2019 Challenge

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1 Introduction

The problem, as presented by Dwarka (2019), is as follows:

> Consider the digits 2, 0, 1, and 9. Using each digit only once, create a formula which generates all positive integers from 1 to 100. Any mathematical operator is allowed, including concatenation of digits.

In this paper, we solve this problem with elementary operations and also provide a general solution using more advanced and esoteric mathematical operators. For our purposes, an "elementary" operator is one of the basic infix operations: addition ($+$), subtraction ($-$), multiplication ($\times$), and division ($\div$). Concatenation of digits (i.e. using 2 and 0 to form 20) and exponentiation (including radicals) are also allowed.

The "advanced" operations are those which would be familiar to high-school level math. This includes: logarithms, basic and inverse trigonometric functions, and the probability functions ($x!$, $\binom{n}{r}$ or ${}^nC_r$, and ${}^nP_r$).

Finally, the "esoteric" level is all other operations that might aid in reducing the number of operations to find shortest solutions. This contains functions like the floor function ($\lfloor x \rfloor$) or the cis function ($\text{cis}(x) = \cos(x) + i\sin(x)$).

Every solution is assigned an operational complexity l. This value is defined as the number of operations required to reach the solution. For example, $\frac{a+b}{c} = (a+b) \div c$ has a complexity of 3. Concatenation does not count as an operation, so $20 + 19$ has a complexity of 1.

2 Specific Solutions

The following are the solutions we could find with the lowest operational complexity. In the problem, all digits do not have to be used, however, we chose to use all digits for all problems. Wherever possible, an effort is made to preserve the order of the digits as 2, 0, 1, 9 or its reverse.

Advanced and esoteric solutions are only shown if they are shorter or match the 2, 0, 1, 9 order where the elementary solution does not.

n	Elementary	(l)	Advanced	(l)	Esoteric	(l)
0	0×219	1				
1	$20 - 19$	1				
2	$2 + 0^{19}$	2			$\lfloor 2.019 \rfloor$	1
3	$12 - 09$	1				
4	$\sqrt{9} + 1^{20}$	3				
5	$2 + 01 + \sqrt{9}$	3				
6	$9 - 2 - 01$	2				
7	$9 - 2 \times 01$	2				
8	$9 - 1^{02}$	2				
9	$9 + 0^{12}$	2				
10	$9 + 1^{02}$	2				
11	$2^{01} + 9$	2				
12	$21 - 09$	1				
13	$12 + 9^0$	2				
14	$12 + 9^0$	2				
15	$12 + \sqrt{09}$	2	$(2 + 01)! + 9$	3		
16	$2^{01+\sqrt{3}}$	3	$2 \times 9 - 0! - 1$	4		
17	$2 \times 09 - 1$	2				
18	$2 \times 01 \times 9$	2				
19	$29 - 10$	1				
20	$2(01 + 9)$	2			$\lfloor 20.19 \rfloor$	1
21	$12 + 09$	1				
22	$20 - 1 + \sqrt{9}$	3	$2 + 0! + 19$	3	$\lceil 21.90 \rceil$	1
23	$20 + 1\sqrt{9}$	3			$\lceil 201 \div 9 \rceil$	2
24	$20 + 1 + \sqrt{9}$	3	$(2 + 0!)1\sqrt{9}$	5		
25	$.20^{1-\sqrt{9}}$	5	$\left(0! + 1 + \sqrt{9}\right)^2$	5		
26	$2(10 + \sqrt{9})$	3	$20^1\sqrt{9}!$	4		
27	$(2 + 01) \cdot 9$	2	$29 - 0! - 1$	3		
28	$29 - 01$	1				
29	$20^1 + 9$	2			$\lfloor 29.10 \rfloor$	1

n	Elementary	(l)	Advanced	(l)	Esoteric	(l)
30	$21+09$	1				
31			$29+0!+1$	3		
32	$2+10\sqrt{9}$	3				
33			$(2+0!+1)!+9$	5		
34	$102\div\sqrt{9}$	2				
35	$9/.2-10$	3			${}_9C_2-01$	2
36			$(2+0!+1)9$	4		
37			$2\times 19-1$	2		
38	2×019	1				
39	$20+19$	1				
40	$120\div\sqrt{9}$	2	$2(0!+19)$	3		
41	$1/.02-9$	3				
42			$21(\sqrt{9}-0!)$	4		
43			$9/.2-0!-1$	5		
44	$9/.2-01$	3				
45	$10/2\cdot 9$	2				
46	$9/.2+01$	3				
47	$10/.2-\sqrt{9}$	4			$\lfloor\ln 201^9\rfloor$	3
48			$(2+0!)!(-1+9)$	6	$\lceil\ln 201^9\rceil$	3
49			$(9-1-0!)^2$	4		
51	$1/.02+.\bar{9}$	5			$\lfloor 2^9\div 10\rfloor$	3
52			$-2+(.\bar{1})^{-0!}\cdot\sqrt{9}!$	10	$\lceil 2^9\div 10\rceil$	3
53	$10/.2-\sqrt{9}$	4				
54			$(2+01)!\times 9$	3	$\lfloor 109\div 2\rfloor$	2
55	$9/.2+10$	3			$\lceil 109\div 2\rceil$	2
56			$.02^{-1}+\sqrt{9}!$	6	${}_{9-1}P_{02}$	2
57	$(20-1)\sqrt{9}$	3				
58			$29(0!+1)$	3	$\lfloor\ln 19^{20}\rfloor$	3
59	$20\times\sqrt{9}+1$				$\lceil\ln 19^{20}\rceil$	3
60	$20\times 1\sqrt{9}$	3				

n	Elementary	(l)	Advanced	(l)	Esoteric	(l)
61	$20\sqrt{9}+1$	3				
62			$21\sqrt{9}-0!$	4		
63	$21\sqrt{09}$	2				
64			$2^{(0!+1)\sqrt{9}}$	5		
65			$(9-1)^2+0!$			
66			$(21+0!)\sqrt{9}$	4		
67	$201 \div \sqrt{9}$	2				
68						
69	$90-21$	1				
70	$210 \div \sqrt{9}$	2				
71	$91-20$	1				
72	$9(10-2)$	2				
73			$12(\sqrt{9}!)+0!$	5		
74			$9/.12-0!$	4		
75	$09/.12$	2				
76			$9/.12+0!$	4		
77			$-2-0!+.\bar{1}((\sqrt{9}!)!)$	10		
78	$90-12$	1				
79			$9^2-0!-1$	4		
80	$20(1+\sqrt{9})$	3				
81	$9^2+0(1)$	3				
82	$92-10$	1				
83			$9^2+0!+1$	4		
84			$90-(2+1)!$	3		
85	$90-1/.2$	3				
86			$91-0!/.2$	4		
87	$90-1-2$	2				
88	$9 \cdot 10-2$	2				
89	$91-02$	1				
90			$92-0!-1$	3		

n	Elementary	(l)	Advanced	(l)	Esoteric	(l)
91	$92 - 01$	1				
92	$2 + 10 \cdot 9$	2			$\lceil 91.02 \rceil$	1
93	$91 + 02$	1				
94			$2 + 0! + 91$	3		
95	$190 \div 2$	1				
96			$(2 + 1)! + 90$	3		
97	$10^2 - \sqrt{9}$	3				
98			$(-.2 + 0! + 9)/.1$	7		
99	$102 - \sqrt{9}$	2	$9(12 - 0!)$	3		
100	$(9 + 1)^{02}$	2				

3 General Formulae

All the general formulae below make use of "esoteric" level mathematical operations and functions.

3.1 Creating e

Throughout this section, we will use the combination of $\ln(x)$ and e to create numbers. e can be created from each of the four digits with the function $\exp(x) = e^x$. $\exp(1) = e$, so any digit can be first transformed to 1 and then to e.

Alternatively, if exp is considered cheating, the complex function $\text{cis}(\theta) = \cos(\theta) + i\sin(\theta)$ along with Euler's identity can be used. Once again, any digit can be transformed to 1, then i, then $\text{cis}(-i) = e^{i(-i)} = e$.

3.2 Integers

The general solution for we present and extend below is adapted from Wang (2019).

It is inspired by the famous four fours problem, which states that any positive whole number can be created with just four 4s, relying on the use of square roots:

$$n = -\log_{\frac{\sqrt{4}}{4}} \left(\log_4 \underbrace{\sqrt{\sqrt{\cdots \sqrt{\sqrt{4}}}}}_{n} \right) \tag{1}$$

where n = number of radicals. We can extend this to the general case by recognizing that the only requirement is that the base of the outer logarithm has to be two and the base of the inner logarithm has to be the same as the radicand. We can prove this:

$$
\begin{aligned}
n &= -\log_2 \left(\log_a \underbrace{\sqrt{\sqrt{\cdots \sqrt{\sqrt{a}}}}}_{k} \right) \\
&= -\log_2 \left(\log_a \left(a^{0.5^k} \right) \right) \\
&= -\log_2 \left(\frac{1}{2^k} \right) \\
&= -(-k) \\
&= k
\end{aligned}
$$

We can further simplify this expression by switching into base e. We can use this to write any integer with only two numbers:

$$
\begin{aligned}
n &= -\log_2 \left(\ln \sqrt{\cdots \sqrt{e}} \right) \\
&= -\log_2 \left(\ln \sqrt{\cdots \sqrt{\exp 1}} \right)
\end{aligned}
$$

This formula holds for all positive integers $n \in \mathbb{Z}^+$. Zero can be created with no square roots. By removing the leading negative, we can create all integers $n \in \mathbb{Z}$. To enforce the use of all four digits, simply add a $+0^9$ term to the end.

$$
n = -\log_2 \left(\ln \sqrt{\cdots \sqrt{\exp 1}} \right) + 0^9 \tag{2}
$$

The operational complexity for this formula is $l = n + 6$ for $n > 0$ and $l = n + 5$ for $n \leq 0$.

3.3 Rationals

Since we can represent any integer with two numbers, we can represent any rational number with two sets of two numbers. For any $n = \frac{a}{b}$, where $n \in \mathbb{Q}$:

$$\begin{aligned} n &= \frac{a}{b} \\ &= \frac{\log_2 \left(\ln \underbrace{\sqrt{\cdots \sqrt{\exp 1}}}_{a} \right)}{\log_2 \left(\ln \underbrace{\sqrt{\cdots \sqrt{\exp 1}}}_{b} \right)} \\ &= \log_{\ln \underbrace{\sqrt{\cdots \sqrt{\exp 1}}}_{b}} \left(\ln \underbrace{\sqrt{\cdots \sqrt{\exp 1}}}_{a} \right) \end{aligned}$$

Once again, applying to 2019:

$$n = \log_{\ln \sqrt{\cdots \sqrt{\exp(2^0)}}} \left(\ln \sqrt{\cdots \sqrt{\exp(1^9)}} \right) \tag{3}$$

Following the same logic as for integers, a negative sign can be prepended for negative rationals. Thus, the operational complexity is $l = 7 + a + b$ for $n \geq 0$ and $l = 8 + a + b$ for $n < 0$.

3.4 Complex

3.4.1 Gaussian Integers

Very similar to how we created the set of rationals, all Gaussian integers $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ can created as the sum of integers. For any $n \in \mathbb{Z}[i]$:

$$\begin{aligned} n &= a + bi \\ &= -\log_2 \left(\ln \underbrace{\sqrt{\cdots \sqrt{\exp 1}}}_{a} \right) - \log_2 \left(\ln \underbrace{\sqrt{\cdots \sqrt{\exp 1}}}_{b} \right) i \\ &= -\log_2 \left(\ln \sqrt{\cdots \sqrt{\exp 1}} \times \ln^i \left(\sqrt{\cdots \sqrt{\exp 1}} \right) \right) \\ &= -\log_2 \left(\ln \sqrt{\cdots \sqrt{\exp 1}} \times \ln^{\sqrt{-1}} \left(\sqrt{\cdots \sqrt{\exp 1}} \right) \right) \end{aligned}$$

Applying to 2019:

$$n = -\log_2 \left(\ln \sqrt{\cdots \sqrt{\exp 0!}} \times \ln^{\sqrt{-1}} \left(\sqrt{\cdots \sqrt{\exp \lceil \sin 9 \rceil}} \right) \right) \tag{4}$$

The above equation has a operational complexity of $l = 13+a+b$. This works for all Gaussian integers with positive a and b. Apply the complex conjugate operator $\bar{n}$ to create $a - bi$ ($l = 14 + a + b$). Remove the leading negative to create numbers of the form $-a - bi$ ($l = 12 + a + b$). Finally, for those of the form $-a + bi$, do both ($l = 13 + a + b$).

3.4.2 Gaussian Rationals

Real rationals, like real integers, can be created with just two numbers. From this, we can create the set of Gaussian rationals $\mathbb{Q}[i] = \{\frac{a}{b} \mid a, b \in \mathbb{Z}[i]\}$. Every $n \in \mathbb{Q}[i]$ can be expressed as the sum $q_1 + q_2 i$ where $q_1, q_2 \in \mathbb{Q}$.

$$\begin{aligned} n &= q_1 + q_2 i \\ &= \frac{a_1}{b_1} + \frac{a_2}{b_2} i \end{aligned}$$

where $a_1, a_2, b_1, b_2 \in \mathbb{Z}$. It is important that our fractions contain the same denominator. Let $d = \operatorname{lcm}(b_1, b_2)$, $k_1 = \frac{d}{a_1}$, $k_2 = \frac{d}{a_2}$.

$$\begin{aligned} &= \frac{k_1}{d} + \frac{k_2}{d} i \\ &= \log_{\ln \underbrace{\sqrt{\cdots\sqrt{\exp 1}}}_{d}} \left(\ln \underbrace{\sqrt{\cdots\sqrt{\exp 1}}}_{k_1} \right) + \log_{\ln \underbrace{\sqrt{\cdots\sqrt{\exp 1}}}_{d}} \left(\ln \underbrace{\sqrt{\cdots\sqrt{\exp 1}}}_{k_2} \right) i \\ &= \log_{\ln \underbrace{\sqrt{\cdots\sqrt{\exp 1}}}_{d}} \left(\ln \underbrace{\sqrt{\cdots\sqrt{\exp 1}}}_{k_1} \times \ln^i \underbrace{\sqrt{\cdots\sqrt{\exp 1}}}_{k_2} \right) \end{aligned}$$

Substituting in the digits of 2019:

$$n = \log_{\ln \sqrt{\cdots\sqrt{\exp\lceil \sin 2 \rceil}}} \left(\ln \sqrt{\cdots\sqrt{\exp \sqrt{0!}}} \times \ln^{\sqrt{-1}} \sqrt{\cdots\sqrt{\exp\lceil \sin 9 \rceil}} \right) \tag{5}$$

There are a lot of square root chains here. The total number of variadic square roots in this expression is

$$n_{sqrt} = d + k_1 + k_2 = \operatorname{lcm}(b_1, b_2) \times \left(1 + \frac{1}{a_1} + \frac{1}{a_2}\right),$$

making the operational complexity $l = 15 + n_{sqrt}$. Like the last equation, the three other sign combinations can be created with the complex conjugate operator and/or a leading negative.