On regions of convergence of quaternionic hyperholomorphic functions.

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Abstract. We show that the natural region of convergence of quaternionic hyperholomorphic functions is not a sphere but a bigger region. We extend the theorem of Cauchy-Hadamard and the theorem of Abel on convergence of series to quaternionic analysis.

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1. Introduction

Let $\mathbb{H}$ denote the quaternion numbers. A function $f : \mathbb{H} \to \mathbb{H}$ is said to be (left) hyperholomorphic in a neighborhood V of the origin, if f is real differentiable on V and if $Df = 0$ when D is the Cauchy-Riemann-Fueter operator:

$$\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}.$$

Hyperholomorphic functions are zeros of the 4-th dimensional Laplacian as $\Delta = \bar{D}D$ where $\bar{D} = \frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}$.

Definition 1.0.1. *The Fueter's basis* [2] *is given by the hyperholomorphic functions* $\zeta_n : \mathbb{H} \to \mathbb{H}, n \in \{1, 2, 3\}$, *defined by*

$$\begin{aligned} \zeta_1(h) &= h_1 - ih_0, \\ \zeta_2(h) &= h_2 - jh_0, \\ \zeta_3(h) &= h_3 - kh_0. \end{aligned}$$

where $h = h_0 + ih_1 + jh_2 + kh_3$.

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The Taylor expansion of a (left) hyperholomorphic function f at the origin is given in terms of non-commutative polynomials

$$f(x) = \sum_{0}^{\infty} \sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} P_\nu a_\nu,$$

where

$$P_\nu = \frac{1}{n!} \sum_{(i_1,\cdots,i_n)\in A_\nu} \zeta_{i_1}\cdots\zeta_{i_n},$$

and the sum is over elements of A_ν, the set of all possible ways to multiply n_1 copies of ζ_1, n_2 copies of ζ_2 and n_3 copies of ζ_3 [4].

Definition 1.0.2. *We denote by* $||\,||' : \mathbb{H} \to \mathbb{R}$ *the norm*

$$||x_0 + ix_1 + jx_2 + kx_3||' = \max\{||x_0 + ix_1||, ||x_0 + jx_2||, ||x_0 + kx_3||\},$$

where $||r + ls|| = \sqrt{r^2 + s^2}$ *is the euclidean distance. We call* $||\,||'$ *the wispy norm.*

The corresponding balls $B(0, r) := \{x \in \mathbb{H}|\, ||x|| < r\}$ and $B'(0, r) := \{x \in \mathbb{H}|\, ||x||' < r\}$ satisfy

$$B(0, r) \subset B'(0, r).$$

Here $||\cdot||$ works on the underlying 4-dimensional vector space. On $i\mathbb{R}+j\mathbb{R}+k\mathbb{R}$ the norm $||\,||'$ becomes the maximum norm.

In this note we extend the following two theorems to quaternionic analysis:

Theorem 1.0.3. *(Quaternionic Cauchy-Hadamard Theorem)*

$$f(x) = \sum_{0}^{\infty} \sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} P_\nu a_\nu, \tag{1}$$

converges compactly on $B'(0, \frac{1}{\rho})$, *where*

$$\rho = \limsup_{k\to\infty}(\max_{||\nu||=k} ||a_\nu||)^{\frac{1}{k}}.$$

Proof. It follows from Lemma 4.0.3 and Lemma 4.0.4. □

Theorem 1.0.4. *(Quaternionic Abel Theorem) Suppose that there are constants* $r_0, M \in \mathbb{R}, N_0 \in \mathbb{R}$, *such that for all* $n > N_0$ *and multi indexes* ν *with* $||\nu|| = n$ *we have the bound* $||a_\nu||r_0^n \leq M$. *Under this hypothesis the series* (1) *converges compactly on* $B'(0, r_0)$.

Proof. See 4.0.1 □

2. Relation with other work

In this work we introduce the Wispy norm. It is motivated by the the algebraic structure of the Fueter basis. In page 180 of [3] they use the same coefficients as us to obtain the radius of convergence of a hyperholomorphic function but they use the inequality $||\zeta_{i_1}\cdots\zeta_{i_n}(x)|| \leq ||x||^n$. Since our norm is different, we both obtain the same radius but our region is bigger. We were able to generalize two classical theorems on convergence of series to the quaternionic setting, the closest result is in [4] where the weak Abel theorem is proven.

By choosing a different Fueter basis, the coefficients of the Taylor series change in a non trivial way, see [1]. We describe the importance of the basis on the last section.

3. Quaternionic Taylor Series

Given an infinite differentiable function $f : \mathbb{H} \to \mathbb{H}$, and a quaternion $h = h_0 + ih_1 + jh_2 + kh_3$, we formally consider the series:

$$T(f)(h) = \sum_{n=0}^{\infty} \frac{1}{n!}(h_0\frac{\partial}{\partial x_0} + h_1\frac{\partial}{\partial x_1} + h_2\frac{\partial}{\partial x_2} + h_3\frac{\partial}{\partial x_3})^n f|_{(0)}$$

Hyperholomorphy means that f satisfies:

$$\frac{\partial}{\partial x_0}f = -(i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3})f$$

and we can rewrite the n-derivative as:

$$\frac{1}{n!}\left((h_1 - ih_0)\frac{\partial}{\partial x_1} + (h_2 - jh_0)\frac{\partial}{\partial x_2} + (h_3 - kh_0)\frac{\partial}{\partial x_3}\right)^n f|_{(0)}$$

Here it is handy to use Fueter's basis and combinatorics:

$$\begin{aligned}\frac{1}{n!}\left(\zeta_1(h)\frac{\partial}{\partial x_1} + \zeta_2(h)\frac{\partial}{\partial x_2} + \zeta_3(h)\frac{\partial}{\partial x_3}\right)^n f|_{(0)} &= \\ &= \frac{1}{n!}\sum_{|\nu|=n}\sum_{(i_1,\cdots,i_n)\in A_\nu}\zeta_{i_1}\cdots\zeta_{i_n}a_\nu \\ &= \frac{1}{n!}\sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} P_\nu a_\nu,\end{aligned}$$

If we consider any complex analytic function $g(z) : \mathbb{C} \to \mathbb{C}$, then the Cauchy-Riemann-Fueter equation restricts to the usual Cauchy-Riemann equation and we get $\zeta_1 = y - ix = (x+iy)(-i) = z(-i)$,

$$\begin{aligned} T(g)(z) &= \sum_{n=0}^{\infty} \frac{1}{n!}(h_0 \frac{\partial}{\partial x} + h_1 \frac{\partial}{\partial y})^n g|_{(0)} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\zeta_1(h) \frac{\partial}{\partial y}\right)^n g|_{(0)} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(z(-i) \frac{\partial}{\partial y}\right)^n g|_{(0)} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(z \frac{\partial}{\partial z}\right)^n g|_{(0)} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} z^n \frac{\partial^n}{\partial z^n} g|_{(0)} \end{aligned}$$

4. Wispy norm.

The norm $||\,||' : \mathbb{H} \to \mathbb{R}$ defined as

$$||x_0 + ix_1 + jx_2 + kx_3||' = \max\{||x_0 + ix_1||, ||x_0 + jx_2||, ||x_0 + kx_3||\},$$

is motivated by the Fueter basis, it distingluish the nature of the input. For purely imaginary values we obtain $||xi + yj + zk||' = max\{|x|, |y|, |z|\}$ while $||r + xi||' = \sqrt{r^2 + x^2}$. So according to the real coefficients we interpolate between the Manhatan norm and the euclidean norm (poly cylinder). The shape of $\{x|\ ||x||' < 1\}$ is a 4 dimensional object whose 3-dim boundary contains cubes and poly cylinders.

From the following examples:

$$\begin{aligned} ||i(1+2j)||' &= ||i + 2k||' \\ &= 2 \\ &< ||i||'||1+2j||' \\ &= \sqrt{5} \\ &= ||1+2j||' \\ &= ||i^{-1}(i+2k)||' \\ &> ||i^{-1}||'||1+2k||' \\ &= 2 \end{aligned}$$

we conclude that in general $||xy||'$ cannot be compared with $||x||'||y||'$ as $||i(1+2j)||' < ||i||'||1+2j||'$ and $||i^{-1}(i+2k)||' > ||i^{-1}||'||1+2k||'$. We will only use the Wispy norm on cuaternions in the 2 dimensional subspaces generated by the real variable and one of the i, j, k imaginary variables. We won't have to work with the norm of a product of quaternions.

While studying domains of convergence, we frequently have to deal with the expression

$$\frac{||a_\nu||}{n!} \sum_{(i_1,\cdots,i_n)\in A_\nu} ||\zeta_{i_1}\cdots\zeta_{i_n}(x)||,$$

in [4] it is pointed that $||\zeta_{i_1}\cdots\zeta_{i_n}(x)|| \leq ||x||^n$. We obtain more precise results by considering $||\zeta_{i_1}\cdots\zeta_{i_n}(x)|| \leq (||x||')^n$, the norm $||\,||'$ allow us to follow naturally the Weierstrass approach and show that hyperholomorphic functions converge in a bigger domain.

Definition 4.0.1. *We say that series*

$$f(x) = \sum_0^\infty \sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} P_\nu a_\nu,$$

converges in the wispy sense at h if

$$Nf(x) = \lim_{n\to\infty} \sum_0^n \sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} ||\frac{a_\nu}{n!}|| \sum ||\zeta_{i_1}\cdots\zeta_{i_n}(x)|| < \infty.$$

For example, given

$$P_\nu = \frac{1}{n!} \sum_{(i_1,\cdots,i_n)\in A_\nu} \zeta_{i_1}\cdots\zeta_{i_n},$$

$$\begin{aligned} NP_{n_1,n_2,n_3}(x) &= \frac{1}{n_1+n_2+n_3\,!} \sum ||\zeta_{i_1}\cdots\zeta_{i_n}(x)|| && (2)\\ &\leq \frac{(||x||')^{n_1+n_2+n_3}}{n_1+n_2+n_3\,!} \binom{n_1+n_2+n_3}{n_1,n_2,n_3}. && (3) \end{aligned}$$

Theorem 4.0.2. *If $f(x) = \sum_0^\infty \sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} P_\nu a_\nu$, converges on the wispy sense at h then it converges compactly on $\{x|\ ||\zeta_1(x)|| \leq ||\zeta_1(h)||, ||\zeta_2(x)|| \leq ||\zeta_2(h)||, ||\zeta_3(x)|| \leq ||\zeta_3(h)||\}$*

Proof. It follows from Weiestrass M-test, see [4].

□

4.0.1. Examples. It is important to work with open convergence domains. As any holomorphic function induces a hyperholomorphic function, the series $\sum \zeta_1^n n^n + 2$ converges only on the plane $j\mathbb{R} + k\mathbb{R}$, where it has the constant value 2.

Lemma 4.0.3. *If $\rho = \limsup_{k\to\infty}(\max_{||\nu||=k} ||a_\nu||)^{\frac{1}{k}} = 0$ then (1) converges compactly for all $\mathbb{H}$.*

Proof. Let $x \in \mathbb{H} - \{0\}$. There is $N_0 \in \mathbb{N}$ such that for all $n > N_0$

$$(\max_{||\nu||=n} ||a_\nu||)^{\frac{1}{n}} \leq \frac{1}{3||x||'}.$$

Then for all $n > N_0$

$$\begin{aligned}\sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} N(P_\nu a_\nu) &\leq \sum_{n_1+n_2+n_3=n} \frac{||x||'^n}{n!(3||x||')^n}\binom{n}{n_1,n_2,n_3}\\ &= \frac{1}{3^n n!}\sum_{n_1+n_2+n_3=n}\binom{n}{n_1,n_2,n_3}\\ &= \frac{1}{n!}.\end{aligned}$$

On the first line we used (3). We conclude that

$$NFf(x) \leq NF(\sum_0^{N_0} \sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} P_\nu(x)a_\nu) + e.$$

Compactly convergence follows from convergence on the wispy sense according to Theorem 4.0.2. ☐

Lemma 4.0.4. *If* $0 < \rho < \infty$ *then* (1) *converges compactly for all* $h \in B'(0, \frac{1}{\rho})$.

Proof. Let $x \in B'(0, \frac{1}{\rho})$. Let $\theta = \sqrt{||x||'\rho} < 1$, then

$$\frac{\theta}{||x||'} = \frac{\rho}{\theta} > \rho,$$

we conclude that there is $N_0 \in \mathbb{N}$ such that for all $n > N_0$

$$||a_\nu||^{\frac{1}{n}} \leq \frac{\theta}{||x||'},\ ||\nu|| = n.$$

Then for $n > N_0$

$$\begin{aligned}\sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} N(P_\nu a_\nu) &\leq \sum_{n_1+n_2+n_3=n} \frac{||x||'^n}{n!}\frac{\theta^n}{||x||'^n}\binom{n}{n_1,n_2,n_3}\\ &= \frac{\theta^n}{n!}\sum_{n_1+n_2+n_3=n}\binom{n}{n_1,n_2,n_3}\\ &= \frac{(3\theta)^n}{n!}.\end{aligned}$$

We conclude that

$$NFf(x) \leq NF(\sum_0^{N_0} \sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} P_\nu(x)a_\nu) + e^3.$$

☐

From this two cases we conclude that:

Theorem. *(Quaternionic Cauchy-Hadamard Theorem)*

$$f(x) = \sum_{0}^{\infty} \sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} P_\nu a_\nu, \tag{4}$$

converges compactly on $B'(0, \frac{1}{\rho})$, *where*

$$\rho = \limsup_{k\to\infty} (\max_{||\nu||=k} ||a_\nu||)^{\frac{1}{k}}.$$

Theorem. *(Quaternionic Abel Theorem) Suppose that there are constants* $r_0, M \in \mathbb{R}, N_0 \in \mathbb{R}$, *such that for all* $n > N_0$ *and multi indexes* ν *with* $||\nu|| = n$ *we have the bound* $||a_\nu|| r_0^n \leq M$. *Under this hypothesis the series* $f(x) = \sum_0^\infty \sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} P_\nu a_\nu$ *converges compactly on* $B'(0, r_0)$.

Proof. Let $x \in \mathbb{H}$ with $||x||' = r < r_0$. Then for $n > N_0$

$$\begin{aligned}
\sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} N(P_\nu a_\nu) &\leq \sum_{n_1+n_2+n_3=n} \frac{(||x||')^n}{n!} ||a_\nu|| \binom{n}{n_1, n_2, n_3} \\
&\leq \sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} \frac{M}{n!} (\frac{r}{r_0})^n \binom{n}{n_1, n_2, n_3} \\
&\leq M \frac{3^n}{n!}.
\end{aligned}$$

Which give us

$$NFf(x) \leq NF(\sum_{0}^{N_0} \sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} P_\nu(x) a_\nu) + Me^3.$$

□

4.1. Domains of Convergence

For the function $f(x) = \sum \zeta_1^n a_n + \sum \zeta_2^n b_n + \sum \zeta_3^n c_n$ we consider $\rho_1 = \limsup_{k\to\infty} (||a_k||)^{\frac{1}{k}}, \rho_2 = \limsup_{k\to\infty} (||b_k||)^{\frac{1}{k}}, \rho_3 = \limsup_{k\to\infty} (||c_k||)^{\frac{1}{k}}$, then Theorem 1.0.3 guarantees convergence on $B'(0, \frac{1}{s}), s = \max\{\rho_1, \rho_2, \rho_3\}$. A bigger domain of convergence is

$$\{||x_0 + ix_1|| < \frac{1}{\rho_1}, ||x_0 + jx_2|| < \frac{1}{\rho_2}, ||x_0 + kx_3|| < \frac{1}{\rho_3}\}.$$

The domain of convergence of the function $f(x) = \sum \zeta_1^{2^n} + \sum \zeta_2^{2^n} + \sum \zeta_3^{2^n}$ is $B'(1)$. In general, if we change the basis of Fueter and consider the Taylor expansion we will find the radius of the largest ball B' that can be embedded in the domain of convergence.

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