My note taking app erased all of my exam 3 notes. These are the notes you provided to the class as a place holder and what I intend to reference after the course ends.

The application is called Drawboard.

Googling "Drawboard Recover Annotations" gives results on their support forum with threads and comments about losing file changes from at least 2014 to others posted earlier this year.

Hopefully I can find a different Notability equivalent for Windows to use next semester.

- **Why another transform?**
  - **In solving a Differential Equation, we never considered the initial conditions.**
  - **The Fourier Transform only works on *energy signals* and *periodic signals*. Therefore, generally it cannot be used to analyze unstable systems.**

- **The Laplace Transform can address both the problems from above**

$$L\{x(t)\} = \Im\{x(t)e^{-\sigma t}u(t)\}$$

$$L\{x(t)\} = \int_{-\infty}^{\infty}[x(t)e^{-\sigma t}u(t)]e^{-j2\pi ft}dt$$

which gives

$$L\{x(t)\} = \int_{0}^{\infty}x(t)e^{-(\sigma+j2\pi f)t}dt$$

Now, defining *complex frequency* to be $s = \sigma + j2\pi f$

$$L\{x(t)\} = \int_{0}^{\infty}x(t)e^{-st}dt = X(s)$$

$$x(t) \longrightarrow \underset{\textbf{LTI}}{\boxed{h(t)}} \longrightarrow y(t) = x(t)*h(t) = \int_{-\infty}^{\infty}x(\lambda)h(t-\lambda)d\lambda$$

Causal if: $h(t) = 0$ **for all** $t < 0$

Stable if: $\int_{-\infty}^{\infty}|h(t)|dt = \text{finite}$

TD

FD

$$H(s) = \int_{0}^{\infty}h(t)e^{-st}dt$$

$h(t)$

- **Can be energy, power, or neither**
- **Must be causal**
- **Can be unstable**
- **Can account for initial conditions:**

$$L\left\{\frac{d}{dt}h(t)\right\} = sH(s) - h(0^-)$$

$$H(f) = \int_{-\infty}^{\infty}h(t)e^{-j2\pi ft}dt$$

$h(t)$

- **Must be energy or periodic signal**
- **Can be non-causal**
- **Must be stable**
- **Cannot account for initial conditions:**

$$\Im\left\{\frac{d}{dt}h(t)\right\} = (j2\pi f)H(f)$$

❖ **Suppose an LTI system is to be built - it is only necessary to define its Transfer Function.**

If $H(s)$ is designed in the Laplace Transform domain

The system must be *causal*

- **If $H(s)$ is designed so that $H(f)$ exists**

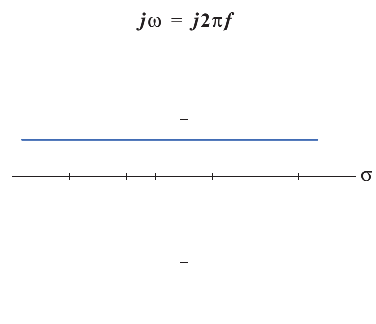
The system must be *stable*

❖ **Conclusion:**

An LTI system that is designed in the Laplace Transform domain for which $H(f)$ exists must be both *stable* and *causal*.

❖ ***Complex frequency* is a mathematical abstraction going beyond the notion of negative frequency**

s- plane



❖ **Each point on this plane defines a specific value of complex frequency s**

❖ **For example, where is $s_1 = 2 + j3$?**

❖ **For this value of s, what is the value of f?**

❖ **What is the value of f if $s_2 = 1 + j3$?**

❖ **How does s_1 and s_2 compare in terms of f?**

- **The inverse Laplace can be shown to be**

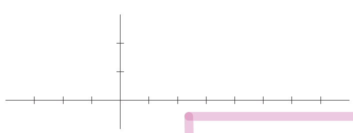
$$L^{-1}\{X(s)\} = \frac{1}{2\pi j}\oint_{(\sigma - j\infty)}^{(\sigma + j\infty)} X(s)e^{st}ds$$

This is not easy to evaluate - requires residue theory

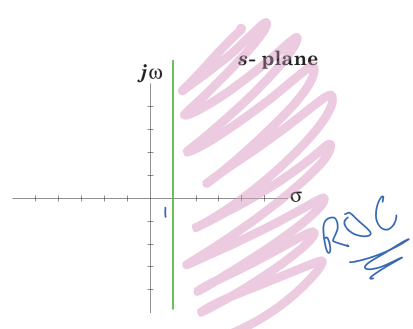
- **In this course, we will *not* evaluate the contour integral shown above.**

- **We will evaluate the inverse Laplace Transform using pattern matching with a known table of Laplace Transform pairs.**

- **If an LTI system has $h(t) = e^t u(t)$, find $L\{h(t)\}$**



$$\boxed{\mathcal{L}\{e^t u(t)\} = \frac{1}{s-1} \quad , \sigma > 1}$$



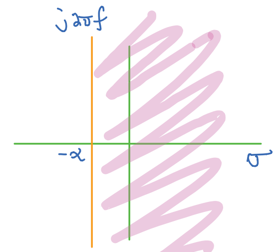
- **The Region of Convergence (ROC) is the values of s, expressed as a region in the s-plane, where the Laplace Transform *integral converges***

Time-Domain	Frequency-Domain	ROC
$\delta(t)$	1	All s
$u(t)$	$1/s$	$\text{Re}\{s\} > 0$
$\dfrac{t^{n-1}}{(n-1)!}u(t)$	$1/s^n$	$\text{Re}\{s\} > 0$
$e^{-\alpha t}u(t)$	$\dfrac{1}{\alpha + s}$	$\text{Re}\{s\} > -\alpha$
$te^{-\alpha t}u(t)$	$\dfrac{1}{(\alpha + s)^2}$	$\text{Re}\{s\} > -\alpha$
$\dfrac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\dfrac{1}{(\alpha + s)^n}$	$\text{Re}\{s\} > -\alpha$
$\cos(\omega_0 t)u(t)$	$\dfrac{s}{s^2 + \omega_0^2}$	$\text{Re}\{s\} > 0$
$\sin(\omega_0 t)u(t)$	$\dfrac{\omega_0}{s^2 + \omega_0^2}$	$\text{Re}\{s\} > 0$
$e^{-\alpha t}\cos(\omega_0 t)u(t)$	$\dfrac{s + \alpha}{(s + \alpha)^2 + \omega_0^2}$	$\text{Re}\{s\} > -\alpha$
$e^{-\alpha t}\sin(\omega_0 t)u(t)$	$\dfrac{\omega_0}{(s + \alpha)^2 + \omega_0^2}$	$\text{Re}\{s\} > -\alpha$

$\text{Re}\{s\} > ?$

$\text{Re}\{\sigma + j2\pi f\} > ?$

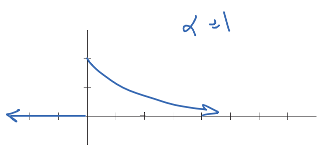
$\sigma > -\alpha$

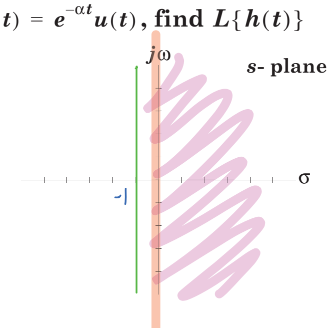


$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt = e^{-st}\Big|_{t=0} = 1$$

$$\frac{N(s)}{D(s)}$$

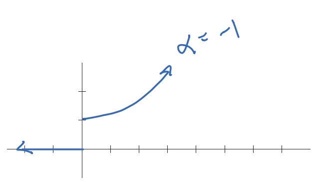
❖ **If an LTI system has $h(t) = e^{-\alpha t}u(t)$, find $L\{h(t)\}$**

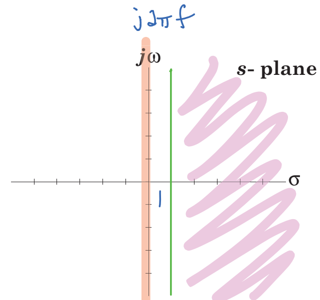




$$\mathcal{L}\{e^{-\alpha t}u(t)\} = \frac{1}{\alpha + s}, \quad \sigma > -\alpha$$
$$= \frac{1}{1+s}, \quad \sigma > -1$$
$$\mathcal{F}\{e^{-\alpha t}u(t)\} = \frac{1}{1+s}\Big|_{s=j2\pi f} = \frac{1}{1+j2\pi f}$$

j2πf





$$\mathcal{L}\{e^{-\alpha t}u(t)\} = \frac{1}{\alpha + s}, \quad \sigma > -\alpha$$
$$= \frac{1}{-1+s}, \quad \sigma > 1$$

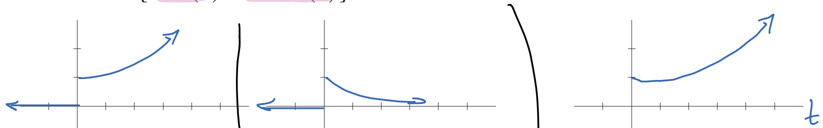
❖ **Which system is *stable*? Which system possesses a *Tranfer Function* $H(f)$?**

ROC DOES NOT INCLUDE THE BOUNDARY

Region Of Convergence (ROC) Determination Example

❖ Find $L\{e^{t}u(t) + e^{-2t}u(t)\}$ $= \mathcal{L}\{e^{t}u(t)\} + \mathcal{L}\{e^{-2t}u(t)\}$

$h(t)$



$\frac{1}{-1+s}$ $\quad \sigma > 1$

$\frac{1}{2+s}$ $\quad \sigma > -2$

$H(s) = \frac{1}{-1+s} + \frac{1}{2+s} = \frac{2+s+(-1+s)}{(-1+s)(2+s)} = \frac{2s+1}{(s-1)(s+2)}$

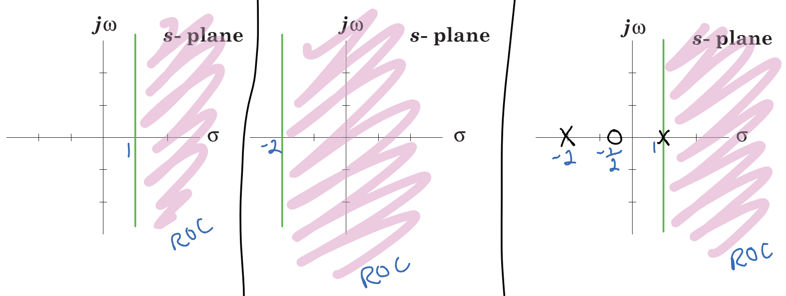
"AND"

ZERO'S

N(s) FACTORS $s = -\frac{1}{2}$

D(s) FACTORS $s = 1, -2$

POLES



$|H(s)|$

POLE-ZERO CONSTELLATION

RIGHT-MOST POLE DETERMINES THE ROC

$H(s) = \frac{K[s-(-\frac{1}{2})]}{(s-1)(s-(-2))}$

Poles and Zeros

❖ The Laplace Transform of $h(t)$ takes the form: $H(s) = \frac{N(s)}{D(s)}$

❖ Values of s where $H(s) = 0$ are known as the *zeros* of $H(s)$. These include the roots of $N(s)$

❖ Values of s where $|H(s)| \to \infty$ are known as the *poles* of $H(s)$. These include the roots of $D(s)$

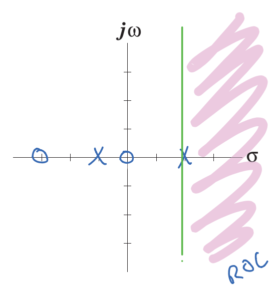
❖ For example, plot the poles and zeros of

$$H(s) = \frac{s(s+3)}{(s+1)(s-2)}$$

ZEROS $s = 0, -3$

POLES $s = -1, 2$

$H(f)$ EXIST? NO



❖ For example, plot the poles and zeros of

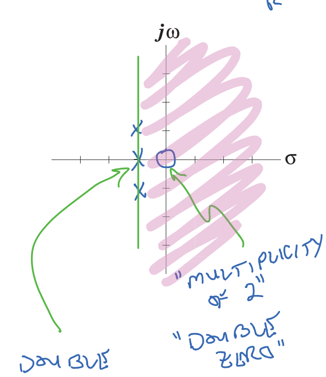
$$H(s) = \frac{s^2}{(s+1)^2(s^2+2s+2)}$$

ZEROS $s = 0, 0$ $\quad s = \infty, \infty$

POLES $s = -1, -1$ $\quad s = -1 \pm j$

$s^2 + 2s + 2 = 0$

$s = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(2)}}{2} = -1 \pm j$



❖ Fact: The ROC is to the right of the right-most *pole*

❖ $H(f) = H(s)|_{s = j2\pi f}$ **but only if it *exists*!**

❖ **If $H(f)$ *exists*, then the system must be *stable*.** ALL POLES MUST LIE IN LEFT-HALF PLANE
$\mathrm{Re}\{\text{POLES}\} < 0$

❖ **$H(f)$ only exists if the ROC includes the set of points $s = j2\pi f$ (the imaginary axis)**

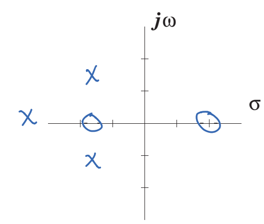
❖ **Conclusion:**

An LTI system that is designed in the Laplace Transform domain to give $H(s)$, and for which the ROC includes the imaginary axis, must be both *stable* and *causal*.

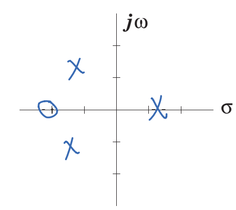
❖ **If $h(t)$ is the impulse response of an LTI system, and if $H(s) = L\{h(t)\}$, then the ROC determines whether the system is stable.**

❖ **3 Possibilities:**

- **System is stable**

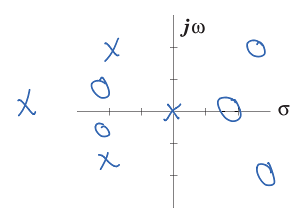


- **System is not stable**



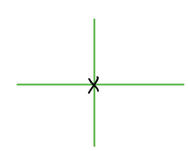
- **System is marginally stable**

IF ONE POLE AT $s=0$
AND NO OTHER POLE ON $j\omega$ AXIS



$x(t) \rightarrow \boxed{\int} \rightarrow y(t)$
LTI
$h(t) = u(t)$

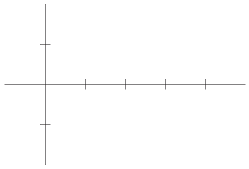
$\mathcal{L}\{\ \} \rightarrow H(s) = \frac{1}{s}$



❖ **Find** $L\{\cos(\omega_0 t)u(t)\} = \mathcal{L}\left\{\left[\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right]u(t)\right\}$

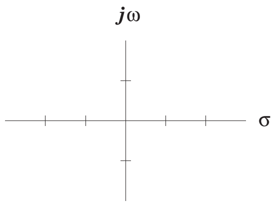
$$\mathcal{L}\{e^{-\alpha t}u(t)\} = \frac{1}{\alpha+s}, \quad \sigma > -\alpha$$

ALSO WORKS IF α IS COMPLEX



$$\frac{1}{2}\mathcal{L}\left\{\left[e^{j\omega_0 t}+e^{-j\omega_0 t}\right]u(t)\right\}$$

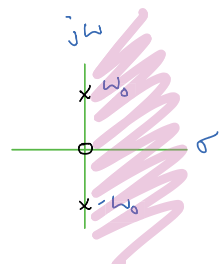
$$= \frac{1}{2}\left[\frac{1}{-j\omega_0+s}\right] + \frac{1}{2}\left[\frac{1}{j\omega_0+s}\right]$$



$$= \frac{1}{2}\left[\frac{\cancel{j\omega_0}+s+(\cancel{-j\omega_0}+s)}{(-j\omega_0+s)(j\omega_0+s)}\right]$$

$$= \frac{1}{\cancel{2}}\frac{\cancel{2}s}{\omega_0^2+s^2}$$

$$= \frac{s}{\omega_0^2+s^2}$$



UNSTABLE SYSTEM IF $H(s) = \frac{s}{\omega_0^2+s^2}$

h(t) REAL
A POLE & ITS CONJUGATE
MUST EXIST.

❖ **Given a set of poles & zeros, it is possible to construct $H(s)$.**

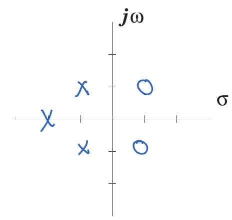
If $H(s)$ has poles at $\{p_1, p_2, p_3, \ldots\}$ and zeros at $\{q_1, q_2, q_3, \ldots\}$ then

$$H(s) = \frac{(s-q_1)(s-q_2)(s-q_3)\ldots}{(s-p_1)(s-p_2)(s-p_3)\ldots}$$

❖ **For example, what is $H(s)$ if it has ...**

Zeros: $1 \pm 2j$

Poles: $-2 \pm j,\ -1$



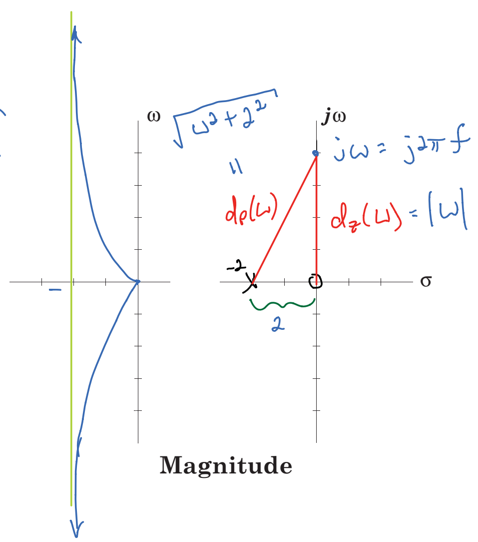
$$H(s) = \frac{N(s)}{D(s)} =$$

What is the ROC?

$$H(s) = \frac{s}{s+2}$$

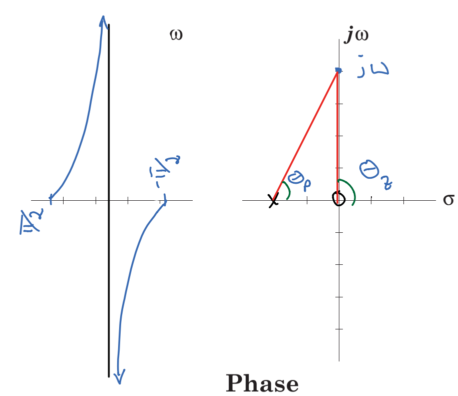
$H(\omega) = \frac{j\omega}{j\omega+2}$, $|H(\omega)| = \frac{|\omega|}{\sqrt{\omega^2+2^2}}$

$|H(\omega)| = \frac{d_z(\omega)}{d_p(\omega)} = \frac{|\omega|}{\sqrt{\omega^2+2^2}}$



Magnitude

$\angle H(\omega) = \theta_z - \theta_p$



Phase

1. ***Linearity***: $L\{a_1x_1(t) + a_2x_2(t)\} = a_1X_1(s) + a_2X_2(s)$

2. ***Time Delay***: $L\{x(t-t_0)u(t-t_0)\} = \left[e^{-st_0}\right]X(s)$

3. ***Frequency Translation***: $L\{x(t)e^{-\alpha t}\} = X(s+\alpha)$

4. ***Convolution***: $L\{x_1(t) \otimes x_2(t)\} = X_1(s)X_2(s)$

5. ***Multiplication***: not easy...

6. ***Differentiation***: $L\left\{\frac{d}{dt}x(t)\right\} = sX(s) - x(0^-)$

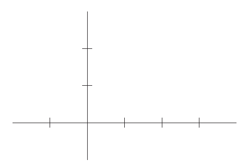
7. ***Integration***: $L\left\{\int x(t)\right\} = \frac{X(s)}{s} + \frac{K}{s}$ where $K = \int_{-\infty}^{0^-} x(\lambda)d\lambda$

8. ***Scale***: $L\{x(at)\} = (1/a)X(s/a)$, where $a > 0$ is a constant

NO DUALITY

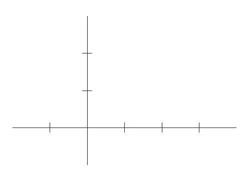
❖ ***Initial Value Theorem:*** $\lim_{t \to 0^+} h(t) = \lim_{s \to \infty} sH(s)$ (if the limit exists)

The limit exists if the numerator degree is strictly less than the denominator degree of $H(s)$



❖ ***Final Value Theorem:*** $\lim_{t \to \infty} h(t) = \lim_{s \to 0} sH(s)$ (if the limit exists)

The limit exists if the system is *stable* or *marginally stable.*



❖ **Find the Laplace Transform of:** $b(t) = \frac{d}{dt}[\underbrace{3e^{2t}u(t)}_{a(t)}]$

- **Solution #1: Differentiate first, then Laplace**

$$\frac{d}{dt}[3e^{2t}u(t)] = 6e^{2t}u(t) + 3\delta(t)$$

$$\xrightarrow{\mathcal{L}} \quad \frac{6}{-2+s} + 3 = \boxed{\frac{3s}{s-2} = B(s)}$$

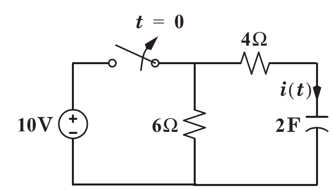
- **Solution #2: Use the differentiation theorem**

$$\mathcal{L}\left\{\frac{d}{dt}a(t)\right\} = sA(s) - a(0^-) = s\left[\frac{3}{-2+s}\right] - 0 = \boxed{\frac{3s}{s-2} = B(s)}$$

$$b(t) = \frac{d}{dt}[\underbrace{3e^{-2t}}_{a(t)}] = -6e^{-2t} \xrightarrow{\mathcal{L}} \frac{-6}{2+s} = \boxed{\frac{-6}{s+2} = B(s)}$$

$\frac{d}{dt}$ THEOREM $\quad B(s) = s\underbrace{\left[\frac{3}{2+s}\right]}_{A(s)} - \underbrace{3}_{a(0^-)} = \frac{3\cancel{s}}{s+2} - \frac{3(\cancel{s}+2)}{s+2} = \boxed{\frac{-6}{s+2} = B(s)}$

❖ **Find the capacitor current** $i(t)$ **in the circuit shown**



- **Basic Laws:**
  - **Kirchoff's Current Law (KCL)**
  - **Kirchoff's Voltage Law (KVL)**
  - **Ohm's Law**
- **Circuit Analysis Methods:**
  - **Nodal Analysis**
  - **Mesh Analysis**
- **Analysis of Op-Amp Circuits**

- **Find** $L^{-1}\left\{\dfrac{10}{(s+1)(s+2)}\right\}$

$$\boxed{\frac{10}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}}$$

$$\frac{10}{s^2+3s+2}$$

$A = 10$

$B = -10$

$$\xrightarrow{\mathcal{L}^{-1}}$$

$10e^{-t}u(t) - 10e^{-2t}u(t)$

- ❖ First, check to make sure the numerator polynomial has a degree that is strictly *less* than the denominator polynomial (USE POLYNOMIAL DIVISION AS NEEDED)
- ❖ Factor the denominator polynomial
- ❖ Write the function in terms of partial fractions with unknown numerator coefficients
- ❖ Solve for the coefficients using heaviside's formula or any other valid method
- ❖ Perform the inverse Laplace Transform by table look-up

Inverse Laplace Transform Example

❖ Find $L^{-1}\left\{\dfrac{\cancel{s}(s+3)}{(s^2+s)(s+2)^2}\right\}$

$$\frac{s+3}{s(s+1)(s+2)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{(s+2)^2} \quad \text{OR} \quad = \frac{A}{s} + \frac{B}{s+1} + \frac{E}{s+2} + \frac{F}{(s+2)^2}$$

$$A = \left.\frac{s+3}{(s+1)(s+2)^2}\right|_{s=0} = 3/4$$

$$B = \left.\frac{s+3}{s(s+2)^2}\right|_{\substack{s+1=0\\ s=-1}} = -2$$

$$F = \left.\frac{s+3}{s(s+1)}\right|_{\substack{s+2=0\\ s=-2}} = 1/2$$

$$\frac{d}{ds}\left[\frac{s+3}{s(s+1)} = \left(\frac{A}{s}+\frac{B}{s+1}\right)(s+2)^2 + E(s+2) + F\right]$$

$$\left.\frac{d}{ds}\left[\frac{s+3}{s(s+1)}\right]\right|_{s=-2} = \left.\left[\left(\frac{A}{s}+\frac{B}{s+1}\right)2(s+2) + (\cdots)(s+2)^2 + E\right]\right|_{s=-2}$$

$$E = \left.\frac{d}{ds}\left[\frac{s+3}{s(s+1)}\right]\right|_{s=-2} = 5/4$$

CHECK THIS!

$$L^{-1}\left\{\frac{3/4}{s} - \frac{2}{s+1} + \frac{5/4}{s+2} + \frac{1/2}{(s+2)^2}\right\}$$

$$\boxed{\frac{3}{4}u(t) - 2e^{-2t}u(t) + \frac{5}{4}e^{-2t}u(t) + \frac{1}{2}te^{-2t}u(t)}$$

Inverse Laplace Transform Example

❖ Find $L^{-1}\left\{\dfrac{s+\cancel{2}3}{s^2+4s+13}\right\}$

$$s = \frac{-4 \pm \sqrt{4^2-4(13)}}{2}$$

ROOTS ARE COMPLEX

$(s+?)(s+?)$ **NO**

COMPLETE SQUARE!

$$(s+\alpha)^2 + k = s^2 + 2\alpha s + \alpha^2 + k = s^2 + 4s + 13$$

$$\alpha = \frac{4}{2} = 2$$

$$\alpha^2 + k = 13$$

$$k = 13 - 2^2 = 9$$

$$(s+2)^2 + 9$$

$$\therefore L^{-1}\left\{\frac{s+3}{(s+2)^2+9}\right\} = L^{-1}\left\{\frac{s+2}{(s+2)^2+9} + \frac{1}{3}\frac{3}{(s+2)^2+9}\right\}$$

$$= e^{-2t}\cos(3t)\,u(t) + \frac{1}{3}e^{-2t}\sin(3t)\,u(t)$$

$$\frac{1}{s^2(s+\alpha)^2(s^2+\alpha s+\beta)(s^2+\omega_0^2)}$$

$\frac{A}{s} + \frac{B}{s^2}$

$\frac{C}{s+\alpha} + \frac{D}{(s+\alpha)^2}$

FIRST TRY TO FACTOR INTO 2 LINEAR WITH REAL COEFFICIENTS

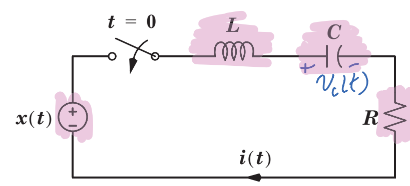
COMPLETE SQUARE $\frac{Es+F}{(s+?)^2+?}$

$\frac{Ks+L}{s^2+\omega_0^2}$

$s^2+2s+1 = (s+1)^2$

$s^2-s-6 = (s-3)(s+2)$

- ❖ **It is possible to move an electric circuit directly into the frequency domain.**



CASE I: $t = 0^-$, $i(0^-) = 0$, GIVEN $v_c(0^-) = V_0$

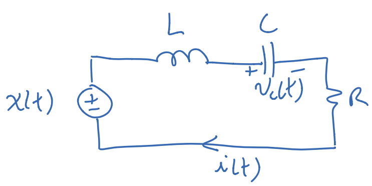
CASE II: $t \geq 0$, INDUCTOR CURRENTS DON'T JUMP, CAPACITOR VOLTAGES " "

$i(0^+) = 0$, $v_c(0^+) = V_0$

$x(t) \rightarrow \boxed{\text{??}} \rightarrow i(t)$

$i(t) = C\frac{dv_c(t)}{dt}$

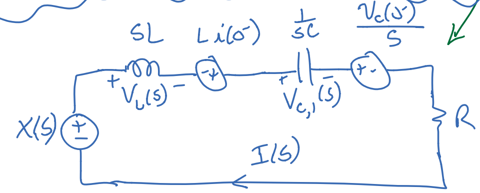
$v_c(t) = \left[\frac{1}{C}\int_0^{\infty} i(t)\right] + V_0$



KVL GIVES SYSTEM EQ $-x(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t) + Ri(t) = 0$

$\mathcal{L}$

...



$V_L(s) = (sL)I(s)$

$V_{c,1}(s) = \left(\frac{1}{sC}\right)I(s)$

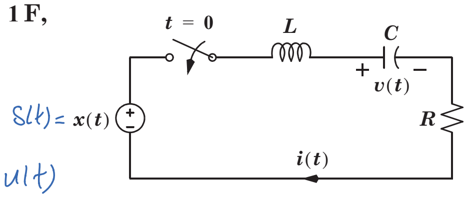
KVL: ...

SOLVE FOR $I(s)$

PARTIAL FRACTIONS TO FIND $i(t)$

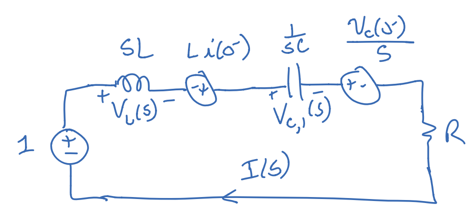
$i(t)$ L T.D. $v(t)$ $v(t) = L\frac{di(t)}{dt}$ $\mathcal{L}$ $V(s) = L\left[sI(s) - i(0^-)\right]$ SL $Li(0^-)$ $I(s)$ $V(s)$	$i(t)$ C $v(t)$ $i(t) = C\frac{dv(t)}{dt}$ $\mathcal{L}$ $I(s) = C\left[sV(s) - v(0^-)\right]$ $V(s) = \left(\frac{1}{sC}\right)I(s) + \frac{v(0^-)}{s}$ $I(s)$ $\frac{1}{sC}$ $\frac{v(0^-)}{s}$ $V(s)$	$i(t)$ R $v(t)$ $v(t) = Ri(t)$ $\mathcal{L}$ $V(s) = RI(s)$ R $I(s)$

❖ **Find the current $i(t)$ in the circuit shown. Assume $L = 1\,\mathrm{H}$, $C = 1\,\mathrm{F}$, $R = 1\,\Omega$, and $v(0^-) = V_0$.** , $i(0^-) = 0$



$$i(t) = e^{-\frac{1}{2}t}\cos\left(\sqrt{\tfrac{3}{4}}\,t\right)u(t) - \frac{1}{2}e^{-\frac{1}{2}t}\sin\left(\sqrt{\tfrac{3}{4}}\,t\right)u(t) - V_0\sqrt{\tfrac{4}{3}}\,e^{-\frac{1}{2}t}\sin\left(\sqrt{\tfrac{3}{4}}\,t\right)u(t)$$

KVL: $-1 + (sL)I(s) - L(0) + \left(\frac{1}{sC}\right)I(s) + \frac{V_0}{s} + RI(s) = 0$

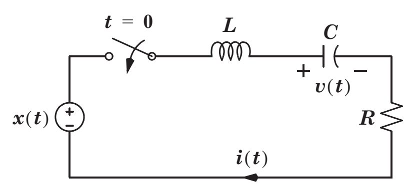


$$I(s) = \frac{s - V_0}{s^2 + s + 1}$$

$\vdots$

- **Classification Type I**
  - **Zero input response:** OUTPUT DUE TO INITIAL CONDITIONS (IC) ONLY (INPUT ZERO)
  - **Zero state response:** OUTPUT DUE TO INPUT ONLY (IC ZERO)
- **Classification Type II**
  - **Steady State response:** OUTPUT TERMS THAT DON'T GO TO ZERO AS $t \to \infty$
  - **Transient response:** OUTPUT TERMS THAT GO TO ZERO AS $t \to \infty$

- **Find the current $i(t)$ in the circuit shown. Assume $L = 1\,\text{H}$, $C = 1\,\text{F}$, $R = 1\Omega$, $v(0^-) = 1\text{V}$, and $x(t) = 0$.**



- **Impulse Response, $h(t)$: Zero state response to an input of $\delta(t)$**

LTI

$\delta(t) \rightarrow \boxed{\mathcal{H}} \rightarrow h(t)$

NO I.C. CONSIDERED

ZERO I.C.

$x(t) \rightarrow \boxed{\mathcal{H}} \rightarrow y(t) = x(t) * h(t)$

ZERO STATE RESPONSE

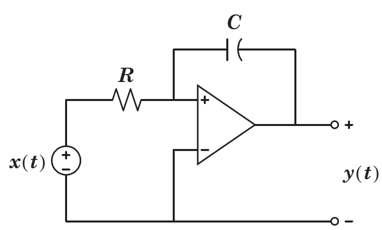
$H(f)$

IMPLIES ZERO I.C.

- **Transfer Function: $H(s) = L\{h(t)\}$**

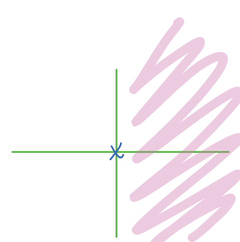
SYSTEM FUNCTION
OR
TRANSFER FUNCTION
IMPLIES ZERO I.C.

- **Is the following system stable? Where are the poles and zeros?**



$h(t) = \frac{-1}{RC} u(t)$

$H(s) = \frac{-1}{RC} \cdot \frac{1}{s}$



MARGINALLY STABLE

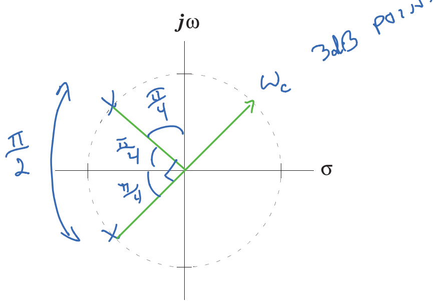
The Butterworth Lowpass Filter: Conceptual Design

FIR = FINITE IMPULSE RESPONSE (FOURIER DOMAIN)
IIR = INFINITE " " (LAPLACE DOMAIN)

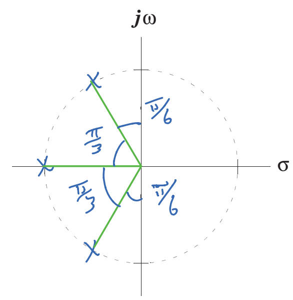
❖ **Design Rules:**

- **The filter is designed using only poles.**
- **The poles are evenly spaced around a semi-circle in the Left-Half-Plane (LHP).**
- **The circle radius gives the cutoff frequency ω_c rad/sec.**

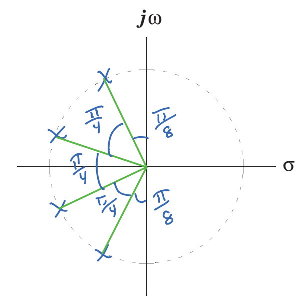
$N = 2$



$N = 3$



$N = 4$



LOWPASS FILTER

Butterworth LPF Pole Polynomials

❖ **Butterworth LPF Transfer Function**

$$H(s) = \frac{1}{B_N(s)}$$

- ***N* even**

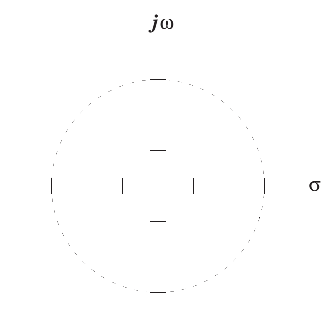
$$B_N(s) = \prod_{k=1}^{N/2}\left[s^2 - 2s\cos\left(\frac{2k+N-1}{2N}\pi\right)+1\right]$$

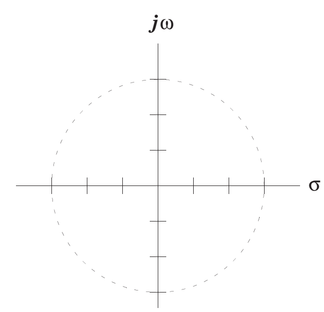
- ***N* odd**

$$B_N(s) = (s+1)\prod_{k=1}^{(N-1)/2}\left[s^2 - 2s\cos\left(\frac{2k+N-1}{2N}\pi\right)+1\right]$$

$N = 1$	$(s+1)$
2	$s^2 + \underset{\sqrt{2}}{1.4142}s + 1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.7654s+1)(s^2+1.8478s+1)$
5	$(s+1)(s^2+0.618s+1)(s^2+1.618s+1)$
6	$(s^2+0.5176s+1)(s^2+1.4142s+1)(s^2+1.9319s+1)$

$\omega_c = 1$

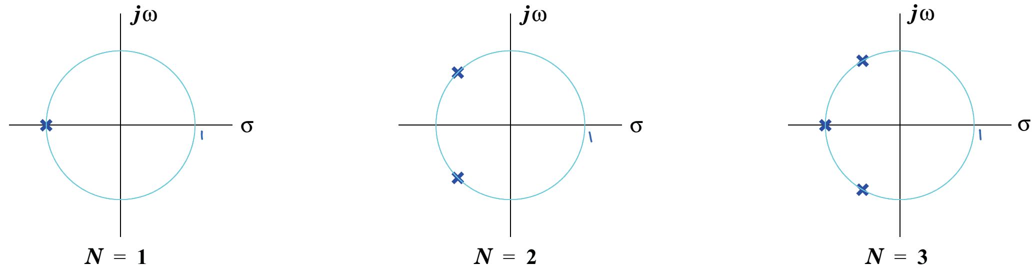


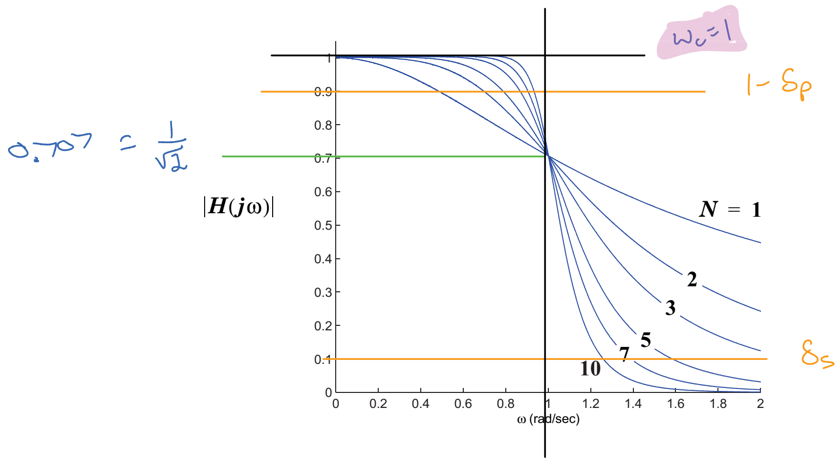


SCALE UP TO ARBITRARY ω_c

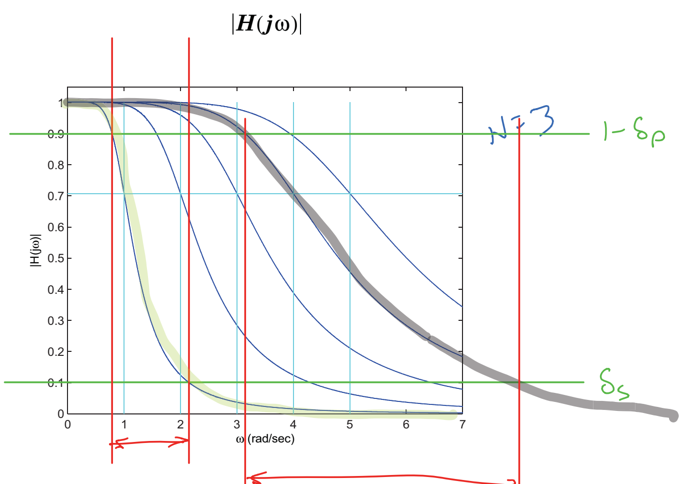
$$\left(\frac{s}{\omega_c}\right)^2 + \sqrt{2}\left(\frac{s}{\omega_c}\right) + 1 \Rightarrow H(s) = \frac{1}{\left(\frac{s}{\omega_c}\right)^2 + \sqrt{2}\left(\frac{s}{\omega_c}\right) + 1}$$

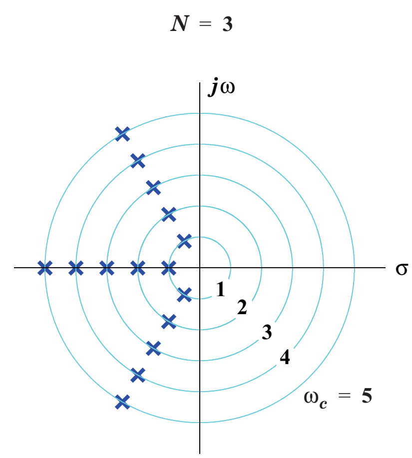
Butterworth LPF Magnitude Response vs. Order





Butterworth LPF Magnitude Response vs. Cutoff Frequency



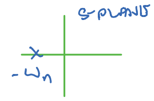


- ❖ Circle radius determines the cutoff frequency, ω_c rad/sec.
- ❖ For a given set of thresholds, δ_p and δ_s, the transition band grows with the cutoff frequency.

The Bode Plot and The Real Pole

CORNER FREQUENCY

$$H(s) = \frac{1}{\left(\frac{s}{\omega_n}\right) + 1}$$



$$H(j\omega) = \frac{1}{\left(\frac{j\omega}{\omega_n}\right) + 1} = \boxed{\frac{1}{\sqrt{\left(\frac{\omega}{\omega_n}\right)^2 + 1^2}}}\angle -\mathrm{atan}\left(\frac{\omega}{\omega_n}\right)$$

* Small ω

$\omega \ll \omega_n$

$H(j0) \to 1$

$|H(j0)|_{\mathrm{dB}} = 0$

$\angle H(j0) = 0$

$|H(\omega)|_{dB} = 20 \log_{10} |H(\omega)|$

* Corner Frequency $\omega = \omega_n$

$$H(j\omega_n) = \frac{1}{j+1} = \left(\frac{1}{\sqrt{2}}\right)e^{j(-\pi/4)}$$

$$|H(j\omega_n)|_{\mathrm{dB}} = 20\log_{10}\left(\frac{1}{\sqrt{2}}\right) = -3.01 \text{ dB} \approx -3\text{dB}$$

$\angle H(j\omega_n) = -\pi/4$

* Large $\omega \gg \omega_n$

$$H(j\omega) \to \frac{\omega_n}{j\omega} = \left(\frac{\omega_n}{\omega}\right)e^{j(-\pi/2)}$$

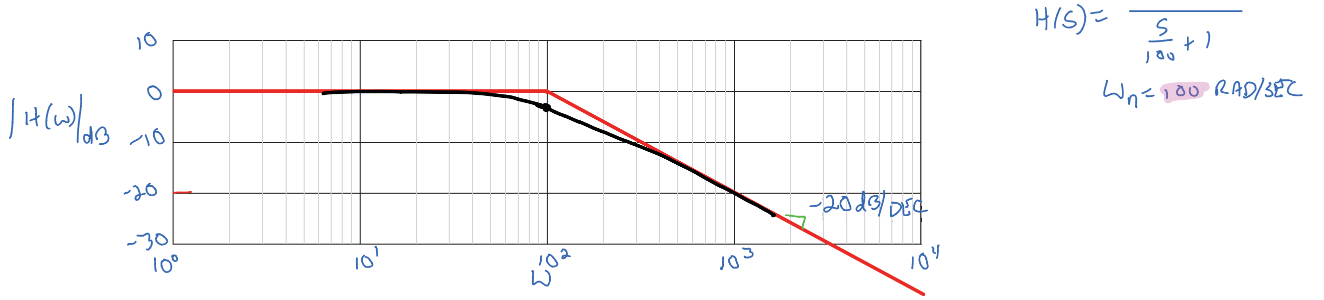
$$|H(j\omega)|_{\mathrm{dB}} = 20\log_{10}\omega_n \underbrace{-20}_{\text{SLOPE}}(\log_{10}\omega)$$

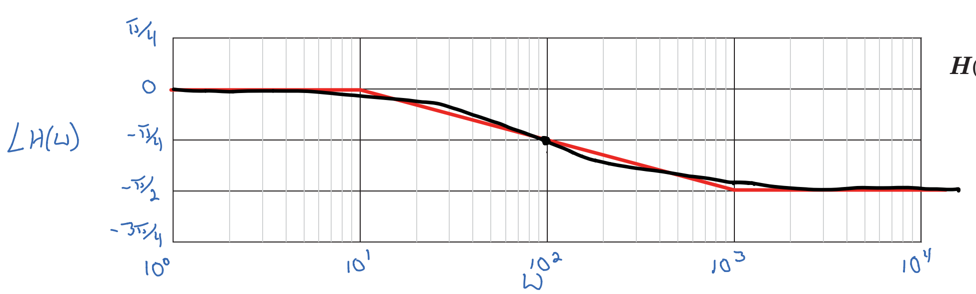
$\angle H(j\omega) = -\pi/2$

The Bode Plot Magnitude: Single Pole Example

$H(s) = \frac{1}{\frac{s}{100} + 1}$

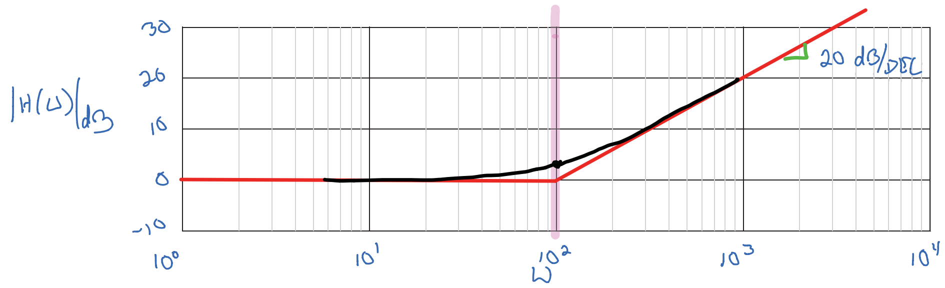
$\omega_n = 100$ RAD/SEC





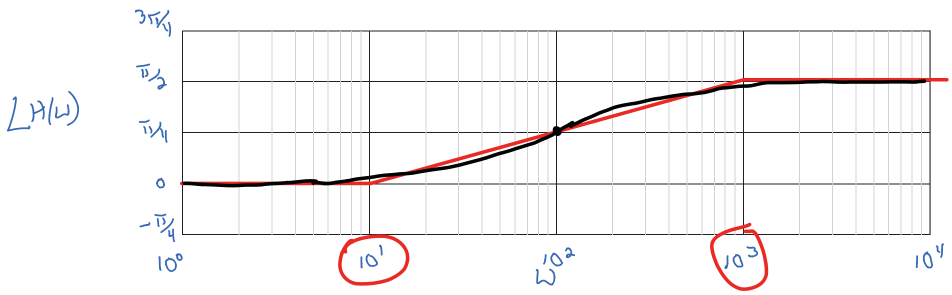
$$H(s) = \frac{1}{\left(\frac{s}{100} + 1\right)}$$

The Bode Plot Phase: Single Pole/Zero Example



$$H(s) = \left(\frac{s}{100} + 1\right)$$

$\omega_n = 100$

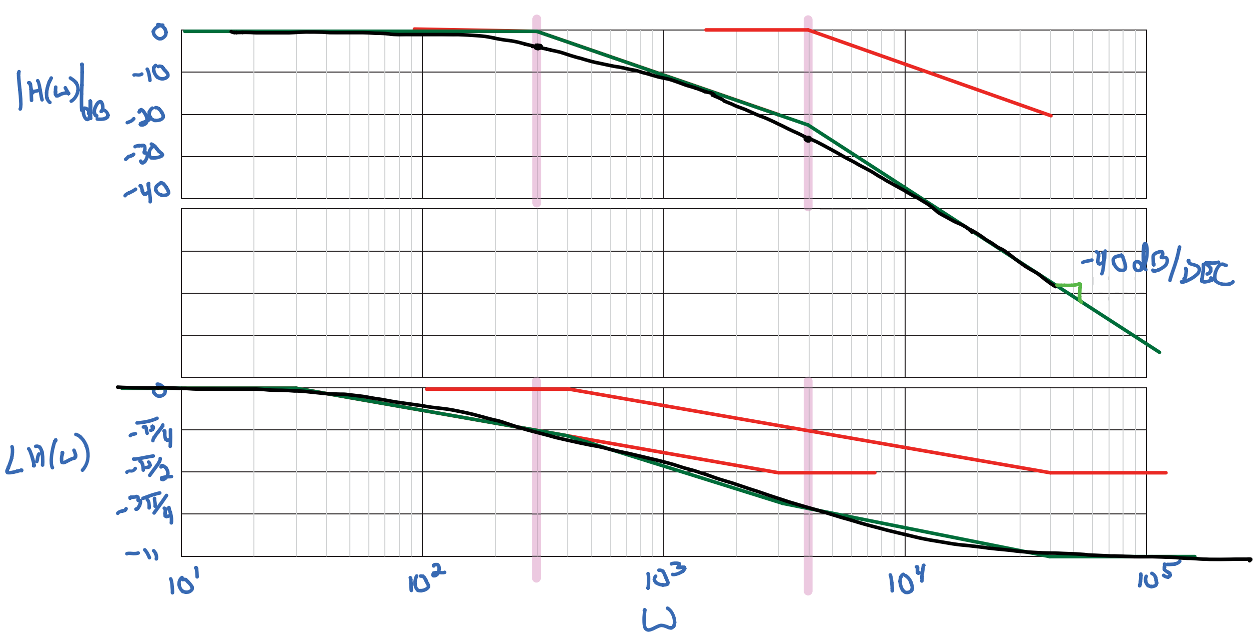


The Bode Plot Magnitude: Two Pole Example

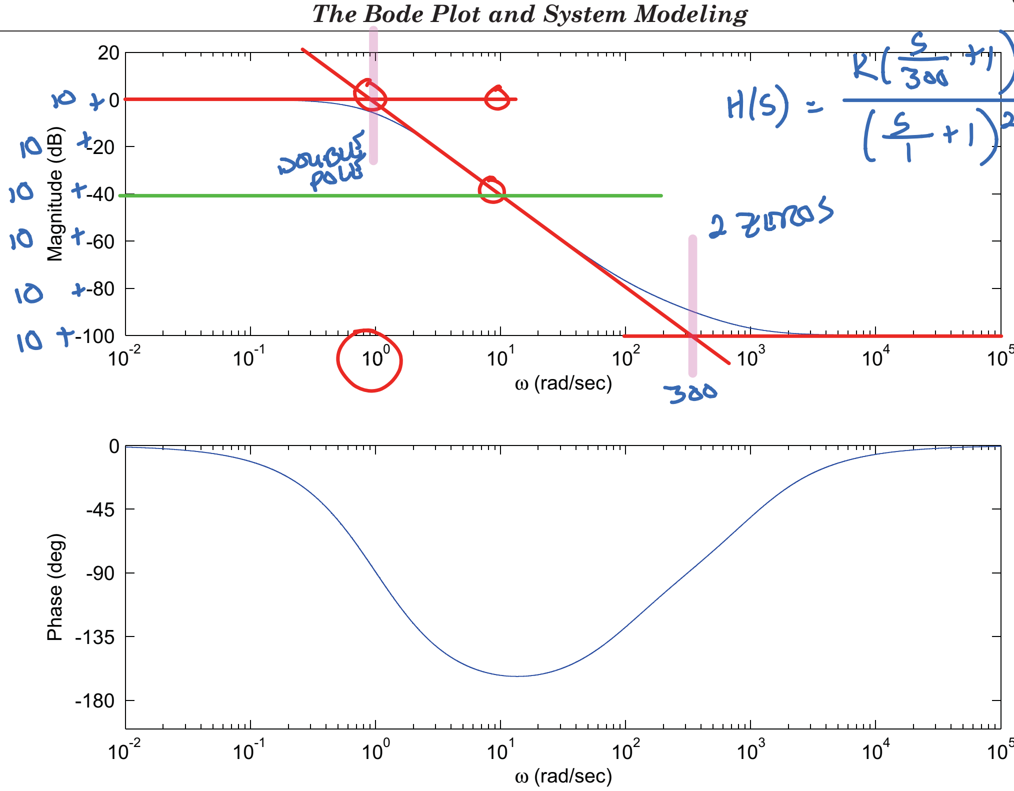
$\text{Log}\, ab = \text{Log}\, a + \text{Log}\, b$

$$H(s) = \frac{1}{\left(\frac{s}{300} + 1\right)\left(\frac{s}{4000} + 1\right)}$$

$\omega_{n,1}$ $\omega_{n,2}$



The Bode Plot and System Modeling



$$H(s) = \frac{K\left(\frac{s}{300}+1\right)^2}{\left(\frac{s}{1}+1\right)^2}$$

$K \Rightarrow 10\text{dB}$

$20\log_{10} K = 10$

$K = 10^{\frac{10}{20}}$

$= \sqrt{10} \approx 3.1$

The Bode Plot and Gain

$$H(s) = K \qquad\qquad H(s) = -K$$

$$H(j\omega) = K = (K)e^{j(0)}$$

$$|H(j\omega)|_{dB} = 20(\log_{10}K)$$

$$\angle H(j\omega) = 0$$

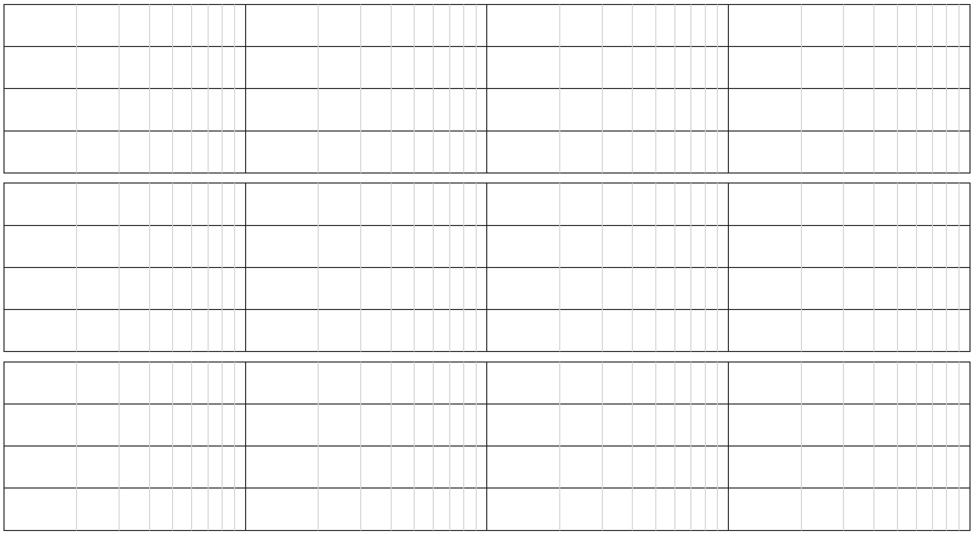
$$H(j\omega) = -K = (K)e^{j(\pi)}$$

$$|H(j\omega)|_{dB} = 20(\log_{10}K)$$

$$\angle H(j\omega) = \pi$$

The Bode Plot Magnitude: One Pole / One Zero, Gain

$$H(s) = \frac{400\left(\frac{s}{4000}+1\right)}{\left(\frac{s}{300}+1\right)}$$

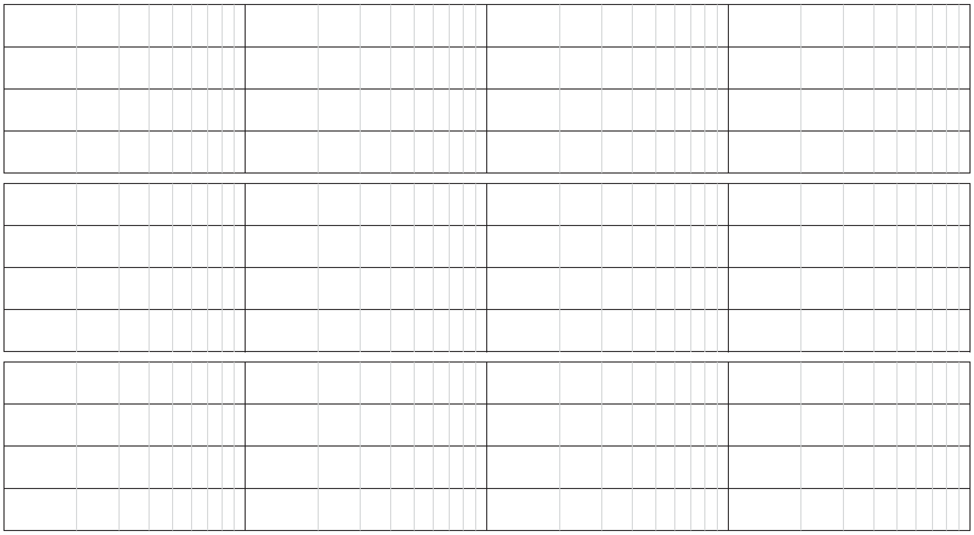


The Bode Plot: A Pole or Zero at the Origin

Pole at the origin	Zero at the origin
$H(s) = \frac{1}{s}$	$H(s) = s$
$H(j\omega) = \frac{1}{j\omega} = \left(\frac{1}{\omega}\right)e^{j(-\pi/2)}$	$H(j\omega) = j\omega = (\omega)e^{j(\pi/2)}$
$\lvert H(j\omega)\rvert_{\mathrm{dB}} = -20(\log_{10}\omega)$	$\lvert H(j\omega)\rvert_{\mathrm{dB}} = 20(\log_{10}\omega)$
$\lvert H(j1)\rvert_{\mathrm{dB}} = 0$	$\lvert H(j1)\rvert_{\mathrm{dB}} = 0$
$\angle H(j\omega) = -\pi/2$	$\angle H(j\omega) = \pi/2$

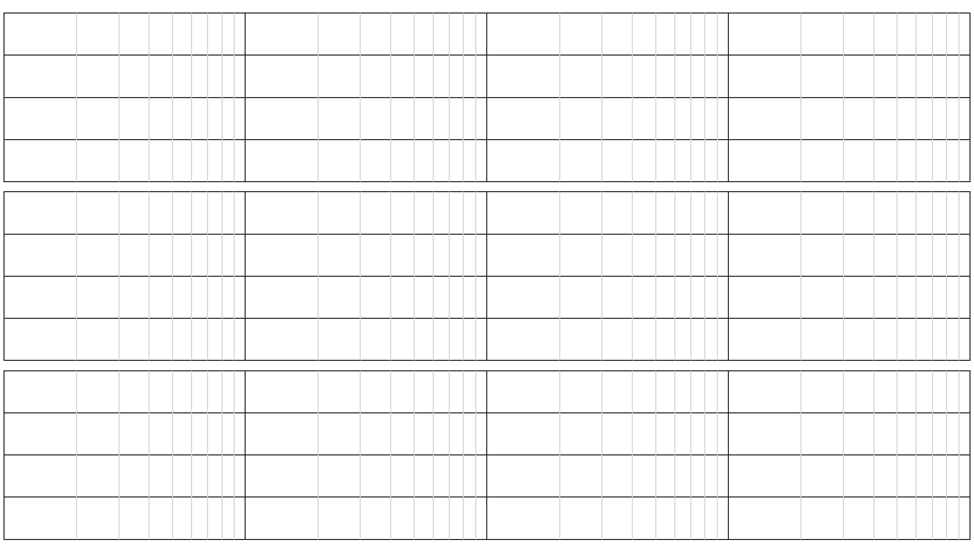
The Bode Plot Magnitude: One Pole / One Zero, Gain

$$H(s) = \frac{400(s)}{\left(\frac{s}{100}+1\right)}$$



The Bode Plot Phase: Two Pole Example

$$H(s) = \frac{1}{\left(\frac{s}{300}+1\right)\left(\frac{s}{4000}+1\right)}$$



Inverse Laplace Transform Procedure

- **First, check to make sure the numerator polynomial has a degree that is strictly *less* than the denominator polynomial** (USE POLYNOMIAL DIVISION AS NEEDED)
- **Factor the denominator polynomial**
- **Write the function in terms of partial fractions with unknown numerator coefficients**
- **Solve for the coefficients using heaviside's formula or any other valid method**
- **Perform the inverse Laplace Transform by table look-up**

$$H(s) = \frac{2s^2+3}{s(s+1)} = \cancel{\frac{A}{s} + \frac{B}{s+1}}$$

$$= \cancel{\frac{A(s+1)+Bs}{s(s+1)}}$$

DOES NOT WORK

$$s^2+s \,\overline{)\,2s^2+3\,} \quad 2 + \frac{3-2s}{s^2+s}$$

$$\frac{2s^2+2s}{3-2s}$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{2 + \frac{3-2s}{s(s+1)}\right\}$$

$$= 2\delta(t) + \mathcal{L}^{-1}\left\{\frac{3-2s}{s(s+1)}\right\}$$

Denominator Factors

$$\frac{1}{s^2(s+\alpha)^2(s^2+\alpha s+\beta)(s^2+\omega_0^2)}$$

$$\frac{A}{s} + \frac{B}{s^2}$$

$$\frac{C}{s+\alpha} + \frac{D}{(s+\alpha)^2}$$

FIRST TRY TO FACTOR INTO 2 LINEAR WITH REAL COEFFICIENTS

COMPLETE SQUARE $\frac{Es+F}{(s+?)^2+?}$

$$\frac{Ks+L}{s^2+\omega_0^2}$$

$$s^2+2s+1 = (s+1)^2$$

$$s^2-s-6 = (s-3)(s+2)$$

$$H(s) = \frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

$$A = \left.\frac{1}{s^2+4}\right|_{s=0} = \boxed{\frac{1}{4} = A}$$

$$(Bs+C)\Big|_{s=\pm j2} = \frac{1}{s}\Big|_{s^2+4=0}$$

$$s^2 = -4$$

$$s = \pm j2$$

$$Bj2 + C = \frac{1}{j2} = \frac{-1}{2}j + 0$$

$$2B = -\frac{1}{2} \Rightarrow \boxed{B = -\frac{1}{4}}, \quad \boxed{C=0}$$

$$\left[\frac{1}{s(s^2+4)} = \frac{1}{s} + \frac{Bs+C}{s^2+4}\right](s^2+4)$$

$$\left.\frac{1}{s}\right|_{s^2+4=0} = \left[\frac{1}{s}(s^2+4) + Bs+C\right]\Bigg|_{s^2+4=0}$$

$$j^2 = -1$$

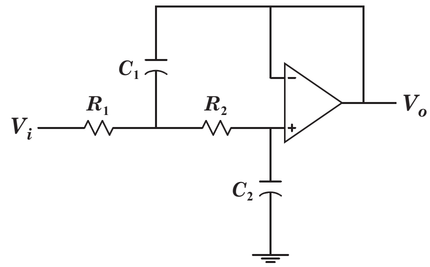
$$j = -\frac{1}{j}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{4}\cdot\frac{1}{s} + \frac{-\frac{1}{4}s}{s^2+4}\right\} = \boxed{\frac{1}{4}u(t) - \frac{1}{4}\cos(2t)\,u(t)}$$

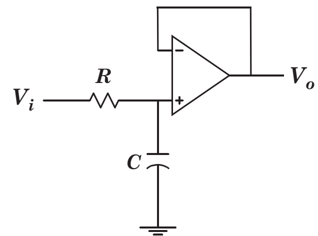
$$\omega_0^2 \Rightarrow \omega_0 = \sqrt{4} = 2$$

❖ Sallen-Key Topology

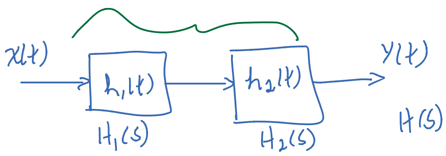
$$H(s) = \frac{1}{1 + C_2(R_1 + R_2)s + C_1C_2R_1R_2s^2}$$



$$H(s) = \frac{1}{1 + CRs}$$

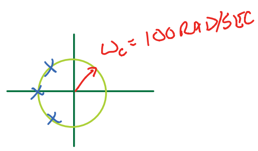


$h(t) = h_1(t) * h_2(t)$



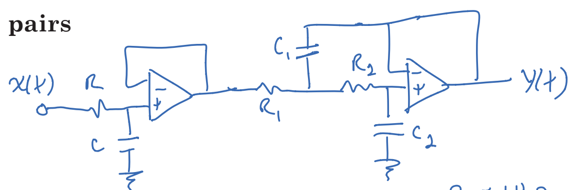
$H(s) = H_1(s)\,H_2(s)$

❖ Cascade sections to build the complex conjugate pole pairs



$$H(s) = \frac{1}{\left(\frac{s}{100}+1\right)\left[\left(\frac{s}{100}\right)^2 + \frac{s}{100} + 1\right]}$$

$$= \left[\frac{1}{\left(\frac{s}{100}+1\right)}\right]\left[\frac{1}{\left(\frac{s}{100}\right)^2 + \frac{s}{100} + 1}\right]$$



$CR = \frac{1}{100}$ $\quad R = 1000\,\Omega = 1\,k\Omega$

$C = 10 \times 10^{-6} = 10\,\mu F$

$C_2(R_1 + R_2) = \frac{1}{100}$

$C_1C_2R_1R_2 = \left(\frac{1}{100}\right)^2$

$R_1 = R_2 = 1\,k\Omega$

$C_2 =$

$C_1 =$