Magnitude Response for a Given Frequency

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Learning Objectives

By the end of this topic, you should be able to:

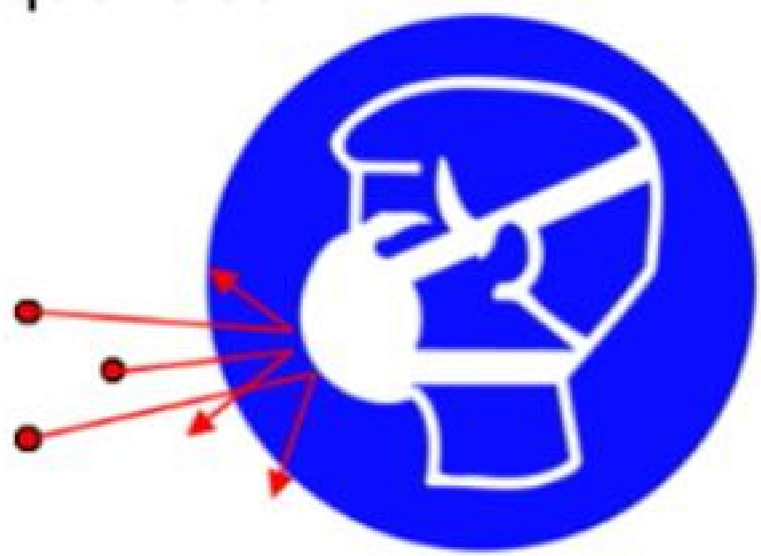
- Analyse a Filter's Magnitude Response;
- Explain the Role of Magnitude Response for a Given Frequency; and
- Solve Magnitude Response for a Given Frequency.

Analyse a Filter's Magnitude Response

What is a Filter?

A filter selectively passes some components but stops other components.

In Signal Processing, a filter is a Linear Time Invariant (LTI) system that selectively passes some frequencies, stops other frequencies.

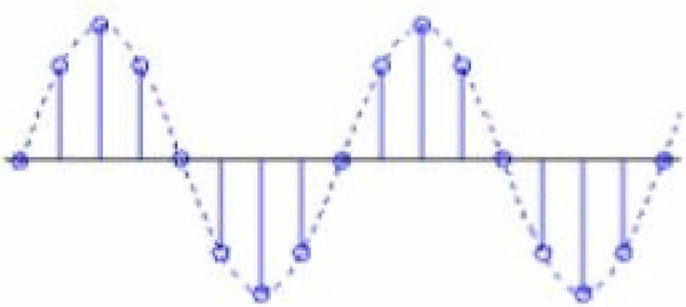


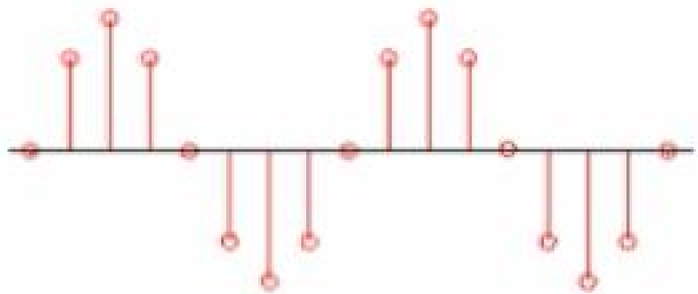
Analyse a Filter's Magnitude Response

- Consider a filter that passes the frequency $\pi/4$ but stops the frequency $\pi/2$.

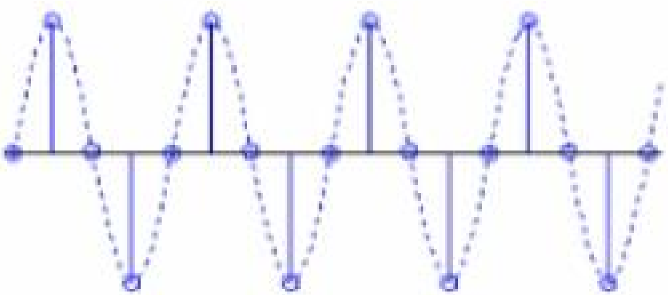
Click to explore

$h[n] = [0.7071\quad \underset{\uparrow}{0}\quad 0.7071]$ is one such filter.





Input = $\sin(0.25\pi n)$ → filter → Output = $\sin(0.25\pi n)$



Input = $\sin(0.5\pi n)$ → filter → Output = 0

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Analyse a Filter's Magnitude Response

How is the filter of example 1 found?

Let $h[n] = [h_{-1} \;\; h_0 \;\; h_1]$

Then $H(e^{j\omega}) = h_{-1}e^{j\omega} + h_0 + h_1 e^{-j\omega}$

Find a filter that passes the frequency $\pi/4$ but stops the frequency $\pi/2$

At $\omega = \pi/2$, $H(e^{j\pi/2}) = 0$

$\Rightarrow jh_{-1} + h_0 - jh_1 = 0$

$\Rightarrow h_0 = 0,\ h_{-1} = h_1$

At $\omega = \pi/4$, $|H(e^{j\pi/4})| = 1$

$\Rightarrow |h_1 e^{j\pi/4} + 0 + h_1 e^{-j\pi/4}| = 1$

$\Rightarrow h_1 = 1/2\cos(\pi/4) = 1/\sqrt{2}$

Thus $h[n] = \left[\frac{1}{\sqrt{2}} \;\; 0 \;\; \frac{1}{\sqrt{2}}\right]$

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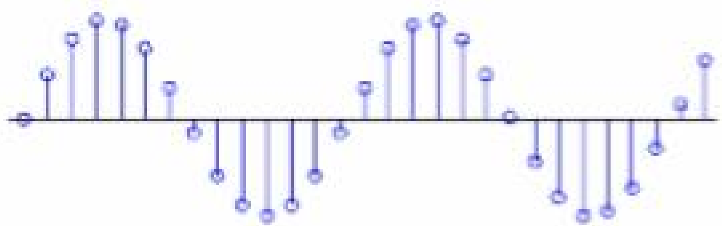
Role of Magnitude Response for a Given Frequency

- Consider the magnitude response of the filter:

$$X(e^{j\omega}) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow |Y(e^{j\omega})| = |H(e^{j\omega})|.|X(e^{j\omega})|$$

Input $x[n]$ is a sinusoid of frequency ω_0



→ $X(e^{j\omega})$ is nonzero only at $\pm\,\omega_0$

$$|Y(e^{j\omega})| = |H(e^{j\omega_0})|.|X(e^{j\omega})|$$
$$= k.|X(e^{j\omega})|$$

where $k = |H(e^{j\omega_0})|$ is a constant

→ output = input scaled by k

→ output $y[n]$ is scaled sinusoid of frequency ω_0

Role of Magnitude Response for a Given Frequency

- k provides amplitude scaling to a sinusoid.
- E.g. $k = |H(e^{j\omega_0})| = 2.5$ means amplitude of the output sinusoid is 2.5 times the amplitude of the input sinusoid.

  $k = |H(e^{j\omega_0})| \approx 1$ means to pass the frequency

  $k = |H(e^{j\omega_0})| \approx 0$ means to stop the frequency

  k more than 1 means gain or amplification

  k less than 1 means loss or attenuation
- Therefore, a filter's characteristics may be found from evaluating $|H(e^{j\omega})|$ at various frequencies.
- Here, the values of the magnitude response at frequencies 0.25π and 0.5π for the filter $h[n]$ = [0.7071 0 0.7071] may be verified as $k = |H(e^{j\pi/4})| = 1$, $k = |H(e^{j\pi/2})| = 0$.

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Role of Magnitude Response for a Given Frequency

- Consider a filter's frequency response at frequency ω_0 is:

$$H(e^{j\omega_0}) = 0.54 - j0.72$$

Find the magnitude response at this frequency and determine if the filter passes or stops this frequency.

The magnitude response at frequency ω_0 is $\boxed{}$

The filter $\boxed{}$ the frequency ω_0.

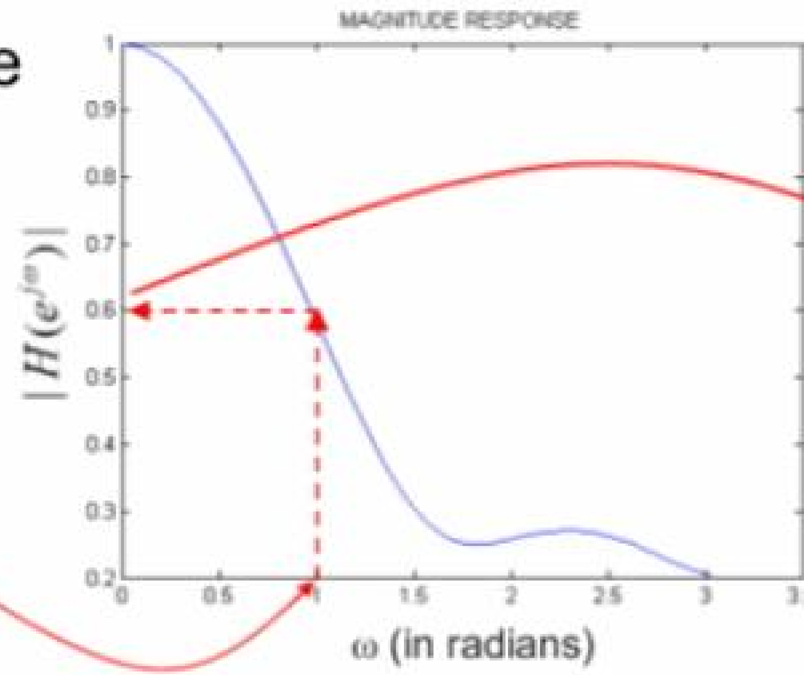
Submit | Correct Answer

Solve Magnitude Response for a Given Frequency

- E.g. frequency response plot for a filter $h[n]$=[0.4 0.3 0.2 0.1]

- Magnitude response tells us the scaling at different frequencies.

Choose a frequency $\omega = 1$



At $\omega = 1$, $|H(e^{j\omega})| = 0.6$

- Input amplitude of sinusoid is scaled by 0.6 at the filter output.

- Recall that magnitude response is symmetric, $|H(e^{j\omega})| = |H(e^{-j\omega})|$
- Phase response is anti-symmetric, $\angle H(e^{j\omega}) = -\angle H(e^{-j\omega})$
- Therefore, plotting the positive frequency, 0 to π, is enough.

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Summary

By now, you should be able to:

- Analyse a filter's Magnitude Response where it selectively passes some frequencies, but stops other frequencies;
- Explain the Role of Magnitude Response for a Given Frequency where the magnitude response of a filter is the magnitude of $H(e^{j\omega})$; and
- Solve Magnitude Response for a Given Frequency.

Frequency-Selective Filters

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Learning Objectives

By the end of this topic, you should be able to:

- Explain the operation of a Filter with General Input; and
- Distinguish 4 types of Ideal Frequency-Selective Filters, Lowpass, Highpass, Bandpass and Bandstop Filters.

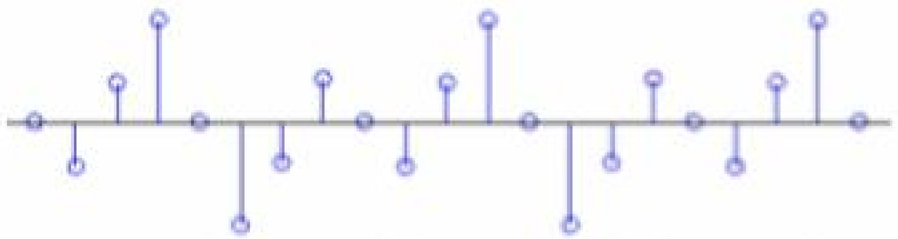
Filter with General Input

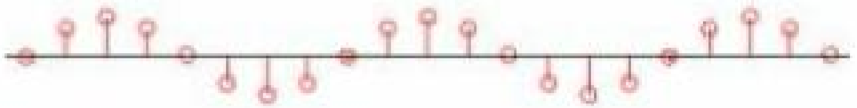
- E.g. Consider the filter of example:
  passes frequency $\pi/4$, stops frequency $\pi/2$.

Input consists of both frequencies → from linearity of LTI systems, output is the sum of individual outputs

Input = $A\sin(0.25\pi n) + B\sin(0.5\pi n)$ → filter → output = $A\sin(0.25\pi n)$

A = 5
B = -9





- Discrete Time Fourier Transform (DTFT) shows that any input consists of weighted sum of many sinusoids.
- Each sinusoid is scaled according to the magnitude response of the filter.
- Output is the sum of all these scaled sinusoids.

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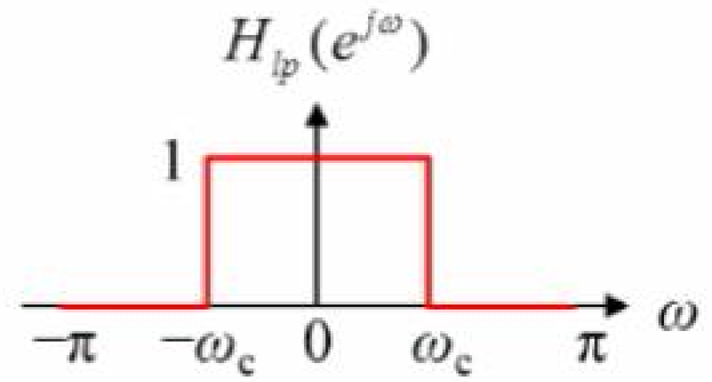
Ideal Frequency-Selective Filters

(a) Ideal Lowpass Filter:

low frequencies from 0 to ω_c passed, rest stopped.
(ω_c = cut-off frequency)

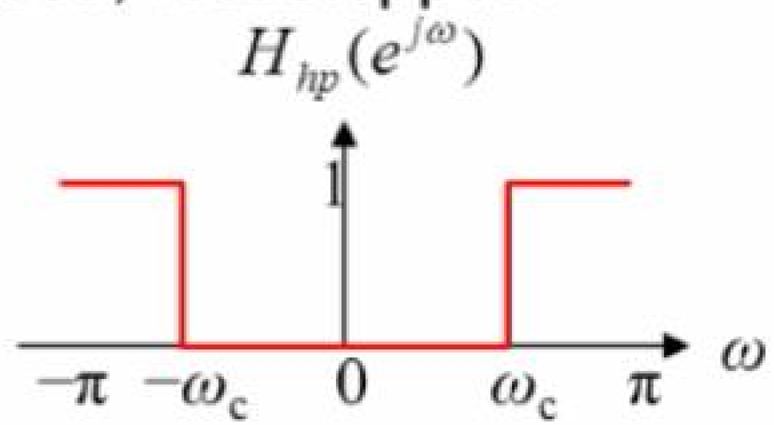
$$H_{lp}(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$



(b) Ideal Highpass Filter:

high frequencies from ω_c to π passed, rest stopped.

$$H_{hp}(e^{j\omega}) = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & \omega_c < |\omega| \le \pi \end{cases}$$



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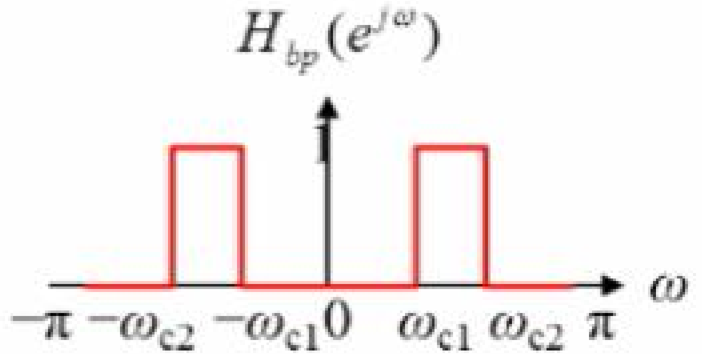
Ideal Frequency-Selective Filters

(c) Ideal Bandpass Filter:

middle frequencies from ω_{c1} to ω_{c2} passed, rest stopped.

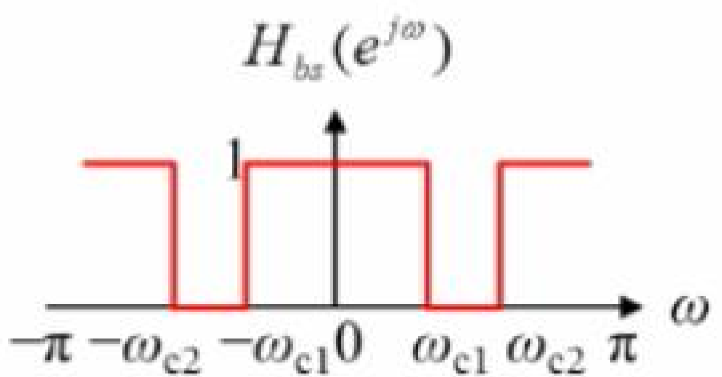
$$H_{bp}(e^{j\omega}) = \begin{cases} 0 & |\omega| < \omega_{c1} \\ 1 & \omega_{c1} < |\omega| < \omega_{c2} \\ 0 & \omega_{c2} < |\omega| \le \pi \end{cases}$$



(d) Ideal Bandstop Filter:

frequencies from 0 to ω_{c1} and from ω_{c2} to π passed, rest stopped.

$$H_{bs}(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_{c1} \\ 0 & \omega_{c1} < |\omega| < \omega_{c2} \\ 1 & \omega_{c2} < |\omega| \le \pi \end{cases}$$



- Ideal filters are non-causal, impulse response extends from $-\infty$ to ∞ $\rightarrow$ not realisable.

Work Example on Ideal Frequency-Selective Filters

- An ideal low pass filter has a cut-off frequency 1.2 rad, find its frequency response.

$$H(e^{j\omega}) = \begin{cases} \boxed{} & -\pi \le \omega < -1.2 \\ \boxed{} & -1.2 < \omega < 1.2 \\ 0 & \boxed{} < \omega \le \pi \end{cases}$$

Submit Correct Answer

Summary

By now, you should be able to:

- Explain that the overall output of a filter with general input is the sum of the individual outputs for each input sinusoid. This is because a filter is a linear time invariant system and it satisfies linearity property; and
- Distinguish 4 types of Ideal Frequency-Selective Filters:
  - Lowpass: A filter that stops all frequencies above ω_c
  - Highpass: A filter that stops all frequencies below ω_c
  - Bandpass: A filter with a passband from ω_{c1} to ω_{c2} and stops all remaining frequencies
  - Bandstop: A filter that passes low and high frequencies, but stops the middle frequencies.

Phase Response

Learning Objectives

By the end of this topic, you should be able to:

- Explain the Role of Phase Response;
- Solve Phase and Time Delay;
- Differentiate between Group Delay and Time Delay;
- Explain Group Delay;
- Identify the Desirable Phase Response of Filter; and
- Give examples of Application of Filtering.

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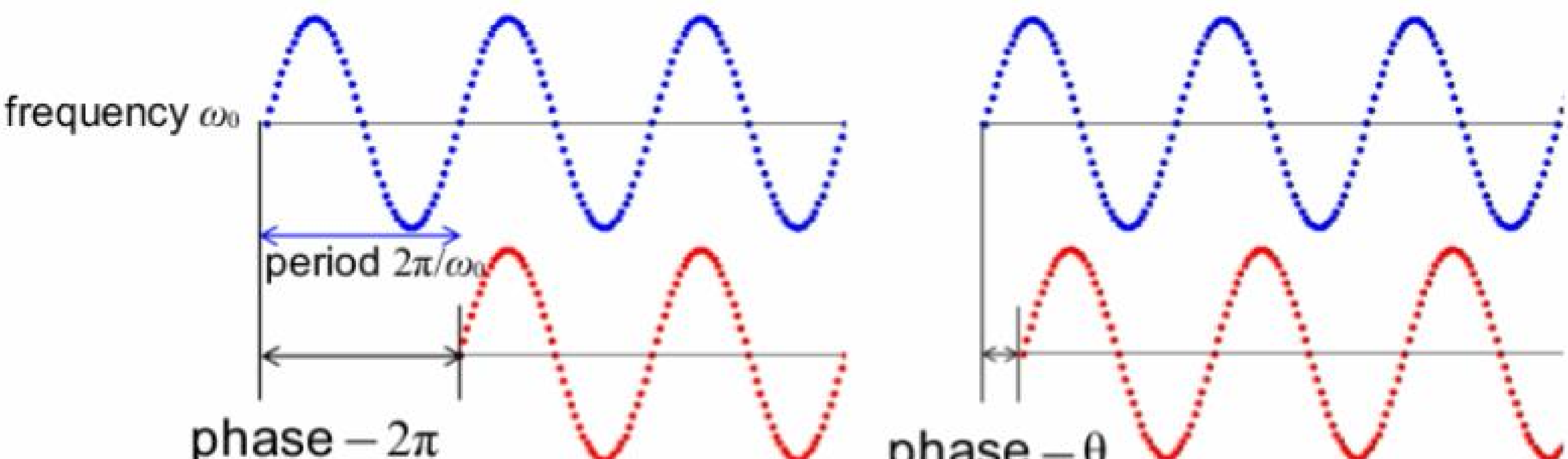
Role of Phase Response for a Given Frequency

- Let a filter's frequency response be $H(e^{j\omega}) = e^{j.f(\omega)}$, so that the phase response is $\angle H(e^{j\omega}) = f(\omega)$
- What is the output for a single sinusoid input $\sin(\omega_0 n)$ to this filter?
- It may be shown that the output is $\sin\left(\omega_0 n + f(\omega_0)\right)$
- Thus, the phase response introduces a phase shift of $f(\omega_0) = \angle H(e^{j\omega_0})$ to a sinusoid input.

Phase and Time Delay

- Typically $\angle H(e^{j\omega_0})$ is negative, so phase shift results in a time shift towards right, or a delay. How much is this delay?



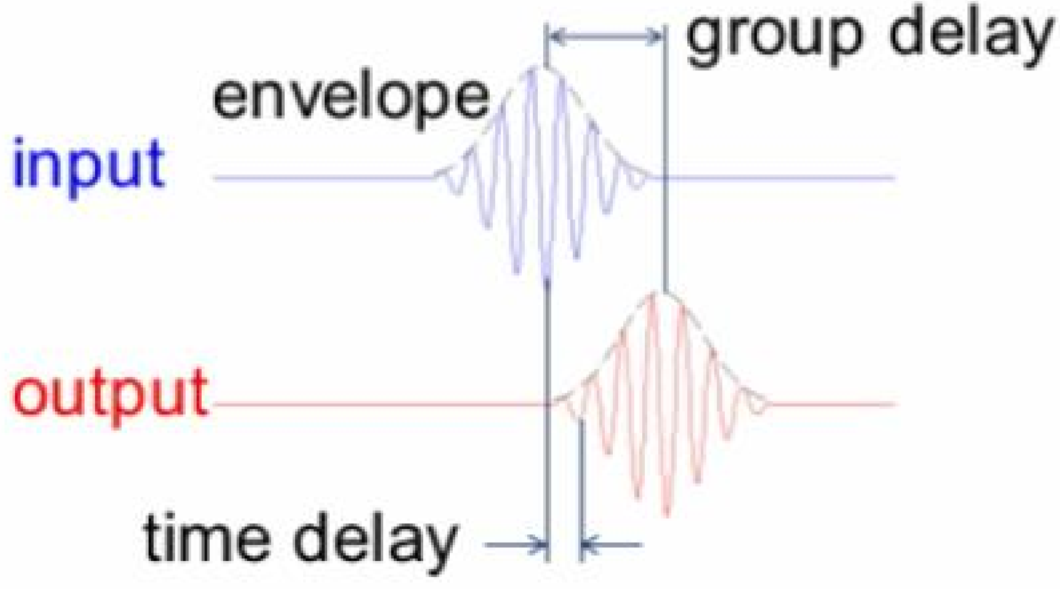
phase -2π = time delay $2\pi/\omega_0$

phase $-\theta$ = time delay θ/ω_0

- Therefore, phase $\angle H(e^{j\omega_0})$ = time delay $-\dfrac{\angle H(e^{j\omega_0})}{\omega_0}$

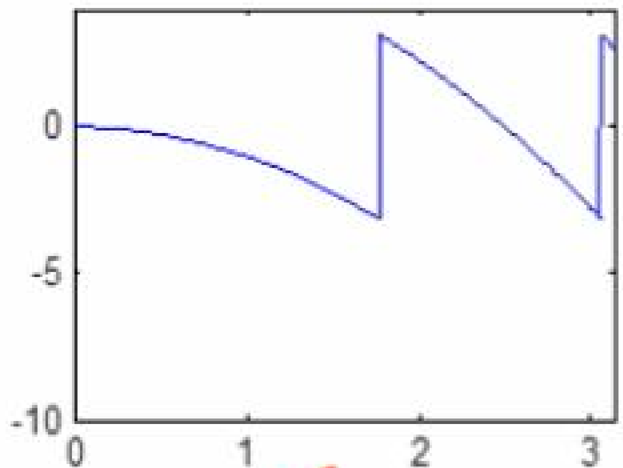
Group Delay and Time Delay

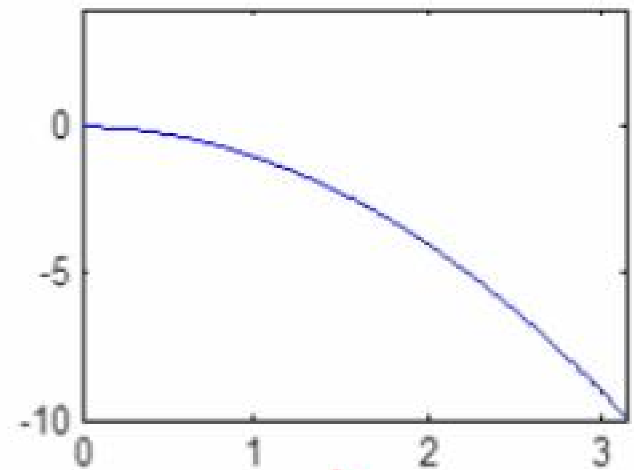
- Normally, a sinusoid signal is present at all times.
- However, let's consider a finite-duration burst of sinusoid as the input to a filter.
- Time delay of the filter is how much the sinusoid is shifted at its output.
- Group Delay of the filter is how much the envelope of the burst is shifted at its output.



Group Delay

- Group Delay of a filter is defined as $\tau(\omega) = -\frac{d}{d\omega}\left\{\angle H(e^{j\omega})\right\}$ using the continuous phase.





Principal phase is defined between $-\pi$ and π, which may result in discontinuous phase response.

Continuous phase (also known as unwrapped) is a continuous function.

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Group Delay

- For some filters, group delay is equal to time delay.
- Consider a delay of *m* samples: $x[n] \longrightarrow \boxed{z^{-m}} \longrightarrow x[n\text{-}m]$
  system function $H(z) = z^{-m}$:
  - Frequency response $H(e^{j\omega}) = e^{-j\omega m}$
  - Phase response $\angle H(e^{-j\omega}) = -\omega m$
  - Group delay $\tau(\omega) = -\frac{d}{d\omega}\left\{\angle H(e^{-j\omega})\right\} = m$ for any frequency
  - Time delay $-\frac{\angle H(e^{-j\omega})}{\omega} = m$ for any frequency
- Thus, for this filter, group delay = time delay.
- In general, group delay = time delay for filters with linear phase response of the form $\angle H(e^{j\omega}) = \text{constant}\,.\omega$

Work Example on Group Delay

- The phase response of a filter is $\angle H(e^{j\omega}) = \frac{\pi}{2} - 2\omega$

Time delay =	$\square$	$2 - \frac{\pi}{2\omega}$
Group delay =	$\square$	2

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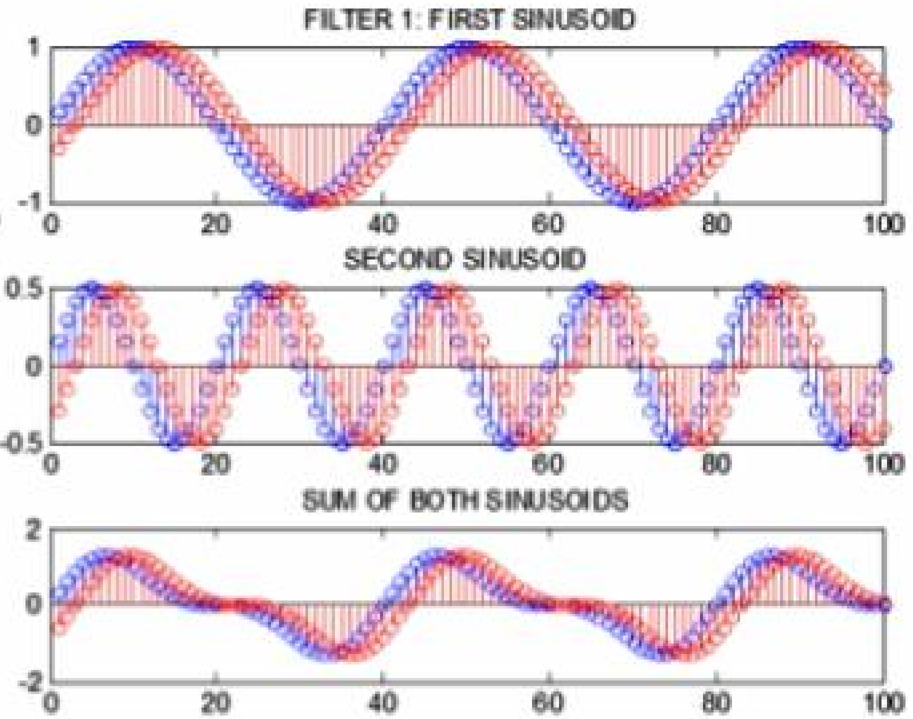
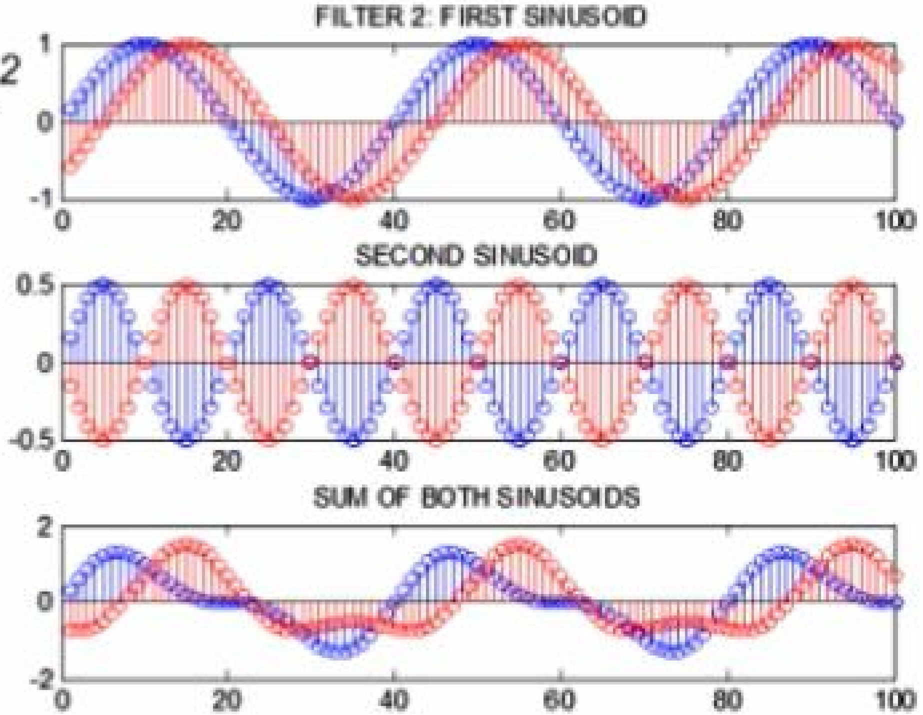
Desirable Phase Response of a Filter

- A Phase Response decides the time delay of the output.
- E.g. comparing outputs of two filters with different phase response.
- Input = sum of two sinusoids.
- Filter 1: $|H(e^{j\omega})| = 1$, $\angle H(e^{j\omega}) = -3\omega$ (linear phase response = constant group delay).
- Time delay = $-\angle H(e^{j\omega})/\omega$ = 3 samples (constant time delay for any frequency).



Desirable Phase Response of a Filter

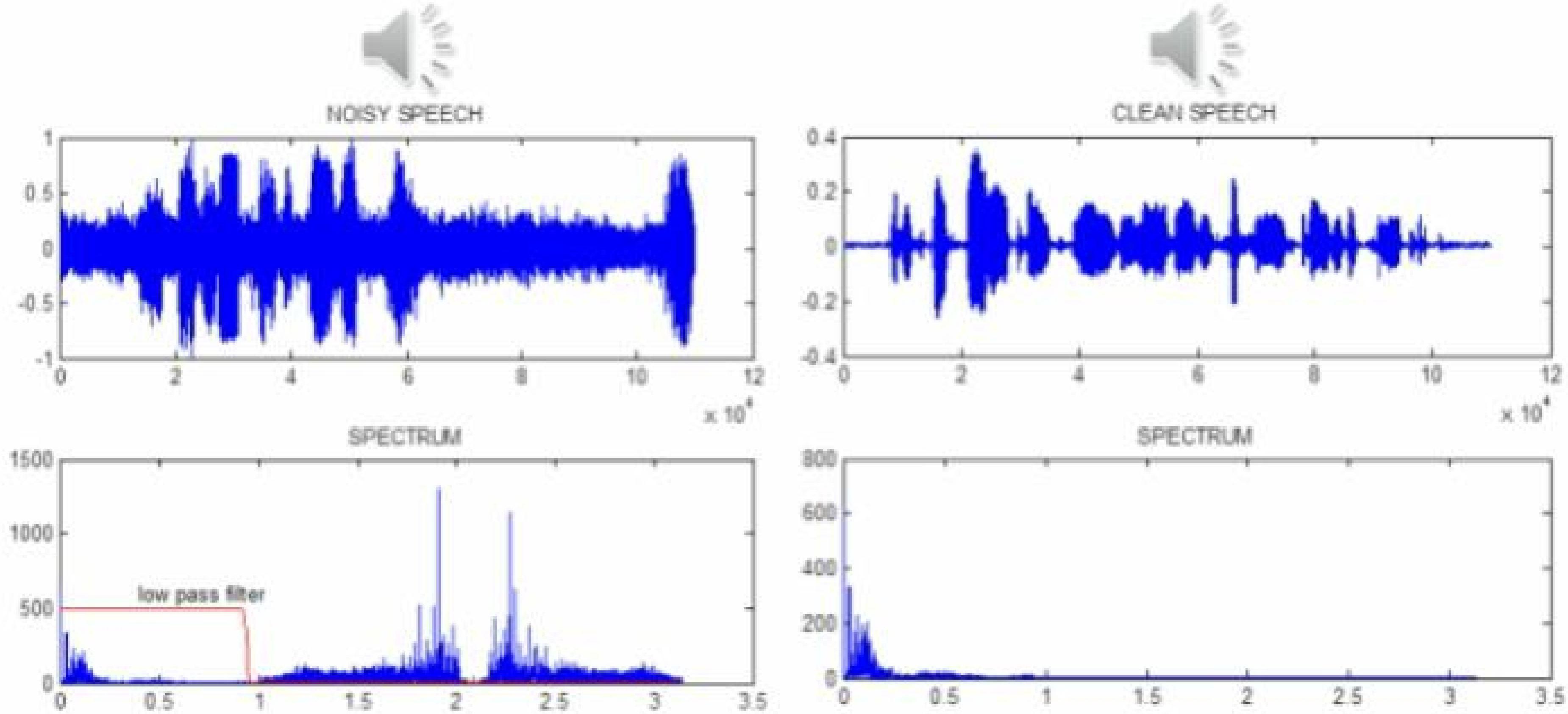
- Filter 2: $|H(e^{j\omega})| = 1$, $\angle H(e^{j\omega}) = -32\omega^2$ (non-linear phase response).
- Time delay $= -\angle H(e^{j\omega})/\omega = 32\omega$ (time delay varying with frequency).
- Since magnitude response is 1, both sinusoids are passed with an amplitude scaling of 1.



- However, Filter 1 output retains the input shape, only delayed.
- Filter 2 output distorts the input shape.
- Therefore, in order to not distort the signal waveform, filters should have linear phase response: $\angle H(e^{j\omega}) = \text{constant} \cdot \omega$

Application of Filtering

- Filtering is used to pass the desired part of the input which is the signal and to stop the undesired part of the input which is the noise.
- E.g. noisy speech enhancement using filtering.



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Summary

By now, you should be able to:

- Explain that the Phase Response will only affect the phase of the input sinusoid;
- Solve Phase and Time Delay;
- Differentiate that Group Delay of the filter gives how much the envelope of the burst is shifted in time while Time Delay of the filter is how much the sinusoid is shifted at its output;
- Explain that Group Delay in general is equal to time delay for filters with linear phase response and is found by taking the negative derivative of the continuous phase;
- Describe the Desirable Phase Response of a Filter where it does not distort any output; and
- Explain that an Application of Filtering is to pass the desired part of the input which is the signal, and to stop the undesired part of the input which is the noise.

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Frequency Response of a Zero

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System Function

- (Recap): A filter or system may be specified by its system function.
- System function gives poles and zeros.
- Infinite Impulse Response (IIR) Filter → system function is rational:
  - Poles are present, zeros may or may not be present:

$$H(z)=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_Mz^{-M}}{a_0+a_1z^{-1}+a_2z^{-2}+\cdots+a_Nz^{-N}}$$

- Finite Impulse Response (FIR) Filter → system function is polynomial:
  - Only zeros are present, no nonzero pole

$$H(z)=b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_Mz^{-M}$$

- Real Coefficient Filter → poles and zeros are in conjugate pairs (else, complex coefficient).
- (Causal) Stable Filter → poles inside unit circle (else, unstable).

Learning Objectives

By the end of this topic, you should be able to:

- Explain the Role of Frequency Response of a Single Zero or Pole; and
- Analyse the Frequency Response of a Single Zero.

Frequency Response of a Single Zero or Pole

- System function in factored form to show the poles and zeros:

$$H(z)=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_Mz^{-M}}{a_0+a_1z^{-1}+a_2z^{-2}+\cdots+a_Nz^{-N}}=\frac{b_0\prod_{k=1}^{M}(1-z_kz^{-1})}{a_0\prod_{k=1}^{N}(1-p_kz^{-1})}$$

- $p_1, p_2, \ldots p_N$ are the poles and $z_1, z_2, \ldots z_M$ are the zeros (ignoring poles/ zeros at 0 since they have no effect on magnitude response).
- Magnitude response of a filter:
  - Product of magnitude responses of each of its pole/ zero
  - Sum of magnitude responses in dB of each of its pole/ zero
- Similarly, phase response of a filter:
  - Sum of phase responses of each of its pole/ zero
- Group delay of a filter:
  - Sum of group delays of each of its pole/ zero
- Therefore, we study the frequency response of a single pole/ zero.

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Frequency Response of a Single Zero 1

- A zero at $z = re^{j\theta}$ $H(z) = 1 - re^{j\theta}z^{-1}$ (ignoring a pole at 0).
- Since $|H(e^{j\omega})|^2 = H(e^{j\omega})\{H(e^{j\omega})\}^*$:

$$= (1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{j\omega})$$

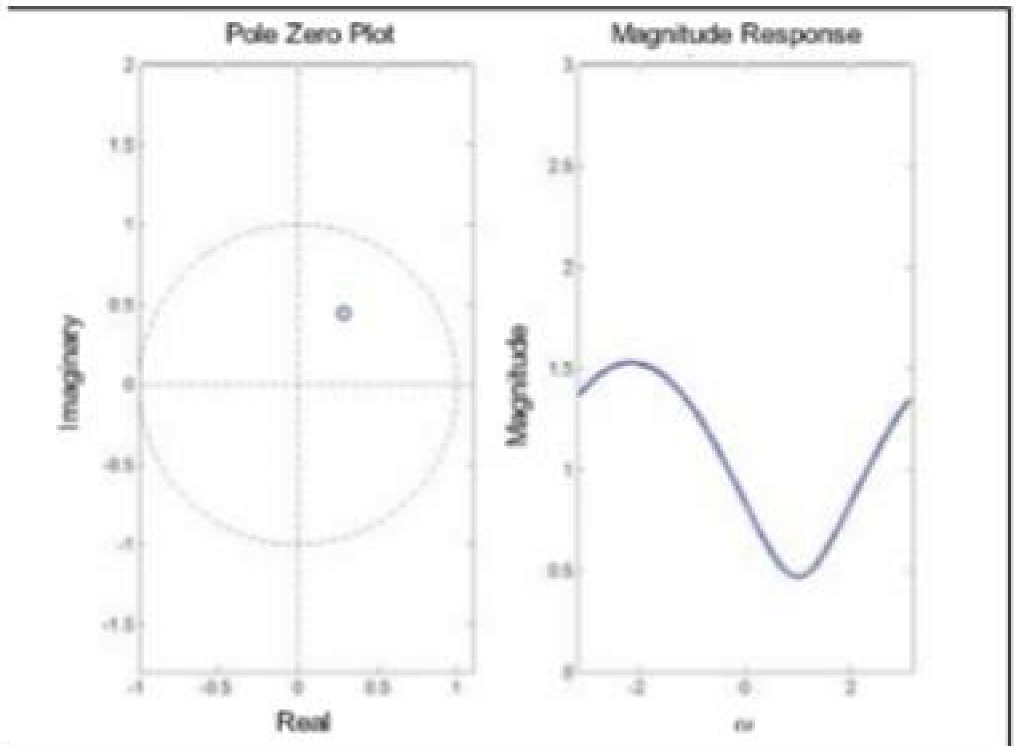
$$= 1 - re^{-j(\omega-\theta)} - re^{j(\omega-\theta)} + r^2$$

$$= 1 - 2r\cos(\omega - \theta) + r^2$$

- Magnitude response $|H(e^{j\omega})| = \sqrt{1 + r^2 - 2r\cos(\omega - \theta)}$

Frequency Response of a Single Zero 1

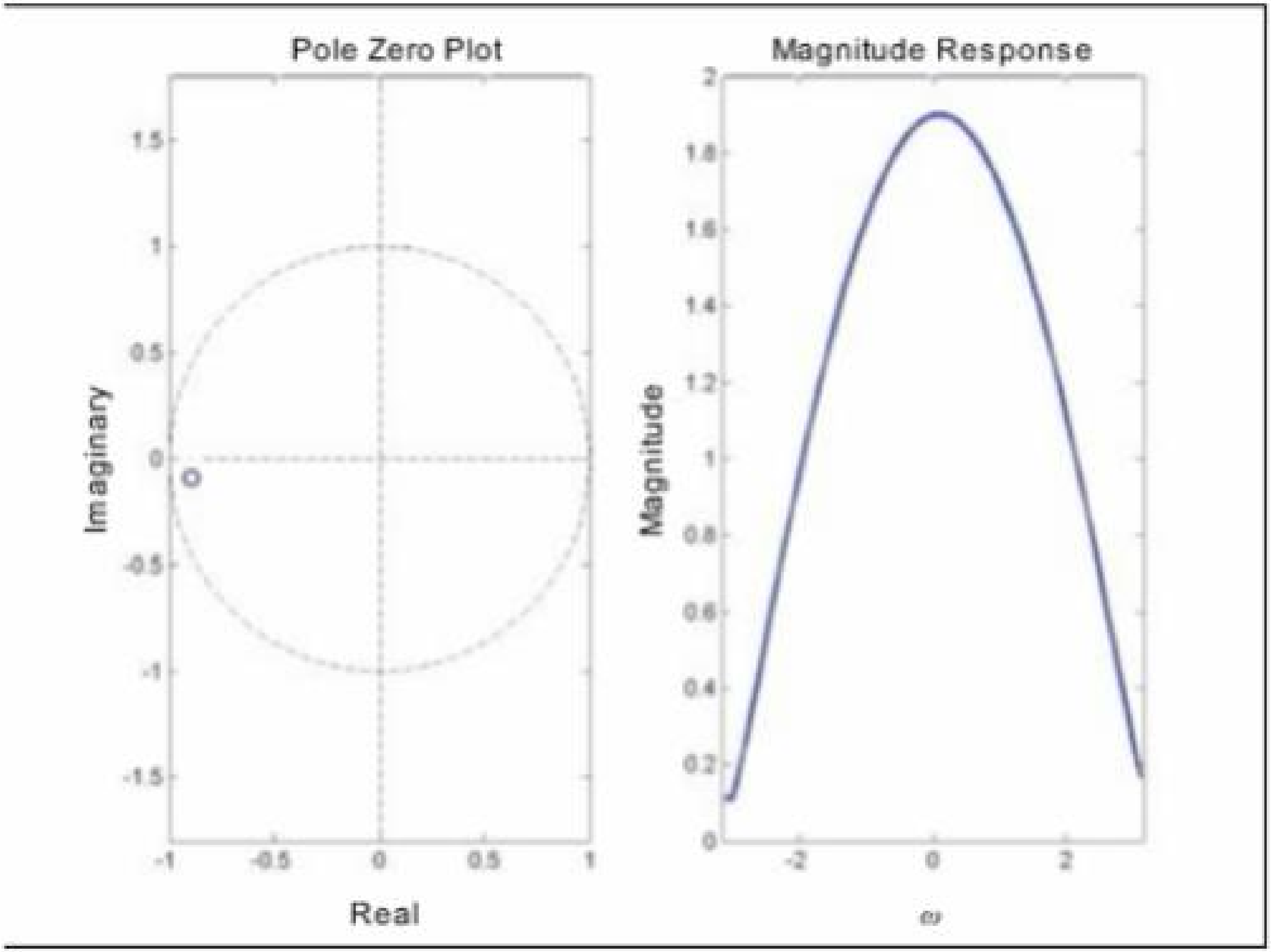
Click to play video of magnitude response for changing r, fixed $\theta = 1$.



- At $r = 1$(on unit circle), magnitude response becomes 0 at frequency θ.
- As r moves away from 1 (away from unit circle) either way, the minimum of magnitude response at frequency θ moves higher.

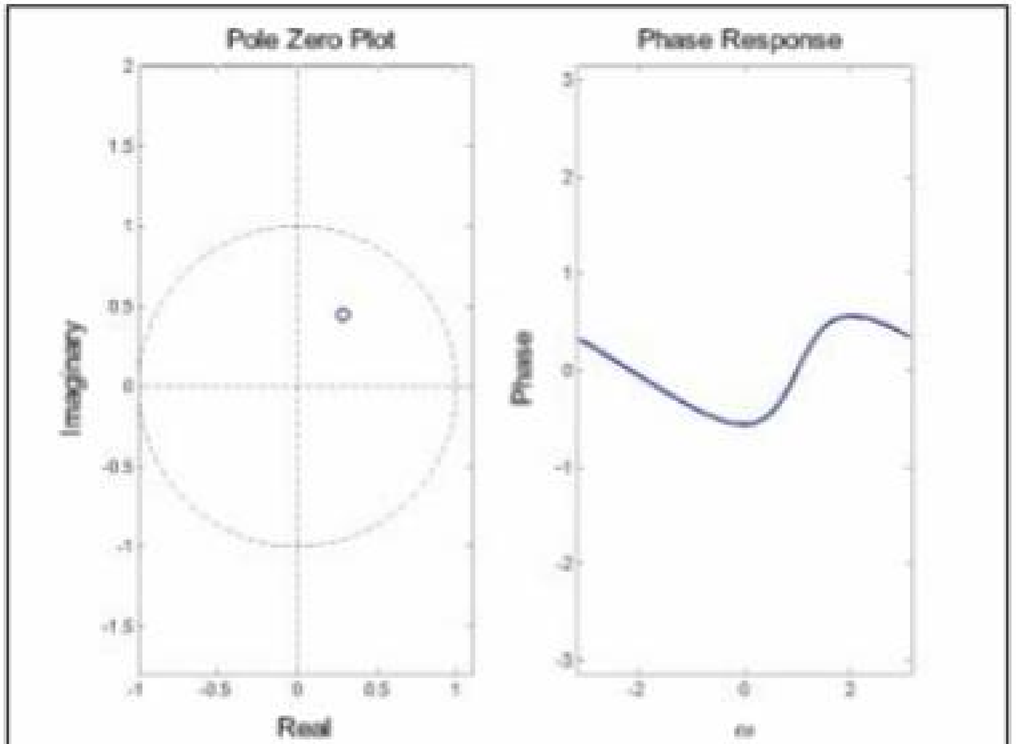
Frequency Response of a Single Zero 2

Click to play video of magnitude response for changing θ, fixed $r = 0.9$.



Frequency Response of a Single Zero 3

Click to play video of phase response for changing *r*, fixed θ = 1.



Frequency Response of a Single Zero 3

- Phase response:

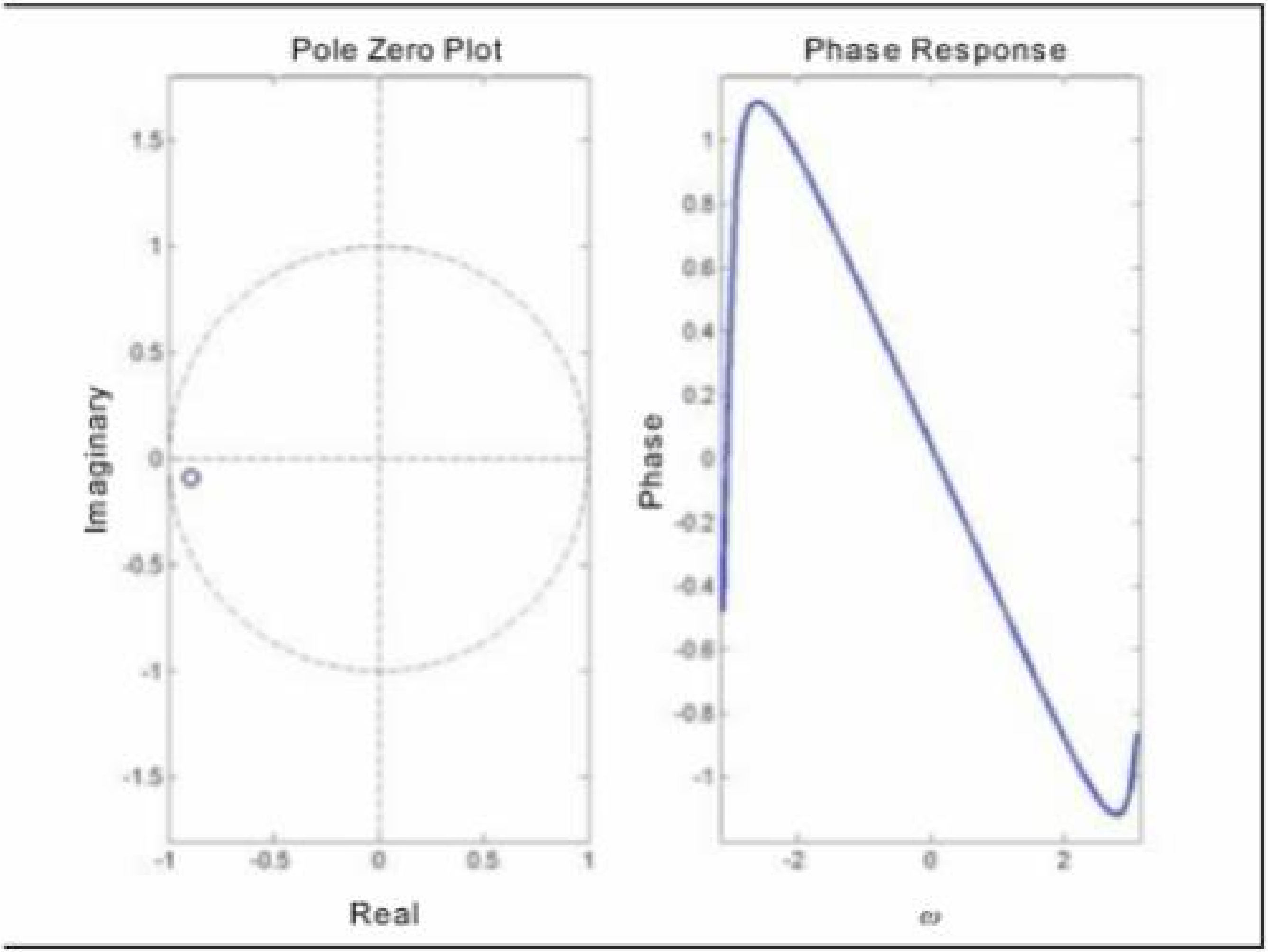
$$H(e^{j\omega}) = 1 - re^{-j(\omega-\theta)}$$

$$= \{1 - r\cos(\omega-\theta)\} + jr\sin(\omega-\theta)$$

- Phase response $\angle H(e^{j\omega}) = \arctan\left[\dfrac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]$

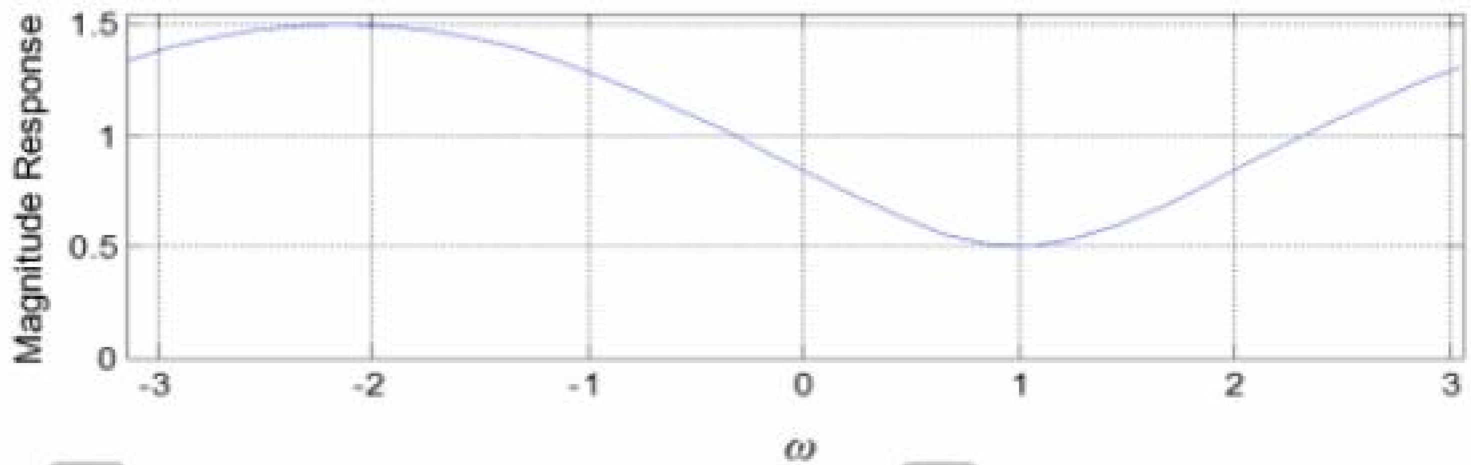
Frequency Response of a Single Zero 4

Click to play video of phase response for changing θ, fixed $r = 0.9$.



Work Example on Magnitude Response of a Zero

- The magnitude response of a single zero at $re^{j\theta}$ is shown below. Determine the values of r and θ. Select the correct answers for r on the left and θ on the right.



1. A. r is equal to 1
   B. r is not equal to 1
   C. r is less than 1
   D. r is more than 1

2. A. θ is equal to 1 radian
   B. θ is equal to 1.5 radians
   C. θ is equal to -2 radians
   D. θ can not be determined

Submit Correct Answer

Summary

By now, you should be able to:

- Recall the concept of System Function;
- Explain the Role of Frequency Response of a Single Zero or Pole; and
- Analyse the Frequency Response of a Single Zero.

Frequency Response of a Pole

Learning Objectives

By the end of this topic, you should be able to:

- Analyse the Frequency Response of a Single Pole; and
- Explain the Frequency Response of a filter by considering the Frequency Responses of each zero and each pole.

Frequency Response of a Single Pole

- A pole at $z = re^{j\theta}$ $H(z) = \dfrac{1}{1 - re^{j\theta} z^{-1}}$ (ignoring a zero at 0).

- From earlier result, magnitude response:

$$| H(e^{j\omega}) | = \frac{1}{\sqrt{1 + r^2 - 2r\cos(\omega - \theta)}}$$

- From earlier result, phase response:

$$\angle H(e^{j\omega}) = -\arctan\left[\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right]$$

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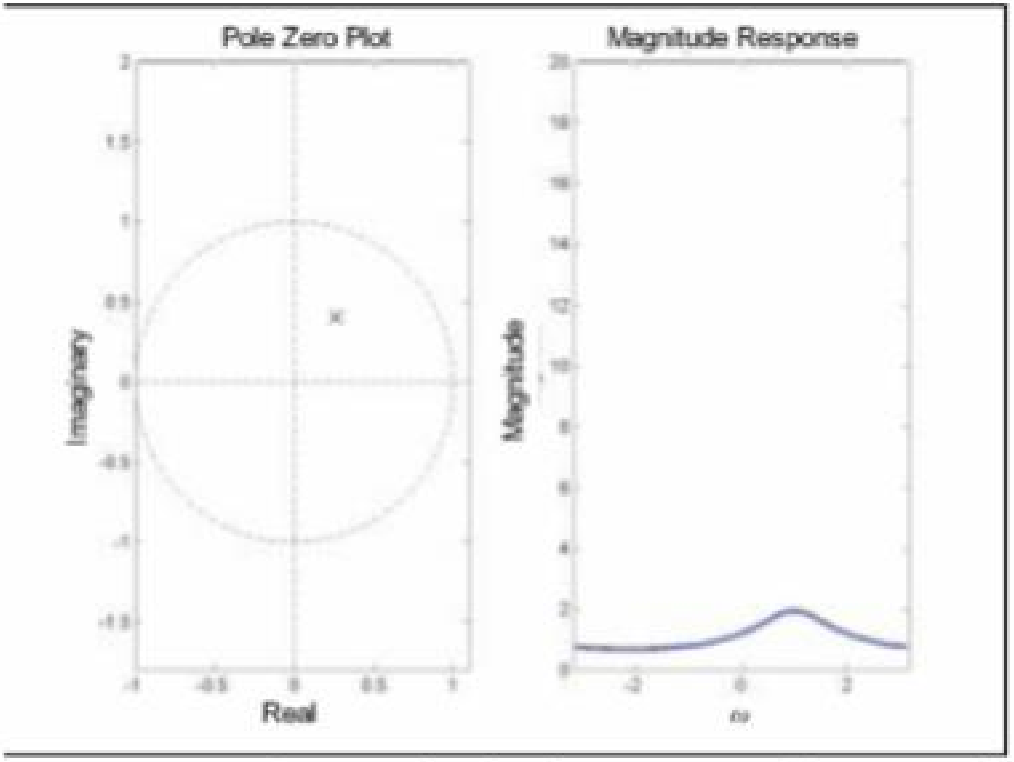
Frequency Response of a Single Pole

Click to play video of magnitude response for changing *r*, fixed $\theta = 1$.



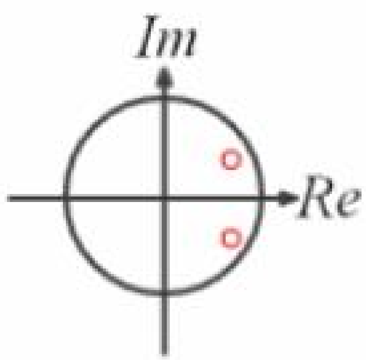
- Magnitude response of a pole is inverse of magnitude response of a zero → minimum at frequency θ becomes maximum.
- As r increase to 1, maximum moves higher.
- $r < 1$ for stability ($r = 1$, or on the unit circle, it becomes infinity).

Complex Plane

- Recap:

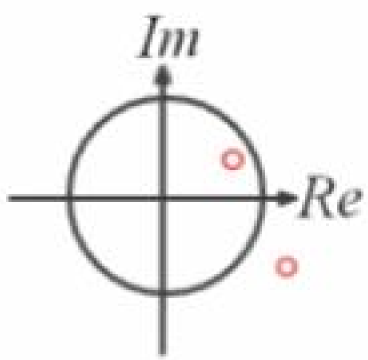
Conjugate

$$\left(re^{j\theta}\right)^* = re^{-j\theta}$$



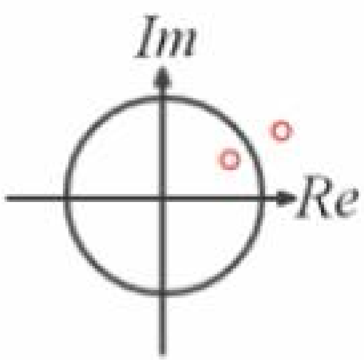
Reciprocal

$$\frac{1}{re^{j\theta}} = \frac{1}{r}e^{-j\theta}$$



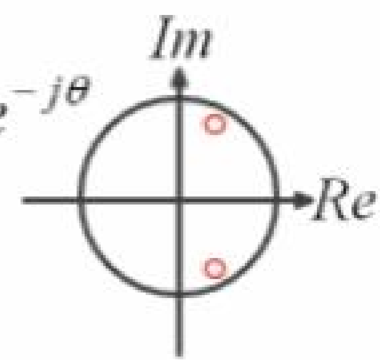
Conjugate reciprocal

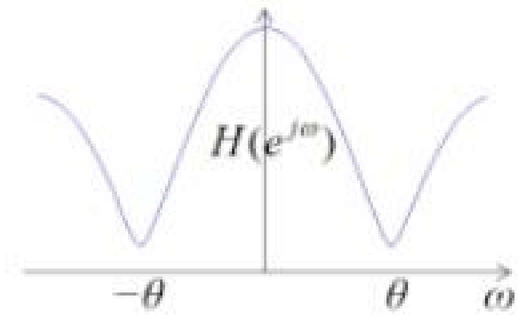
$$\left(\frac{1}{re^{j\theta}}\right)^* = \frac{1}{r}e^{j\theta}$$



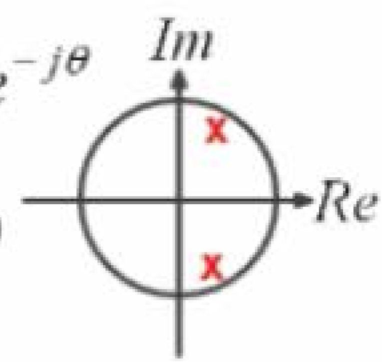
Frequency Response of Multiple Poles and Zeros

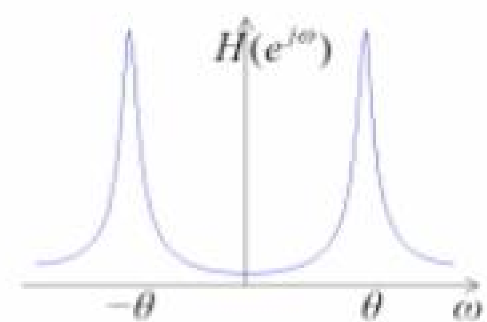
- Real coefficient filters have complex conjugate zeros and poles.
- Conjugate zeros at $re^{j\theta}$ and $re^{-j\theta}$ $\rightarrow$ symmetric frequency response with two minima at θ and $-\theta$.





- Conjugate poles at $re^{j\theta}$ and $re^{-j\theta}$ $\rightarrow$ symmetric frequency response with two maxima at θ and $-\theta$.





- To design a filter:
  - Place zeros along the frequencies to be stopped
  - Places poles along the frequencies to be passed

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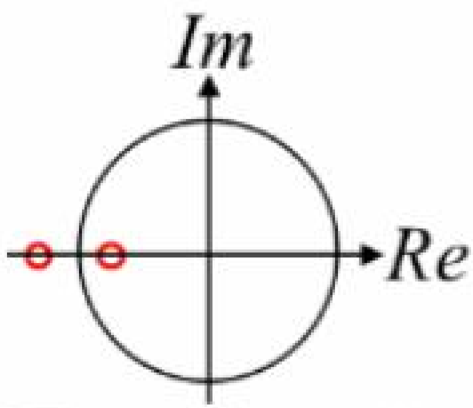
Matlab Video

Select to Play Video

Note: To proceed, do remember to close the pop-up window at the end of the video

Work Example on the Relationships of Zeros

- For each of the following cases, drag and drop the appropriate description of how the pair of zeros are related.



(i) Zeros shown in above figure:

(ii) Zeros $4e^{j}$ and $0.25e^{j}$:

Conjugate Pairs

Reciprocal Pairs

Conjugate Reciprocal Pairs

No Special Relationship

Submit

Correct Answer

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Summary

By now, you should be able to:

- Analyse the Frequency Response of a Single Pole; and
- Explain the Frequency Response of a filter by considering the Frequency Responses of each zero and pole.

Allpass Systems

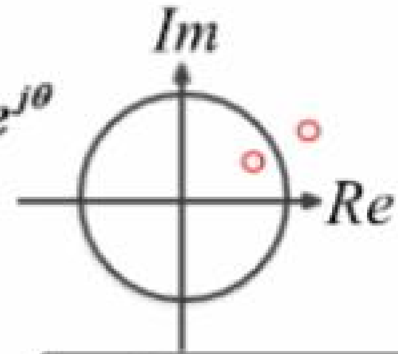
Learning Objectives

By the end of this topic, you should be able to:

- Explain the properties of Allpass Systems; and
- Identify an Allpass System.

Allpass Systems

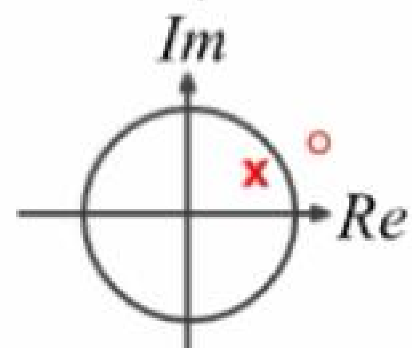
- Equivalence of zero at $re^{j\theta}$ and $\frac{1}{r}e^{j\theta}$



- 1 - $re^{j\theta}z^{-1}$ has magnitude response $\sqrt{1+r^2-2r\cos(\omega-\theta)}$
- With scaling, $r(1-(1/r)e^{j\theta}z^{-1})$ has magnitude response:

$$r\sqrt{1+\frac{1}{r^2}-\frac{2}{r}\cos(\omega-\theta)} = \sqrt{1+r^2-2r\cos(\omega-\theta)}$$

- They have identical magnitude response (but different phase response).
- Zero at $re^{j\theta}$ and pole at $\frac{1}{r}e^{j\theta}$ have reciprocal magnitude response.



- A filter with such a zero and a pole will have magnitude response = 1.

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Allpass Systems

- System function $H_{ap}(z) = \dfrac{1 - re^{j\theta}z^{-1}}{r\left(1 - \frac{1}{r}e^{j\theta}z^{-1}\right)} \cdot (-e^{j\theta})$ (to simplify, scale by $-e^{j\theta}$)

$$= \frac{1 - re^{j\theta}z^{-1}}{-re^{-j\theta} + z^{-1}}$$

$$= \frac{1 + az^{-1}}{a^* + z^{-1}} \quad \text{with } a = -re^{j\theta}$$

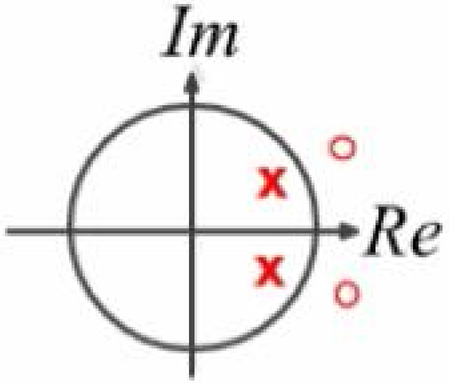
- Since $|H_{ap}(e^{j\omega})| = 1$, it is called an allpass system (a filter that passes all frequencies).
- Numerator coeffs = (1, a) denominator coeffs = (a^*, 1) = (a, 1)*
- Denominator is the conjugate of mirror-image polynomial of numerator.

Allpass Systems

- Such construction of all pass filter may be extended using pairs of zero and pole:

$$H_{ap}(z) = \frac{1 + a_1 z^{-1} + \cdots + a_{M-1} z^{-M+1} + a_M z^{-M}}{a_M + a_{M-1} z^{-1} + \cdots + a_1 z^{-M+1} + z^{-M}}$$

- The denominator is the mirror-image polynomial of the numerator (real coefficients).



- Stability → poles inside unit circle → zeros outside unit circle.
- Real coefficient → zeros in conjugate pair → zeros and poles in conjugate + reciprocal set of 4.
- Allpass filters find application in compensation of phase distortion, etc.

Work Example on Allpass Filter

- For each of the filters below, drag and drop the correct type in the empty boxes provided.

$$H(z) = \frac{2 - 3z^{-1}}{1 - 3z^{-1} + 2z^{-2}}$$

All pass

$$H(z) = \frac{0.4 - 0.7z^{-1} + z^{-2}}{1 - 0.7z^{-1} + 0.4z^{-2}}$$

Not allpass

Submit

Correct Answer

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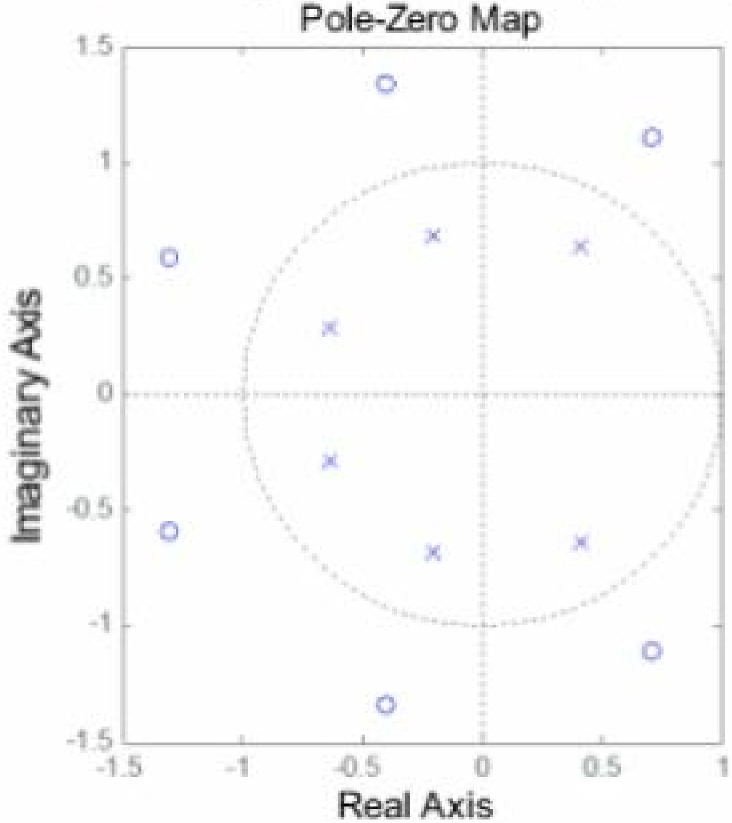
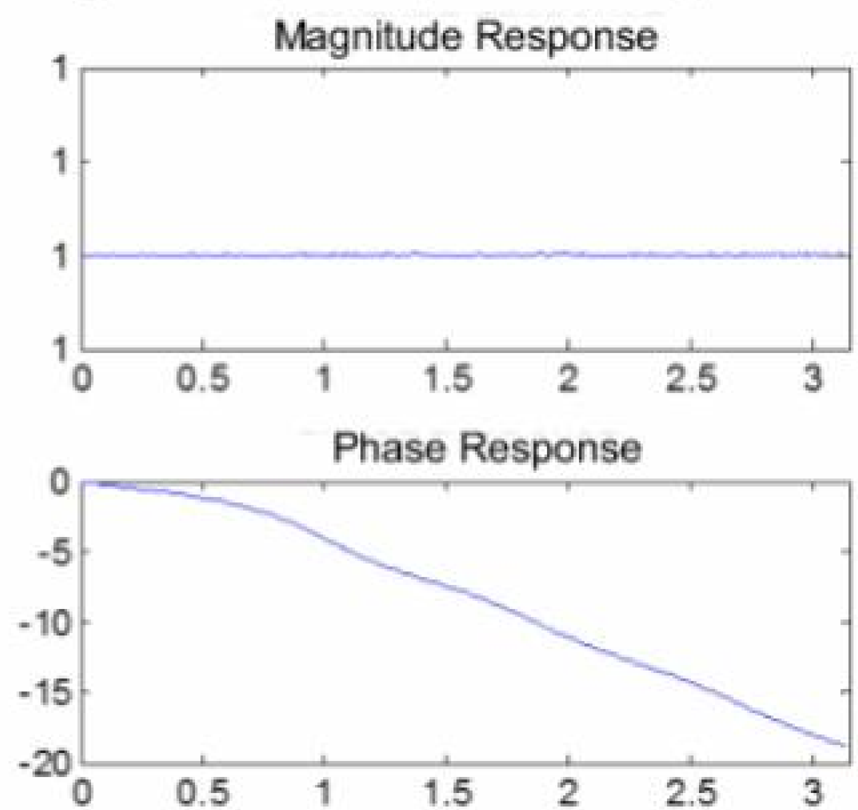
Example of Allpass Systems

- Example: pole zero plot and frequency response plot for allpass filters.
  - Choose any numerator, say $1+2z^{-1}+3z^{-2}+4z^{-3}+5z^{-4}+6z^{-5}+7z^{-6}$
  - Construct an allpass filter, $H(z)=\dfrac{1+2z^{-1}+3z^{-2}+4z^{-3}+5z^{-4}+6z^{-5}+7z^{-6}}{7+6z^{-1}+5z^{-2}+4z^{-3}+3z^{-4}+2z^{-5}+z^{-6}}$
  - Using Matlab, its poles, zeros, magnitude, and phase response are:





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Summary

By now, you should be able to:

- Explain the properties of Allpass Systems; and
- Identify an Allpass System.

Minimum and Maximum Phase Systems

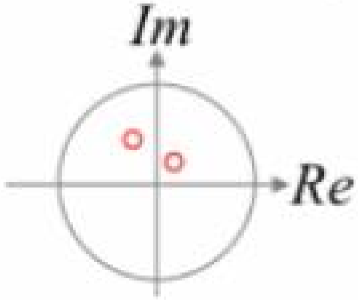
Learning Objectives

By the end of this topic, you should be able to:

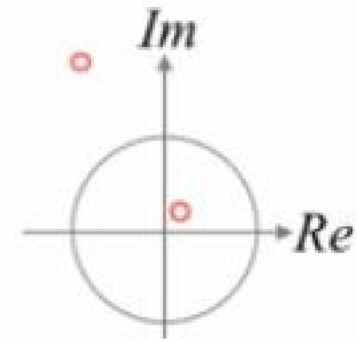
- Explain the properties of Minimum Phase and Maximum Phase; and
- Identify if a filter is Minimum Phase or Maximum Phase.

Minimum Phase and Maximum Phase Systems

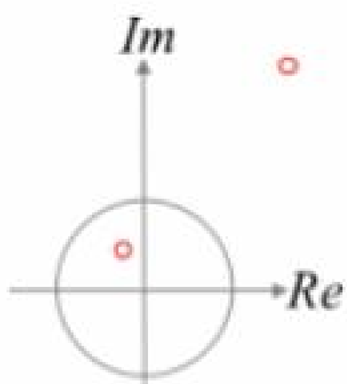
- Consider 4 systems:



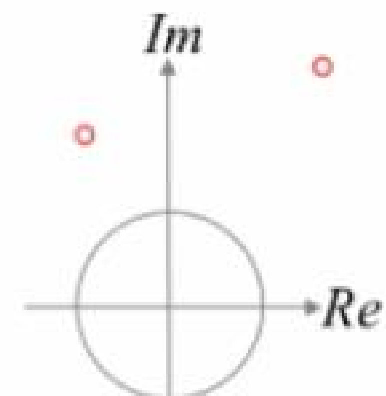
$$H_1(z) = (1 - 0.25e^{j\pi/3}z^{-1})(1 - 0.5e^{j2\pi/3}z^{-1})$$



$$H_2(z) = (1 - 0.25e^{j\pi/3}z^{-1})0.5(1 - 2e^{j2\pi/3}z^{-1})$$



$$H_3(z) = 0.25(1 - 4e^{j\pi/3}z^{-1})(1 - 0.5e^{j2\pi/3}z^{-1})$$



$$H_4(z) = 0.25(1 - 4e^{j\pi/3}z^{-1})0.5(1 - 2e^{j2\pi/3}z^{-1})$$

- From equivalence of zeros, these 4 systems have identical magnitude response: $|H_1(e^{j\omega})| = |H_2(e^{j\omega})| = |H_3(e^{j\omega})| = |H_4(e^{j\omega})|$

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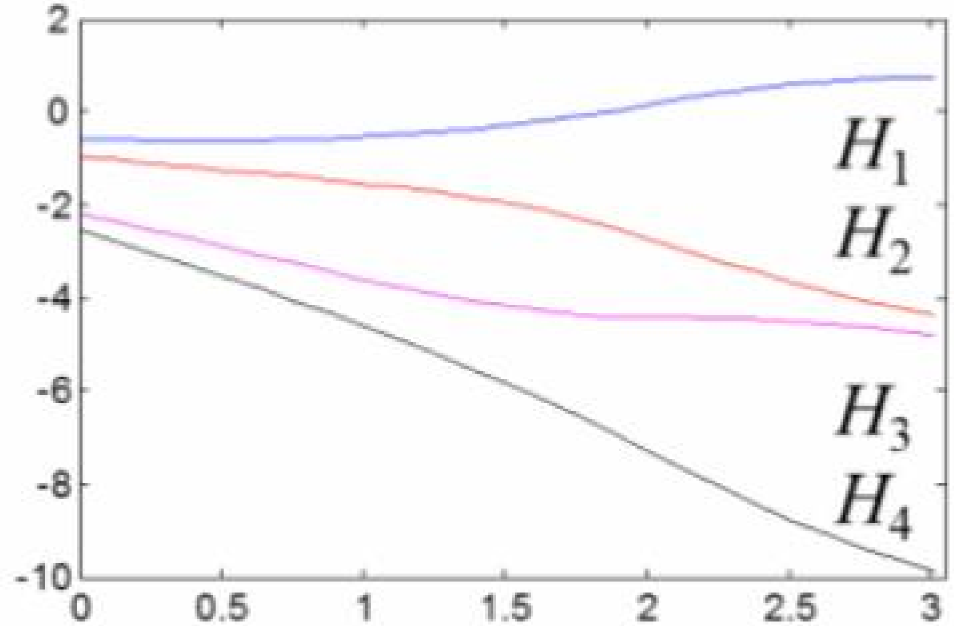
Minimum Phase and Maximum Phase Systems



- Compare their phase responses:
  → $H_1(z)$ has minimum phase lag (least time delay), $H_4(z)$ has maximum phase lag.
- This is because, zeros of $H_1(z)$ are inside the unit circle, zeros of $H_4(z)$ are outside the unit circle.
- Generalising the above example:
  - All zeros of a system are inside the unit circle
  → minimum phase system (has least time delay)
  - All zeros of a system are outside the unit circle
  → maximum phase system (has highest time delay)

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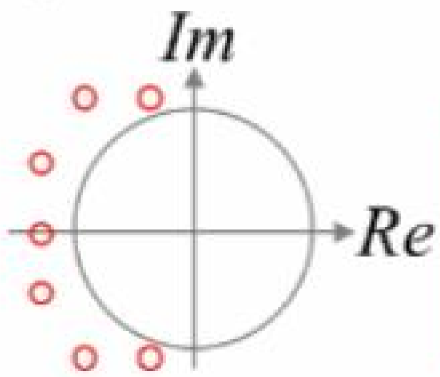
Work Example on Minimum/ Maximum Phase

- For each of the filters below, drag and drop the correct type in the empty boxes provided.

Pole zero plot of the first filter is



The second filter has zeros at

$-0.7e^{j0.5}$, $-0.7e^{-j0.5}$, $2e^{j0.9}$, $2e^{-j0.9}$

Minimum Phase

Maximum Phase

Neither Minimum nor Maximum Phase

Submit

Correct Answer

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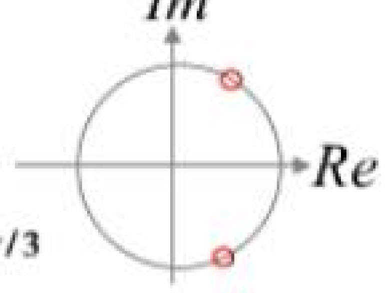
Application on Notch Filter

- From the mains power supply, an ADC output $x[n]$ picks up noise at 50 Hz. Sampling frequency is 300 Hz. Design a length 3 real coefficient FIR filter for $x[n]$ that removes its 50 Hz noise, and also keeps the dc value of $x[n]$ unchanged.

- To remove 50 Hz frequency, place a zero on the unit circle ($r = 1$) at 50 Hz .

- Sampling frequency 300 Hz (continuous-time frequency) = 2π radians (discrete-time frequency).

  $\rightarrow$ 50 Hz = $\pi/3$ radians $\rightarrow$ place a zero at $\omega = \pi/3$



- Real coefficient $\rightarrow$ conjugate pair of zeros at $1\cdot e^{j\pi/3}, 1\cdot e^{-j\pi/3}$

$$\rightarrow H(z) = a(1 - e^{j\pi/3}z^{-1})(1 - e^{-j\pi/3}z^{-1})$$
$$= a(1 - z^{-1} + z^{-2})$$

Application on Notch Filter

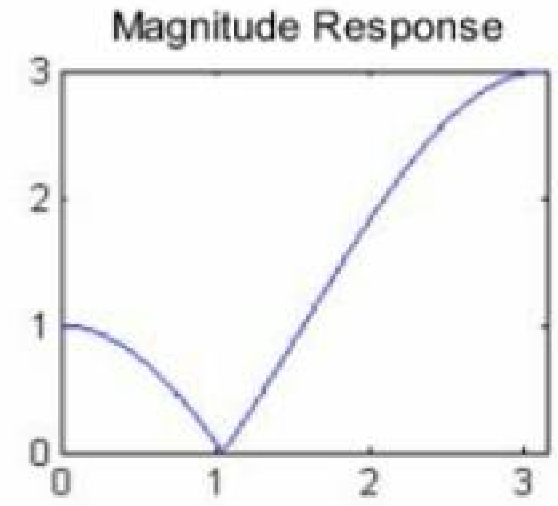
- dc = 0 frequency → at 0 frequency, magnitude response should be 1.

  ω = 0 → $z = e^{j\omega} = 1$

  → $H(z)|_{z=1} = 1$

  → a(1 – 1 + 1) = 1 → a = 1

- Therefore, the filter is $H(z) = 1 - z^{-1} + z^{-2}$.



Summary

By now, you should be able to:

- Explain the properties of Minimum Phase and Maximum Phase; and
- Identify if a filter is Minimum Phase or Maximum Phase.

Linear Phase Systems

Learning Objectives

By the end of this topic, you should be able to:

- Explain Zero Phase property; and
- Explain Linear Phase property.

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Zero Phase System

- Phase response $\angle H(e^{j\omega}) = 0$
- Frequency response $H(e^{j\omega})$ = magnitude response $|H(e^{j\omega})|$ or, frequency response is real and non-negative.

Example: $H(z) = -z^2 + 3 - z^{-2} \rightarrow H(e^{j\omega}) = -e^{j2\omega} + 3 - e^{-j2\omega}$

$$= 3 - (e^{j2\omega} + e^{-j2\omega})$$

$$= 3 - 2\cos 2\omega$$

- Note that it is real and positive, since $-2\cos 2\omega \geq -2$
- Zero phase filters are non-causal → not realisable.

Linear Phase System

- (Strictly) Linear phase = phase response is linear.
- Phase response $\angle H(e^{j\omega}) = -\omega\alpha$

  $\rightarrow$ Frequency response $H(e^{j\omega}) = |H(e^{j\omega})| e^{-j\omega\alpha}$
  $\rightarrow$ Group delay $\tau(\omega) = \alpha$

Linear Phase System

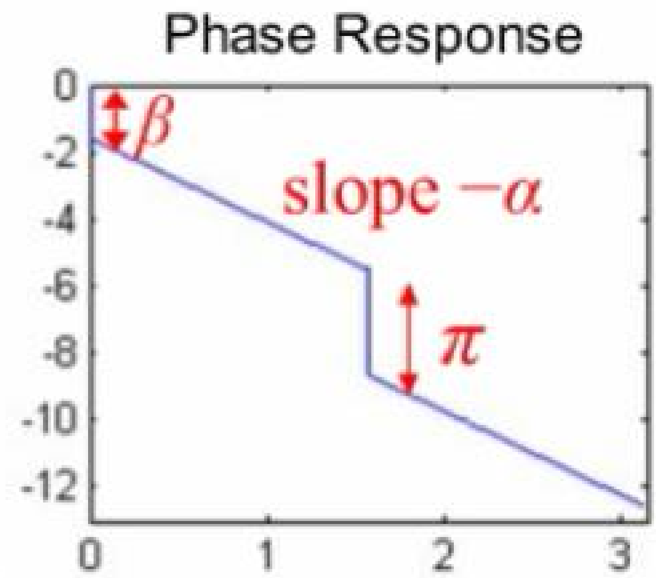
- (Generalised) Linear Phase = group delay is constant.
- Frequency response $H(e^{j\omega}) = A(e^{j\omega})e^{-j\omega\alpha + j\beta}$ where α and β are constants and $A(e^{j\omega})$ is a real function of ω (possibly negative for some frequencies).

$A(e^{j\omega})$ = negative → an additional phase of π radians

- Phase response:

$$\angle H(e^{j\omega}) = \beta - \omega\alpha \quad \text{or} \quad \beta - \omega\alpha + \pi$$

Example:



- Group delay $\tau(\omega) = \alpha$
- All stable linear phase filters are FIR (IIR stable filters can not be linear phase).

Example of Linear Phase

- Example of (strictly) Linear Phase:

$$H(z) = z^{-1} + 2z^{-2} + z^{-3} \xrightarrow{\text{problem 2, week 8}} H(e^{j\omega}) = e^{-j2\omega}(2 + 2\cos\omega)$$

linear phase

non - negative real

- Example of (generalised) Linear Phase:

Click to see Phase Response

$$H(z) = 1 - z^{-1} \xrightarrow{z = e^{j\omega}} H(e^{j\omega}) = 1 - e^{-j\omega}$$

$$= e^{-j\omega/2}\{e^{j\omega/2} - e^{-j\omega/2}\}$$

$$= e^{-j\omega/2}\{2\,j\,\sin(\omega/2)\}$$

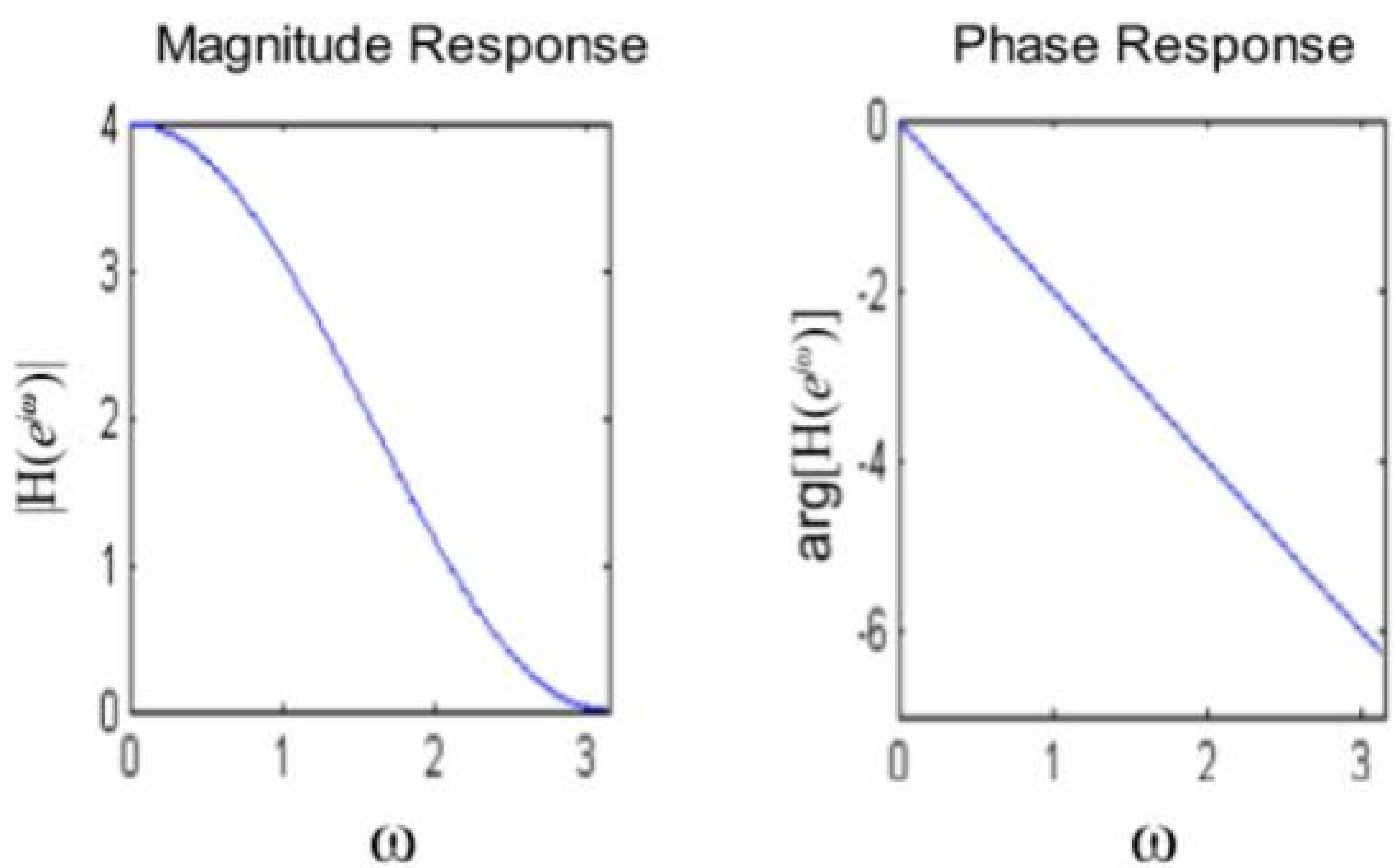
$$= e^{-j\omega/2} \cdot e^{j\pi/2}\{2\,\sin(\omega/2)\}$$

$$= e^{-j\omega/2 + j\pi/2} .2\,\sin(\omega/2)$$

generalised linear phase

real positive or negative

Example of Linear Phase



egative
al
e

generalised linear phase

real positive or negative

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Interactive Exercise: Facts about Linear Phase Filter

- One of the following facts about linear phase filters is false. Spot the false fact.

  1. Linear phase filters have constant group delay.
  2. Linear phase filters may be FIR or IIR.
  3. Linear phase filters may be causal or non-causal.

Submit   Correct Answer

Summary

By now, you should be able to:

- Explain Zero Phase property where its phase response is zero; and
- Explain Linear Phase property where its group delay is a constant.

Types of Linear Phase Systems

Learning Objectives

By the end of this topic, you should be able to:

- Categorise 4 types of Linear Phase Filters; and
- Identify if a filter is Linear Phase from its coefficient.

Types of Causal Linear Phase Systems

- A causal FIR filter of length $M+1$:

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + \cdots + h[M]z^{-M}$$

is linear phase if its impulse response $h[n]$ is symmetric or anti-symmetric.

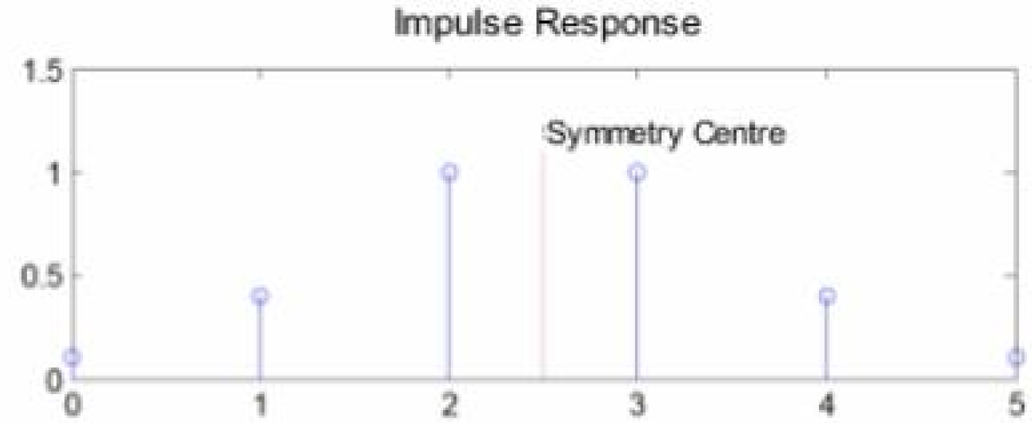
Symmetric Filters

$$h[n] = h[M-n] \quad \text{for } 0 \le n \le M$$

- Assume length is even, or $M = 2m + 1$
  then $h[0] = h[M]$, $h[1] = h[M\text{-}1]$, …, $h[m] = h[m+1]$

Example 7: Symmetric filter, length = 6 (even)

$$H(z) = 0.1 + 0.4z^{-1} + z^{-2} + z^{-3} + 0.4z^{-4} + 0.1z^{-5}$$



Symmetric Filters

$$H(e^{j\omega}) = h[0] + \cdots + h[m]e^{-j\omega m} + h[m+1]e^{-j\omega(m+1)} + \cdots + h[M]e^{-j\omega M}$$

$$= h[0] + \cdots + h[m]e^{-j\omega m} + h[m]e^{-j\omega(m+1)} + \cdots + h[0]e^{-j\omega M}$$

(using the symmetry)

$$= h[0](1 + e^{-j\omega m}) + h[1](e^{-j\omega} + e^{-j\omega(m-1)}) + \cdots + h[m](e^{-j\omega m} + e^{-j\omega(m+1)})$$

$$= e^{-j\omega M/2}\{\, h[0](e^{j\omega M/2} + e^{-j\omega M/2}) + h[1](e^{j\omega(M/2-1)} + e^{-j\omega(M/2-1)})$$
$$+ \cdots + h[m](e^{j\omega/2} + e^{-j\omega/2})\,\}$$

$$= e^{-j\omega M/2}\,\{\, h[0]2\cos(\omega\cdot\tfrac{M}{2}) + h[1]2\cos(\omega(\tfrac{M}{2}-1)) + \cdots + h[\tfrac{M-1}{2}]2\cos\,(\omega\cdot\tfrac{1}{2})\,\}$$

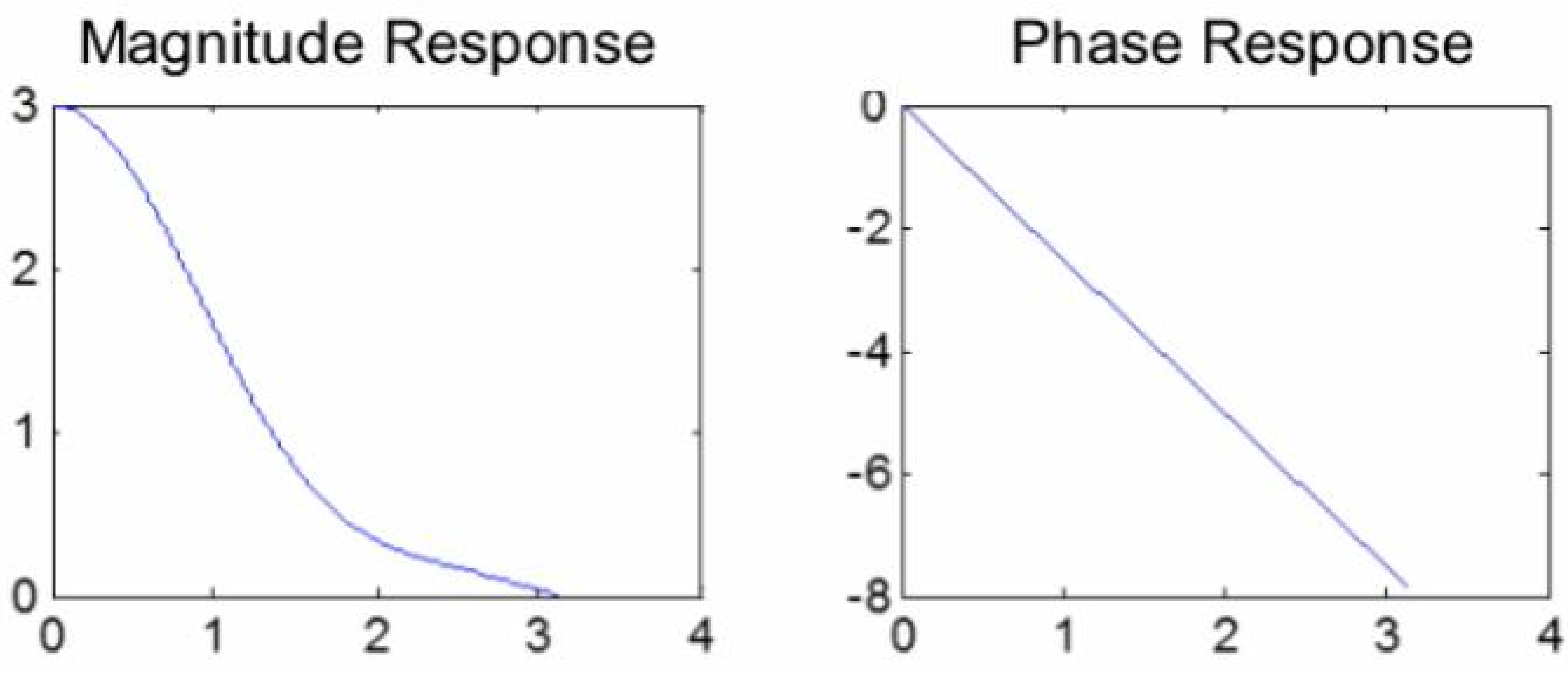
$$= e^{-j\omega M/2}A(e^{j\omega})$$

$A(e^{j\omega})$ = a function of $\cos(\omega/2)$, $\cos(3\omega/2)$, etc. $\rightarrow$ real

$e^{-j\omega M/2}$ shows that the phase is linear, and group delay is $M/2$

Symmetric Filters

Example 7(continued): symmetric filter, length = 6



Symmetric Filters

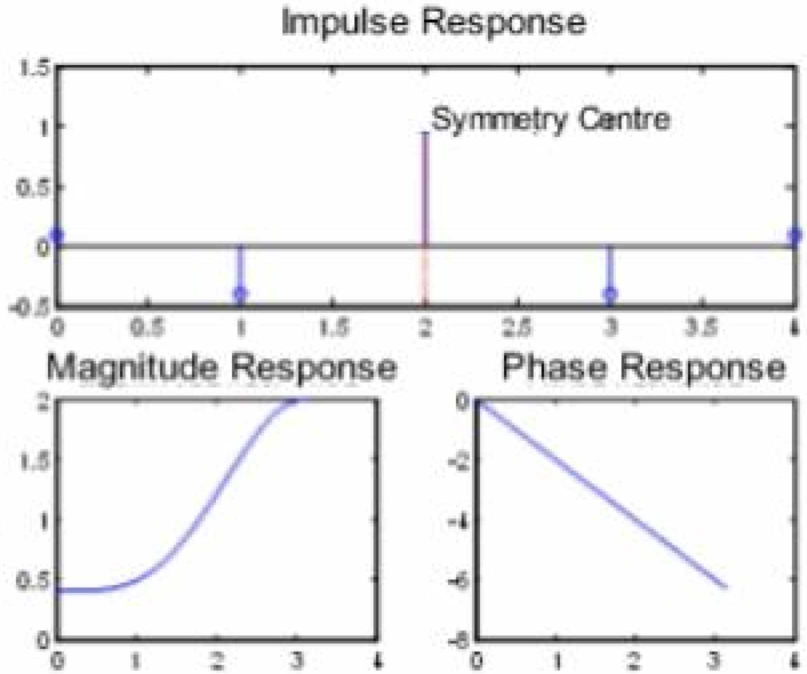
- Now assume length is odd, or $M = 2m$ then $h[0] = h[M]$, $h[1] = h[M\text{-}1]$, …, $h[m\text{-}1] = h[m\text{+}1]$ but $h[m]$ has no corresponding coefficient.
- It may similarly be shown that
$$H(e^{j\omega}) = e^{-j\omega M/2}\{h[0]2\cos(\omega\cdot\tfrac{M}{2}) + \cdots + h[\tfrac{M}{2}-1]2\cos(\omega) + h[\tfrac{M}{2}]\}$$

Click to see Derivation of Frequency Response

Note the linear phase, and same group delay of $M/2$

Example 8:
symmetric filter, length = 5(odd)

$$H(z) = 0.1 - 0.4z^{-1} + z^{-2} - 0.4z^{-3} + 0.1z^{-4}$$



Symmetric Filters

Symmetric odd length filter has linear phase

$$H(e^{j\omega}) = h[0] + \cdots + h[m-1]e^{-j\omega(m-1)} + h[m]e^{-j\omega m} + h[m+1]e^{-j\omega(m+1)} + \cdots + h[M]e^{-j\omega M}$$

$$= h[0] + \cdots + h[m-1]e^{-j\omega(m-1)} + h[m]e^{-j\omega m} + h[m-1]e^{-j\omega(m+1)} + \cdots + h[0]e^{-j\omega M}$$

$$= h[0](1+e^{-j\omega M}) + \cdots + h[m-1](e^{-j\omega(m-1)} + e^{-j\omega(m+1)}) + h[m]e^{-j\omega m}$$

middle term $h[m]$ has no other term to pair with

$$= e^{-j\omega M/2}\{h[0](e^{j\omega M/2} + e^{-j\omega M/2}) + \cdots + h[m-1](e^{j\omega} + e^{-j\omega}) + h[m]\}$$

$$= e^{-j\omega M/2}\{h[0]2\cos(\omega \cdot \tfrac{M}{2}) + \cdots + h[\tfrac{M}{2}-1]2\cos(\omega) + h[\tfrac{M}{2}]\}$$

m changed to $M/2$

$$= e^{-j\omega M/2} A(e^{j\omega})$$

Anti-Symmetric Filters

- $h[n] = -h[M-n] \quad \text{for } 0 \le n \le M$

  Assume length is even, or $M = 2m + 1$

  then $h[0] = -h[M]$, $h[1] = -h[M-1]$, …, $h[m] = -h[m+1]$

- It may similarly be shown that:

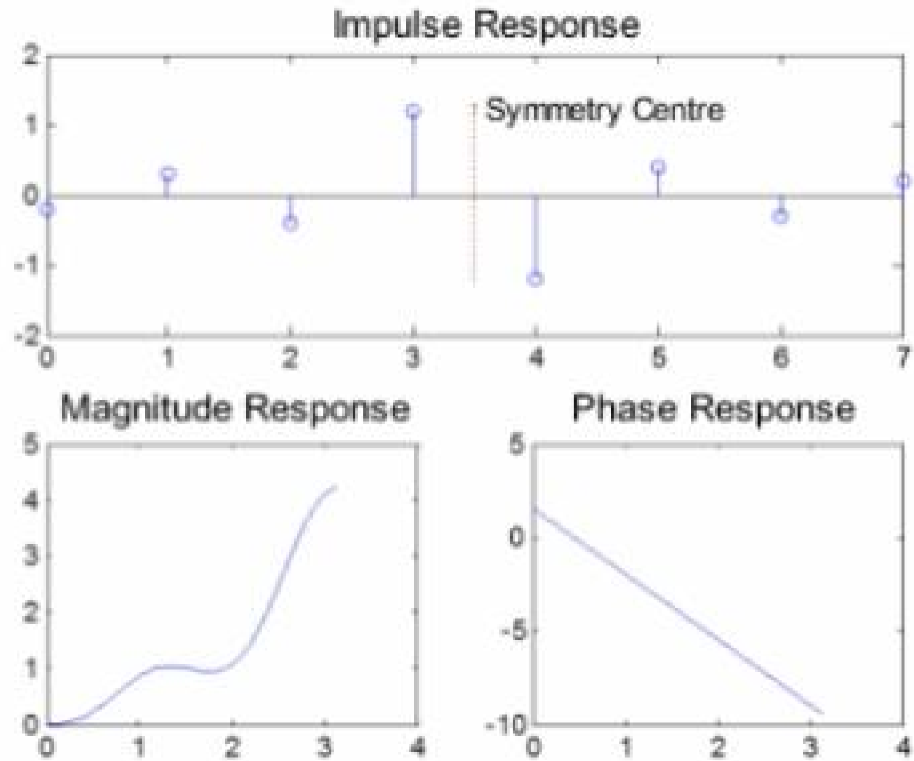
$$H(e^{j\omega}) = e^{-j\omega M/2 + j\pi/2}\left\{2h[0]\sin(\omega \cdot \tfrac{M}{2}) + 2h[1]\sin(\omega(\tfrac{M}{2}-1)) + \cdots + 2h[\tfrac{M-1}{2}]\sin(\omega \cdot \tfrac{1}{2})\right\}$$

Click to see Derivation of Frequency Response

The phase is linear, and group delay is $M/2$

Example 9: anti-symmetric filter, length = 8 (even)

$$H(z) = -0.2 + 0.3z^{-1} - 0.4z^{-2} + 1.2z^{-3} - 1.2z^{-4} + 0.4z^{-5} - 0.3z^{-6} + 0.2z^{-7}$$



Anti-Symmetric Filters

Anti-symmetric even length filter has linear phase

$$H(e^{j\omega}) = h[0] + \cdots + h[m]e^{-j\omega m} + h[m+1]e^{-j\omega(m+1)} + \cdots + h[M]e^{-j\omega M}$$

$$= h[0] + \cdots + h[m]e^{-j\omega m} - h[m]e^{-j\omega(m+1)} - \cdots - h[0]e^{-j\omega M}$$

$$= h[0](1 - e^{-j\omega M}) + \cdots + h[m](e^{-j\omega m} - e^{-j\omega(m+1)})$$

$$= e^{-j\omega M/2}\{h[0](e^{j\omega M/2} - e^{-j\omega M/2}) + \cdots + h[m](e^{j\omega/2} - e^{-j\omega/2})\}$$

$$= e^{-j\omega M/2}\{h[0]2j\sin(\omega \cdot \tfrac{M}{2}) + \cdots + h[\tfrac{M-1}{2}]2j\sin(\tfrac{\omega}{2})\}$$

$$= e^{-j\omega M/2} j\{2h[0]\sin(\omega \cdot \tfrac{M}{2}) + \cdots + 2h[\tfrac{M-1}{2}]\sin(\tfrac{\omega}{2})\}$$

$$= e^{-j\omega M/2} e^{j\pi/2}\{2h[0]\sin(\omega \cdot \tfrac{M}{2}) + \cdots + 2h[\tfrac{M-1}{2}]\sin(\tfrac{\omega}{2})\}$$

$$= e^{-j\omega M/2 + j\pi/2} A(e^{j\omega})$$

1

Anti-Symmetric Filters

- Assume length is odd, or $M = 2m$
  then $h[0] = -h[M]$, $h[1] = -h[M-1]$, …., $h[m-1] = -h[m+1]$
  and $h[m] = -h[m] = 0$

- It may similarly be shown that:

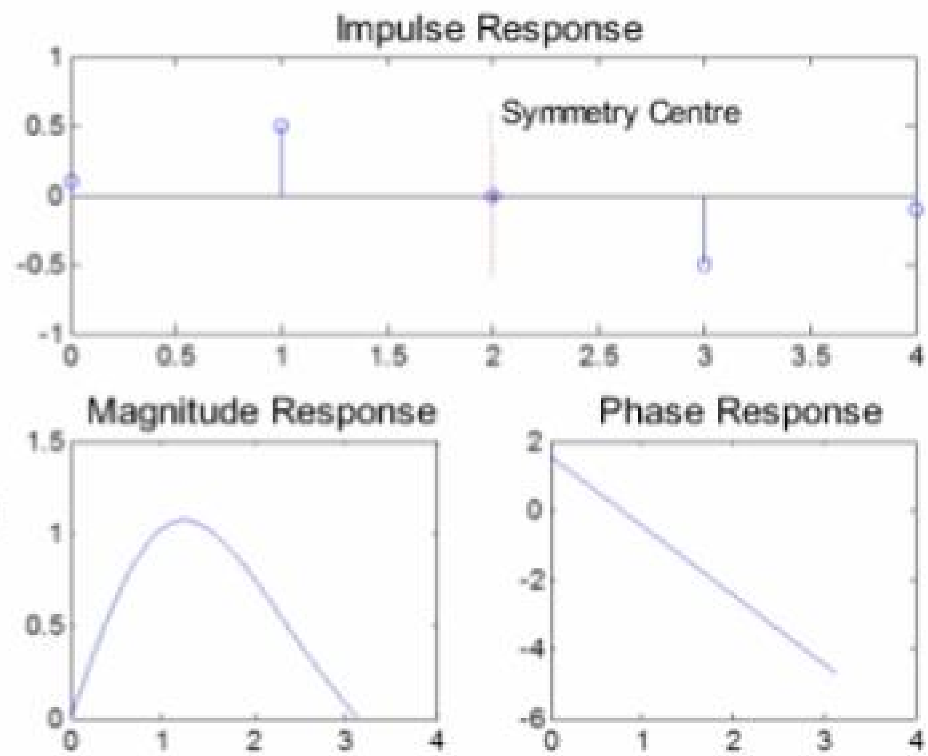
$$H(e^{j\omega}) = e^{-j\omega M/2 + j\pi/2}\left\{2h[0]\sin(\omega \cdot \tfrac{M}{2}) + 2h[1]\sin(\omega(\tfrac{M}{2}-1)) + \cdots + 2h[\tfrac{M}{2}-1]\sin(\omega)\right\}$$

Click to see Derivation of Frequency Response

Same linear phase and group delay

Example 10: anti-symmetric filter, length = 5 (odd)

$$H(z) = 0.1 + 0.5z^{-1} - 0.5z^{-3} - 0.1z^{-4}$$



Anti-Symmetric Filters

Anti-symmetric odd length filter has linear phase

$$
\begin{aligned}
H(e^{j\omega}) &= h[0] + \cdots + h[m-1]e^{-j\omega(m-1)} + h[m]e^{-j\omega m} + h[m+1]e^{-j\omega(m+1)} + \cdots + h[M]e^{-j\omega M} \\
&= h[0] + \cdots + h[m-1]e^{-j\omega(m-1)} + 0 - h[m-1]e^{-j\omega(m+1)} - \cdots - h[0]e^{-j\omega M} \\
&= h[0](1-e^{-j\omega M}) + \cdots + h[m-1](e^{-j\omega(m-1)} - e^{-j\omega(m+1)}) \\
&= e^{-j\omega M/2}\{h[0](e^{j\omega M/2} - e^{-j\omega M/2}) + \cdots + h[m-1](e^{j\omega} - e^{-j\omega})\} \\
&= e^{-j\omega M/2}\{h[0]2j\sin(\omega\cdot\tfrac{M}{2}) + \cdots + h[\tfrac{M}{2}-1]2j\sin(\omega)\} \\
&= e^{-j\omega M/2}j\{2h[0]\sin(\omega\cdot\tfrac{M}{2}) + \cdots + 2h[\tfrac{M}{2}-1]\sin(\omega)\} \\
&= e^{-j\omega M/2}e^{j\pi/2}\{2h[0]\sin(\omega\cdot\tfrac{M}{2}) + \cdots + 2h[\tfrac{M}{2}-1]\sin(\omega)\} \\
&= e^{-j\omega M/2+j\pi/2}A(e^{j\omega})
\end{aligned}
$$

1

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Interactive Exercise: Spot the Linear Phase Filter

- One of the following 3 filters is linear phase. Spot it.

1. $H(z) = 2 - 3z^{-1} + 4z^{-2} + 3z^{-3} - 2z^{-4}$
2. $H(z) = 1 + 5z^{-1} + 5z^{-2} + z^{-4}$
3. $H(z) = -3 + 7z^{-1} - 6z^{-2} + 7z^{-3} - 3z^{-4}$

Submit   Correct Answer

Summary

By now, you should be able to:

- Categorise 4 types of Linear Phase Filters; and
- Identify if a filter is Linear Phase from its coefficients.

Zeros of Linear Phase Systems

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Learning Objectives

By the end of this topic, you should be able to:

- Explain Zero Positions of Linear Phase Filters; and
- Identify if a Filter is Linear Phase from its Zeros.

Zero Positions for FIR Linear Phase Systems

- Take a symmetric coefficient filter:

$$H(z) = h[0] + h[1]z^{-1} + \cdots + h[M-1]z^{-M+1} + h[M]z^{-M}$$
$$= h[0] + h[1]z^{-1} + \cdots + h[1]z^{-M+1} + h[0]z^{-M}$$

- Replace z by z^{-1} $H(z^{-1}) = h[0] + h[1]z + \cdots + h[1]z^{M-1} + h[0]z^{M}$
- Multiply by z^{-M} $z^{-M}H(z^{-1}) = h[0]z^{-M} + h[1]z^{-M+1} + \cdots + h[1]z^{-1} + h[0]$
- Therefore $H(z) = z^{-M}H(z^{-1})$
- Let z_0 be a zero of $H(z)$ $\Rightarrow H(z_0) = 0$

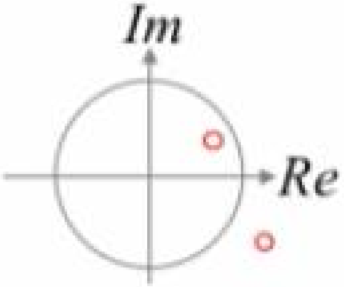
$$\Rightarrow z_0^{-M}H(z_0^{-1}) = 0$$

$\Rightarrow H(z_0^{-1}) = 0$ or, z_0^{-1} is also a zero of $H(z)$

- Similarly, for an anti-symmetric coefficient filter, $H(z) = -z^{-M}H(z^{-1})$ therefore, if z_0 is a zero of $H(z)$, then so is z_0^{-1}

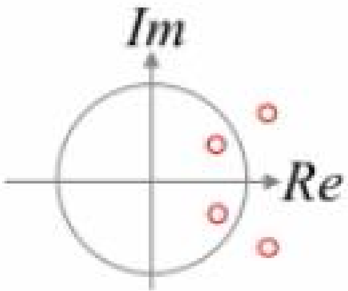
Zero Positions for FIR Linear Phase Systems

- Zeros occur in reciprocal pairs $re^{j\theta}$ and $\frac{1}{r}e^{-j\theta}$ in any linear phase filter.

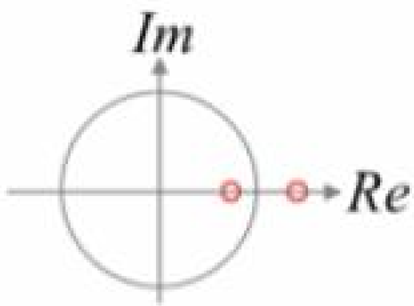


- Since complex zeros occur in conjugate pairs $re^{j\theta}$ and $re^{-j\theta}$ in any real coefficient filter,
  → Complex zeros occur in a set of four in a linear phase filter:

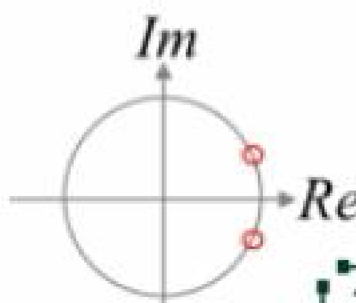
$$re^{j\theta}, re^{-j\theta}, \frac{1}{r}e^{j\theta}, \frac{1}{r}e^{-j\theta}$$



Exceptions: Real zeros occur in reciprocal pairs: $r, 1/r$

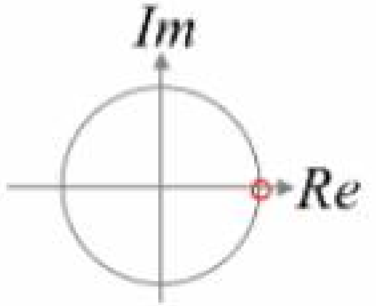


Zeros on the unit circle occur in conjugate pairs: $e^{j\theta}, e^{-j\theta}$



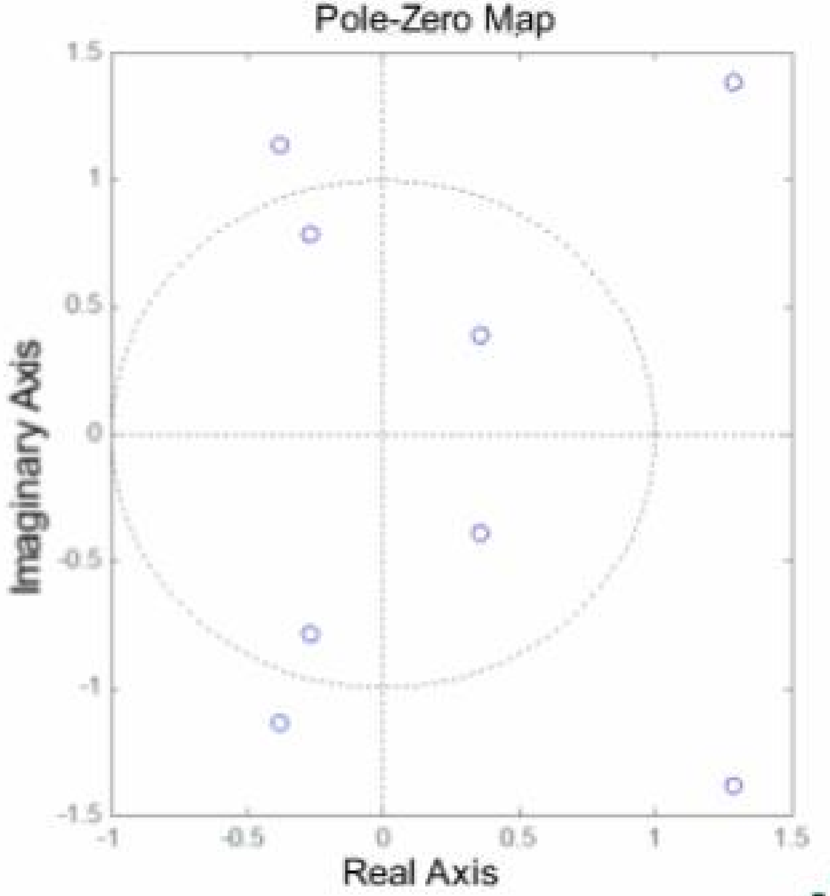
Zero Positions for FIR Linear Phase Systems

- Zero at +1 or –1 may occur alone, since it is reciprocal and conjugate of itself.



Example 11:
Pole zero plot of a linear phase filter

$$H(z) = 1 - 2z^{-1} + 4z^{-2} - 3z^{-3} + 8z^{-4} - 3z^{-5} + 4z^{-6} - 2z^{-7} + z^{-8}$$



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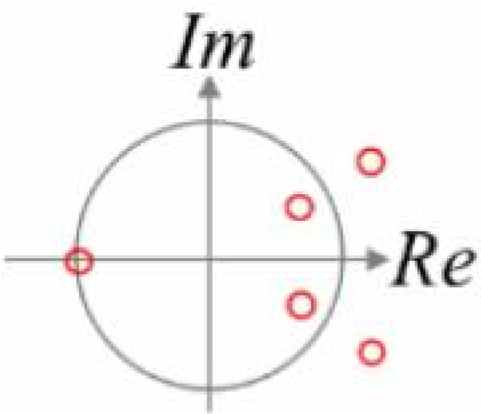
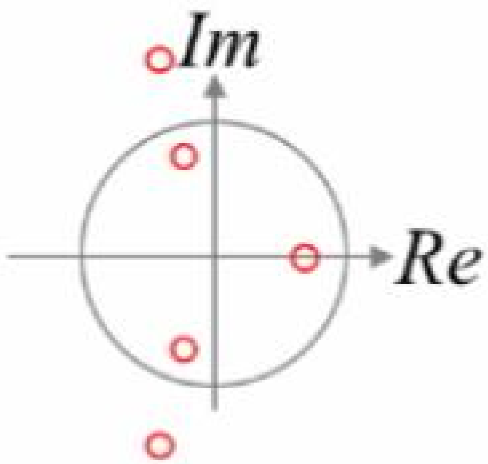
Interactive Exercise: Spot the Linear Phase Filter

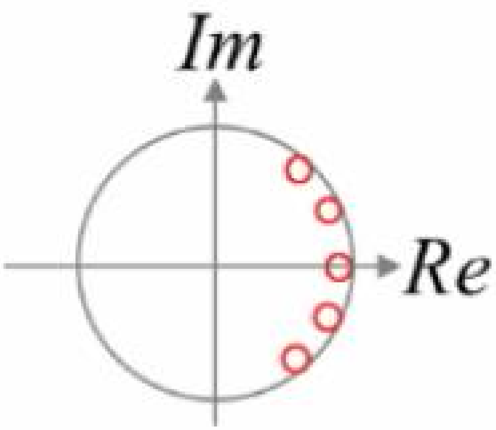
- Pole-zero plot of 3 filters are shown. One of them is linear phase. Spot it.



1.



2.



3.

Submit   Correct Answer

Summary

By now, you should be able to:

- Explain Zero Positions of Linear Phase Filters; and
- Identify if a Filter is Linear Phase from its Zeros.

Filter Types from Poles and Zeros

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Learning Objectives

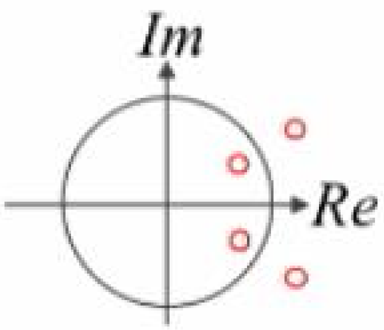
By the end of this topic, you should be able to:

- Identify various filter types from Pole and Zero Positions, such as:
  - FIR or IIR
  - Real coefficient
  - Stable
  - Allpass
  - Minimum or Maximum Phase
  - Linear Phase

Summary of Filter Type from Poles and Zeros

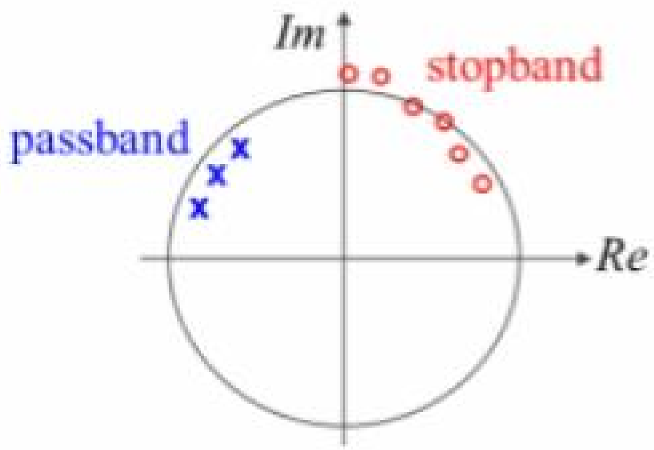
(neglect pole/zero at 0 or ∞)

Filter	Zero	Pole
FIR	Yes	No
IIR	Maybe	Yes
Real coefficient	If present, then in conjugate pairs	If present, then in conjugate pairs
Causal stable	Maybe	If present, then inside unit circle
Allpass	Yes	Conjugate reciprocal of zero
Minimum phase	Inside unit circle	*(ignore)*
Maximum phase	Outside unit circle	*(ignore)*
Stable linear phase	In conjugate+reciprocal of 4	No

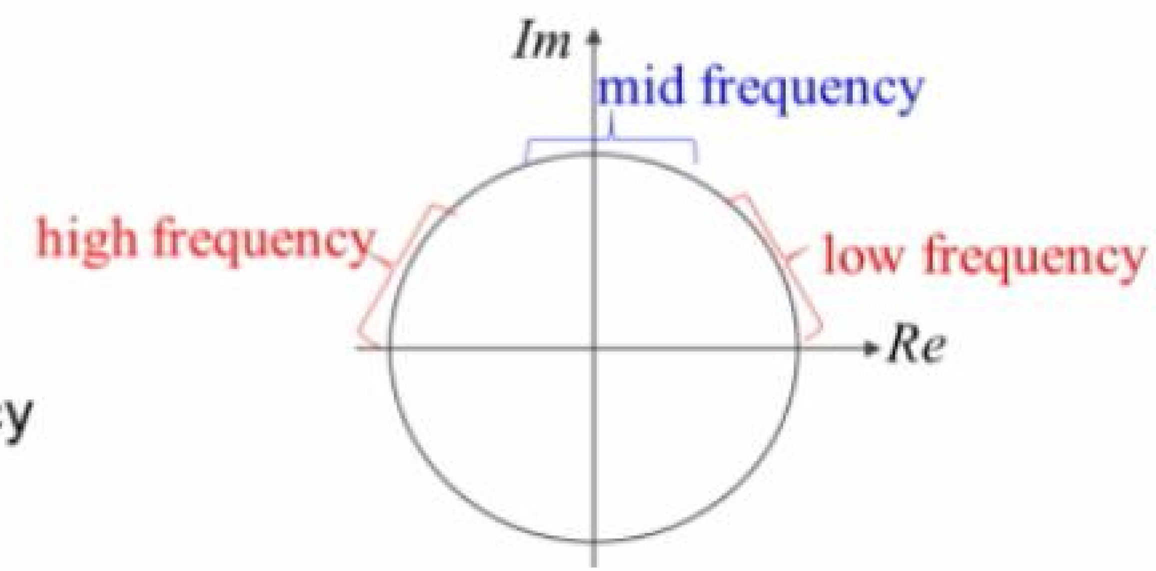


Summary of Filter Type from Poles and Zeros

- Further, low pass/ highpass/ band pass/ band stop nature of a filter may also be guessed from the position of its poles and zeros.



Poles near unit circle = passband
Zeros on/near unit circle = stopband



Low, mid, and high frequency sectors of the unit circle

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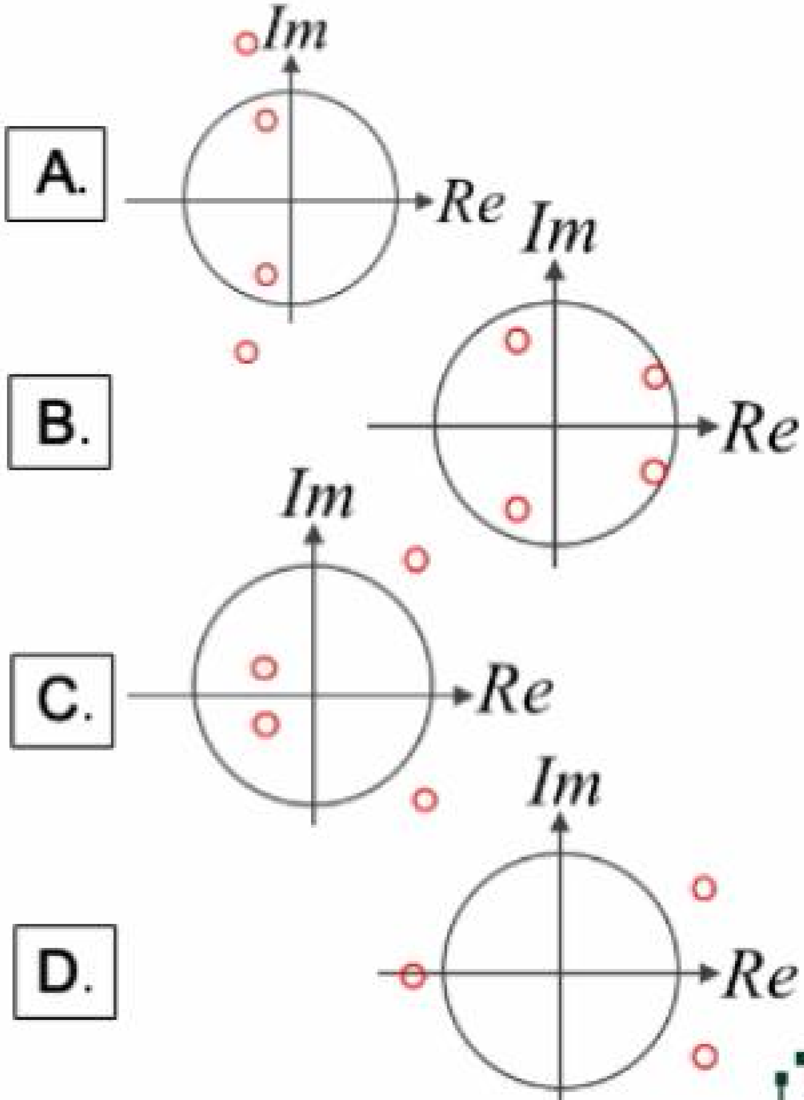
Interactive Exercise: Match a Filter with its Pole Zero Plot

- Match the filter types on the left with the pole-zero plots on the right.

1. Minimum phase filter
2. Maximum phase filter
3. Linear phase filter
4. Filter has no special phase (neither minimum, nor maximum, nor linear phase)



Correct Answer

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Application Example: Motion Blur

- Assume $x[n]$ to be an image (photograph) taken by a stationary digital camera. Assume $y[n]$ to be the same image taken from a moving vehicle. Due to motion, the second image gets blurred, since each pixel $y[n]$ now moves in front of several original pixel positions $x[n]$, $x[n\text{-}1]$, etc., and becomes equal to the average of these original pixels.

- Each pixel of the camera has a span of 0.1mm.
  The image is taken for an aperture time of 0.0015sec.
  The speed of the vehicle is 0.2m/sec.

  → In 0.0015sec, the camera moves 0.0003m = 0.3mm = 3 pixels

  → $y[n]$ is the average of three $x[n]$'s → $y[n] = \dfrac{x[n]+x[n-1]+x[n-2]}{3}$

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Application Example: Motion Blur

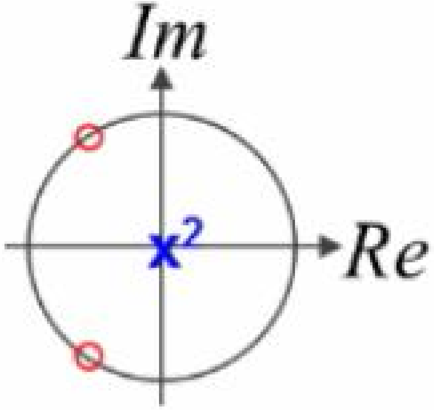
Photo taken from a stationary camera = original image $x[n]$

Photo taken from a moving camera = blurred image $y[n]$

Application Example: Motion Blur

- Express the above motion blur as a filtering operation on the original image $x[n]$. How does such filtering affect the original image? Is it possible to use another filter to get back $x[n]$?
- Difference equation $y[n] = \frac{1}{3}x[n] + \frac{1}{3}x[n-1] + \frac{1}{3}x[n-2]$

→ System function $H(z) = \frac{1}{3} + \frac{1}{3}z^{-1} + \frac{1}{3}z^{-2}$

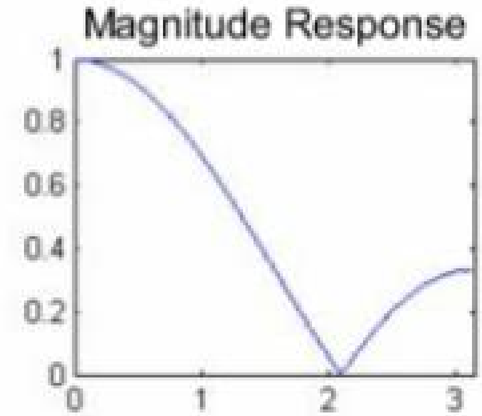


- Symmetric filter → linear phase, no phase distortion.
- Factorising, $H(z) = \frac{1}{3}(1 - e^{j2\pi/3}z^{-1})(1 - e^{-j2\pi/3}z^{-1})$

→ Two zeros at $r = 1$, $\theta = \pm 2\pi/3$

- Since these zeros are on the unit circle, they completely stop the frequencies of ± 2π/3 radians.

Motion blur
= low pass filter



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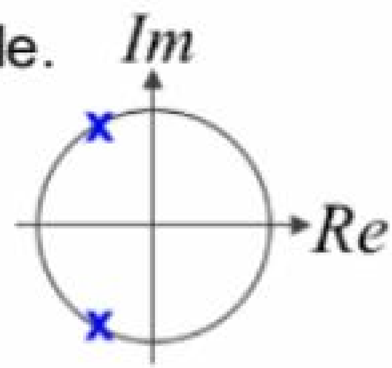
Application Example: Motion Blur

- To get back $x[n]$, the magnitude distortion has to be compensated. Exactly inverse magnitude response of a zero may be obtained by a pole at the same location.
- Unfortunately, for $H(z)$, both zeros are on the unit circle.

$\rightarrow$ If we design another filter with exactly inverse magnitude response, the poles will also be on the unit circle.

$\rightarrow$ Filter will be unstable.



So, the exact original image $x[n]$ can not be obtained from the blurred image $y[n]$.

(However, $y[n]$ may be improved significantly using a stable filter).

Summary

By now, you should be able to:

- Identify various filter types from Pole and Zero Positions, such as:
  - FIR or IIR
  - Real coefficient
  - Stable
  - Allpass
  - Minimum or Maximum Phase
  - Linear Phase

Direct Form I Stucture

for IIR Filters

Learning Objectives

By the end of this topic, you should be able to:

- Describe the Building Blocks used in the filter structures;
- Sketch the Direct Form I Structure for any IIR filter from its system function/ CCDE; and
- Identify the system function/ CCDE of any IIR filter from its Direct Form I Structure.

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Building Blocks in a Structure

- Difference equation (scale to make $a_0 = 1$):

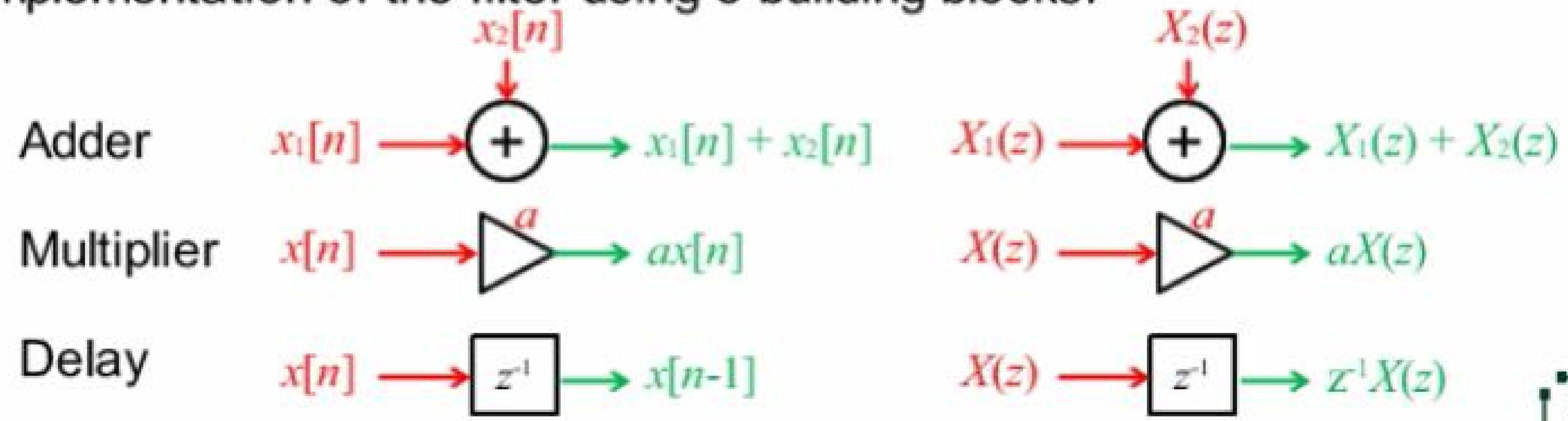
$$y[n]-a_1 y[n-1]-\cdots-a_N y[n-N]=b_0 x[n]+b_1 x[n-1]+\cdots+b_M x[n-M]$$

- Take z-transform of both sides:

$$Y(z)-a_1 z^{-1}Y(z)-\cdots-a_N z^{-N}Y(z)=b_0 X(z)+b_1 z^{-1}X(z)+\cdots+b_M z^{-M}X(z)$$

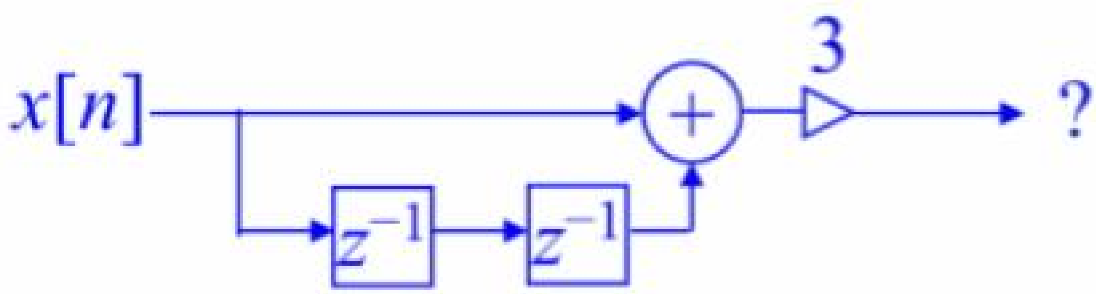
- System function: $H(z)=\dfrac{Y(z)}{X(z)}=\dfrac{b_0+b_1 z^{-1}+b_2 z^{-2}+\cdots+b_M z^{-M}}{1-a_1 z^{-1}-a_2 z^{-2}-\cdots-a_N z^{-N}}$

- Implementation of the filter using 3 building blocks:



Work Example on Building Blocks

- Here we have the outputs for the following interconnection of building blocks.



1. $x[n] + 2z^{-1} + 3$
2. $3x[n] + 3z^{-2}$
3. $3x[n] + 3x[n-2]$
4. $x[n] + 3z^{-2}x[n]$

Submit

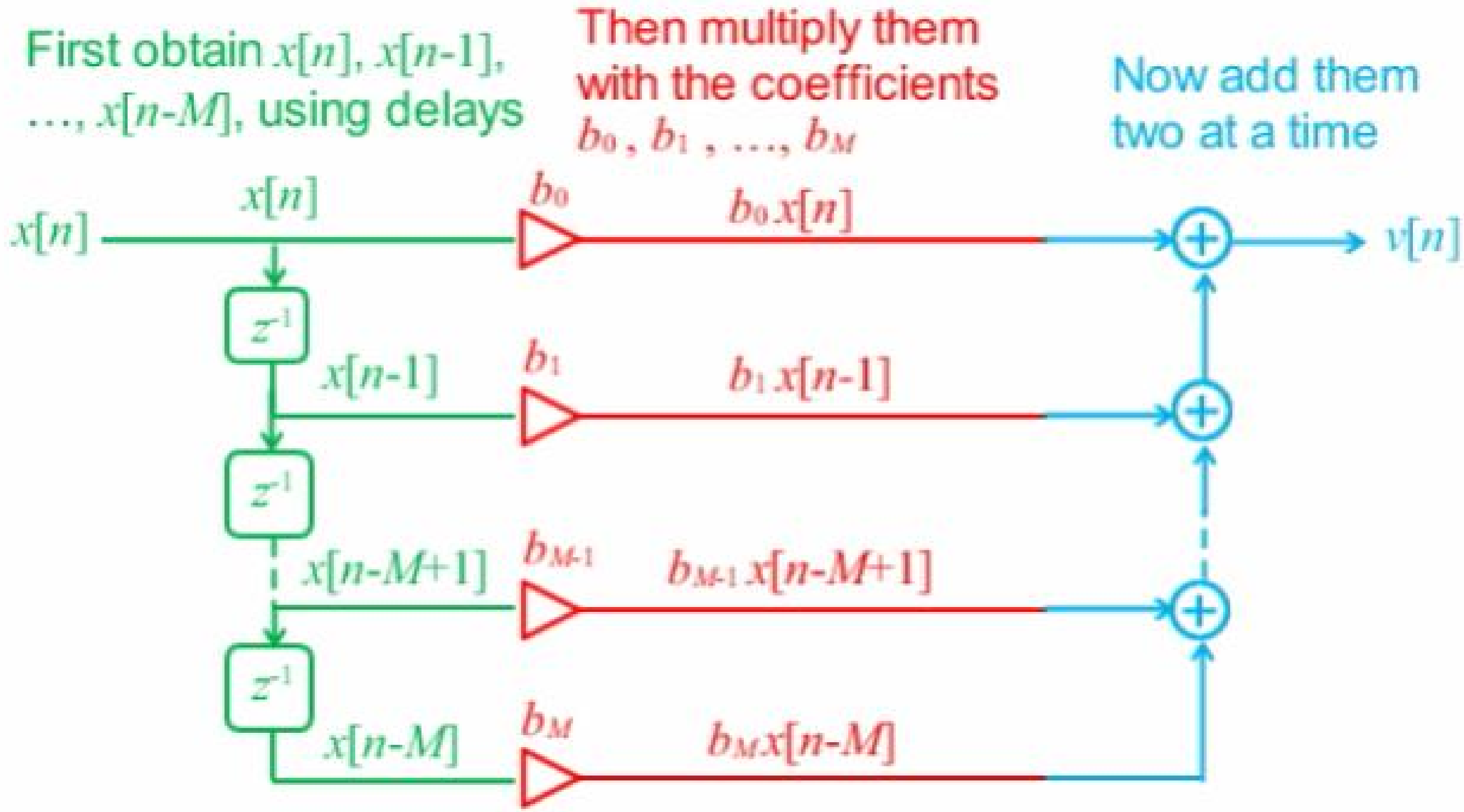
Correct Answer

Direct Form I Structure for IIR Filters: First Half

- Difference equation:

$$y[n] = a_1 y[n-1] + \cdots + a_N y[n-N] + b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

- Implementation: $v[n] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$

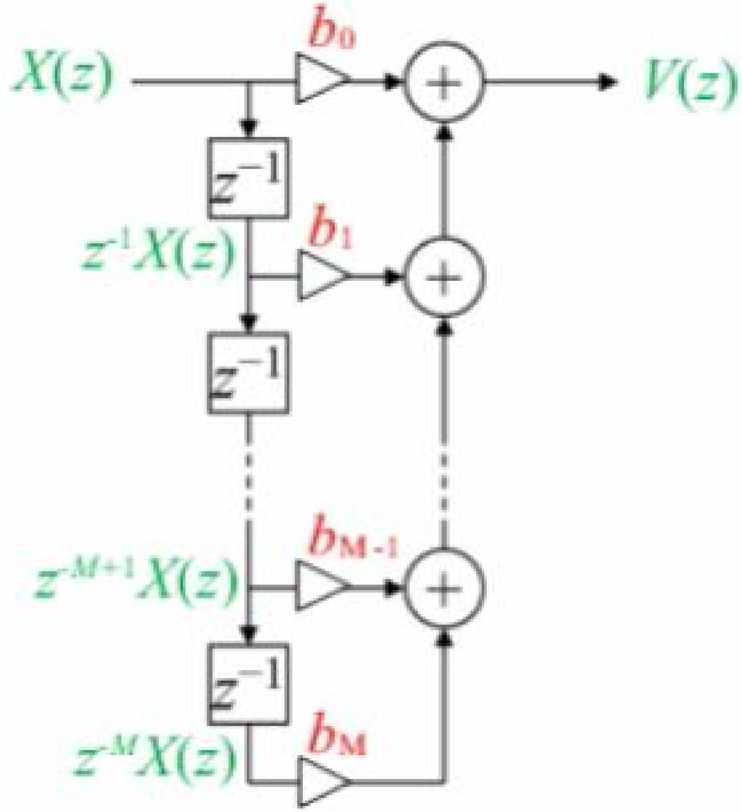


Direct Form I Structure for IIR Filters: First Half

- The structure may also be explained in z-domain from

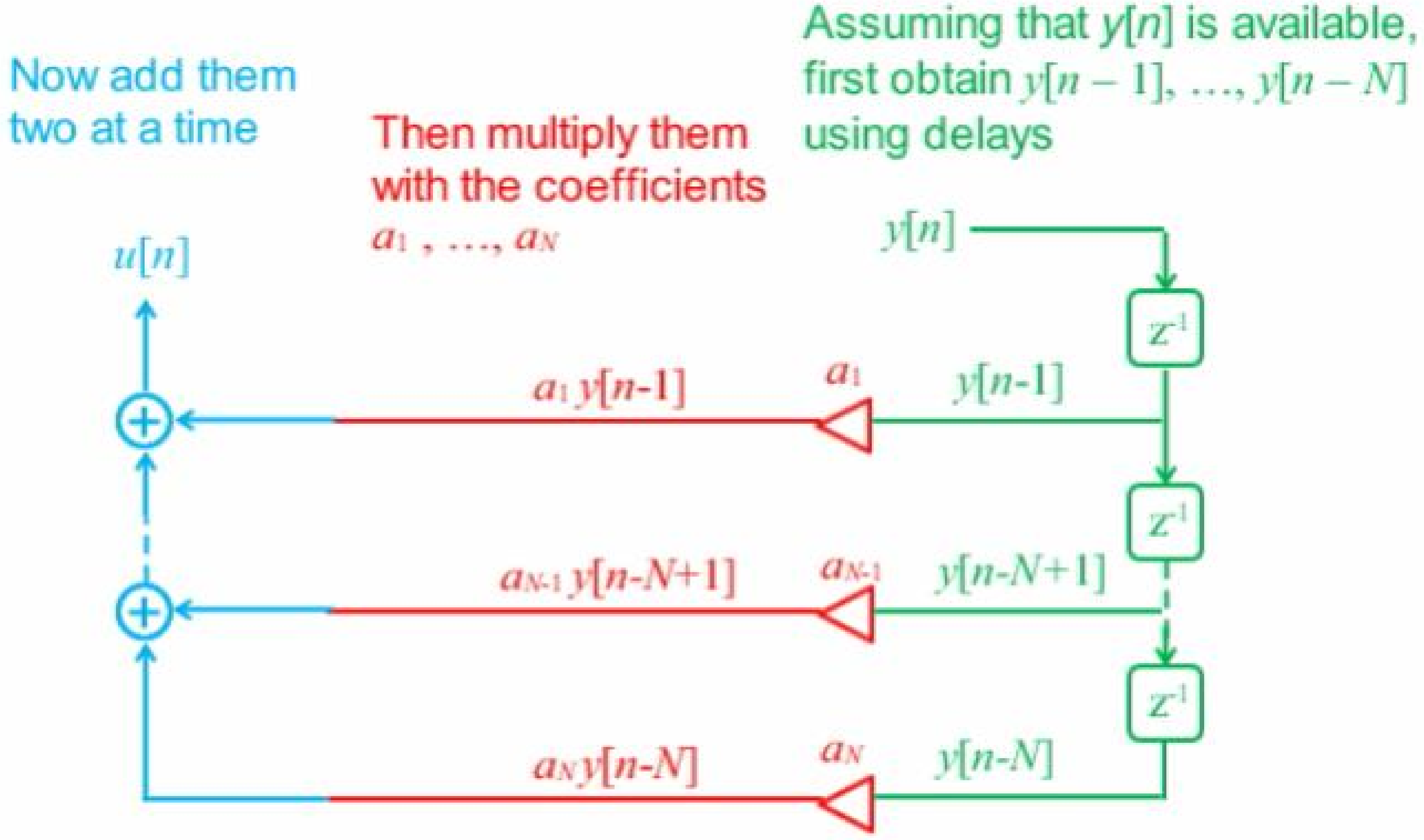
$$v[n] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

- We obtain $V(z) = b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z)$



Direct Form I Structure for IIR Filters: Second Half

- Similarly, implement: $u[n] = a_1 y[n-1] + \cdots + a_N y[n-N]$

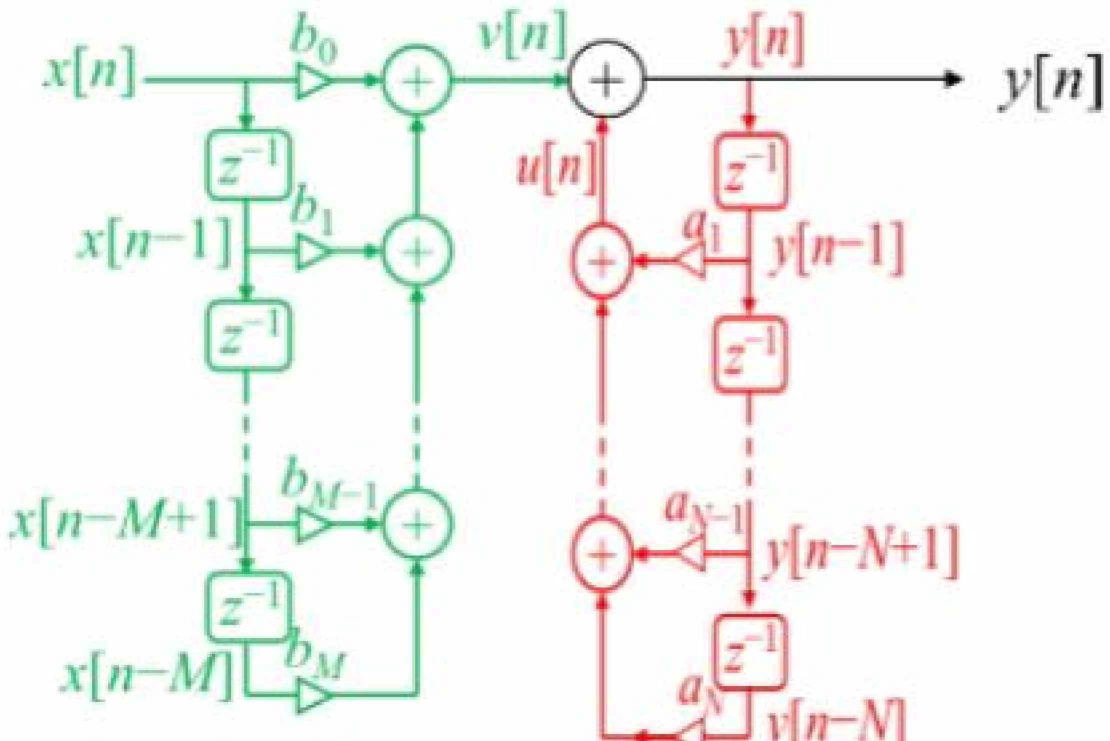


Direct Form I Structure for IIR Filters: Second Half

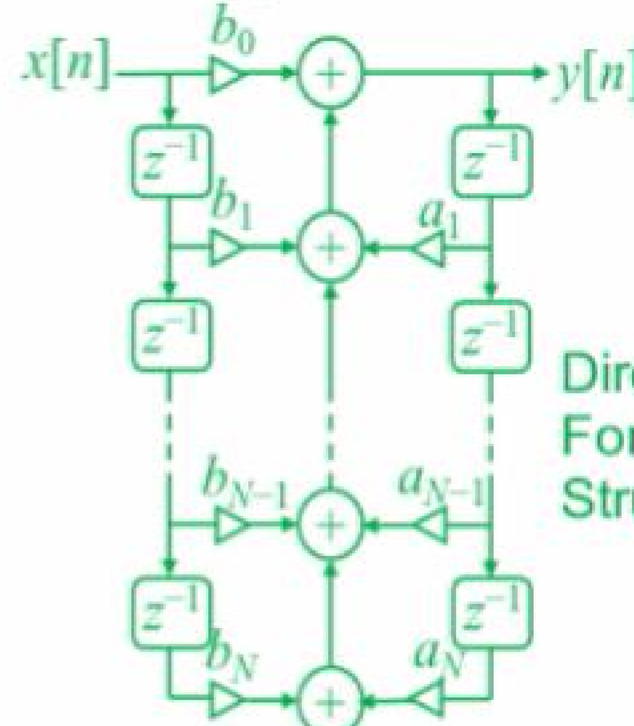
- Difference equation:

$$y[n]=a_1y[n-1]+\cdots+a_Ny[n-N]+b_0x[n]+b_1x[n-1]+\cdots+b_Mx[n-M]$$

- Now reduces to $y[n]=u[n]+v[n]$
- Step 1: implement $v[n]$
- Step 2: implement $u[n]$
- Step 3: add $v[n]$ and $u[n]$



- Simplify the structure by combining the adder chain



Direct Form I Structure

for $M = N$, $2N$ adders, $2N + 1$ multipliers, $2N$ delays

Summary

By now, you should be able to:

- Describe the Building Blocks used in the filter structures;
- Sketch the Direct Form I Structure for any IIR filter from its system function/ CCDE; and
- Identify the system function/ CCDE of any IIR filter from its Direct Form I Structure.

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Direct Form II and

Direct Form FIR Structures

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Learning Objectives

By the end of this topic, you should be able to:

- Sketch the Direct Form II Structure for any IIR filter from its system function/ CCDE;
- Identify the system function/ CCDE of any IIR filter from its Direct Form II Structure;
- Sketch the Direct Form Structure for any FIR filter from its system function/ CCDE; and
- Identify the system function/ CCDE of any FIR filter from its Direct Form Structure.

Direct Form II Structure for IIR Filters

- Direct form I implements the filter in 2 halves:

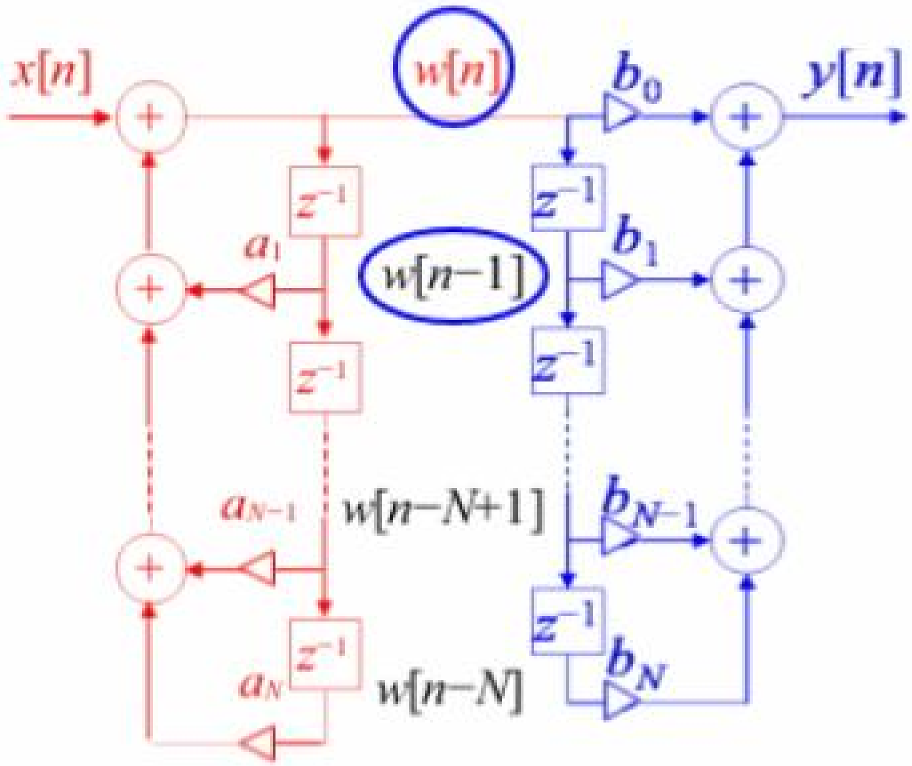
$$X(z) \longrightarrow \boxed{\frac{V(z)}{X(z)} = b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}} \xrightarrow{V(z)} \boxed{\frac{Y(z)}{V(z)} = \frac{1}{1 - a_1 z^{-1} - \cdots - a_N z^{-N}}} \longrightarrow Y(z)$$

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 - a_1 z^{-1} - \cdots - a_N z^{-N}}$$

- Change the order of these two halves:

$$X(z) \longrightarrow \boxed{\frac{W(z)}{X(z)} = \frac{1}{1 - a_1 z^{-1} - \cdots - a_N z^{-N}}} \xrightarrow{W(z)} \boxed{\frac{Y(z)}{W(z)} = b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}} \longrightarrow Y(z)$$

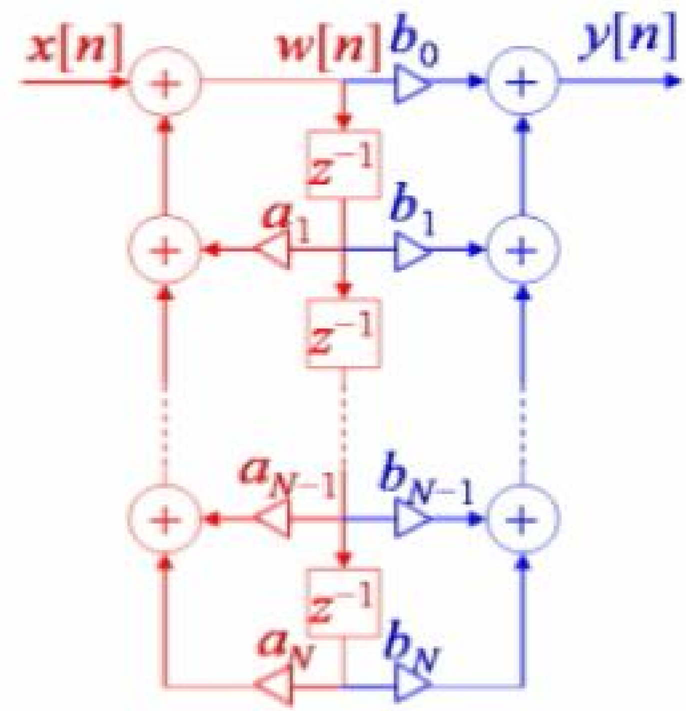
Direct Form II Structure for IIR Filters



Assume $M = N$

- Note that the delayed signals in both halves are the same.

- Simplify the structure by combining the delay chains.

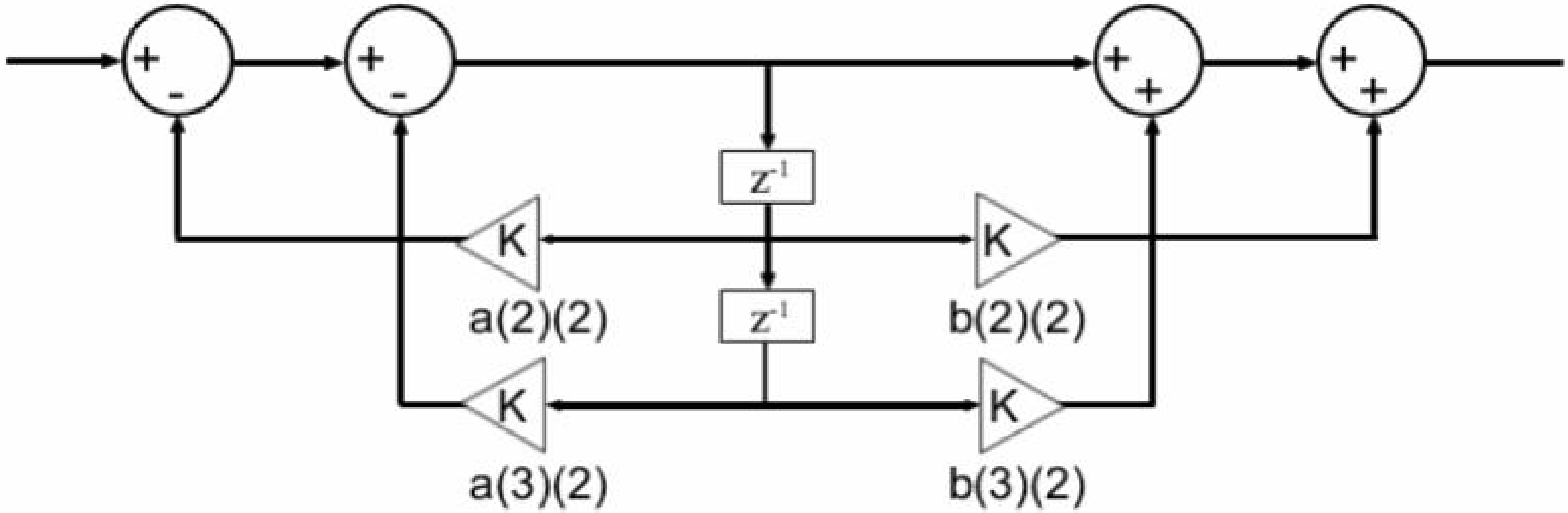


Direct Form II Structure

- Requires $2N$ adders, $2N$+1 multipliers, N delays.

Example on Direct form II structure of a filter using Matlab 'fdatool'

$$H(z) = \frac{1 + b_{22}z^{-1} + b_{32}z^{-2}}{1 + a_{22}z^{-1} + a_{32}z^{-2}}$$



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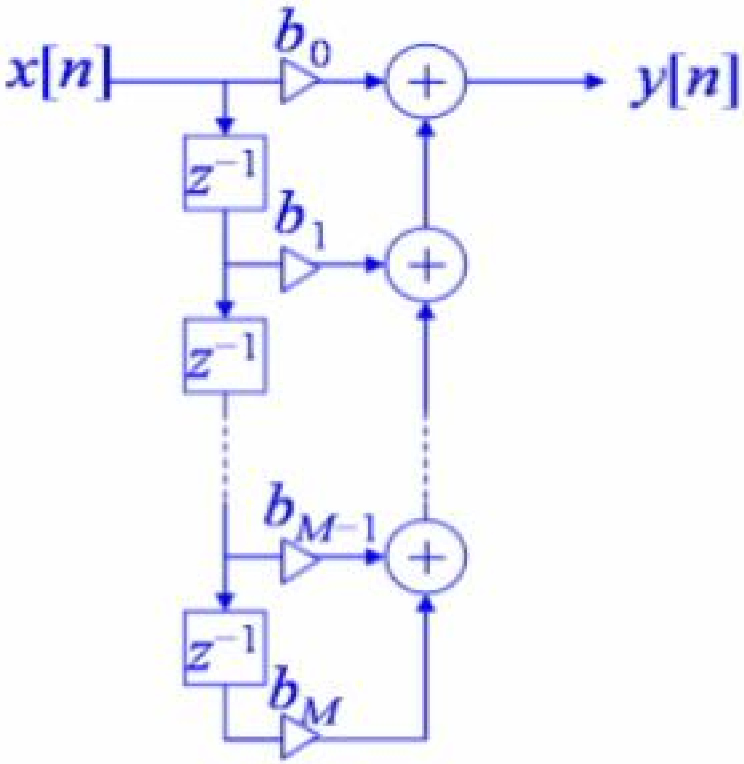
Direct Form Structure for FIR Filters

- Difference equation $y[n] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$
- System function $H(z) = \dfrac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}$
- FIR structure is obtained by making all $a_i = 0$ (removing them) in the IIR structure.



- Direct Form Structure is also known as tapped delay line structure and as transversal filter structure.
- It requires M adders, M+1 multipliers, M delays.
- Note that both direct form I and direct form II structures become identical since one half is removed.

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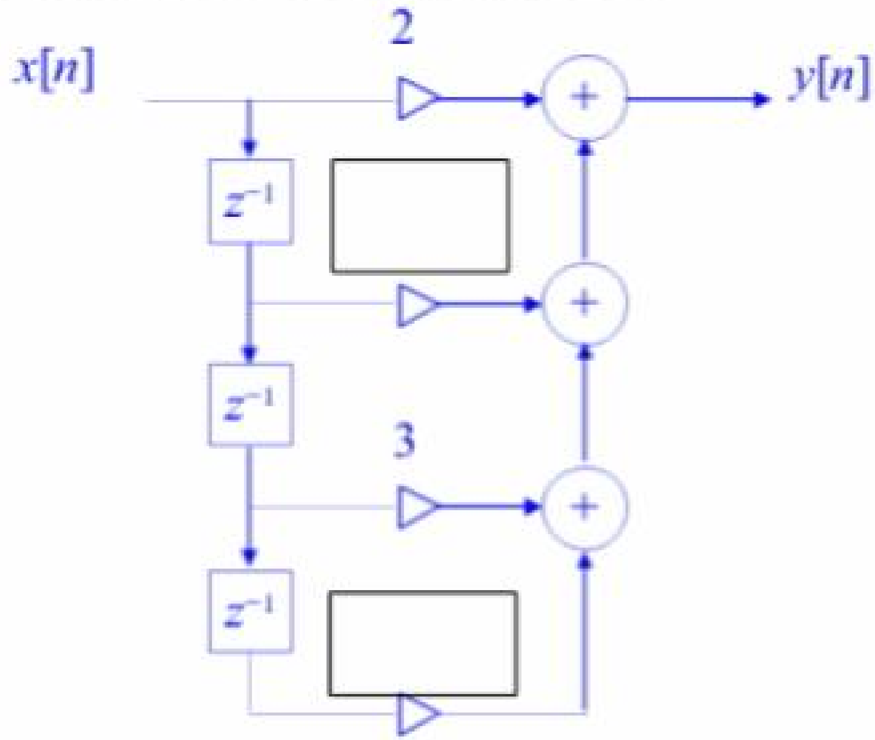
Work Example on Missing Multipliers

- The following direct form structure implements an FIR filter with system function:

$$H(z) = 2 - 5z^{-1} + 3z^{-2} + 7z^{-3}$$

- Find the missing values of the multipliers.



Submit

Correct Answer

Summary

By now, you should be able to:

- Sketch the Direct Form II Structure for any IIR filter from its system function/ CCDE;
- Identify the system function/ CCDE of any IIR filter from its Direct Form II Structure;
- Sketch the Direct Form Structure for any FIR filter from its system function/ CCDE; and
- Identify the system function/ CCDE of any FIR filter from its Direct Form Structure.

Transposed Form Structures

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Learning Objectives

By the end of this topic, you should be able to:

- Sketch the Transposed Direct form I and II Structures for any IIR filter from its system function/ CCDE;
- Identify the system function/ CCDE of any IIR filter from its Transposed Direct form I and II Structures;
- Sketch the Transposed Direct Form Structure for any FIR filter from its system function/ CCDE; and
- Identify the system function/ CCDE of any FIR filter from its Transposed Direct Form Structure.

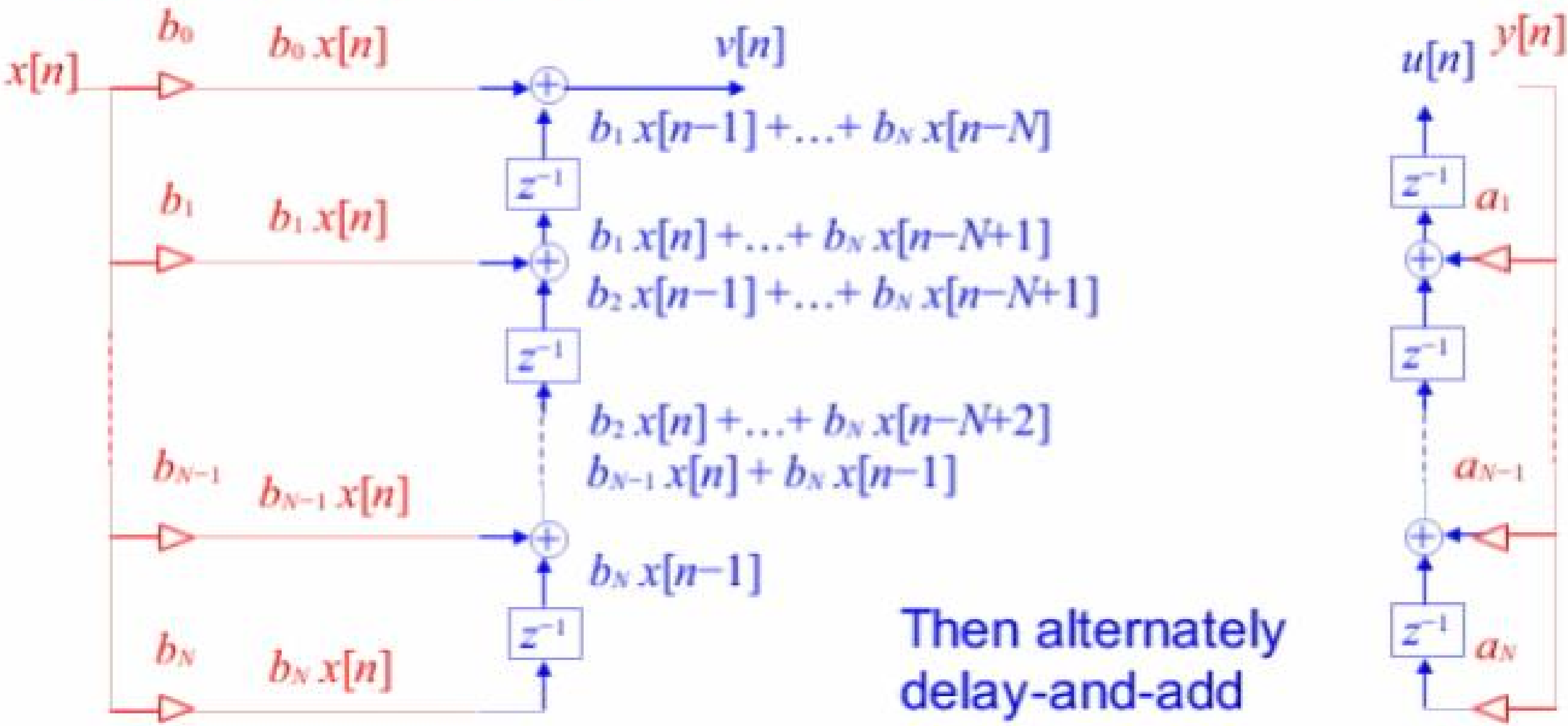
Transposed Direct Form II Structure for IIR Filters

- In direct form, each half was implemented using delay, multiply and add. We now consider a different sequencing to implement:

$$v[n] = b_0 x[n] + b_1 x[n-1] + \cdots + b_N x[n-N]$$

- Similarly, use the multiply and delay-and-add sequence to implement the other half: $u[n] = a_1 y[n-1] + \cdots + a_N y[n-N]$

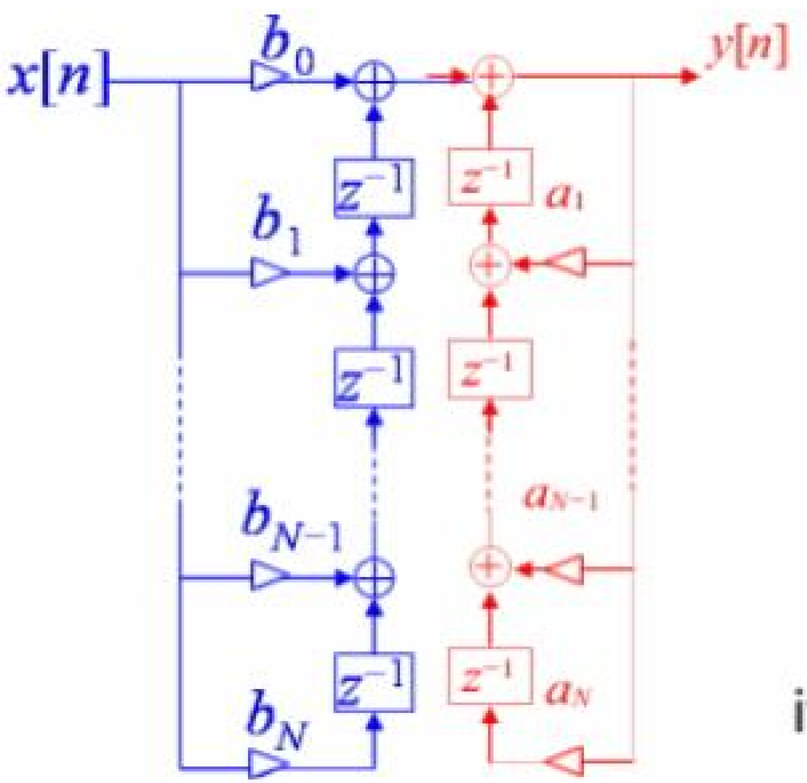
First multiply $x[n]$ with the coefficients b_0 , b_1 , …, b_N



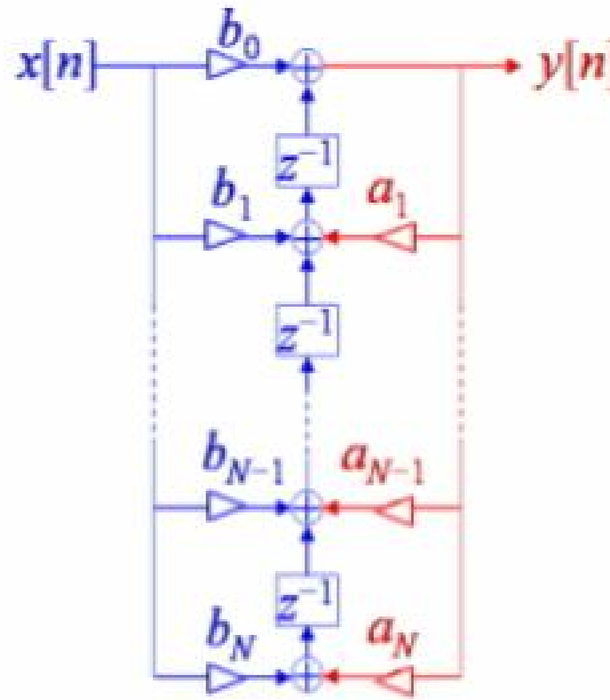
Then alternately delay-and-add

Transposed Direct Form II Structure for IIR Filters

- Similar to direct form, these two halves may be implemented in any order to give two structures.
- As in direct form I, implement the numerator $b_0 + b_1 z^{-1} + \ldots + b_N z^{-N}$ first, and then the denominator $\dfrac{1}{1 - a_1 z^{-1} - \ldots - a_N z^{-N}}$ of the system function to obtain.



simplify to →



Transposed direct form II structure

it requires $2N$ adders, $2N+1$ multipliers, N delays

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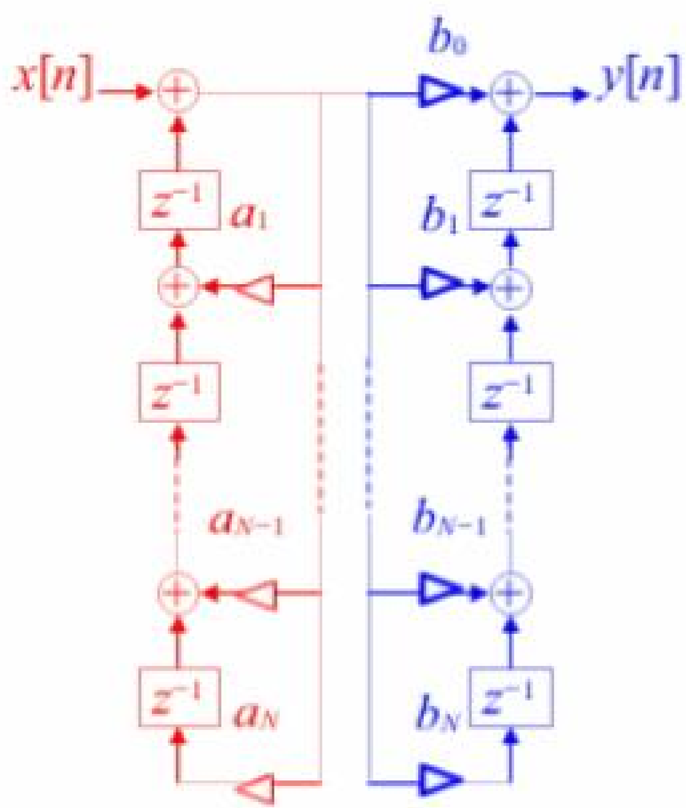
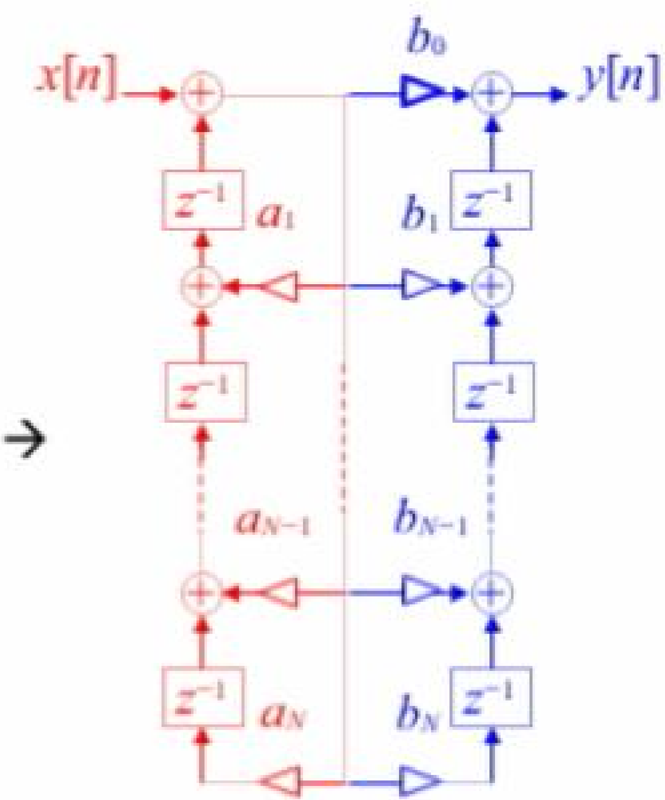
Transposed Direct Form I Structure for IIR Filters

- As in direct form II, implement the denominator $\dfrac{1}{1 - a_1 z^{-1} - \ldots - a_N z^{-N}}$ first,and then the numerator $b_0 + b_1 z^{-1} + \ldots + b_N z^{-N}$ of the system function to obtain.



simplify to →



Transposed Direct Form I Structure

It requires $2N$ adders, $2N+1$ multipliers, $2N$ delays

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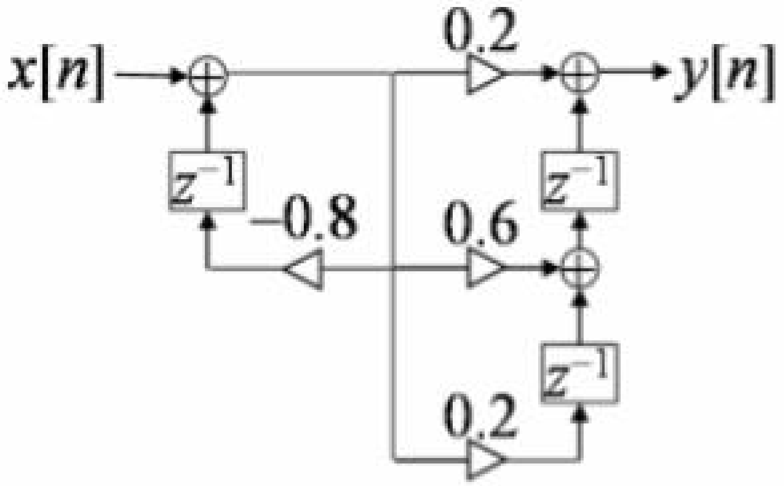
Work Example on System Function

- The structure below implements a filter with system function $H(z)$, which is partly shown. Complete $H(z)$ by writing the missing coefficients in the boxes provided.



$$H(z) = \frac{0.2 + \boxed{}\, z^{-1} + 0.2z^{-2}}{1 + \boxed{}\, z^{-1}}$$

Submit **Correct Answer**

4 IIR Structures

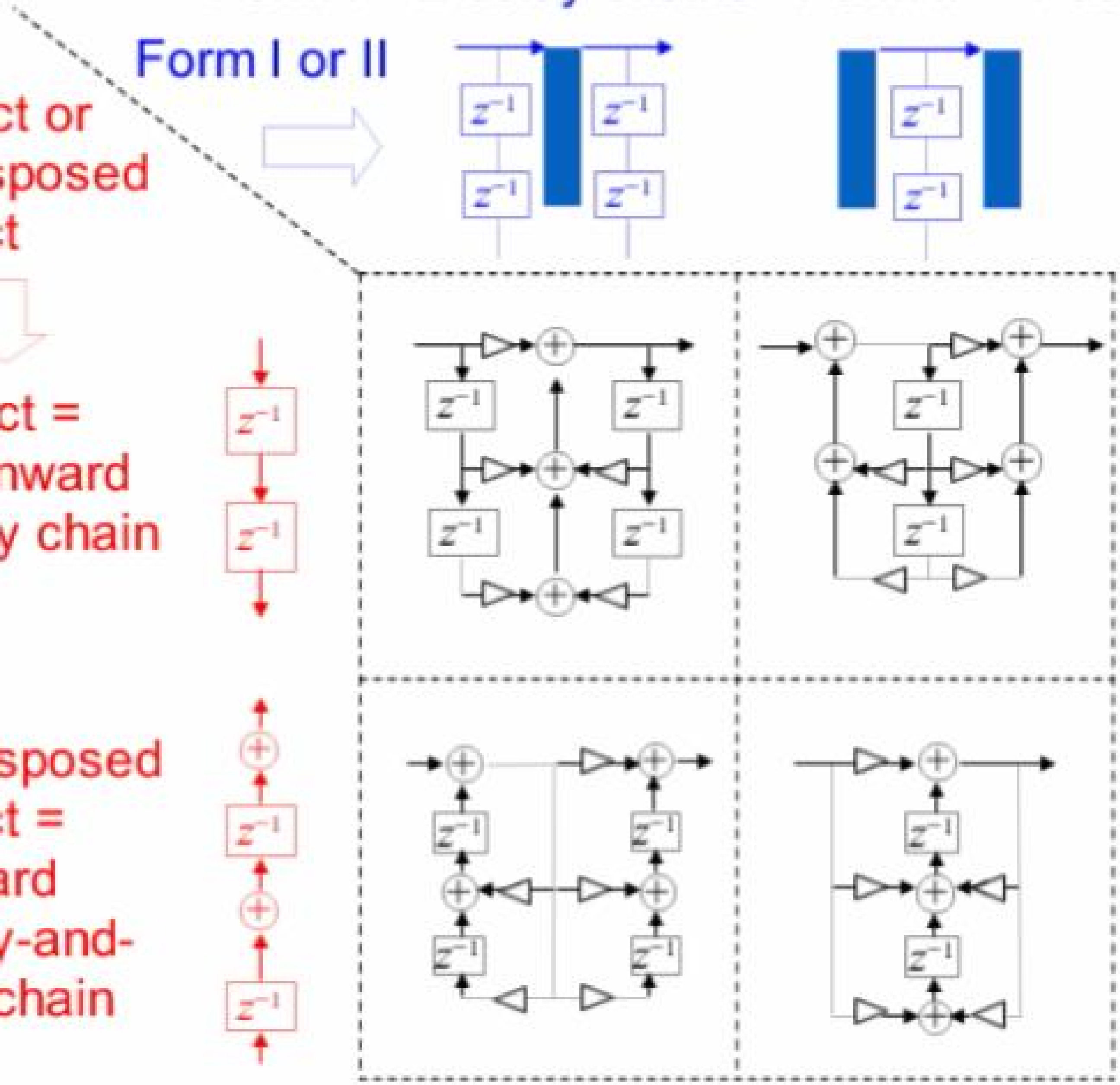
Form I = 2 delay chains Form II = 1 delay chain

Form I or II

Direct or transposed direct

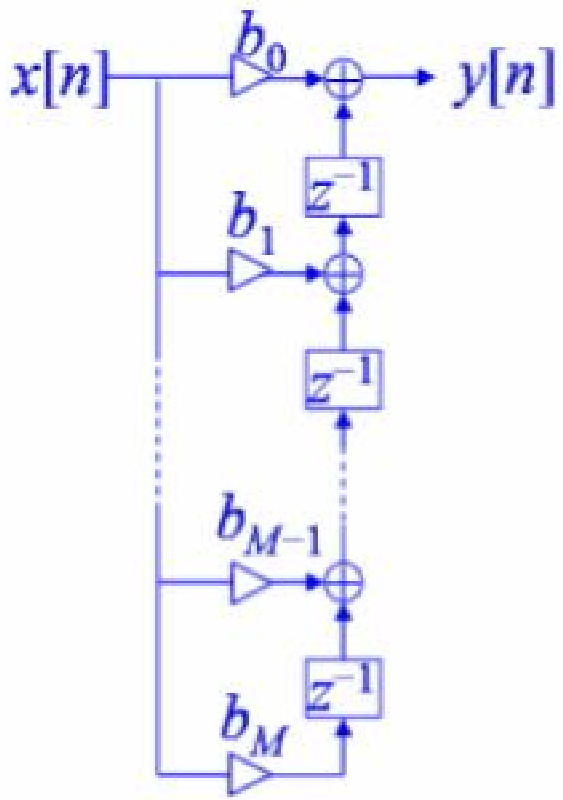
Direct = downward delay chain

Transposed direct = upward delay-and-add chain



Transposed Direct Form Structure for FIR Filters

- System function only has the numerator part, $b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}$
- Therefore, FIR structure is obtained by making all $a_i = 0$ (removing them) in the IIR structure.



Transposed Direct Form Structure

- It requires M adders, M+1 multipliers, M delays.
- Again, both transposed direct form I and transposed direct form II structures become identical since one half is removed.

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Summary

By now, you should be able to:

- Sketch the Transposed Direct form I and II Structures for any IIR filter from its system function/ CCDE;
- Identify the system function/ CCDE of any IIR filter from its Transposed Direct form I and II Structures;
- Sketch the Transposed Direct Form Structure for any FIR filter from its system function/ CCDE; and
- Identify the system function/ CCDE of any FIR filter from its Transposed Direct Form Structure.

Cascade and Parallel Form Structures

Learning Objectives

By the end of this topic, you should be able to:

- Sketch the Cascade Form Structure for any IIR/ FIR filter from its system function/ CCDE;
- Identify the system function/ CCDE of any IIR/ FIR filter from its Cascade Form Structure;
- Sketch the Parallel Form Structure for any IIR filter from its system function/ CCDE; and
- Identify the system function/ CCDE of any IIR filter from its Parallel Form Structure.

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Cascade Form Structure for IIR Filters

- Factorise the numerator and denominator polynomials of the system function.

$$H(z)=\frac{b_0+b_1z^{-1}+\cdots+b_Nz^{-N}}{1-a_1z^{-1}-\cdots-a_Nz^{-N}}$$

$$=\left(\frac{b_{01}+b_{11}z^{-1}+b_{21}z^{-2}}{1-a_{11}z^{-1}-a_{21}z^{-2}}\right)\times\left(\frac{b_{02}+b_{12}z^{-1}+b_{22}z^{-2}}{1-a_{12}z^{-1}-a_{22}z^{-2}}\right)\times\cdots\times\left(\frac{b_{0N_s}+b_{1N_s}z^{-1}+b_{2N_s}z^{-2}}{1-a_{1N_s}z^{-1}-a_{2N_s}z^{-2}}\right)$$

where $N_s=\frac{N}{2}$

- Implement individual second-order sections using any of the earlier structures, then cascade each of these sections to obtain the overall structure.
- In practice, cascade form may be attractive due to its:
  - Modular structure
  - Ease of stability check
  - Less effect of coefficient quantisation

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Cascade Form Structure for FIR Filters

- Factorise the system function polynomial:

$$
\begin{aligned}
H(z) &= b_0 + b_1 z^{-1} + \cdots + b_M z^{-M} \\
&= \left(b_{01} + b_{11} z^{-1} + b_{21} z^{-2}\right) \times \left(b_{02} + b_{12} z^{-1} + b_{22} z^{-2}\right) \times \cdots \times \left(b_{0M_s} + b_{1M_s} z^{-1} + b_{2M_s} z^{-2}\right)
\end{aligned}
$$

where $M_s = \dfrac{M}{2}$

- Implement individual second-order sections using any of the earlier structures, cascade each of these sections to obtain the overall structure.

Work Example on Facts about Cascade Form Structure

- Only one of the following facts about cascade form structure is true. Spot the true fact.

  - ○ 1. The overall system function in cascade form is the sum of the system function of each section.
  - ○ 2. If the order of the sections in a cascade form is changed, the overall system function also changes.
  - ○ 3. Cascade form uses second order sections because second order sections are easier to implement.
  - ○ 4. Cascade form uses second order sections because second order sections give real coefficients

Submit Correct Answer

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Parallel Form Structure for IIR Filters

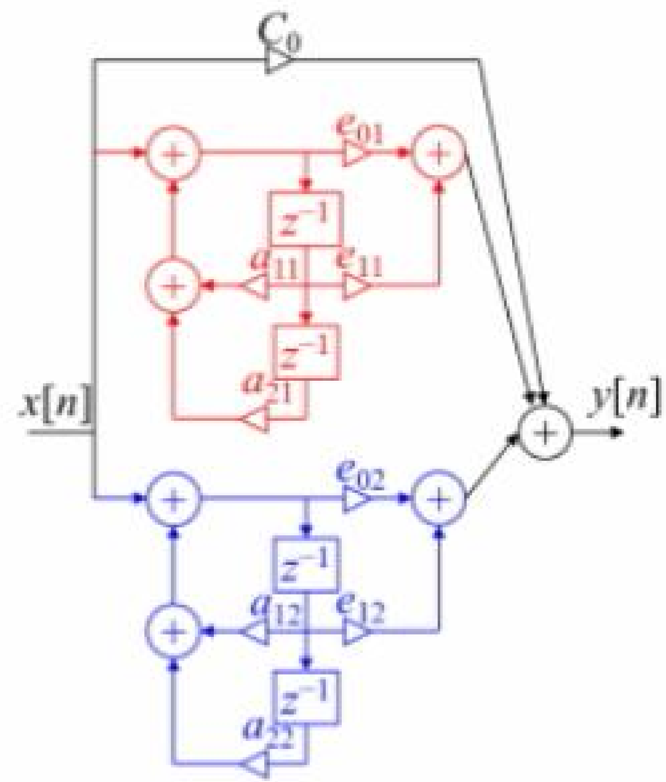
- Partial fraction expansion of the system function:

$$H(z)=\frac{b_0+b_1z^{-1}+\cdots+b_Nz^{-N}}{1-a_1z^{-1}-\cdots-a_Nz^{-N}}$$

$$=C_0+\left(\frac{e_{01}+e_{11}z^{-1}}{1-a_{11}z^{-1}-a_{21}z^{-2}}\right)+$$

$$\cdots+\left(\frac{e_{0N_s}+e_{1N_s}z^{-1}}{1-a_{1N_s}z^{-1}-a_{2N_s}z^{-2}}\right)$$

where $N_s=\frac{N}{2}$



- Implement individual second-order sections using any of the earlier structures.
- Parallelly connect all these sections to obtain the overall structure.

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Summary

By now, you should be able to:

- Sketch the Cascade Form Structure for any IIR/ FIR filter from its system function/ CCDE;
- Identify the system function/ CCDE of any IIR/ FIR filter from its Cascade Form Structure;
- Sketch the Parallel Form Structure for any IIR filter from its system function/ CCDE; and
- Identify the system function/ CCDE of any IIR filter from its Parallel Form Structure.

Structures for

Linear Phase FIR Filters

Learning Objectives

By the end of this topic, you should be able to:

- Sketch the Linear Phase Structure for any linear phase filter from its system function/ CCDE; and
- Identify the system function/ CCDE of any linear phase filter from its Linear Phase Structure.

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Structures for Linear Phase FIR Filters

- Since linear phase filter impulse response coefficients satisfy symmetry condition:

$$h[M-n] = \pm h[n]$$

- The number of required multipliers may be reduced by 50%.

- Example: Symmetric even length filter difference equation:

$$y[n] = h[0]x[n] + h[1]x[n-1] + \cdots + h[M-1]x[n-M+1] + h[M]x[n-M]$$
$$= h[0]x[n] + h[1]x[n-1] + \cdots + h[1]x[n-M+1] + h[0]x[n-M]$$

add

$$= h[0]\left(x[n] + x[n-M]\right) + h[1]\left(x[n-1] + x[n-M+1]\right) + \cdots + h\left[\tfrac{M-1}{2}\right]\left(x\left[n-\tfrac{M-1}{2}\right] + x\left[n-\tfrac{M+1}{2}\right]\right)$$

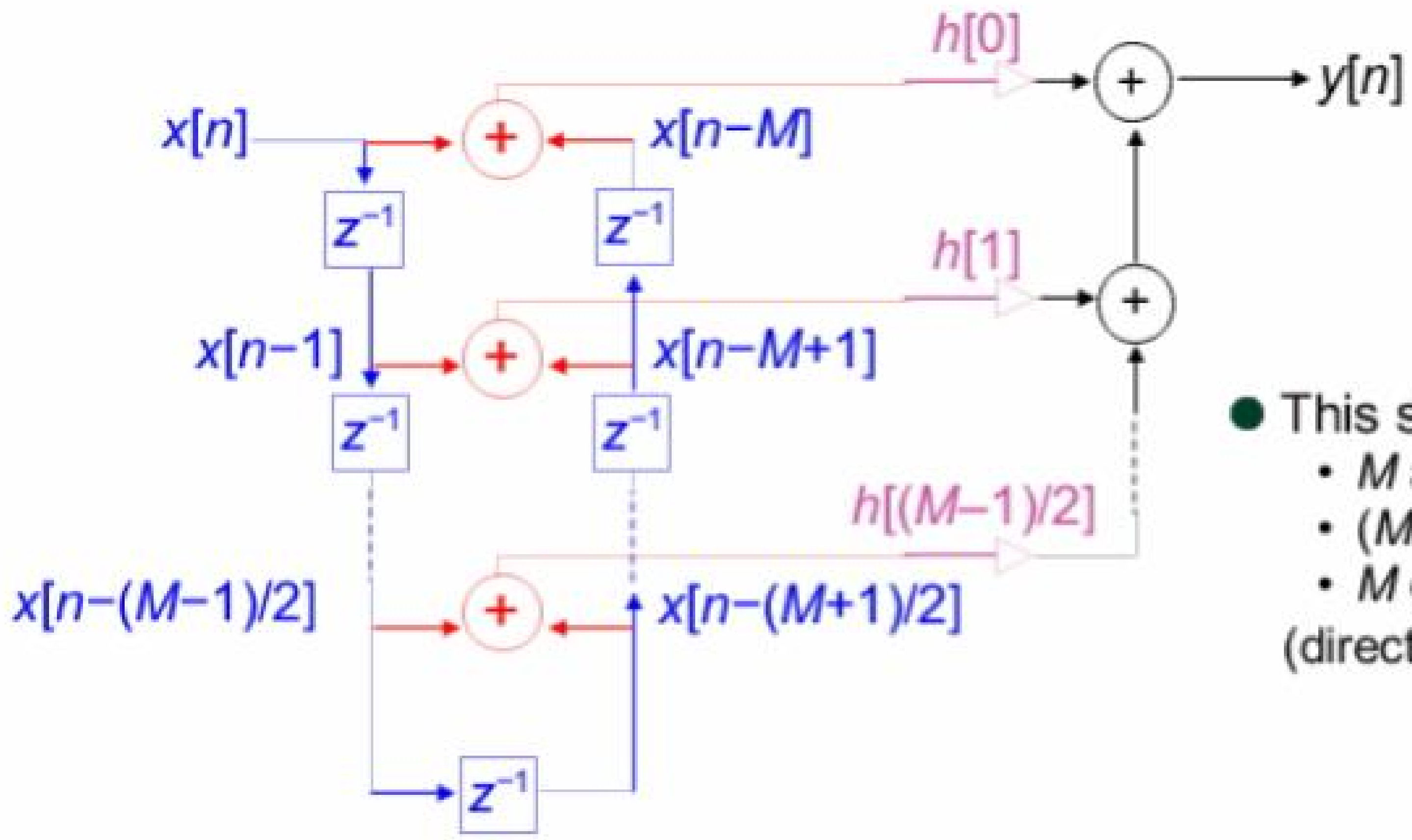
Structures for Linear Phase FIR Filters

First obtain $x[n]$, $x[n-1]$, …, $x[n-M]$, using delays

Then add them pairwise, such as $x[n] + x[n-M]$

Then multiply them with the coefficients $h[0]$, h[1], …, $h[(M-1)/2]$

Finally add them two at a time



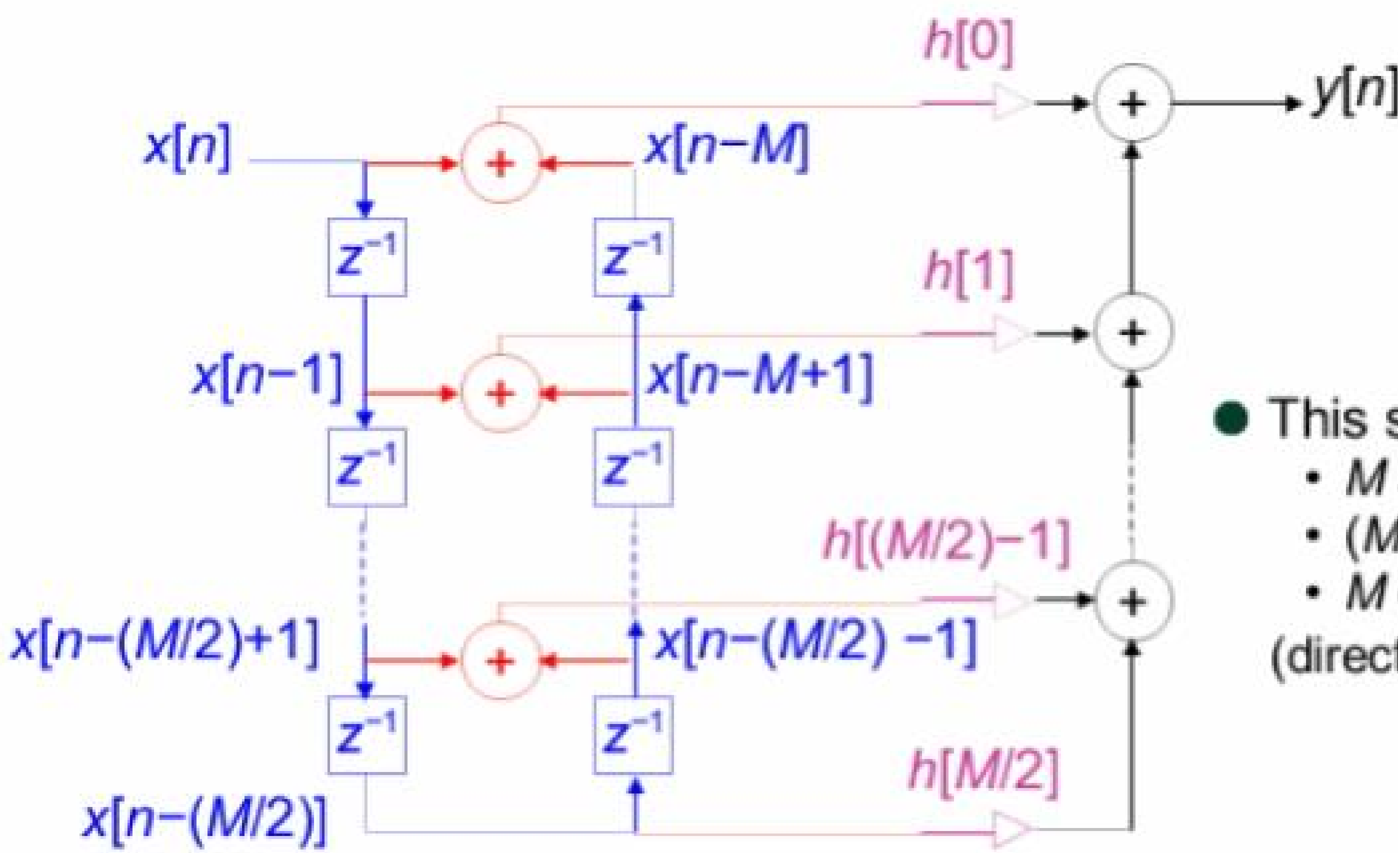
- This structure requires:
  - M adders,
  - $(M+1)/2$ multipliers,
  - M delays

  (direct form requires $M+1$ multipliers)

Structures for Linear Phase FIR Filters

- Example: Symmetric odd length filter:

$$y[n] = h[0]\big(x[n] + x[n-M]\big) + \cdots + h\left[\tfrac{M}{2}-1\right]\left(x\left[n-\tfrac{M}{2}+1\right] + x\left[n-\tfrac{M}{2}-1\right]\right) + h\left[\tfrac{M}{2}\right]x\left[n-\tfrac{M}{2}\right]$$



- This structure requires:
  - M adders,
  - $(M/2)+1$ multipliers,
  - M delays

  (direct form requires M+1 multipliers)

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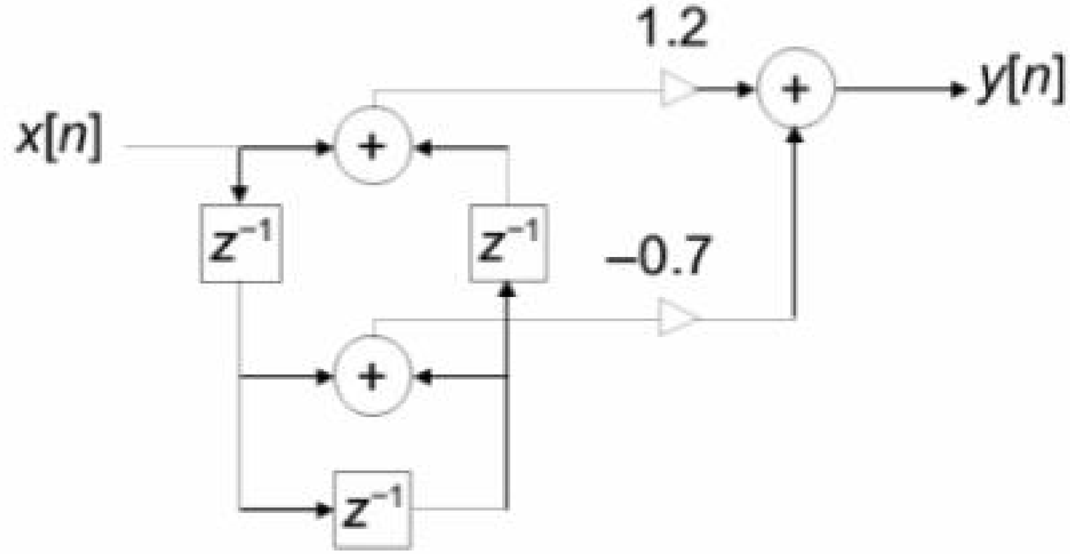
Work Example on Number of Multipliers

- Shown below is a structure implementing a linear phase filter.



- How many multipliers are required in this structure?
- If the same filter were implemented using the direct form structure, then how many multipliers were required?

Submit

Correct Answer

Summary

By now, you should be able to:

- Sketch the Linear Phase Structure for any linear phase filter from its system function/ CCDE; and
- Identify the system function/ CCDE of any linear phase filter from its Linear Phase Structure.

Filter Specifications and Continuous-Time Filters

Learning Objectives

By the end of this topic, you should be able to:

- Describe the Filter Specifications used to design filters;
- Recall Continuous-Time Filters; and
- Outline some Continuous-Time Filters used to design discrete-time filters.

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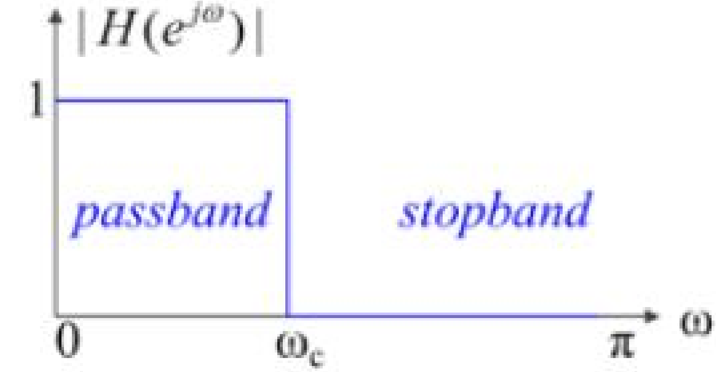
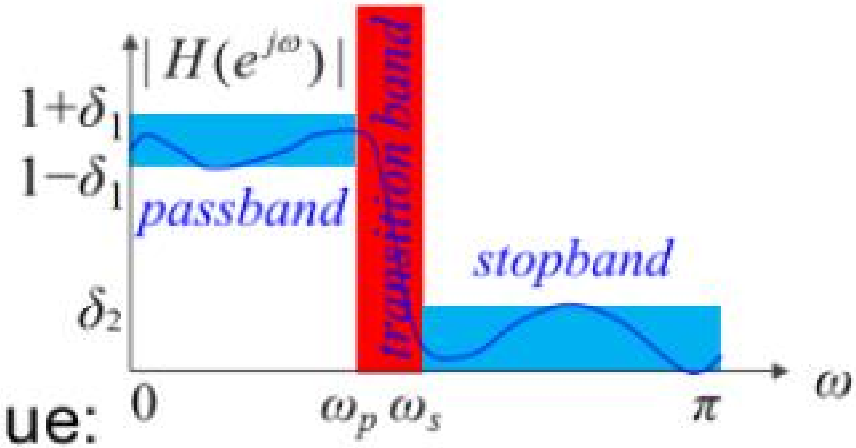
Filter Specifications

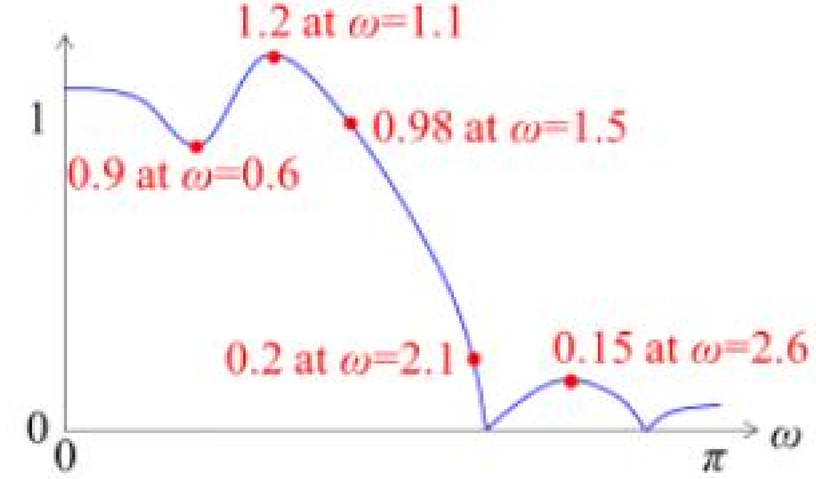
- The ideal magnitude response of a Low Pass Filter.
- Since such magnitude response can not be achieved, provide some tolerance in the specification.
- Passband $1-\delta_1 \le |H(e^{j\omega})| \le 1+\delta_1$, $|\omega| \le \omega_p$
- Stopband $|H(e^{j\omega})| \le \delta_2$, $\omega_s \le |\omega| \le \pi$
- Transition band $\omega_p < |\omega| < \omega_s$, any value:
  - δ_1 = maximum passband error
  - δ_2 = maximum stopband error
  - ω_p = passband cutoff frequency
  - ω_s = stopband cutoff frequency





Interactive Exercise: Filter Specifications

- The required specifications of a low pass filter are:
  - Max passband error $\delta_1 = 0.2$
  - Max stopband error $\delta_2 = 0.16$
  - Passband cutoff $\omega_p = 1.5$ rad
  - Stopband cutoff $\omega_s = 2.1$ rad



- A filter shown on the right is designed. It's magnitude response values at some frequencies are shown.

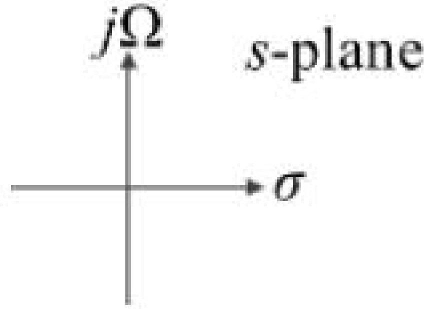
Does its passband meet the specification? ☐ Yes ☐ No

Does its stopband meet the specification? ☐ Yes ☐ No

Submit   Correct Answer

Continuous-Time Filters: Quick Recap

- Why are continuous-time filters relevant in DSP?
- Good continuous-time filter designs were available.
- To design IIR filters, transform continuous-time filters to discrete-time filters.
- Recall continuous-time filter basics:
  - System function $H_c(s)$ is a function of s, the Laplace transform variable:
    $s = \sigma + j\Omega$ where Ω = continuous-time frequency
- Poles at $\sigma_i + j\Omega_i$:
  - Stable if poles have $\sigma_i < 0$
    (all poles in the left-half plane)



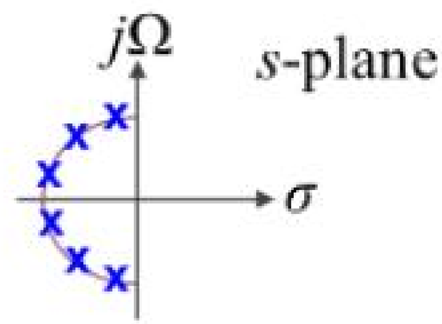
- Frequency response = $H_c(s)|_{s=j\Omega} = H_c(j\Omega)$
- Magnitude response = $|H_c(j\Omega)|$

Continuous-Time Filters: Butterworth Filter

- N-th order Butterworth filter with cut-off frequency Ω_c :

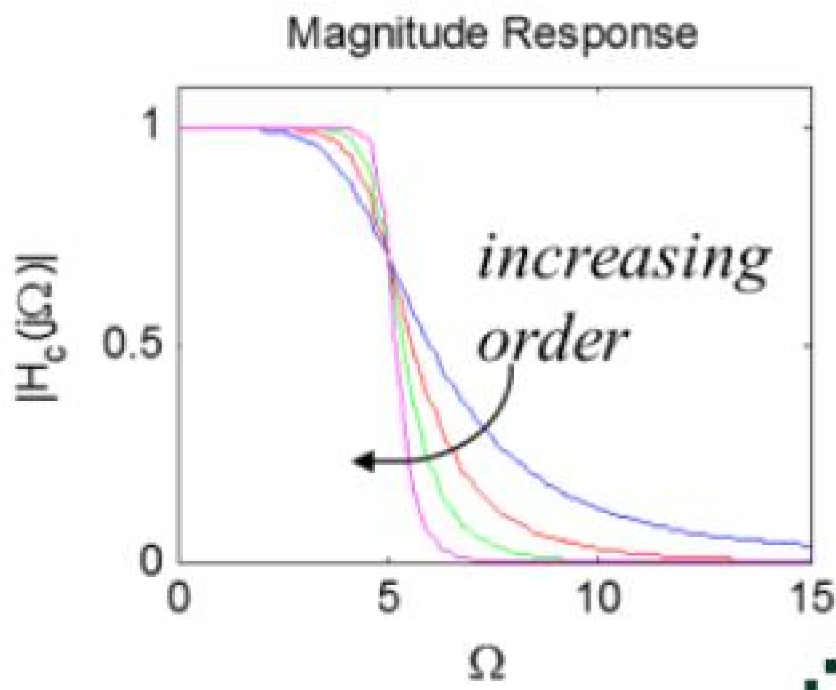
$$H_c(s) = \frac{\Omega_c^N}{(s-\Omega_c e^{j\pi(N+1)/2N})(s-\Omega_c e^{j\pi(N+3)/2N})\cdots(s-\Omega_c e^{j\pi(3N-1)/2N})}$$

- N poles uniformly placed on a left-plane semi-circle of radius Ω_c :

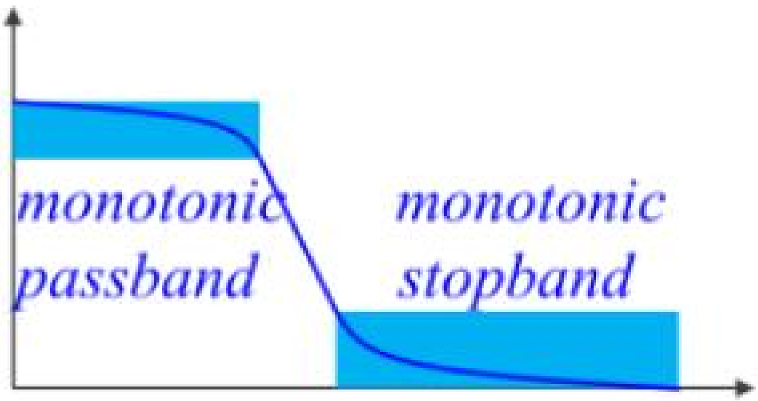


- Magnitude response:

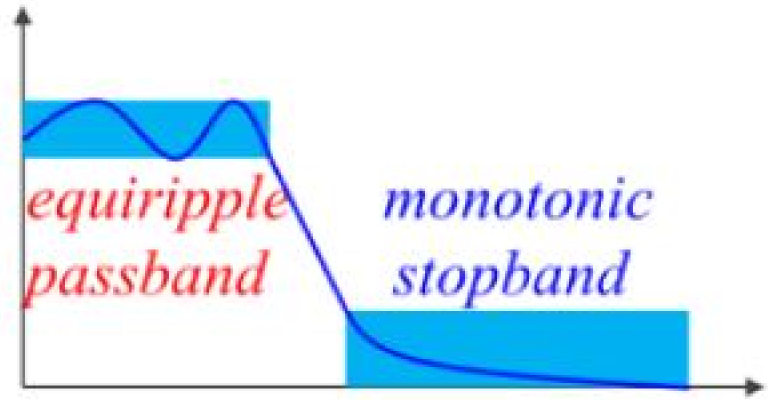
$$|H_c(j\Omega)| = \sqrt{\frac{1}{1+(\Omega/\Omega_c)^{2N}}}$$



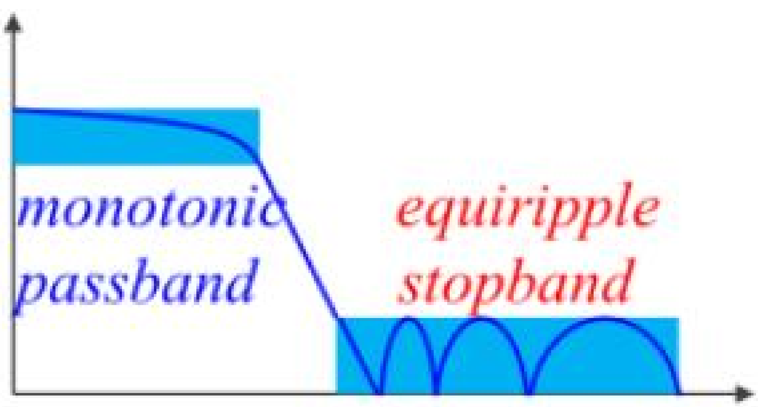
Other Continuous-Time Filters



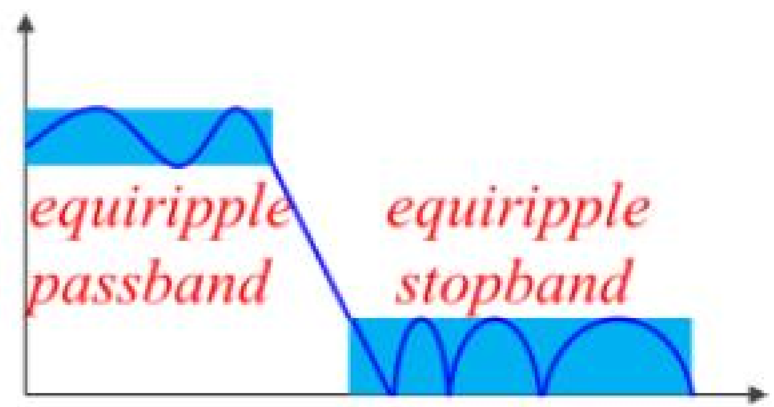
Butterworth



Type I Chebyshev



Type II Chebyshev



Elliptic
(*smallest possible order*)

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Summary

By now, you should be able to:

- Describe the Filter Specification used to design filters;
- Recall Continuous-Time Filters; and
- Outline some Continuous-Time Filters used to design discrete-time filters.

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Theory of Impulse Invariance

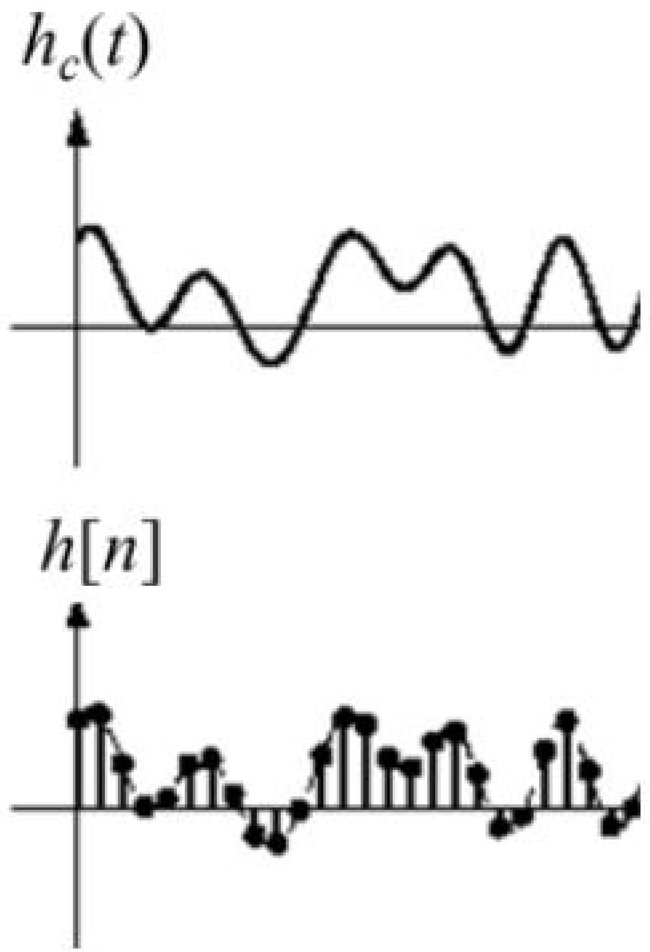
Learning Objectives

By the end of this topic, you should be able to:

- Explain Impulse Invariance mapping from continuous-time to discrete-time;
- Describe S to Z Mapping;
- Explain the Desirable Properties of Impulse Invariance; and
- Explain the Undesirable Properties of Impulse Invariance.

Impulse Invariance

- Sample the continuous-time filter's impulse response $h_c(t)$ to obtain the discrete-time filter impulse response $h[n]$
- Let T_d be the sampling interval:
  $h[n] = T_d h_c(nT_d)$
- T_d has no role in filter design. In the following, we choose $T_d = 1$ for convenience.
- Frequency responses:
  $h_c(t) \leftrightarrow H_c(j\Omega) \quad h[n] \leftrightarrow H(e^{j\omega})$



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S to Z Mapping

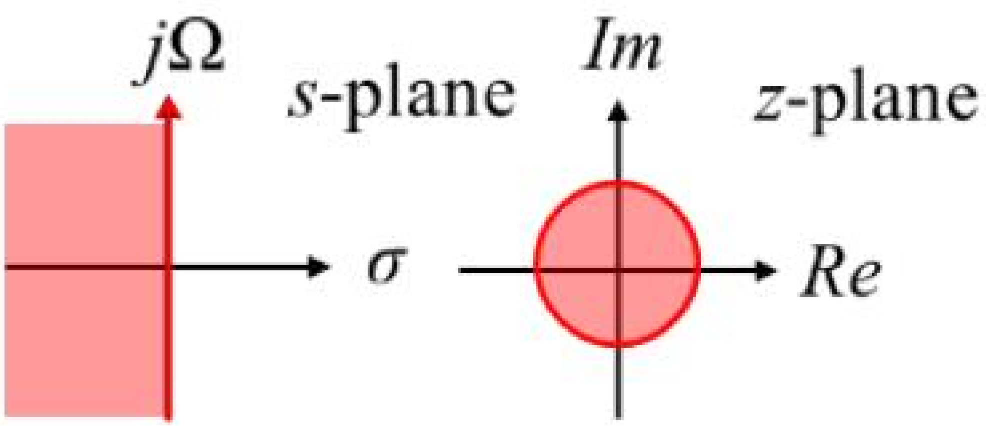
- What mapping from s-domain to z-domain will result in such sampling of the impulse response?
- Let $H_c(s) = \dfrac{1}{s - s_1}$ where s_1 is a pole of $h_c(t) = e^{s_1 t} u(t)$

  (Any $H_c(s)$ may be expressed as sum of first order factors).
- After sampling: $h[n] = e^{s_1 n} u[n] \longrightarrow H(z) = \dfrac{1}{1 - e^{s_1} z^{-1}}$
- Therefore, an s-domain pole s_1 gives a z-domain pole e^{s_1}
  - the mapping is $z = e^{s}$
- Keeping T_d, it may be shown that the mapping is:

  $z = e^{sT_d}$ or $s = \dfrac{1}{T_d} \ln z$

Desirable Properties of Impulse Invariance

- Two desirable properties of any mapping for filter design are:
- Frequency response = $j\Omega$ axis in s-plane
  = unit circle in z-plane
  - map $j\Omega$ axis to unit circle
    mathematically, $\sigma = 0 \longrightarrow |z| = 1$
- Stable if poles in left half of s-plane equals inside unit circle in z-plane:
  - map left half to inside unit circle
    mathematically, $\sigma < 0 \longrightarrow |z| < 1$



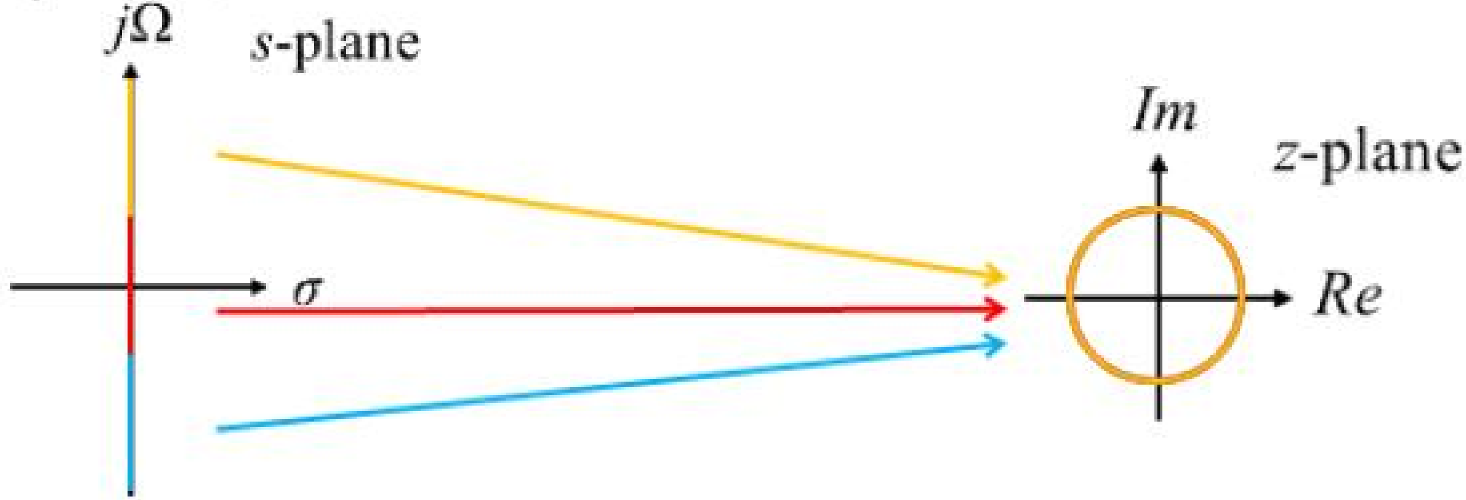
- For this mapping:

$$\sigma = 0 \Rightarrow z = e^{j\Omega} \quad \text{or} \quad |z| = 1$$
$$\sigma < 0 \Rightarrow z = e^{\sigma} e^{j\Omega} \quad \text{or} \quad |z| < 1 \quad \text{since} \quad e^{\sigma} < 1$$

Undesirable Properties of Impulse Invariance

- It is a many-to-one mapping: we have seen $s_1 \rightarrow z_1 = e^{s_1}$

  Also, $s_1 + j2\pi k \rightarrow z_1$ since $e^{s_1 + j2\pi k} = e^{s_1} e^{j2\pi k} = z_1$
- All points$\ldots, s_1 - j2\pi,\ s_1,\ s_1 + j2\pi,\ s_1 + j4\pi, \ldots$map to the same point z_1



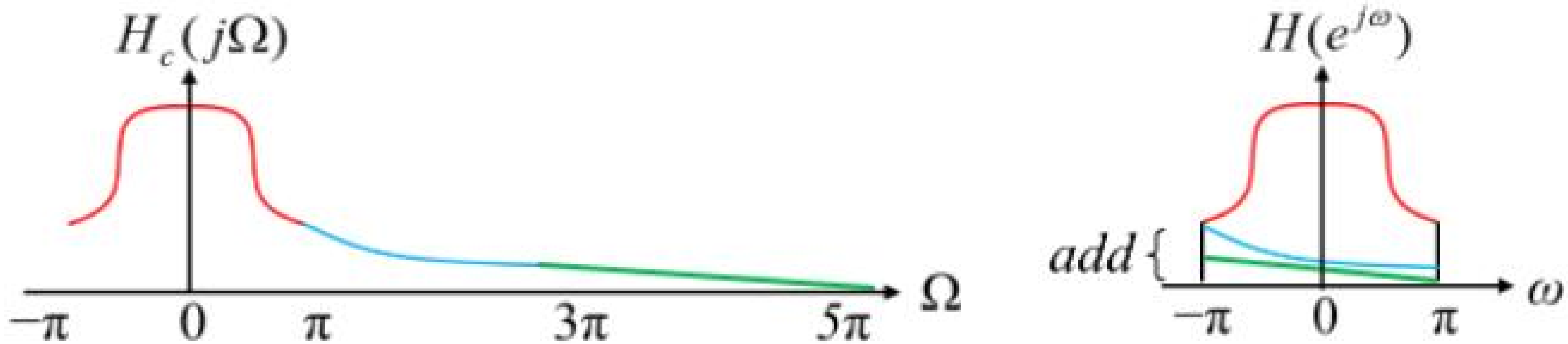
- Frequency mapping: $\omega = \Omega$ in $-\pi \le \Omega \le \pi$
- All segments$\ldots, -\pi \le \Omega \le \pi,\quad \pi \le \Omega \le 3\pi, \ldots$map to unit circle.

Undesirable Properties of Impulse Invariance

- Recall that, sampled spectrum equals sum of infinite shifted copies of original spectrum:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\omega + j2\pi k\right)$$

- Therefore, $H(e^{j\omega})$ is the sum of shifted copies of $H_c(j\Omega)$



- Other copies will also be added in to the response, resulting in aliasing.
- Other copies are from stopband tail.
- Therefore, continuous-time filter response should be very small above frequency π

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Interactive Element: Impulse Invariance Mapping

- Fill in the boxes with appropriate values.
- In the impulse invariance method, a point s = $\boxed{}$ in the s-plane maps to a point $z = e^2$ in the z-plane.
- In the impulse invariance method, a continuous-time frequency $\Omega = 0.8$ radian maps to a discrete-time frequency ω = $\boxed{}$ radian.

Submit   Correct Answer

Summary

By now, you should be able to:

- Explain Impulse Invariance mapping from continuous-time to discrete-time;
- Describe S to Z Mapping;
- Explain the Desirable Properties of Impulse Invariance; and
- Explain the Undesirable Properties of Impulse Invariance.

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Design of Impulse Invariance

Learning Objectives

By the end of this topic, you should be able to:

- Explain the design steps involved in IIR filter design using Impulse Invariance.

Design Step 1: Specification

- Input the specifications.
- Note that the specifications are same for $H(z)$ and $H_c(s)$ since the magnitude response values do not change, and $\omega = \Omega$

- Lowpass Butterworth IIR filter design using impulse invariance.

Let $\delta_1 = 0.05, \delta_2 = 0.1, \omega_p = 1, \omega_s = 1.8$

$\rightarrow$specifications for $H(z)$

$$1 \le |H(e^{j\omega})| \le 0.95, \qquad |\omega| \le 1$$

$$|H(e^{j\omega})| \le 0.1, \qquad 1.8 \le |\omega| \le \pi$$

$\rightarrow$specifications for $H_c(s)$

$$1 \le |H_c(j\Omega)| \le 0.95, \qquad |\Omega| \le 1$$

$$|H_c(j\Omega)| \le 0.1, \qquad 1.8 \le |\Omega| \le \infty$$

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Interactive Exercise: Specifications using Impulse Invariance

- The specifications of a discrete-time filter has to be converted to the specifications for continuous-time filter for a design using impulse invariance method. The stopband specification for the discrete-time filter is: $|H(e^{j\omega})| \leq 0.001,\ 1.2\ \mathrm{rad} \leq |\omega| \leq \pi\ \mathrm{rad}$
- Which of the following should be changed to specify the stopband of the continuous-time filter? You may choose multiple options.

☐ A. Change 0.001

☐ B. Change ω

☐ C. Change 1.2 rad

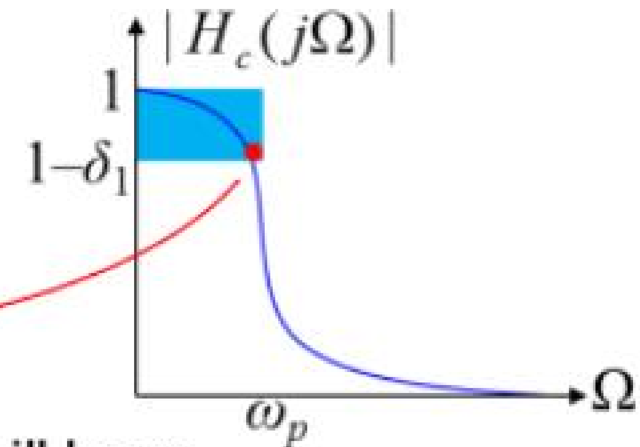
☐ D. Change π rad

☐ E. Change $H(e^{j\omega})$

Submit Correct Answer

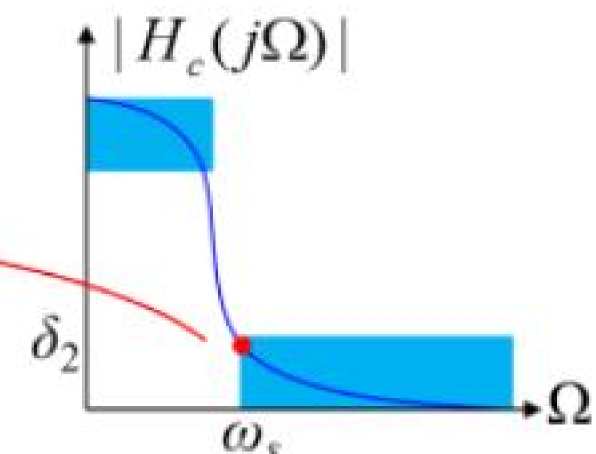
Design Step 2: Design $H_c(s)$

- Butterworth filter magnitude response $|H_c(j\Omega)| = \sqrt{\dfrac{1}{1+(\Omega/\Omega_c)^{2N}}}$
- Since its passband is monotonic decreasing, it will have minimum value at the passband cutoff.



$$\sqrt{\frac{1}{1+(\omega_p/\Omega_c)^{2N}}} = 1-\delta_1$$

- Since its stopband is monotonic decreasing, it will have maximum value at the stopband cutoff.

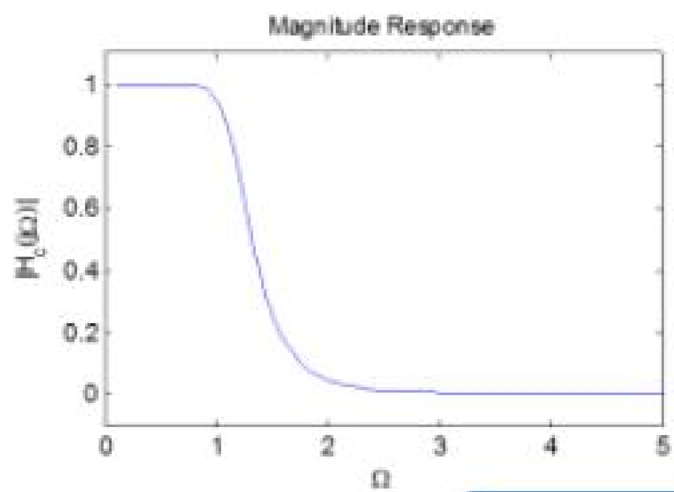


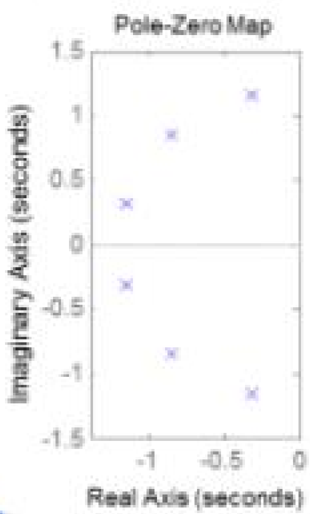
$$\sqrt{\frac{1}{1+(\omega_s/\Omega_c)^{2N}}} = \delta_2$$

Design Step 2: Design $H_c(s)$

- Solve two equations for unknowns N and Ω_c ; if N is not an integer, take the next integer as N
- Choose Ω_c satisfying one of the two equations exactly.
- The other equation will be over-designed since the integer N is more than what's needed.
- Since impulse invariance methods have aliasing, let the stopband specification be over-designed.
- Choose the passband (first equation).

- Example:
  $N = 5.80, \Omega_c = 1.21 \rightarrow \text{integer } N = 6$
  $\rightarrow \Omega_c = 1.20$
- Design Butterworth filter
  $$H_c(s) = \frac{3.04}{s^6 + 4.65s^5 + \cdots + 3.04}$$
- Poles $-0.31+j1.16$, $-0.31-j1.16$, …





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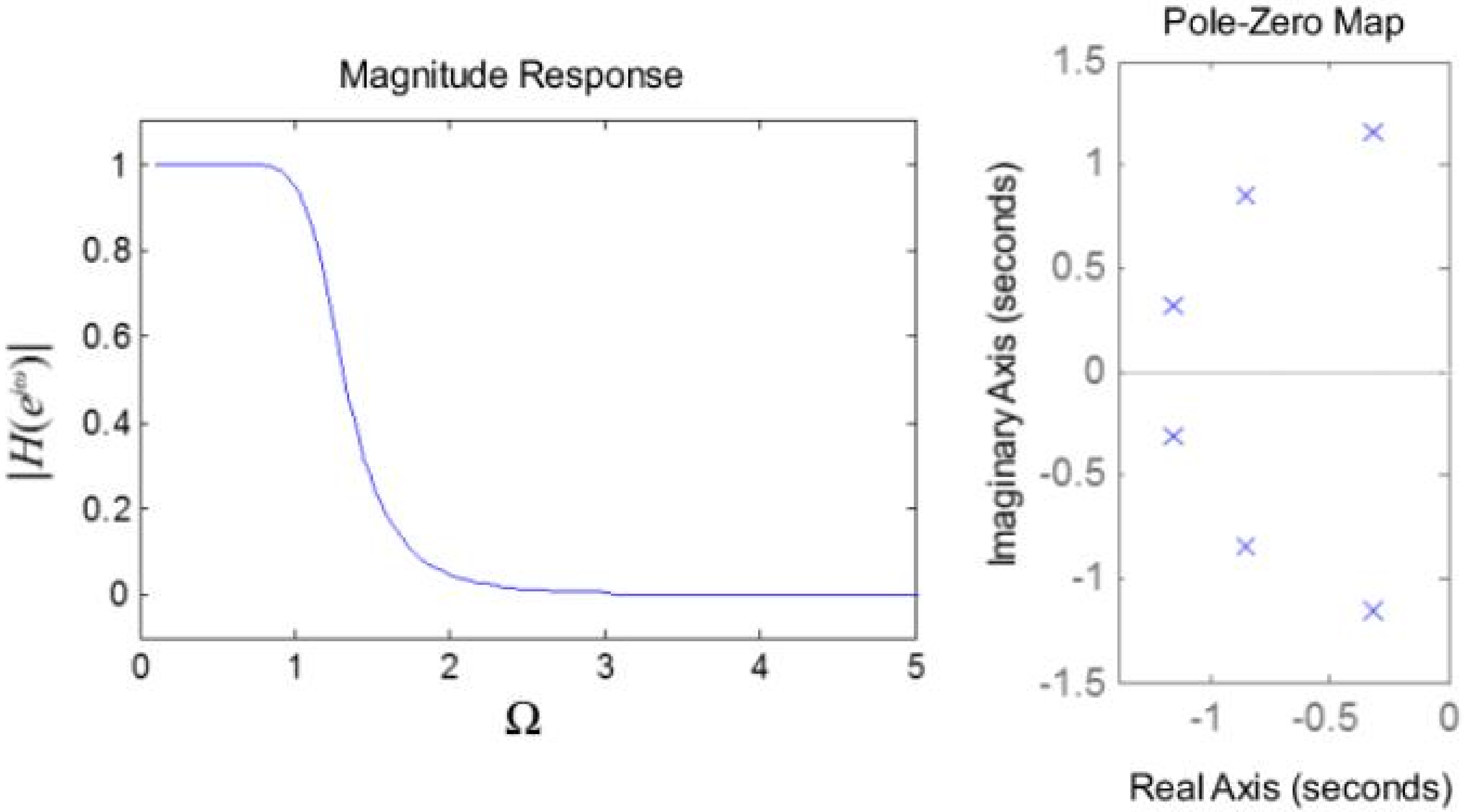
Design Step 2: Design $H_c(s)$



Design Step 3: Transform $H_c(s)$ to $H(z)$

- How to obtain $H(z)$ from $H_c(s)$?
- The mapping is $s = \ln z$ which can not be directly substituted in $H_c(s)$
- Instead, express $H_c(s)$ in terms of a partial fraction expansion:

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

$$H_c(s) = \frac{0.25 + j0.43}{s + 0.31 - j1.16} + \cdots$$

Design Step 3: Transform $H_c(s)$ to $H(z)$

- We have already seen that each factor $\dfrac{1}{s-s_k}$ transforms to $\dfrac{1}{1-e^{s_k}z^{-1}}$

$$H(z)=\sum_{k=1}^{N}\frac{A_k}{1-e^{s_k}z^{-1}}$$

- First partial fraction = $\dfrac{0.25+j0.43}{s+0.31-j1.16}$
- First pole $-0.31+j1.16$ maps to: $e^{-0.31+j1.16}=0.29+j0.67$
- First partial fraction transforms to: $\dfrac{0.25+j0.43}{1-(0.29+j0.67)z^{-1}}$

$$H(z)=\frac{0+0.01z^{-1}+\cdots+0.0005z^{-5}}{1-1.74z^{-1}+\cdots+0.01z^{-6}}$$

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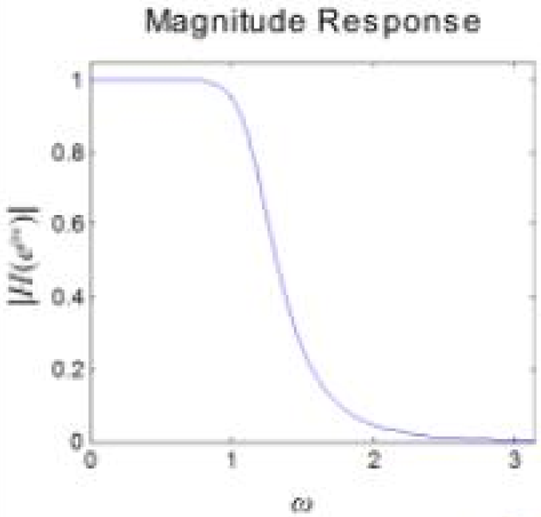
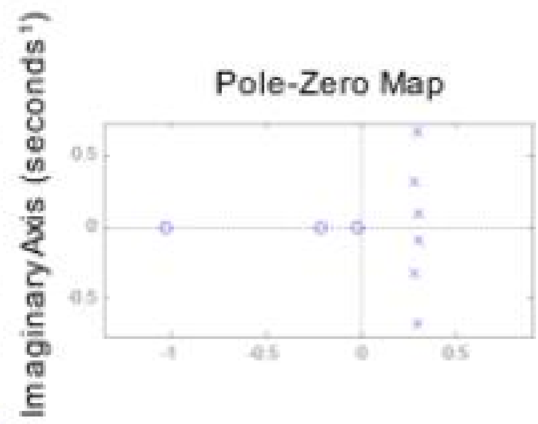
Design Step 4: Check

- Check filter response; if magnitude response is beyond the specifications in any band, aliasing is too much.

- Magnitude response is numerically found to be:

$$\left.\begin{array}{l}\text{Passband } 1 \le |H(e^{j\omega})| \le 0.9501 \\ \text{Stopband } |H(e^{j\omega})| \le 0.089\end{array}\right\} \text{Specifications satisfied}$$





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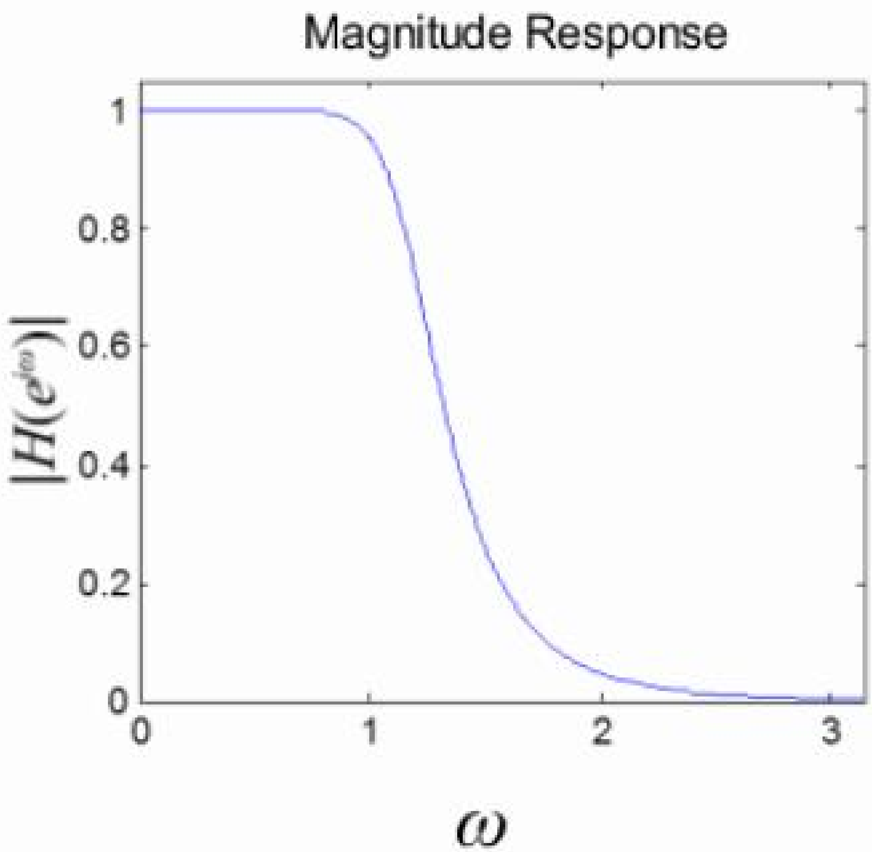
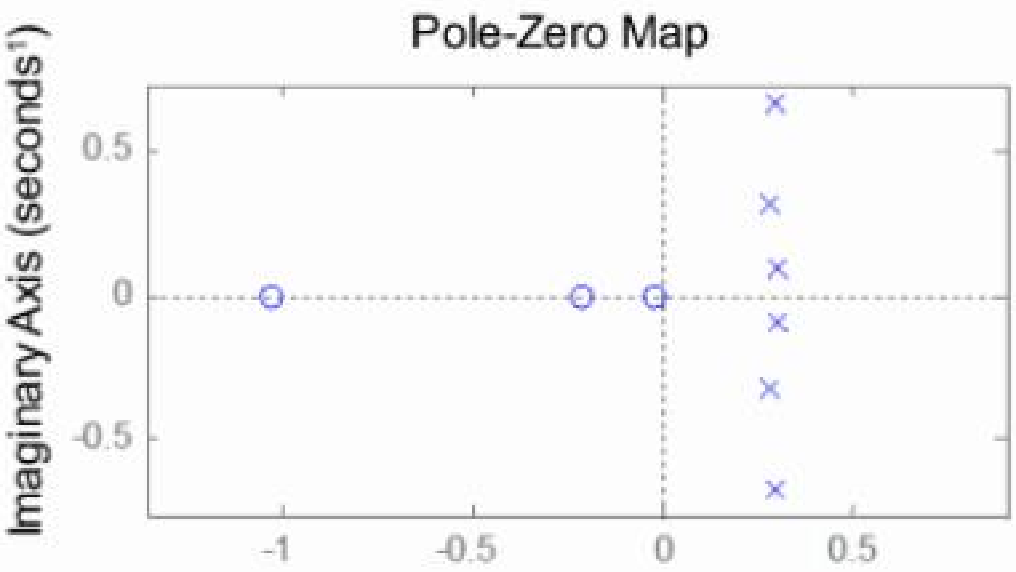
Design Step 4: Check





Summary

By now, you should be able to:

- Explain the design steps involved in IIR filter design using Impulse Invariance.

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Theory and Design of Bilinear Transformation

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Learning Objectives

By the end of this topic, you should be able to:

- Explain the Bilinear Transformation mapping from continuous-time to discrete-time; and
- Explain the steps involved in IIR filter design using Bilinear Transformation.

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Bilinear Transformation: S to Z Mapping

- Design a continuous-time filter $H_c(s)$
- Transform $H_c(s)$ to discrete-time filter $H(z)$ using a mapping from s-domain to z-domain.
- Infinite-length $j\Omega$ axis maps to finite-length unit circle.
  - If linear mapping is used, only a finite segment of $j\Omega$ axis will be mapped to the unit circle, so many-to-one mapping (aliasing) has to be used
  - If non-linear mapping is used, then an infinite axis can be mapped to the unit circle, using one-to-one mapping (no aliasing)
- Mapping $s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ where, as before, T_d has no role, so we take $T_d = 2$ from now on.
- Hence, $s = \frac{1-z^{-1}}{1+z^{-1}}$ or $z = \frac{1+s}{1-s}$

Mapping: Desirable Properties

- The $j\Omega$ axis maps to unit circle: $\sigma = 0 \Rightarrow z = \dfrac{1 + j\Omega}{1 - j\Omega}$

$$= e^{2j\arctan\Omega}$$

- Therefore: $|z| = 1$
- Also, the left-half plane maps to inside unit circle:

$$\sigma < 0 \Rightarrow z = \frac{1+\sigma+j\Omega}{1-\sigma-j\Omega} \quad \text{or} \quad |z| = \sqrt{\frac{(1+\sigma)^2+\Omega^2}{(1-\sigma)^2+\Omega^2}}$$

$$< 1 \quad \text{since} \quad 1+\sigma < 1-\sigma$$

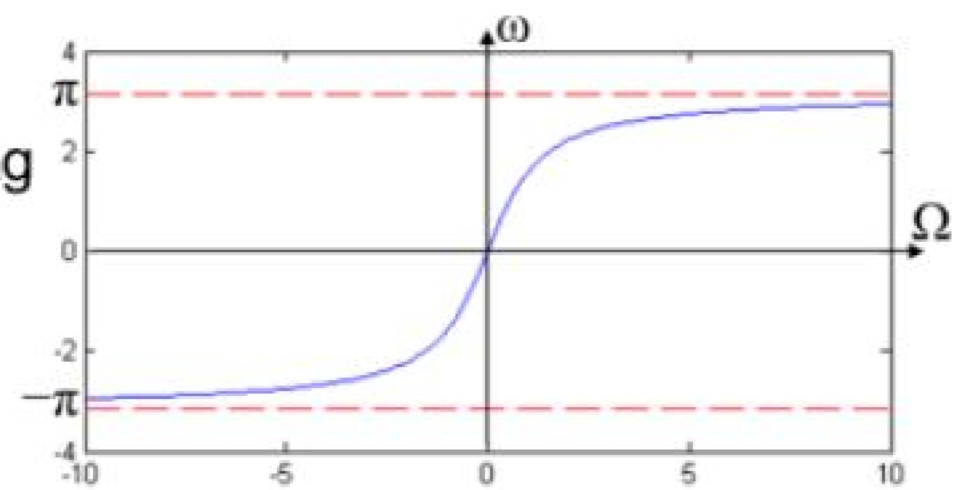
Mapping: Desirable Properties

- It is a one-to-one mapping:

- Frequency mapping: put $s = j\Omega$ and $z = e^{j\omega}$

$$e^{j\omega} = \frac{1 + j\Omega}{1 - j\Omega} = e^{2j\arctan\Omega} \rightarrow \quad \omega = 2\arctan\Omega \qquad \Omega = \tan(\omega/2)$$

- Non-linear mapping:
  - Introduces frequency warping
  - Only piecewise constant magnitude response



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Interactive Exercise: Bilinear Transformation Mapping

- Fill in the boxes with appropriate values.
- In the bilinear transformation method, a point $s = 0.5$ in the s-plane maps to a point $z = \boxed{}$ in the z-plane.
- In the bilinear transformation method, a continuous-time frequency $\Omega = \boxed{}$ radians maps to a discrete-time frequency $\omega = \pi/2$ radians.

Submit

Correct Answer

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Design Step 1: Specification

- Input the specifications.
- Note that the specifications are not the same for $H(z)$ and $H_c(s)$ since $\Omega = \tan(\omega/2)$

- Lowpass Butterworth IIR filter design using bilinear transformation.

Let $\delta_1 = 0.08, \delta_2 = 0.05, \omega_p = 2, \omega_s = 2.3$

$\rightarrow$ specifications for $H(z)$

$$1 \le |H(e^{j\omega})| \le 0.92, \qquad |\omega| \le 2$$
$$|H(e^{j\omega})| \le 0.05, \qquad 2.3 \le |\omega| \le \pi$$

$$\omega_p = 2 \Rightarrow \Omega_p = \tan\frac{\omega_p}{2} = 1.56, \text{ etc.}$$

$\rightarrow$ specifications for $H_c(s)$

$$1 \le |H_c(j\Omega)| \le 0.92, \qquad |\Omega| \le 1.56$$
$$|H_c(j\Omega)| \le 0.05, \qquad 2.23 \le |\Omega| \le \infty$$

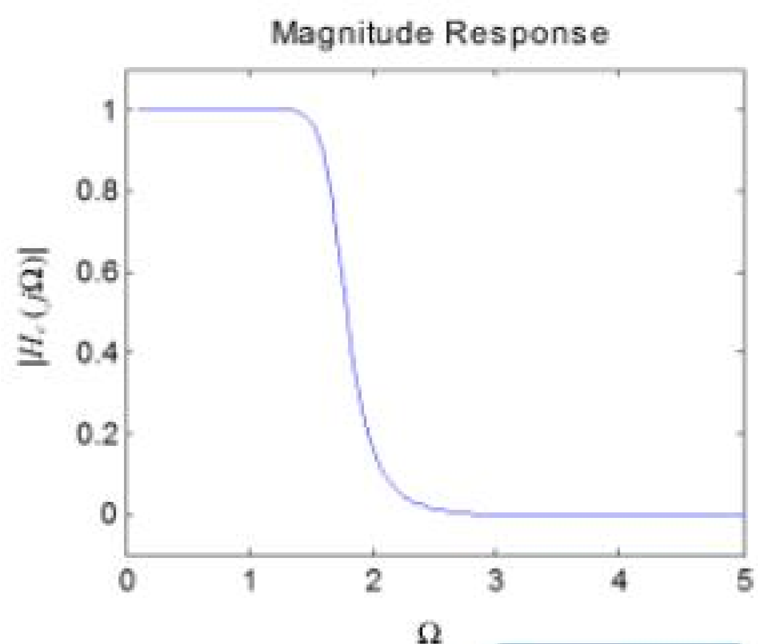
Design Step 2: Design $H_c(s)$

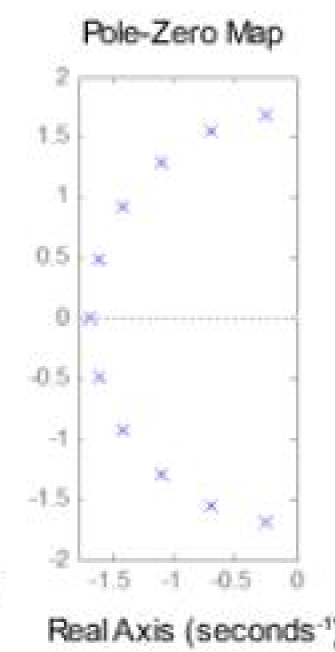
- Same method as impulse invariance is used to design $H_c(s)$
- May choose Ω_c satisfying any one of the two equations exactly (no aliasing).

$$N = 10.66 \rightarrow \text{integer } N = 11$$
$$\rightarrow \Omega_c = 1.70$$

- Design Butterworth filter:

$$H_c(s) = \frac{347.13}{s^{11} + 11.96 s^{10} + \cdots + 347.13}$$





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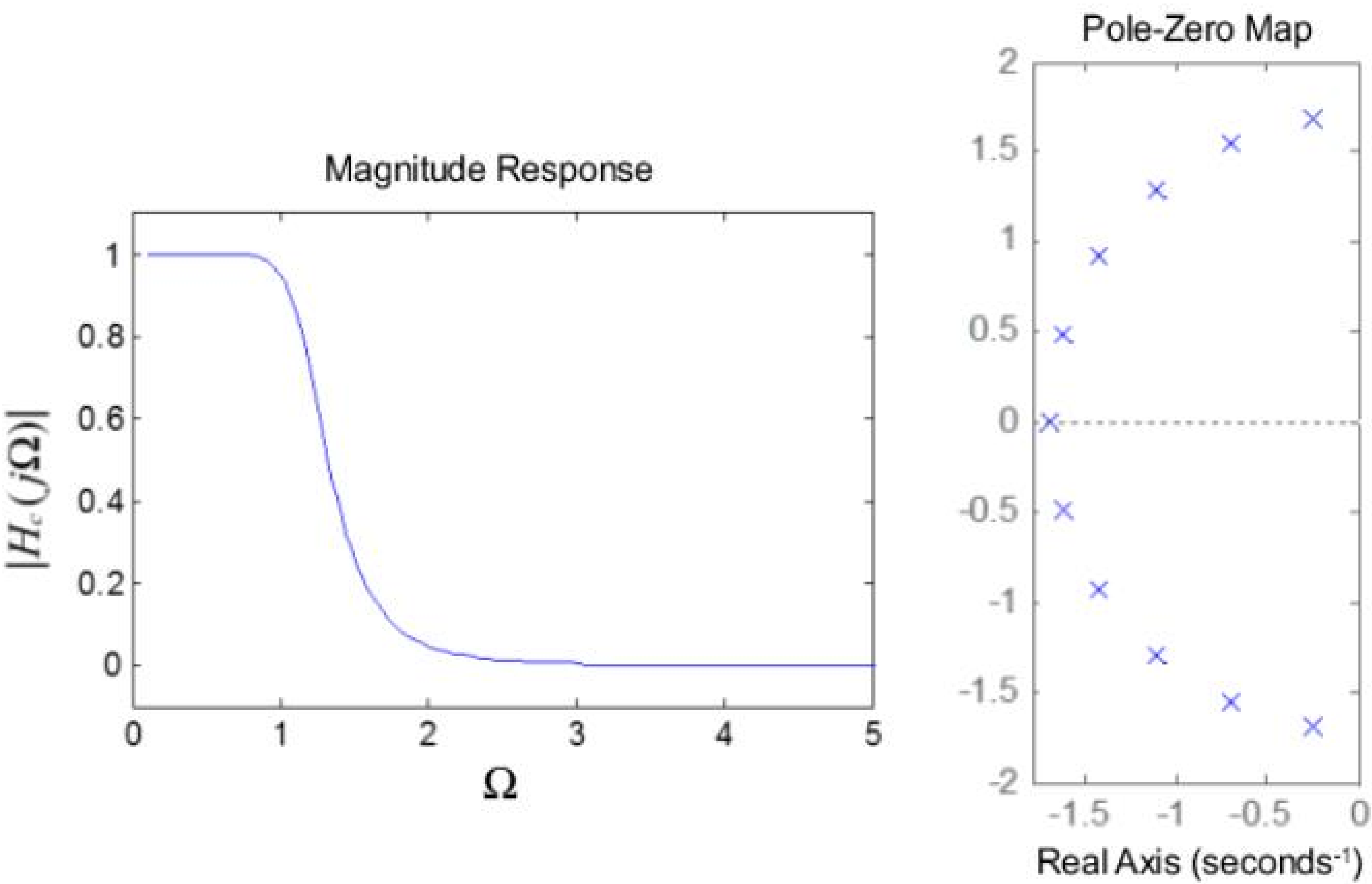
Design Step 2: Design $H_c(s)$



Design Step 3: Transform $H_c(s)$ to $H(z)$

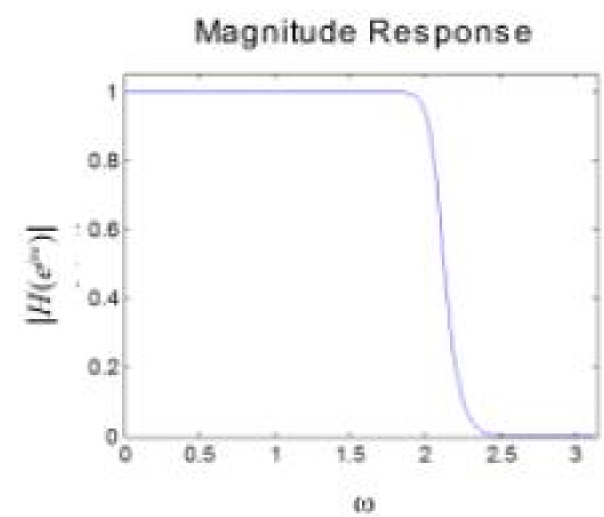
- Substitute $s = \dfrac{1-z^{-1}}{1+z^{-1}}$

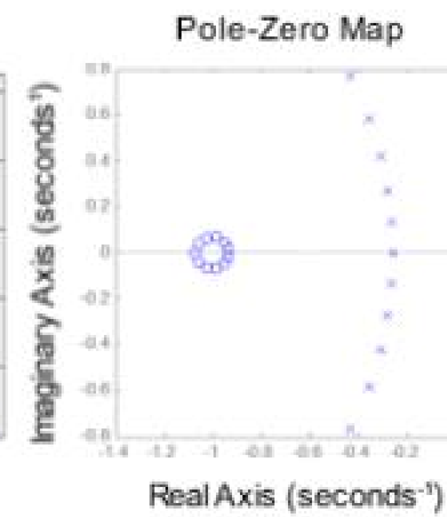
$$H(z) = \frac{347.13}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^{11} + 11.96\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^{10} + \cdots + 347.13}$$

$$= \frac{0.02 + 0.20z^{-1} + \cdots + 0.02z^{-11}}{1 + 3.55z^{-1} + \cdots + 0.0003z^{-11}}$$

Passband $1 \le |H(e^{j\omega})| \le 0.94$

Stopband $|H(e^{j\omega})| \le 0.05$





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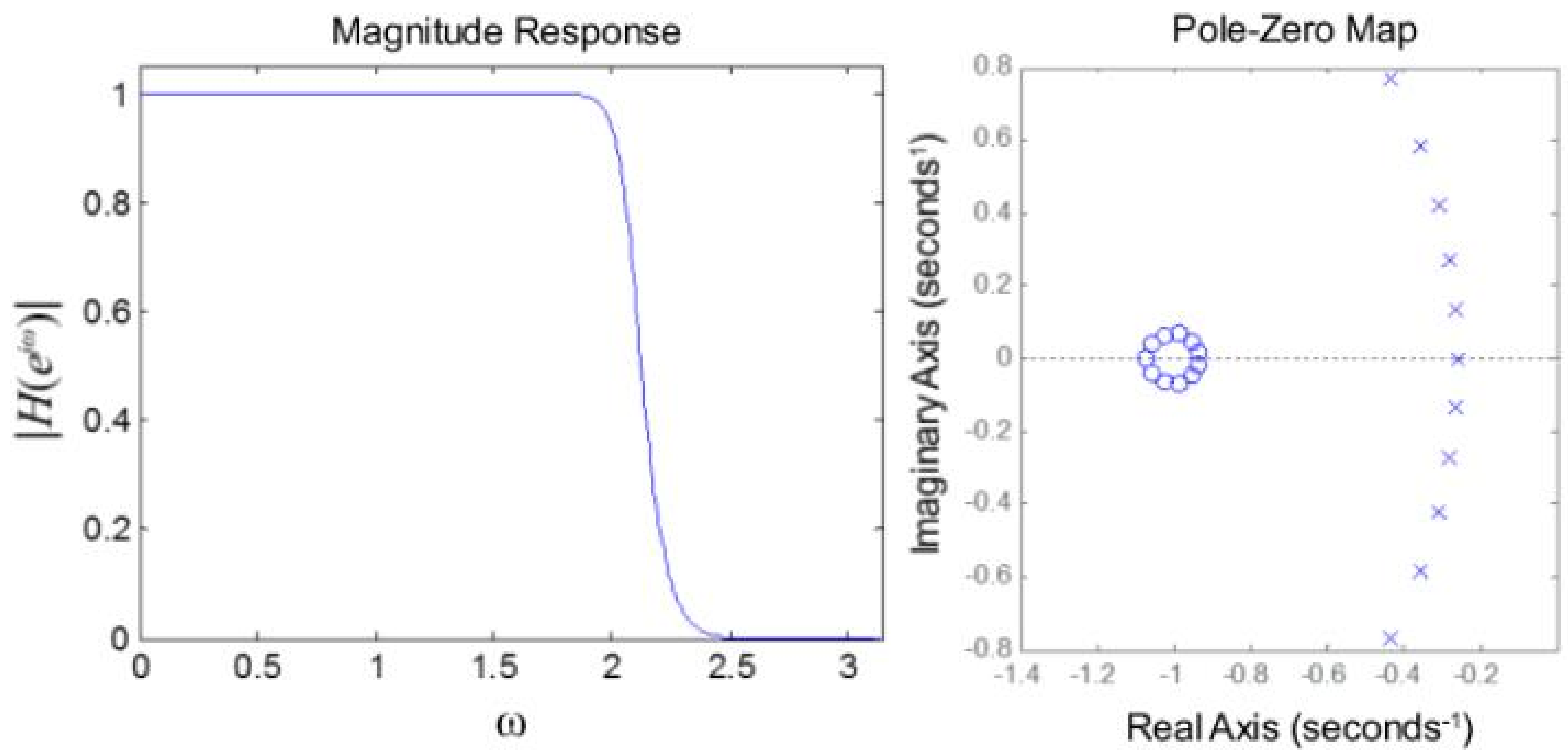
Design Step 3: Transform $H_c(s)$ to $H(z)$



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Summary

By now, you should be able to:

- Explain the Bilinear Transformation mapping from continuous-time to discrete-time; and
- Explain the steps involved in IIR filter design using Bilinear Transformation.

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FIR Filter Design by Windowing

Learning Objectives

By the end of this topic, you should be able to:

- Explain the steps involved in FIR filter Design using Windowing;
- Identify Windows used to design FIR filters; and
- Explain Filter Designs by Windowing.

Design using Rectangular Window

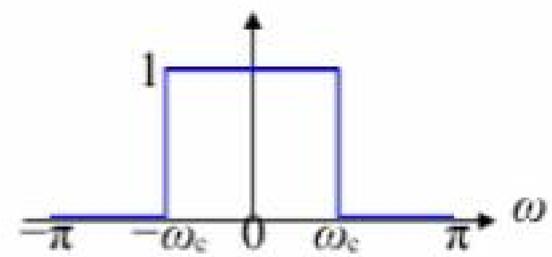
- Unlike IIR filters, FIR filters are designed entirely in discrete-time.
  - Desired frequency response $H_d(e^{j\omega})$
  - Take inverse DTFT to obtain impulse response:

$$h_d[n] = \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} \frac{d\omega}{2\pi}$$

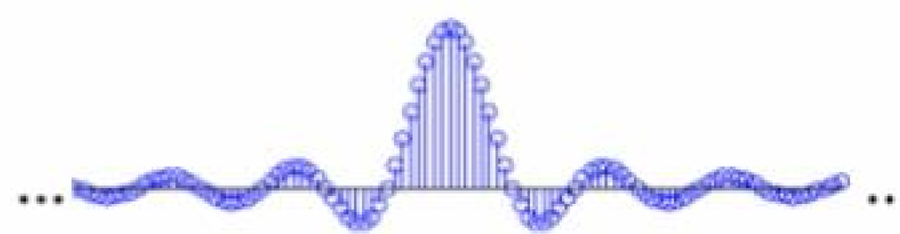
For ideal desired response, the impulse response is infinitely long.

- Lowpass filter design:

$$H_d(e^{j\omega}) = \begin{cases} 1 & -\omega_c < \omega < \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$



- Impulse response: $h_d[n] = \dfrac{\sin \omega_c n}{\pi n}$



Design using Rectangular Window

- Approximate impulse response by truncation:

$$h[n] = \begin{cases} h_d[n] & m_1 \le n \le m_2 \\ 0 & \text{otherwise} \end{cases}$$

- To obtain a causal FIR filter, delay the impulse response to start at n=0

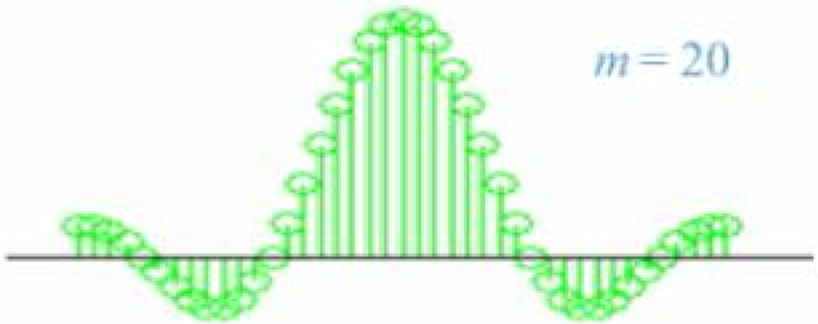
- Truncation equal to multiply by a finite-duration window:

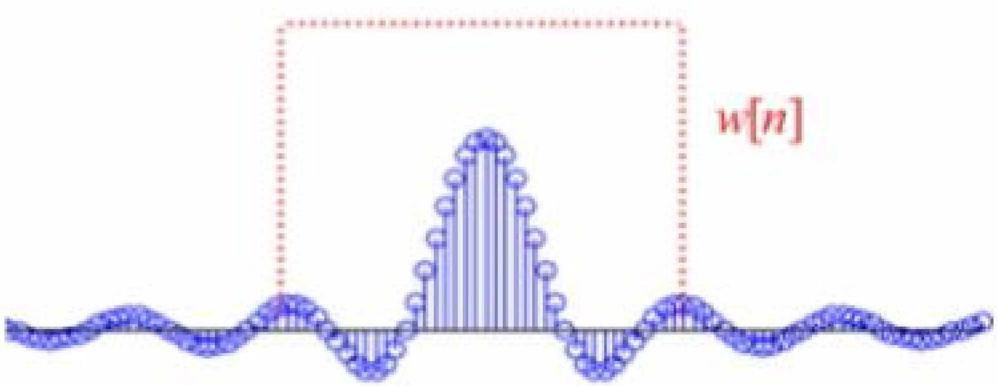
$$h[n] = h_d[n] \cdot w[n]$$

$w[n]$ = rectangular window

$$w[n] = \begin{cases} 1 & m_1 \le n \le m_2 \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} \sin(\omega_c n) / \pi n & -m \le n \le m \\ 0 & \text{otherwise} \end{cases}$$

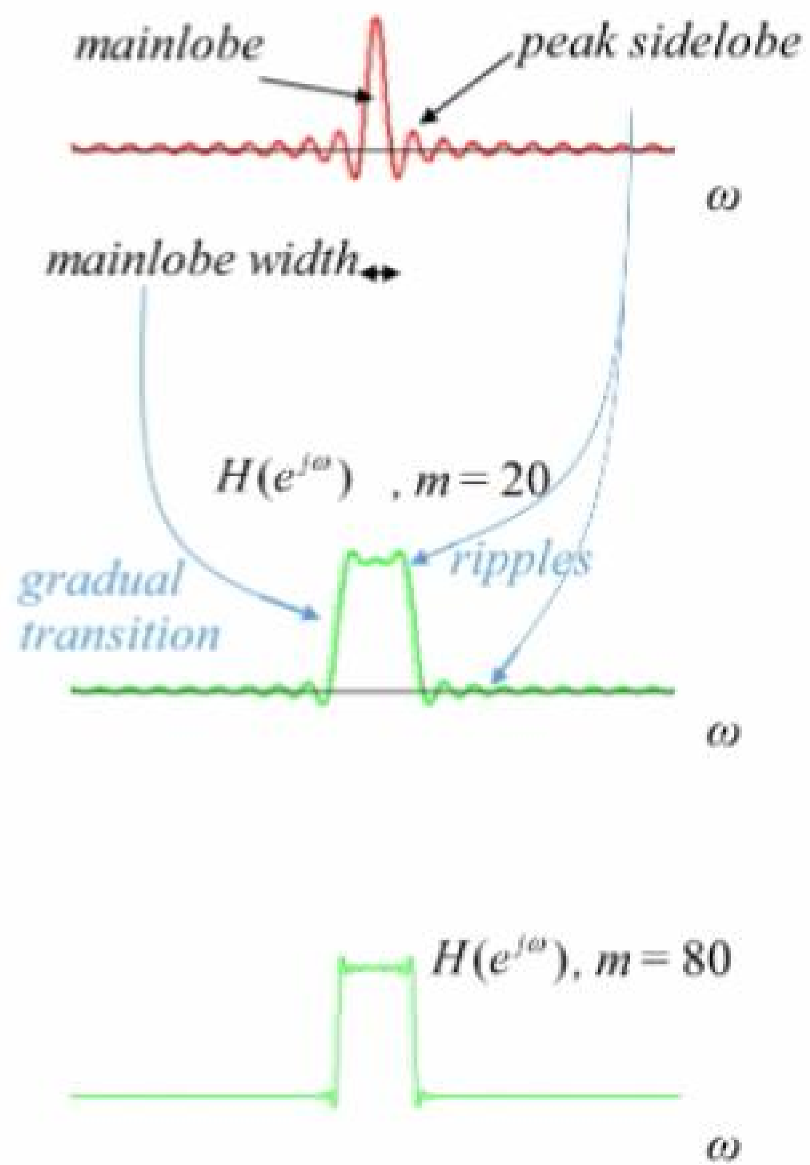




Design using Rectangular Window

- Multiplication in time equals to periodic convolution in frequency:
  $$H(e^{j\omega}) = H_d(e^{j\omega}) \circledast W(e^{j\omega})$$
- If $W(e^{j\omega})$ were an impulse there will be no difference between $H_d(e^{j\omega})$ and $H(e^{j\omega})$
- Due to windowing, the designed frequency response is smeared non-zero mainlobe width causing the designed filter to have a gradual drop or transition.
- Due to the peak side-lobe which is not zero, the designed filter has ripples in passband and stopband.
- As m increases, mainlobe width decreases, while peak sidelobe remains same. And this is known as the Gibbs Phenomenon.

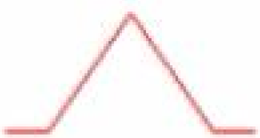
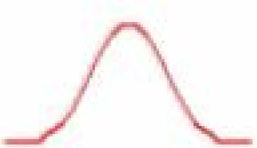
$$W(e^{j\omega}) = \frac{\sin[\omega(2m+1)/2]}{\sin(\omega/2)}$$



Window Designs

	$w[n]$	Peak sidelobe	Mainlobe width	
Rectangular window	$w[n]=\begin{cases}1 & 0\le n\le M\\ 0 & \text{otherwise}\end{cases}$	−13 dB	$\frac{4\pi}{M+1}$	

- Rectangular window has smallest possible mainlobe width. Peak sidelobe may be reduced (at the expense of mainlobe width) by tapering the window smoothly to zero at each end.

Bartlett (triangular) window	$w[n]=\begin{cases}\frac{2n}{M} & 0\le n\le \frac{M}{2}\\ 2-\frac{2n}{M} & \frac{M}{2}< n\le M\\ 0 & \text{otherwise}\end{cases}$	−25 dB	$\frac{8\pi}{M}$	
Hann(ing) window	$w[n]=\begin{cases}0.5-0.5\cos\frac{2\pi n}{M} & 0\le n\le M\\ 0 & \text{otherwise}\end{cases}$	−31 dB	$\frac{8\pi}{M}$	
Hamming window	$w[n]=\begin{cases}0.54-0.46\cos\frac{2\pi n}{M} & 0\le n\le M\\ 0 & \text{otherwise}\end{cases}$	−41 dB	$\frac{8\pi}{M}$	

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Interactive Exercise: Identify the Window

- Identify the name of the window for each case:

1) This window looks like:

2) This window has a different mainlobe width than Hamming window.

Rectangular

Bartlett

Hann

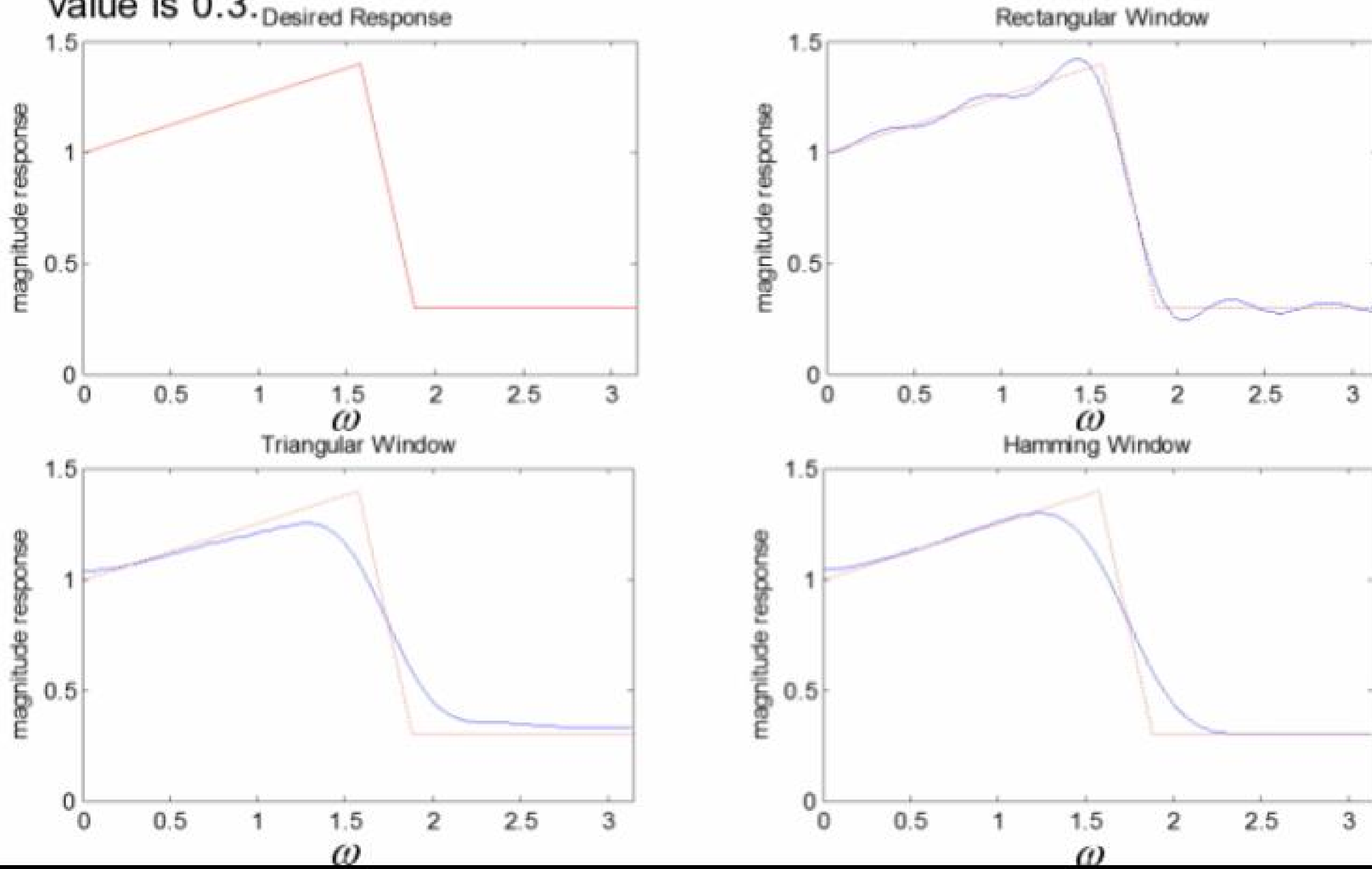
Hamming

Submit

Correct Answer

Filter Designs by Windowing

- Length 20, desired passband value increases from 1 to 1.4, desired stopband value is 0.3.



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Summary

By now, you should be able to:

- Explain the steps involved in FIR filter Design using Windowing;
- Identify Windows used to design FIR filters; and
- Explain Filter Designs by Windowing.

Theories of Parks-McClellan

FIR Filter Design

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Learning Objectives

By the end of this topic, you should be able to:

- Describe Optimal Approximation of FIR Filters;
- Generate the Weighting Function used in Parks-McClellan FIR Filter Design; and
- Explain the theory of Parks-McClellan FIR Filter Design.

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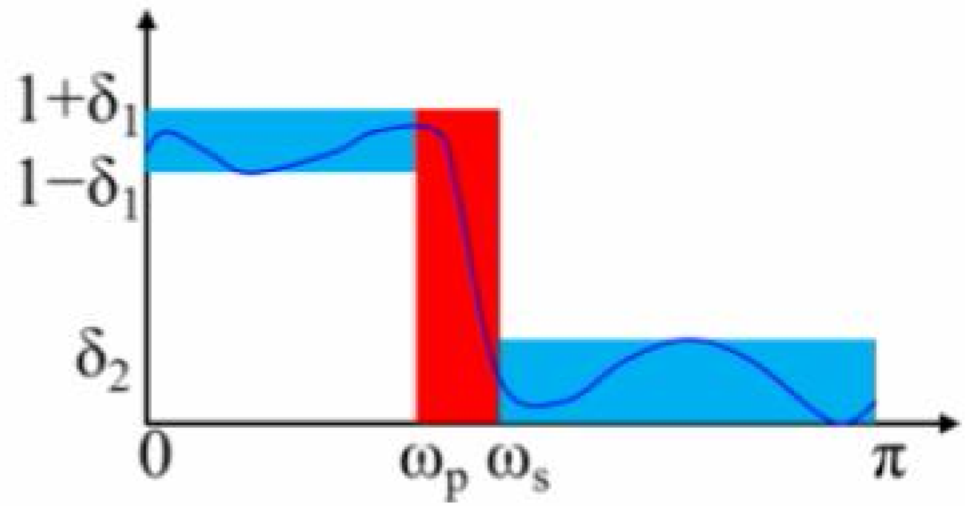
Optimal Approximation of FIR Filters

- Optimal equals minimum L for a given specification δ_1, δ_2, ω_p, ω_s (where $L = M/2$, and M = order)



- However, order has to be made an integer.
- Optimal = minimum δ_1 (or δ_2) for a given specification L, ratio $K = \delta_1/\delta_2$, ω_p, ω_s

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Weighting Function

- Let $A(e^{j\omega})$ = frequency response of the FIR filter to be designed
  $H_d(e^{j\omega})$ = desired frequency response
- Since error in different bands have different tolerance, define a weighting function.

$$W(\omega) = \begin{cases} 1/K & 0 \le \omega \le \omega_p \quad \textit{error weighted by 1/K in passband} \\ 0 & \omega_p < \omega < \omega_s \quad \textit{error weighted by 0 in transition band (don't care)} \\ 1 & \omega_s \le \omega \le \pi \quad \textit{error weighted by 1 in stopband} \end{cases}$$

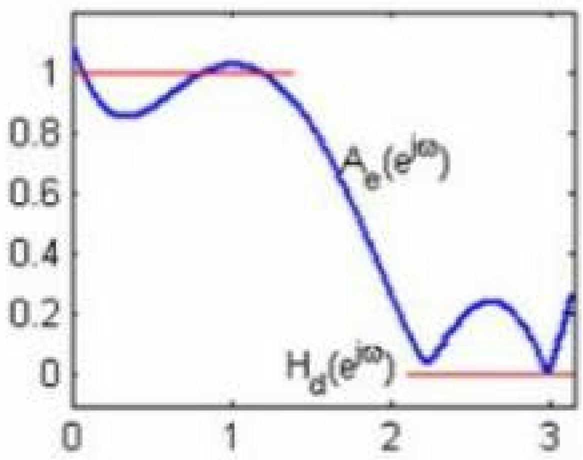
- Error function $E(\omega) = W(\omega)[H_d(e^{j\omega}) - A(e^{j\omega})]$

  minimax or *Chebyshev* criterion = minimise the maximum absolute error, $\min\limits_{h[n]}\left(\max |E(\omega)|\right)$

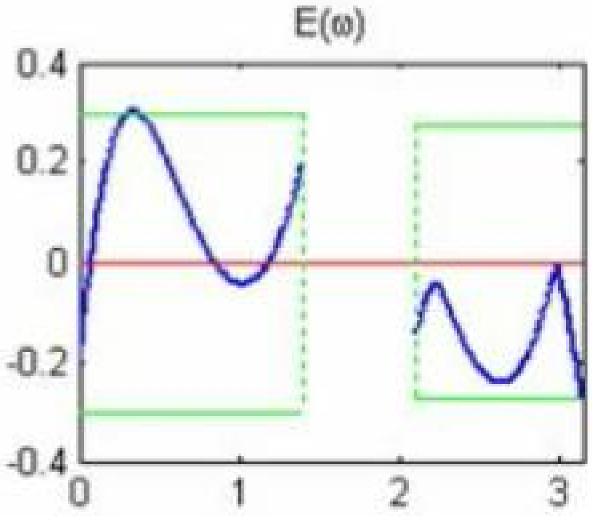
  (minimisation is done over all impulse response coefficient $h[n]$)

Weighting Function

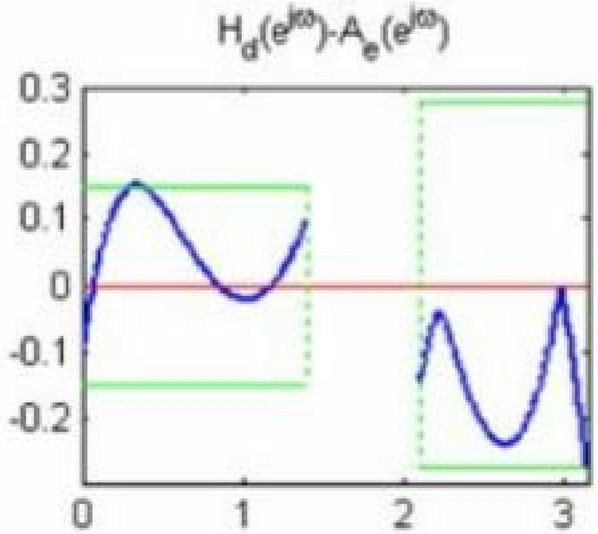
- Filter



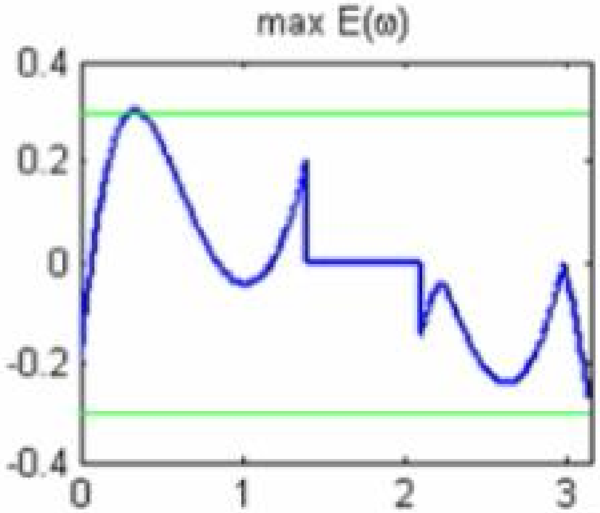
- Error function for K =0.5 (δ_1 is half of δ_2)



- Error before weighting



- Maximum error = 0.3



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Interactive Exercise: Find the Weighting Function

- Parks-McClellan design has been used to design a filter. Its passband has maximum error of 0.065, while its stopband has maximum error of 0.0013. Which of the following weighting function has been used to design this filter?

A) $W(\omega)=\begin{cases} 0.065 & \text{passband} \\ 0 & \text{transition band} \\ 0.0013 & \text{stopband} \end{cases}$

B) $W(\omega)=\begin{cases} 50 & \text{passband} \\ 1 & \text{transition band} \\ 1 & \text{stopband} \end{cases}$

C) $W(\omega)=\begin{cases} 0.02 & \text{passband} \\ 0 & \text{transition band} \\ 1 & \text{stopband} \end{cases}$

D) $W(\omega)=\begin{cases} 50 & \text{passband} \\ 0 & \text{transition band} \\ 1 & \text{stopband} \end{cases}$

Submit   Correct Answer

Parks-McClellan Design: Theory

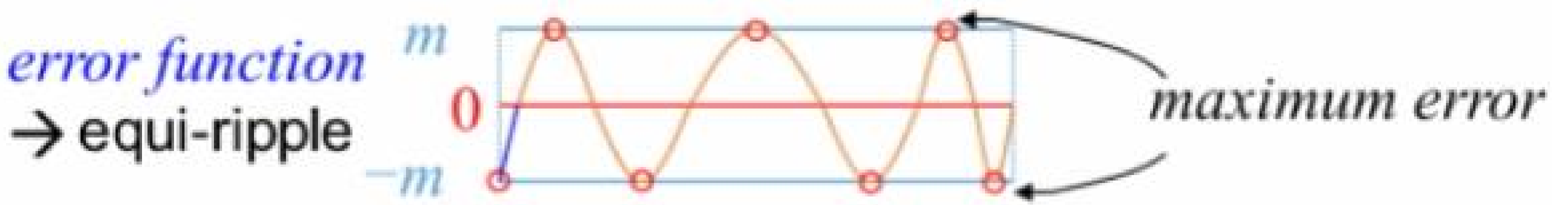
- We have seen, minimum order IIR filter is the elliptic filter which is equi-ripple in both bands.
- Parks-McClellan FIR filter is also equi-ripple in both bands.
- However, IIR filter design uses a closed-form solution. Parks-McClellan FIR filter design uses a computer-based iterative algorithm.
- Alternation Theorem:
  - A polynomial $a_0 + a_1 x + a_2 x^2 + \cdots + a_r x^r$ is optimum using the minimax criterion (minimizes the maximum absolute weighted error from a desired function) if and only if it has at least r+2 values of x where the absolute error is maximum and they occur with alternate signs of error

Parks-McClellan Design: Theory

- Let maximum absolute error = m

  then, these r+2 values have error = either m or $-m$

- Since the error alternates between these values, if error at first value is m, then error at second value is $-m$, error at third value is m, etc



Parks-McClellan Design: Theory

- The Parks-McClellan design is used for linear phase FIR filters.
- Let the filter be symmetric with odd length.
- Due to symmetry its frequency response simplifies to:

$$H(e^{j\omega}) = e^{-j\omega M/2} A(e^{j\omega})$$

$$A(e^{j\omega}) = h[0]2\cos(\omega \cdot \tfrac{M}{2}) + \cdots + h[\tfrac{M}{2}-1]2\cos(\omega) + h[\tfrac{M}{2}]$$

- For any integer n, $\cos(n\omega)$ = an nth-order polynomial of $(\cos\omega)$

  for example: $\cos(2\omega) = 2(\cos\omega)^2 - 1$

$$\cos(3\omega) = 4(\cos\omega)^3 - 3(\cos\omega)$$

- $A(e^{j\omega}) = a_0 + a_1(\cos\omega) + \cdots + a_{M/2}(\cos\omega)^{M/2}$ is a polynomial of order $M/2$ (substitute $x = \cos\omega$). Thus alternation theorem is applicable.

- The above argument is extendible to all types of linear phase filters of any length and any phase. Thus, the design works for any linear phase filter.

Summary

By now, you should be able to:

- Describe Optimal Approximation of FIR Filters;
- Generate the Weighting Function used in Parks-McClellan FIR Filter Design; and
- Explain the theory of Parks-McClellan FIR Filter Design.

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Algorithm and Examples of

Parks-McClellan FIR Filter Design

Learning Objectives

By the end of this topic, you should be able to:

- Explain the steps involved in Parks-McClellan Algorithm;
- Explain the iterations of Parks-McClellan Algorithm; and
- Describe the Parks-McClellan FIR Filter Design.

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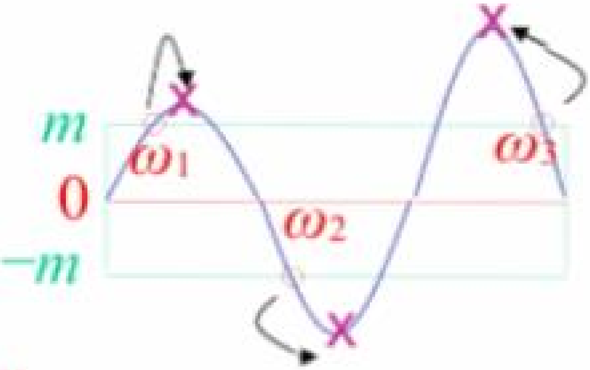
Parks-McClellan Algorithm

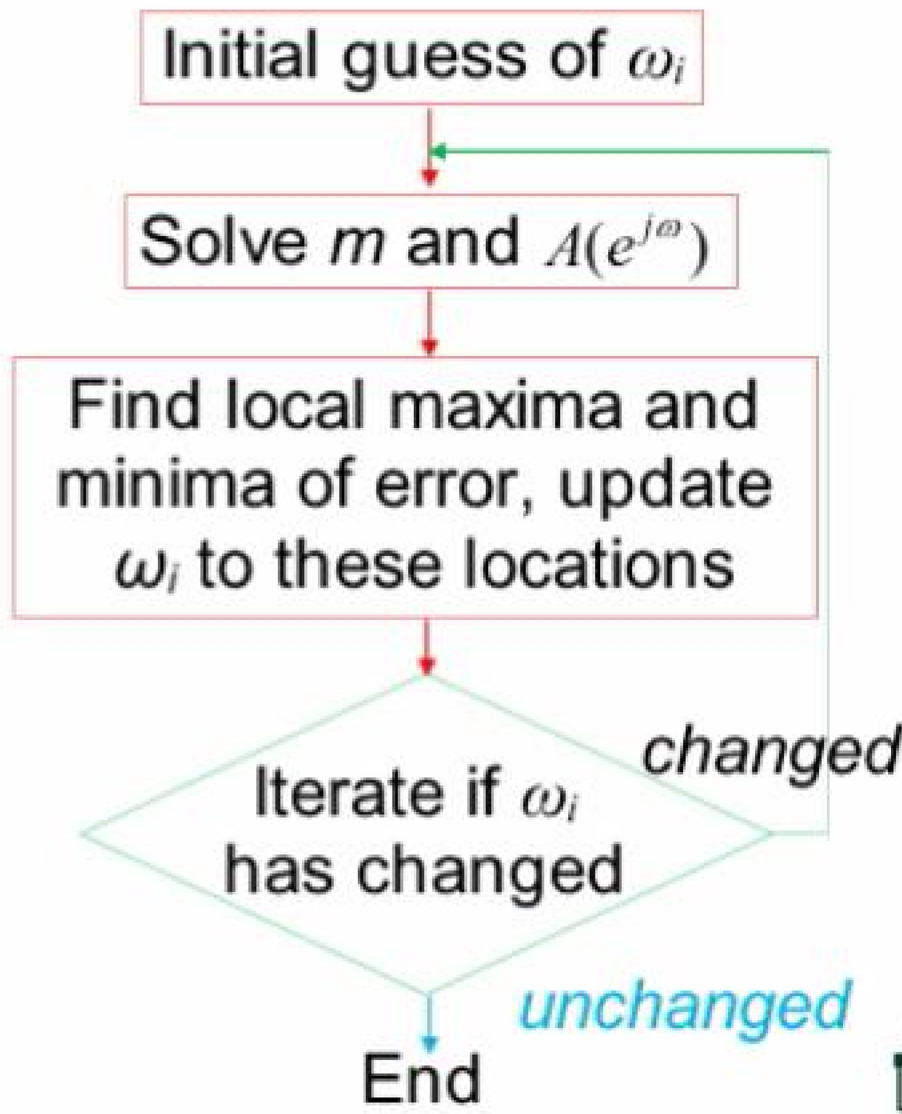
- Assume maximum error locations ω_i of the optimal filter are known, we can find a_i (hence, the filter) by solving the following equations:

$$W(\omega_i)[H_d(e^{j\omega_i}) - A(e^{j\omega_i})] = (-1)^{i+1} m$$



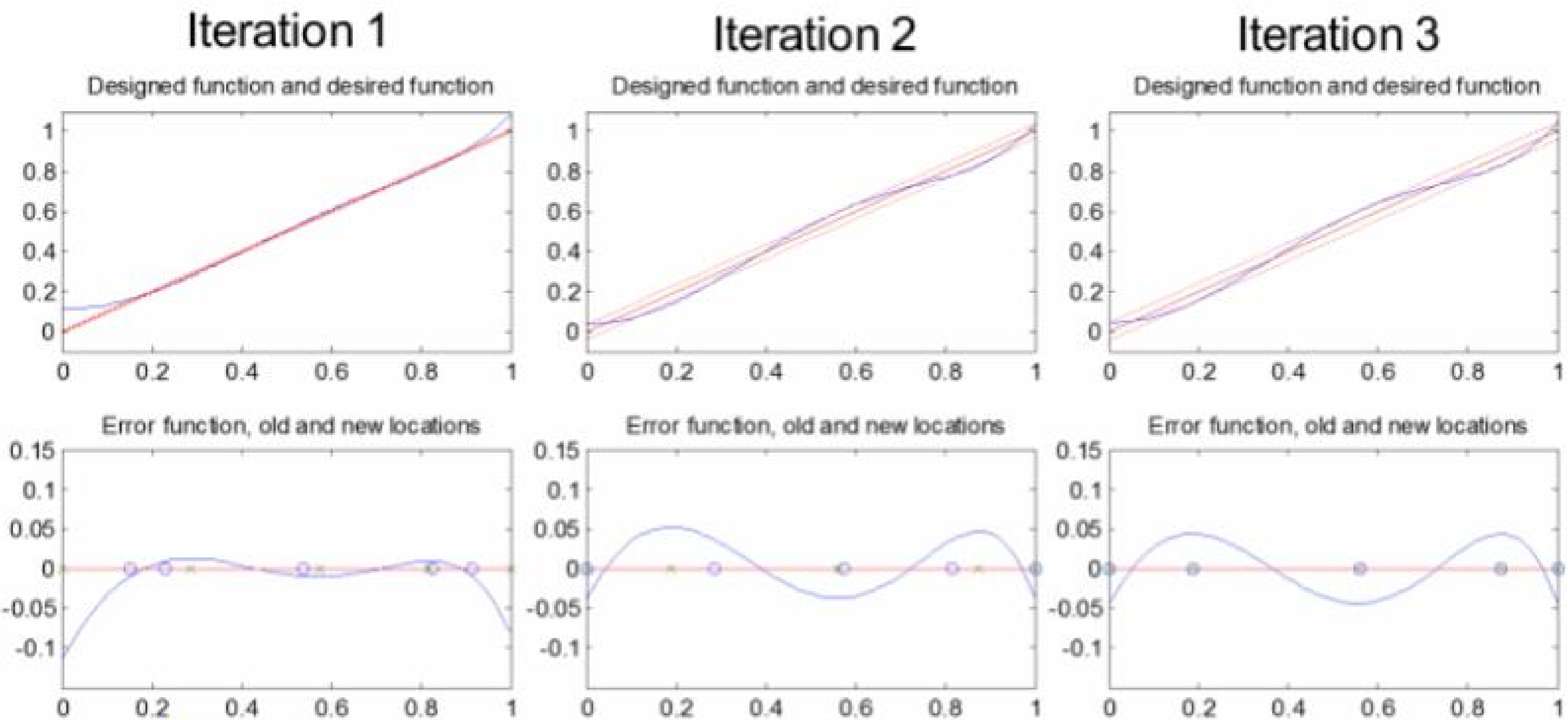
○ *old locations*

x *new locations*



Iterations of Parks-McClellan Algorithm

- Order r = 3, no of locations r + 2 = 5, desired function = red inclined line:

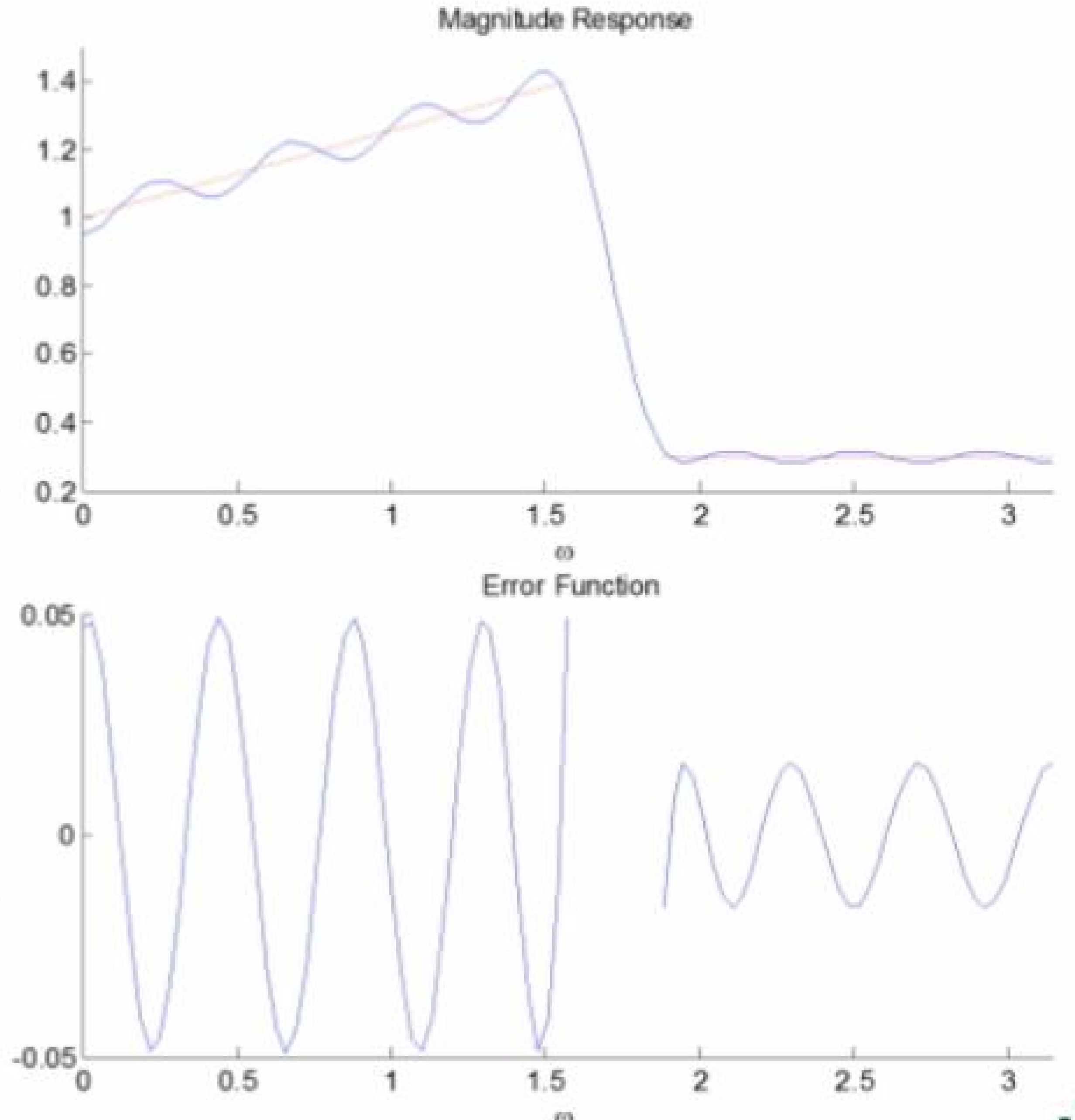


○ *old locations*

x *new locations*

Parks-McClellan FIR Filter Design

- Desired response is the same as FIR Filter Design by Windowing example.
- Filter order = 30.
- Weighting function = 1/3 in passband, 1 in stopband.
- Error function before weighting is shown below the filter response.
- Note error in passband is 3 times error is stopband.



Work Example on Updating Maximum Error Locations

- You begin running Parks-McClellan algorithm with the following initial guesses of maximum error locations:

  $\omega_1 = 0.3$, $\omega_2 = 1.1$, $\omega_3 = 1.6$, $\omega_4 = 2.0$, $\omega_5 = 2.9$ radians
- During the iteration, the filter is found to have the following minima and maxima locations:

  minima at 0, 1.2, 2.8 radians, maxima at 0.5, 2.2 radians
- How should you update ω_2 and ω_3?

Update ω_2 to ☐ radians, and ω_3 to ☐ radians.

Submit Correct Answer

Matlab Video

Select to Play Video

Note: To proceed, do remember to close the pop-up window at the end of the video

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Summary

By now, you should be able to:

- Explain the steps involved in Parks-McClellan Algorithm;
- Explain the iterations of Parks-McClellan Algorithm; and
- Describe the Parks-McClellan FIR Filter Design.