PS0

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Question 1

(a)

$$\nabla f(x) = Ax + b$$

(b)

$$\nabla f(x) = g'(h(x))\nabla_x h(x)$$

(c)

$$\nabla^2 f(x) = \nabla(\nabla f(x))^T = [\nabla A_1^T x ... \nabla A_n^T x] = A^T = A$$

where A_i is the ith row vector of matrix A
(d)

$$\nabla f(x) = g'(a^T x)a$$

$$\nabla^2 f(x) = [\nabla g'(a^T x)a_1 ... \nabla g'(a^T x)a_n] = g''(a^T x)aa^T$$

Question 2

(a) Clearly $A = zz^T$ is symmetric. Now let x$\in$R^n

$$x^T Ax = (x^T z)(z^T x) = (x \cdot z)^2 \geq 0$$

(b) From the previous result it follows that:

$$ker(A) = span(z)^\perp$$

where the $.^\perp$ means "the orthogonal space of".
Hence, using the rank-nullity theorem, $rank(A) = n - (n-1) = 1$.

(c) Let x $\in$R^n and $y = Bx$

$$x^T BAB^T x = (Bx)^T A(Bx) = y^T Ay \geq 0$$

Question 3

(a)

$$A = T\Lambda T^{-1} \Leftrightarrow AT = T\Lambda$$

$$AT = [At^{(0)}...At^{(n-1)}] = T\Lambda = [\lambda_0 t^{(0)}...\lambda_n t^{(n-1)}]$$

(b) Same answer as (a), replace U^T with U^{-1} since we suppose U is orthogonal

(c) Let $A \in R^{nxn}$ be a PSD matrix. By the spectral theorem, $\exists U, \Lambda$ such that $A = U\Lambda U^T$. Let $x \in R^n$ be the i'th column vector of U.
We set $y = U^T x$.
By this definition, we have $(y)_i = 1$ and $(y)_j = 0$ for $j \neq i$. Hence:

$$x^T Ax = x^T U\Lambda U^T x = (U^T x)^T \Lambda(U^T x) = y^T \Lambda y = \lambda_i \geq 0$$