TEST III HOMEWORK

1. What frequency f, in units of Hz, are represented by ~~the following~~ complex frequencies.

   a. $s = -500 + j1000\pi$ $\quad j\cancel{1000}^{50}\pi = j\cancel{2\pi}f$ $\quad \boxed{f = 50}$

   b. $s = -20$ $\quad j2\pi f = 0$ $\quad \boxed{f = 0}$

   c. $s = 200 - j50\pi$ $\quad -j\cancel{50\pi}^{25} = j\cancel{2\pi}f$ $\quad \boxed{-25 = f}$

2. Classify each problem below in terms of which transform can be applied to solve it. For each problem below, select one of:

   ☐ Fourier ☐ Laplace ☐ Both ☐ Neither

   Clearly explain why you chose your answer. You do NOT need to solve the problem.

   a. Solve for $y(t)$ in the differential equation given by: (assume this is stable)

   no initial conditions (causal)

   $$\frac{d}{dt}y(t) - y(t) = \cos(\pi t) \qquad \boxed{\text{Both}}$$

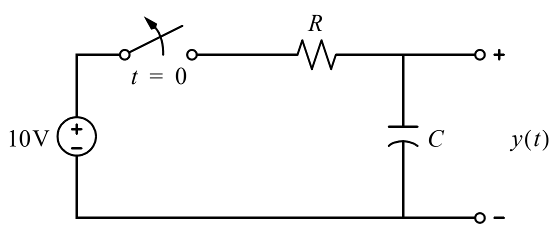
   b. Solve for $y(t)$ in the differential equation given by: (assume this is stable)

   $$\frac{d}{dt}y(t) - y(t) = r(t)$$

   with $y(0^-) = 2$. initial condition $\quad \boxed{\text{Laplace only}}$

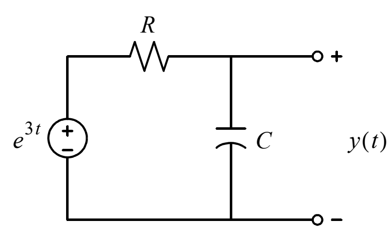
   c. Solve for the output voltage $y(t)$ in the following circuit if $y(0^-) = 5\text{V}$

   initial condition

   $\boxed{\text{Laplace only}}$

   d. Solve for the output voltage $y(t)$ in the following circuit:

3. Use the defining integral to find the Laplace Transform of the following signals

   a. $x_a(t) = u(t)$

   $\int_0^\infty u(t)e^{-st}dt$ $\quad u(t)=1$ for all $t>0$ $\quad \int_0^\infty e^{-st}dt = -\frac{e^{-st}}{s}\Big|_0^\infty = \frac{\cancel{e^{-s\infty}}}{-s} + \frac{e^0}{+s} = \boxed{\frac{1}{s}}$

   b. $x_b(t) = e^t + e^{-t}$

   $\int_0^\infty e^{t}e^{-st}dt + \int_0^\infty e^{-t}e^{-st}dt = \int_0^\infty e^{-t(s-1)}dt + \int_0^\infty e^{-t(1+s)}dt = \frac{e^{-t(1-s)}}{-(1-s)}\Big|_0^\infty + \frac{e^{-t(1+s)}}{-(1+s)}\Big|_0^\infty$

   $\frac{\cancel{e^{-\infty}}}{-(1-s)} - \frac{\cancel{e^{0}}}{+(1+s)} + \frac{\cancel{e^{-\infty}}}{-(1+s)} + \frac{\cancel{e^{0}}}{+(1+s)} = -\frac{1}{s-1} + \frac{1}{s+1}$

   $= \frac{-1+s+1-s}{s^2-1} = \frac{-2s}{s^2-1} = \boxed{\frac{-2s}{s^2-1}}$

4. Use the Laplace Transform pair $L\{e^{-\alpha t}u(t)\} = \frac{1}{\alpha+s}$, $\sigma > -\alpha$ to find each Laplace Transform given below. Be sure the state the region of convergence (ROC) for each. Note that the transform given above is valid even when α is complex.

   a. $h_a(t) = u(t)$ $\quad \mathcal{L}\{e^{-0t}u(t)\} = \frac{1}{s}, \sigma > 0$

   b. $h_b(t) = \cos(\omega_0 t)u(t)$

   Using $\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$

   $= \frac{1}{2}\mathcal{L}\{u(t)e^{j\omega_0 t}\} + \frac{1}{2}\mathcal{L}\{u(t)e^{-j\omega_0 t}\} = \frac{1}{2}\left[\frac{1}{-j\omega_0+s} + \frac{1}{j\omega_0+s}\right] = \frac{1}{2}\left[\frac{\cancel{j\omega_0}+s-\cancel{j\omega_0}+s}{\omega_0^2+s^2}\right] = \frac{1}{\cancel{2}}\left[\frac{\cancel{2}s}{\omega_0^2+s^2}\right] = \boxed{\frac{s}{\omega_0^2+s^2}}$ for $\sigma>0$

   c. $h_c(t) = \sin(\omega_0 t)u(t)$

   $= \frac{1}{2}\mathcal{L}\{u(t)e^{j\omega_0 t}\} - \frac{1}{2}\mathcal{L}\{u(t)e^{-j\omega_0 t}\} = \frac{1}{2}\left[\frac{1}{-j\omega_0+s} - \frac{1}{j\omega_0+s}\right] = \frac{1}{2}\left[\frac{j\omega_0+s-s+j\omega_0}{\omega_0^2+s^2}\right] = \frac{1}{\cancel{2}}\left[\frac{\cancel{2}j\omega_0}{\omega_0^2+s^2}\right] = \boxed{\frac{j\omega_0}{\omega_0^2+s^2}}$ for $\sigma>0$

   d. $h_d(t) = e^{-\alpha t}\cos(\omega_0 t)u(t)$

   $= \frac{1}{2}\mathcal{L}\{e^{-\alpha t}e^{j\omega_0 t}u(t)\} + \frac{1}{2}\mathcal{L}\{e^{-\alpha t}e^{-j\omega_0 t}u(t)\} = \frac{1}{2}\mathcal{L}\{e^{t(j\omega_0-\alpha)}u(t)\} + \frac{1}{2}\mathcal{L}\{e^{t(-j\omega_0-\alpha)}u(t)\} = \frac{1}{2}\left[\frac{1}{-j\omega_0+\alpha+s} + \frac{1}{j\omega_0+\alpha+s}\right] = \dots$

   $= \frac{s+\alpha}{\omega_0^2+2s\alpha+\alpha^2+s^2} = \boxed{\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2} \text{ for } \sigma > -\alpha}$

5. For each Transfer Function given below, (1) state the poles and zeros (both finite and infinite), state the *region of convergence* (ROC), and state whether the system is *stable* or not.

   a. $H_a(s) = \frac{s}{s+6}$ $\quad$ Pole: $s=0$; Zero: $s=-6$ $\quad$ ROC $\sigma > -6$; Stable

   b. $H_b(s) = \frac{s+2}{s-3}$ $\quad$ Pole: $s=-2$; Zero: $s=+3$ $\quad$ ROC $\sigma > 3$; Not stable

   c. $H_c(s) = \frac{s}{s^2+5s+6}$ $\quad$ $(s+2)(s+3)$ $\quad$ Zeros: $s=0,\infty$; Poles: $s=-3,-2$ $\quad$ ROC: $\sigma > -2$; Stable

   d. $H_d(s) = \frac{s-1}{s^2-5s+6}$ $\quad$ $(s-3)(s-2)$ $\quad$ Zeros: $s=+1,\infty$; Poles: $s=3,2$ $\quad$ ROC $\sigma > 3$; Not stable

   e. $H_e(s) = \frac{s+2}{s^3-5s^2+6s}$ $\quad$ Zeros: $s=-1,\infty,\infty$; Poles: $s=0,2,3$ $\quad$ ROC $\sigma > 3$; Not Stable

   $s(s^2-5s+6)$

   $s(s-3)(s-2)$

6. For each Transfer Function $H(s)$ given below, (1) state the *pole* and *zero* locations, both finite and infinite, and (2) determine whether the system is *stable* and clearly state why it is, or is not, stable. Hint: use the matlab **`roots`** function to factor the polynomials that are large.

a. $H_a(s) = \dfrac{s^2 + 2s + 2}{s^4 - 10s^3 + 35s^2 - 50s + 24}$

Zeros: $s = -1 \pm j, \infty, \infty$ Not Stable, poles > (s=0)
Poles: 4,3,2,1

b. $H_b(s) = \dfrac{3}{s^3 + s^2 + s - 1}$

Poles: $s = -.772 \pm j1.12$; .5437 Not stable, pole > (s=0)
Zeros: $s = \infty, \infty, \infty$

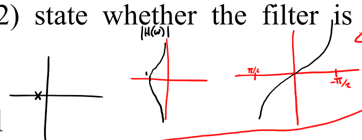
c. $H_c(s) = \dfrac{s^2 - 3s + 6}{s^4 + 2s^3 - 6s^2 + 2s + 3}$

Zeros: $s = 1.5 \pm j1.94, \infty, \infty$ Not Stable, Pole > (s=0)
Poles: -3.71, $1.12 \pm j.513$, -.534

7. The following filters are described by their pole-zero constellation in the s-plane. For each one, (1) use the graphical method to make a reasonable sketch of the Transfer Function *magnitude* $|H(f)|$ and *phase* $\angle H(f)$, (2) state whether the filter is *best* described as lowpass, bandpass, highpass, or bandreject.

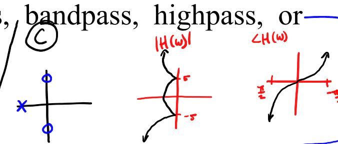
$|H(\omega)| = \frac{d_z}{d_p}$
$\angle H(\omega) = \theta_z - \theta_p$

a. pole: $s = -1$

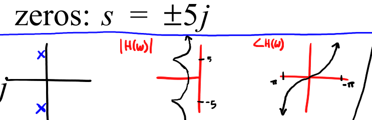


b. pole: $s = -1$, zero: $s = 0$

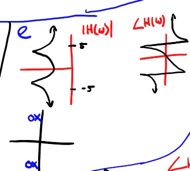
c. poles: $s = -5, -5$, zeros: $s = \pm 5j$



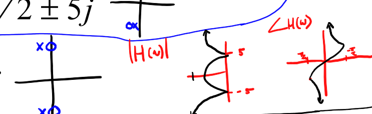
d. poles: $s = -1/4 \pm 5j$



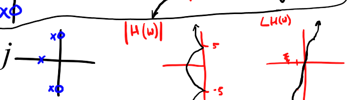
e. poles: $s = -1/4 \pm 5j$, zeros: $s = -1/2 \pm 5j$



f. poles: $s = -1/4 \pm 5j$, zeros: $s = \pm 5j$



g. poles: $s = -1, -1/4 \pm 5j$, zeros: $s = \pm 5j$



8. Reflect on your answers for the previous problem. Specifically, consider the type of filter created by each pole-zero constellation and the corresponding type of filter. Try to create a set of generalized criteria that can be used to determine the filter type given the pole-zero constellation. Your criteria could include the relative number of poles and zeros (same or different), it could also consider the placement of a closely-spaced poles-zero pair, and can also include the overall local placement of poles or zeros.

LPF: #P > #Z, has p near s=0
BPF: #P > #Z, Poles ±jω
HPF: #P = #Z, Poles around s=0
BRF: #P = #Z, Poles around s = ±jω

$\lim_{t\to 0^+} h(t) = \lim_{s\to\infty} s H(s)$

$\lim_{t\to\infty} h(t) = \lim_{s\to 0} s H(s)$

9. Use the *initial and final value theorems* to find $h(0^+)$ and $h(\infty)$ for each system described by the following Transfer Functions. For each problem, be sure to determine *if* the solution exists.

a. $H_a(s) = \dfrac{3(s+1)}{s^2+2s}$

°Num < °Den

$h(0^+) = \lim_{s\to\infty} s \cdot \frac{3(s+1)}{s^2+2s} = \lim_{s\to\infty} \cancel{s}\frac{3(s+1)}{\cancel{s}(s+2)} = \frac{3}{1} = \boxed{3}$

System is stable

$h(\infty) = \lim_{s\to 0} \frac{3(s+1)}{(s+2)} = \boxed{3/2}$

b. $H_b(s) = \dfrac{s^2-1}{s^2+4s+13}$

z $s = \pm 1$

p $s = -2 \pm 3j$

°N = °D Initial Value does not apply

System is stable $h(\infty) = \lim_{s\to 0} s \frac{s^2-1}{s^2+4s+13} = \boxed{\frac{-1}{13}}$

c. $H_c(s) = \dfrac{5s+1}{s^2-4}$

$s = -\frac{1}{5}$

$s = \pm 2$

°Num < °Den

$\lim_{s\to\infty} s \frac{5s+1}{s^2+4} = \text{DNE}$

Not Stable

Final Value does not apply

10. Using the theorem and transform pairs, find the Laplace Transform for the following signals.

a. $\frac{d}{dt}x_a(t)$, where $x_a(t) = e^{-2t}\cos(3t)u(t)$

$X(s) = \frac{s+2}{(s+2)^2+3^2}$ $\quad x(0^-) = [e^0 \cos(0)\, u(0^-)] = 0$

$\mathcal{L}\{\frac{d}{dt}x(t)\} = s\mathcal{L}\{x(t)\} - x(0) = \boxed{\left(\frac{s+2}{(s+2)^2+9}\right)s}$

b. $\frac{d}{dt}x_b(t)$, where $x_b(t) = e^{-2t}\cos(3t)$ ... this answer is different than part (a)!

$X(s) = \frac{s+2}{(s+2)^2+9}$ $\quad x(0^-) = e^0\cos(0) = 1$

$\mathcal{L}\{\frac{d}{dt}x(t)\} = sX(s) - x(0) = \boxed{s\frac{s+2}{(s+2)^2+9} - 1}$

c. $x_c(t) = e^{-2(t-1)}\cos(3t-3)u(t-1)$

$\boxed{X(s) = e^{-s}\frac{s+2}{(s+2)^2+9}}$

d. $x_d(t) = \Pi(t-1/2)$ (hint: write this in terms of unit step functions)

$x(t) = u(t) - u(t-1)$

$\mathcal{L}\{x(t)\} = \frac{1}{s} - \frac{e^{-s}}{s}$

$\boxed{= \frac{1}{s}[1-e^{-s}]}$

11. Find the Fourier Transform of $\Pi(t-1/2)$ and then show that your answer to part (d) in the previous problem reduces to this Fourier Transform when setting $s = j2\pi f$.

$\mathcal{F}\{\Pi(t-\frac{1}{2})\} = \mathcal{F}\{\Pi(t)\}e^{-j2\pi f(1/2)} = \text{sinc}(f)\,e^{-j\pi f}$

$\frac{1}{s}[1-e^{-s}]\Big|_{s=j2\pi f} = \frac{1}{j2\pi f}[1-e^{-j2\pi f}] \cdot \frac{e^{j\pi f}}{e^{j\pi f}} = \frac{e^{-j\pi f}}{j2\pi f}[e^{j\pi f} - e^{-j\pi f}]$

$= \frac{e^{-j\pi f}}{\pi f}\left[\frac{e^{j\pi f} - e^{-j\pi f}}{2j}\right] = e^{-j\pi f}\left[\frac{\text{sinc}(\pi f)}{\pi f}\right] = e^{-j\pi f}\text{sinc}(f)$

12. Find the inverse Laplace Transform for each Transfer Function given below.

a. $H_a(s) = \dfrac{3(s-1)}{s^2+s}$

$A = \frac{3(s-1)}{s+1}\Big|_{s=0} = \frac{-3}{1} = -3$ $\quad B = \frac{3(s-1)}{s}\Big|_{s=-1} = \frac{-6}{-1} = 6$

$= \frac{A}{s} + \frac{B}{s+1} = 6\frac{1}{s+1} - 3\frac{1}{s} = \boxed{6e^{-t}u(t) - 3u(t)}$

b. $H_b(s) = \dfrac{3}{s^2+2s+1}$

$(s+1)^2$

$= 3t$ $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$ $\boxed{= 3te^{-t}u(t)}$

c. $H_c(s) = \dfrac{5s}{s^3-s}$

$\frac{5}{s^2-1} = 5\frac{1}{j^2+s^2} = 5\sin(jt)u(t)$

$s^2 + 2 = As^2 + 2As + 5A + Bs^2 + Cs + Bs + C$

s^2 term: $1 = A + B$, $B = 1 - A$, $B = \frac{1}{4}$

s^1 term: $0 = 2A + B + C$, $0 = 2A + 1 - A + 2 - 5A$, $-3 = -4A$, $\frac{3}{4} = A$

s^0 term: $2 = C + 5A$, $C = 2 - 5A$, $= -\frac{7}{4}$

d. $H_d(s) = \dfrac{s^2+2}{s^3+3s^2+7s+5}$

$= \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+5}$ $\quad s^2+2s+5 = (s+1)^2+4$

$= \boxed{\frac{3/4}{s+1}} + \boxed{\frac{\frac{1}{4}s - \frac{7}{4}}{s^2+2s+5}}$ $\quad \frac{1}{4}\left[\frac{s-7}{(s+1)^2+4}\right] = \frac{1}{4}\left[\frac{s+1}{(s+1)^2+2^2} + \frac{-8}{(s+1)^2+2^2}\right]$

$\frac{3}{4}e^{-t}u(t)$ $\quad \mathcal{L}^{-1}$: $\boxed{\frac{1}{4}e^{-t}\cos(2t)u(t) - e^{-t}\sin(2t)u(t)}$

$\boxed{= u(t)\left[\frac{3}{4}u(t) + \frac{1}{4}e^{-t}\cos(2t) - e^{-t}\sin(2t)\right]}$

e. $H_e(s) = \dfrac{s+1}{s^3-s^2+4s-4}$

$\frac{s+1}{(s^2+4)(s-1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4}$ $\quad A = \frac{s+1}{s^2+4}\Big|_{s=1} = \frac{2}{5}$ $\quad s^2(s-1) + 4(s-1)$, $(s-1)(s^2+4)$

$s+1 = As^2 + Bs^2 + Cs - Bs - C + 4A$

s^2: $0 = A + B$, $B = -A$, $B = -\frac{2}{5}$

s: $1 = C - B$

s^0: $1 = 4A - C$, $1 = \frac{8}{5} - C$, $C = \frac{8}{5} - 1 = \frac{3}{5}$

$= \frac{2}{5}\frac{1}{s-1} + \frac{-\frac{2}{5}s + \frac{3}{5}}{s^2+4}$

$= u(t)\left[\frac{2}{5}e^{t} - \frac{2}{5}\cos(2t) + \frac{3}{10}\sin(2t)\right]$

13. Solve for the complete solution for each system given below.

a. Solve for $y(t)$ given the differential equation $\frac{d}{dt}y(t) - 2y(t) = u(t)$ and the initial condition $y(0^-) = 3$.

$sY(s) - y(0^-) - 2Y(s) = \frac{1}{s}$

$Y(s)[s-2] = \frac{1}{s} + 3$

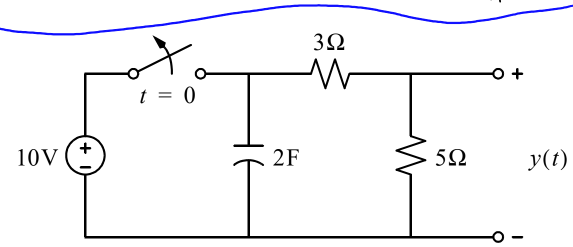
$Y(s) = \frac{\frac{1}{s}+3}{s-2} = \frac{1+3s}{s^2-2s} = \frac{3s+1}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2} = -\frac{1}{2}\frac{1}{s} + \frac{7}{2}\frac{1}{s-2}$

$\boxed{Y(t) = -\frac{1}{2}u(t) + \frac{7}{2}e^{2t}u(t)}$

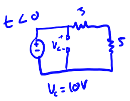
0 input

Steady State

b. Solve for $y(t)$ in the circuit shown. Be sure to state $y(t)$ for all time values.

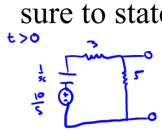


$t < 0$



$V_c = 10V$

$t > 0$



KVL: $-\frac{10}{s} + \frac{1}{2s}I(s) + 8I(s) = 0$

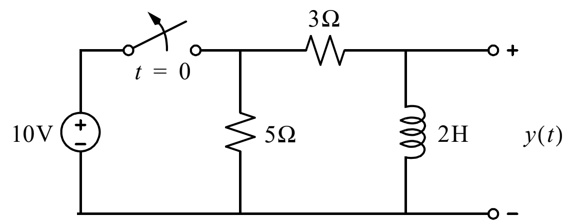
$I(s)\left[\frac{1}{2s} + 8\right] = \frac{10}{s}$

$I(s) = \frac{10/s}{\frac{1}{2s} + 8} \cdot \frac{2s}{2s} = \frac{20}{1+16s} = \frac{20}{16}\left[\frac{1}{s+\frac{1}{16}}\right] \Rightarrow \boxed{i(t) = \frac{5}{4}e^{-\frac{1}{16}t}u(t)}$

Zero State

transient

c. Solve for $y(t)$ in the circuit shown. Be sure to state $y(t)$ for all time.



14. This problem refers to the previous problem.

a. Identify the zero-input response and zero-state response for each of the three systems given.

b. Identify the steady-state response and transient response for each of the three systems given.

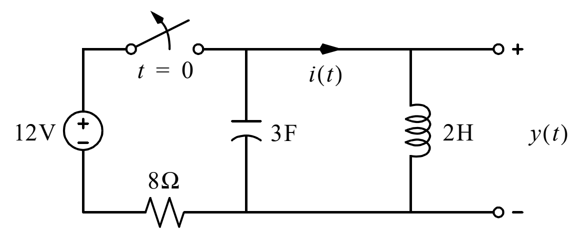
15. For the circuit shown at the right

$V_c(0^-) = 0$

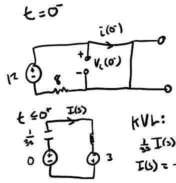
$i(0^-) = \frac{12}{8} = \frac{3}{2}$

a. Find the current $i(t)$ for all values of time.

b. What part of the solution is the transient response and what part is the steady-state response?



$t = 0^-$



$t \le 0^+$

KVL:

$\frac{1}{3s}I(s) + 2sI(s) - 3 = 0$

$I(s) = \frac{3}{\frac{1}{3s} + 2s} = \frac{9s}{1+6s^2}$

$\frac{9}{6}\left[\frac{s}{\frac{1}{6}+s^2}\right]$

$i(t) = \frac{3}{2}\cos\left(\frac{1}{\sqrt{6}}t\right)u(t), t > 0$ Steady State.

16. Write an expression for the transfer function for a butterworth lowpass filter for each set of parameters given below.

a. $\omega_c = 1000$ rad/sec and 3rd order.

b. $\omega_c = 5000$ rad/sec and 6th order.

17. A 3rd order butterworth lowpass filter with cutoff frequency $\omega_c = 1000$ rad/sec is to be designed using the Sallen-Key topology. Draw the realization and identify numeric values for all resistors and capacitors to complete the design.

18. Use matlab to design an 8th order butterworth lowpass filter with cutoff frequency $f_c = 1000$ Hz by completing the following steps:

```
fs = 100000;            % define the sample frequency
fc = 1000;              % define the cutoff frequency (Hz)
N = 8;                  % define the filter order

wn = fc/(fs/2);         % calculate the normalized cutoff frequency
[b,a] = butter(N,wn);   % calculate the filter coefficients

T = 1/fs;               % Find the sample time
t = -T:T:1;             % define an appropriate time vector
x = dc(t);              % construct the delta function

h = filter(b,a,x);      % find the impulse response (filter output)

myFT(t,h,'plot','frange',[0,2000]);  % find the transfer function

subplot(3,1,1);  ax = axis;  axis([0 .005 ax(3:4)]);  % zoom in h(t)
```

Try increasing the filter order **N** to 9, 10, and 11. Does the filter "break down" at some point? For what order does matlab give a reasonably correct filter? Try changing the cutoff frequency and see if the filter transfer function appears to be correct.

Is the transfer function rolloff symmetric like it was for the window method?

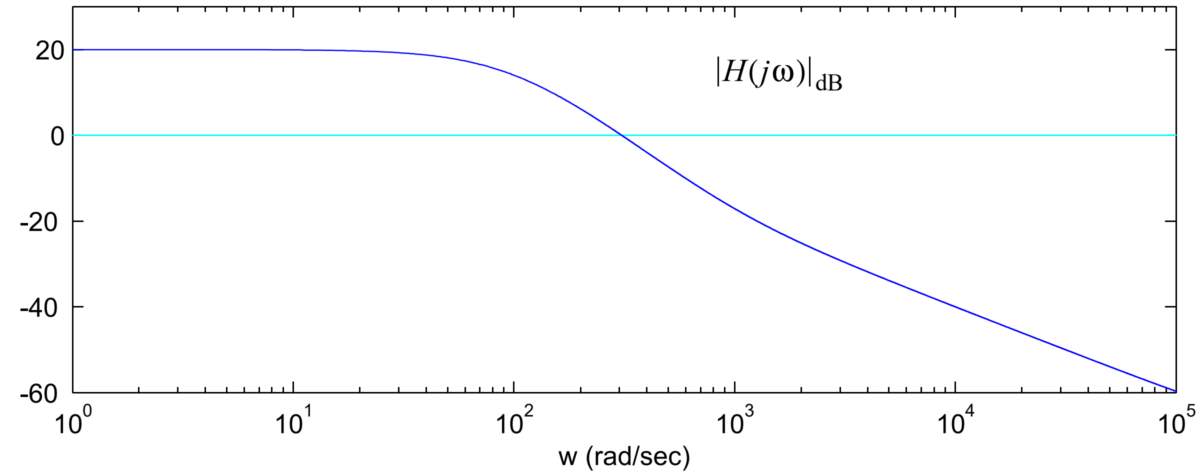
19. Draw Bode Plots, both the magnitude and phase, for each system given. Blank bode plot forms are given on the website.

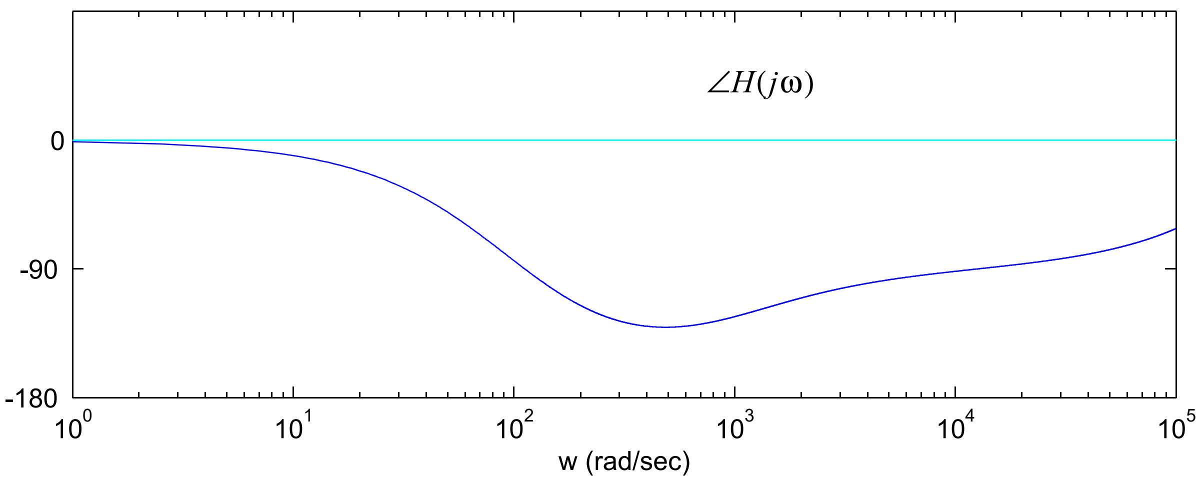
a. $H(s) = s + 400$

b. $H(s) = \dfrac{200}{s + 10}$

c. $H(s) = \dfrac{s + 1000}{s + 50000}$

20. The bode plot magnitude and phase are shown below for system $H(s)$.





a. State the numeric values for the poles and zeros for this transfer function.

b. Find the gain factor K for this transfer function.

c. Write a complete expression for the transfer function $H(s)$.