Draft:On regions of convergence of quaternionic hyperholomorphic functions.

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Abstract. We show that the natural region of convergence of quaternionic hyperholomorphic functions is not a sphere but a bigger region. We extend the theorem of Cauchy-Hadamard and the theorem of Abel on convergence of series to quaternionic analysis.

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1. Introduction

Let $\mathbb{H}$ denote the quaternion numbers. A function $f : \mathbb{H} \to \mathbb{H}$ is said to be (left) hyperholomorphic in a neighborhood V of the origin, if f is real differentiable on V and if $Df = 0$ when D is the Cauchy-Riemann-Fueter operator:

$$\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}.$$

Definition 1.0.1. *The Fueter's basis* [2] *is given by the hyperholomorphic functions* $\zeta_n : \mathbb{H} \to \mathbb{H}, n \in \{1, 2, 3\},$ *defined on* $h = h_0 + ih_1 + jh_2 + kh_3$ *by*

$$\begin{aligned} \zeta_1(h) &= h_1 - ih_0, \\ \zeta_2(h) &= h_2 - jh_0, \\ \zeta_3(h) &= h_3 - kh_0. \end{aligned}$$

The Taylor expansion of a (left) hyperholomorphic function f at the origin is given in terms of non-commutative polynomials

$$f(x) = \sum_0^{\infty} \sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} P_\nu a_\nu,$$

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where

$$P_\nu = \frac{1}{n!}\sum_{(i_1,\cdots,i_n)\in A_\nu} \zeta_{i_1}\cdots\zeta_{i_n},$$

and the sum is over elements of A_ν, the set of all possible ways to multiply n_1 copies of ζ_1, n_2 copies of ζ_2 and n_3 copies of ζ_3 [3]. In [1] we can find a review of the recent work about the Taylor series in quaternionic analysis.

We denote by $||\,||'$ the norm

$$||x_0 + ix_1 + jx_2 + kx_3||' = \max\{||x_0 + ix_1||, ||x_0 + jx_2||, ||x_0 + kx_3||\},$$

the corresponding balls $B(0,r) := \{x \in \mathbb{H}|\, ||x|| < r\}$ and $B'(0,r) := \{x \in \mathbb{H}|\, ||x||' < r\}$ satisfy

$$B(0,r) \subset B'(0,r).$$

On $i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$ the norm $||\,||'$ becomes the maximum norm.

In this note we extend the following two theorems to quaternionic analysis:

Theorem 1.0.2. *(Quaternionic Cauchy-Hadamard Theorem)*

$$f(x) = \sum_0^\infty \sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} P_\nu a_\nu, \tag{1}$$

converges compactly on $B'(0,\frac{1}{\rho})$, *where*

$$\rho = \limsup_{k\to\infty}(\max_{||\nu||=k} ||a_\nu||)^{\frac{1}{k}}.$$

Theorem 1.0.3. *(Quaternionic Abel Theorem) Suppose that there are constants* $r_0, M \in \mathbb{R}, N_0 \in \mathbb{R}$, *such that for all* $n > N_0$ *and multi indexes* ν *with* $||\nu|| = n$ *we have the bound* $||a_\nu||r_0^n \leq M$. *Under this hypothesis the series* (1) *converges compactly on* $B'(0,r_0)$.

2. Wispy norm.

While studying domains of convergence, we frequently have to deal with the expression

$$\frac{||a_\nu||}{n!}\sum_{(i_1,\cdots,i_n)\in A_\nu} ||\zeta_{i_1}\cdots\zeta_{i_n}(x)||,$$

in [3] and [4] it is used that $||\zeta_{i_1}\cdots\zeta_{i_n}(x)|| \leq ||x||^n$. Instead we propose to consider $||\zeta_{i_1}\cdots\zeta_{i_n}(x)|| \leq ||x||'^n$, and our goal is to show that the norm $||\,||'$ better fits the Weierstrass approach [5].

Definition 2.0.1. *We say that series* (1) *converges in the wispy sense at* h *if the series*

$$Nf(x) = \lim_{n\to\infty}\sum_0^n \sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} ||\frac{a_\nu}{n!}||\sum ||\zeta_{i_1}\cdots\zeta_{i_n}(x)||.$$

satisfies $Nf(h) < \infty$.

For example:

$$NP_{n_1,n_2,n_3}(x) \leq \frac{||x||'}{n!}\binom{n}{n_1, n_2, n_3}. \tag{2}$$

Theorem. [4] *If* (1) *converges on the wispy sense at h then it converges compactly on* $\{x| \, ||\zeta_1(x)|| \leq ||\zeta_1(h)||, ||\zeta_2(x)|| \leq ||\zeta_2(h)||, ||\zeta_3(x)|| \leq ||\zeta_3(h)||\}$

Proof. It follows from Weiestrass M-test.

□

2.0.1. Examples. Any holomorphic function induces a hyperholomorphic function, in particular the series $\sum \zeta_1^n n^n + 2$ converges only on the plane $j\mathbb{R} + k\mathbb{R}$. From now on, we restrict our study to regions (open sets) of convergence of hyperholomorphic functions.

For the function $f(x) = \sum \zeta_1^n a_n + \sum \zeta_2^n b_n + \sum \zeta_3^n c_n$ we consider $\rho_1 = \limsup_{k\to\infty}(||a_k||)^{\frac{1}{k}}, \rho_2 = \limsup_{k\to\infty}(||b_k||)^{\frac{1}{k}}, \rho_3 = \limsup_{k\to\infty}(||c_k||)^{\frac{1}{k}}$, then the function converges on

$$\{||x_0 + ix_1|| < \frac{1}{\rho_1}, ||x_0 + jx_2|| < \frac{1}{\rho_2}, ||x_0 + kx_3|| < \frac{1}{\rho_3}\}.$$

In particular, $\sum \zeta_1^{2^n} + \sum \zeta_2^{2^n} + \sum \zeta_3^{2^n}$ convergences on $\{x \in \mathbb{H}| \, ||x||' < 1\}$, and diverges else where.

Lemma 2.0.2. *If* $\rho = \limsup_{k\to\infty}(\max_{||\nu||=k} ||a_\nu||)^{\frac{1}{k}} = 0$ *then* (1) *converges compactly for all* $\mathbb{H}$.

Proof. Let $x \in \mathbb{H} - \{0\}$. There is $N_0 \in \mathbb{N}$ such that for all $n > N_0$

$$(\max_{||\nu||=n} ||a_\nu||)^{\frac{1}{n}} \leq \frac{1}{3||x||'}.$$

Then for all $n > N_0$

$$\begin{aligned}\sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} N(P_\nu a_\nu) &\leq \sum_{n_1+n_2+n_3=n} \frac{||x||'^n}{n!(3||x||')^n}\binom{n}{n_1, n_2, n_3} \\ &= \frac{1}{3^n n!}\sum_{n_1+n_2+n_3=n}\binom{n}{n_1, n_2, n_3} \\ &= \frac{1}{n!}.\end{aligned}$$

On the first line we used (2). We conclude that

$$NFf(x) \leq NF(\sum_0^{N_0} \sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} P_\nu(x)a_\nu) + e.$$

Compactly convergence follows from convergence on the wispy sense according to Lemma 2. □

Lemma 2.0.3. *If* $0 < \rho < \infty$ *then* (1) *converges compactly for all* $h \in B'(0, \frac{1}{\rho})$.

Proof. Let $x \in B'(0, \frac{1}{\rho})$. Let $\theta = \sqrt{||x||'\rho}$, then

$$\frac{\theta}{||x||'} = \frac{\rho}{\theta} > \rho,$$

we conclude that there is $N_0 \in \mathbb{N}$ such that for all $n > N_0$

$$||a_\nu||^{\frac{1}{n}} \leq \frac{\theta}{||x||'}, \ ||\nu|| = n.$$

Then for $n > N_0$

$$\begin{aligned} \sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} N(P_\nu a_\nu) &\leq \sum_{n_1+n_2+n_3=n} \frac{||x||'^n}{n!} \frac{\theta^n}{||x||'^n} \binom{n}{n_1, n_2, n_3} \\ &= \frac{\theta^n}{n!} \sum_{n_1+n_2+n_3=n} \binom{n}{n_1, n_2, n_3} \\ &= \frac{(3\theta)^n}{n!}. \end{aligned}$$

We conclude that

$$NFf(x) \leq NF(\sum_0^{N_0} \sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} P_\nu(x) a_\nu) + e^3.$$

□

This concludes the proof of theorem 1.0.2.

Proof of Theorem 1.0.3. Let $x \in \mathbb{H}$ with $||x||' < r_0$. Then for $n > N_0$

$$\begin{aligned} \sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} N(P_\nu a_\nu) &\leq \sum_{n_1+n_2+n_3=n} \frac{||x||'^n}{n!} ||a_\nu|| \binom{n}{n_1, n_2, n_3} \\ &\leq \sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} \frac{1}{n!} (\frac{r}{r_0})^n \binom{n}{n_1, n_2, n_3} \\ &\leq M \frac{3^n}{n!}. \end{aligned}$$

Which give us

$$NFf(x) \leq NF(\sum_0^{N_0} \sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} P_\nu(x) a_\nu) + Me^3.$$

□

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