Draft DG Swiss Cheese Conjecture

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1 Introduction

There is an operad E_d called the little disk operad such that for every n, $E_d(n)$ is the configuration space of embeddings of n disjoint disks into the unit disk. Composition is given by rescaling d-disks and plugging them into the unit d-disk. In his thesis [10], Justin Thomas solved the following problem:

- Given an algebra A over E_{d-1}, define its Hochschild space H.
- Find an algebraic structure that acts on the pair (H, A).

The above mentioned algebraic structure turns out to be given by the action of an operad SC_d called the Swiss Cheese operad.

This operad has two types of arguments, called full disks f and half disks h. The operations on full disks, $SC_d(n,0)^f$ coincide with $E_d(n)$. We also have operations that take r full disks and s half disks as arguments and a half disk as a result. They are parametrized by the space $SC_d(r,s)^h$ of disjoint embeddings of $2r+s$ disks in such a way that the reflection about the last coordinate fixes each one of the s disks and permutes the $2r$ disks. Equivalently, we can consider disjoint embeddings of r disks and s half disks into the half unit disk, in such a way that the flat boundaries land on the flat boundary of the unit half disk, see 2.1.

According to [10] this operad acts on the pair (H, A) where H is of type full disk, and A is of type half disk.[1]

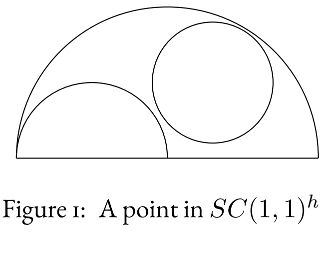


Figure 1: A point in $SC(1,1)^h$.

In our thesis we prove a DG analog of this result. We follow the approach of [10], however there are several technical issues to be solved, for example we know that the Kontsevich version of the Swiss Cheese operad is not formal, so we cannot work with the smaller homology operad, more over the natural adjunction between simplicial and chain operads doesn't preserve symmetry.

[1] This was a conjecture of Kontsevich.

This DG version of Thomas Theorem has an application to the Kontsevich Soibelman theorem on the action of the Cylinder operad on the pair (Hochschild cochains(A), Hochschild chains(A)) as observed by Geoffroy Horel [5], we discuss this application in the concluding section of our thesis.

We begin with the detailed formulation of the problem. The next section is devoted to its proof. Lastly, we discuss the application to Kontsevich Soibelman problem.

2 Statement of the problem

2.1 Topological Swiss Cheese operad.

In this section we are going to define a two colored operad SC_d. The set of colors is $\{f, h\}$ where f stands for full and h for half. The operations are of the following types

- There are n arguments of type f and the result is of type f. The corresponding operadic space is denoted by $SC_d(n, 0)^f$, by definition $SC_d(n, 0)^f = E_d(n)$.
- There are n arguments of type f and s arguments of type h, and the result is of type h. The corresponding operadic space is denoted by $SC_d(n, s)^h$. There are two types of composition, namely:

$$SC_d(n, s)^h \otimes SC_d(n_1, 0)^f \otimes \cdots \otimes SC_d(n_r, 0)^f \to SC_d(\sum n_i, s)^h,$$

which means that for the disks contained in the upper half disk we can use the E_d composition.

$$SC_d(n, s)^h \otimes SC_d(n_1, t_1)^h \otimes \cdots \otimes SC_d(n_t, t_s)^h \to SC_d(n + \sum n_j, \sum t_j)^h,$$

here we rescale the s target half disks and compose the corresponding embeddings of half disks into a half disk.

2.2 DG Swiss Cheese operad

We fix a field k of characteriztic zero. We denote by $c_*(X)$ the normalized singular chains of a topological space X with coefficients in k. We have the Eilenberg Zilber lax symmetric monoidal structure on the functor c_*. Therefore, given a topological operad O, we have an induced DG operad structure on $c_{-*}O$. In particular we get a DG operad $c_{-*}SC_d$.

2.2.1 Homology operad and its algebras.

As an illustration, consider the example of $H_*(E_2)$ the *homology* Swiss Cheese operad in dimension two. It is generated by $\mu \in H_0(E_2(2))$ and $\{,\} \in H_1(E_2(2))$. We can give a visual proof that the bracket satisfies

$$\{ab, c\} = \pm a\{b, c\} \pm \{a, c\}b,$$

this is called Leibnitz relation, where $ab := \mu(a, b)$. Gerstenhaber proved that for an associative algebra A, Hochschild cohomology $HH^*(A)$ has a commutative product (the cup product), and a Lie braket which satisfy Leibnitz relation. In other words, $HH^*(A)$ is an algebra over $H_*(E_2)$.

What about algebras over the homology of the Swiss Cheese operad? They are pairs (B, A) where $H_*(E_2)$ acts on B, $H_*(E_1)$ acts on A and there is a central morphism h :

$B \to A$. An example of an algebra over $H_*(SC_2)$ is the pair $(HH^*(A), A)$ where A is an associative algebra, the central morphism $h(f)$ is multiplication by $Z(A)$ if $f \in HH^0(A)$ or zero otherwise.

Definition 2.2.1. *A d-Gerstenhaber algebra is an algebra over the Gerstenhaber operad Ger that is generated by $\mu \in Ger(2)^0$ and $\{,\} \in Ger(2)^{d-1}$, where μ is an commutative product and $\{,\}$ is a d-Poisson bracket.*

A H_*SC_d algebra is given by a d-Ger algebra $(B, \mu_d, \{\}_d)$ and a $d-1$-Ger algebra $(A, \mu_{d-1}, \{\}_{d-1})$ together with a central morphisms $h : B \to A$.

2.2.2 Formality.

Recall that if a DG operad O is weakly equivalent to $H_*(O)$ then O is called formal. It turns out that $c_{-*}SC_d$ is not formal whereas $c_{-*}E_d$ is formal [6]. Let us illustrate the non formality of $c_{-*}SC_d$.

Definition 2.2.2. *Consider the following chains*

- $a \in c_{-*}SC_2(0,2)^h[0]$,
- $f \in c_{-*}SC_2(1,0)^h[0]$,
- $n_1 \in c_{-*}SC_2(0,2)^h \otimes c_{-*}SC_2(1,0)^h[1] \subset c_{-*}SC_2(1,1)^h[1]$,
- $l \in c_{-*}SC_2(2,0)^f[1]$,
- $n_2 \in c_{-*}SC_2(2,0)^h[2]$.

a, f, l are cycles and we have

$$\partial n_1 = a \circ_1 f(21) - a \circ_2 f,$$

that is, if we have $a \circ_1 f$ to be the chain associated to having a disk in the left quadrant while the half disk on the right quadrant, and $a \circ_1 f(21)$ is the chain with the disk on the right quadrant and the half disk on the left quadrant.

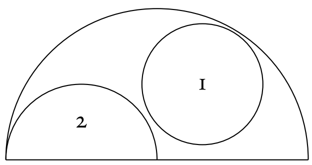


Figure 2: $a \circ_1 f(21) \in c_{-*}\mathtt{SC}_2(1,1)^h$

n_1 is the path that moves the half disk from right to left and the disk from left to right. l is given by a loop of the two disk in the clockwise direction. There is a homotopy $n_2 \in c_{-*}SC_2(2,0)^h[2]$ so that

$$\partial n_2 \quad = \quad n_1 \circ_{(1,0)} f + n_1 \circ_{(1,0)} f(21) - f \circ_{(0,1)} l, \tag{1}$$

this is called the “Eye relation”, those maps come from a construction on the topological Swiss Cheese operad.

Remark 2.2.3. *In [6] it is proven that the Kontsevich version of the $c_{-*}SC_d$ operad is not formal.*

The Eye relation induces explicit Massey products on $c_{-*}(SC_d)$ that does not vanish and so, the algebra structure of $c_{-*}(SC_d)$ and $H_*(SC_d)$ are not quasi isomorphic, even if the underlying complexes are quasi isomorphic.

2.3 Higher Hochschild complexes.

2.3.1 A bimodule of an algebra over an operad.

Let $\mathcal{O}$ be an operad in a certain symmetric monoidal category and A an $\mathcal{O}$-algebra. An $\mathcal{O}$-A-bimodule structure on an object B is a collection of maps

$$\mathcal{O}(n+1) \otimes B \otimes A^{\otimes n} \to B$$

which is invariant under the S_n-action on the left hand side and the following diagrams commute:

$$\begin{tikzcd}
\mathcal{O}(n+2) \otimes \mathcal{O}(m) \otimes B \otimes A^{\otimes m} \otimes A^{\otimes n} \arrow[r] \arrow[d] & \mathcal{O}(n+2) \otimes B \otimes A \otimes A^{\otimes n} \arrow[d] \\
\mathcal{O}(n+m+1) \otimes B \otimes A^{\otimes m+n} \arrow[r] & B
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{O}(n+1) \otimes \mathcal{O}(m+1) \otimes B \otimes A^{\otimes n+m} \arrow[r] \arrow[d] & \mathcal{O}(n+1) \otimes B \otimes A^{\otimes n} \arrow[d] \\
\mathcal{O}(n+m+1) \otimes B \otimes A^{\otimes n+m} \arrow[r] & B
\end{tikzcd}$$

By a $(d-1)$-algebra we mean a complex A with a map of operads $c_{-*}(E_{d-1}) \to End_A$.

Definition 2.3.1. *The category $c_{-*}(E_{d-1}) - Mod - A$ has as objects $c_{-*}(E_{d-1}) - A$ bimodules, morphisms are required to commute with the bimodule structure.*

A is an example of an $c_{-*}(E_{d-1}) - A$ bimodule. Due to the fact that A is a $c_{-*}(E_{d-1})$-algebra, we already have maps

$$c_{-*}(E_{d-1})(n) \otimes A \otimes A^{n-1} \to A.$$

2.3.2 Definition of Higher Hochschild complex.

The category $c_{-*}(E_{d-1}) - Mod - A$ has a projective closed model structure. Let $RA \to A$ be a cofibrant replacement of A. The complex

$$\hom(RA, A) \tag{2}$$

is called the higher Hochschild complex of A, denoted by $\mathrm{Hoch}(A)$. In other words, the Hochschild object is the derived endomorphism space of a $c_{-*}(E_{d-1})$-A bimodule A. Let V a vector space, then End_V is an associative algebra. We wonder if given a $(d-1)$-algebra A we can reformulate the definition of the Hochschild complex of A so that it carries a natural d-algebra structure.

For A associative we obtain $\mathrm{Hoch}(a) \simeq C^*(A)$, the Hochschild cochain complex of A.

Deligne conjecture says:

Theorem 2.3.2. *$C^*(A)$ is a homotopy $Wc_{-*}E_d$ algebra.*

See 3.6.6.

2.4 Swiss Cheese version of Deligne conjecture.

Kontsevich introduced the notion of an action of an $d+1$ algebra on a d algebra and he conjectured that for any d algebra A, on the homotopy category of pairs (B, ρ) given by B a $d+1$ algebra and an action of B on A, there is a final object. The DG case for SC_2 was proven in

Theorem. *[2] Let A be an associative algebra. The natural operad on the pair $(Hoch(A), A)$ is quasi isomorphic to $c_{-*}SC_2$, the induced action on homology recovers the structure previously defined.*

Informally, the topological higher analog of the previous theorem is:

Theorem. *[10] Let A be an algebra over the topological E_{d-1} operad. There exists an object Hoch(A), and a model $SC^\infty \rtimes E$, $E \simeq E_d$ of SC_d such that $Hoch(A)$ is the universal E algebra actiong on A through $SC^\infty \rtimes E_d$.*

In this thesis I prove a dg version of this theorem:

Theorem. *Let A be an algebra over $c_{-*}E_{d-1}$. There is an object Hoch(A) and a model $FSC \rtimes Wc_{-*}E_d$ of $c_{-*}SC_d$, such that $Hoch(A)$ is the universal $c_{-*}E_d$ algebra actiong on A through $FSC \rtimes Wc_{-*}E_d$.*

3 Proving DG Swiss Cheese conjecture.

We follow Justin Thomas's approach. Let's begin by reviewing the main ideas. Given a $(d-1)$-algebra A, its Hochschild complex is defined as follows. Let's consider the homotopy category of SC_d-algebras (B, A) which extend the existing $(d-1)$-algebra structure on A. The goal is to show that this category has a final object (H, A) and to show that H is homotopy equivalent to the Hochschild complex as defined in (2). This implies the Swiss Cheese conjecture.

The key idea is to consider the following 'partial' operad structures (we still assume two colors: full and half disk):

1) a half disk output operad: it only has those operadic spaces whose output color is 'half disk' and all the composition maps of these spaces. We denote the category of such partial operads by O_1.

2) at most one full disk input, half disk output operad: it has at most one full disk as the argument and the output of the type half disk, denote the resulting category by O_2.

Let also S be the category of all operads of SC type. We have forgetful functors

$$S \xrightarrow{F_1} O_1 \xrightarrow{F_2} O_2.$$

Both F_1 and F_2 have left adjoints, to be denoted by G_1, G_2. Denote by W the cofibrant resolution functor. We then have a natural map in O_1:

$$G_2WF_2F_1SC_d \quad \to \quad F_1SC_d. \tag{3}$$

The key observation by Thomas is that this map is a weak equivalence. In 3.5.3 we prove a DG version of this fact. Quoting [10] "The theorem amounts to the fact that the Swiss Cheese operad is generated up to homotopy by its degress 0 and 1 pieces ".

The problem now reduces to showing the existence of the universal $F_2F_1SC_d$-algebra (H, A) compatible with the $(d-1)$-algebra structure on A, this is done in 3.6.

The methods used in section 3.4 are mostly categorical, meaning that we rely on universal constructions rather than set theory, with the hope that the methods can be extended to operads on other closed symmetric monoidal categories.

In the first section of this Chapter we review the W construction for reduced operads, the main reference is [1]. The next section is devoted to the definition of the category of forests and the colored version of the category of forests. We give a modified definition that implies the associativity property of the grafting operation. In Section 3.3 we define several partial operad structures. Section 3.4 introduces the notion of Woods. In Section 3.5 we show that 3 is a cofibrant replacement. In Section 3.6 we show that a model for the Higher Hochschild complex satisfy the Swiss Cheese version of Deligne conjecture.

3.1 W Construction.

3.1.1 Category of Forest.

An n-corolla is a planar graph with one vertex, n edges called leaves, and one fixed edge called root. We denote n-corolla by $(n), n \geq 0$.

Definition 3.1.1. *The category* $(\mathcal{FOREST}, \cup)$ *is defined as:*

Objects: disjoint union of n*-corollas. We define the set* $Hom_{\mathcal{FOREST}}(Y, (n))$ *as instructions to use the pieces of* Y *to graft a tree with* n *leaves. We extend this operation to disjoint union of corollas.*

The grafting operation is given by identifying a root in one corolla with a leave on another corolla. We introduce this category to keep tract of the way we compose operations. There is an action of S_n on (n) and the grafting operation is compatible with this action. For a detailed definition see 3.2.1.

Remark 3.1.2. *Given* $W, X, Y, Z \in \mathcal{FOREST}, e : W \to X, f : X \to Y, g : Y \to Z$, *we consider the composition:* $f \to gf$ *as taking the trees that compose* f *and assembling them to form the trees* gf. *As for the precomposition* $f \to fe$, *we let the tree* f *to grow, in the process new vertex may be formed, some may be unary.*

3.1.2 Functor W.

Given $T \in \mathcal{FOREST}$, the comma category $\mathcal{FOREST}/T$ is the category whose objects are pairs $\{S, \phi | S \in \mathcal{FOREST}, \phi : S \to T\}$ and morphisms $(U, \psi) \mapsto (S, \phi)$ are commutative triangles

$$\begin{tikzcd} U \arrow[r] \arrow[dr, "\psi"'] & S \arrow[d, "\phi"] \\ & T \end{tikzcd}$$

For example, in the case $T = (n)$, elements of $\mathcal{FOREST}/(n)$ can be seen as all possible trees with n leaves obtained by grafting.

The nerve construction associates a simplicial set to a category, in the case of the comma categories, the nerve construction can be described as $Nerve(n) = \{f_n \circ \cdots \circ f_1 | f_i \in$

$\mathcal{FOREST}/T\}$, each set $Nerve(n)$ has boundary (given partial composition) and degeneration maps (given by inserting identities). $Nerve(0) = \{$ objects of $\mathcal{FOREST}/T\}$ which are corollas than can be assembled to T, $Nerve(1)$ is the set of oriented paths $s \to t$ between $s, t \in Nerve(0)$, basically means that among the moves to assemble T out of s, there is a subset of those graftings that assembles t.

From the simplicial abelian group $k[Nerve(*)]$ and then we apply Dold Khan to we obtain a chain complex. Finally we normalize this complex. We denote this composition by $(N(), \sum(-1)^i \partial_i)$.

The compositions on the category of Forest now give us the following data. For P, Q such that $P \circ_i Q = R \in \mathcal{FOREST}/T$ we have a grafting arrow

$$\circ_i : P \cup Q \to R,$$

this arrow give us two 0- simplex $N(P) \otimes N(Q), N(R)$ and one 1-simplex $N(\circ_i) =: N(P) \otimes N(Q)[1]$ satisfying

$$\partial_0(N(\circ_i)) = N(R),$$
$$\partial_1(N(\circ_i)) = N(P) \otimes N(Q).$$

This is a contravariant functor.

If P happens to be decomposable as $P_1 \circ_j P_2$ then the map

$$\begin{tikzcd} P_1 \circ_j P_2 \circ_i Q \arrow[r] \arrow[dr] & P \circ_i Q \arrow[d] \\ & R \end{tikzcd}$$

will give us the term $N(\circ_{ji}) := N(P_1) \otimes N(P_2) \otimes N(Q)[2]$. Remembering that for a n-simplex the i-boundary is obtained by omitting the i-vertex we deduce that:

$$\begin{aligned} \partial_0 N(\circ_{ji}) &= N(P) \otimes N(Q)[1], \\ \partial_0 \partial_0 N(\circ_{ji}) &= N(R), \\ \partial_1 \partial_0 N(\circ_{ji}) &= N(P) \otimes N(Q), \end{aligned}$$

second boundary

$$\begin{aligned} \partial_1 N(\circ_{ji}) &= N(P_1) \otimes N(P_2) \otimes N(Q)[1], \\ \partial_0 \partial_1 N(\circ_{ji}) &= N(R), \\ \partial_1 \partial_1 N(\circ_{ji}) &= N(P_1) \otimes N(P_2) \otimes N(Q), \end{aligned}$$

and third boundary

$$\begin{aligned} \partial_2 N(\circ_{ji}) &= N(P_1) \otimes N(P_2) \otimes N(Q)[1]. \\ \partial_0 \partial_2 N(\circ_{ji}) &= N(P) \otimes N(Q), \\ \partial_1 \partial_2 N(\circ_{ji}) &= N(P_1) \otimes N(P_2) \otimes N(Q). \end{aligned}$$

Another way to think of the normalized chain functor is by considering

$$N(\Delta[1]) = Z\{\gamma_0, \gamma_1, \gamma | \partial\gamma = \gamma_0 - \gamma_1\},$$

γ_0, γ_1 are zero simplex and γ is a one simplex. Then for any $(U, \phi) \in \mathcal{FOREST}^{op}/T$ we associate the chain complex given by

$$(N(\Delta[1]))^{\otimes\{\text{internal edges of } U\}}.$$

We introduced homotopies between the composition $N(T_1 \circ T_2)$ were we label the new edge γ_0, and a formal composition $N(T_1) \circ N(T_2) = N(T_1) \otimes N(T_2)$ labeled with γ_1. When we labeled the new edge with γ we considered it as a homotopy between the formal composition and the old one.

So for any possible grafting of trees $\cup_i S_i$ that assembles to T we have an object $\otimes_i N(S_i)[\#\{i\}]$ and a path/sequence of homotopies between $\otimes_i N(S_i)$ and $N(T)$. If we choose a forest S with k-vertices, then there are paths of length $k-1$ from S to C_n for some n, namely by grafting internal edges one by one.

3.1.3 Operads.

Rather than thinking on operads as parametrized by natural numbers union $\{0\}$, we want to parametrize them by trees, to remember the way composition is done.

Definition 3.1.3. *An operad O in the category of chain complexes is a symmetric monoidal functor from $\mathcal{FOREST}$ to C, where C is a symmetric monoidal category enriched on $C_{-*}(k)$.*

We denote by $O(n, s) = \oplus_{T=\cup n_i} \otimes_i O(n_i)$ where the sum is over forest T with n leaves and s roots.

Definition 3.1.4. *Given an operad O there exist a cofibrant resolution WO defined by the coend $WO(T) = O \otimes_{\mathcal{FOREST}/T} N$.*

An intuitive way to think of the W construction is to draw Trees with vertices labeled by elements of our operad, and edges labeled with the normalized interval $N(\Delta[1])$, subject to the following relations:

Remark 3.1.5. *If an edge is labeled with γ_0 we contract the edge and apply the operadic composition of the two terms that label the vertices of that edge, if the edge is labeled with γ_1 it stands for the formal composition of the two trees obtained by cutting E. If an edge is labeled by γ we shift by $[1]$ the chain associated to the tree.*

Given $W, X, Y, Z \in \mathcal{FOREST}, e : W \to X, f : X \to Y, g : Y \to Z$, we have an induced map $Wf \to W(gf)$ and according to 3.1.2 this corresponds to the the operadic composition, in this case we label the new edge with γ_1. For $W(f) \to W(fe)$ we label the new edges with γ_0. If we insert unary vertex then think of this as precomposition with identity, in which case the new vertex creates a new edge below the original vertex, or we postcomposition with identity in which case the new edge is above the orignal vertex. The following diagram commutes:

$$\begin{tikzcd} W(f) \arrow[r] \arrow[d] & W(fe) \arrow[d] \\ W(gf) \arrow[r] & W(fge) \end{tikzcd}$$

3.1.4 Equivalent definitions.

We can organize all possible trees with edges labeled with elements of $N(\Delta[1])$ as formal compositions of trees with edges of length γ. We use [1] convention $A[i]_j = A_{j-i}$ so we are shifting to the right. For an arbitrary tree T, we label its vertex v_i with elements of $O(|v_i|)[1]$, here we shift because every vertex (but the root) is the target vertex of an internal edge, then we obtain an element of $\mathcal{F}(O(T)[1])$ which is located at the wrong degree. We consider this for all possible T with grafting those trees using the formal composition operation, then the corrected object to consider is $WO = F(F(O[1])[-1])$.

Given any tree $S \in \mathcal{FOREST}/T$, we can describe the trees that belong to the path from S to T : For any corolla of T, there is a subtree of S on its preimage, then any inter middle tree between S and T is giving by collapsing subtrees that are preimages of corollas.

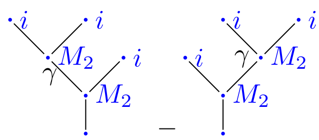
Another way to count is to fix a tree T and consider all possible forest obtained after we label the edges of T with λ, by varying T we get

$$WO \quad = \quad \bigoplus_{T} \bigoplus_{E_T \subset T} FO(T)[|E_T|]. \tag{4}$$

Definition 3.1.6. *A complex $WO(T)$ is connected if no edge of T is labeled with γ_1.*

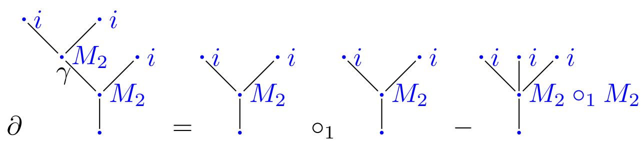
3.1.5 Example: W Associative

We denote by μ_n the generator of $Assoc(n)$ as free S_n-module. For n fixed, $WAssoc$ has a basis M_n, with $M_2 = \mu_2$. For $n = 3$ we consider the element M_3 :



For any i, j the operation $\mu_i\{\mu_j\}$ is given by the sum of all possible ways of grafting μ_j on μ_i, we denote by $\{\mu_T, \mu_S\} := \mu_T\{\mu_S\} - (-1)^{|\mu_T||\mu_S|}\mu_S\{\mu_T\}$, S, T forest.

From



we conclude that $\partial M_3 = \frac{\{\mu_2,\mu_2\}}{2}$.

We denoted by $\mathcal{FOREST}^2$ the subcategory of trees with only binary vertex. Now we can describe the basis elements as

$$M_n = \sum_{T \in \mathcal{FOREST}^2(n)} sign(T)\mu_T[n-2] \in WAssoc(n).$$

Where the sum is over all binary trees with n-leaves, with label γ on every internal edge. The function $sign(T)$ comes from giving an orientation to the associahedron K_n, whose 1-strata is precisely elements of $\mathcal{FOREST}^2$. It can be shown that they satisfy the relations $dM_n = \sum_{i=2}^{n-1} \frac{\{M_{n+1-i}, M_i\}}{2}$. This is the usual resolution of associative algebras.

Let V a vector space and (A, μ) an associative algebra. We consider that A has the trivial $WAssoc$ structure where $\mu_n = 0, n > 2.$. suppose V is a homotopy retract of A :

$$\begin{tikzcd} A \arrow[loop left, "h"] \arrow[r, bend left, "p"] & V \arrow[l, bend left, "i"] \end{tikzcd} ,$$

where $dh + hd = ip - id$, and i is a quasi isomorphism. The typical example is when A is a dg associative algebra, and $V = H_*(A)$.

Theorem 3.1.7 (Martin Markl [9].). *There are explicit formulas for the following objects:*

- *a $WAssoc$ structure on V,*
- *a $WAssoc$ map $\phi : A \to V$, (as $WAss$ algebras).*
- *a $WAssoc$ map $\psi : V \to A$, (a quasi isomorphism of $WAss$ algebras).*
- *a $WAssoc$ homotopy H between $\psi\phi$ and Id.*

such that $\phi_1 = p, \psi_1 = i$ and $H_1 = h$.

The action of $WAssoc$ on V is given by labeling every leaf with the map i, every root with the map p, and interpreting: every edge labeled with γ as the map h, when an edge is labeled with γ_1 we split the edge and relabel with the composition pi. When a vertex is labeled with γ_0 we use the Associative structure on A.

Remark 3.1.8. *Assume V is an homotopy retract and a subcomplex of A, it defines the automorphism $\alpha : i \circ p$. For an arbitrary α, $dh + hd = id - \alpha$, there is a $WAssoc$ structure on A and the previous theorem implies that h extends to a homotopy among the two $WAssoc$ structures.*

3.2 Colored operads.

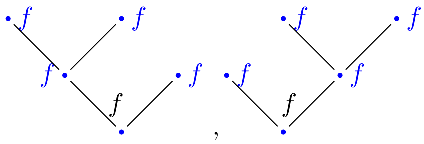
Lets consider the set $K = \{f, h\}$ where f stands for full and h for half. Here we use the notation $(n, m)^r$ to denote a corolla with root of color $r \in K$, n leaves of color f and m leaves of color h[2]. We call grafting the operation that identifies a root of a corolla with a leave of another corolla, obtaining a tree with at least one internal vertex.

Definition 3.2.1 ([10] 6.1.1). *The category of K-$\mathcal{FOREST}$ is defined as:*

Objects: disjoint union of corollas $(n, m)^r$. There is an object denoted by $(,)$ called the empty tree. We don't allow stumps $(0, 0)^f, (0, 0)^h$.

For a corolla $(n, m)^r$, we define the set $Hom_{K\text{-}\mathcal{FOREST}}(Y, (n, m)^r)$, whose elements are called Trees, by using all pieces of Y to graft a tree with n leaves of color h, m of color h and whose root has color r. Here the grafting operation plugs a root on a leave of the same color. If there is no coherent way to make the grafting using every corolla of Y then $Hom_{K\text{-}\mathcal{FOREST}}(Y, (n, m)^r) = \emptyset$.

For example



$\in Hom_{K\text{-}\mathcal{FOREST}}((2, 0)^f \cup (2, 0)^f, (3, 0)^f)$.

[2] We follow the notation of Computer Science, to draw trees $(a, b)^r$ with roots at the top and leaves at the bottom, while our diagrams follow Nature.

The category K-$\mathcal{FOREST}$ has a symmetric monoidal structure by considering disjoint union of trees with $(,)$ as the monoidal unit. There is an action of $S_{n,m}$ on $(n,m)^r$ and the grafting operation is compatible with this action.

This definition is equivalent to the definition on [10]. We write it this way because it is clear how to compose morphisms, and that composition is associative. We extend morphisms from corollas to all objects in the following way. Homomorphisms from Y to Z consist of all possible ways to use the corollas of Y to graft as many trees as corollas in Z. Formally, let

$$\cup_0^{r(z)}(n_i,m_i)^{r_i} = Z,$$

we consider P_Z to be all partitions $\cup_0^{r(z)} T_j = Y$ into $r(z)$ subsets where

$$Hom_{K\text{-}\mathcal{FOREST}}(T_i,(n_i,m_i)^{r_i})$$

is non empty. Then

$$Hom_{K\text{-}\mathcal{FOREST}}(Y,Z) = \bigcup_{\cup_0^{r(z)} T_j \in P_Z} \{Hom_{K\text{-}\mathcal{FOREST}}(T_i,(n_i,m_i)^{r_i})\}_i.$$

This can be viewed as an extension from operads to props.

The only arrows with source the empty tree are disjoint union of :

$$\begin{array}{l} \bullet\, f \\ | \\ \bullet\, f \end{array} \in Hom_{K\text{-}\mathcal{FOREST}}((,),(1,0)^f),$$

$$\begin{array}{l} \bullet\, h \\ | \\ \bullet\, h \end{array} \in Hom_{K\text{-}\mathcal{FOREST}}((,),(0,1)^h).$$

Definition 3.2.2. *For a symmetric monoidal category $(C,\otimes)$ we define a K-colored operad in C as a symetric monoidal functor $O : (K\text{-}Forest, \cup) \to (C,\otimes)$. A morphism of operads is a natural transformation between functors.*

Remark 3.2.3. *If you restrict to a particular color, for example f, then we have maps*

$$(n,0)^f \cup (a_1,0)^f \cup \cdots \cup (a_n,0)^f \to (\sum a_1, 0)^f,$$

by grafting the leaves of the corolla $(n,0)^f$ with the other corollas.This way our definition coincides with the usual definition of operad.

Notation 3.2.4. *We denote by $O(n,m;r,s) = \oplus_{T=\cup T_i} \otimes O(T_i)$ where the sum is over all Forests T with n leaves of color f, m leaves of color h, r roots of color f and s roots of color h.*

If $(C,\otimes)$ is a SM category, for $A,B \in C$, we define the Endomorphism operad/functor

$$End_{(B,A)}(n,m)^h = Hom_C(A^{\otimes n} \otimes B^{\otimes m}, A),$$

$$End_{(B,A)}(n,m)^f = Hom_C(A^{\otimes n} \otimes B^{\otimes m}, B).$$

The following definition is motivated by the nature of the SC_d operad.

Definition 3.2.5. *Let SC-$Forest$ be the subcategory of K-$\mathcal{FOREST}$ consisting of all objects $(n,0)^f$, $(n,m)^h$, and their tensor products, where we forbid a color h root to be grafted on a color f leaf.*

When we see the operad of topological spaces SC_d and the operad of chains $c_{-*}SC_d$ as Symmetric Monoidal functors, both have $SC\text{-}Forest$ as the domain category:

$$SC_d : SC\text{-}Forest \to (Top, \times),$$

Here we enrich Hom_{Top} by giving it the compact open topology, and

$$c_*SC_d : SC\text{-}Forest \to (C(R), \otimes),$$

is enriched on chain complexes by considering $innerHom_{C(R)}$. R is a fixed ground ring of charo.

Definition 3.2.6. *Let C be enriched on chain complexes. Given $A, B \in C$, a $c_{-*}SC_d$-algebra on (B, A) is given by a natural transformation $c_{-*}SC_d \to SC\text{-}End_{(B,A)}$,*

Here $SC\text{-}End$ is End restricted to $SC - Forest$. Basically we associate f to B and h to A, and we have maps of chain complexes

$$c_{-*}(SC)(n, 0)^f \to End_{(B,A)}(n, 0)^f,$$

$$c_{-*}(SC)(n, m)^h \to End_{(B,A)}(n, m)^h.$$

In particular A is a $c_{-*}E_{d-1}$-algebra via $c_{-*}(SC)(0, m)^h \to End_{(B,A)}(0, m)^h$, similarly B is a $c_{-*}E_d$-algebra.

3.3 The category of one full argument operations.

We want to study the suboperad of $c_{-*}SC_d$ generated by one full argument operations, this should be a partially defined operads generated by operations $(1, m)^h$, $(0, n)^h$.

Definition 3.3.1. *The base subcategory. Objects $(1, m)^h \cup \cup_i (0, n_i)^h$ of the base subcategory have roots of color h and at most one disk as leaf. The morphisms have the form*

$$(1, m_1)^h \cup (0, m_{s+1})^h \cup \cdots \cup (0, m_r)^h \to (1, m)^h,$$

or

$$(0, m_{s+1})^h \cup \cdots \cup (0, m_r)^h \to (0, m)^h.$$

This is the smallest subcategory $D \subset SC_d\text{-}\mathcal{FOREST}$ with

$$D/(1, m)^h = SC_d\text{-}\mathcal{FOREST}^h/(1, m)^h$$

and

$$D/(0, n)^h = SC_d\text{-}\mathcal{FOREST}^h/(0, n)^h$$

for all n, m. We denote the base subcategory by $SC_d\text{-}\mathcal{FOREST}^{h\prime} \subset SC_d\text{-}\mathcal{FOREST}$.

Remark 3.3.2. *A functor from $SC_d\text{-}\mathcal{FOREST}^{h\prime} \to C$ does not define an operad since the base subcategory is not symmetric monoidal.*

Definition 3.3.3. *Given a symmetric monoidal subcategory $U \subset SC_d\text{-}\mathcal{FOREST}$ and an U-operad $O : U \to C$, we define the W construction restricted to U as: $W_U O(T) := W \otimes_{U/T} O$.*

Explicitly [8]

$$\begin{aligned} W_U O(T) &= \int^{R\in U/T} W(R)\otimes O(R) \\ &= coeq\left\{ \bigoplus_{f:\alpha\to\beta | f\in Hom_{U/T}} W(\beta)\otimes O(\alpha) \rightrightarrows \right. \\ &\left. \rightrightarrows \bigoplus_{\gamma\in U/T} W(\gamma)\otimes O(\gamma) \right\} \end{aligned}$$

Where for $(s,r)\in W(\beta)\otimes O(\alpha)$,

$$f_*(s,r) = (s, O(f)(r)) \in W(\beta)\otimes O(\beta),$$

$O(f)(r)$ is given by using the operadic composition, and

$$f^*(s,r) = (W(f)s, r) \in W(\alpha)\otimes O(\alpha),$$

$W(f)s$ labels with γ_0 new internal edges.

Definition 3.3.4. *Let's consider SC_d-$\mathcal{FOREST}^{h0\prime}$ the smallest symmetric monoidal subcategory of SC_d-$\mathcal{FOREST}$ containing the base category, its objects are unions of $(1,m)$, $(0,n)$, and the morphisms have the form*

$$\cup_i (1,m_i)^h \cup \cup_j (0,n_j)^h \to \cup_k (1,r_k)^h,$$

or

$$(0,m_{s+1})^h \cup\cdots\cup (0,m_r)^h \to (0,m)^h.$$

Remark 3.3.5. *The operad $W \otimes_{SC_d\text{-}\mathcal{FOREST}^{h0\prime}} SC_d$ is a partially defined operad with domain SC_d-$\mathcal{FOREST}^{h0\prime}$. It is defined on a symmetric monoidal category, but it lacks what we consider operad structure since it contains only permutations of one full argument operations.*

Definition 3.3.6. *We consider*

- *the symmetric monoidal subcategory of SC_d-$\mathcal{FOREST}$ which contains trees that have all roots and vertices of color h. Objects are unions of $(n,m)^h$, and morphisms are of the form*

$$(n_1,m_1)^h \cup\cdots\cup (n_s,m_s)^h \to (n,m)^h,$$

  This category is denoted by SC_d-$\mathcal{FOREST}^h$.

- *the symmetric monoidal subcategory of one full argument operations. It has the same objects as SC_d-$\mathcal{FOREST}^h$. We keep only the morphisms of the form*

$$(1,m_1)^h \cup\cdots\cup (1,m_s)^h \cup (0,m_{s+1})^h \cup\cdots\cup (0,m_r)^h \to (s,m)^h,$$

  In other words, vertex can have at most one disk as input. We denote the subcategory of one full argument by SC_d-$\mathcal{FOREST}^{h1\prime} \subset SC_d$-$\mathcal{FOREST}^h$.

We will use the subcategory of one full argument to parametrize operations that are composition of one full argument operations.

Definition 3.3.7. *We denote by WSC' the left Kan extension of $W \otimes_{SC_d\text{-}\mathcal{FOREST}^{h0\prime}} SC_d$ to SC_d-$\mathcal{FOREST}^{h1\prime}$ and by FSC we denote the left Kan extension of WSC' to SC_d-$\mathcal{FOREST}^h$.*

$$\begin{tikzcd}
SC_d\text{-}\mathcal{FOREST}^h \arrow[rr, "FSC"] & & C \\
SC_d\text{-}\mathcal{FOREST}^{h1\prime} \arrow[u] \arrow[urr, "WSC'"] & & \\
SC_d\text{-}\mathcal{FOREST}^{h0\prime} \arrow[u] \arrow[uurr, "W\otimes_{SC_d\text{-}\mathcal{FOREST}^{h0\prime}}SC_d"'] & &
\end{tikzcd} \tag{5}$$

FSC_d contains the suboperad WE_{d-1} and one full argument operations. Explicitly:

$$\begin{aligned}
FSC_d(T) &= \int^{R\in SC_d\text{-}\mathcal{FOREST}^{h\prime}/T} Hom_{SC_d\text{-}\mathcal{FOREST}^h}(R,T)\otimes W_{SC_d\text{-}\mathcal{FOREST}^{h\prime}}c_{-*}SC_d(R) \\
&= \int^{R\in SC_d\text{-}\mathcal{FOREST}^{h\prime}/T} Hom_{SC_d\text{-}\mathcal{FOREST}^h}(R,T)\otimes \int^{S\in SC_d\text{-}\mathcal{FOREST}^{h\prime}/R} W(S)\otimes c_{-*}SC_d(S)
\end{aligned}$$

$$= coeq\left\{ \bigoplus_{f\in Hom_{SC_d\text{-}\mathcal{FOREST}^{h\prime}/(T)}(\alpha,\beta)} Hom_{SC_d\text{-}\mathcal{FOREST}^h}(\beta,T)\otimes Wc_{-*}SC_{d'}(\alpha) \rightrightarrows \right.$$

$$\left. \rightrightarrows \bigoplus_{\gamma\in SC_d\text{-}\mathcal{FOREST}^{h\prime}/(T)} Hom_{SC_d\text{-}\mathcal{FOREST}^h}(\gamma,T)\otimes Wc_{-*}SC_{d'}(\gamma) \right\}$$

Where for $(s',r')\in Hom_{SC_d\text{-}\mathcal{FOREST}^h}(\beta,T)\otimes Wc_{-*}SC_d(\alpha)$,

$$f_*(s',r') = (s', Wc_{-*}SC_d(f)r')\in Hom_{SC_d\text{-}\mathcal{FOREST}^h}(\beta,T)\otimes Wc_{-*}SC_d(\beta),$$

$Wc_{-*}SC_d(f)r'$ is given by using the formal operadic composition (labeling new edges with γ_1), and

$$f^*(s',r') = (f^*(s'),r')\in Hom_{SC_d\text{-}\mathcal{FOREST}^h}(\alpha,T)\otimes Wc_{-*}SC_d(\alpha),$$

since SC_d-$\mathcal{FOREST}^{h\prime}\subset SC_d$-$\mathcal{FOREST}^h$, the map

$$f^*: Hom_{SC_d\text{-}\mathcal{FOREST}^h}(\beta,T)\to Hom_{SC_d\text{-}\mathcal{FOREST}^h}(\alpha,T)$$

is give by precomposition with f.

Lemma 3.3.8. *The left Kan extension WSC' of $W\otimes_{SC_d\text{-}\mathcal{FOREST}^{h0\prime}}SC_d$ along i_0 coincides with $W\otimes_{SC_d\text{-}\mathcal{FOREST}^{h1\prime}}SC_d$.*

Proof. Notice that for the diagram

$$\begin{tikzcd}
SC_d\text{-}\mathcal{FOREST}^{h1\prime} \arrow[rr, "W\otimes_{SC_d\text{-}\mathcal{FOREST}^{h1\prime}}SC_d"'] & & C \\
SC_d\text{-}\mathcal{FOREST}^{h0\prime} \arrow[u, "i_0"] \arrow[urr, "W\otimes_{SC_d\text{-}\mathcal{FOREST}^{h0\prime}}SC_d"'] & &
\end{tikzcd} \tag{6}$$

there is a natural transformation

$$L : W \otimes_{SC_d\text{-}\mathcal{FOREST}^{h0\prime}} SC_d \to W \otimes_{SC_d\text{-}\mathcal{FOREST}^{h1\prime}} SC_d \circ i_0,$$

namely, the identity transformation. For any other functor

$$g : SC_d\text{-}\mathcal{FOREST}^{h1\prime} \to C,$$

and natural transformation

$$L_g : W c_{-*} SC'_d \to g \circ i_0$$

we compose with the identity to obtain

$$W \otimes_{SC_d\text{-}\mathcal{FOREST}^{h1\prime}} SC_d \circ i_0 \to g \circ i_0.$$

□

3.4 Woods.

We want to prove that WSC' is a cofibrant operad. We modify Justin Thomas approach, incidentally our approach also help us to interpolate between the categories WSC' and FSC.

Definition 3.4.1. *Woods are symmetric monoidal subcategories $U, i : U \subset (SC\text{-}Forest^h, \cup)$, with the following property:*

- *There is a family of trees $V \subset U$ so that for all $T \in V, U/T = SC_d\text{-}\mathcal{FOREST}^h/T$.*

According to the dictionary, woods are smaller than a forest and they are covered with growing trees.

Lemma 3.4.2. *Let U be woods and let V the defining family. Then for any $T \in V$*

$$W \otimes_U c_{-*} SC^h(T) = W \otimes_{SC_d\text{-}\mathcal{FOREST}^h} c_{-*} SC^h(T).$$

Proof. This is because the W construction is determined by the comma category. □

Definition 3.4.3. *Let's consider the woods*

$$SC_d\text{-}\mathcal{FOREST}^{h1\prime} \subset SC_d\text{-}\mathcal{FOREST}^{h2\prime} \subset \cdots \subset SC_d\text{-}\mathcal{FOREST}^h,$$

those categories have all the same objects. On $SC_d\text{-}\mathcal{FOREST}^{hk\prime}$ we keep only the morphisms of the form

$$(a_1, m_1)^h \cup \cdots \cup (a_r, m_r)^h \to (s, m)^h,$$

$0 \le a_1, \cdots, a_s \le k$. *For every k we denote the inclusion by*

$$i_k : SC_d\text{-}\mathcal{FOREST}^{hk\prime} \subset SC_d\text{-}\mathcal{FOREST}^{h(k+1)\prime}.$$

and we denote $W \otimes_{SC_d\text{-}\mathcal{FOREST}^{hi\prime}} SC_d$ by WSC^{hi}.

The family

$$\{(a, m)^h, a \le k\} \subset SC_d\text{-}\mathcal{FOREST}^{hk\prime}$$

is the defining Wood structure of $SC_d\text{-}\mathcal{FOREST}^{hk\prime}$.

Theorem 3.4.4. *The map $FSC \to W \otimes_{SC_d\text{-}\mathcal{FOREST}^h} c_{-*}SC^h$ is a cofibrant replacement.*

Proof. We prove in lemma 3.4.6 that the Left Kan extension of $WSC_d^{hk\prime}$ along i_k is a direct summand of $WSC_d^{h(k+1)\prime}$. The cokernel is given by operations with $k+1$ corollas and so it is fibrant, we conclude that i_{k*} is a cofibration for all k. If we consider

$$\begin{tikzcd}
SC_d\text{-}\mathcal{FOREST}^h \arrow[r] & C \\
\vdots \arrow[u] & \\
SC_d\text{-}\mathcal{FOREST}^{h3\prime} \arrow[u] \arrow[uur] & \\
SC_d\text{-}\mathcal{FOREST}^{h2\prime} \arrow[u, "i_2"] \arrow[uuur] & \\
SC_d\text{-}\mathcal{FOREST}^{h1\prime} \arrow[u, "i_1"] \arrow[uuuur] &
\end{tikzcd}$$

The map $i : SC_d\text{-}\mathcal{FOREST}^{h1\prime} \to SC_d\text{-}\mathcal{FOREST}^h$ is sequence of inclusions $\cdot \circ i_2 \circ i_1$, and so each one of the following maps is a cofibration

$$\begin{aligned}
FSC &= i_* W c_{-*} SC'_d \\
&= \cdots i_{3*} \circ i_{2*} \circ i_{1*} W c_{-*} SC'_d \\
&\to \cdots i_{3*} \circ i_{2*} WSC_d'^{2h} \\
&\to \cdots i_{3*} WSC_d'^{3h} \\
&\to \varinjlim_k i_{k*} WSC'^{kh} \\
&= W \otimes_{SC_d\text{-}\mathcal{FOREST}^h} SC_d.
\end{aligned}$$

□

Lemma 3.4.5.
$$\varinjlim_k i_{k*} WSC'^{kh} = W \otimes_{SC_d\text{-}\mathcal{FOREST}^h} SC_d.$$

Proof. We will show that $W \otimes_{SC_d\text{-}\mathcal{FOREST}^h} SC_d$ satisfies the universal property of the colimit.

Let T be a tree with n leaves. Then $WSC'^{kh}(T) = W \otimes_{SC_d\text{-}\mathcal{FOREST}^h} SC_d(T)$ for all $k \geq n$. In particular

$$i_{k*} WSC'^{kh}(T) = W \otimes_{SC_d\text{-}\mathcal{FOREST}^h} SC_d(T),$$

identity maps give us a sequence of compatible maps, and this defines a map $\varinjlim_k i_{k*} WSC'^{kh} \to W \otimes_{SC_d\text{-}\mathcal{FOREST}^h} SC_d$.

Let O be another operad, with a sequence of compatible maps

$$\begin{tikzcd}
WSC'^{kh} \arrow[r] \arrow[dr] & i_{k*} WSC'^{kh} \arrow[r] \arrow[d] & WSC'^{(k+1)h} \arrow[dl] \\
& O &
\end{tikzcd}$$

Let T be a tree with n leaves, then for all $k \geq n$ the previous diagram factors through identity morphisms

$$\begin{tikzcd}
WSC'^{kh}(T) \arrow[r] \arrow[dr] & i_{k*}WSC'^{kh}(T) \arrow[r] \arrow[d] & WSC'^{(k+1)h}(T) \arrow[dl] \\
& W \otimes_{SC_d\text{-}\mathcal{FOREST}^h} SC_d(T) \arrow[d] & \\
& O(T) &
\end{tikzcd}$$

□

Lemma 3.4.6. *The Left Kan extension of $SC_d^{hk\prime}$ along i_k is a direct summand of*

$$W \otimes_{SC_d\text{-}\mathcal{FOREST}^{h(k+1)\prime}} SC_d.$$

Proof. The second equality is due to the fact that the subcategories are woods.

$$\begin{aligned}
i_{k*}W \otimes_{SC_d\text{-}\mathcal{FOREST}^{hk\prime}} c_{-*}SC^h(n,m)^h &= \varinjlim_{SC_d\text{-}\mathcal{FOREST}^{hk\prime}/(m,n)^h} W \otimes_{SC_d\text{-}\mathcal{FOREST}^{hk\prime}} c_{-*}SC^h \\
&= \varinjlim_{SC_d\text{-}\mathcal{FOREST}^{hk\prime}/(m,n)^h} W \otimes_{SC_d\text{-}\mathcal{FOREST}^{h(k+1)\prime}} c_{-*}SC^h \\
&\to \varinjlim_{SC_d\text{-}\mathcal{FOREST}^{h(k+1)\prime}/(m,n)^h} W \otimes_{SC_d\text{-}\mathcal{FOREST}^{h(k+1)\prime}} c_{-*}SC^h \\
&= WSC_d^{h(k+1)\prime}(n,m)^h.
\end{aligned}$$

The map in the third line is given by allowing $k+1$ corollas. □

Definition 3.4.7. *For any corolla $(n,m)^h$, the length of a tree $\cup_{0\leq i\leq s}(a_i,b_i)^h \to (n,m)^h$ is equal to s. For a general tree $T = \cup_i(a_i,b_i)^h \to \cup(n_j,m_j)^h$ the length of T is the maximum among the length of the connected subtrees.*

Lemma 3.4.8. *WSC' is a cofibrant operad.*

Proof. This is proven in the same way as in [1] and [10]. In fact the Woods were inspired by those proofs. In our case $c_{-*}SC_d$ is a reduced operad, and we can simplify $Wc_{-*}SC_d = \bar{k} \oplus W^{Ps}c_{-*}SC_d$ where we consider $W^{Ps}c_{-*}SC_d$ as a Pseudo operad, and $\bar{k}$ as a complex concentrated on degree 0 and 1 with k in both entries and identity boundary. Working with the W construction of Pseudo operads have the advantage that we don't need to worry about making identifications for units.

The woods U_n with $V_n = \{(a,b), a+b \leq n, a \leq 1\}$ help us to focus only on operations with corollas of at most n inputs.

We further subdivide the woods U_n into subcategories $U_{n,i}$ where morphisms have length less equal to i. $W \otimes_{U_{n,i}} Sc_d$ is freely generated by forest with corollas of less than n leaves and length less than i. $U_{1,1}$ contains unions of

$$(), \to (0,1)^h, (), \to (1,0)^h.$$

and

$$(1,0)^h \to (1,0)^h, (0,1)^h \to (0,1)^h.$$

We then compare $i_* W \otimes_{U_{n,i}} Sc_d \to W \otimes_{U_{n,i+1}} Sc_d$ and see that we are freely adding new terms to kill cycles, so the map is a cofibration and we conclude that the colimit object is itself cofibrant, (see3.4.4).

□

Lemma 3.4.9. $W \otimes_{SC_d\text{-}\mathcal{FOREST}^h} SC_d \simeq W \otimes_{SC_d\text{-}\mathcal{FOREST}|_h} SC_d$.

Proof. We use a filtration by length of edges of color f, and notice that two consecutive filtered operads are homotopic.

□

Corollary 3.4.10. *The functor that assigns to $B \in C$ the set of FSC actions on the pair (B, A) is representable by H.*

Proof. According to 3.3.6 $End_{SC_d\text{-}\mathcal{FOREST}^{h'}}(A, B) = O(End_{SC_d\text{-}\mathcal{FOREST}^h}(A, B))$, where O is the forgetful functor from operads on SC_d-$\mathcal{FOREST}^h$ to operads on SC_d-$\mathcal{FOREST}^{h'}$. By adjunction we have

$$\begin{aligned} Hom_{SC_d\text{-}\mathcal{FOREST}^h}(FSC, End(A, B)) &= Hom_{SC_d\text{-}\mathcal{FOREST}^{h'}}(SC'_d, O(End(A, B)) \\ &= \{B \to H\}. \end{aligned}$$

Where the last equality follows from 3.6.5. □

3.5 Weakly Equivalence

3.5.1 Shrunk disk.

Definition 3.5.1. *The category of SC_d^*-$\mathcal{FOREST}$, is the category of forest colored by the set $K \cup K_*, K_* = \{h_*, f_*\}$, f_* stands for the shrunken disk and h_* for a shrunken half disk. We only allow composition of a shrunken disk f_* into a half disk or a shrunken half disk. A shrunken half disk can only be located on the flat part of the vertex.*

The idea is that if we reduce the radius of a half disk, and it contains a disk inside, then the disk will shrink as well.

We have new objects

$$\begin{tikzcd} f^* \arrow[d] & f^* \arrow[d] & h^* \arrow[d] \\ h & h^* & h \end{tikzcd} , \tag{7}$$

and there is a new arrow allowed

$$\begin{tikzcd} h^* \arrow[d] & & f^* \arrow[d] \\ h^* & \longrightarrow & h \end{tikzcd} , \tag{8}$$

with corresponding $FSC_d^*(1, 0|n, m), Wc_{-*}SC_d^{h*}(1, 0|n, m)$ operads that satisfy

$FSC_d(n+1, m) \simeq FSC_d^*(1, 0|n, m), Wc_{-*}SC_d^h(n+1, m) \simeq Wc_{-*}SC_d^{h*}(1, 0|n, m)$.

There is an inclusion functor $SC_d\text{-}\mathcal{FOREST} \to SC_d^*\text{-}\mathcal{FOREST}$, at the level of operads there is a restriction functor $O \to O|_{SC_d\text{-}\mathcal{FOREST}}$, and a pruning functor $O \to PO$, $PO : SC_d\text{-}\mathcal{FOREST} \to C$ that we define in the following lines.

Definition 3.5.2 (Pruning.). *The Pruning functors $FP : FSC_d^*(1,0|n,m) \to FSC_d(n,m)$ and $WP : Wc_{-*}SC_d^{h*}(1,0|n,m) \to Wc_{-*}SC_d^h(n,m)$, remove any vertex, leaf or root of color in K^*. For any map $f : X \to Y$, there is a well defined subjactent map $P(f) : P(X) \to P(Y)$.*

Prunning of maps is well defined since a young forest with root of color h^* can only have a leaf of color f^* due to (7). In the case of the $c_{-*}SC_d$, the maps of topological spaces

$$SC_d(1,1|n,m)^h, SC_d(1,0|n,m)^h \quad \to \quad SC_d(n,m)^h \tag{9}$$

induce maps of chains

$$c_{-*}SC_d(1,1|n,m)^h, c_{-*}SC_d(1,0|n,m)^h \quad \to \quad c_{-*}SC_d(n,m)^h \tag{10}$$

with the condition

$$c_*SC_d(1,0|0,0)^{h^*} \to 0.$$

The main theorem of this section is the following

Theorem 3.5.3. *$Wc_{-*}SC^h$ and FSC are weakly equivalent operads.*

Proof. Given the following diagram of chain complexes

$$\begin{tikzcd}
FSC_d^h(n+1,m) \arrow[r] \arrow[d, "i"] & FSC_d^{*h}(1,0|n,m) \arrow[r, "p_1"] \arrow[d] & FSC_d^h(n,m) \arrow[d, "i"] \\
Wc_{-*}SC_d^h(n+1,m) \arrow[r] & Wc_{-*}SC_d^h(1,0|n,m) \arrow[r, "p"] & Wc_{-*}SC_d^h(n,m)
\end{tikzcd}$$

If we can show that for $\alpha \in FSC_d^h(n,m), p_1^{-1}(\alpha) \simeq p^{-1}(i\alpha)$, then we can use the long exact sequence of homotopy and by induction over n the number of disk we conclude that all chain complex's maps i are quasi isomorphisms. The case $n > 1$ follows from 3.5.4 and 3.5.5.

Case $n = 1$,

$$\begin{tikzcd}
SC_d\text{-}\mathcal{FOREST}^h \arrow[rr, "FSC"] & & C \\
SC_d\text{-}\mathcal{FOREST}^{h1'} \arrow[u] \arrow[urr, "WSC'"] & &
\end{tikzcd} \tag{11}$$

$$\begin{aligned}
FSC_d(1,n)^h &= \int^{R \in SC_d\text{-}\mathcal{FOREST}^{h'}/(1,n)^h} Hom_{SC_d\text{-}\mathcal{FOREST}^h}(R,(1,n)^h) \otimes W_{SC_d\text{-}\mathcal{FOREST}^{h'}} c_{-*}SC_d(R) \\
&= \int^{R \in SC_d\text{-}\mathcal{FOREST}^{h'}/(1,n)^h} Hom_{SC_d\text{-}\mathcal{FOREST}^h}(R,(1,n)^h) \otimes W_{SC_d\text{-}\mathcal{FOREST}^h} c_{-*}SC_d(R) \\
&= \int^{R \in SC_d\text{-}\mathcal{FOREST}^h/(1,n)^h} Hom_{SC_d\text{-}\mathcal{FOREST}^h}(R,(1,n)^h) \otimes W_{SC_d\text{-}\mathcal{FOREST}^h} c_{-*}SC_d(R) \\
&= W \otimes_{SC_d\text{-}\mathcal{FOREST}^h} c_{-*}SC_d(1,n)^h.
\end{aligned}$$

The second line is due to lemma 3.4.2, the third line because $SC_d\text{-}\mathcal{FOREST}^{h'}$ are woods with $k = 1$, and the last equality is the Yoneda lemma.

□

3.5.2 Case $FP : FSC_d^*(1,0|1,m) \to FSC_d(1,m)$

Let $\alpha \in FSC_d(1,m)^h$, we want to study its fiber under the map $FP : FSC_d^*(1,0|1,m) \to FSC_d(1,m)$. at the level of trees we have the following diagram:

$$\begin{tikzcd}
(1,0|0,0)^{h^*} \cup (0,1|0,0)^h \cup (0,0|1,m)^h \arrow[rr, "s"] \arrow[dr, "P"'] & & (1,0|0,0)^h \cup (0,0|1,m)^h \arrow[dl, "P"] \\
& (1,m)^h &
\end{tikzcd}$$

In terms of chains, figure 3 parametrizes the possible fibers.

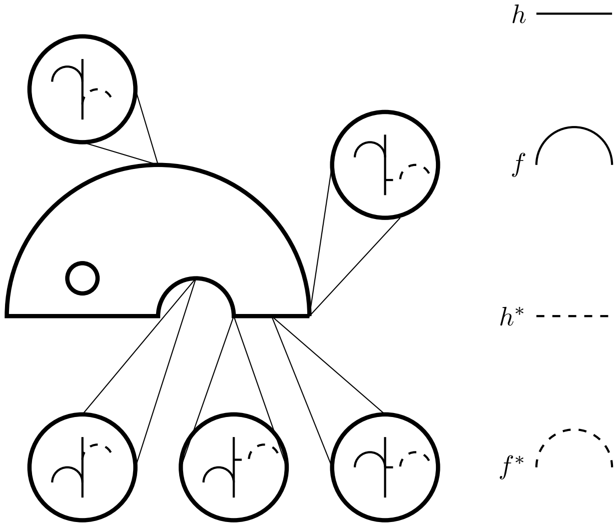
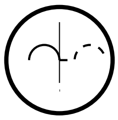
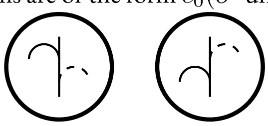
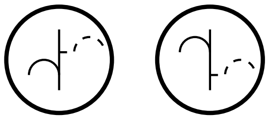


Figure 3: Possible fiber of an element of $SC_d^h(1,1)$ on $FSC_d^*(1,0|1,1)$. Notation of [10]

 represents terms parametrized by forest $(1,0|0,0)^{h^*} \cup (0,1|0,0)^h \cup (0,0|1,m)^h$. Since a shrunken half disk can only be composed on the flat boundary of a half disk those terms are of the form $c_0(\partial^h \text{unital half disk})) \otimes c_{-*}SC^h(0,0|1,m)^h$.[3]

 stands for trees parametrized by $(1,0|0,0)^h \cup (0,0|1,m)^h$. A shrunken disk can be composed on a shrunken half disk or a half disk. In the category SC_d^*-$\mathcal{FOREST}^h$ a shrunken disk cannot be composed into a half disk that already contains a full disk. In this case we precompose (or post compose) our operations with an unary operation (elements of $(0,1)^h$), we think of this as inserting a vertex on an edge. Then we compose the shrunken disk on the half disk that parametrizes that new edge. This are chains $c_0(\partial^f \text{unital half disk}) \otimes c_{-*}SC_d^h((0,0|1,m)^h$.

[3] For the unital half d-disk H, we denote its intersection with R^{d-1} by $\partial^h H$ and we called it the flat boundary. We denote its intersection with S^d by $\partial^f H$ and we call it the round boundary.

 are chains on the intersection of the flat and the round borders. They exist because the map (8) induces a homotopy between the trees $(1,0|0,0)^{h^*} \cup (0,1|0,0)^h$ and $(1,0|0,0)^h$.

3.5.3 Case $WP : Wc_{-*}SC_d^{h*}(1,0|1,m) \to Wc_{-*}SC_d^h(1,m)$,

Now we are parametrized by the category SC_d-$\mathcal{FOREST}^h$, here we allow a shrank full disk to be inserted into a vertex that already have a full disk, so we need to consider the new case 4.

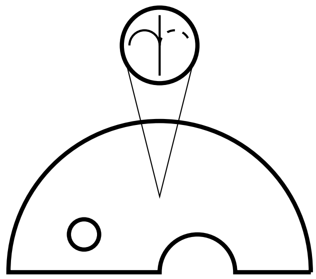


Figure 4: Possible fiber of an element of $SC_d^h(1,1)$ on $Wc_{-*}SC_d^{h*}(1,0|1,1)$. Notation of [10]

Now that we understand the fibers, we retract the fiber of $SC_d^h(1,1)$ to the boundary, meaning, we consider the subspace $\partial P^{-1}(i(\alpha))$ where we don't allow a collapsed disk and a disk.

Formally, lets consider the maps $id : P^{-1}(i(\alpha)) \to P^{-1}(i(\alpha)), i_* : \partial P_1^{-1}(\alpha) \to P^{-1}(i(\alpha)),\ p_* : P^{-1}(i(\alpha)) \to \partial P_1^{-1}(\alpha),\ \alpha \in FSC_d(1,m)^h$, we can find a homotopy that expands the disk into the boundary, while fixing the boundary. This homotopy of topological spaces induces a homotopy of chain complexes between Id and $i_* \circ p_*$. There exist $\nu : P^{-1}(i(\alpha)) \otimes \Delta[1] \to P^{-1}(i(\alpha))$ with: $\partial\nu + \nu\partial = id - i_* \circ p_*$.

Lemma 3.5.4. *Let's consider* $\alpha \in FSC_d(1,m)^h$,

$$\begin{tikzcd} FSC_d^*(1,0|1,m) \arrow[r, "P_1"] \arrow[d] & FSC_d^*(1,m) \arrow[d, "i"] \\ Wc_{-*}SC_d^h(1,0|1,m) \arrow[r, "P"] & Wc_{-*}SC_d^h(1,m) \end{tikzcd} \tag{12}$$

Then $P_1^{-1}(\alpha) \simeq P^{-1}(i\alpha)$.

We retract the fiber $P^{-1}(i\alpha)$ into $p_*P^{-1}(i\alpha)$. Then we keep only trees labeled with γ_0, this subcomplex is homotopy equivalent to $P^{-1}(i\alpha)$. It is parametrized by a space homotopy equivalent to a wedge of copies of S^d. Similarly we contract the edges of $P_1^{-1}(\alpha)$ and this two topological spaces are homotopic to each other.

Corollary 3.5.5. *Let* $\delta \in FSC_d^*(n,m)$, *where*

$$\begin{tikzcd} FSC_d^{*h}(1,0|n,m) \arrow[r, "p_1"] \arrow[d] & FSC_d^{*h}(n,m) \arrow[d, "i"] \\ Wc_{-*}SC_d^h(1,0|n,m) \arrow[r, "p"] & Wc_{-*}SC_d^h(n,m) \end{tikzcd} \tag{13}$$

Then $p_1^{-1}(\delta) \simeq p^{-1}(i\delta)$.

$FSC = i_{1*}(W \otimes SC')$, so δ is parametrized by some $T \in SC_d^*\text{-}\mathcal{FOREST}^{h*\prime}$. This tell us that every vertex is of the form $(1,m)$ or $(0,n)$ and from lemma 3.5.4 it follows that each of these spaces are homotopy equivalent to a wedge of spheres. The total number of spheres is the number of disk in T.

3.5.4 Recovering the E_d action

$Wc_{-*}SC^h$ is generated by trees were only leaves can have color f. By acting on the leaves we have an action of $c_{-*}E_d$ on $Wc_{-*}SC^h$ (we use the γ_0 action). We want to further extend this to an action of $Wc_{-*}E_d$ on FSC_d.

Theorem 3.5.6. *We have maps of* $c_{-*}E_{d-1}$*-algebras* $p : Wc_{-*}SC_d^h \to FSC_d$, *and* $i : FSC_d \to Wc_{-*}SC_d^h$, *together with a homotopy* k *such that* $\partial k = id - i \circ p$.

We consider the following diagram:

$$\begin{tikzcd} FSC_d \arrow[r] \arrow[d, "i"] & FSC_d \arrow[d, "j"] \\ Wc_{-*}SC_d^h \arrow[r] \arrow[ur, dashed, "p"] & 0 \end{tikzcd} \tag{14}$$

Here i is an acyclic cofibration and j is a fibration, it follows that there is a lifting p, now by Quillen SM7 we have that $map(Wc_{-*}SC_d^h, Wc_{-*}SC_d^h) \to map(FSC_d, Wc_{-*}SC_d^h)$ is an acyclic fibration, and since $id_{Wc_{-*}SC_d^h}$ and $i \circ p$ satisfy :

$$id_{Wc_{-*}SC_d^h} \circ i = i = i \circ p \circ i,$$

then they should be homotopic, we conclude that there is a homotopy of operads k so that $dk = pi - id_{Wc_{-*}SC_d}$.

Corollary 3.5.7. *[7] The map* $i : FSC \to Wc_{-*}SC^h$ *induces a* $Wc_{-*}E_d$ *action on* FSC.

Proof. A map $FSC(n,m)^h \otimes WE_d(r,n) \to FSC(r,m)$ is given by a forest with inputs labeled by i, internal vertex labeled with elements of WE_d, internal edges labeled by k and roots labeled by p. We start at the leaves and follow let the gravity send us to the roots. We first send our elements of FSC to $Wc_{-*}SC^h$ via i, then we let $Wc_{-*}E_d$ act, then we use the homotopy k act, and so on, we continue until we reach the root and apply p to recover a term in FSC. □

3.6 A model of the Higher Hochschild Complex

As far as I know, the following is an idea of Costello. The algebra A can be replaced by

$$c_{-*}SC_d(1,)^h \otimes_{c_{-*}(E_{d-1})} A^{\otimes},$$

this is the coend of

$$\oplus_m c_{-*}SC_d(1,n)\otimes c_{-*}E_{d-1}(n,m)\otimes A^{\otimes m}\rightrightarrows \oplus_n c_{-*}SC_d(1,n)\otimes A^{\otimes n}, \tag{15}$$

where the first map is the action of E_{d-1} on SC_d and the second is the action of E_{d-1} on A. In the coend we keep the left action of $c_{-*}E_{d-1}$ on A and the right action on $c_{-*}SC_d(1,)^h$ compatible. The left action of $c_{-*}E_{d-1}$ on the coefficients $c_{-*}SC_d(1,n)$ commutes with both maps of 15 and induces an action on the coend. Similarly, there is an right action of A on the coend, and this two actions commute. We conclude that $c_{-*}SC_d(1,)^h\otimes_{c_{-*}(E_{d-1})}A^{\otimes}$ is a $c_{-*}E_{d-1}-A$ bimodule.

The action of $c_{-*}E_{d-1}$ on $c_{-*}SC_d(1,)^h\otimes_{c_{-*}(E_{d-1})}A^{\otimes}$ induces an action on the endomorphisms object

$$Hom_{c_{-*}(E_{d-1})-A\ Mod}(c_{-*}SC_d(1,)^h\otimes_{c_{-*}(E_{d-1})}A^{\otimes},A).$$

via precomposition, it inherits a structure of $c_{-*}E_{d-1}$-$c_{-*}E_{d-1}$ bimodule.

Remark 3.6.1. *From now on we replace all our operads by cofibrant operads via the W construction [1], see 3.1.1.*

We consider the operad of normalized chains of SC_d which is a reduced operad. We work in Char 0.

Remark 3.6.2. *We fix a model of the Higher Hochschild complex*

$$H=Hom_{c_{-*}(E_{d-1})-A}(c_{-*}SC_d(1,)^h\otimes^L_{c_{-*}(E_{d-1})}A^{\otimes},A).$$

H is a $Wc_{-*}(E_{d-1})-A$ bimodule, where the map

$$Wc_{-*}(E_{d-1})(n)\otimes H\otimes A^{n-1}\to H,$$

uses the left action of $Wc_{-*}(E_{d-1})$ on $Wc_{-*}SC_d(1,)^h$ and the $Wc_{-*}(E_{d-1})$ module structure on A .

Definition 3.6.3. *We consider the (partial) operad $Wc_{-*}SC'_d$, whose formal definition is contained in 3.3.7. An algebra over $Wc_{-*}SC'_d$ is given by a pair (B,A) where A is an $Wc_{-*}(E_{d-1})$ algebra, B is an $Wc_{-*}(E_{d-1})-A$ bimodule and the two structures are compatible via*

$$\begin{aligned} Wc_{-*}E_{d-1}(m)\otimes A^m &\to A, && (16)\\ Wc_{-*}E_{d-1}(m)\otimes B\otimes A^{n-1} &\to B, && (17)\\ Wc_{-*}SC'_d(1,n)^h\otimes A^n &\to B, && (18)\\ Wc_{-*}SC'_d(1,n)^h\otimes B\otimes A^n &\to A, && (19)\end{aligned}$$

where the following diagrams commutes

$$\begin{tikzcd} Wc_{-*}SC'_d(1,n)^h\otimes Wc_{-*}(E_{d-1})(m,n)^h\otimes B\otimes A^m \arrow[r] \arrow[d] & Wc_{-*}SC'_d(1,m)^h\otimes B\otimes A^m \arrow[d] \\ Wc_{-*}SC'_d(1,n)^h\otimes B\otimes A^n \arrow[r] & A \end{tikzcd} \tag{20}$$

$$\begin{tikzcd} Wc_{-*}(E_{d-1})(m) \otimes Wc_{-*}SC'_d(1,n;0,m)^h \otimes B \otimes A^n \arrow[r] \arrow[d] & Wc_{-*}SC'_d(1,n)^h \otimes B \otimes A^n \arrow[d] \\ Wc_{-*}(E_{d-1})(m) \otimes A^m \arrow[r] & A \end{tikzcd} \tag{21}$$

Lemma 3.6.4. *The pair (H, A) has an action of $Wc_{-*}SC'_d$.*

Proof. A is an $Wc_{-*}(E_{d-1})$ algebra, H is an $Wc_{-*}(E_{d-1}) - A$ bimodule and the evaluation map $SC'_d(1,n)^h \otimes H \otimes A^n \to A$ satisfies (20) and (21). □

Given (B, A) a SC'_d algebra, from property 19 we construct a map by adjuntion:

$$B \to Hom(Wc_{-*}SC'_d(1,n)^h \otimes A^n, A), \tag{22}$$

Diagrams 20 and 21 implies that the actions of $Wc_{-*}(E_{d-1})$ on A and on $Wc_{-*}SC'_d$ are compatible, for that reason we can instead consier $\otimes_{Wc_{-*}(E_{d-1})}$, and if we assemble the diagrams for all n then equation (22) factors through

$$B \to Hom(Wc_{-*}SC'_d(1,)^h \otimes^L_{Wc_{-*}(E_{d-1})} A^{\otimes}, A), \tag{23}$$

this is the derived tensor product because W provide us of a cofibrant replacement. Since the actions are compatible (23) factors through

$$B \to Hom_{Wc_{-*}(E_{d-1})-AMod}(Wc_{-*}SC'_d(1,)^h \otimes^L_{Wc_{-*}(E_{d-1})} A^{\otimes}, A), \tag{24}$$

(see [10]). We can rephrase equation 24 as:

Theorem 3.6.5. *The functor that assigns to $B \in C$ the set of $Wc_{-*}SC'_d$ actions on the pair (B, A) is representable by H.*

Proof. This is because given a map $B \to H$, by adjuntion, it gives us a sequence of maps 19. □

The $c_{-*}E_{d-1}$-$c_{-*}E_{d-1}$ bimodule $c_{-*}SC_d(1,)^h \otimes^L_{c_{-*}(E_{d-1})} A^{\otimes}$ can be constructed as the colimit over the base subcategory of the functor

$$F : \mathcal{FOREST}^{h\prime} \to c_{-*}E_{d-1}\text{-}c_{-*}E_{d-1} \text{ bimodules},$$

F is defined by

$$\begin{array}{rcl} (1,0)^h & \to & c_{-*}SC(1,0)^h, \\ (0,1)^h & \to & A, \\ (1,n)^h & \to & c_{-*}SC(1,0)^h \otimes_{c_{-*}E_{d-1}} A. \end{array}$$

For any of the woods defined on 3.4.3

$$SC_d\text{-}\mathcal{FOREST}^{h1\prime} \subset SC_d\text{-}\mathcal{FOREST}^{h2\prime} \subset \cdots \subset SC_d\text{-}\mathcal{FOREST}^{h},$$

we have $c_{-*}E_{d-1}$-$c_{-*}E_{d-1}$ bimodules $c_{-*}SC_d(n,)^h \otimes^L_{c_{-*}(E_{d-1})} A^{\otimes}$, and we define

$$H^n = Hom_{c_{-*}(E_{d-1})-A}(c_{-*}SC_d(n,)^h \otimes^L_{c_{-*}(E_{d-1})} A^{\otimes}, A),$$

Corollary 3.6.6. *H is a $Wc_{-*}E_d$ algebra.*

Proof. The inclusion SC_d-$\mathcal{FOREST}^{h\prime} \subset SC_d$-$\mathcal{FOREST}^{h1\prime}$ induces a homotopy action $WSC_d^{h1\prime} \to End_{(H,A)}$, by adjuntion we have

$$H^{\otimes n} \to Hom_{c_{-*}(E_{d-1})-A}(c_{-*}SC_d(n,)^h \otimes^L_{c_{-*}(E_{d-1})} A^{\otimes}, A) = H^n,$$

that defines the first map of the composition

$$H^{\otimes n} \otimes Wc_{-*}E_d(n,1) \to H^n \otimes Wc_{-*}E_d(n,1) \to H$$

while the last one is the homotopy action 3.5.7 on $c_{-*}SC_d(n,)^h$. □

Elements of $F(SC[1])$ are parametrized with trees whose internal vertex have all lenght γ, now the lower degree we can obtain is 1 given by a term of degree 0 parametrized by a corolla, and a composition of s corollas is shifted s terms to the right. On $F(F(c_{-*}SC[1])[-1])$ we consider trees with edges labeled with γ_0 and vertex labeled with elements of $F(c_{-*}SC[1])$ shifted to the left once. The terms of lower grade are still located at degree 0. We conclude that WSC is still an operad of bounded complexes.

3.6.1 Homotopically sound operads.

Consider $Alg_O(C(k))$ the category of O-algebras, when O is an operad over $C(k)$. We have the natural functors $F_O : C(k)^O \to Alg_O(C(k))$ free O-algebra and the forgetful functor. In good cases this functors induce a model structure on algebras from the projective model structure of $C(k)$, if not we need to replace our operad O by an equivalent and better behaved one R whose algebras admits the model structure. In those cases we want to make sure the model category is independent of the choice of the replacement, this is true for homotopic sound operads.

Definition 3.6.7. *An Operad over $C(k)$ is called admissible if the category of algebras over the operad admits the following model structure*

- *$f : A \to B$ fibration if $G(f) : G(A) \to G(B)$ is a fibration.*
- *$f : A \to B$ weak equivalence if $G(f) : G(A) \to G(B)$ is a weak equivalence.*
- *$f : A \to B$ cofibration if it admits the left lifting property with respect to acyclic fibrations.*

Definition 3.6.8. *An operad O is Σ-cofibrant if for each corolla $(c)_r$, the complex $O(c)_r$ is a cofibrant complex of $Aut(c)_r$-modules. An Operad is homotopically sound if it is admissible and Σ-cofibrant.*

Theorem. *6.1 [4] If $Q \subset R$, then for operads on $C(R)$ we have*

- *If the components $O(c,d)$ of the operad O are cofibrant complexes, O is homotopical sound.*

Due to the argument on 3.4.8, we apply this theorem and conclude that the category of $Alg_{c_*SC_d}$ admits a model structure. We restrict our attention to categories C so that the natural forgetful functor and free algebra functor

$$Mod_{c_*(E_{d-1})-A}(C) \leftrightarrows Alg_{c_*SC_d}(C)$$

induce a Quillen equivalence.

Now we show that $c_{-*}SC_d(1,)' \otimes^L_{c_{-*}E_{d-1}} A^{\otimes} \to A$ is a trivial cofibration in $Mod_{(E_{d-1})-A}(C)$. It is enough to pick a point $b \in SC_d(1,1)$, this will be a generator since any zero chain is determined by the center of the disk and they are all connected.

The map that forgets the center of the disk

$$\oplus c_{-*}SC_d(1,n) \otimes A^{\otimes n} \to \oplus c_{-*}E_{d-1}(n) \otimes A^{\otimes n},$$

and the half disk composition with b map

$$c_{-*}E_{d-1}(n) \otimes A^{\otimes n} \to^{b \otimes_h} \oplus c_{-*}SC_d(1,n) \otimes A^{\otimes n},$$

induce a retraction between A and $SC_d(1,)' \otimes^L_{c_* E_{d-1}} A$.

3.6.2 Model of $c_{-*}SC_d$.

We define a two colored operad WSC_d

- $WSC_d(n,0)^f = Wc_{-*}E_d(n)$,
- $WSC_d(n,m)^h = FSC_d(n,m)^h$.

as the semidirect product of operads $W(c_{-*}E_d) \ltimes FSC$ where on the compositions

$$WSC_d(n,0)^f \circ^f WSC_d(r,s)^h \to WSC_d(n+r-1,s)^h,$$

we use the transferred $Wc_{-*}E_d$ algebra structure on FSC. We have the following sequence of quasi isomorphisms:

$$SC'_d \longleftarrow W(c_{-*}E_d) \rtimes \otimes_{SC_d\text{-}\mathcal{FOREST}^h} c_{-*}SC_d \longrightarrow W(c_{-*}E_d) \rtimes W \otimes_{SC_d\text{-}\mathcal{FOREST}} c_{-*}SC_d|_h.$$

Theorem 3.6.9. *Let B be an $c_{-*}(E_{d+1})$ algebra and A be a $c_{-*}(E_d)$ algebra. Then the functor that associates*

$$B \to \left\{ \begin{array}{c} WSC_d \textit{ actions} \\ \textit{on the pair } (B,A) \textit{ compatible} \\ \textit{with the } C_{-*}E_{d-1}, C_{-*}E_d \textit{ algebra} \\ \textit{structure on } A \,\&\, B \textit{ respectively.} \end{array} \right\}$$

is representable by H.

Proof. The proof follows [10].

$$\begin{tikzcd}
 & & W(c_{-*}E_d)(n)WSC^h(1,m)B^nA^m \arrow[rr] \arrow[dll] \arrow[d, dashed] & & WSC^h(n,m)B^nA^m \arrow[dl] \arrow[dd] \\
W(c_{-*}E_d)(n)WSC^h(1,m)H^nA^m \arrow[rrr] \arrow[dd] & & & WSC^h(n,m)H^nA^m \arrow[dd] & \\
 & & WSC^h(1,m)BA^m \arrow[rr, dashed, no head] \arrow[dll, dashed, no head] & & A \arrow[dl] \\
WSC^h(1,m)HA^m \arrow[rrr] & & & A &
\end{tikzcd}$$

The front face commutes because H has an action of $WSC_d \rtimes Wc_{-*}E_d$.

Given a map $B \to H$ The top commutes trivially. The base and right face commutes by theorem 3.6.5, and if the map is a map of $Wc_{-*}E_d$ algebras, the left face commutes. We conclude that the back face commutes and so we recover an action of $WSC_d \rtimes Wc_{-*}E_d$ on the pair (B, A).

Conversely, if the pair (B, A) have an action of $WSC_d \rtimes Wc_{-*}E_d$ then the left face commutes because all the other faces commutes, and that implies that the map $B \to W$ preserves the $Wc_{-*}E_d$ structure. $\square$

4 Applications.

In this section we explain how the work of Geoffroy Horel and our result we can construct an algebra over a generalization of Kontsevich Soibelman operad.

4.1 The operad of Cylinders

Let's consider cylinders $S^1 \times [a, b]$ with marked points $I \in S^1 \times a, O \in S^1 \times b$. We can decorate the sides of the cylinder with little disk, and we can also combine two cylinders into a big cylinder, we just rotate the upper cylinder so that the O point of it coincides with the I point of the cylinder below. We denote by $Cyl(n, 0) = E_2(n)$, by $Cyl(0, 1)$ the space of configuration of cylinders with market points I, O, and by $Cyl(n, 1)$ the space of configuration of n disk on the surface of a labeled disk.

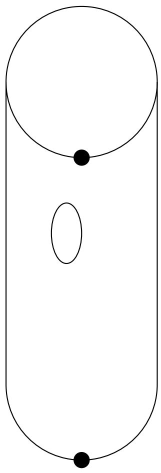


Figure 5: An example of a term in $c_0Cyl(1, 1)$.

We have the usual operations $\mu, \{,\}$ coming from $H_*(E_2)$, an operation i on $H_0(cyl(1, 1))$ that inserts a disk on the surface of the cylinder. $l \in H_1(cyl(1, 1))$ which represents the cycle that rotates a disk on the cylinder about the vertical axis. There is also an operation $\delta \in H_1(Cyl(0, 1))$ that rotates the base of the cylinder.

As example l can be achieved by rotating both the disk and the base point O to the right (δi) and then rotating the base point to the left ($-i\delta$), we conclude that $l = \delta i - i\delta$. l is called the Lie derivative. Similarly we can verify that:

- $i_{[a,b]} = [i_a, l_b]$,
- $l_{\mu(a,b)} = l_a i_b + i_a l_b$,
- $\delta^2 = 0$.

For more details see [3].

Algebras over H_*Cyl are pairs (V, W) were V is a Gerstenhaber algebra with the operations $\mu, \{,\}$; we have $i : V \otimes W \to W, l : V[-1] \otimes W$, and $\delta : W \to W$ satisfying the previous conditions. $(HH^*(A), HH_*(A))$ is an example of an algebra over the homology of the cylinder operad, in the case $A = C^*(X)$, this corresponds to $(\oplus \wedge^n Vect(X), \Omega(X))$.

4.1.1 smooth operads.

We consider the space $Emb(\cup_n D, D)$ with the weak C^1 topology. We defined the operad E_d by $E_d(n) = Emb_l(\cup D^{\cup n}, D)$, where Emb_l is the sub space of rectilinear embeddings. Now to any embedding of the disk into itself that fixes the origin, we associate an element of $Gl(d)$ by considering the derivative at origin.

Theorem. *The space of self embeddings of D that send the origin to itself is weak equivalent to $GL(d)$.*

In fact it is proven that the derivative at origin map is a Hurewicz fibration.

Theorem. *Given M a manifold, there is a map*

$$Emb(D, M) \to Fr(TM),$$

which is a weak equivalence and a Hurewicz fibration.

We can consider the case of manifolds with boundary, and embeddings of H as well.

Theorem. *Given M a manifold, the derivative at the origin map*

$$Emb(D^{\cup n} \cup H^P \cup m, M) \to Fr(TConf(n, M - \partial M)) \times Fr(TConf(m, \partial M)),$$

is a weak equivalence and a Hurewicz fibration.

Instead we consider framed embeddings of manifolds with boundary $(\phi, p) \in Emb^f(M, N)$, defined as an embedding $\phi : M \to N$ and locally a homotopy between the trivialization of M and the induced trivialization comming from N. This map should vary smoothly and respect boundary conditions, for an explicit construction and a precise definition see page 17,[5].

Theorem. *Given M a manifold, the evaluation at the center induces a weak equivalence*

$$Emb^f(D^{\cup n} \cup H^P \cup m, M) \to Conf(n, M - \partial M) \times Conf(m, \partial M).$$

Definition 4.1.1. *The operad $\mathcal{E}_d$ is defined as $\mathcal{E}_d(n) = Emb^f(D^{\cup n}, D)$.*

This is a smooth version of the little disk operad, and in fact it is weakly equivalent to it.

Definition 4.1.2. *The operad $\mathcal{SC}_d$ is defined as*

$$\mathcal{SC}_d(n, 0)^f = \mathcal{E}_d(n),$$
$$(SC)_d(n, m)^h = Emb^f(D^{\cup n} \cup H^{\cup m}, H).$$

It turns out that $\mathcal{SC}_d$ and SC_d are weakly equivalent.

4.1.2 Kontsevich-Soibelman operad.

Definition 4.1.3. *The operad of Kontsevich-Soibelman is a two colored operad, with colors $\{a, m\}$ defined by:*

$$KS(a^n; a) = \mathcal{E}_2(n),$$
$$KS(a^n \cup m; m) = Emb^f(S^1 \times [0, 1) \cup D^{\cup n}, S^1 \times [0, 1)).$$

We previously referred to this operad as Cylinder operad. There is a higher dimensional version of this operad.

Definition 4.1.4. *Given a* $d-1$*-manifold* M *with a framing* τ*, the operad* $M_\tau^{\circlearrowleft}\mathbf{Mod}$ *is the two colored operad given by*

$$S_\tau^{\circlearrowleft}\mathbf{Mod}(a^n, a) = \mathcal{E}_d(n),$$

$$S_\tau^{\circlearrowleft}\mathbf{Mod}(a^n \cup m, m) = Emb^f(M \times [0,1) \cup D^{\cup n}, M \times [0,1)).$$

Definition 4.1.5. *The operad* $f\mathcal{MAN}$ *has as colors framed d-manifolds embedded in* R^∞*, and* $f\mathcal{MAN}(N_1 \cup \cdots \cup N_l; N) = Emb^f(N_1 \cup \cdots \cup N_l, N)$.

We defined an operad as a symmetric monoidal functor $O : Forest \to C$, in [5] to any operad O it is assigned the free monoidal category O generated by the colors subcategory of O. In our setting that corresponds to subcategory $O(Forest) \subset C$. The subcagory $E_d \subset$ fMAN is the full subcategory with objects disjoint union of framed disk, embedded in R^∞.

Definition 4.1.6. *Let* A *a* $C_{-*}(\mathcal{E}_d)$ *algebra, factorization homology is defined as:*

$$\int_M A := C_{-*}(Emb^f(,M)) \otimes^L_{C_{-*}E_d} A.$$

The following statement is proved in 9.6 [5].

Theorem. *Let (B,A) be an algebra over* $C_{-*}(\mathcal{SC}_d)$ *in the category* $C_*(R)$*. Let* M *be a framed* $(d-1)$*-manifold and* τ *be the product framing on* $TM \oplus R$*. The pair* $(B, \int_M A)$ *is weakly equivalent to an algebra over the operad* $C_{-*}(M_\tau^{\circlearrowleft}\mathbf{Mod})$.

With our result we conclude that

Corollary 4.1.7. *Let* A *be an algebra over* $C_{-*}(\mathcal{E}_{d-1})$ *in the category* $C_*(R)$*. Let* M *be a framed* $(d-1)$*-manifold and* τ *be the product framing on* $TM \oplus R$*. The pair* $(Hoch(A), \int_M A)$ *is weakly equivalent to an algebra over the operad* $C_{-*}(M_\tau^{\circlearrowleft}\mathbf{Mod})$.

References

[1] Berger, C., and Moerdijk, I. The boardman-vogt resolution of operads in monoidal model categories. *Topology 45*, September (2006), 807–849.

[2] Dolgushev, V., Tamarkin, D., and Tsygan, B. The homotopy gerstenhaber algebra of hochschild cochains of a regular algebra is formal. 21.

[3] Dolgushev, V., Tamarkin, D., and Tsygan, B. Formality of the homotopy calculus algebra of hochschild (co)chains. 1–66.

[4] Hinich, V. Rectification of algebras and modules.

[5] Horel, G. Factorization homology and calculus \'a la kontsevich soibelman. 28.

[6] Livernet, M. Non-formality of the swiss-cheese operad. 10.

[7] Loday, J.-L., and Vallette, B. *Algebraic operads*. Springer-Verlag Berlin Heidelberg, 2012.

[8] Loregian, F. This is the (co)end, my only (co)friend. 39.

[9] Markl, M. Transferring a_∞ (strongly homotopy associative) structures. 14.

[10] Thomas, J. D. Kontsevich's swiss cheese conjecture. 33.

A Relation with Topological Quantum Field Theory.

In the topological quantum field theory we have a functor F that assigns to S^1 a vector space $F(S^1)$. The cobordism of the pair of pants defines a (commutative) product $F(S^1)^{\otimes 2} \to F(S^1)$.

I want to describe some operads that help us understand the algebraic structure that $F(S^1)$ inherits. We will restrict to cobordisms from $\sqcup_n S^1$ into S^1 of genus 0. This is necessary to avoid maps of the form $V \to V^{\otimes m}$. A general cobordism can be described as a composition of cobordisms of the form $\sqcup_n S^1 \to S^1$ with a cobordism $S^1 \to S^1$, which possibly has higher genus.

An open TQFT assigns to the interval I operations $F(I)^{\otimes n} \to F(I)$ induced by cobordisms from disjoint copies of the interval to an interval, here we work with cobordisms without holes. There is no orientation preserving transformation between the cobordisms in figure 6, which implies that the product on $F(I)$ may not be commutative.

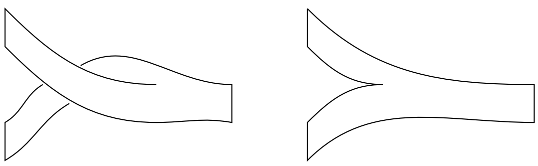


Figure 6: $F(I)$ may not be commutative.

In the previous paragraphs we have some vector space V together with elements of $Hom(V^{\otimes n}, V)$ that satisfy rules of composition. An operad O is a sequence of vector spaces $O(n)$ (or topological spaces or chain complexes, etc), together with compositions that satisfy some conditions that we will study later. It is enough to say that $End_v(n) = Hom(V^{\otimes n}, V)$ is our model of how does an operad behave.

We are considering only cobordisms $\cup_n S^1 \to S^1$, from now on instead of drawing cobordisms we imagine that we look through the waist. What we will see is n disks inside of the waist. This is compatible with what we will see if we do the composition of cobordisms.

A.1 Open-close cobordism.

Let's consider the cobordism $\sqcup_3 S^1 \to S^1$ that is symmetric with respect to the z-coordinate. Because of symmetry we can just cut the cobordism along the horizontal line. If we situate our eyes on the waist of the cobordism and cut the cobordism we will something like a slice of Swiss Cheese.

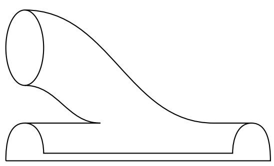


Figure 7: Here inputs are displayed on the left side and the output to on right side.

We deduce that we have a map $h : F(S^1) \to ZF(I)$ where $ZF(I)$ is the center of the algebra $F(I)$. As the pair $(C * (A), A)$ is a homotopy algebra over SC_2, one may ask if this structure extends to a TQFT. At least one needs a Frobenius algebra structure on A.