❖ **Why do we need yet another transform?**

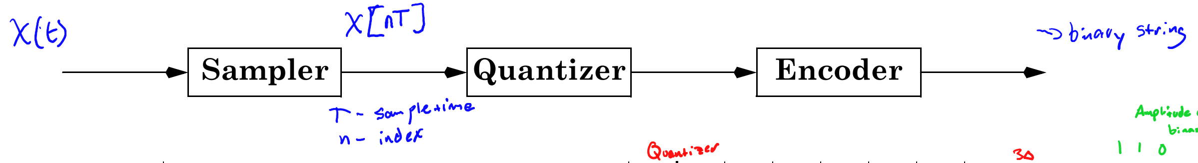
Continuous-Time World ====> **Digital Processing**

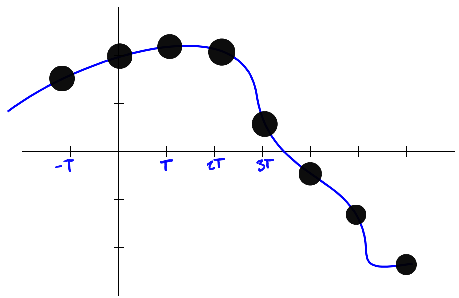
the majority of processing

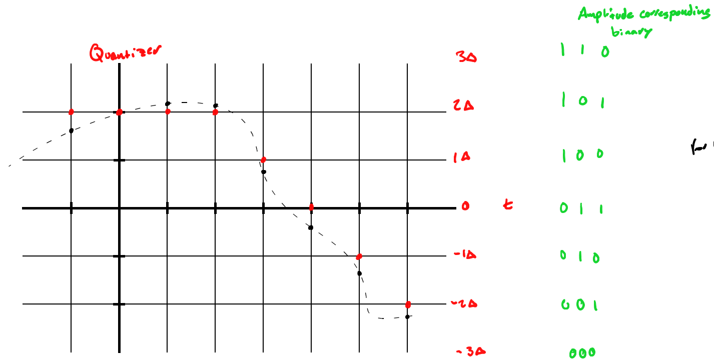
❖ **BME Signals (Continuous-Time)**
- **Naturally Occurring Signals**
  - → **EKG**
  - → **Action Potential at a synaps**
  - → **Diffusion of a chemical through a membrane**
  - → **Movement of the human body signal (arm, leg, torso, head, etc.)**
  - → **Vocal track signal (sound)**
- **Externally Generated Signals**
  - → **Electrical pulse propagating down a body**
  - → **Ultrasound imaging**
  - → **Magnetic Resonance Imaging (MRI)**

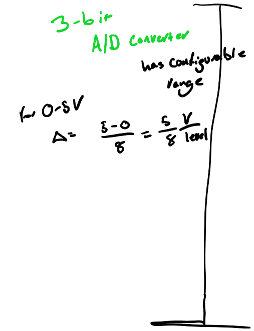
❖ **ECE Signals (Continuous-Time)**
- **Communications**
  - → **FM Signal Reception**
  - → **Cell phone signal (Xmit, Recv)**
  - → **Computer modem signal (Xmit, Recv)**
- **Controls**
  - → **Thrust**
  - → **Airplane Aeleron Actuator**

Analog-to-Digital Conversion

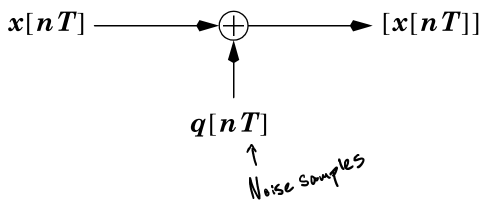




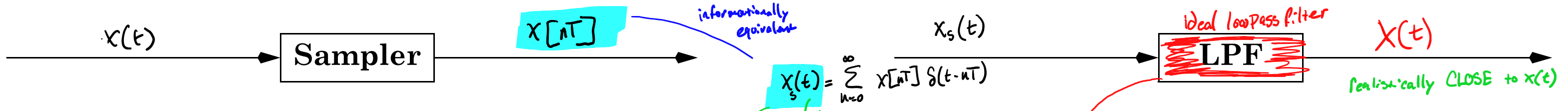




Quantizer



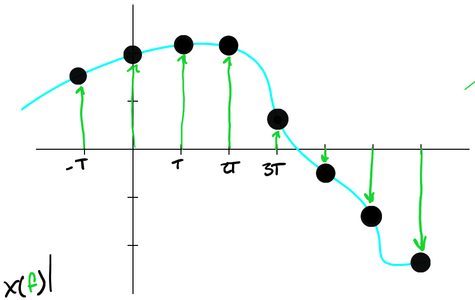
Reconstruction from a Sampled Signal

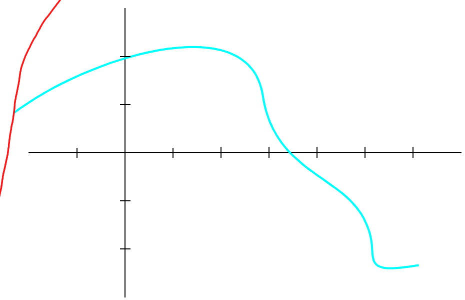


$$x_s(t) = \sum_{n=0}^{\infty} x[nT]\,\delta(t-nT)$$

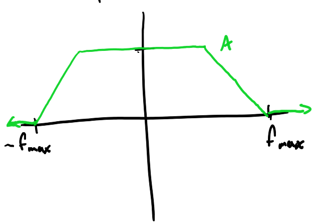
$\mathcal{F}$

$$X_s(f) = \mathcal{F}\{x_s(t)\}$$
$$= \mathcal{F}\Big\{\sum_{n=0}^{\infty} \cancel{x[nT]}\,\delta(t-nT)\Big\}$$
x(t)
$$= \mathcal{F}\Big\{x(t)\cdot\sum_{n=0}^{\infty}\delta(t-nT)\Big\}$$
$$= X(f) * \frac{1}{T}\sum_{k=-\infty}^{\infty}\Big(f-\frac{k}{T}\Big)$$
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\Big(f-\frac{k}{T}\Big)$$



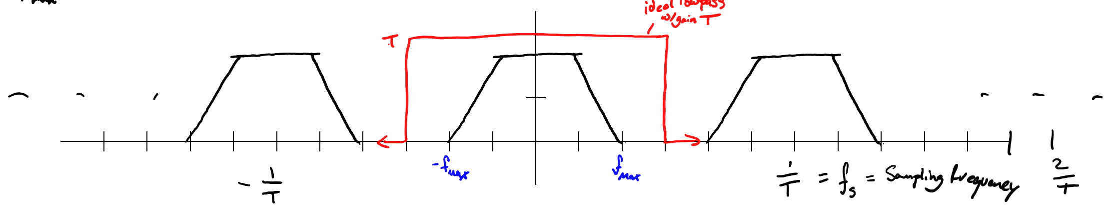


$|X(f)|$



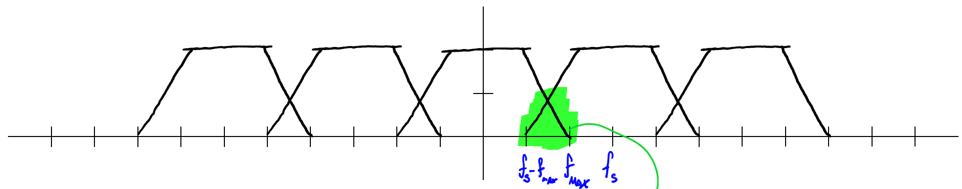
T = sample time $\left(\frac{\text{sec}}{\text{sample}}\right)$

$|X_s(f)|$



if $T\uparrow$

$f_s\downarrow$



Known as aliasing

for no Aliasing

$f_s - f_{max} \geq f_{max}$

$\vdots$

Nyquist Sampling Criteria

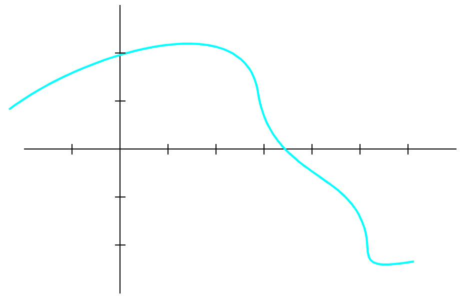
$$\boxed{f_s > 2 f_{max}}$$

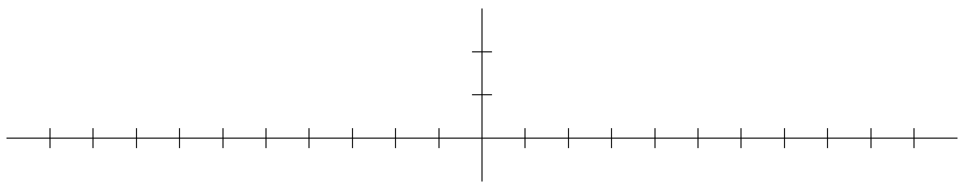
CD Quality

44. KHz

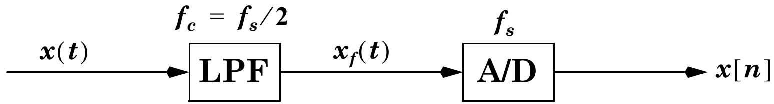
(2 · Max human hearing) + rolloff

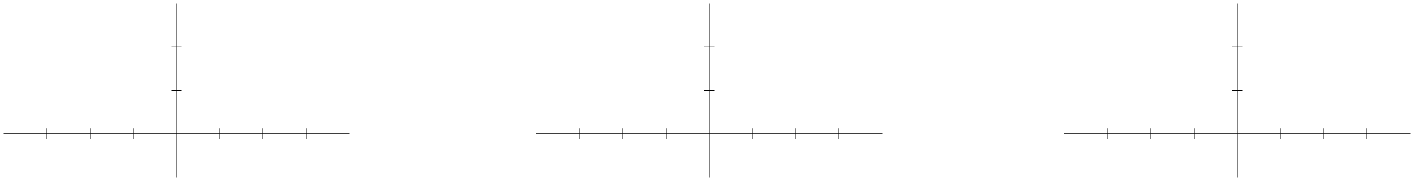

Sampler





- **When acquiring a real-world, continuous-time signal using an A/D convertor, it is important to always use an *antialiasing filter*.**



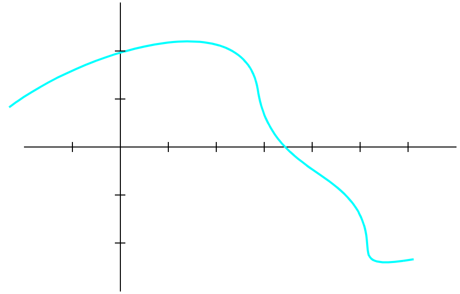


$f_s = \frac{1}{T}$

$T = \frac{1}{f_s}$

❖ **Given the signal**

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT)$$



The spectrum of $x_s(t)$ is given by

$$L\{x_s(t)\} = L\left\{\sum_{n=-\infty}^{\infty} x(t)\delta(t-nT)\right\} = \int_0^{\infty}\left[\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)\right]\underbrace{e^{-st}}_{(e^{sT})^{-n}}dt$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\int_0^{\infty}\delta(t-nT)e^{-st}dt = \sum_{n=0}^{\infty} x(nT)e^{-snT}$$

$s = \sigma + j2\pi f$

$z = e^{(\sigma+j2\pi f)T}$

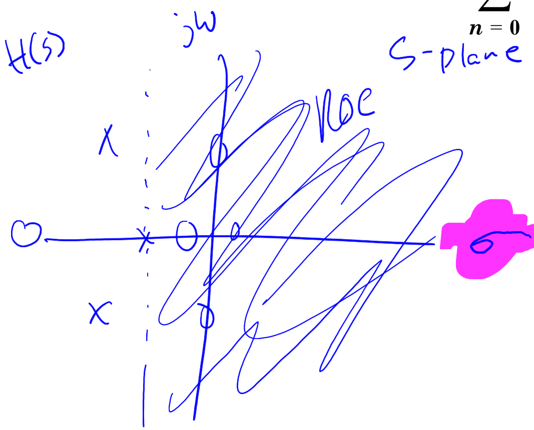
$= \underbrace{e^{\sigma T}}_{\text{magnitude}} e^{j2\pi(fT)}$

$r = \frac{f}{f_s}$

Normalized freq. btwn $-\frac{1}{2}$ and $\frac{1}{2}$

Now *define* the frequency variable $z = e^{sT}$, then

$$Z\{x[nT]\} = Z\{x[n]\} = \sum_{n=0}^{\infty} x[n]z^{-n} = X(z)$$



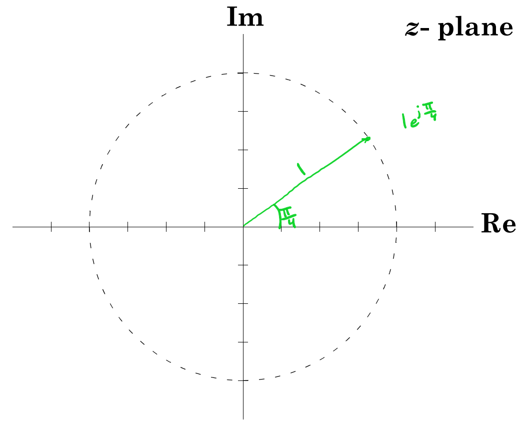
The Z-Transform is a Laplace Transform for a Discrete-Time Signal

(expressed using delta functions)

All your Laplace Transform concepts carry over to the Z-Transform

What is Complex Frequency z?

- ***Complex frequency*** **and the z-plane:**



- **Each point on this plane defines a specific value of complex frequency z**
- **For example, where is $z_1 = e^{j(\pi/4)}$?**
- **For this value of z, what is the value of f?**

$e^{j\frac{\pi}{4}} = \underbrace{e^{\sigma T}}\, e^{j\underbrace{2\pi\left(\frac{f}{f_s}\right)}}$

$\sigma = 1$

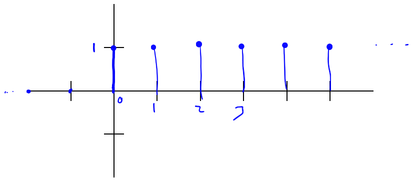
$2\pi \frac{f}{f_s} = \frac{\pi}{4}$

$\boxed{f = \frac{f_s}{8}}$

Z-Transform Example

❖ **What is the z-transform of the signal?**

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



$$\mathcal{Z}\{u[n]\} = \sum_{n=0}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$$= 1 \cdot z^{-0} + 1 \cdot z^{-1} + 1 \cdot z^{-2} + \ldots$$

$$Q = 1 \cdot z^{-0} + \cancel{1 \cdot z^{-1}} + \cancel{1 \cdot z^{-2}} + \ldots$$

$$Qz^{-1} = \cancel{z^{-1}} + \cancel{z^{-2}} + \cancel{z^{-3}} + \ldots$$

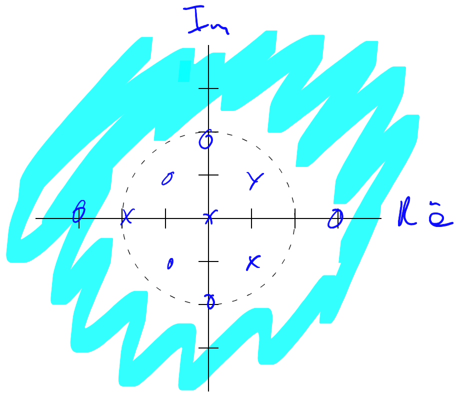
$$Q - Qz^{-1} = 1$$

$$Q(1 - z^{-1}) = 1$$

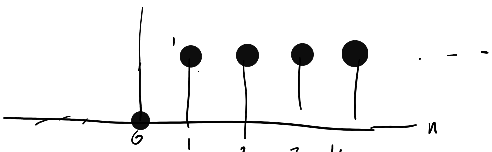
$$Q = \frac{1}{1 - z^{-1}} = \mathcal{Z}\{u[n]\}$$

For convergence

$$|z| > 1$$



Q: **What is the time-domain significance of z^{-1}?**



u[n-1]

$$\mathcal{Z}\{u[n-1]\} = \sum_{n=0}^{\infty} u[n-1] z^{-n} = \sum_{n=1}^{\infty} z^{-n}$$

$$= z^{-1} + z^{-2} + z^{-3} + \ldots$$

$$= z^{-1}(1 + z^{-1} + z^{-2} + \ldots)$$

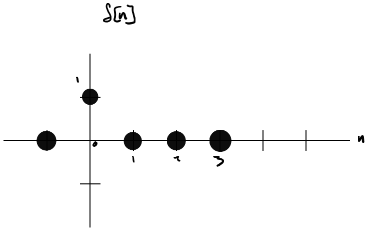
$$= z^{-1}\left(\sum_{n=0}^{\infty} z^{-n}\right)$$

$$= z^{-1}\, \mathcal{Z}\{u[n]\}$$

❖ **Given** $X(z) = \dfrac{z^{-1}}{1 - z^{-1}}$**, what is** $x[n]$**? Use long division ...**

Another Z-Transform Example

❖ **What is the z-transform of the unit-sample (dirac delta function)?**

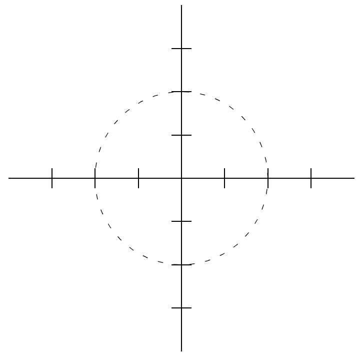


$$x[n] = \delta[n]$$

$$Z\{\delta[n]\} = \sum_{n=0}^{\infty} \delta[n] z^{-n}$$

$$= \underbrace{\delta(0)}_{1} z^{0} + \underbrace{\delta(1)}_{0} z^{-1} + \underbrace{\delta(2)}_{0} z^{-2} + \underbrace{\delta(3)}_{0} z^{-3} + \dots$$

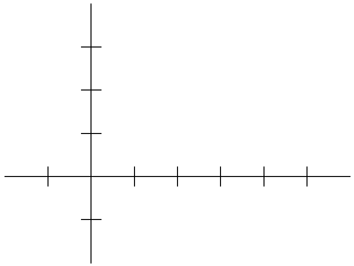
$Z\{\delta[n]\} = 1$ ROC for all z

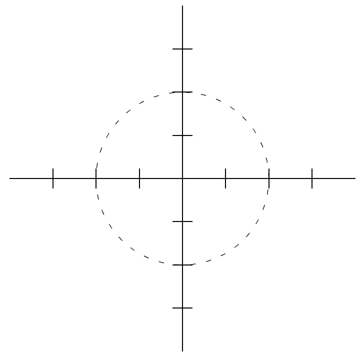


Another Z-Transform Example

❖ **What is the z-transform of the signal**

$$x[n] = \begin{cases} e^{-\alpha n T} & n \geq 0 \\ 0 & n < 0 \end{cases}$$





Z-Transform Theorems

1. ***Linearity*:** $Z\{K_a a[n] + K_b b[n]\} = K_a A(z) + K_b B(z)$

2. ***Time Delay*:** $Z\{x[n - n_0]\} = X(z)z^{-n_0}$

3. ***Convolution*:** $Z\{a[n] \otimes b[n]\} = A(z)B(z)$

4. ***Differentiation*:** $\underbrace{\overbrace{Z\{x[n] - x[n-1]\}}^{\text{approximates derivatives w/ rise/run}} =}_{\text{Known as 1st order Backward Difference}} (1 - z^{-1})X(z)$

5. ***Initial Value Th*:** $x(0) = \lim_{z \to \infty} X(z)$

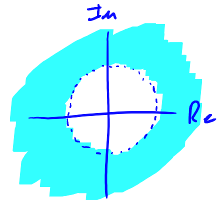
6. ***Final Value Th*:** $x(\infty) = \lim_{z \to 1} [(1 - z^{-1})X(z)]$

Z-Transform Pairs

has many similarities to Laplace pairs

$\delta[n]$	1	All z
$u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\delta[n-m]$	z^{-m}	$\lvert z\rvert > 0$
$a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n\cos(\omega_0 n)u[n]$	$\dfrac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
$r^n\sin(\omega_0 n)u[n]$	$\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Region of Convergence
(does not include boundary)



Inverse Z-Transform Example

Note: With discrete time, T is an implied input (if it exists).

❖ **Find** $h[n]$ **if** $H(z) = \dfrac{30}{(z-1/2)(z-1/3)}$ $= \dfrac{A}{z-\frac{1}{2}} + \dfrac{B}{z-\frac{1}{3}}$

$A = \left.\dfrac{30}{z-\frac{1}{3}}\right|_{z=\frac{1}{2}} = 180 = A$

$B = \left.\dfrac{30}{z-\frac{1}{2}}\right|_{z=\frac{1}{3}} = -180$

$Z^{-1}\left\{\dfrac{180}{z-\frac{1}{2}} - \dfrac{180}{z-\frac{1}{3}}\right\}$

$180\, Z^{-1}\left\{\dfrac{1}{z-\frac{1}{2}}\right\} - 180\, Z^{-1}\left\{\dfrac{1}{z-\frac{1}{3}}\right\}$

$180\, Z^{-1}\left\{\dfrac{z^{-1}}{1-\frac{1}{2}z^{-1}}\right\} - 180\, Z^{-1}\left\{\dfrac{z^{-1}}{1-\frac{1}{3}z^{-1}}\right\}$

z^{-1} F.D. $\downarrow$ T.D.

Extra Z^{-1} means time delay

$Z\{x[n-1]\} = z^{-1} Z\{x[n]\}$

So $Z^{-1}\left\{\dfrac{1}{1-\frac{1}{2}z^{-1}}\right\} - Z^{-1}\left\{\dfrac{1}{1-\frac{1}{3}z^{-1}}\right\}$

$\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{3}\right)^n u[n]$

$h[n] = 180\left(\frac{1}{2}\right)^n u[n]\Big|_{n-1 \leftarrow n} - 180\left(\frac{1}{3}\right)^n u[n]\Big|_{n-1 \leftarrow n}$

$$\boxed{h[n] = 180\left(\frac{1}{2}\right)^{n-1} u[n-1] - 180\left(\frac{1}{3}\right)^{n-1} u[n-1]}$$

❖ **What is the difference equation corresponding to this** $H(z)$**?**

$H(z) = \dfrac{30}{(z-\frac{1}{2})(z-\frac{1}{3})} \cdot \left(\dfrac{z^{-2}}{z^{-2}}\right) = \dfrac{30z^{-2}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}$

$= \dfrac{30z^{-2}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}} = \dfrac{Y(z)}{X(z)}$

Applies to all Z-transforms we know.

$30z^{-2}X(z) = Y(z)\left[1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}\right]$

FD TD

$30\,x[n-2] = y[n] - \frac{5}{6}y[n-1] - \frac{1}{6}y[n-2]$

Could generate the output in MATLAB

$y[n] = 30\,x[n-2] + \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2]$

Inverse Z-Transform Example Revisited

❖ **Find $h[n]$ if** $H(z) = \dfrac{30}{(z-1/2)(z-1/3)}$ $\left(\dfrac{z^{-2}}{z^{-2}}\right) = \dfrac{30z^{-2}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}$ $\cancel{\overset{?}{\Rightarrow} \dfrac{A}{(1-\frac{1}{2}z^{-1})} + \dfrac{B}{(1-\frac{1}{3}z^{-1})}}$

Cannot make a z^{-2} in numerator.

$$= z^{-1}\left(\underbrace{\frac{30z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}}_{\text{this does work}}\right)$$

$$\frac{A}{(1-\frac{1}{2}z^{-1})} + \frac{B}{(1-\frac{1}{3}z^{-1})}$$

$$A = \left.\frac{30z^{-1}}{1-\frac{1}{3}z^{-1}}\right|_{z^{-1}=2} = \frac{30(2)}{1-\frac{1}{3}(2)} = 180$$

$$B = \left.\frac{30z^{-1}}{1-\frac{1}{2}z^{-1}}\right|_{z^{-1}=3} = -180$$

$$H(z) = z^{-1}\left(\frac{180}{1-\frac{1}{2}z^{-1}} - \frac{180}{1-\frac{1}{3}z^{-1}}\right)$$

$\downarrow$ FD / TD

$$h[n] = \left.\left[180\left(\tfrac{1}{2}\right)^n u[n] - 180\left(\tfrac{1}{3}\right)^n u[n]\right]\right|_{n-1 \leftarrow n}$$

$$h[n] = 180\left(\tfrac{1}{2}\right)^{n-1} u[n-1] - 180\left(\tfrac{1}{3}\right)^{n-1} u[n-1]$$

❖ ***Difference equations*** **describe discrete-time systems:**

$$y[n] - Ky[n-1] = x[n]$$

- **Is this system *Linear*?**

- ***Time-Invariant*?**

- ***Causal*?**

- ***Stable*?**

Let $x[n] = u[n]$

$y[n] = u[n] + Ky[n-1]$

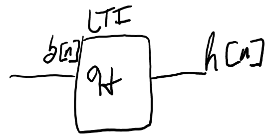
n	$x[n]$	$y[n-1]$	$y[n]$
-1	0	0	0
0	1	0	1
1	1	1	$1+K$
2	1	$1+K$	$1+K(1+K)$
3	1	$1+K(1+K)$	$1+K(1+K+K^2) = 1+K+K^2+K^3$

Series converges if $|K| < 1$

$$y[n] - Ky[n-1] = x[n]$$

❖ **What is the *impulse response* for this system? Method #1: Create a table ...**

n	$x[n] = \delta[n]$	$y[n-1]$	$y[n]$
-1	0	0	0
0	1	0	1
1	0	1	K
2	0	K	K^2
3	0	K^2	K^3



$\therefore\ h[n] = K^n u[n]$

❖ **Is this system *stable*?**

FACT: a discrete-time LTI system is stable if $\boxed{\displaystyle\sum_{n=-\infty}^{\infty} |h[n]| < \infty}$ "it's finite"

$$\sum_{n=0}^{\infty} |K^n| = \sum_{n=0}^{\infty} |K|^n =$$

$$y[n] - Ky[n-1] = x[n]$$

❖ **What is the *impulse response* for this system? Method #2: Use the *Z*-transform ...**

$h[n] = K^n u[n]$

z

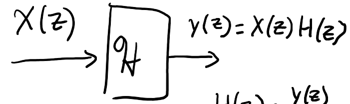
z^{-1}

$Y(z) - KY(z)z^{-1} = X(z)$

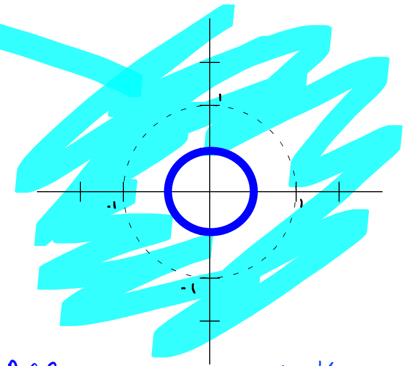
$Y(z)\left(1 - Kz^{-1}\right) = X(z)$

$\dfrac{Y(z)}{X(z)} = \dfrac{1}{1 - Kz^{-1}}$

ROC: $|z| > |K|$



$H(z) = \dfrac{Y(z)}{X(z)}$



Since ROC includes unit circle, then system is stable for $K < 1$

$K = 1/2$

❖ **Find $h[n]$ if the system is described by the following difference equation**

$$6y[n] - 5y[n-1] + y[n-2] = 30x[n-1]$$

$h[n] = -10\left(\frac{1}{3}\right)^{n-1} u[n-1] + 15\left(\frac{1}{2}\right)^{n-1} u[n]$

$6Y(z) - 5z^{-1}Y(z) + z^{-2}Y(z) = 30z^{-1}X(z)$

$\frac{Y(z)}{X(z)} = H(z) = \frac{30z^{-1}}{6 - 5z^{-1} + z^{-2}}$

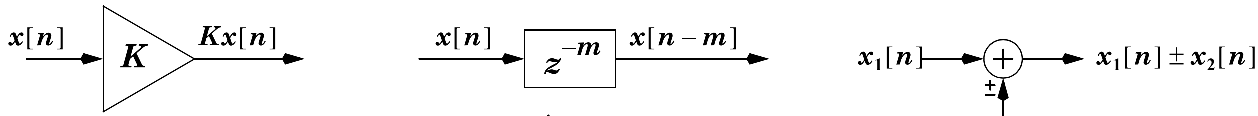
⋮

do $Z^{-1}\{\}$

- Given a system described by the *Difference Equation*

$$6y[n] - 5y[n-1] + y[n-2] = 30x[n-1]$$

Synthesize the system using only the following systems

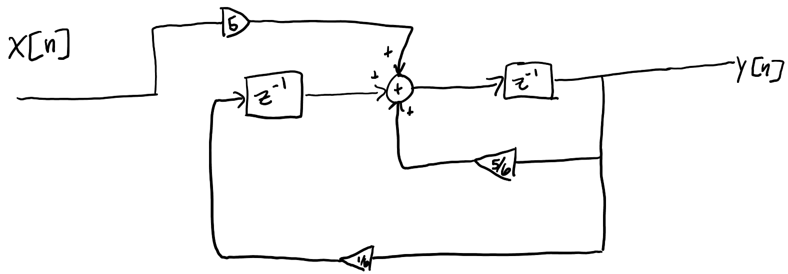


$\approx \frac{\partial}{\partial x}$

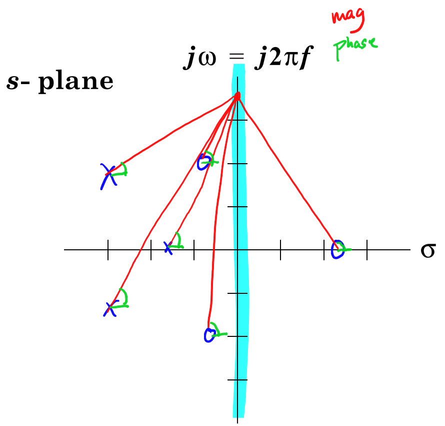
- **Procedure**
  a. Solve the DE for the *lowest* delay of $y[n]$
  b. Combine all like delay terms
  c. Draw the output and input
  d. Construct the right-hand side of the resulting equation

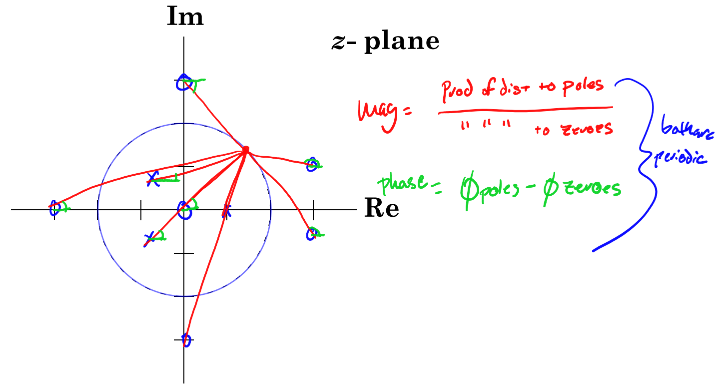
like terms

(a) $y[n] = \frac{1}{6}\left(\overset{5}{30x[n-1]} + \overset{5/6}{5y[n-1]} \overset{1/6}{- y[n-2]}\right)$



$$e^{sT} = z$$





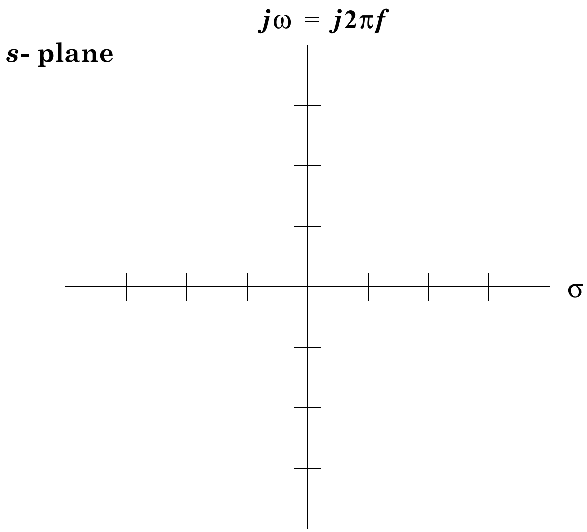
$$s = \sigma + j2\pi f$$

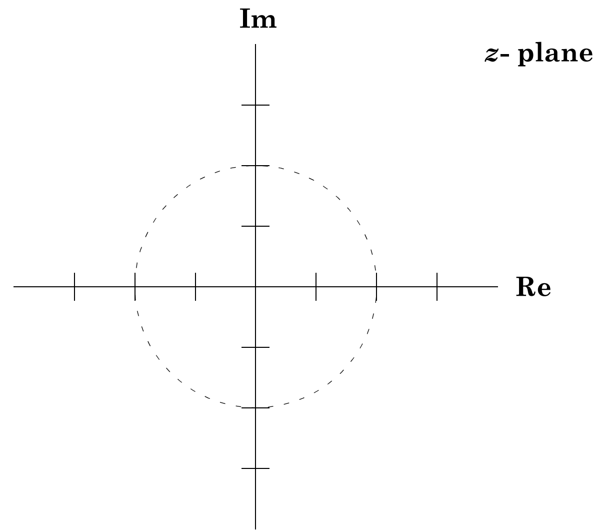
$$z = e^{\sigma T} e^{j2\pi f T}$$

❖ **Where must the discrete-time system *poles* lie for the system to be *stable*?**

$H(f)$ and Poles/Zeros: Graphical Approach

$$e^{sT} = z$$





$$s = \sigma + j2\pi f$$

$$z = e^{\sigma T} e^{j2\pi f T}$$

Response to a Sinusoidal Input (Discrete-Time)

$$x[nT] \longrightarrow \boxed{H(z)} \longrightarrow y[nT]$$

$$f_s = 10 \text{ Hz}$$
$$A = 1$$
$$f_0 = 1 \text{ Hz}$$
$$x[nT] = A\cos[2\pi f_0(nT)]$$

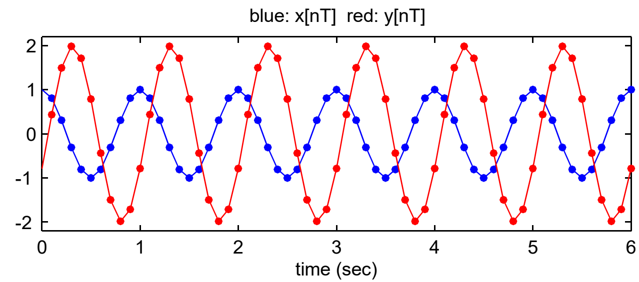
$$z_0 = e^{j2\pi f_0(T)} = e^{j\frac{2\pi}{10}}$$

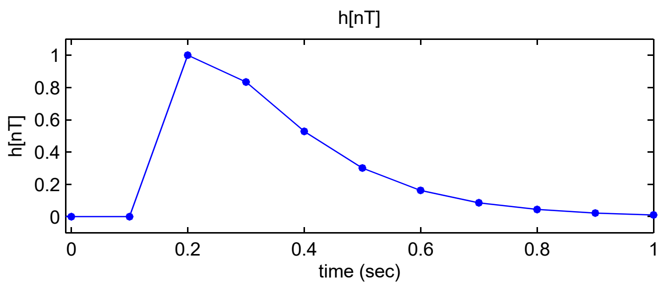
$$H(z_0) = \frac{6}{(2z_0 - 1)(3z_0 - 1)} = 1.99\angle{-113^\circ}$$

$$y[nT] = A|H(z_0)|\cos[2\pi f_0(nT) + \angle H(z_0)]$$
$$= 1.99\cos[2\pi(n/10) - 113^\circ]$$

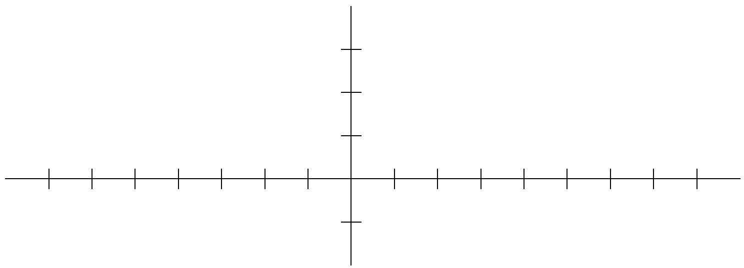
$$H(z) = \frac{6}{(2z-1)(3z-1)}$$

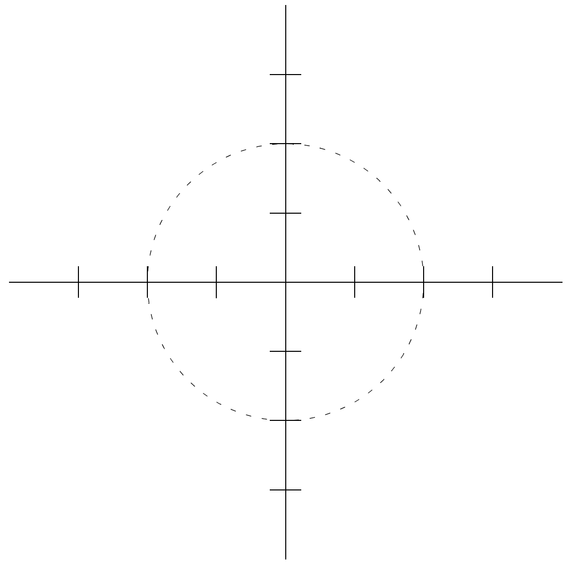
$$h[nT] = \left[6\left(\frac{1}{2}\right)^{n-1} - 6\left(\frac{1}{3}\right)^{n-1}\right]u[n-1]$$





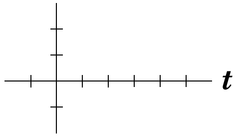
$$z = e^{sT} = e^{j2\pi fT}$$





Continuous-Time vs Discrete-Time System Comparison

Continuous-Time



$$y(t) - K\frac{d}{dt}y(t) = x(t)$$

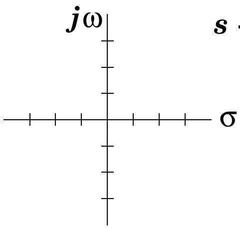
$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) \qquad \delta(t)$$

$$y(t) = x(t) \otimes h(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

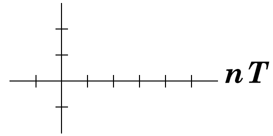
$$X(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s) = H(s)X(s)$$

$$L\{x(t)\} = \int_0^{\infty} x(t)e^{-st}dt = X(s) = \frac{N(s)}{D(s)}$$

***s* - plane**



Discrete-Time



$$y[n] - Ky[n-1] = x[n]$$

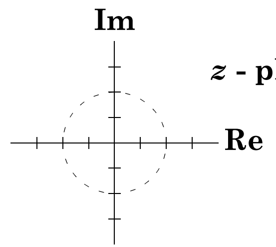
$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] \qquad \delta[n]$$

$$y[nT] = x[nT] \otimes h[nT] = \sum_{k=-\infty}^{\infty} x[kT]h[(n-k)T]$$

$$X(z) \longrightarrow \boxed{H[z]} \longrightarrow Y(z) = H(z)X(z)$$

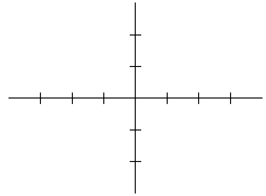
$$Z\{x[n]\} = \sum_{n=0}^{\infty} x[n]z^{-n} = X(z) = \frac{N(z)}{D(z)}$$

***z* - plane**



$$x(t) \longrightarrow \boxed{h(t) = e^{-\alpha t}u(t)} \longrightarrow y(t) = x(t) \otimes h(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

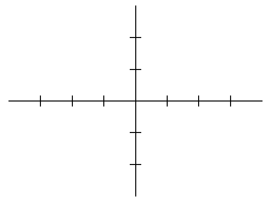
$$X(s) \longrightarrow \boxed{H(s) = \frac{1}{s+\alpha}} \longrightarrow Y(s) = X(s)H(s)$$



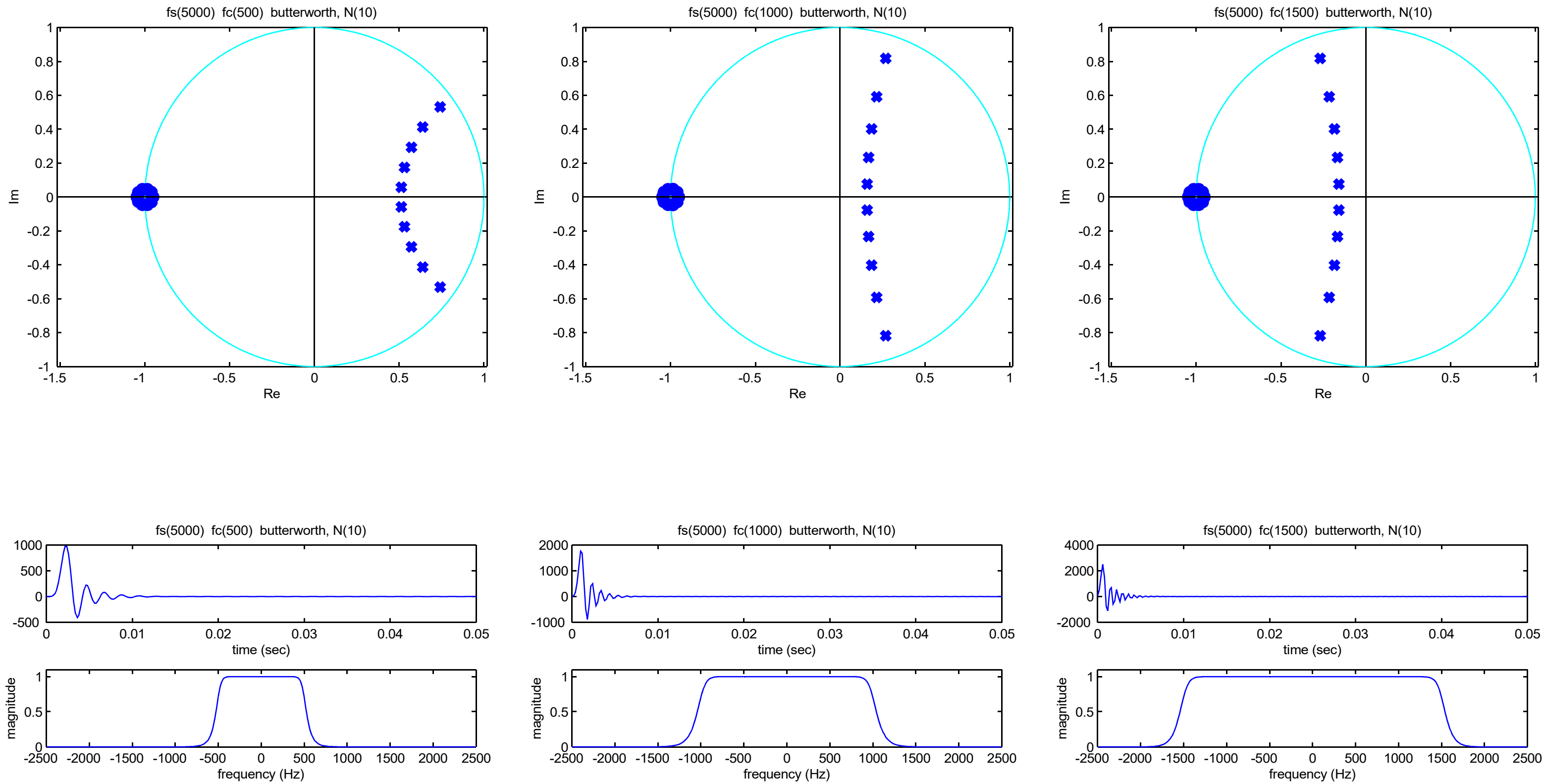
ROC: $\mathrm{Re}\{s\} > -\alpha$

$$x[nT] \longrightarrow \boxed{h[nT] = e^{-\alpha nT}u[nT]} \longrightarrow y[nT] = x[nT] \otimes h[nT] = \sum_{k=-\infty}^{\infty} x[kT]h[(n-k)T]$$

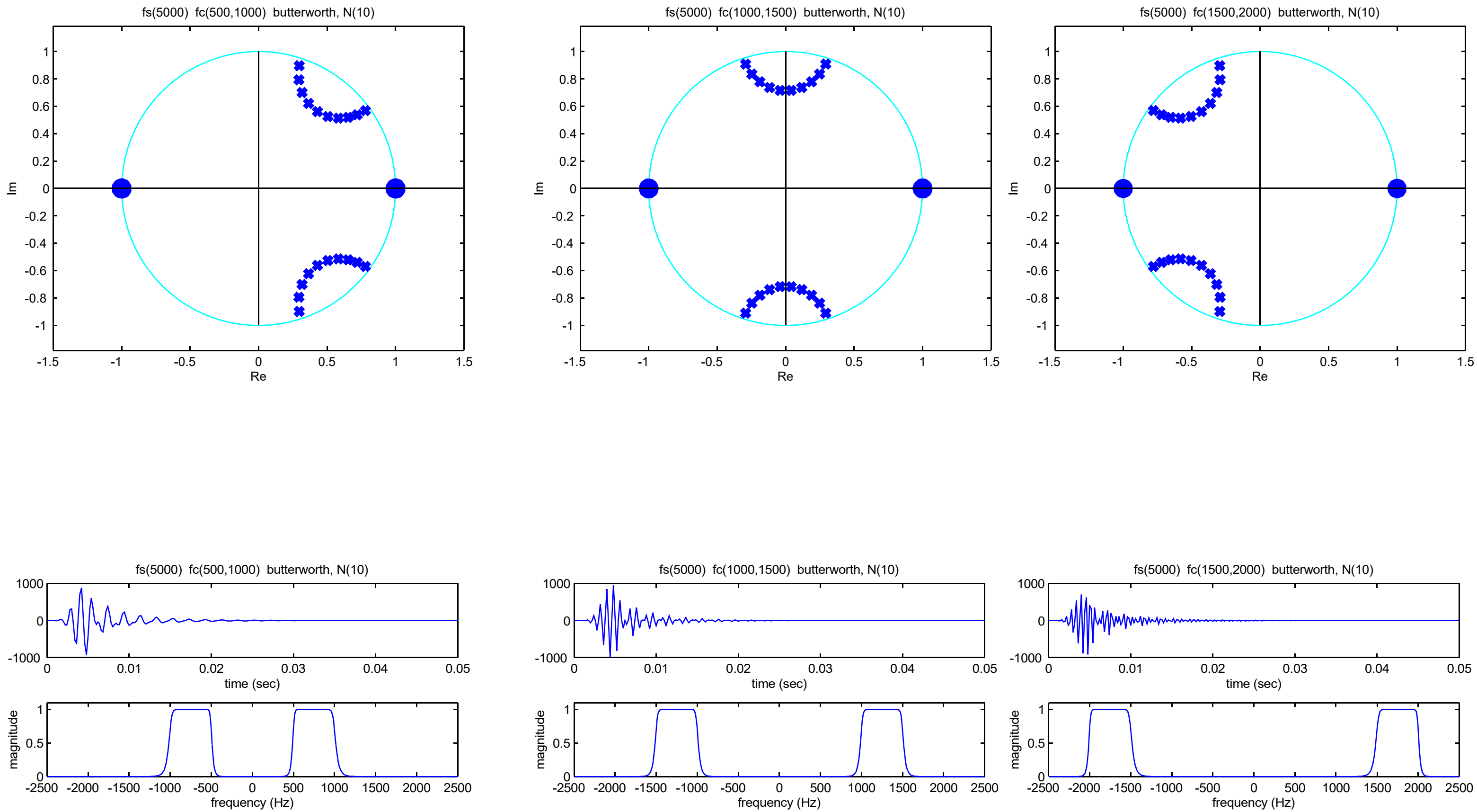
$$X(z) \longrightarrow \boxed{H(z) = \frac{1}{1-e^{-\alpha T}z^{-1}}} \longrightarrow Y(s) = X(s)H(s)$$

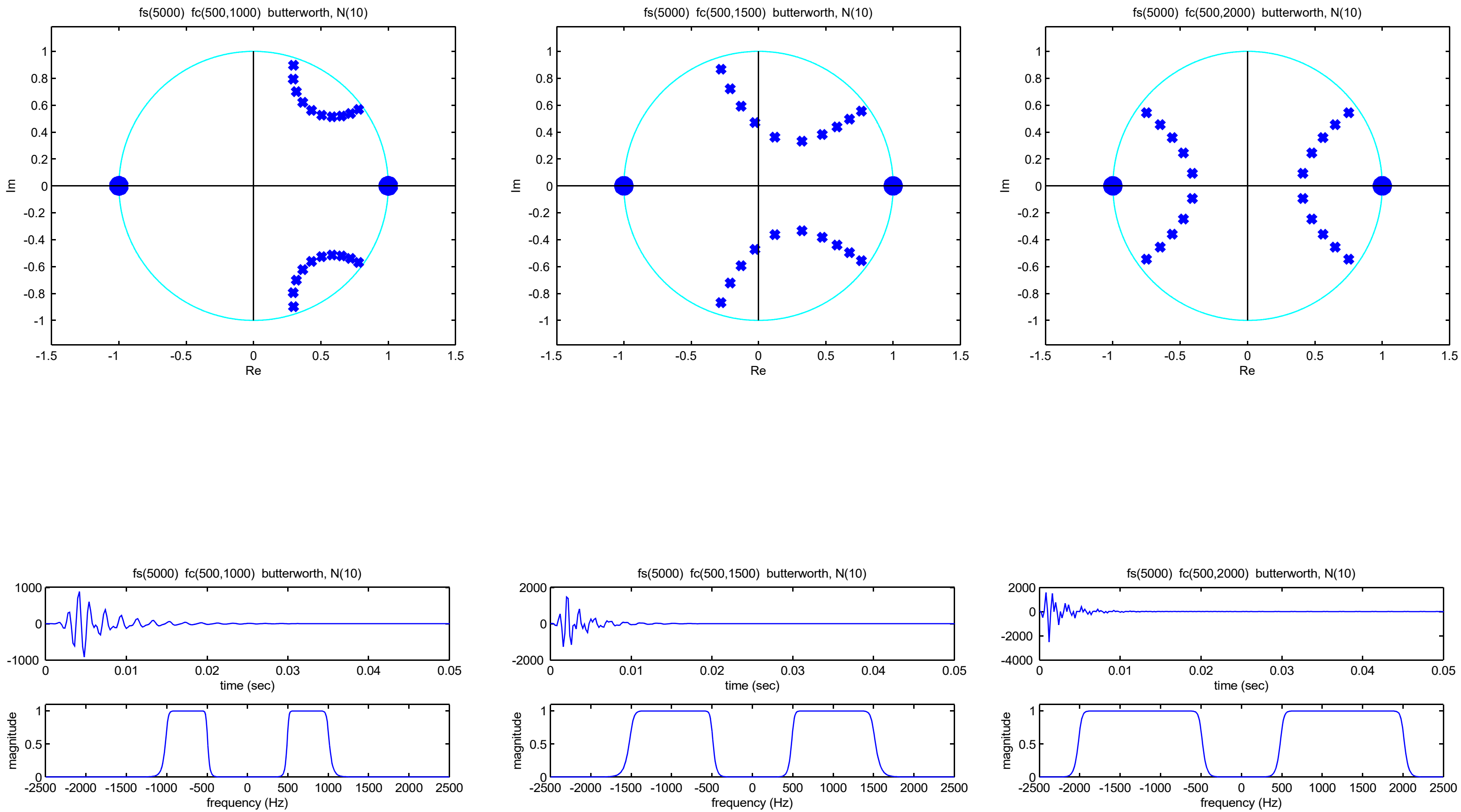


ROC: $|z| > \left|e^{-\alpha T}\right|$


fs(5000) fc(500) butterworth, N(10)
fs(5000) fc(1000) butterworth, N(10)
fs(5000) fc(1500) butterworth, N(10)
Im
Re
time (sec)
magnitude
frequency (Hz)

The Butterworth BPF: Z-Plane Pole-Zero Constellations




fs(5000) fc(500,1000) butterworth, N(10)
fs(5000) fc(500,1500) butterworth, N(10)
fs(5000) fc(500,2000) butterworth, N(10)
Im
Re
fs(5000) fc(500,1000) butterworth, N(10)
fs(5000) fc(500,1500) butterworth, N(10)
fs(5000) fc(500,2000) butterworth, N(10)
time (sec)
magnitude
frequency (Hz)