A Combinatorial proof of a binomial identity

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1 Introduction

In [JAAN21] the following identity was proven multiplying power series. Let $n = n_1 + \cdots + n_k$. Then for $v \geq n + k - 1$

$$\left(\!\!\binom{v-k+1}{n}\!\!\right) = \binom{k-1}{0} \sum_{\substack{\sum v_i = v \\ v_i \geq 1}} \prod_{i=1}^{k} \left(\!\!\binom{v_i}{n_i}\!\!\right) - \binom{k-1}{1} \sum_{\substack{\sum v_i = v-1 \\ v_i \geq 1}} \prod_{i=1}^{k} \left(\!\!\binom{v_i}{n_i}\!\!\right) + \cdots$$

$$+ (-1)^{k-2} \binom{k-1}{k-2} \sum_{\substack{\sum v_i = v-(k-2) \\ v_i \geq 1}} \prod_{i=1}^{k} \left(\!\!\binom{v_i}{n_i}\!\!\right) + \binom{k-1}{k-1} \sum_{\substack{\sum v_i = v-(k-1) \\ v_i \geq 1}} \prod_{i=1}^{k} \left(\!\!\binom{v_i}{n_i}\!\!\right) \tag{1}$$

Here we provide a combinatorial proof.

Theorem 1.1. *Let $n = n_1 + \cdots + n_k$ and $v \geq n + k - 1$. Equation 1 counts the number of ways to choose n points out of $v - k + 1$ with repetitions, by:*

- *first choosing n_1 points with repetitions,*
- *then choosing n_2 points with values greater or equal to those chosen for the first n_1 points,*
- *and so on until we choose n_k points with values greater or equal to those already chosen.*

To prove this, we first need to remember some definitions.

Consider $Poset+$ the category where objects are finite posets and morphisms f preserve order and satisfy that $x < y$ implies $f(x) \leq f(y)$.

We denote by $< n >$ the n-chain $1 < 2 < 3 \cdots < n$.

Lemma 1.2.

$$\#Hom_{Poset+}(< s >, < m >) = \left(\!\!\binom{m}{s}\!\!\right).$$

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Proof. Every map is determined by the values of the points. Any map give us a way to choose with repetitions s points out of m points. $\square$

Given a pair of elements f, g where $f \in Hom_{Poset+}(< s_1 >, < m_1 >)$ and $g \in Hom_{Poset+}(< s_2 >, < m_2 >)$ we construct an element

$$f *_+ g \in Hom_{Poset+}(< s_1 + s_2 >, < m_1 + m_2 - 1 >).$$

The rule is, we identify the largest point of m_1 with the minimum point of m_2.

Proof of Theorem 1.1. For every possible way to split the chain $< v >$ in k parts, the term $\prod_{i=1}^{k} \left(\!\binom{v_i}{n_i}\!\right)$ counts ways to choose n_i points out of v_i with repetitions. For each one of this choices we have an element in

$$Hom_{Poset+}(< v - k + 1 >, < n >).$$

However, note that will have repeated counting. To see this take the case $n = 2$, and $v_1 > 1, v_2 > 1$. In this case the pair of functions

$$f_1(1) = v_1, f_2(1) = 2$$

and the pair of functions

$$g_1(1) = v_1 - 1, g_2(1) - 1$$

satisfy

$$f_1 *_+ f_2 = g_1 *_+ g_2.$$

We now remove this redundancies: consider all k partitions of $v - 1$. choose $i \in \{1, \cdots k-1\}$ and we assume that there is repeating counting between $< v_i >$ and $< v_{i+1} >$. The term

$$-\binom{k-1}{1} \sum_{\substack{\sum v_i = v-1 \\ v_i \geq 1}} \prod_{i=1}^{k} \left(\!\binom{v_i}{n_i}\!\right)$$

stands for those functions with at least one redundancy. The remaining of the formula follows by inclusion-exclusion. $\square$

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References

[JAAN21] Marko Berghoff Jose Antonio Arciniega-Nevarez, Eric Dolores-Cuenca. Power series representing posets. 2021.