On regions of convergence of quaternionic hyperholomorphic functions.

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Abstract. We extend the theorem of Cauchy-Hadamard and the theorem of Abel on convergence of series to quaternionic analysis. We find a region of convergence, a polycylinder, that is more than two times bigger than the previously reported in the literature.

1. Introduction

Let $\mathbb{H}$ denote the quaternion numbers. A function $f : \mathbb{H} \to \mathbb{H}$ is said to be (left) hyperholomorphic in a neighborhood V of the origin, if f is real differentiable on V and if $Df = 0$ when D is the Cauchy-Riemann-Fueter operator:

$$\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}.$$

Hyperholomorphic functions are zeros of the 4-th dimensional Laplacian as $\Delta = \bar{D}D$ where $\bar{D} = \frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}$.

Definition 1.0.1. *The Fueter's basis* [2] *is given by the hyperholomorphic functions* $\zeta_n : \mathbb{H} \to \mathbb{H}, n \in \{1, 2, 3\}$, *defined by*

$$\begin{aligned} \zeta_1(h) &= h_1 - ih_0, \\ \zeta_2(h) &= h_2 - jh_0, \\ \zeta_3(h) &= h_3 - kh_0, \end{aligned}$$

where $h = h_0 + ih_1 + jh_2 + kh_3$.

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Mathematics subject classification 30G35, 32A30 and 32A07.
Keywords. Fueter variables, hyperholomorphic functions, Taylor series, regions of convergence, quaternionic analysis.

The Taylor expansion of a (left) hyperholomorphic function f at the origin is given in terms of non-commutative polynomials

$$f(x) = \sum_{0}^{\infty} \sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} P_\nu a_\nu,$$

where

$$P_\nu = \frac{1}{n!} \sum_{(i_1,\cdots,i_n)\in A_\nu} \zeta_{i_1}\cdots\zeta_{i_n},$$

and the sum is over $\vec{\nu} \in A_\nu$, the set of all possible ways to multiply n_1 copies of ζ_1, n_2 copies of ζ_2 and n_3 copies of ζ_3, see [4].

Definition 1.0.2. *We denote by* $||\,||' : \mathbb{H} \to \mathbb{R}$ *the norm*

$$||x_0 + ix_1 + jx_2 + kx_3||' = \max\{||x_0 + ix_1||, ||x_0 + jx_2||, ||x_0 + kx_3||\},$$

where $||r + ls|| = \sqrt{r^2 + s^2}, l \in \{i, j, k\}$ *is the euclidean distance. We call* $||\,||'$ *the Wispy norm.*

The corresponding regions

$$B(0, r) := \{x \in \mathbb{H}| \, (\sum_{0}^{3} x_i^2)^{1/2} < r\}, B'(0, r) := \{x \in \mathbb{H}| \, ||x||' < r\}$$

satisfy

$$B(0, r) \subset B'(0, r).$$

For example $.9i + .9j \in B'(0, 1)$ but $.9i + .9j \notin B(0, 1)$. In this note we extend the following theorems to quaternionic analysis:

Theorem. *(Quaternionic Abel Theorem) Suppose that there are constants* $r_0, M \in \mathbb{R}, N_0 \in \mathbb{R}$, *such that for all* $n > N_0$ *and multi indexes* ν *with* $||\nu|| = n$; *we have the bound* $||a_\nu||r_0^n \leq M$. *Under this hypothesis the series*

$$f(x) = \sum_{0}^{\infty} \sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} P_\nu a_\nu, \tag{1}$$

converges compactly on $B'(0, r_0)$.

Proof. See Theorem 5.0.3 □

Theorem. *(Quaternionic Cauchy-Hadamard Theorem)*
Let

$$\sigma = \limsup_{n} (\sum_{||\nu||=n, \vec{\nu}\in A_\nu} ||a_{\vec{\nu}}/n!||)^{1/n}$$

then (1) *converges compactly for all* $h \in B'(0, \frac{1}{\sigma})$.

Proof. See Theorem 5.0.5. □

In some situations, it is better to use the following version to compute regions of convergence.

Theorem. *(Weaker Quaternionic Cauchy-Hadamard Theorem)*
Let

$$\rho = \limsup_{k\to\infty} (\max_{\|\nu\|=k} \|a_\nu/n!\|)^{\frac{1}{k}}$$

and

$$\tau = \limsup_{n} (\#\{a_{\vec{\nu}} \neq 0\}_{\|\nu\|=n, \vec{\nu}\in A_\nu})^{1/n},$$

then (1) converges compactly on $B'(0, \frac{1}{\tau\rho})$.

Proof. See Lemma 5.0.4 and Theorem 5.0.6. □

2. Notation

Given n we consider indexes $\nu = (r, s, t) \in R^3$ with $|\nu| = r+s+t = n$. For each $\nu = (r, s, t)$ we consider $A_\nu \in R^{r+s+t}$ the set of indexes that parametrizes words with r variables ζ_1, s variables ζ_2 and t variables ζ_3. An element of A_ν is denoted by $\vec{\nu} = (i_1, i_2, \cdots, i_{r+s+t})$, with $(i_1, i_2, \cdots, i_{r+s+t})$ a permutation of $(1, \ldots, 1, 2, \ldots, 2, 3, \ldots, 3)$ with r 1's, s 2's and t 3's.

$\{a_\nu\}_{\{\|\nu\|=k\}}$ are the coefficients of homogeneous polynomials P_ν of degree k. Sometimes, it will be necessary to decompose $P_\nu a_\nu$ into

$$\frac{1}{n!} \sum_{\vec{\nu}=(i_1,\cdots,i_n)\in A_\nu} \zeta_{i_1}\cdots\zeta_{i_n} a_{\vec{\nu}},$$

here we denote by $a_{\vec{\nu}}$ the copy of a_ν that is coefficient of $\zeta_{i_1}\cdots\zeta_{i_n}$ when $\vec{\nu} = (i_1, \cdots, i_n)$. We denote by $\{a_{\vec{\nu}}\}_{\{\|\nu\|=n, \vec{\nu}\in A_\nu\}}$ the set of all the coefficients of the monomials (in $\zeta's$) of degree n.

3. Relation with other work

As far as the author is aware, Theorem 1 is proven here for the first time in the Quaternionic sense. A weak version of Abel lemma, which is closely related, has been proved in [[4], equation (18)] using estimations in terms of $\|\zeta_i\|$.

Our radius of convergence differs from [[3], page 168] because we consider the wispy norm $\|x\|'$ and the coefficients of the products of Fueter variables while they consider the norm of the quaternion $\|x\|$ and the coefficients of monomials in real variables. The key difference is in the step $\|\zeta_{i_1}\cdots\zeta_{i_n}\| \leq \sum \|x_{i_1}\cdots x_{i_n}\| \leq 2^n\|x\|^n$, instead we consider $\|\zeta_{i_1}\cdots\zeta_{i_n}\| \leq (\|x\|')^n$. Going from the Fueter variables to the real variables always adds a 2^n coefficient. We conclude that our radius is twice as big. Even for the same radius our norm gives larger regions as we will exemplify in section 5.1.4.

In [5] they introduce the GHR calculus, by parametrizing derivatives with ortonormal basis of quaternions. They work with arbitrary real differentiable functions and they derive a Taylor Series in powers of the variable q. By using those Taylor coefficients we can get information about convergence in Euclidean Balls. In our case, we restrict to the kernel of the Cauchy-Riemann

Operator, $\frac{\partial_r f}{\partial q^{\{1,i,j,k\}*}} = 0$ in their notation, we use a Taylor decomposition in terms of the Basis of Fueter and this lead us to a detailed description of regions of convergence and a computation of a radious of convergence in terms of the wispy norm.

4. CRF Taylor Series

Given an infinite differentiable function $f : \mathbb{H} \to \mathbb{H}$, and a quaternion $h = h_0 + ih_1 + jh_2 + kh_3$, we formally consider the Cauchy-Riemman Fueter series:

$$T(f)(h) = \sum_{n=0}^{\infty} \frac{1}{n!}(h_0 \frac{\partial}{\partial x_0} + h_1 \frac{\partial}{\partial x_1} + h_2 \frac{\partial}{\partial x_2} + h_3 \frac{\partial}{\partial x_3})^n f|_{(0)}.$$

Hyperholomorphy means that f satisfies:

$$\frac{\partial}{\partial x_0} f = -(i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} + k \frac{\partial}{\partial x_3}) f$$

and we can rewrite the n-derivative as:

$$\frac{1}{n!} \left((h_1 - ih_0) \frac{\partial}{\partial x_1} + (h_2 - jh_0) \frac{\partial}{\partial x_2} + (h_3 - kh_0) \frac{\partial}{\partial x_3} \right)^n f|_{(0)}.$$

Here, it is handy to use Fueter's basis and combinatorics:

$$\begin{aligned} \frac{1}{n!} \left(\zeta_1(h) \frac{\partial}{\partial x_1} + \zeta_2(h) \frac{\partial}{\partial x_2} + \zeta_3(h) \frac{\partial}{\partial x_3} \right)^n f|_{(0)} &= \\ = \frac{1}{n!} \sum_{|\nu|=n} \sum_{\vec{\nu}=(i_1,\cdots,i_n)\in A_\nu} \zeta_{i_1} \cdots \zeta_{i_n} a_{\vec{\nu}} & \\ = \sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} P_\nu a_\nu. & \end{aligned}$$

5. Wispy norm.

The norm $|| \, ||' : \mathbb{H} \to \mathbb{R}$ is defined as

$$||x_0 + ix_1 + jx_2 + kx_3||' = \max\{||x_0 + ix_1||, ||x_0 + jx_2||, ||x_0 + kx_3||\},$$

this is motivated by the Fueter basis. For purely imaginary values, we obtain $||xi + yj + zk||' = max\{|x|, |y|, |z|\}$ while for complex numbers, we obtain $||r + xi||' = \sqrt{r^2 + x^2}$. The shape of $\{x| \; ||x||' < 1\}$ is a 4 dimensional object whose 3-dim boundary contains cubes and polycylinders.

From the following examples:

$$\begin{aligned} ||i(1+2j)||' &= ||i+2k||' \\ &= 2 \\ &< ||i||'||1+2j||' \\ &= \sqrt{5} \\ &= ||1+2j||' \\ &= ||i^{-1}(i+2k)||' \\ &> ||i^{-1}||'||1+2k||' \\ &= 2 \end{aligned}$$

we conclude that in general $||xy||'$ cannot be compared with $||x||'||y||'$ as $||i(1+2j)||' < ||i||'||1+2j||'$ and $||i^{-1}(i+2k)||' > ||i^{-1}||'||1+2k||'$. This won't affect our calculations because in this paper we won't have to work with the Wispy norm of a product of quaternions.

Definition 5.0.1. *We say that the series*

$$f(x) = \sum_0^{\infty} \sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} P_\nu a_\nu,$$

converges in the wispy sense at h if

$$Nf(x) = \lim_{n\to\infty} \sum_0^{n} \sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} ||\frac{a_\nu}{n!}|| \sum ||\zeta_{i_1}\cdots\zeta_{i_n}(x)|| < \infty.$$

For example, given

$$n!P_\nu = \sum_{(i_1,\cdots,i_n)\in A_\nu} \zeta_{i_1}\cdots\zeta_{i_n},$$

$$N(n!P_{n_1,n_2,n_3}(x)) = \sum ||\zeta_{i_1}\cdots\zeta_{i_n}(x)|| \tag{2}$$

$$\leq (||x||')^{n_1+n_2+n_3} \binom{n_1+n_2+n_3}{n_1,n_2,n_3}. \tag{3}$$

Theorem 5.0.2. *If $f(x) = \sum_0^\infty \sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} P_\nu a_\nu$ converges on the wispy sense at h then it converges compactly on*

$$\{x|\ ||\zeta_1(x)|| \leq ||\zeta_1(h)||, ||\zeta_2(x)|| \leq ||\zeta_2(h)||, ||\zeta_3(x)|| \leq ||\zeta_3(h)||\}.$$

Proof. It follows from Weiestrass M-test, see [[4], Theorem 3].

□

Theorem 5.0.3. *(Quaternionic Abel Theorem) Suppose that there are constants $r_0, M \in \mathbb{R}, N_0 \in \mathbb{R}$, such that for all $n > N_0$ and multi indexes ν with $||\nu|| = n$ we have the bound $||a_\nu||r_0^n \leq M$. Under this hypothesis the series $f(x) = \sum_0^\infty \sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} P_\nu a_\nu$ converges compactly on $B'(0,r_0)$.*

Proof. Let $x \in \mathbb{H}$ with $||x||' = r < r_0$. Then for $n > N_0$ and by using (3):

$$\begin{aligned}
\sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} N(P_\nu a_\nu) &\leq \sum_{n_1+n_2+n_3=n} \frac{(||x||')^n}{n!}||a_\nu||\binom{n}{n_1,n_2,n_3} \\
&\leq \sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} \frac{r^n}{n!}(\frac{M}{r_0^n})\binom{n}{n_1,n_2,n_3} \\
&\leq M\frac{3^n}{n!}.
\end{aligned}$$

Which give us

$$NFf(x) \leq NF(\sum_0^{N_0} \sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} P_\nu(x)a_\nu) + Me^3.$$

□

The previous theorem depends on the properties of $\|a_\nu\|$ while the following theorems depend on the properties of $\|a_\nu/n!\|$ leading to more general results.

Lemma 5.0.4. *If*

$$\rho = \limsup_{n\to\infty}(\max_{||\nu||=n} ||a_\nu/n!||)^{\frac{1}{k}} = 0$$

then

$$f(x) = \sum_0^\infty \sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} P_\nu a_\nu,$$

converges compactly for all $\mathbb{H}$.

Proof. Let $x \in \mathbb{H} - \{0\}$. There is $N_0 \in \mathbb{N}$ such that for all $n > N_0$

$$(\max_{||\nu||=n} ||a_\nu/n!||)^{\frac{1}{n}} \leq \frac{1}{6||x||'}.$$

Then for all $n > N_0$:

$$\begin{aligned}
\sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} N(P_\nu a_\nu) &\leq \sum_{n_1+n_2+n_3=n} \frac{||x||'^n}{(6||x||')^n}\binom{n}{n_1,n_2,n_3} \\
&= \frac{1}{6^n}\sum_{n_1+n_2+n_3=n}\binom{n}{n_1,n_2,n_3} \\
&= 1/2^n.
\end{aligned}$$

We conclude that

$$NFf(x) \leq NF(\sum_0^{N_0} \sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} P_\nu(x)a_\nu) + 2.$$

Compactly convergence follows from convergence on the wispy sense according to Theorem 5.0.2. □

The following theorem is useful in cases where the magnitude of the coefficients of the homogeneous components of the Taylor series vary in magnitude or several of them have absolute values smaller than 1.

In the following results, we consider the sum $\sum_{|\nu|=n,\vec{\nu}\in A_\nu} a_{\vec{\nu}}$, which means that we consider the coefficient a_ν for every product of zetas in P_ν. For $6P_{(1,1,0)} = 3(\zeta_1\zeta_2 + \zeta_2\zeta_1)$, we are considering the coefficient 3 twice, a copy for $\zeta_1\zeta_2$ and another for $\zeta_2\zeta_1$.

Theorem 5.0.5. *(Quaternionic Cauchy-Hadamard Theorem)*

Let

$$\sigma = \limsup_n \Big(\sum_{|\nu|=n,\vec{\nu}\in A_\nu} |a_{\vec{\nu}}/n!| \Big)^{1/n}$$

then

$$f(x) = \sum_0^\infty \sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} P_\nu a_\nu,$$

converges compactly for all $h \in B'(0, \frac{1}{\sigma})$.

Proof. If $\sigma = 0$ then $\rho = 0$, thus we can apply Lemma (5.0.4). Assuming $\sigma \neq 0$, let $x \in B'(0, \frac{1}{\sigma})$ and let $\theta = \sqrt{||x||'\sigma} < 1$, then

$$\frac{\theta}{||x||'} = \frac{\sigma}{\theta} > \sigma,$$

we conclude that there is $N_0 \in \mathbb{N}$ such that for all $n > N_0$:

$$\Big(\sum_{|\nu|=n} ||a_\nu/n!|| \Big)^{\frac{1}{n}} \leq \frac{\theta}{||x||'}.$$

Then for $n > N_0$:

$$\begin{aligned} \sum_{a_\nu\neq 0,|\nu|=n} N(P_\nu a_\nu) &\leq ||x||'^n \sum_{a_\nu\neq 0,|\nu|=n} \frac{||a_\nu||}{n!} \\ &\leq ||x||'^n \frac{\theta^n}{||x||'^n} \\ &= \theta^n. \end{aligned}$$

We conclude that

$$NFf(x) \leq NF\Big(\sum_0^{N_0} \sum_{\substack{\nu=(n_1,n_2,n_3)\\ n_1+n_2+n_3=n}} P_\nu(x)a_\nu\Big) + 1/(1-\theta).$$

□

To consider the sum of the absolute values of the coefficients is convenient when some of those coefficients are very small. For example, at each degree we can ignore those smaller than a certain threshold. The next theorem is a more practical way to compute the radius when it is easier to consider the maximum absolute value of the coefficients multiplied by the number of non zero coefficients.

We introduce

$$\tau = \limsup_{n} (\#\{a_{\vec{\nu}} \neq 0\}_{|\nu|=n, \vec{\nu} \in A_\nu})^{1/n},$$

For polynomials $\tau = 0$. We are interested in the case $1 \leq \tau \leq 3$. Any holomorphic function is an example of a series with $\tau = 1$. The hyperholomorphic function (4) has $\tau = 3$.

Theorem 5.0.6. *(Weak Quaternionic Cauchy-Hadamard Theorem) Let*

$$\rho = \limsup_{n \to \infty} (\max_{||\nu||=n} ||a_\nu / n!||)^{\frac{1}{k}}, 0 \leq \rho < \infty$$

and

$$\tau = \limsup_{n} (\#\{a_{\vec{\nu}} \neq 0\}_{|\nu|=n, \vec{\nu} \in A_\nu})^{1/n}$$

then

$$f(x) = \sum_{0}^{\infty} \sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} P_\nu a_\nu,$$

converges compactly for all $h \in B'(0, \frac{1}{\tau\rho})$.

Proof. Lemma (5.0.4) considers the case $\rho = 0$. Let $x \in B'(0, \frac{1}{\tau\rho})$. Let $\theta = \sqrt{||x||'\tau\rho} < 1$, then

$$\frac{\theta}{\tau ||x||'} = \frac{\rho}{\theta} > \rho,$$

we conclude that there is $N_0 \in \mathbb{N}$ such that for all $n > N_0$

$$||a_\nu / n!||^{\frac{1}{n}} \leq \frac{\theta}{\tau ||x||'}, \ ||\nu|| = n.$$

Since

$$\frac{\tau}{\theta^{1/2}} > \tau,$$

we can find $M_0 > 0$ so that for all $M > M_0$:

$$\#\{a_{\vec{\nu}} | a_{\vec{\nu}} \neq 0, |\nu| = M, \vec{\nu} \in A_\nu\} < (\tau / \theta^{1/2})^M.$$

Then for $n > \max\{N_0, M_0\}$:

$$\begin{aligned}
\sum_{a_\nu \neq 0, |\nu|=n} N(P_\nu a_\nu) &\leq \sum_{a_\nu \neq 0, |\nu|=n} ||x||'^n \frac{||a_\nu||}{n!} \\
&\leq \sum_{a_\nu \neq 0, |\nu|=n} ||x||'^n \frac{\theta^n}{||x||'^n \tau^n} \\
&= (\tau/\theta^{1/2})^n (\theta/\tau)^n \\
&= \theta^{n/2}.
\end{aligned}$$

We conclude that

$$NFf(x) \leq NF(\sum_0^{N_0} \sum_{\substack{\nu=(n_1,n_2,n_3) \\ n_1+n_2+n_3=n}} P_\nu(x) a_\nu) + 1/(1-\sqrt{\theta}).$$

□

5.1. Concluding remarks

5.1.1. Hyperholomorphic and Holomorphic Taylor expansions. If we consider any complex analytic function $g(z) : \mathbb{C} \to \mathbb{C}$, then the Cauchy-Riemann-Fueter equation restricts to the usual Cauchy-Riemann equation and we get $\zeta_1 = y - ix = (x+iy)(-i) = z(-i)$,

$$\begin{aligned}
T(g)(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} (h_0 \frac{\partial}{\partial x} + h_1 \frac{\partial}{\partial y})^n g|_{(0)} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\zeta_1(h) \frac{\partial}{\partial y} \right)^n g|_{(0)} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(z(-i) \frac{\partial}{\partial y} \right)^n g|_{(0)} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(z \frac{\partial}{\partial z} \right)^n g|_{(0)} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} z^n \frac{\partial^n}{\partial z^n} g|_{(0)}
\end{aligned}$$

5.1.2. Change of basis. Different basis can be used to obtain a Taylor expansion of a hyperholomorphic function, see for example [1]. In terms of theorem 5.0.6, a change of Fueter basis will vary the parameter τ because in the new basis we will obtain a rotated Wispy ball and that region should be contained in the maximal region of convergence. The radius ρ will decrease or increase accordingly. For example, let $\sum (x_i - kx_j)^n = \sum (\zeta_i - \zeta_j k)^n$. This series converges on a 'tubular' region of those quaternions with $\|x_i + kx_j\| < 1$. Notice that $B'(0,1)$ contains points outside of $\|x_i + kx_j\| < 1$ as $.9j + .9k$. If we write $\sum (x_i + kx_j)^n$ in the Fueter basis then the number of coefficients $a_{\vec{\nu}}$ increases and so does τ, allowing $B'(0, 1/\tau)$ to be contained in $\|x_i + kx_j\| < 1$.

Note that in the computation of t, for every degree, we work with a linear combination of binomial coefficients.

The expression $\sum(\zeta_i + \zeta_j k)^n$ contains terms of the form $\zeta_i k \zeta_j$, which are not hyperholomorphic. Thus, by only expanding $\sum(\zeta_i + \zeta_j k)^n$ will not lead to the hyperholomorphic expression. With some algebra we can find $(\zeta_i + \zeta_j k)^2 = \zeta_i^2 - \zeta_j^2 + (\zeta_i\zeta_j + \zeta_j\zeta_i)k$.

5.1.3. Examples of domains. It is important to work with open domains. As any holomorphic function induces a hyperholomorphic function, the series $\sum \zeta_1^n n^n + 2$ converges only on the plane $j\mathbb{R} + k\mathbb{R}$, where it has the constant value 2.

Consider $\sum \zeta_1^n a_n + \sum \zeta_2^n b_n + \sum \zeta_3^n c_n$ with $\rho_1 = \limsup_{k\to\infty}(||a_k||)^{\frac{1}{k}}$, $\rho_2 = \limsup_{k\to\infty}(||b_k||)^{\frac{1}{k}}$, $\rho_3 = \limsup_{k\to\infty}(||c_k||)^{\frac{1}{k}}$; then Theorem 5.0.6 guarantees that $f(x) = \sum \zeta_1^n a_n + \zeta_2^n b_n + \zeta_3^n c_n$ convergences on $B'(0, \frac{1}{s})$, $s = \max\{\rho_1, \rho_2, \rho_3\}$. On the other hand, Theorem 5.0.2 give us a bigger domain of convergence

$$\{||x_0 + ix_1|| < \frac{1}{\rho_1}, ||x_0 + jx_2|| < \frac{1}{\rho_2}, ||x_0 + kx_3|| < \frac{1}{\rho_3}\}.$$

Here is an example when the domain of convergence of the function is exactly a poly-cylinder $f(x) = \sum \zeta_1^{2^n} + \zeta_2^{2^n} + \zeta_3^{2^n}$. And here is an example when the radius of convergence is not rational: let's consider $\nu_k = (4k, k, k)$, then using Stirling formula we obtain

$$\lim_{k\to\infty} \binom{6k}{4k, k, k}^{\frac{1}{6k}} = \frac{3}{2^{\frac{1}{3}}},$$

and so the series $\sum_{k=0}^{\infty} P_{\nu_k}$ has radius of convergence $2^{1/3}/3$.

5.1.4. Explicit comparison with the current methods. In [[3], page 168] the radius of convergence of a hyperholomorphic function is given in terms of the real variables $|x_i|$, meaning that we consider the function

$$\sum_n \sum_{|\nu|=n, \vec{\nu}\in A_\nu} \|a_{\vec{\nu}}\| |\vec{x}_\nu|,$$

where ν runs over all words with size n and 3 letters, $\vec{x}_\nu = \prod_{i\in\nu} x_i$ and $|\vec{x}_\nu| = |\prod_{i\in\nu} x_i|$ with x_j real variables. Then, if $A_n = \sum_{|\nu|=n,\vec{\nu}\in A_\nu} \|a_{\vec{\nu}}\|$, they deduce that

$$\|f(x)\| \leq \sum_n \sum_{|\nu|=n,\vec{\nu}\in A_\nu} \|a_{\vec{\nu}}\| |\vec{x}_\nu| \leq \|x\|^k \sum_n A_k$$

as $|x_i| \leq \|x\|$. Convergence is guarantee on the euclidean ball $B(\rho)$ where $1/\rho = \limsup_{k\to\infty} \|A_k\|^{1/k}$.

For example given the function

$$\sum_0^\infty \sum_{|\nu|=n} n! P_\nu = \sum_0^\infty \sum_{|\nu|=n} \sum_{(i_1,\cdots,i_n)\in A_\nu} \zeta_{i_1} \cdots \zeta_{i_n} \tag{4}$$

we formally test their procedure by rewriting each homogeneous polynomial in real variables x_i, by noticing that $|x_i| \leq \|x\|$ and $\|\zeta_{i_1}\cdots\zeta_{i_n}\| \leq \sum |x_{i_1}\cdots x_{i_n}| \leq 2^n\|x\|^n$.

$$\|\sum_0^\infty \sum_{|\nu|=n} \sum_{(i_1,\cdots,i_n)\in A_\nu} \zeta_{i_1}\cdots\zeta_{i_n}\| \leq \sum_0^\infty \sum_{|\nu|=n} \sum_{(i_1,\cdots,i_n)\in A_\nu} \|\zeta_{i_1}\cdots\zeta_{i_n}\| \quad (5)$$

$$\leq \sum_0^\infty \sum_{|\nu|=n} \sum_{(i_1,\cdots,i_n)\in A_\nu} 2^n\|x\|^n \quad (6)$$

$$\leq \sum_0^\infty \sum_{|\nu|=n,\nu=(r,s,t)} \binom{n}{r\,s\,t} 2^n\|x\|^n \quad (7)$$

$$\leq \sum_0^\infty 3^n 2^n\|x\|^n, \quad (8)$$

their procedure give us the bound $\sum \|X\|^n 6^n$ and so $\rho = 1/\limsup_{k\to\infty} |6^n|^{1/n} = 1/6$ while Theorem 5.0.6 determines the ball $B'(1/3) \supset B(1/6)$ with $\rho = 1, \tau = 3$, and Theorem 5.0.5 also gives $B'(1/3)$ since it works at the level of (5).

6. Acknowledgements

The author gratefully acknowledge the support from Dr. Mykhaylo Shapiro Fishman and from Dr. Jose Mendoza-Cortez, who reviewed this paper. I am responsible for mistakes still present in this version.

The author thank CONACyT for providing the undergraduate Fellowship PIFI project cgip 20070253.

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