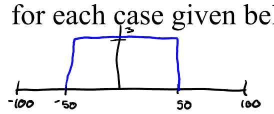
LAST HOMEWORK SET

1. Draw the Fourier Spectral *magnitude*, over the frequency range $[-100, 100]$ Hz, for the sampled signal

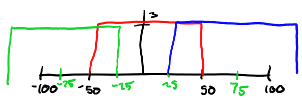
$$x_s(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT)$$

where $x(t) = 3\mathrm{sinc}(100t)$ for each case given below.

a. $T = \dfrac{1}{150}$ seconds



b. $f_s = 75\mathrm{Hz}$



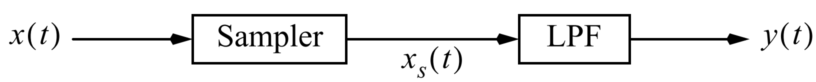
2. What is the *minimum* sample frequency, in Hz, that can be used to sample each signal given below so that the original signal can be perfectly reconstructed from the sample signal. If it is not possible to reconstruct the original signal from the sampled signal, explain why.

a. $x(t) = 10\mathrm{sinc}(50t)$ $f_s \geq 50$

b. $x(t) = 5\cos(1000\pi t + \pi/3)$ $f_s \geq 1000$

c. $x(t) = 2\Pi(t)$

3. The signal $x(t) = 5\cos(1000\pi t + \pi/3)$ is pass through a sampler, using sample frequency of $f_s = 800\mathrm{Hz}$, and an *ideal* LPF with cutoff frequency $f_c = 450\mathrm{Hz}$ as shown below. Describe the



signal at the output, $y(t)$.

4. Using the definition, find the z-transform of the following discrete-time signals in *closed-form*. To find the closed-form representation, use the fact that some of these series are *geometric* (see the Trig formula sheet on the course website).

a. $\delta[nT] = 1$ for all z

b. $\delta[(n-1)T] = z^{-1}\cdot 1 = z^{-1}$ for $|z|>0$

c. $u[nT] = \frac{1}{1-z^{-1}}$ for $|z|>1$

d. $u[(n-1)T] = z^{-1}\mathcal{Z}\{u[nT]\} = \frac{z^{-1}}{1-z^{-1}}$ for $|z|>1$

e. $\left(\frac{1}{2}\right)^n u[n]$ $= \frac{1}{1-\frac{1}{2}z^{-1}}$ for $|z| > \frac{1}{2}$

5. A signal is sampled with a sample time of $T = 10^{-5}$

   a. What is the sample frequency f_s? $= 10^5$

   b. What values of frequency f and σ, correspond to the complex frequency $z = e^{j(\pi/4)}$? $2\cancel{\pi}\frac{f}{f_s} = \frac{\cancel{\pi}}{4}$

      $e^{\sigma T} = 1$, $\sigma = 0$ $f = \frac{10^{-5}}{8}$ $r = \frac{1}{8}$

   c. What values of frequency f and σ, correspond to the complex frequency $z = (1/2)e^{j(\pi/4)}$?

      $e^{\sigma T} = \frac{1}{2} \Rightarrow \sigma = 10^5 \ln(\frac{1}{2})$ $f = \frac{10^{-5}}{8}$ $r = \frac{1}{8}$

   d. What values of frequency f and σ, correspond to the complex frequency $z = e^{-j(\pi/4)}$?

      $\sigma = 0$ $2\pi\frac{f}{f_s} = \frac{\pi}{4}$ $f = -\frac{10^{-5}}{8}$ $r = -\frac{1}{8}$

   e. What values of frequency f and σ, correspond to the complex frequency $z = e^{j(3\pi/4)}$?

      $\sigma = 0$ $2\pi\frac{f}{f_s} = \frac{3\pi}{4}$ $f = \frac{3(10^{-5})}{8}$ $r = \frac{3}{8}$

   f. What values of frequency f and σ, correspond to the complex frequency $z = e^{j(5\pi/4)}$?

      $f = \frac{5(10^{-5})}{8}$ $r = \frac{5}{8}$

   g. What values of frequency f and σ, correspond to the complex frequency $z = 1 + j$?

6. What is the *normalized frequency*, r, for parts (b) through (g) in the previous problem?

   r Values in green.

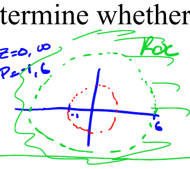
7. For each transfer function given below, find the poles and zeros, plot the poles/zeros, find the Region-of-Convergence (ROC), and determine whether the ROC includes the unit circle or not.

   a. $H(z) = \frac{z^{-1}}{1 - 5z^{-1} - 6z^{-2}}$

      $(1+z^{-1})(1-6z^{-1})$

      $z^{-1} = 1$, $z = 1$ $z^{-1} = \pm\frac{1}{6}$, $z = \pm 6$

      $z = 0, \infty$
      $p = -1, 6$



      Unit Circle not included

   b. $H(z) = \frac{z+1}{z^2 + 2z + 1}$ $= \frac{1}{z+1}$

      $\frac{\cancel{z+1}}{(z+1)\cancel{(z+1)}}$

      $P = -1$ $Z = \infty$

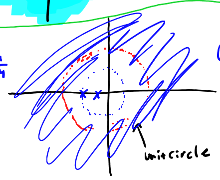
      ROC includes unit Circle.
      Stable

   c. $H(z) = \frac{z^{-1} + 1}{8 + 6z^{-1} + z^{-2}}$

      $(2+z^{-1})(4+z^{-1})$

      $z^{-1} = -2$, $z = -\frac{1}{2}$ $z^{-1} = -4$, $z = \frac{1}{4}$

      $z = 1$
      $p = -\frac{1}{2}, -\frac{1}{4}$



      Unit Circle included
      Stable

8. Use long division to find the first three samples of the impulse response of the systems defined in the previous problem. The first three samples are $h[0]$, $h[1]$, and $h[2]$.

   (a) $1 - 5z^{-1} - 6z^{-2} \,\overline{)\, z^{-1}}$, quotient $0 + z^{-1} + 5z^{-2}$

   $0 - 0 - 0$
   $- \; z^{-1} - 5z^{-2} - 6z^{-3}$
   $+\cancel{5z^{-2}} + 6z^{-3}$
   $- \; \cancel{5z^{-2}} - 25z^{-3} - 30z^{-4}$

   $h[0] = 0$
   $h[1] = 1$
   $h[2] = 5$

   (b) $1 + z \,\overline{)\, 1}$, quotient $1 - z + z^2$

   $-(1+z)$
   $0 - \cancel{z}$
   $- \; -\cancel{z} - z^2$
   $- \; \cancel{z^2} + z^3$

   $h[0] = 0$
   $h[1] = +1$
   $h[2] = -1$

   (c) $8 + 6z^{-1} + z^{-2} \,\overline{)\, 1 + z^{-1}}$, quotient $\frac{1}{8} + \frac{1}{32}z^{-1} - \frac{5}{128}$

   $-1 - \frac{6}{8}z^{-1} - \frac{1}{8}z^{-2}$
   $\cancel{\frac{1}{4}z^{-1}} - \frac{1}{8}z^{-2}$
   $+ \; -\cancel{\frac{1}{4}z^{-1}} - \frac{3}{16}z^{-2} - \frac{1}{32}z^{-3}$
   $\rightarrow -\frac{5}{16}z^{-2} - \frac{1}{32}z^{-3}$

   $h[0] = \frac{1}{8}$
   $h[1] = \frac{1}{32}$
   $h[2] = -\frac{5}{128}$

9. What is the Transfer Function $H(z)$ given the impulse response $h[n]$.

   a. $h[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$ $\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{3}z^{-1}}$

   b. $h[n] = \left(\frac{1}{2}\right)^n u[n-1]$ $\frac{1}{1-\frac{1}{2}z^{-1}} \cdot \frac{z^{-1}}{2}$

$2z^{-1}\,Z[\delta[n]] = 2z^{-1}$

$2Z[\delta[n-1]]$

c. $h[n] = \delta[n] - 2\delta[n-1]$ $\quad 1 - 2z^{-1}$

d. $h[n] = \cos(10\pi n)u[n]$ $\quad \dfrac{1-\cos(10\pi)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$

e. $h[n] = \sin[(\pi/8)n - \pi/4]u[n]$

$\left(\sin(\tfrac{\pi}{8}n)\cos(\tfrac{\pi}{4}) - \cos(\tfrac{\pi}{8}n)\sin(\tfrac{\pi}{4})\right)u[n] = \tfrac{\sqrt{2}}{2}u[n]\left(\sin(\tfrac{\pi}{8}n) - \cos(\tfrac{\pi}{8}n)\right) = \tfrac{\sqrt{2}}{2}\left[\dfrac{\sin(\frac{\pi}{8})z^{-1}}{\quad} - \dfrac{1-\cos(\frac{\pi}{8})z^{-1}}{1-2\cos(\frac{\pi}{8})z^{-1}+z^{-2}}\right]$

10. Find the inverse z-transform for the following.

a. $X(z) = \dfrac{3}{z-1/4} = \dfrac{3z^{-1}}{1-\frac{1}{4}z^{-1}} \Rightarrow 3u[n-1]\,Z^{-1}\left[\dfrac{1}{1-\frac{1}{4}z^{-1}}\right] = 3\left[\tfrac{1}{4}\right]^n u[n-1]$

b. $X(z) = \dfrac{3}{(z-1/4)(z-1/5)} = \dfrac{A}{z-\frac{1}{4}} + \dfrac{B}{z-\frac{1}{5}}$

$Az - \tfrac{1}{5}A + Bz - \tfrac{1}{4}B$

$0 = A+B$; $A = -B$; $A = 60$

$3 = -\tfrac{1}{5}A - \tfrac{1}{4}B$; $60 = -4A - 5B$; $60 = 4B - 5B$; $-60 = B$

$\left[\dfrac{60}{z-\frac{1}{4}} - \dfrac{60}{z-\frac{1}{5}}\right]\dfrac{z^{-1}}{z^{-1}}$

$60z^{-1}\left[\dfrac{1}{1-\frac{1}{4}z^{-1}} - \dfrac{1}{1-\frac{1}{5}z^{-1}}\right]$

$60u[n-1]\left(\left[\tfrac{1}{4}\right]^n - \left[\tfrac{1}{5}\right]^n\right)$

c. $X(z) = \dfrac{10}{2z^2-3z+1} = \dfrac{10}{(2z-1)(z-1)} = \dfrac{A}{2z-1} + \dfrac{B}{z-1}$

$10 = Az + 2Bz - A - B$

$0 = A + 2B$; $A = -2B$; $A = -20$

$10 = -A - B$; $10 = 2B - B = B$

$= \dfrac{10}{z-1} - \dfrac{20}{2z-1}\cdot\dfrac{z^{-1}}{z^{-1}} = 10z^{-1}\dfrac{1}{1-z^{-1}} - 20z^{-1}\dfrac{1}{1-2z^{-1}} = 10u[n-1]u[n] - 20u[n-1]2^n u[n]$

d. $X(z) = \dfrac{z}{z^2 - \sqrt{2}z + 1}$

e. $X(z) = \dfrac{z^{-1} - z^{-2}}{1-(1/2)z^{-1}}$

11. What is the output $y[n]$ of the Linear, Time-Invariant (LTI) discrete-time system with input

$Z\{x[n]\} = \delta[n+1] + 2\delta[n] + 3\delta[n-1]$ $\quad (z + 2 + 3z^{-1})$

and impulse response

$Z\{h[n]\} = 2\delta[n] + 3\delta[n-1] + \delta[n-2]$ $\quad (2 + 3z^{-1} + z^{-2})$

$Y[z] = X[z]\cdot H[z]$

$(z + 2 + 3z^{-1})(2 + 3z^{-1} + z^{-2})$

$2z + 3 + z^{-1} + 4 + 6z^{-1} + 2z^{-2} + 6z^{-1} + 9z^{-2} + 3z^{-3}$

$Y[z] = 2z + 7 + 13z^{-1} + 11z^{-2} + 3z^{-3}$

$y[n] = 2\delta[n+1] + 7\delta[n] + 13\delta[n-1] + 11\delta[n-2] + 3\delta[n-3]$

12. What is the *impulse response* $h[n]$ of an LTI discrete-time system with the following input/output signal pairs?

a. $x_1[n] = \delta[n] + \delta[n-1] \Rightarrow y_1[n] = \delta[n] + 3\delta[n-1] + 2\delta[n-2]$

$1 + z^{-1}$ $\quad 1 + 3z^{-1} + 2z^{-2} = \underbrace{(1+z^{-1})}_{X[z]}\underbrace{(1+2z^{-1})}_{H[z]}$

$h[n] = \delta[n] + 2\delta[n-1]$

b. $x_2[n] = \delta[n] - \delta[n-1] \Rightarrow y_2[n] = \delta[n] + \delta[n-1] - 2\delta[n-2]$

$1 - z^{-1}$ $\quad 1 + z^{-1} - 2z^{-2}$

$(1-z^{-1})\underbrace{(1+2z^{-1})}_{H[z]}$

$h[n] = \delta[n] + \delta[n-1]$

13. Given the LTI discrete-time system with impulse response

$$H[z] = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}$$

$$h_1[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] + \ldots$$

a. What is the output $y[n]$ if the input is

$$x[n] = 2\delta[n] + 3\delta[n-1] + \delta[n-2]$$

$2 + 3z^{-1} + z^{-2}$

$Y[z] = \frac{2+3z^{-1}+z^{-2}}{1-z^{-1}}$

$= 2\frac{1}{1-z^{-1}} + 3z^{-1}\frac{1}{1-z^{-1}} + z^{-2}\frac{1}{1-z^{-1}}$

$y[n] = 2u[n] + 3u[n-1] + u[n-2]$

b. What effect does this system have on the input?

Translates from $\delta[n]$ to $u[n]$, or performs an integration.

14. Given the LTI discrete-time system with impulse response

$$h_2[n] = \delta[n] - \delta[n-1]$$

$1 - z^{-1}$

$= 2\left(\frac{1-z^{-1}}{1-z^{-1}}\right) + 3z^{-1}\frac{1-z^{-1}}{1-z^{-1}} + z^{-2}\frac{1-z^{-1}}{1-z^{-1}}$

$Y[z] = 2 + 3z^{-1} + z^{-2}$

$\boxed{y[n] = 2\delta[n] + 3\delta[n-1] + \delta[n-2]}$

a. If the output from problem #3 above is passed through this system, what is the output?

b. How does the output of this system relate to the input $x[n]$ from problem #3 above?

Differentiates the input.

c. What is the convolution $h_1[n]*h_2[n]$? Does this makes sense? Clearly explain why or why not.

$\delta[n]$, yes, because one integrates while the other differentiates

15. A discrete-time system is described by the following difference equation.

$H[z] = \frac{Y[z]}{X[z]}$

$$y[n] = x[n-1] + \left(\frac{1}{3}\right)y[n-1]$$

$Y[z] = z^{-1}X[z] + \frac{1}{3}z^{-1}Y[z]$

$Y[z] - \frac{1}{3}z^{-1}Y[z] = z^{-1}X[z]$

$Y[z](1-\frac{1}{3}z^{-1}) = z^{-1}X[z]$

$H[z] = \frac{Y[z]}{X[z]} = z^{-1}\left(\frac{1}{1-\frac{1}{3}z^{-1}}\right)$ $\boxed{\Rightarrow h[n] = u[n-1]\left(\tfrac{1}{3}\right)^{n}}$

a. Find the *transfer function* $H(z)$

b. Is this system *stable*? Clearly explain why or why not. Stable, pole exists w/in unit circle.

c. Find the *impulse response* $h[n]$

$(2 - 3z^{-1})$

$z^{-1}\left(\frac{2-3z^{-1}}{1-\frac{1}{3}z^{-1}}\right) = z^{-1}\left[\frac{2}{1-\frac{1}{3}z^{-1}} + z^{-1}\frac{3}{1-\frac{1}{3}z^{-1}}\right]$

$= 2z^{-1}\frac{1}{1-\frac{1}{3}z^{-1}} + 3z^{-2}\frac{1}{1-\frac{1}{3}z^{-1}}$

$= 2u[n-1]\left(\frac{1}{3}\right)^{n} + 3u[n-2]\left(\frac{1}{3}\right)^{n}$

d. Find the *output* $y[n]$ of this system if the input is $x[n] = 2\delta[n] - 3\delta[n-1]$

16. A discrete-time system, with input $x[n]$ and output $y[n]$, is given by the following *difference equation*

$Y[z]\left(2 - 5z^{-1} + 2z^{-2}\right)$ $(2z^{-1}-1)(z^{-1}-2)$

$2Y[z] - 5z^{-1}Y[z] + 2z^{-2}Y[z] = 6X[z] - 6z^{-1}X[z]$

$X[z](6-6z^{-1}) / (2-5z^{-1}+2z^{-2})$

$$2y[n] - 5y[n-1] + 2y[n-2] = 6x[n] - 6x[n-1]$$

$Y[z] = \frac{1}{\cancel{1-z^{-1}}}\cdot 6 \cdot \frac{\cancel{1-z^{-1}}}{(2z^{-1}-1)(z^{-1}-2)}$

$\frac{6}{} = \frac{A}{2z^{-1}-1} + \frac{B}{z^{-1}-2}$ $6 = Az^{-1} + 2Bz^{-1} - 2A - B$

$0 = A + 2B$, $A = -2B$; $6 = -2A - B$, $6 = 3B$, $2 = B$, $A = -4$

$\frac{2}{z^{-1}-2} - \frac{4}{2z^{-1}-1}$

$-1\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right) + 4\frac{1}{1-2z^{-1}}$

$y[n] = -\left(\tfrac{1}{2}\right)^n u[n] + 4(2)^n u[n]$, RoC $|z|>2$

a. What is the output if the input is $x[n] = u[n]$? Express your answer in *closed-form*.

b. Is this system *stable*? Clearly explain why or why not. Unstable, pole outside unit circle

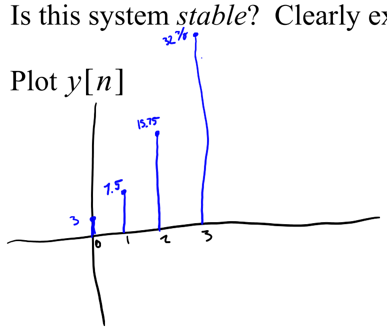
$n=0$: $y = -1 + 4 = 3$

$n=1$: $y = 8 - \frac{1}{2} = 7.5$

$n=2$: $y = 16 - \frac{1}{4} = 15.75$

$n=3$: $y = 32 - \frac{1}{8} = 31\frac{7}{8}$

c. Plot $y[n]$



$H[z] = \frac{Y[z]}{X[z]}$

$Y[z] = H[z]X[z]$

17. A discrete-time system is described by the transfer function

$Y[z] = 6 \cdot \frac{1}{\cancel{1-z^{-1}}} \cdot \frac{\cancel{1-z^{-1}}}{6-5z^{-1}+z^{-2}}$

$(z^{-1}-2)(z^{-1}-3)$

$= \frac{6}{(z^{-1}-2)(z^{-1}-3)} \cdot \frac{z^{2}}{z^{2}}$

$= \frac{6z^{2}}{(1-2z)(1-3z)} =$

$1 + \frac{5}{6}z^{-1} + \frac{25}{36}z^{-2} + \frac{5^3}{6^3}z^{-3}$

$6z^2-5z+1 \,\overline{)\,6z^2}$

$-\,6z^2+5z-1$

$5z-1$

$-5z+\frac{25}{6}-\frac{5}{6}z^{-1}$

$-\frac{25}{6}+\frac{125}{36}z^{-1}-\frac{25}{36}z^{-2}$

$1 - z^{-1}$

$H[z] \cdot \frac{X[z]}{Y[z]} \cdot \frac{z^{-2}}{z^{-2}} = \frac{Y(z)(6-5z^{-1}-z^{-2})}{X(z)(6-6z^{-1})} \Rightarrow 6y[n]-5y[n-1]-y[n-2] = 6x[n]-6x[n-1]$

$$H(z) = \frac{6(z^2 - z)}{6z^2 - 5z + 1} \cdot \frac{z^{-2}}{z^{-2}}$$

$6 - 5z^{-1} + z^{-2}$

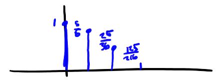
a. What is the output if the input is $x[n] = u[n]$? Express your answer in *closed-form*.

$X[z] = \frac{1}{1-z^{-1}}$

$Y[z] = \sum_{m=0}^{\infty} \frac{5^n}{6^m} z^{-m} = \frac{5}{1-6z^{-1}}$

$y[n] = \left(\frac{5}{6}\right)^n u[n]$

b. Plot the output, $y[n]$



c. Find the *difference equation* associated with this system.

18. A discrete-time system has the transfer function

$$H(z) = \frac{1}{(2z-1)(5z-1)}$$

a. Find the closed-form representation of the impulse response $h[n]$.

b. Find the difference equation that relates the discrete-time input sequence $x[n]$ to the discrete-time output sequence $y[n]$.