

## readme

## Lab 7

Goal: to practice proving asymptotic (big-Oh) results. The class notes from Lecture 20 may come in handy; however, we will try to be even more rigorous here than in lecture.

Recall the definition of big-Oh:

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| O(f(n)) is the SET of ALL functions T(n) that satisfy:
|
|   There exist positive constants c and N such that, for all n >= N,
|               T(n) <= c f(n)
|=====
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## EXAMPLE

Formally prove that  $n^2 + n + 1$  is in  $O(n^2)$ .

Solution:

Let  $T(n) = n^2 + n + 1$ . Let  $f(n) = n^2$ .

Choose  $c = 3$ , and  $N = 1$ . Then, we know  $T(n)$  is in  $O(n^2)$  if we can prove

$$\begin{array}{lcl} T(n) <= c f(n), & & \\ \text{or equivalently, } n^2 + n + 1 <= 3 n^2, & \text{for all } n \geq 1. \end{array}$$

Is this inequality true? Well, for any  $n \geq 1$ , we know that  $1 \leq n \leq n^2$ . Hence, all of the following are true:

$$\begin{array}{l} 1 \leq n^2 \\ n \leq n^2 \\ n^2 = n^2 \end{array}$$

Adding the left and right sides of these inequalities together, we have

$$n^2 + n + 1 \leq 3 n^2, \text{ which completes the proof.}$$

Part I: (1 point)

Formally prove that  $2^n + 1$  is in  $O(4^n - 16)$ .

HINT: You may assert without proof that, for all  $n \geq 1$ ,  $2^n \geq 1$ . (You may also assert without proof that  $4^n$  and  $2^n$  are monotonically increasing, if you find it useful.)

Part II: (1 point)

Formally prove that if  $f(n)$  is in  $O(g(n))$ , and  $g(n)$  is in  $O(h(n))$ , then  $f(n)$  is in  $O(h(n))$ .

NOTE: The values of  $c$  and  $N$  used to prove that  $f(n)$  is in  $O(g(n))$  are not necessarily the same as the values used to prove that  $g(n)$  is in  $O(h(n))$ . Hence, assume that there are positive  $c'$ ,  $N'$ ,  $c''$ , and  $N''$  such that

$$\begin{array}{ll} f(n) \leq c' g(n) & \text{for all } n \geq N', \text{ and} \\ g(n) \leq c'' h(n) & \text{for all } n \geq N''. \end{array}$$

Part III: (2 points)

Formally prove that  $0.01 n^2 - 1$  is NOT in  $O(n)$ .

We need to show that, no matter how large we choose  $c$  and  $N$ , we will never obtain the desired inequality. We cannot prove this by picking a specific value of  $c$  and  $N$ . Instead, we must study how the two functions behave as  $n$  approaches infinity.

Let  $T(n) = 0.01 n^2 - 1$ , and let  $f(n) = n$ . Prove that

$$\lim_{n \rightarrow \infty} \frac{c f(n)}{T(n)} = 0,$$

no matter how large we choose  $c$  to be. You will need to scale both the numerator and the denominator by a well-chosen multiplier to get the result.

Use this result to show that there are no values  $c$ ,  $N$  such that  $T(n) \leq c f(n)$  for all  $n \geq N$ .

Postscript

The functions  $|\cos(n)|$  and  $|\sin(n)|$  are interesting, because neither is dominated by the other. Can you informally suggest why  $|\cos(n)|$  is not in  $O(|\sin(n)|)$ , and  $|\sin(n)|$  is not in  $O(|\cos(n)|)$ ?

How would you prove that, for all  $n \geq 1$ ,  $2^n \geq 1$ ? (Hint: use calculus.)