n = \exp \lim_{n\to\infty} S_n$

Note: P_n can not $\longrightarrow 0$.

auxiliary series = $\frac{1}{n^2}$

Ex Show that $\prod_{n=1}^{\infty}\left[1+\frac{1}{(n+1)(n+3)}\right]$ is conv. & find its value

Hints: $a_n = \frac{1}{(n+1)(n+3)}$. $\prod_{n=1}^{\infty}(1+a_n)$

Auxiliary $\frac{1}{n^2} \longrightarrow$ conv.

$\sum 1/n^p$, Conv, $p>1$
Div, $p \leq 1$

$$P_n = \prod_{k=1}^{n}\left[1+\frac{1}{(k+1)(k+3)}\right]$$

$$= \prod_{k=1}^{n}\left[\frac{k^2+3k+k+3+1}{(k+1)(k+3)}\right]$$

$$= \prod_{k=1}^{n}\left[\frac{k^2+4k+4}{(k+1)(k+3)}\right].$$

$$= \prod_{k=1}^{n}\left[\frac{(k+2)^2}{(k+1)(k+3)}\right]$$

$$= \frac{3^2 . 4^2 . 5^2 \cdots (n+2)^2}{2.4.3.5 \cdots (n+1)(n+3)}$$

$$= \frac{[3.4.5 \cdots (n+2)]^2}{2.4.3.5 \cdots (n+1)(n+3)}$$

$$= \frac{n+2}{2} \cdot \frac{3}{n+3} \qquad \lim_{n\to\infty}\left[\frac{n+2}{2} \cdot \frac{3}{n+3}\right] = 3/2$$

Ex $\prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right)$ is div.

$$= \prod_{k=2}^{n}\left(\frac{k-1}{k}\right) = \frac{(2-1)}{2}\cdot\frac{(3-1)}{3}\cdot\frac{(4-1)}{4}\cdots\left[\frac{(n-1)-1}{(n-1)}\right]\left[\frac{n-1}{n}\right]$$

$$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{4}}\cdot\frac{\cancel{4}}{5}\cdots\frac{(n-1)}{n}$$

$$= \frac{1}{n}$$

On putting $\lim_{n\to\infty}\frac{1}{n} = 0$

= Diverges to zero.

Ex $\prod_{n=1}^{\infty}\exp\left(\frac{1}{n}\right)$

$$\prod_{k=1}^{n}\exp\left(\frac{1}{k}\right) = \prod_{k=1}^{n} e^{1/k}$$

harmonic series

$$= e^{(1+1/2+1/3+\cdots+1/n)}$$

$$\lim_{n\to\infty} e^{\bullet} = 1$$

$\Rightarrow$ Div.

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Theorem: If there exists +ve constant M_n, Such that $\sum_{n=1}^{\infty} M_n$ is conv. and $|a_n(z)| < M_n$, $\forall z$, in closed region R then the product $\prod_{n=1}^{\infty} [1 + a_n(z)]$ is uniform conv. in R.

Proof: As given that $\sum M_n$ is conv. $M_n > 0$.

$$\Rightarrow \prod_{n=1}^{\infty} (1 + M_n) \text{ is Conv.}$$

and $\lim_{n\to\infty} \prod_{k=1}^{n} (1 + M_n)$ exists.

for $\epsilon > 0$, there exist n_0 (no.) Such that

$$\prod_{k=1}^{n_0+p} (1 + M_k) - \prod_{k=1}^{n_0} (1 + M_k) < \epsilon \text{ where } p > 0.$$

Since $|a_k(z)| < M_k$,

Therefore, $$\left|\prod_{k=1}^{n_0+p} \left(1 + a_k(z)\right) - \prod_{k=1}^{n_0} \left(1 + a_k(z)\right)\right|$$

$$= \left|\prod_{k=1}^{n_0} (1 + a_k(z))\right| \left|\prod_{k=n_0+1}^{n_0+p} (1 + a_k(z)) - 1\right|$$

$$\leq \prod_{k=1}^{n_0} (1 + M_k) \left[\prod_{k=n_0+1}^{n_0+p} [1 + M_k] - 1\right] < \epsilon$$

Hence proved.

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Ex Show that the product $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots$ conv. to ½

Solⁿ $P_n = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\ldots$??

$= \left(\frac{2^2-1}{2^2}\right)\left(\frac{3^2-1}{3^2}\right)\left(\frac{4^2-1}{4^2}\right)\ldots\left(\frac{(n+1)^2-1}{(n+1)^2}\right)$

$= \frac{1.3}{2^2}\ \frac{2.4}{3^2}\ \frac{3.5}{4^2}\ \ldots\ \frac{(n+1-1)(n+1+1)}{(n+1)^2}$

$= \frac{1}{2}\left[\frac{1.3}{2}\cdot\frac{2.4}{3^2}\ \frac{3.5}{4^2}\ \ldots\ \frac{n(n+2)}{(n+1)^2}\right]$

$\lim_{n\to\infty} P_n = \frac{1}{2}$

Ex $\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\ldots\ldots$

Case: I (n is odd)

$P_n = \left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdot\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\ldots\left(1+\frac{1}{n+1}\right)$

$P_n = \frac{3}{2}\cdot\frac{2}{3}\cdot\frac{5}{4}\cdot\frac{4}{5}\ldots\left(\frac{n+2}{n+1}\right)$

$\lim_{n\to\infty} P_n = 1.$

Case II: (n is even).

$P_n = \left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\ldots\left(1+\frac{1}{n}\right)\left(1-\frac{1}{n}\right.$

$= \frac{3}{2} \cdot \frac{2}{3} \cdot \frac{5}{4} \cdot \frac{4}{5} \cdots\cdots \left(\frac{n+1}{n}\right)\left(\frac{n+1-1}{n+1}\right)$

$\lim_{n\to\infty} P_n = 1.$

Ex Examine the product

$e^{1+1} \cdot e^{-1-1/2} \cdot e^{1+1/3} \cdot e^{-1-1/4} \cdots\cdots \infty$

Case: I (as per n is even and odd)

$P_n = e^{1+1}\, e^{-1-1/2}\, e^{1+1/3}\, e^{-1-1/4} \cdots\cdots e^{-1-1/n}$

$\log P_n = (1+1)\log e + \left(-1-\frac{1}{2}\right)\log e + \left(1+\frac{1}{3}\right)\log e + \left(-1-\frac{1}{4}\right)\log e + \cdots$

$= (1+1) + \left(-1 - \frac{1}{2}\right) + \left(1+\frac{1}{3}\right) + \left(-1-\frac{1}{4}\right) + \cdots$

$= (1-1+1-1+1\cdots) \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots\right)$

$= 0 + \log_e 2$

$= 1 + \log e^2$

$\Rightarrow$ Not conv.

$\Rightarrow$ Oscillating. Series.

Date:

Ex. Discuss the convergence of the unif. product $\prod_{n=1}^{\infty} \frac{z+z^{2n}}{1+z^{2n}}$

Sol^n Since $1+a_n(z) = \frac{z+z^{2n}}{1+z^{2n}}$

$$a_n(z) = \frac{z+z^{2n}}{1+z^{2n}} - 1 = \frac{z+\cancel{z^{2n}}-1-\cancel{z^{2n}}}{1+z^{2n}} = \frac{z-1}{1+z^{2n}}$$

Here, note that $\sum z^{-2n}$ is convergent for $|z|>1$

$$\Rightarrow |a_n(z)| = \left|\frac{z-1}{1+z^{2n}}\right| < \left|\frac{1}{z^{2n}}\right|$$

Therefore, $|a_n(z)|$ is absolute conv.

for $|z|<1$

$$\lim_{n\to\infty} [1+a_n(z)] = \lim_{n\to\infty} \frac{z[1+z^{2n-1}]}{1+z^{2n}} = z$$

$\Rightarrow$ The product as $a_n(z)$ does not $\longrightarrow 0$ as $n\to\infty$.

$\Rightarrow$ Div.

for $z=1$

$$|a_n(z)| = \left|\frac{z-1}{1+z^{2n}}\right| \geq \left|\frac{|z|-1}{1+z^{-2n}}\right|$$

$0 \geq 0 \qquad = 0$

$\Rightarrow$ Div.

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(a) $\log\left(1-\frac{1}{k+1}\right) = -\left[\frac{1}{(k+1)}+\frac{1}{(k+1)^2}\cdot\frac{1}{2}+\frac{1}{(k+1)^3}\cdot\frac{1}{3}\cdots\right]$

Euler or Mescheroni Constant (γ):

$$\gamma = \lim_{n\to\infty}(H_n - \log n) \quad \text{where } H_n = \sum_{k=1}^{n}\frac{1}{k}$$

Let $A_n = H_n - \log n$

$A_{n+1} = H_{n+1} - \log(n+1)$

Now, $A_{n+1} - A_n = (H_{n+1} - H_n) - (\log(n+1) - \log n)$

$$= \frac{1}{1+n} + \log\left(\frac{n}{n+1}\right)$$

$$= \frac{1}{1+n} + \log\left(1 - \frac{1}{n+1}\right)$$

$$= -\sum_{k=1}^{n}\frac{1}{(k+1)(n+1)^{k+1}} < 0$$

Note: Let $\frac{1}{k} \le \int_{k-1}^{k}\frac{dt}{t} < \frac{1}{k-1}$

$$\frac{1}{k} \le \int_1^2\frac{dt}{t} + \int_2^3\frac{dt}{t} + \cdots \int_{n-1}^{n}\frac{dt}{t} < \frac{1}{k-1}$$

$$(H_n - 1) \le \int_1^2\frac{dt}{t} + \int_2^3\frac{dt}{t} + \cdots \int_{n-1}^{n}\frac{dt}{t} < H_{n-1}$$

$$\Rightarrow (H_n - 1) \le \log n < H_{n-1}$$

$$-1 \le \log n - H_n < \left(\frac{-1}{n}\right)$$

$$-1 \le -H_n + \log n < 1/n$$

A_n lies b/w 0 and 1

$n=2$ | 1 $\Rightarrow A_n > 1/n$ $\Rightarrow$

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$\frac{1}{\Gamma -1}$ - not define $\Gamma 0$ = no definite →Singularities ?

finite no of terms = pole

or $\gamma > 0$ (non-negative value)

Gamma function: Weierstrass

check $\frac{1}{\Gamma z} = e^{-\gamma z} \prod \left(1 + z/n\right) e^{-z/n}$

$$\Gamma z = \int_0^\infty e^{-t} t^{z-1} dt \quad Re(z) \geq 0$$

(i) Analytic for all finite value of z.

(ii) Zeros are simple at $z = 0, -1, -2, -3, \ldots$

Essential Singularity : at $z = \infty$.

$\frac{1}{\Gamma z}$ is never zero. because $\frac{1}{\Gamma z}$ has no pole

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$$\frac{1}{\Gamma z} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) \exp\left(-\frac{z}{n}\right) \quad (1)$$

On putting z = 1 in (1)

$$\frac{1}{\Gamma 1} = e^{\gamma} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) \exp\left(-\frac{1}{n}\right)$$

$$= e^{\gamma} \lim_{n \to \infty} \prod_{K=1}^{n} \left(1 + 1/K\right) \exp\left(\frac{-1}{K}\right)$$

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$\rightarrow \frac{k+1}{1} = \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{n}{(n-1)}\left(\frac{n+1}{n}\right) = (n+1)$ | $e^{-1} e^{-1/2} e^{-1/3}\cdots$

$H_n = \sum_{k=1}^{n} 1/k.$ | $\rightarrow e^{-(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots)} \rightarrow H_n$

$\gamma = H_n - \log n$

or $e^{\gamma} \lim_{n\to\infty} \prod_{k=1}^{n}\left(\frac{k+1}{k}\right)\exp\left(-\frac{1}{k}\right)$ | $-\log n = \log\frac{1}{n}$

$= e^{\gamma} \lim_{n\to\infty} (n+1)\exp\left(-\frac{1}{n}\right)$

or $e^{\gamma} \lim_{n\to\infty} (n+1)\exp\left[-\gamma - \log n\right]$

$\cancel{e^{\gamma}} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\cancel{e^{-\gamma}} = 1$ Here $\Gamma 1 = 1$

take log in eq①

We can write ① as,

$$\log\Gamma z = \log z + \gamma z + \left[\sum_{n=1}^{\infty}\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right] \quad — ②$$

On differentiating ②, w.r.t "z", we get

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma - \left[\sum_{n=1}^{\infty}\frac{1}{(1+z/n)}\cdot\frac{1}{n} - \frac{1}{n}\right]$$

$\frac{1}{(n+2)} \cdot \frac{1}{n} - \frac{1}{n}$

$\frac{n-(n+2)}{(1+2)n}$

$\frac{-z}{(1+z)n}$

$$\text{or } \frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma + \left[\sum_{n=1}^{\infty}\frac{z}{n(z+n)}\right] \quad — ③$$

On putting $z=1$, in ③, this gives.

$$\frac{\Gamma' 1}{\Gamma 1} = -1 - \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= -1 - \gamma + \lim_{n\to\infty}\left[1 - \frac{1}{n+1}\right]$$

$\left[1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} + \cdots = 1\right]$

$\Gamma' 1 = -\gamma$

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<u>Note:</u> $$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) \exp\left(\frac{-z}{n}\right) \quad \text{—— (1)}$$

From (1), we have

$$z\Gamma(z) = \frac{\exp(-\gamma z)}{\prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) \exp\left(-z/n\right)}$$

$$z\Gamma(z) = \exp(-\gamma z) \lim_{n\to\infty} \prod_{k=1}^{n} \left(1+\frac{z}{k}\right)^{-1} \exp\left(\frac{z}{k}\right) \quad \text{—— (2)}$$

Since $\gamma = \lim_{n\to\infty} \left[H_n - \log n\right] = \lim_{n\to\infty} \left[H_n - \log(n+1)\right]$

$$\gamma = \lim_{n\to\infty} \left[H_n - \sum_{k=1}^{n} \log\left(\frac{k+1}{k}\right)\right]$$

Therefore, $\exp(-\gamma z) = \left[\exp\left\{\lim_{n\to\infty}\left\{-zH_n + z\sum_{k=1}^{n} \log\left(\frac{k+1}{k}\right)\right\}\right\}\right]$

$$= \lim_{n\to\infty} \left[\exp\left(\frac{-z}{k}\right)\left(\frac{k+1}{k}\right)^{z}\right]$$

$$= \lim_{n\to\infty} \prod_{k=1}^{n} \left(\frac{k+1}{k}\right)^{z} \exp\left(\frac{-z}{k}\right) \quad \text{—— (3)}$$

From (2) and (3), we get,

$$z\Gamma(z) = \lim_{n\to\infty} \prod_{k=1}^{n} \left(1+\frac{z}{k}\right)^{-1} \cancel{\exp\left(\frac{z}{k}\right)} \cdot \left(\frac{k+1}{k}\right)^{z} \cancel{\exp\left(-\frac{z}{k}\right)}$$

Date :

From (1), We have

$$z\Gamma(z) = \lim_{n\to\infty} \prod_{k=1}^{n} \left(\frac{k+1}{k}\right)^{z} \left(1+\frac{z}{k}\right)^{-1}$$

$$z\Gamma(z) = \lim_{n\to\infty} \frac{(n+1)^{z}\,[1.2.3.4.5.6\ldots n]}{(z+1)(z+2)(z+3)\ldots(z+n)}$$

$$z\Gamma(z) = \lim_{n\to\infty} \frac{\lfloor n\,.(n+1)^{z}\ldots}{(z+1)(z+2)(z+3)\ldots(z+n)}$$

<u>or</u> $$\Gamma(z) = \frac{\lfloor(n-1)\,n^{z}}{z(z+1)(z+2)\ldots(z+n-1)}$$

$$= \frac{\lfloor(n-1)\,n^{z}}{z(z+1)(z+2)\ldots(z+n-1)}$$

$z(z+1)(z+2)(z+3)\ldots(z+n-1) = (z)_n \rightarrow$ Factorial function.

$$\lfloor(n) = (1)_n = 1.2.3\ldots n$$
$$(1+0)(1+1)(1+2)\ldots(1+n-1)$$
$$z(z+1)(z+2)\ldots(z+n-1)$$

15/1 $$\Gamma(z+1) = z\Gamma(z)$$

Since $$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\exp\left(\frac{-z}{n}\right)$$

We have

$$\frac{\Gamma(z+1)}{\Gamma(z)} = \frac{z e^{\gamma z} \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\exp\left(\frac{-z}{n}\right)}{(z+1)\,e^{\gamma(z+1)} \prod_{n=1}^{\infty}\left(1+\frac{z+1}{n}\right)\exp\left(-\frac{z+1}{n}\right)}$$

$\exp(\gamma) = e^{-1/n}$ $\quad (1+1/n)^n \quad e^{x} = (1+\frac{1}{x})^{x}$

$\gamma = H_n - \log n$ $\quad e^{\gamma} = \lim_{n\to\infty} (1+\frac{1}{n})^{n} = e^{1}$

$$= \left(\frac{z}{z+1}\right) \frac{1}{e^{\gamma}} \frac{\prod_{n=1}^{\infty} \left(\frac{n+z}{n}\right)}{\prod_{n=1}^{\infty} \left(\frac{n+z+1}{n}\right) \operatorname{Exp}(-1/n)}$$

$$= \left(\frac{z}{z+1}\right) \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)\left(1+\frac{z+1}{n}\right)^{-1}\left(1+\frac{1}{n}\right)$$

$$\gamma = H_n - \log n$$
$$e^{\gamma} = e^{(H_n - \log n)}$$
$$e^{-\gamma} = e^{-(H_n + \log n)}$$

$$1 = \frac{e^{H_n} e^{-\log n}}{e^{\gamma}}$$
$$=$$

$$e^{-\log n} = \left(1+1/n\right)$$

$$= \frac{z}{z+1} \lim_{n\to\infty} \prod_{k=1}^{n} \cdot \left(\frac{k+1}{k}\right) \frac{(k+z)}{(1+z+k)}$$

$$= \frac{z}{z+1} \lim_{n\to\infty} \left(\frac{n+1}{1}\right) \frac{(1+z)}{(n+z+1)}$$

$$= z$$

$$\frac{\Gamma(z+1)}{\Gamma(z)} = z$$

$$\Rightarrow \Gamma(z+1) = z\,\Gamma(z)$$

Since $\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) \exp\left(\frac{-z}{n}\right)$

If $z = m \in I^{+}$ then

$$\Gamma(m+1) = \underset{\substack{\uparrow \\ \text{factorial}}}{\lfloor m}$$

Note: If $\lim_{z \to c} \frac{f(z)}{g(z)} = 0$ in region R of complex plane

$\Rightarrow f(z) = o[g(z)]$,

$\uparrow$ small o (order of)

& $\lim_{z \to c} \frac{f(z)}{g(z)}$ = constant or bounded

then $f(z) = O[g(z)]$.

$\uparrow$ Big O

Since $\frac{1}{\Gamma z} = z e^{\gamma z}$

$\lim_{z \to 0} \frac{\sin^2 z}{z} = 0 \Rightarrow \sin^2 z = o(z)$

$\lim_{z \to 0} \frac{\sin z}{z} \Rightarrow \sin z = O(z)$

If x is Real and $x \to 0$ $|\cos x| \le 1$
then $\cos x - 4x \Rightarrow O(x)$; $x \to 0$ in $x \in R$.

MIMP 18 Marks

Prove that $\Gamma z = \int_0^\infty e^{t} \cdot t^{z-1} \, dt$, $Re(z) > 0$.

Lemma (1): If $0 \le \alpha < 1$, then $(1+\alpha) \le \exp(\alpha) \le (1-\alpha)^{-1}$

Proof: Let $\alpha = 1+\alpha - (1)$, $\exp(\alpha) = 1+\alpha+\sum_{n=2}^{\infty} \frac{\alpha^n}{\lfloor n}$

$= (1+\alpha) \le \exp(\alpha)$ — (2)

$(1-\alpha)^{-1} = (1+1\alpha + \sum_{n=2}^{\infty} \alpha^n$ — (3)

From ①, ② & ③
we get,
$$(1+\alpha) \le \exp(\alpha) < (1-\alpha)^{-1}$$

Lemma 2 If $0 \le \alpha < 1$, then
$$(1-\alpha)^n \ge 1 - n\alpha, \; n \in I^+$$

proof: for $n = 1$
$$(1-\alpha)^1 = 1 - 1.\alpha$$

for $n = k$
$$(1-\alpha)^k \ge 1 - k.\alpha$$

$$(1-\alpha)^{k+1} \ge (1-k\alpha)(1-\alpha)$$
$$= 1-(k+1)\alpha + k\alpha^2$$ neglecting higher terms

$$(1-\alpha)^{k+1} \ge (1-(k+1)\alpha)$$

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Lemma 1: If $0 \le \alpha < 1$, then $1+\alpha \le \exp(\alpha) \le (1-\alpha)^{-1}$ — ①

Lemma 2: If $0 \le \alpha < 1$, then $(1-\alpha)^n \ge 1 - n\alpha$, $n \in I^+$ — ②

Lemma 3: If $0 \le t < n$, $n \in I^+$, then $0 \le e^{-t} - (1 - t/n)^n \le \frac{t^2}{n}$ — ③*

proof: On putting $\alpha = t/n$ in lemma I, we get
$$\left(1 + t/n\right) \le \exp\left(t/n\right) \le \left(1 - \alpha/n\right)^{-1} \quad \text{— ③}$$

$e^{\frac{t}{n}\cdot n}$

$$\Rightarrow \left(1+\frac{t}{n}\right)^{n} \le \exp(t) \le \left(1-\frac{t}{n}\right)^{-n} \quad \text{---} \ (4)$$

or $$\left(1+\frac{t}{n}\right)^{-n} \ge \exp(-t) \ge \left(1-\frac{t}{n}\right)^{n} \quad \text{---} \ (5)$$

From (5), $e^{-t} - \left(1-\frac{t}{n}\right)^{n} \ge 0$.

$$\Rightarrow e^{-t} - \left(1-\frac{t}{n}\right)^{n} = e^{-t}\left[1 - e^{t}\left(1-\frac{t}{n}\right)^{n}\right] \quad \text{---} \ (6)$$

From (4), $e^{t} \ge \left(1+\frac{t}{n}\right)^{n}$ --- (7)

From (6) & (7), we get,

$$e^{-t} - \left(1-\frac{t}{n}\right)^{n} \le e^{-t}\left[1-\left(1-\frac{t^2}{n^2}\right)^{n}\right] \quad \text{---} \ (8)$$

From Lemma (2) and (8), we get,

$$e^{-t} - \left(1-\frac{t}{n}\right)^{n} \le e^{-t}\left[\frac{t^2}{n}\right]$$

$$\Rightarrow 0 \le e^{-t} - \left(1-\frac{t}{n}\right)^{n} \le \frac{e^{-t}\cdot t^2}{n} \longrightarrow \text{Lemma-(3)}^{*}$$

Lemma-4 $$\Gamma z = \lim_{n\to\infty} \int_0^n \left(1-\frac{t}{n}\right)^{n} t^{z-1}\, dt, \text{ where } n \in I^{+} \quad \text{---} \ (4)$$

proof. On putting $t = n\beta$ is in (4) $\Rightarrow dt = n\, d\beta$ then

On Solving $\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\, dt$, by putting $t = n\beta$,

we get $n^{z} \int_0^1 (1-\beta)^{n} \beta^{z-1}\, d\beta$.

We get, $\frac{n}{z}\int_0^1 (1-\beta)^{n-1}\cdot \beta^z \, d\beta$.

$\vdots$

$$= \frac{n(n-1)(n-2)(n-3)\dots}{z(z+1)(z+2)\dots(z+n-1)}\int_0^1 \beta^{z+n-1}\, d\beta.$$

Finally we get

$$\lim_{n\to\infty} \frac{\lfloor n \; n^z}{z(z+1)(z+2)(z+3)\dots(z+n)} = \Gamma z$$

Now, we prove, $\Gamma z = \int_0^\infty e^{-t}\cdot t^{z-1}\, dt$, $Re(z) > 0$

$$\Rightarrow \int_0^\infty e^{-t}\cdot t^{z-1}\, dt - \Gamma z = \lim_{n\to\infty}\left[\int_0^\infty e^{-t}\, t^{z-1}\, dt - \int_0^n \left(1-\tfrac{t}{n}\right)^n t^{z-1}\, dt\right]$$

(on using lemma 4)

$$= \lim_{n\to\infty}\left[\int_0^n \left\{e^{-t} - \left(1-\tfrac{t}{n}\right)^n\cdot t^{z-1}\, dt\right\} + \int_n^\infty e^{-t}\cdot t^{z-1}\, dt\right]$$

Here $\lim_{n\to\infty} \int_n^\infty e^{-t}\, t^{z-1}\, dt \to 0$.

$$\Rightarrow \int_0^\infty e^{-t}\, t^{z-1}\, dt - \Gamma z.$$

$$= \lim_{n\to\infty}\left[\int_0^n \left[e^{-t} - \left(1-\tfrac{t}{n}\right)^n\right] t^{z-1}\, dt + 0\right.$$

On solving

$$\left|\int_0^\infty \left[e^{-t} - \left(1-\frac{t}{n}\right)^n\right] t^{z-1}\, dt\right| \le \int_0^n \frac{t^2 e^{-t}}{n}\, t^{Re(z)-1}\, dt$$

From lemma (4)

Since $|t^z| = t^{Re(z)}$

Since $\int_0^n \frac{e^{-t}}{n}\, t^{Re(z)+1}\, dt$ is bounded &

$$\lim_{n\to\infty} \int_0^n \frac{e^{-t}}{n}\, t^{Re(z)+1}\, dt \to 0$$

$$\boxed{\Rightarrow \int_0^\infty e^{-t}\, t^{z-1}\, dt = \Gamma z}$$

20) $\Gamma n = \int_0^\infty e^{-t}\, t^{n-1}\, dt$ — (i)

put $t = r^2$, $dt = 2r\,dr$.

$$\Gamma n = 2\int_0^\infty e^{-r^2}\, r^{2n-1}\, dr \quad \text{— (ii)}$$

$$\text{If } \left.\begin{aligned} \Gamma p &= \int_0^\infty e^{-u}\, u^{p-1}\, du \\ \Gamma q &= \int_0^\infty e^{-v}\, v^{q-1}\, dv \end{aligned}\right] \text{— (iii)}$$

From (iii), we have

$$\Gamma p\, \Gamma q = \int_0^\infty \int_0^\infty e^{-(u+v)}\, u^{p-1}\, v^{q-1}\, du\, dv.$$

On setting $u = x^2$, $v = y^2$, then (from - ii)

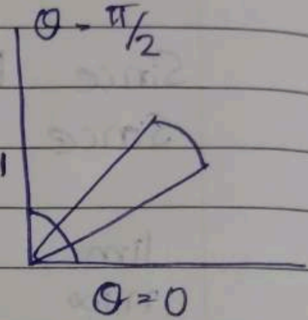
$$\Gamma p \, \Gamma q = 2\int_0^{\infty} e^{-x^2} x^{2p-1} \, dx \cdot 2\int_0^{\infty} e^{-y^2} y^{2q-1} \, dy$$

$$= 4 \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} x^{2p-1} \cdot y^{2q-1} \, dx \, dy$$

Let $x = r\cos\theta$

$y = r\sin\theta$, then

$\Rightarrow dx\,dy = r\,dr\,d\theta$



$$\Gamma p \, \Gamma q = \int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} (r\cos\theta)^{2p-1} \cdot (r\sin\theta)^{2q-1} \, r\,dr\,d\theta$$

$$\Gamma p \, \Gamma q = \int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} (r\cos\theta)^{2p-1} (r\sin\theta)^{2q-1} \, r\,dr\,d\theta$$

From (iii), we have

$$\Gamma p \, \Gamma q = 2\int_0^{\infty} e^{-r^2} r^{2(p+q)-1} \, dr \int_{\theta=0}^{\pi/2} (\cos^{2p-1}\theta)(\sin^{2q-1}\theta) \, d\theta$$

$$\Gamma p \, \Gamma q = \Gamma(p+q) \, B(p,q) \quad \text{—— (iv)}$$

Since, $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} \, dx$

On putting $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta \, d\theta$

$$\Rightarrow B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}\theta \cdot \cos^{2q-1}\theta \, d\theta$$

Note : $\beta(p,q) = \beta(q,p)$

Finally, we arriave at,

$$\beta(p,q) = \frac{\Gamma p \cdot \Gamma q}{\Gamma p+q} \quad ; p>0, q>0$$

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\, dx$$

On putting $x = \frac{1}{1+y} \Rightarrow 1+y = 1/x$

$$dx = \frac{-1}{(1+y)^2} dy.$$

$$\beta(p,q) = \int_0^\infty \left(\frac{1}{1+y}\right)^{p-1}\left(\frac{y}{1+y}\right)^{q-1}\left(\frac{1}{1+y}\right)^2 dy.$$

$$B(p,q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\, dy \quad \text{—— (5)}$$

From, (5)

we have, $B(p,q) = \underbrace{\int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}} dy}_{(i)} + \underbrace{\int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}} dy}_{(ii)}$

R.H.S of

from (ii) part

i.e $\int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}} dy,$

If we put $y = 1/z \Rightarrow dy = \frac{-1}{z^2} dz,$

$$= \int_0^1 \frac{(1/z)^{q-1}}{\left(1+\frac{1}{z}\right)^{p+q}} \cdot \frac{1}{z^2}\, dz.$$

$$\Rightarrow \int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy = \int_0^1 \frac{1}{z^{q-1}} \cdot \frac{z^{p+q}\, z^{-2}}{\left(1+\frac{1}{z}\right)^{p+q}}\,dz$$

$$= \int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}}\,dz$$

On using the property of definite integral, we can write

$$\int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}}\,dz = \int_0^1 \frac{y^{p-1}}{(1+y)^{p+q}}\,dy.$$

Therefore, $B(p,q) = \int_0^1 \left[\frac{y^{q-1} + y^{p-1}}{(1+y)^{p+q}}\right] dy.$

Since, $B(p,q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy.$

Now, $B(1-q;q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{1}}\,dy$

$$B(p,q) = \frac{\Gamma p \cdot \Gamma q}{\Gamma(p+q)}$$

$$B(1-q,q) = \frac{\Gamma(1-q)\,\Gamma q}{\Gamma\left(1 \cancel{-q} \cancel{+q}\right)} = \Gamma(1-q)\,\Gamma q.$$

$$\Rightarrow \int_0^\infty \frac{y^{n-1}}{(1+y)}\,dy = \frac{\pi}{\sin n\pi}$$

$$\Rightarrow \int_0^\infty \frac{y^{n-1}}{(1+y)}\,dy = \Gamma(1-n)\,\Gamma(n) = \frac{\pi}{\sin n\pi}$$

On setting, $n = \frac{1}{2}$, then

$$\int_0^\infty \frac{y^{-1/2}}{(1+y)}\,dy = \Gamma\left(1-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{\pi}{\sin \pi/2} = \pi$$

$$\Rightarrow \left[\Gamma\tfrac{1}{2}\right]^2 = \pi$$

$$\Rightarrow \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

* Factorial function:

Since $\lfloor n = n(n-1)(n-2)\ldots\ldots 2\,1$

or $= 1.2.3\ldots\ldots(n-1)n$

$(n-1)$
$\downarrow$
$1+n-2$
$n-1$

$(1)_n = (1+0)(1+1)(1+2)\ldots\ldots(1+n-1)$

$(a)_n = (a+0)(a+1)(a+2)\ldots\ldots(a+n-1)$

$(a)_n = a(a+1)(a+2)(a+3)\ldots\ldots(a+n-1)$

$\Gamma(n+1) = n\,\Gamma n = n(n-1)\Gamma(n-1)$

$= 1\,\Gamma 1 = n\overset{(n-2)}{(n-1)}\Gamma(n-2)$

$= n(n-1)(n-2)(n-3)\Gamma(n-3)$

$= n(n-1)(n-2)\ldots\ldots 2.1$

$\therefore \lfloor n$

$$\Rightarrow \Gamma(n+1) = \lfloor n \Rightarrow n \in I^+$$

$$\frac{\Gamma(n+1)}{\Gamma n} = n$$

21/1

$\frac{\Gamma a+0}{\Gamma a} = 1$ | $(0)_n$ does not exists
$(0)_0 = \frac{\Gamma 0+0}{\Gamma 0}$ = morning less

$$(a_n) = \prod_{k=1}^{n} (a+k-1)$$

$$\begin{aligned}\Gamma 1+n &= n\,\Gamma (n-1)+1 \\ &= n(n-1)\,\Gamma n-1 \\ &= n(n-1)(n-2)\,\Gamma n-2 \\ &\;\vdots\end{aligned}$$

We have

$$\Gamma a+n = (a+n-1)(a+n-2)(a+n-3)\cdots a\,\Gamma a$$

$$\Rightarrow \frac{\Gamma (a+n)}{\Gamma a} = a(a+1)(a+2)\cdots(a+n-1) = (a)_n$$

$$\Rightarrow (a)_n = \frac{\Gamma (a+n)}{\Gamma a}$$

$$\begin{aligned}(a)_{2n} &= a(a+1)(a+2)\cdots(a+2n-1) \\ &= 2\cdot\left(\frac{a}{2}\right)\cdot 2\cdot\left(\frac{a+1}{2}\right)\cdot 2\left(\frac{a+2}{2}\right)\cdots 2\left(\frac{a+2n-1}{2}\right) \\ &= 2^{2n}\left(\frac{a}{2}\right)_n \left(\frac{a+1}{2}\right)_n\end{aligned}$$

Since,

$$\Gamma z = \lim_{n\to\infty} \frac{\lfloor (n-1)\cdot n^z}{z(z+1)(z+2)\cdots(z+n-1)}$$

$$\Gamma z = \lim_{n\to\infty} \frac{\lfloor (n-1)\cdot n^z}{(z)_n}$$

$$\cancel{\Gamma z} = \lim_{n\to\infty} \frac{\lfloor (n-1)\cdot n^z\cdot \cancel{\Gamma z}}{\Gamma z+n} \qquad z_n = \frac{\Gamma z+n}{\Gamma z}$$

$$1 = \lim_{n\to\infty} \frac{n^z \lfloor(n-1)}{\lceil z+n}$$

We have,

$$(a)_{2n} = 2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n$$

On putting $a = 2z$,

$$(2z)_{2n} = 2^{2n}\left(\frac{\cancel{2}z}{\cancel{2}}\right)_n\left(\frac{2z+1}{2}\right)_n$$

$$\text{or } \frac{\lceil 2z+2n}{\lceil 2z} = 2^{2n}\frac{\lceil z+n}{\lceil z}\cdot\frac{\lceil z+\frac{1}{2}+n}{\lceil z+\frac{1}{2}}$$

$$= 2^{2n}\frac{\lceil z+n}{\lceil z}\cdot\frac{\lceil z+\frac{1}{2}+n}{\lceil z+\frac{1}{2}}$$

$$\Rightarrow \frac{\lceil(2z+2n)}{2^{2n}\lceil z+n\cdot\lceil z+\frac{1}{2}+n} = \frac{\lceil 2z}{\lceil z\cdot\lceil z+\frac{1}{2}}$$

On taking the limit $n\to\infty$

$$\text{or } \frac{\lceil 2z}{\lceil z\lceil z+\frac{1}{2}} = \lim_{n\to\infty}\frac{\lceil(2z+2n)}{\lceil(z+n)\cdot\lceil z+\frac{1}{2}+n}$$

Now, we use the formula,

$$\lim_{n\to\infty}\frac{\lfloor(n-1)n^z}{\lceil z+n} = 1$$

$$\left[\lim_{n\to\infty}\frac{\lfloor(2n-1)(2n)^{2z}}{\lceil(2z+2n)} = 1\right.$$

$$\left.\lim_{n\to\infty}\frac{\lfloor(n-1)\cdot n^{z+\frac{1}{2}}}{\lceil z+\frac{1}{2}+n} = 1\right.$$

We get

$$\frac{\lceil 2z}{\lceil z\lceil(z+\frac{1}{2})} = \lim_{n\to\infty}$$

we get

$$\frac{\Gamma 2z}{\Gamma z\,\Gamma z+\frac{1}{2}} = \lim_{n\to\infty}\left[\frac{\Gamma 2z+2n}{\Gamma 2n-1\,(2n)^{2z}}\cdot\lfloor(2n-1)(2n)^{2z}\cdot\frac{\lfloor(n-1).n^{z}}{\Gamma(z+n)}\cdot\frac{1}{\lfloor(n-1).n^{z}}\cdot\frac{\lfloor(n-1).n^{z+1/2}}{\Gamma z+\frac{1}{2}+n}\cdot\frac{1}{\lfloor(n-1).n^{z+1/2}}\cdot\frac{1}{2^{2n}}\right]$$

$$\frac{\Gamma 2z}{\Gamma z\,\Gamma z+\frac{1}{2}} = \lim_{n\to\infty}\frac{1}{[\lfloor n-1]^2}\cdot\frac{1}{n^{2z}}\cdot\frac{1}{n^{1/2}}\cdot\frac{1}{2^{2n}}\cdot(\lfloor 2n-1).2^{2z}.n^{2z} = c$$

On putting $z=\frac{1}{2}$ then

$$c = \frac{\Gamma 2\cdot\frac{1}{2}}{\Gamma\frac{1}{2}\cdot\Gamma\frac{1}{2}+\frac{1}{2}}$$

$$c = \frac{\Gamma 1}{\Gamma\frac{1}{2}\,\Gamma 1} = \frac{1}{\sqrt{\pi}}$$

$$\frac{\Gamma 2z}{\Gamma z\,\Gamma z+\frac{1}{2}} = \frac{1}{\sqrt{\pi}}$$

Lengendre's duplication formula.

4.

From Legendre's duplication formula,

$$\sqrt{\pi}\cdot\Gamma 2z = 2^{2z-1}\,\Gamma z\,\Gamma z+\tfrac{1}{2}$$

taking a log,

$$\log\sqrt{\pi} + \log\Gamma 2z = (2z-1)\log 2 + \log\Gamma z + \log\Gamma(z+\tfrac{1}{2})$$

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On diff. w.r.t "z", we get

$$0 + 2\frac{\Gamma'(2z)}{\Gamma(2z)} = 2\log 2 + \frac{\Gamma'(z)}{\Gamma(z)} + \frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})}$$

$$\Rightarrow 2\frac{\Gamma'(2z)}{\Gamma(2z)} - \frac{\Gamma'(z)}{\Gamma(z)} - \frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})} = 2\log 2 \quad \text{--- (1)}$$

On putting $z = \frac{1}{2}$ in (1)
we get

$$2\frac{\Gamma'(2\cdot\frac{1}{2})}{\Gamma(2\cdot\frac{1}{2})} - \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} - \frac{\Gamma'(\frac{1}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})} = 2\log 2$$

$$2\cdot\frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(\frac{1}{2})}{\sqrt{\pi}} - \frac{\Gamma'(1)}{\Gamma(1)} = 2\log 2 \qquad \Gamma'(1) = -\gamma$$

$$\Gamma'(1) - \frac{\Gamma'(\frac{1}{2})}{\sqrt{\pi}} = 2\log 2$$

$$\boxed{-(\gamma + 2\log 2)\sqrt{\pi} = \Gamma'(\tfrac{1}{2})}$$

$$\Rightarrow (\alpha)_{n+k} = \alpha(\alpha+1)(\alpha+2)(\alpha+3)\cdots\cdots(\alpha+n+k-1)$$

$$= \underbrace{\alpha(\alpha+1)(\alpha+2)(\alpha+3)\cdots\cdots(\alpha+n-1)}(\alpha+n)(\alpha+n+1)\cdots(\alpha+n+k-1)$$

$$\boxed{(\alpha)_{n+k} = (\alpha)_n(\alpha+n)_k}$$

$$(\alpha)_{n-k} = \alpha(\alpha+1)(\alpha+2)(\alpha+3)\cdots\cdots(\alpha+n-k-1)$$

$$= \frac{(-1)^k (\alpha)_n}{(1-\alpha)_k}$$

$$(5)_3 = (5)_{5-2}$$

$$= 5(5+1)(5+2) \qquad = \frac{5(5+1)(5+2)(5+3)(5+4)}{(1-5)_2}$$

$$= 5\times 6\times 7 \qquad = \frac{5\times6\times7\times8\times9}{4\times3}$$

$$(1-5)_2 = (-4)(1-5+1)$$
$$= (-4)(-3)$$

$$= \frac{(-1)^2\, 5(5+1)(5+2)(5+3)(5+4)}{(1-5-5)(2-5-5)}$$
$$(-9)\quad(-8)$$

$$= 5(5+1).(5+2)$$

$$(\alpha)_{n-k} = \frac{\Gamma(\alpha+n-k)}{\Gamma(\alpha)}, \text{ since } (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

$$= \frac{\Gamma(\alpha+n-k)}{\Gamma((\alpha-1)+1)} = \frac{\Gamma(\alpha+n-k)}{(\alpha-1)\Gamma(\alpha-1)}$$

$$= \frac{\Gamma(\alpha+n-k)}{(\alpha-1)(\alpha-2)\Gamma(\alpha-2)} = \frac{\Gamma(\alpha+n-k)}{(\alpha-1)(\alpha-2)(\alpha-3)\cdots(\alpha-(n-k))\Gamma(\alpha-(n-k))}$$

$$= \alpha+n-k+1-1$$
$$= (\alpha+n-k-1)\Gamma(\alpha+n-k)$$

$$\Gamma(\alpha+n) = (\alpha+n-1)(\alpha+n-2)\cdots\cdots\alpha\Gamma(\alpha)$$

$$\Gamma(\alpha+1)$$

$$\alpha\Gamma(\alpha)$$

Date :

$$= \Gamma(\alpha+n)\,(-1)(1-\alpha)(-1)(2-\alpha)(-1)(3-\alpha)\cdots(-1)$$

$$(\alpha)_{n-k} = \frac{(\alpha)_n(-1)^k}{(1-\alpha-n)_k}$$

$$\alpha=1,\quad (1)_{n-k} = \lfloor n-k = \frac{(1)_n(-1)^k}{(1-1-n)_k}$$

$$(1)_{n-k} = \lfloor n-k = \frac{\lfloor n\,(-1)^k}{(-n)_k}$$

$$(\alpha)_{n-k} = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n+k-1)$$

$$= \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+n-k-1).(\alpha+n-k)(\alpha+n-k+1)\cdots(\alpha+n-1)}{(\alpha+n-1)(\alpha+n-2)\cdots(\alpha+n-k)}$$

$$= \frac{(\alpha)_n}{(-1)(1-\alpha-n).(-1)(2-\alpha-n)\cdots(-1)(k-\alpha-n)}$$

$$= \frac{(\alpha)_n(-1)^k}{(1-\alpha-n)_k}$$

Ex

$$\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = (1-\alpha)_{-n}$$

$$= \frac{(-1)^n}{(1-1+\alpha)_n} = \frac{(-1)^n}{(\alpha)_n}$$

$$\Gamma(1-\alpha) = -\alpha\,\Gamma(-\alpha)$$

$$= -\alpha\,\Gamma(-\alpha+1-1)$$

$$= (-\alpha)(-\alpha-1)(-\alpha-2)\cdots(-\alpha-n+1)\,\Gamma(1-\alpha-n)$$

$$= (-1)^n(\alpha)(\alpha+1)(\alpha+2)(\alpha+3)\cdots(\alpha+n-1)\,\Gamma(1-\alpha-n)$$

$$\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = \frac{\Gamma(1-\alpha-n)}{(-1)^n(\alpha)_n\,\Gamma(1-\alpha-n)} \Rightarrow \frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = \frac{(-1)^n}{(\alpha)_n}$$

$$(\alpha)_{2n} = 2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n$$

$$(\alpha)_{kn} = k^{kn}\prod_{s=1}^{k}\left(\frac{\alpha+s-1}{k}\right)_n \quad \text{—— (1)}$$

$$\frac{\Gamma(\alpha+kn)}{\Gamma\alpha} = k^{kn}\prod_{s=1}^{k}\frac{\Gamma\left(\frac{\alpha+s-1}{k}+n\right)}{\Gamma\left(\frac{\alpha+s-1}{k}\right)}$$

Put $\alpha = kz$, then we get,

$$\frac{\Gamma(kz+kn)}{\Gamma kz} = k^{kz}\prod_{s=1}^{k}\frac{\Gamma\left(\frac{kz+s-1}{k}+n\right)}{\Gamma\left(\frac{kz+s-1}{k}\right)}$$

$$\Rightarrow \frac{\Gamma kz}{\prod_{s=1}^{k}\Gamma\left(z+\frac{s-1}{k}\right)} = \frac{\Gamma(kz+kn)}{k^{kn}\prod_{s=1}^{k}\Gamma\left(z+n+\frac{s-1}{k}\right)}$$

$$\frac{\Gamma kz}{\prod_{s=1}^{k}\Gamma\left(z+\frac{s-1}{k}\right)} = \lim_{n\to\infty}\frac{\Gamma(kz+kn)}{k^{kn}\prod_{s=1}^{k}\Gamma\left(z+n+\frac{s-1}{k}\right)} \quad \text{—— (2)}$$

Since, $\lim_{n\to\infty}\frac{(n-1)!\,n^{z}}{\Gamma(z+n)} = 1$

$$\Rightarrow \lim_{n\to\infty}\frac{(kn-1)!\,(kn)^{kz}}{\Gamma(kz+kn)} = 1$$

and

$$\lim_{n\to\infty} \left[\frac{\Gamma(n-1) \cdot n^{z+\frac{s-1}{k}}}{\Gamma\left(z+n+\frac{s-1}{k}\right)} = 1\right\} \quad \text{— (3)}$$

From (2) and (3); we get,

$$\frac{\Gamma(kz)}{\prod_{s=1}^{k}\Gamma\left(z+\frac{s-1}{k}\right)} = \lim_{n\to\infty} \frac{\Gamma(kn-1)\,(kn)^{kz}}{k^{kn}\prod_{s=1}^{k}[\Gamma(n-1)]^{k} \cdot n^{kz+\frac{1}{2}(k-1)}} = C$$

$$\Rightarrow \frac{1}{k^{kz}}\,\frac{\Gamma(kz)}{\prod_{s=1}^{k}\Gamma\left(z+\frac{s-1}{k}\right)} = C \quad \text{— (4)}$$

On putting $z = 1/k$, then

$$\frac{1}{kC} = \prod_{s=1}^{k-1}\Gamma\left(\frac{s}{k}\right) = \prod_{s=1}^{k-1}\Gamma\left(\frac{k-s}{k}\right) \quad \text{— (5)}$$

From (5), we have

$$\frac{1}{k^2C^2} = \prod_{s=1}^{k-1}\Gamma\left(\frac{s}{k}\right)\Gamma\left(1-s/k\right)$$

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

$$\frac{1}{k^2C^2} = \prod_{s=1}^{k-1}\frac{\pi}{\sin \pi s/k} \quad \text{— (6)}$$

Now, we find the value of $\prod_{s=1}^{k-1}\sin\frac{\pi s}{k}$, $k \geq 2$

Let $\alpha = \exp\left(\frac{2\pi i}{k}\right)$ be the k^{th} root of unity

(Imp)

$x^2-1 = (x-1)(x+1)$

$\sin 2\theta = \frac{e^{+x} - e^{-x}}{2x}$

We consider $(x^k - 1) = (x-1) \prod_{s=1}^{k-1} (x - \alpha^s)$ — (7)

On differentiating (7), w.r.t "x", we find that,

$$k x^{k-1} = \prod_{s=1}^{k-1} (x - \alpha^s) + \underbrace{(x-1)\, g(x)}_{} \quad \text{— (8)}$$

$g(x)$: poly of degree $(k-2)$

On putting $x = 1$, in (8).

$$k = \prod_{s=1}^{k-1} (1 - \alpha^s) \quad \text{— (9)}$$

Since

$$1 - \alpha^s = 1 - \exp\left(\frac{2\pi i s}{k}\right)$$

$$= -\exp\left(\frac{\pi i s}{k}\right)\left[\exp\left(\frac{\pi i s}{k}\right) - \exp\left(-\frac{\pi i s}{k}\right)\right]$$

$2i$ multiply and divided

$$1 - \alpha^s = -2i \exp\left(\frac{\pi i s}{k}\right) \cdot \sin\left(\frac{\pi s}{k}\right)$$

Therefore,

$$k = (-2i)^{k-1} \exp\left[\tfrac{1}{2}\pi i (k-1)\right] \prod_{s=1}^{k-1} \sin \frac{\pi s}{k}$$

IMP Exam

or
$$\frac{k}{2^{k-1}} = \prod_{s=1}^{k-1} \sin \frac{\pi s}{k}$$

$$\Rightarrow k^2 c^2 \cdot \pi^{k-1} = \frac{k}{2^{k-1}}$$

$$\Rightarrow c = (2\pi)^{-1/2} (k-1)^{-1/2}$$

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Hypergeometric function

$$z(1-z)w'' + [c-(a+b-1)z]w' - abw = 0,$$

$$w' = \frac{dw}{dz} \quad \& \quad w'' = \frac{d^2w}{dz^2}.$$

↓

Hypergeometric differential equation

Singularities at $z = 0, 1 \ \& \ \infty$

$$p(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0.$$

$P_0(x) \neq 0$, at $x = a$, called Ordinary point.

$P_0(x) = 0$ at $x = a$, called Singularites point.

⇒ whenever singularities aries → Series solution.

$${}_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix}\ z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n \lfloor n} \rightarrow \text{exponential series} \rightarrow \frac{z^n}{\lfloor n}$$

↑
Parameter
(shape and size should be change]

extension of ~~hyper~~ geometrical series / function = Hypegem.

Hypergeometric → we always need convergent series

$${}_0F_0\left[\begin{matrix} -, -; \\ -; \end{matrix}\ z\right] = \sum_{n=0}^{\infty} \frac{z^n}{\lfloor n} = e^z$$

↳ extension of exponential series

$${}_1F_0\left[\begin{matrix} a, -; \\ -; \end{matrix}\ z\right] = \sum_{n=0}^{\infty} (a)_n \frac{z^n}{\lfloor n}.$$

$$\frac{(-n)(-n-1)(-n-2)\cdots(-1)^n x^n}{\lfloor n}$$

Series is terminate = poly.
Series ∞ = function

$$_1F_0 \Rightarrow (1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(-a)(-a-1)(-a-2)\cdots(-1)^n a^n}{\lfloor n} = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{\lfloor n}$$

$$(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{\lfloor 2} + n(n-1)(n-2)\frac{x^3}{\lfloor 3} + \cdots$$

$$(1-x)^{-n} =$$

converges for exponential series.
↳ always convergent for the all the finit value of x

if infinit → divergent.

$$u_n = \frac{(a)_n (b)_n z^n}{(c)_n \lfloor n} \cdot \frac{(a)_{n+1} (b)_{n+1} z^{n+1}}{(c)_{n+1} \lfloor n+1} = u_{n+1}$$

$\boxed{\Gamma(n+1) = n\Gamma n}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(a)_{n+1}(b)_{n+1} z^{n+1}}{(c)_{n+1}\lfloor n+1} \cdot \frac{(c)_n \lfloor n}{(a)_n (b)_n z^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{\Gamma(a+n+1)}{\Gamma a} \cdot \frac{\Gamma(b+n+1)}{\Gamma b} \frac{\Gamma c}{\Gamma(c+n+1)} \frac{\Gamma a}{\Gamma(a+n)} \frac{\Gamma b}{\Gamma(b+n)} \cdot \frac{\Gamma(c+n)}{\Gamma c} \cdot \frac{z^n \cdot z \cdot \lfloor n}{z^n \lfloor n+1}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(a+n)\cancel{\Gamma(a+n)} \cdot (b+n)\cancel{\Gamma(b+n)}\, \cancel{\Gamma c}\, \cancel{\Gamma a}\, \cancel{\Gamma b}\, \cancel{\Gamma(c+n)} \cdot \cancel{\lfloor n}}{\cancel{\Gamma a}\, \cancel{\Gamma b}\, (c+n)\cancel{\Gamma(c+n)}\, \cancel{\Gamma(a+n)}\, \cancel{\Gamma(b+n)}\, \cancel{\Gamma c}\, (n+1)\cancel{\lfloor n}} \cdot z\right|$$

$$= \lim_{n\to\infty} \left|\frac{(a+n)(b+n)}{(c+n)(n+1)} \cdot z\right| = |z| < 1 \text{ conv.}$$
$$> 1 \text{ Div.}$$

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$\sum \frac{1}{n^p}$, $p>1$ conv.
$p<1$ Div

$${}_2F_1\left[\begin{matrix}a, b; & z\\ c; & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n \lfloor n} = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n \lfloor n} \quad \text{— (1)}$$

We have already proved that

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = |z|$$

If $|z|<1$, the series conv.
& $|z|>1$, the series div.

Now, we prove the convergence at $|z| = 1$.

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} = v_n \quad \text{— (2)}$$

$$\boxed{\lim_{n\to\infty} \frac{u_n}{v_n} \Rightarrow \text{exists.}} \quad ?? \quad \text{— (3)}$$

Let $\delta = \frac{1}{2} Re(c-a-b) > 0$.

Now, we have $\lim_{n\to\infty} \left| \frac{(a)_n (b)_n n^{1+\delta}}{(c)_n \lfloor n} \right|$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{\Gamma(a+n)\,\Gamma(b+n)\,\Gamma(c)\, n^{1+\delta}}{\Gamma(c+n)\,\Gamma(a)\,\Gamma(b)\,\lfloor n} \right| \quad \text{— (4)}$$

Note that, $\lim_{n\to\infty} \frac{\lfloor n-1 \; n^z}{\Gamma(z+n)} = 1$ — (5)

From (4) and (5), we have

$$\lim_{n\to\infty} \left| \underbrace{\frac{\Gamma(a+n)}{\lfloor(n-1)\, n^a}}_{=1} \cdot \underbrace{\frac{\lfloor(n-1).n^a}{\Gamma(c+n)}}_{} \cdot \frac{\lfloor(n-1).n^c}{\lfloor(n-1).n^c} \cdot \underbrace{\frac{\Gamma(b+n)}{\lfloor(n-1).n^b}}_{=1} \cdot \frac{\lfloor(n-1).n^b.n^{1+\delta}.\Gamma c}{\Gamma a\, \Gamma b\, \lfloor n} \right|$$

$$= \left| \frac{\Gamma c}{\Gamma a \, \Gamma b} \right| \lim_{n\to\infty} \left| \frac{1}{n^{(c-a-b-\delta)}} \right| = 0$$

$\Rightarrow Re(c-a-b-\delta) > 0$

$2\delta - \delta > 0 \Rightarrow \delta > 0$

since $\frac{1}{2} Re(c-a-b) = \delta$.

meaningless

$$\boxed{Re(c) = Re(a+b)}$$

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since, $_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n \lfloor n} \quad \text{---} (1)$

$$\frac{(b)_n}{(c)_n} = \frac{\Gamma(b+n)\,\Gamma c}{\Gamma b\,\Gamma(c+n)} \cdot \frac{\Gamma(c-b)}{\Gamma(c-b)}$$

$$= \frac{\Gamma c}{\Gamma b\,\Gamma(c-b)} \quad \frac{\Gamma(b+n)\,\Gamma(c-b)}{\Gamma(c+n)} = \int_0^1 x^{b+n-1}(1-x)^{c-b-1}\,dx \quad \text{---} (2)$$

On using the formula

$$B(p,q) = \frac{\Gamma p\,\Gamma q}{\Gamma(p+q)} = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$$

From (1) and (2)

$$\sum_{n=0}^{\infty} \frac{(a_n)(b)_n z^n}{(c)_n \lfloor n} = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{\lfloor n} \int_0^1 x^{b+n-1}(1-x)^{c-b-1}\,dx \quad \text{---} (3)$$

$$\left[(1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{\lfloor n}\right]$$

On using $(1-y)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n y^n}{\lfloor n} \quad \text{---} (4)$

from (3) & (4).

$$_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n \lfloor n} = \left[\sum_{n=0}^{\infty} \int_0^1 x^{b+n-1}(1-x)^{c-b-1}\right.$$

$$= \left[\sum_{n=0}^{\infty} \int_0^1 x^{b+n-1}(1-x)^{c-b-1} \frac{(a)_n z^n}{\lfloor n}\,dx\right] \frac{\Gamma c}{\Gamma b\,\Gamma(c-b)}$$

$$= \left[\sum_{n=0}^{\infty} \int_0^1 x^{b-1}(1-x)^{c-b-1} (a)_n \frac{(zx)^n}{\lfloor n}\,dx\right] \frac{\Gamma c}{\Gamma b\,\Gamma(c-b)}$$

$c > a+b$

$c < a+b$ ↳ div.

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$$\frac{\Gamma c}{\Gamma b\,\Gamma c-b}\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\,dx = {}_2F_1\left[\begin{matrix}a,b;\\ c;z\end{matrix}\right] \quad \text{---} (5)$$

If $z=1$, (in eq 5) then.

$${}_2F_1\left[\begin{matrix}a,b;1\\ c;1\end{matrix}\right] = \left[\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-x)^{-a}\,dx\right]\left[\frac{\Gamma c}{\Gamma b\,\Gamma c-b}\right]$$

or

$${}_2F_1\left[\begin{matrix}a,b;\ 1\\ c;\end{matrix}\right] = \left[\frac{\Gamma c}{\Gamma b\,\Gamma c-b}\int_0^1 x^{b-1}(1-x)^{c-a-b-1}\,dx\right]$$

$$= \frac{\Gamma c}{\cancel{\Gamma b}\,\Gamma c-b}\cdot\frac{\cancel{\Gamma b}\,\Gamma c-a-b}{\Gamma(\cancel{b}+c-a-\cancel{b})}$$

$$= \frac{\Gamma c\,\Gamma(c-a-b)}{\Gamma(c-b)\,\Gamma(c-a)}$$

The Contignous Relation function

$$F = {}_2F_1\left[\begin{matrix}a,b;\ z\\ c;\end{matrix}\right] = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n z^n}{(c)_n\,\lfloor n} = \sum_{n=0}^{\infty}\delta_n$$

$$F(a+) = {}_2F_1\left[\begin{matrix}a+1,b;\\ c;z\end{matrix}\right]$$

$$F(a-) = {}_2F_1\left[\begin{matrix}a-1,b;\ z\\ c;\end{matrix}\right]$$

$$F(b+) = {}_2F_1\left[\begin{matrix}a,b+1;\ z\\ c;\end{matrix}\right]$$

$$F(b-) = {}_2F_1\left[\begin{matrix}a,b-1;\ z\\ c;\end{matrix}\right]$$

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$$F(c+) = {}_2F_1\begin{bmatrix} a, b; & z \\ c+1; & \end{bmatrix}$$

$$F(c^-) = {}_2F_1\begin{bmatrix} a, b; & z \\ c-1; & \end{bmatrix}$$

$${}_2F_1\begin{bmatrix} a+1, b; & z \\ c; & \end{bmatrix} = f(a+)$$

$$\Rightarrow F(a+) = \sum_{n=0}^{\infty} \frac{(a+1)_n (b)_n z^n}{(c)_n \lfloor n} = \sum_{n=0}^{\infty} \left(\frac{a+n}{a}\right) \delta_n$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a+1+n)(b)_n z^n}{\Gamma(a)(c)_n \lfloor n}$$

$$= \sum_{n=0}^{\infty} \frac{(a+n)\Gamma(a+n)(b)_n z^n}{\Gamma(a)(c)_n \lfloor n}$$

$$= \sum_{n=0}^{\infty} \frac{(a+n)(a)_n (b)_n z^n}{(c)_n \lfloor n} = \sum_{n=0}^{\infty} \frac{a+n}{a} \delta_n$$

$$F(a-) = \sum_{n=0}^{\infty} \frac{(a-1)(b)_n z^n}{(c)_n \lfloor n}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a-1+n)(b)_n z^n}{\Gamma(a-1)(c)_n \lfloor n}$$

$$= \sum_{n=0}^{\infty} \frac{(a-1+n)\Gamma(a-1+n)}{(a-1+n)(a-1)\Gamma(a-1)} \quad \frac{\Gamma(a-1)(b)_n z^n}{(c)_n \lfloor n}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)(b)_n (a-1) z^n}{(a-1+n)\Gamma(a)(c)_n \lfloor n}$$

$$F(a-) = \sum_{n=0}^{\infty} \frac{(a-1).(a)_n (b)_n z^n}{(a-1+n)(c)_n \lfloor n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a-1}{a-1+n}\right) \delta_n$$

$$F(b^+) = \sum_{n=0}^{\infty} \left(\frac{b+n}{n}\right) \delta_n$$

$$F(b^-) = \sum_{n=0}^{\infty} \left(\frac{b-1}{b-1+n}\right) \delta_n$$

$$F(c+) = \sum_{n=0}^{\infty} \left(\frac{c}{c+n}\right) \delta_n$$

$$F(c-) = \sum_{n=0}^{\infty} \left(\frac{c-1+n}{c-1}\right) \delta_n$$

Let θ be the operator and define as $\theta = z\dfrac{d}{dz}$.

Then $\theta z^n = \left(z\dfrac{d}{dz}\right) z^n$

$$= z \cdot n z^{n-1} = n z^n$$

$\Rightarrow \theta z^n = n z^n$

$$(\theta + a)F = \theta F + aF = (n+a)F$$

$$z \frac{d}{dz}\, {}_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix}\ z\right] = z\frac{d}{dz} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n \lfloor n}$$

$$= \sum_{n=1}^{\infty} n \frac{(a)_n (b)_n z^n}{(c)_n \lfloor n} = \sum_{n=1} n \cdot \delta_n$$

$$a \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n \lfloor n} \Rightarrow aF(a+)$$ ↳ one line left

$(\theta + c - 1)F = (c-1) f(c-)$ Here, ${}_2F_1 = \left[\begin{matrix} a, b; \\ c; \end{matrix}\ z\right] = W$

i) $(a-b)F = af(a+) - bf(b+)$

ii) $(a-c+1)F = af(a+) - (c-a)F(c-)$

Let $\theta(\theta + c - 1)W = \displaystyle\sum_{n=1}^{\infty} n(n+c-1) \frac{(a)_n (b)_n z^n}{(c)_n \lfloor n}$ ↳ $(n\lfloor n-1)$

$$= \sum_{n=1}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_{n-1} \lfloor n-1}$$

$$= \frac{\cancel{x}}{\cancel{x}} \frac{(n+c-1)}{\lfloor n-1\,(c)_n} = \frac{(n+c-1)\cdot\Gamma c}{\lfloor n-1\,\Gamma c+n}$$

$$= \frac{(n+c-1)\Gamma c}{\lfloor (n-1)\,\Gamma (c+n-1)+1} = \frac{(n+c-1)\Gamma c}{(c+n-1)\,\Gamma c+n-1}$$

$$= \frac{1}{\frac{\Gamma c+n-1}{\Gamma c}} = \frac{1}{(c)_{n-1}}.$$

$$= \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1} z^{n+1}}{(c)_n \lfloor n}$$

$$= z\sum_{n=0}^{\infty} \frac{(a+n)(b+n)(a)_n(b)_n z^n}{(c)_n \lfloor n} \Rightarrow w$$

$$\theta(\theta+c-1)w = z(\theta+a)(\theta+b)w$$

$$\Rightarrow [\theta(\theta+c-1) - z(\theta+a)(\theta+b)]w = 0.$$

12. $[\theta(\theta+c-1) - z(\theta+a)(\theta+b)]w = 0$, $\theta = \frac{d}{dz}$

This reduces to

$$z(1-z)w'' + [c-(a+b+1)z]w' - abw = 0.$$

which is hypergeometric equation.

$F = {}_2F_1\left[\begin{matrix} a, b\,; & z \\ c\,; & \end{matrix}\right]$ is a solution of above said diff. eqn.

Since, ${}_2F_1\left[\begin{matrix} a, b\,; & z \\ c\,; & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n \lfloor n}$

$$\lim_{n\to\infty} \frac{1}{n^2} \to 0 \quad \Big| \quad \lim_{n\to\infty} \frac{1}{n^{-2}} \to \infty$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b+n)\,\Gamma(c)}{\Gamma(a)\cdot\Gamma(b)\,\Gamma(c+n)\, n!} \cdot z^n \qquad \left[\lim_{n\to\infty} \frac{(n-1)!\, n^z}{\Gamma(z+n)} = 1\right]$$

or $$\frac{{}_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix}\; z\right]}{\Gamma(c)} = \sum_{n=0}^{\infty} 1.$$

$$\lim_{n\to\infty} \overbrace{\left[\frac{\Gamma(a+n)}{(n-1)!\, n^a}\right]}^{=1} \left[(n-1)!\, n^a\right]\left[(n-1)!\, n^b\right] \cdot \overbrace{\left[\frac{\Gamma(b+n)}{(n-1)!\, n^b}\right]}^{=1}$$

$$\cdot \left[\frac{(n-1)!\, n^c}{\Gamma(c+n)}\right] \underbrace{\left[\frac{1}{(n-1)!\, n^c}\right]}_{=1} \left[\frac{1}{\Gamma(a)} \cdot \frac{1}{\Gamma(b)} \cdot \frac{1}{n!}\right]$$

$$= \lim_{n\to\infty} \frac{n^{a+b-c}\,(n-1)!}{\Gamma(a)\,\Gamma(b)\cdot n\,(n-1)!}$$

$$= \lim_{n\to\infty} \frac{1}{\Gamma(a)\,\Gamma(b)} \lim_{n\to\infty} \frac{1}{n^{c-a-b+1}} = 0,$$

$$c-a-b+1 > 0$$

$$\Rightarrow \left|\frac{(a)_n (b)_n z^n}{\Gamma(c+n)\, n!}\right| < k \cdot z^{\frac{1}{2}n}$$

$\downarrow$ Convergent $\qquad$ k $\uparrow$ Convergent

Condition: for all finite value of a, b & c.

Series Manipulation

(i) $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n-k)$

(ii) $\sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n+k)$

(iii) $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(n,k) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} A(k, n-2k)$

(iv) $\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n+2k)$

i) Let $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) \cdot t^{n+k}$ $\quad t=1$

$k = j$, & $n = m-j$

$n \geq 0,\ k \geq 0 \Rightarrow j \geq 0$

$\Rightarrow m-j \geq 0 \Rightarrow 0 \leq j \leq m$

$\Rightarrow \sum_{m=0}^{\infty} \sum_{j=0}^{m} A(j, m-j)\, t^{m-\cancel{j}+\cancel{j}}$ or $\sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k,n)$

iii) let $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n)\, t^{n+2k}$

$k = j$, & $n = m-2j$

$n \geq 0,\ k \geq 0 \Rightarrow j \geq 0$

$\Rightarrow m-2j \geq 0 \Rightarrow 0 \leq j \leq m$

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$\lfloor n-k = (1)_{n-k}$

$$(1+x)^n = \sum_{k=0}^{n} \frac{(-n)_k (-1)^k \cdot x^k}{\lfloor k}$$

$$(1+1)^n = \sum_{k=0}^{n} \frac{(-n)_k (-1)^k \cdot 1}{\lfloor k}$$

$$(1-x)^{-a} = \sum (a)_n x^n$$

$$(1)_n (1+n)_{-k} = \frac{\lfloor n \cdot (-1)^k}{(1-k-n)_k} = \frac{\lfloor n (-1)^k}{(-n)_k}$$

$$\Rightarrow \sum_{m=0}^{\infty} \sum_{j=0}^{[m/2]} A(j, m-j) t^{m-\cancel{2j}+\cancel{2j}} \quad \text{or} \quad \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} A(k, n),$$

M.I.M.P

$$\rightarrow e^x \cdot e^x$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{\lfloor n} \sum_{n=0}^{\infty} \frac{x^k}{\lfloor k} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^{n+k}}{\lfloor n \cdot \lfloor k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^{n-\cancel{k}+\cancel{k}}}{\lfloor n-k \lfloor k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^n}{\lfloor n-k \lfloor k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^n}{\lfloor n-k \lfloor k}$$

2 line missing

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-n)_k}{\lfloor n (-1)^k \lfloor k}$$

Assi $e^x \cdot e^{2x} = e^{3x}$

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$(a)_k (a+k)_n$ — ...] differnt
$= (a)_{n+k}$ change the Σ so.

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$$(1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{\lfloor n}$$

$$(a)_{n+k} = (a)_n (a+n)_k$$

$$\text{or} \quad = (a)_{k+n} = (a)_k (a+k)_n.$$

$$(a)_{n-k} = (a)_n (a+n)_{-k}$$

$$= \frac{(a)_n (-1)^k}{(1-a-n)_k}.$$

If $a=1$, then $(1)_{n-k} = \lfloor n-k = \dfrac{\lfloor n (-1)^k}{(-n)_k}.$ — ⊛

$$(a)_{2k} = 2^{2k} \left(\frac{a}{2}\right)_k \left(\frac{a+1}{2}\right)_k.$$

Let $(1-z)^{-a}$, ${}_2F_1 \left[\begin{matrix} a, & c-b; & \dfrac{-z}{1-z} \\ & c; & \end{matrix}\right]$

$$\Rightarrow (1-z)^{-a} \sum_{k=0}^{\infty} \frac{(a)_k (c-b)_k}{(c)_k \lfloor k} \cdot \frac{(-1)^k \cdot z^k}{(1-z)^k}$$

$$\text{or} \sum_{k=0}^{\infty} \frac{(a)_k (c-b)_k (-1)^k \cdot z^k}{(c)_k \lfloor k} (1-z)^{-(a+k)}$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{(a)_k (c-b)_k (-1)^k \cdot z^k}{(c)_k \lfloor k} \sum_{n=0}^{\infty} \frac{(a+k)_n z^n}{\lfloor n}$$

$$\Rightarrow \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_k (a+k)_n (c-b)_k (-1)^k \cdot z^{k+n}}{(c)_k \lfloor k \lfloor n}$$

change the series Σ
$$\Rightarrow \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a)_{n+k} (c-b)_k (-1)^k z^{n+k}}{\lfloor n (c)_k \lfloor k}$$

$(a)_n (a+n)_k = (a)_{n+k}$

$n=n$

$$\left[\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k)\right]$$

$n = n-k$

$$\Rightarrow \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(a)_{n-\cancel{k}+\cancel{k}}\,(c-b)_k\,(-1)^k\,z^{n-\cancel{k}+\cancel{k}}}{\lfloor n-k\;(c)_k\,\lfloor k}$$

$$\Rightarrow \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(a)_n\,(c-b)_k\,\cancel{(-1)^k}\,z^n\,(-n)_k}{\lfloor n\;\cancel{(-1)^k}\,(c)_k\,\lfloor k} \quad [\text{use } (*)]$$

or

$$\Rightarrow \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(a)_n z^n}{\lfloor n}\cdot\frac{(-n)_k\,(c-b)_k}{(c)_k\,\lfloor k}$$

$$\Rightarrow \sum_{n=0}^{\infty} {}_2F_1\left[\begin{matrix}-n,\ c-b;\\ c;\end{matrix}\ 1\right]\frac{(a)_n z^n}{\lfloor n}$$

$$\left[{}_2F_1\left[\begin{matrix}a,\ b;\\ c;\end{matrix}\ 1\right] = \frac{\Gamma c\;\Gamma(c-a-b)}{\Gamma(c-a)\;\Gamma(c-b)}\right]$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{\Gamma c\;\Gamma(\cancel{c}+n-c+b)}{\Gamma(c+n)\;\Gamma(\cancel{c}-\cancel{c}+n)}\cdot\frac{(a)_n z^n}{\lfloor n}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n\,z^n}{(c)_n\,\lfloor n}$$

HA $= {}_2F_1\left[\begin{matrix}a,\ b;\\ c;\end{matrix}\ z\right] = (1-z)^{-a}\,{}_2F_1\left[\begin{matrix}a,\ c-b;\\ c;\end{matrix}\ \frac{-z}{1-z}\right]$

<u>Note:</u> on putting,

$y = \frac{-z}{1-z}$,

then ${}_2F_1\left[\begin{matrix}a,\ c-b;\\ c;\end{matrix}\ y\right] = (1-y)^{-c+b}\left[\begin{matrix}c-a,\ c-b;\\ c;\end{matrix}\ \frac{-y}{1-y}\right]$

and $1-y = (1-z)^{-1}$ & $\frac{-y}{1-y} = z$,

$(a)_{n+1} = (a)_n (a+n)$

or $(a)_{1+n} = (a)_1 (a+1)_n$.

then,

H.A $\quad F\left[\begin{matrix} a, c-b; & \frac{-z}{1-z} \\ c; & \end{matrix}\right] = (1-z)^{c-b} F\left[\begin{matrix} c-a, c-b; & z \\ c; & \end{matrix}\right]$

Ex $\quad \frac{d}{dx}\, {}_2F_1\left[\begin{matrix} a, b; & x \\ c; & \end{matrix}\right] = \frac{ab}{c}\, {}_2F_1\left[\begin{matrix} a+1, b+1; & x \\ c+1; & \end{matrix}\right]$.

$$\frac{d}{dx}\sum_{n=0}^{\infty} \frac{(a)_n (b)_n x^n}{(c)_n \lfloor n} = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n\, n\, x^{n-1}}{(c)_n \lfloor n} \qquad \left[\frac{n}{n\lfloor n-1}\right.$$

$$= \sum_{n=1}^{\infty} \frac{(a)_n (b)_n \cancel{n}\, x^{n-1}}{(c)_n \lfloor n-1 \cdot \cancel{n}}$$

$$= \sum_{n=0}^{\infty} \frac{(a)_{n+1} (b)_{n+1} x^{n-1+1}}{(c)_{n+1} \lfloor n-1+1} = \sum_{n=0}^{\infty} \frac{(a)_{n+1} (b)_{n+1}}{(c)_{n+1} \lfloor n} x^n$$

$$= \frac{ab}{c} \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n x^n}{(c+1)_n \lfloor n}$$

$$= \frac{ab}{c}\, {}_2F_1\left[\begin{matrix} a+1, b+1; & x \\ c+1; & \end{matrix}\right]$$

Ex $\quad {}_2F_1\left[\begin{matrix} -n, b; & 1 \\ c; & \end{matrix}\right] = \frac{(c-b)_n}{(c)_n}$

$$= \frac{\Gamma c\, \Gamma(c+n-b)}{\Gamma(c+n)\, \Gamma(c-b)} = \frac{(c-b)_n}{(c)_n}.$$

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Theore

Proo

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Theorem: If $2b$ is neither zero nor negative integer, $|y|<\frac{1}{2}$, $\left|\frac{y}{1-y}\right|<1$, then $(1-y)^{-a} F\left[\begin{matrix}\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; \\ b+\frac{1}{2};\end{matrix} \frac{y^2}{(1-y)^2}\right]$

$$= {}_2F_1\left[\begin{matrix}a, b; \\ 2b;\end{matrix} 2y\right]$$

Proof: L.H.S:

$$(1-y)^{-a}\sum_{k=0}^{\infty}\frac{(\frac{1}{2}a)_k(\frac{1}{2}a+\frac{1}{2})_k\, y^{2k}}{(b+\frac{1}{2})_k \lfloor k\, (1-y)^{2k}}$$

$$=\sum_{k=0}\frac{(\frac{1}{2}a)_k(\frac{1}{2}a+\frac{1}{2})_k\, y^{2k}(1-y)^{-(a+2k)}}{(b+\frac{1}{2})_k \lfloor k}$$

$$= {}_2F_1\left[\begin{matrix}a, b; \\ 2b;\end{matrix} 2y\right]$$

On using formula $(1-x)^{-a}=\sum_{n=0}^{\infty}\frac{(a)_n x^n}{\lfloor n}$, we get

$$=\sum_{k=0}^{\infty}\frac{(\frac{1}{2}a)_k(\frac{1}{2}a+\frac{1}{2})_k\, y^{2k}}{(b+\frac{1}{2})_k \lfloor k}\sum_{n=0}^{\infty}\frac{(a+2k)_n\, y^n}{\lfloor n}$$

Since $(a)_{2k}=2^{2k}\cdot\left(\frac{a}{2}\right)_k\left(\frac{a+1}{2}\right)_k$

$$\Rightarrow \frac{(a)_{2k}}{2^{2k}}=\left(\frac{a}{2}\right)_k\left(\frac{a+1}{2}\right)_k$$

$$=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{2k}(a+2k)_n\, y^{n+2k}}{2^{2k}(b+\frac{1}{2})_k \lfloor k \lfloor n} \qquad [(a)_{2k}(a+2k)_n=(a)_{n+2k}]$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a)_{n+2k}(y)^{2k}}{2^{2k}(b+\frac{1}{2})_k \lfloor k \lfloor n}$$

$$\left[\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}A(k,n-2k)\right]$$

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$\Gamma 0 = \infty$

[Factorial - value is from of integer.
gamma - value is from of fraction/not integer
↳ not negative.

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(a)_{n-2\cancel{k}+2\cancel{k}}\; y^{n-2k+2k}}{2^{2k}\,(b+\frac{1}{2})_k\, k!\, (n-2k)!}$$

$$\left[(n-2k)! = (1)_{n-2k} = (1)_n (1+n)_{-2k} = \frac{n!\,(-1)^{2k}}{(1-1-n)_{2k}} = \frac{n!}{(-n)_{2k}} \right]$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(a)_n\, y^n\, (-n)_{2k}}{2^{2k}\,(b+\frac{1}{2})_k\, k!\, n!\, (-1)^{2k}}$$

$$\left[\text{Since } (a)_{2k} = 2^{2k} \left(\tfrac{a}{2}\right)_k \left(\tfrac{a+1}{2}\right)_k \right]$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(a)_n\, y^n\, 2^{2k} \left(-\frac{n}{2}\right)_k \left(-\frac{n}{2}+\frac{1}{2}\right)_k}{2^{2k}\,(b+\frac{1}{2})_k\, k!\, n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a)_n\, y^n}{n!} \frac{\left(-\frac{n}{2}\right)_k \left(-\frac{n}{2}+\frac{1}{2}\right)_k (1)^k}{(b+\frac{1}{2})_k\, k!}$$

$$= \sum_{n=0}^{\infty} {}_2F_1\left[\begin{matrix} -\frac{n}{2}, & -\frac{n}{2}+\frac{1}{2} & ; \\ & b+\frac{1}{2} & ; \end{matrix}\; 1\right] \frac{(a)_n\, y^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})\, \Gamma(b+\cancel{\tfrac{1}{2}}+\frac{n}{2}+\frac{n}{2}-\cancel{\tfrac{1}{2}})}{\Gamma(b+\frac{1}{2}+\frac{n}{2})\, \Gamma(b+\cancel{\tfrac{1}{2}}+\frac{n}{2}-\cancel{\tfrac{1}{2}})} \cdot \frac{(a)_n\, y^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(b+n)\, (a)_n\, y^n}{(b+\frac{1}{2})_{n/2}\, \Gamma(b+\frac{n}{2})\, n!} \qquad \left[\frac{\Gamma(a+n)}{\Gamma(a)} = (a)_n \right]$$

or $$= \sum_{n=0}^{\infty} \frac{\Gamma(b+n)\, (a)_n\, \Gamma(b) \cdot y^n}{\Gamma(b)\, (b+\frac{1}{2})_{n/2}\, \Gamma(b+\frac{n}{2})\, n!}$$

$$\left[(2a)_{\left(\frac{2k}{2}\right)} = 2^{2k/2}\left(\frac{2a}{2}\right)_{k/2}\left(\frac{2a+1}{2}\right)_{k/2} = 2^{k}(a)_{k/2}\left(\frac{2a+1}{2}\right)_{k/2} \quad\middle|\quad (2b)_{2n} \to N = 2^{2n}\left(\frac{2b}{2}\right)_{n/2}\left(\frac{2b+1}{2}\right)_{n/2}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n y^n}{(b+1/2)_{n/2}\,(b)_{n/2}\, \lfloor n \cdot 2^n}$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (2y)^n}{(2b)_n \lfloor n} = {}_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix}\; 2y\right].$$

Ex Prove that if $g_n = {}_2F_1\left[\begin{matrix} -n, \alpha; \\ 1+\alpha-n; \end{matrix}\; 1\right]$

α is not an integer, then $g_n = 0$, for $n \geq 1$, $g_0 = 1$

Proof:

$$= \frac{\Gamma(1+\alpha-n)\,\Gamma(1+\cancel{\alpha}-\cancel{n}+\cancel{n}-\cancel{\alpha})}{\Gamma(1+\alpha-n+n)\,\Gamma(1+\alpha-n-\alpha)}$$

$$= \frac{\Gamma(1+\alpha-n)}{\Gamma(1+\alpha)}\,\frac{\Gamma 1}{\Gamma(1-n)}$$

$$= \frac{1}{(1+\alpha)\,\Gamma(1-n)}$$

$$= \frac{(1+1-\alpha)_n}{(-1)^n\,\Gamma 1-n}$$

$$= \frac{(-\alpha)_n}{(-1)^n\,\Gamma 1-n}$$

$$= \frac{\Gamma(n-\alpha)}{\Gamma(-\alpha)(-1)^n\,\Gamma(1-n)}$$

$$\left[\begin{aligned} (a)_{n-k} &= (a)_n (a+n)_k \\ &= \frac{(a)_n (-1)^k}{(1-a-n)_k} \\ (a)_n &= \frac{(-1)^n}{(1-a)_n} \\ &= \frac{(-1)^n \cdot \Gamma 1-a}{\Gamma(1-a+n)} \end{aligned}\right]$$

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IMP

Generalize Hypergeometric function

$$_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix}\ z\right]$$

$$_pF_q\left[\begin{matrix} a_1, a_2 \ldots\ldots a_p; \\ b_1, b_2 \ldots\ldots b_q; \end{matrix}\ z\right]$$

p = parameters (Num.)
q = parameters (Dano.)

IMP

<u>Convergence Conditions</u>

i) If $p \leq q$, then Series will conv. for all finite ~~vector~~ values of z.

$$_1F_3\left[\begin{matrix} a & ; \\ b_1, b_2, b_3 & ; \end{matrix}\ z\right] = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b_1)_n (b_2)_n (b_3)_n \lfloor n}$$

$$e^z = {}_0F_0\left[\begin{matrix} \text{—} & ; \\ \text{—} & ; \end{matrix}\ z\right] = \sum_{n=0}^{\infty} \frac{z^n}{\lfloor n}$$

ii) $p = q+1$, conv. for $|z| < 1$
& div. for $|z| > 1$.

iii) $p > q+1$, the series will div for $z \neq 0$.

iv) $p = q+1$, then absolute conv. at $z = 1$,
if $Re\left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right) > 0$.

$(-n)(n+1)(n+2)+ \cdots (-n+(n-1))(-\cancel{n}+\cancel{n})$

IMP Diff b/w Hypergeometric function & Hypergeometric Polynomials

$${}_0F_0\left[\begin{matrix}-;\\ -;\end{matrix}\, z\right] = e^z = \sum_{n=0}^{\infty} \frac{z^n}{\lfloor n}$$

$${}_1F_0\left[\begin{matrix}a;\\ -;\end{matrix}\, z\right] = (1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{\lfloor n}$$

$${}_0F_1\left[\begin{matrix}-;\\ b;\end{matrix}\, z\right] = \sum_{n=0}^{\infty} \frac{z^n}{(b)_n \lfloor n} \rightarrow \text{family of Bessel function}$$

$${}_2F_1 = \left[\begin{matrix}a, b;\\ c;\end{matrix}\, z\right]$$

IMP

$${}_2F_1\left[\begin{matrix}-n, a;\\ c;\end{matrix}\, z\right] \rightarrow \text{Hypergeometric Polynomials}$$

We have already proved that

$${}_2F_1\left[\begin{matrix}c-a, c-b;\\ c;\end{matrix}\, z\right] = (1-z)^{(a+b-c)}\, {}_2F_1\left[\begin{matrix}a, b;\\ c;\end{matrix}\, z\right]$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(c-a)_n (c-b)_n z^n}{(c)_n \lfloor n} = (1-z)^{-(c-a-b)} \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k \lfloor k}$$

$$= \sum_{n=0}^{\infty} \frac{(c-a-b)_n}{\lfloor n} z^n \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k \lfloor k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(c-a-b)_n (a)_k (b)_k z^{n+k}}{\lfloor n\, (c)_k \lfloor k}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(c-a)(c-b)_n z^n}{\lfloor n\, (c)_n} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(c-a-b)_{n-k} (a)_k (b)_k z^{n-k+k}}{\lfloor n-k \lfloor k\, (c)_k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(c-a-b)_n (c-a-b+n)_{-k} (a)_k (b)_k z^n (-n)_k}{\lfloor n (-1)^k \lfloor k\, (c)_k}$$

since $(1)_{n-k} = \lfloor n-k = \dfrac{\lfloor n (-1)^k}{(-n)_k}$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(c-a-b)_n z^n}{\lfloor n} \frac{\cancel{(-1)^k}}{\cancel{(-1)^k}} \frac{(a)_k (b)_k (-n)_k}{(1-c+a+b-n)_k \lfloor k\, (c)_k}$$

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non-integer → gamma function

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$$\sum_{n=0}^{\infty} \frac{(c-a)_n (c-b)_n z^n}{\lfloor n \, (c)_n} = \sum_{n=0}^{\infty} {}_3F_2\left[\begin{matrix} -n, a, k; \\ -n+1-c+a+b; \end{matrix} \; 1\right] \frac{(c-a-b)_n z^n}{\lfloor n}$$

$$\Rightarrow \boxed{{}_3F_2\left[\begin{matrix} -n, a, b; \\ 1-a+b+c-n; \end{matrix} \; 1\right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}}$$

SAALSCHITZ THEOREM.

Special case:

OR ${}_3F_2\left[\begin{matrix} -n, a+n, \frac{1}{2}+\frac{1}{2}a-b ; \\ 1+a-b ; \frac{1}{2}+\frac{1}{2}a ; \end{matrix} \; 1\right] = \frac{(b)_n}{(1+a-b)_n}$

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Saalschitz Theorem:

$${}_3F_2\left[\begin{matrix} -n, a, b \\ c, 1-c+a+b-n ; \end{matrix} \; 1\right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}.$$

$${}_3F_2\left[\begin{matrix} -n, a+n, \frac{1}{2}+\frac{1}{2}a-b ; \\ 1+a-b , \frac{1}{2}a+\frac{1}{2} ; \end{matrix} \; 1\right] = \frac{(b)_n}{(1+a-b)_n}$$

$a = a+n$

$b = \frac{1}{2}+\frac{1}{2}a-b$

$c = 1+a-b;$

L.H.S $1-c+a+b-n = 1-1-a+b+a+n+b-n$

R.H.S. $\frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n} = \frac{(1+\cancel{a}-b-\cancel{a})_n^{-n} (1+a-\cancel{b}-\frac{1}{2}a-\frac{1}{2}+\cancel{b})_n}{(1+a-b)_n (1+\cancel{a}-\cancel{b}-\cancel{a}-n-\frac{1}{2}-\frac{1}{2}a+b)}$

$$= \frac{(1-b-n)_n \left(\tfrac{1}{2}+\tfrac{a}{2}\right)_n}{(1+a-b)_n \left(\tfrac{1}{2}-\tfrac{1}{2}a-n\right)_n}$$

$$= \frac{\Gamma(1-b-\cancel{n}+\cancel{n})\,\Gamma\left(\tfrac{1}{2}+\tfrac{a}{2}+n\right)\Gamma(1+a-b)\,\Gamma\left(\tfrac{1}{2}-\tfrac{1}{2}a-n\right)}{\Gamma(1-b-n)\,\Gamma\left(\tfrac{1}{2}+\tfrac{a}{2}\right)\Gamma(1+a-b+n)\,\Gamma\left(\tfrac{1}{2}-\tfrac{1}{2}a-\cancel{n}+\cancel{n}\right)}$$

$$= \frac{\Gamma(1-b)}{\Gamma(1-b-n)} \cdot \frac{\left(\tfrac{1}{2}-\tfrac{1}{2}a\right)_{+n}}{\left(\tfrac{1}{2}+\tfrac{a}{2}\right)_n (1+a-b)_n}$$

$$= \frac{1 \cdot \cancel{\left(\tfrac{1}{2}-\tfrac{1}{2}a\right)_n}}{(1-b)_n \cancel{\left(\tfrac{1}{2}-\tfrac{a}{2}\right)_n} (1+a-b)_n}$$

$$= \frac{(1-1+b)_n \left(\tfrac{1}{2}+\tfrac{1}{2}a\right)_n}{(-1)^n (1-a}$$

check

$$= \frac{(b)_n}{(1+a-b)_n}$$

Theorem: If n is a non-negative integer and if a and b are independent of n, then

$$ {}_3F_2\left[\begin{matrix}-n, a+n, \frac{1}{2}+\frac{1}{2}a-b; \\ 1+a-b, \frac{1}{2}a+\frac{1}{2};\end{matrix}\ 1\right] = \frac{(b)_n}{(1+a-b)_n}. $$

Proof: Since

$$ {}_2F_1\left[\begin{matrix}a, b; \\ 1+a-b;\end{matrix}\ z\right] = (1-z)^{-a}\, {}_2F_1\left[\begin{matrix}\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}a-b; \\ 1+a-b;\end{matrix}\ \frac{-4z}{(1-z)^2}\right] $$

$$ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(1+a-b)_n \lfloor n} = \sum_{k=0}^{\infty} \frac{(\frac{1}{2}a)_k (\frac{1}{2}+\frac{1}{2}a-b)_k (-4)^k z^k}{\lfloor k\, (1+a-b)_k (1-z)^{2k+a}} $$

$$ = \sum_{k=0}^{\infty} \frac{(\frac{1}{2}a)(\frac{1}{2}+\frac{1}{2}a-b)_k (-1)^k \cdot 2^{2k} \cdot z^k}{\lfloor k\, (1+a-b)_k} \sum_{n=0}^{\infty} \frac{(a+2k)_n z^n}{\lfloor n} $$

num/din ...ly hoga

$$ = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_k \left(\frac{1}{2}+\frac{1}{2}a-b\right)_k (-1)^k 2^{2k} \cdot z^{k+n} (a)_{2k} (a+2k)_n}{\lfloor k\, (1+a-b)_k \lfloor n\, (a)_{2k}} $$

$$ = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{\cancel{(\frac{1}{2}a)_k} (\frac{1}{2}+\frac{1}{2}a-b)_k (-1)^k \cancel{2^{2k}} (a)_{2k+n}\, z^{n+k}}{\lfloor k\, (1+a-b)_k \lfloor n\, \cancel{2^{2k}} \cancel{(\frac{a}{2})_k} (\frac{1}{2}+\frac{a}{2})_k} $$

$$ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(1+a-b)_n \lfloor n} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(a)_{n+k} (\frac{1}{2}+\frac{1}{2}a-b)_k z^n (-1)^k}{\lfloor k\, (1+a-b)_k \lfloor n-k\, (\frac{1}{2}+\frac{1}{2}a)_k} $$

$$ \Rightarrow \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(a)_n (a+n)_k (\frac{1}{2}+\frac{1}{2}a-b) z^n \cancel{(-1)^k} (-n)_k}{\lfloor n\, \cancel{(-1)^k} \lfloor k\, (1+a-n)_k (\frac{1}{2}+\frac{a}{2})_k} $$

$$ \sum_{n=0}^{\infty} \frac{(a)_n z^n}{\lfloor n}\, {}_3F_2\left[\begin{matrix}-n, a+n, \frac{1}{2}+\frac{a}{2}-b; \\ 1+a-b, \frac{1}{2}a+\frac{1}{2};\end{matrix}\ 1\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(1+a-b)_n \lfloor n} $$

$$ \Rightarrow {}_3F_2\left[\begin{matrix}-n, a+n, \frac{1}{2}+\frac{a}{2}-b; \\ 1+a-b, \frac{1}{2}a+\frac{1}{2};\end{matrix}\ 1\right] = \frac{(b)_n}{(1+a-b)_n} $$

Ex $ {}_0F_1\left[\begin{matrix} - ; & x \\ a ; & \end{matrix}\right] \cdot {}_0F_1\left[\begin{matrix} - ; & \\ b ; & x \end{matrix}\right] = {}_2F_3\left[\begin{matrix} \frac{1}{2}+\frac{1}{2}a, \ \frac{1}{2}a+\frac{1}{2}b-\frac{1}{2} ; & 4x \\ a, b, a+b-1 ; & \end{matrix}\right]$

Sol^n $\sum_{n=0}^{\infty} \frac{x^n}{(a)_n \lfloor n} \sum_{k=0}^{\infty} \frac{x^k}{(b)_k \lfloor k} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{x^{n+k}}{(a)_n \lfloor n (b)_k \lfloor k}$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{x^{n-k+k}}{(a)_{n-k} \lfloor n-k \, (b)_k \lfloor k}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{x^n (-n)_k}{(a)_n (a+n)_k \lfloor n (-1)^k (b)_k \lfloor k}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{x^n (1-a-n)_k (-n)_k}{(a)_n (-1)^k \lfloor n (-1)^k (b)_k \lfloor k}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{x^n (-n)_k (1-a-n)_k}{(a)_n \lfloor n \ (b)_k \lfloor k}$$

$$= \sum_{n=0}^{\infty} {}_2F_1\left[\begin{matrix} -n, 1-a-n ; & 1 \\ b ; & \end{matrix}\right] \frac{x^n}{(a)_n \lfloor n}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma b \ \Gamma(b+n-1+a+n)}{\Gamma(b+n) \ \Gamma(b+a+n-1)} \cdot \frac{x^n}{(a)_n \lfloor n}$$

$$= \sum_{n=0}^{\infty} \frac{(a+b-1+n)_n}{(b)_n (a)_n \lfloor n} \cdot x^n$$

not use [$(a+b-1)_{2n} = (a+b-1+n)_n$

$(a+b-1)_{n+n} = (a+b-1)_n (a+b-1+n)_n$

$\Rightarrow \frac{(a+b-1)_{2n}}{(a+b-1)_n} = (a+b-1+n)_n$

1 line left

$$\frac{2^{2n}\left(\frac{a+b-1}{2}\right)_n \left(\frac{a+b-\cancel{1}+\cancel{1}}{2}\right)_n}{(a+b-1)_n}$$

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Whipple's Theorem:

If n is a non-negative integer and if b and c are independent of n, then

$${}_3F_2\left[\begin{matrix}-n, b, c; \\ b-n, 1-c-n;\end{matrix}\ x\right] = (1-x)^n\, {}_3F_2\left[\begin{matrix}-\tfrac12 n, -\tfrac12 n+\tfrac12, 1-b-c-n; \\ 1-b-n, 1-c-n;\end{matrix}\ \frac{-4x}{(1-x)^2}\right]$$

Proof: Let ${}_2F_1\left[\begin{matrix}b, c; \\ b+c;\end{matrix}\ t(1-x+xt)\right]$

Case: I

$$= \sum_{n=0}^{\infty} \frac{(b)_n (c)_n\, k^n [1-x+xt]^n}{(b+c)_n \lfloor n}$$

$$= \sum_{n=0}^{\infty} \frac{(b)_n (c)_n\, t^n (1-x)^n}{(b+c)_n \lfloor n}\left[1+\frac{xt}{1-x}\right]^n$$

On using the formula $(1-z)^{-a} = \sum \frac{(a)_n z^n}{\lfloor n}$

$$= \sum_{n=0}^{\infty} \frac{(b)_n (c)_n\, t^n (1-x)^n}{(b+c)_n \lfloor n} \sum_{k=0}^{n} \frac{(-n)_k (-1)^k x^k t^k}{\lfloor k\, (1-x)^k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(b)_n (c)_n\, t^{n+k} (1-x)^{n-k} (-n)_k}{\lfloor n\, (b+c)_n \lfloor k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(b)_n (c)_n\, t^{n+k} (1-x)^{n-k} \cancel{\lfloor n}\, (-1)^k x^k}{\cancel{\lfloor n}\, (b+c)_n \lfloor k \lfloor n-k}$$

$$\left[\begin{aligned} \text{since } (1)_{n-k} &= \lfloor n-k = (1)_n (1+n)_{-k} \\ &= \frac{\lfloor n\, (-1)^k}{(1-1-n)_k} \\ \text{or } (-n)_k &= \frac{\lfloor n\, (-1)^k}{\lfloor n-k} \end{aligned}\right]$$

$$\left[\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n-k) \\ \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k,n) &= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} A(k, n-k) \rightarrow \text{Imp} \end{aligned}\right.$$

$$\frac{1}{(1-x)^2}$$

$$\left[\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} A(k,n-2k)\right]$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} \frac{(b)_{n-k}(c)_{n-k}\, t^{n-\cancel{k}+k}\, x^k (1-x)^{n-k-k}}{(b+c)_{n-k}\, \lfloor(n-k-k)\, \lfloor k} (-1)^k$$

or $$= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} \frac{(b)_n (b+n)_{-k} (c)_n (c+n)_{-k}\, t^n x^k (1-x)^{n-2k} (-1)^k}{(b+c)_n (b+c+n)_{-k}\, \lfloor n-2k\, \lfloor k}$$

or $$= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} \frac{(b)_n (c)_n (-1)^k (-1)^k (-1)^k\, t^n x^k (1-x)^{n-2k}\, (-n)_{2k}\, (1-b-c-n)_k}{(b+c)_n (1-b-n)_k (1-c-n)_k\, \cancel{(-1)^k}\, \lfloor n\, \lfloor k}$$

$$\left[\begin{aligned}(1)_{n-2k} &= (1)_n (1+n)_{-2k} \\ &= \frac{\lfloor n\, (-1)^{2k}}{(1-1-n)_{2k}} = \frac{\lfloor n}{(-n)_{2k}}\end{aligned}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(b)_n (c)_n t^n (1-x)^n}{(b+c)_n \lfloor n} \sum_{k=0}^{[n/2]} \frac{2^{2k} (-n/2)_k (-n/2+1/2)_k}{(1-b-n)_k (1-c-n)_k \lfloor k}$$

$$= \sum_{n=0}^{\infty} \frac{(b)_n (c)_n t^n (1-x)^n}{(b+c)_n \lfloor n} \sum_{k=0}^{[n/2]} \frac{2^{2k} (-n/2)_k (-n/2+1/2)_k}{(1-b-n)_k (1-c-n)_k \lfloor k} \left\{\frac{4x}{(1-x)^2}\right\}^k$$

$$(-1)^k \cdot (1-b-c-n)_k$$

$$\sum_{n=0}^{\infty} {}_3F_2\left[\begin{matrix}-n/2, -n/2+1/2, 1-b-c-n; \\ 1-b-n, 1-c-n;\end{matrix}\ \frac{-4x}{(1-x)^2}\right] \frac{(b)_n (c)_n t^n (1-x)^2}{(b+c)_n \lfloor n} \quad \text{—— (1)}$$

Case-II:

$${}_2F_1\left[\begin{matrix} b, c; \\ ; \end{matrix}\ t(1-x+xt)\right]$$

or $\quad {}_2F_1\left[\begin{matrix} b, c; & t\{1-x(1-t)\} \\ b+c; & \end{matrix}\right]$

$$= \sum_{n=0}^{\infty} \frac{(b)_n (c)_n\, t^n \{1-x(1-t)\}^n}{\lfloor n \,(b+c)_n}$$

$$= \sum_{n=0}^{\infty} \frac{(b)_n (c)_n\, t^n}{\lfloor n \,(b+c)_n} \sum_{k=0}^{n} \frac{(-n)_k\, x^k (1-t)^k}{\lfloor k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(b)_n (c)_n\, t^n (-1)^k \cancel{\lfloor n}\; x^k (1-t)^k}{\cancel{\lfloor n}\,(b+c)_n \,\lfloor n-k \;\lfloor k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(b)_{n+k} (c)_{n+k}\, t^{n+k} (-1)^k x^k (1-t)^k}{(b+c)_{n+k}\, \lfloor(n+k-k)\; \lfloor k}$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(b)_{k+n}\, (c)_{k+n}\; t^{k+n}\, (-1)^k x^k (1-t)^k}{(b+c)_{k+n}\, \lfloor n \;\lfloor k}$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(b)_k (b+k)_n (c)_k (c+k)_n\, t^{n+k} (-1)^k x^k (1-t)^k}{(b+c)_k (b+c+k)_n \lfloor n \,\lfloor k}$$

$$= \sum_{n=0}^{\infty} \frac{(b+k)_n (c+k)_n\, t^n}{(b+c+k)_n \lfloor n} \sum_{k=0}^{\infty} \frac{(b)_k (c)_k (-xt)^k (1-t)^k}{(b+c)_k \lfloor k}$$

$$\sum_{k=0}^{\infty} {}_2F_2\left[\begin{matrix} b+k, c+k; & t \\ b+c+k; & \end{matrix}\right] \frac{(b)_k (c)_k (-xt)^k (1-t)^k}{(b+c)_k \lfloor k}$$

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Note:

$${}_2F_1\left[\begin{matrix} b+k, c+k ; \\ b+c+k ; \end{matrix} t\right] = (1-t)^{-k}\, {}_2F_1\left[\begin{matrix} b, c ; \\ b+c+k ; \end{matrix} t\right] \text{ — iii)}$$

From (ii) and (iii) we get,

$$= \sum_{k=0}^{\infty} \cancel{(1-t)^{-k}}\, {}_2F_1\left[\begin{matrix} b, c ; t \\ b+c+k \end{matrix}\right] \frac{(b)_k (c)_k (-xt)^k \cancel{(1-t)^k}}{(b+c)_k \lfloor k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(b)_n (c)_n t^n (b)_k (c)_k (-xt)^k}{(b+c+k)_n (b+c)_k \lfloor k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(b)_n (c)_n t^{n+k} (b)_k (c)_k x^k (-1)^k}{(b+c)_{n+k} \lfloor n \lfloor k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(b)_{n-k} (c)_{n-k} \lfloor n-k+k \, (b)_k (c)_k x^k \cancel{(-1)^k}}{(b+c)_{n-k+k} \lfloor n-k \lfloor k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(b)_n (b+n)_k (c)_n (c+n)_k t^n (b)_k (c)_k (-1)^k (x)^k (-n)_k}{(b+c)_n \cdot (-1)^k \lfloor n \lfloor k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(b)_n (c)_n \lfloor n (-1)^k (-1)^k (-n)_k (b)_k (c)_k x^k}{(b+c)_n \lfloor n (1-b-n)_k (1-c-n)_k}$$

$$\Rightarrow \sum_{n=0}^{\infty} {}_3F_2\left[\begin{matrix} -n; b, c; \\ 1-b-n, 1-c-n; \end{matrix} x\right] \frac{(b)_n (c)_n t^n}{(b+c)_n \lfloor n} \text{ —}$$

From (i) and (iv) we get result

$(1+a)_{2k} = \dfrac{\Gamma(1+a+2k)}{\Gamma(1+a)}$

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$${}_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix}\ 1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}$$

Theorem: (for non-terminating Series)

If neither $(a-b)$ nor $(n-c)$ nor a negative integer then
$${}_3F_2\left[\begin{matrix} a, b, c; \\ 1+a-b, 1+a-c; \end{matrix}\ x\right] = (1-x)^{-a}\, {}_3F_2\left[\begin{matrix} \tfrac{1}{2}a, \tfrac{1}{2}a+\tfrac{1}{2}, 1+a-b-c; \\ 1+a-b, 1+a-c; \end{matrix}\ \frac{-4x}{(1-x)^2}\right]$$

Proof: L.H.S.
$${}_3F_2\left[\begin{matrix} a, b, c; \\ 1+a-b, 1+a-c; \end{matrix}\ x\right] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (c)_k\, x^k}{(1+a-b)_k (1+a-c)_k\, k!}$$

$(1+a)_{2k}$ multiply and divide
$$= \sum_{k=0}^{\infty} \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+a+2k)\,(a)_k (b)_k (c)_k\, x^k \cdot \Gamma(1+a-b-c)}{\Gamma(1+a-b+k)\,\Gamma(1+a-c+k)\,\Gamma(1+a)\,(1+a)_{2k}\, k!\, \Gamma(1+a-b-c)}$$

$$= \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (c)_k\, x^k\, \Gamma(1+a+2k)\,\Gamma(1+a-b-c)}{(1+a)_{2k}\, k!\, \Gamma(1+a-b+k)\,\Gamma(1+a-c+k)} \cdot \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)}$$

$$= \underbrace{\frac{\Gamma(1+a-b)\,\Gamma(1+a-c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)}}_{=A} \sum_{k=0}^{\infty} {}_2F_1\left[\begin{matrix} b+k, c+k; \\ 1+a+2k; \end{matrix}\ 1\right] \frac{(a)_k (b)_k (c)_k\, x^k}{(1+a)_{2k}\, k!}$$

$$= A \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (c)_k\, x^k}{(1+a)_{2k}\, k!} \sum_{n=0}^{\infty} \frac{(b+k)_n (c+k)_n}{n!\,(1+a+2k)_n}$$

$$= A \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_k (b)_k (c)_{k+n}\, x^k}{(1+a)_{2k+n}\, k!\, n!}$$

$$= A \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(b)_{n-\cancel{k}+\cancel{k}} (c)_{n-\cancel{k}+\cancel{k}} (a)_k\, x^k}{(1+a)_{n-k+2k}\, (n-k)!\, k!}$$

$$\left[(n-k)! = \frac{(-1)^k\, n!}{(-n)_k} \;\middle|\; (1+a)_{n+k} = (1+a)_n (1+a+n)_k \right]$$

$$= A \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(b)_n (c)_n (-n)_k (a)_k x^k}{(1+a)_n \lfloor n \, (1+a+n)_k \lfloor k \, (-1)^k}$$

$$= A \sum_{n=0}^{\infty} {}_2F_1\left[\begin{matrix} -n, a & ; \\ 1+a+n & ; \end{matrix} -x\right] \frac{(b)_n (c)_n}{(1+a)_n \lfloor n} \quad \text{—— (1)}$$

on using the formula;

$$= A\, {}_2F_1\left[\begin{matrix} -n, a & \\ 1+a+n, & \end{matrix} -x\right] = (1-x)^{-a} {}_2F_1\left[\begin{matrix} \tfrac{1}{2}a, \tfrac{1}{2}a+\tfrac{1}{2} & ; \\ 1+a+n & ; \end{matrix} -\frac{4x}{(1-x)^2}\right]$$

then (1) becomes,

$$= A \sum_{n=0}^{\infty} (1-x)^{-a} {}_2F_1\left[\begin{matrix} \tfrac{1}{2}a, \tfrac{1}{2}a+\tfrac{1}{2} & ; \\ 1+a+n & ; \end{matrix} -\frac{4x}{(1-x)^2}\right] \frac{(b)_n (c)_n}{(1+a)_n \lfloor n}$$

$$= A \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(b)_n (c)_n (\tfrac{1}{2}a)_k (\tfrac{1}{2}a+\tfrac{1}{2})_k (-1)^k 2^{2k} x^k}{(1+a)_n \lfloor n \lfloor k (1+a+n)_k (1-x)^{2k+a}}$$

$$= A \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(b)_n (c)_n (a)_{2k} (-1)^k x^k}{(1+a)_{n+k} \lfloor n \lfloor k (1-x)^{2k+a}}$$

$$\left[\begin{aligned} (1+a)_{n+k} &= (1+a)_{k+n} \\ &= (1+a)_k (1+a+k)_n \end{aligned}\right]$$

$$= A \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{2k} (-1)^k x^k (b)_n (c)_n}{\lfloor k (1+a)_k (1+a+k)_n \lfloor n}$$

$$= A \sum_{k=0}^{\infty} \frac{(a)_{2k} (-1)^k x^k}{(1+a)_k \lfloor k} \cdot {}_2F_1\left[\begin{matrix} b, c & ; \\ 1+a+k & ; \end{matrix} 1\right]$$

$$= A \sum_{k=0}^{\infty} \frac{(a)_{2k}(-1)^k x^k}{(1+a)_k \lfloor k} \frac{\Gamma(1+a+k)\,\Gamma(1-a-b-c+k)}{\Gamma(1+a+k-b)\,\Gamma(1+a-c-k)}$$

$$= (1-x)^{-a} \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)} \sum_{k=0}^{\infty} (\tfrac{1}{2}a)(\tfrac{1}{2}a+\tfrac{1}{2})_k \left\{\frac{-4x}{(1-x)^2}\right\}^k \frac{\quad}{(1+a-b)_k(1+a-c)_k}$$

$$= (1-x)^{-a} \frac{\cancel{\Gamma(1+a-b)}\,\cancel{\Gamma(1+a-c)}}{\cancel{\Gamma(1+a)}\,\cancel{\Gamma(1+a-b-c)}} \sum_{k=0}^{\infty} (\tfrac{1}{2})a(\tfrac{1}{2}a+\tfrac{1}{2})_k \left\{\frac{-4x}{(1-x)^2}\right\}^k \frac{(1+a-b-c)_k}{(1+a-b)_k(1+a-c)_k}$$

$$= (1-x)^{-a}\, {}_3F_2\left[\begin{matrix} \tfrac{1}{2}a,\ \tfrac{1}{2}a+\tfrac{1}{2},\ 1+a-b-c; & \dfrac{-4x}{(1-x)^2} \\ 1+a-b,\ 1+a-c\ ; & \end{matrix}\right]$$