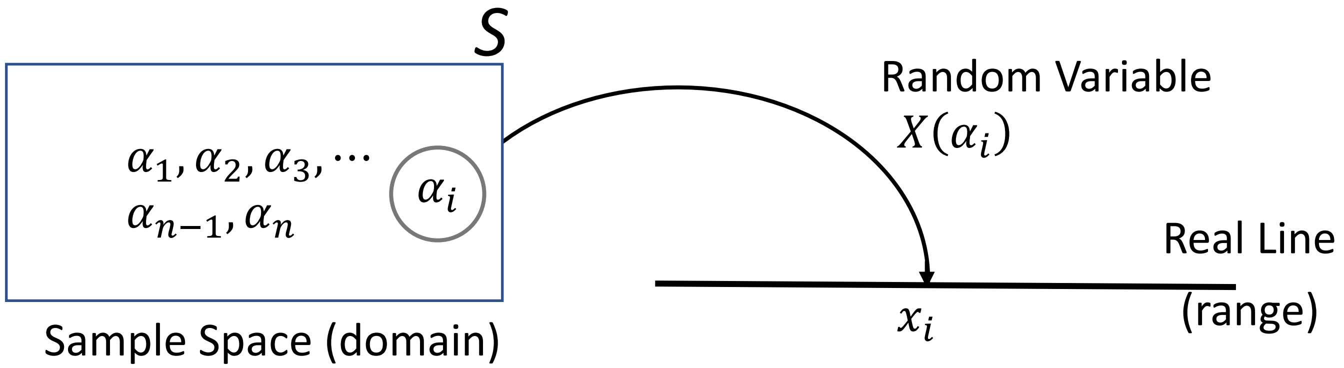
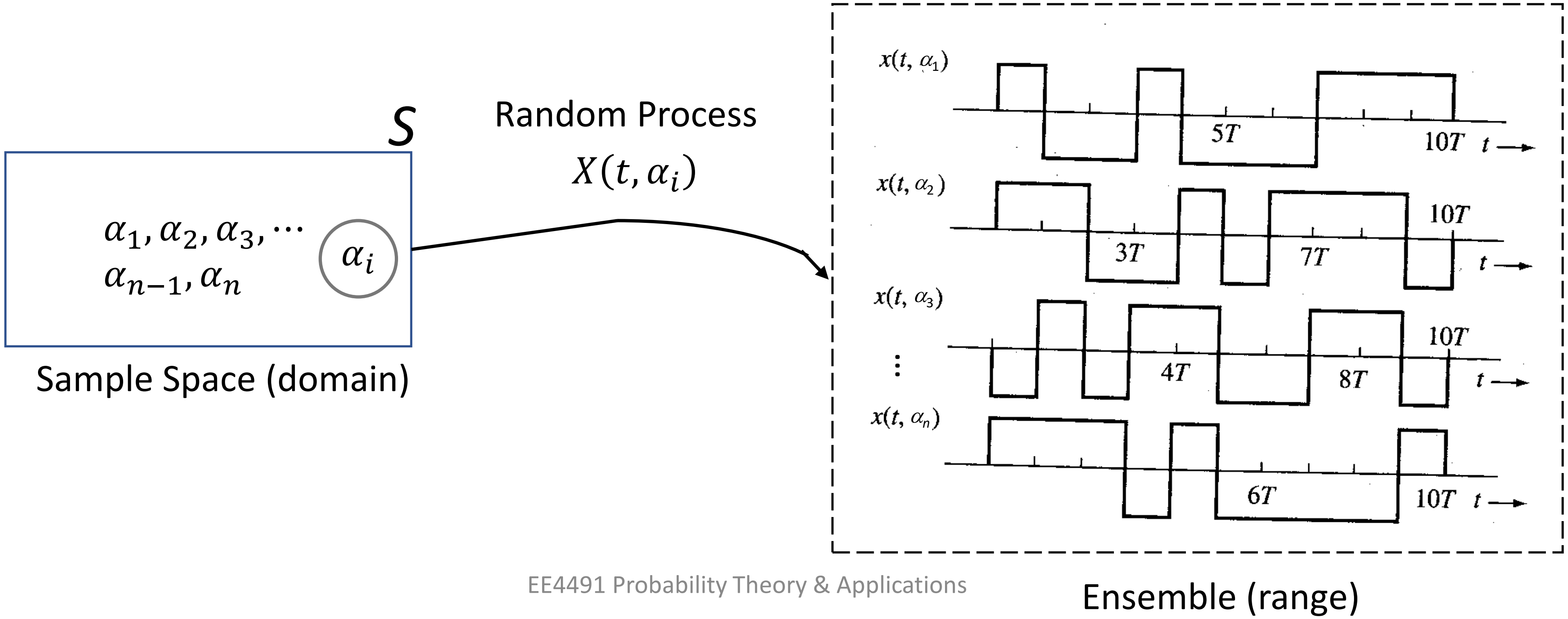
Random Variables and Random Processes



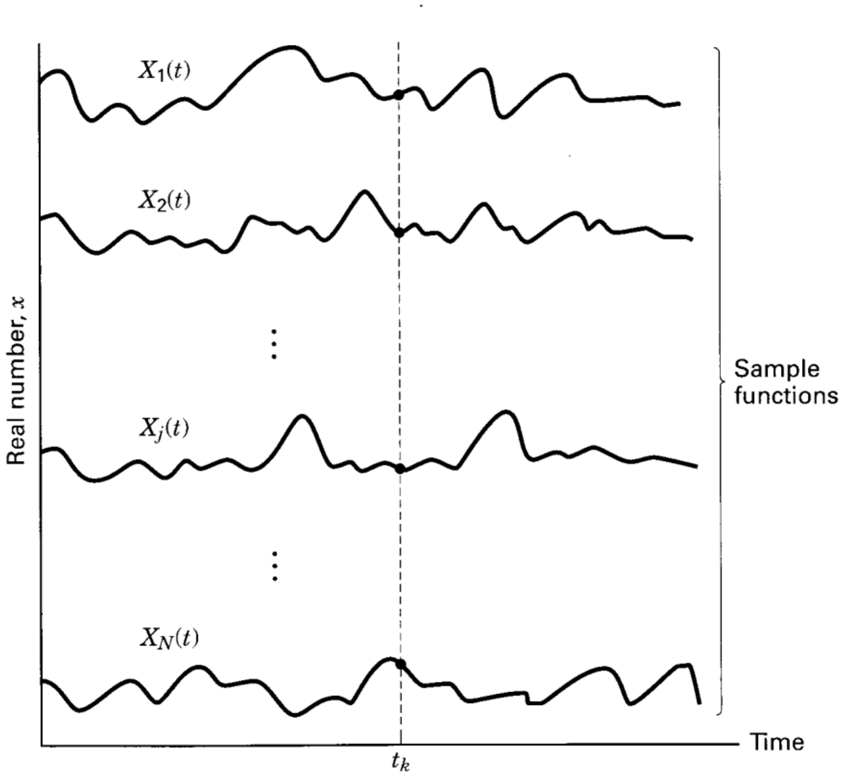


Ensemble and Sample Functions

- A ***random process*** $X(t, \alpha_i)$ can be viewed as a function of two variables: time t and an element $\alpha_i \in S$. The outcome of the mapping gives a ***time function*** $x(t, \alpha_i)$.
- The collection of all possible time functions is known as the ***ensemble***. A time function in the collection is a ***sample function*** or a ***realization***.

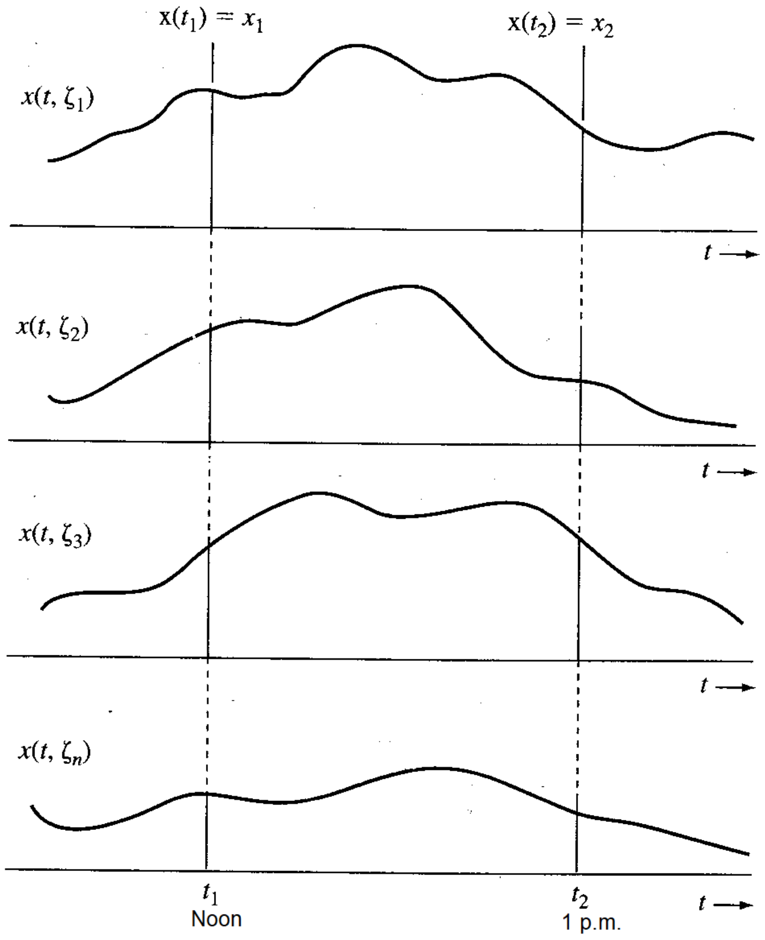
Example 36 – Random Process

- The right figure illustrates a random noise process. There are *N* sample functions of time $\{X(t, A_j)\}$. Each of the sample functions can be regarded as the output of a different noise generator.
- For a specific noise generator A_j, we have a single time function $X(t, A_j) = X_j(t)$. The totality of all sample functions is the ensemble.
- For a specific time t_k, $X(t_k, A)$ is a random variable $X(t_k)$ whose value depends on which noise generator has been chosen.
- For a specific time t_k and a specific A_j, $X(t_k, A_j)$ is simply a real number.



Example 37 – Temperature of a City

- The temperature $X = X(\zeta)$ of a certain city at noon is a random variable and takes on different values every day, where ζ is an element of the sample space S. To get a complete statistics of X, we need to record values at noon over many days and determine the PDF $f_X(x)$.
- However, the temperature is also a function of time. At 1 p.m., for example, the temperature may have an entirely different distribution from that of the temperature at noon. Hence, this random temperature X is also a function of time and can be expressed as $X(t, \zeta)$.



Sample Functions are NOT Random!

- An important point that needs clarification is that the sample functions in the ensemble are not random. They have occurred and are therefore deterministic. Randomness in this situation is associated not with the time functions but with the uncertainty as to which time function would occur in a given trial.
- For notational convenience, we shall designate the random process by $X(t)$ or $Y(t)$, and let the functional dependence on α or ζ be implicit.

Analytical Description

- The next important question is how to describe a random process.
- In some cases, we may be able to describe it analytically.
- Consider a random process described by

$$X(t) = A\cos(\omega_c t + \Theta) \tag{104}$$

where A is a constant, ω_c is the angular carrier frequency and Θ is a random variable uniformly distributed over the range $(0, 2\pi)$.

- This analytical expression completely describes a random process (and its ensemble). Each sample function is a sinusoid with constant amplitude A and constant angular frequency ω_c, but the phase Θ is a random variable uniformly distributed over $(0, 2\pi)$.
- Such an analytical description requires well-defined models such that the random process is characterized by specific parameters that are random variables.

General Description

- Unfortunately, it is not always possible to describe a random process analytically. Without a specific model, we may have just an ensemble obtained experimentally. The ensemble has the complete information about the random process. From this ensemble, we must find some quantitative measure that will characterize the random process.
- In this case, we consider the random process as a random variable $X(t)$ that is a function of time t. Thus, a random process is just a collection of an infinite number of random variables $\{X(t)\}$ for different time instants, which are generally dependent. We know that the complete statistical information of several dependent random variables is provided by the joint CDF or joint PDF of these variables.

Characterization

- Let X_i represent the random variable $X(t_i)$ generated by the amplitudes of the random process at instant $t = t_i$.
- Thus, X_1 is the random variable generated by the amplitudes at $t = t_1$, and X_2 is the random variable generated by the amplitudes at $t = t_2$, and so on.
- The n random variables $X_1, X_2, X_3, \dots, X_n$ generated by the amplitudes at $t = t_1, t_2, t_3, \dots, t_n$, respectively, are dependent in general.
- For the n samples, they are fully characterized by the nth-order joint CDF or the nth-order joint PDF.

Joint CDF and Joint PDF

- The nth order joint CDF

$$F_X(x_1, x_2, \dots, x_n;\ t_1, t_2, \dots, t_n)$$
$$= \Pr\left[X(t_1) \le x_1;\ X(t_2) \le x_2;\ \dots;\ X(t_n) \le x_n\right] \qquad (105)$$

- It leads to the joint PDF

$$f_X(x_1, x_2, \dots, x_n;\ t_1, t_2, \dots, t_n)$$
$$= \frac{\partial^n}{\partial x_1\, \partial x_2 \dots \partial x_n} F_X\,(x_1, x_2, \dots, x_n;\ t_1, t_2, \dots, t_n) \qquad (106)$$

- It can be shown that the random process is ***completely described*** by the nth-order joint CDF or PDF for all n (up to ∞) and for any choice of $t_1, t_2, t_3, \dots, t_n$. This is seen to be a formidable task.
- Fortunately, we shall soon see that when analyzing random signals and noises in conjunction with linear systems, we are often content with the specifications of the first- and second-order statistics.

Lower-Order CDF and PDF

- A higher-order PDF is the joint PDF of the random process at multiple time instants. Hence, we can always derive a lower-order PDF from a higher-order PDF by simple integration. For example,

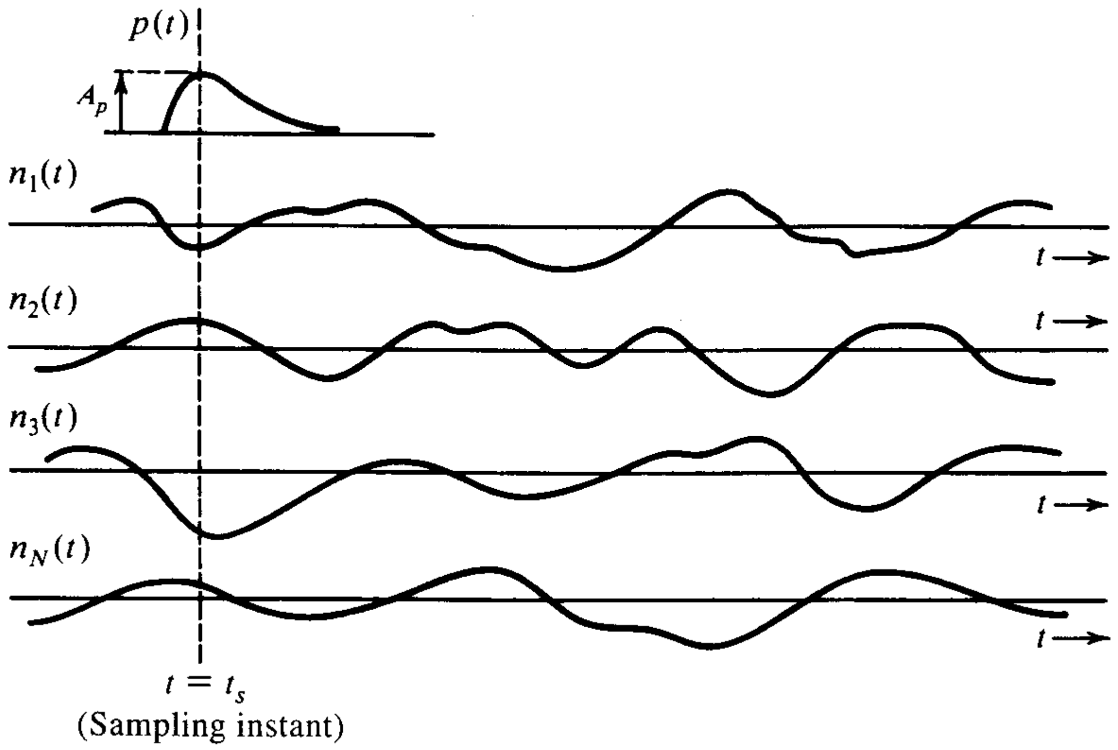
$$f_X(x_1;\ t_1) = \int_{-\infty}^{\infty} f_X(x_1, x_2; t_1, t_2) dx_2$$

- Hence, when the *n*th-order PDF is available, there is no need to specific PDFs of order lower than *n*.
- Similarly, the lower-order CDF can be generated by the higher-order CDF. For example,

$$F_X(x_1; t_1) = F_X(x_1, x_2 = \infty; t_1, t_2)$$

Why Do We Need Ensemble Statistics?

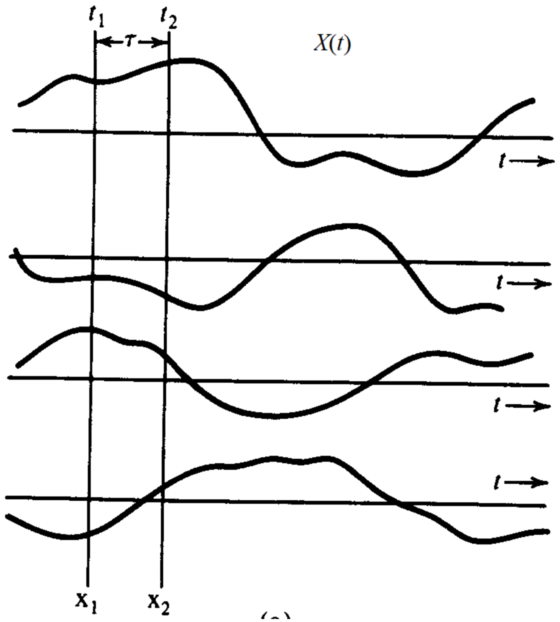
- Consider the problem of threshold detection. A binary 1 is transmitted by a pulse $p(t)$ and a binary 0 is transmitted by $-p(t)$, which is called ***polar signalling***. The peak value of $p(t)$ is A_p at $t = t_s$ and we are going to detect the pulse at this time instant. When a binary 1 is transmitted, the received sample value is $A_p + n$, where n is the noise component.

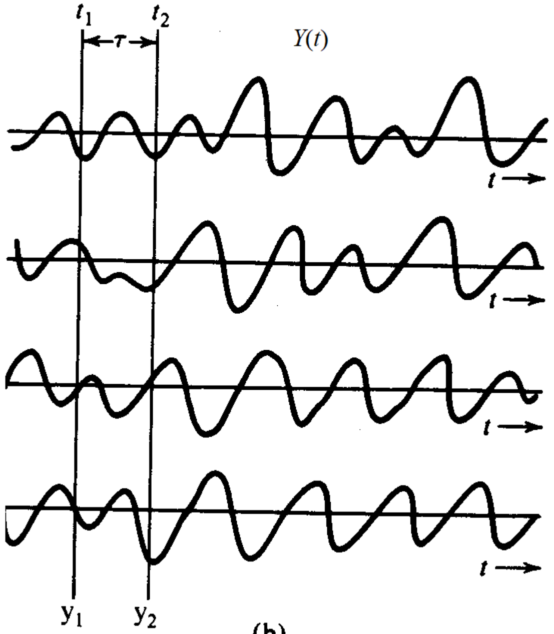


- We shall make a detection error if the noise component at the sampling instant t_s is so negative that makes $A_p + n < 0$. Hence, we need the ensemble statistics of the noise process $n(t)$ at instant t_s.
- When we are dealing with a random process, we do not know which sample function will occur in a given trial. Hence, for calculation the probability of detection error, we need to do ***averaging*** over the entire ensemble. This is the basic reason for the appearance of ensemble statistics in random processes.

Autocorrection Function

- For the purpose of signal analysis, one of the most important characteristics of a random process is its ***autocorrelation function***, which leads to the spectral information of the random process. The spectral content of a process depends on how fast it changes with time. This can be measured by correlating amplitudes at t_1 and $t_2 = t_1 + \tau$.
- On average, the random process $X(t)$ in Fig. (a) is a slowly varying process in comparison to the process $Y(t)$ in Fig. (b). For $x(t)$, the amplitudes at t_1 and $t_1 + \tau$ are similar (see Fig. (a)), that is, have stronger correlation. On the other hand, for $y(t)$, the amplitudes at t_1 and $t_1 + \tau$ have little resemblance (see Fig. (b)), that is, have weaker correlation.



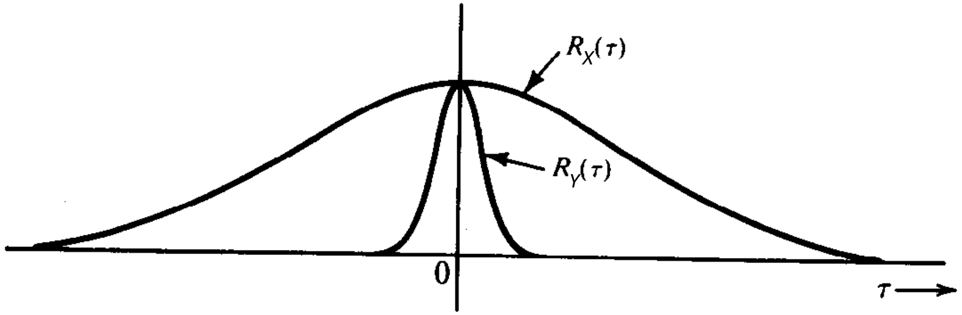


(b)

- Correlation is a measure of the similarity of two random variables. Hence, we can use correlation to measure the similarity of amplitudes at t_1 and $t_2 = t_1 + \tau$. If the random variables $X(t_1)$ and $X(t_2)$ are denoted by X_1 and X_2, respectively, then for a real random process, the autocorrelation function $R_X(t_1, t_2)$ is defined as

$$R_X(t_1, t_2) = E\{X(t_1)X(t_2)\} = E\{X_1X_2\} \tag{107}$$

- It can be seen that for a small τ, the product X_1X_2 will be positive for most sample functions of $X(t)$, but the product Y_1Y_2 is equally likely to be positive or negative. Hence, X_1X_2 will be larger than Y_1Y_2. Moreover, X_1 and X_2 will show correlation for considerably larger values of τ, whereas Y_1 and Y_2 will lose correlation quickly, even for small τ, as shown in Fig. (c).

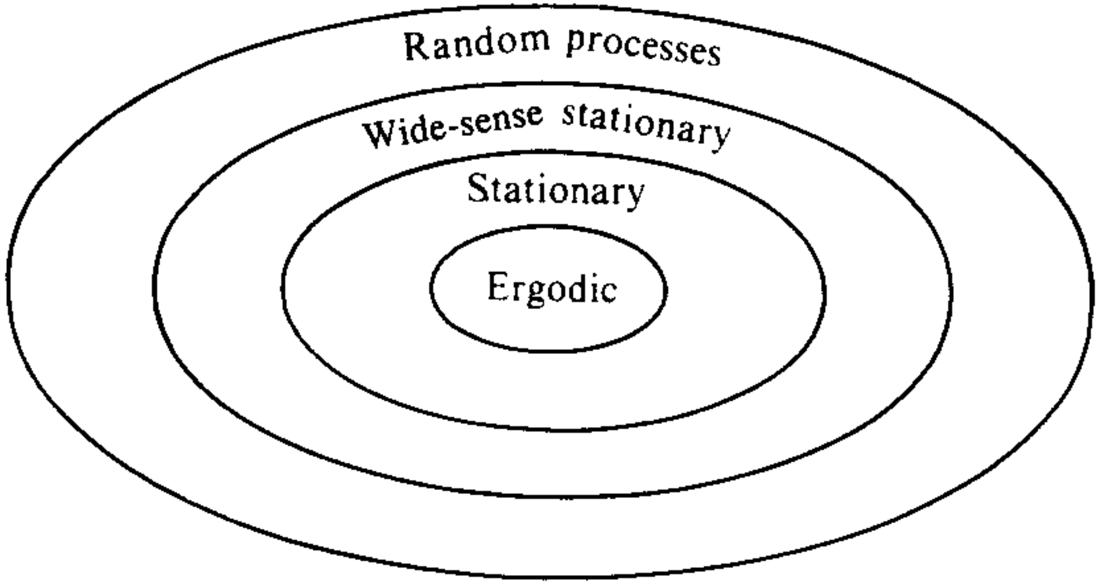


(c)

- Thus, $R_X(t_1, t_2)$, the autocorrelation function of $X(t)$, provides valuable information about the frequency content of the process. In fact, we shall show that the PSD of $X(t)$ is the Fourier transform of its autocorrelation function.

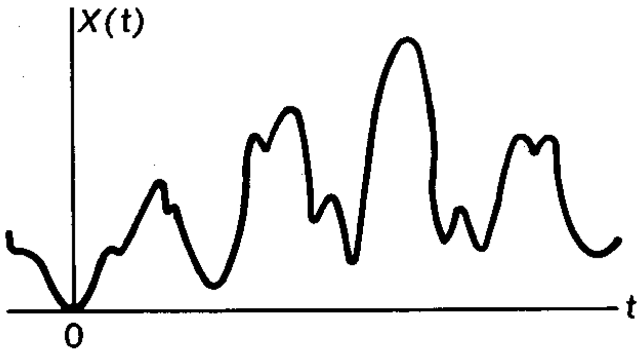
Classification of Random Processes

1) Continuous and Discrete Random Processes
2) Stationary and Nonstationary Random Processes
3) Wide-Sense (or Weakly) Stationary Processes
4) Ergodic Wide-Sense Stationary Processes



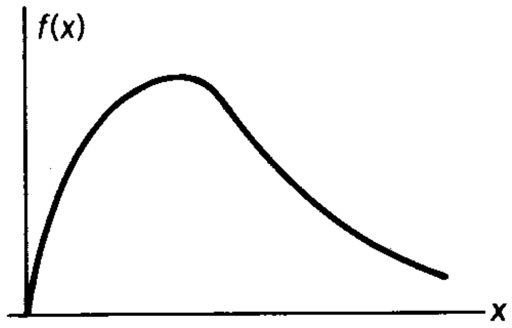
Continuous and Discrete Random Processes

- A ***continuous random process*** is one in which random variables $X(t_1), X(t_2)$, and so on, can assume ***any*** value within a specified range of possible values. This range may be finite, infinite, or semi-infinite.



(a)

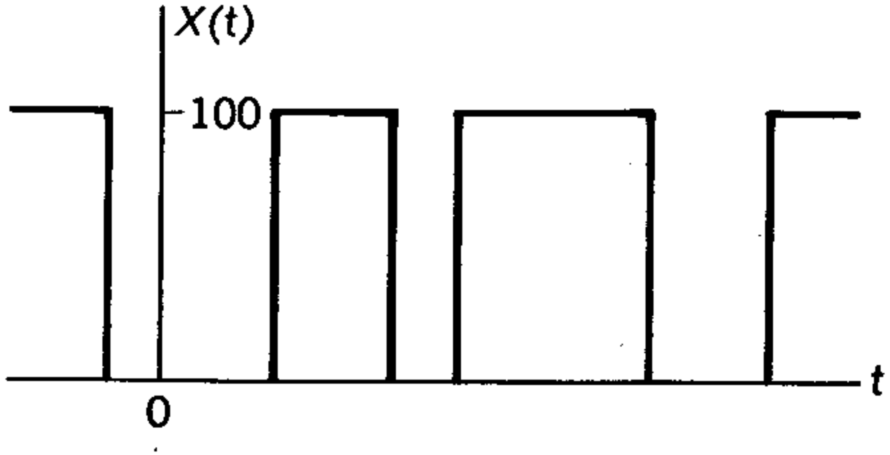
A typical sample function with a semi-definite range.



(b)

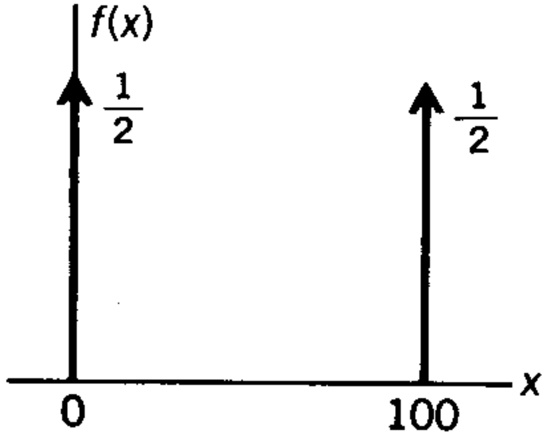
PDF

- A more precise definition for continuous random processes would be that the PDFs are continuous. That is, the PDFs do not contain any $\delta(\cdot)$.
- A ***discrete random processes*** is one in which the random variables can assume only certain isolated values (can be infinite in number) and no other values. The PDFs contain only $\delta(\cdot)$.



(a)

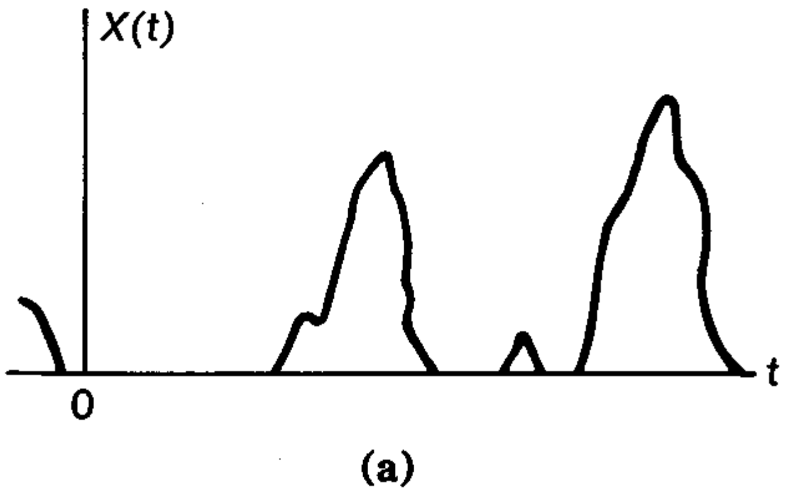
A typical sample function with 2 possible values



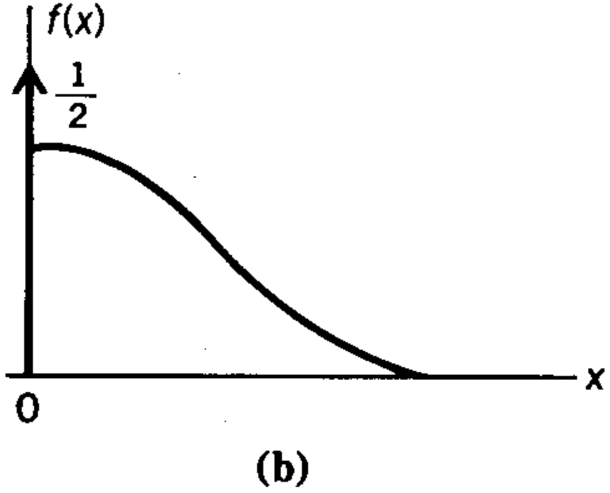
(b)

PDF

- It is also possible to have ***mixed random processes***, which have both continuous and discrete components.



A typical sample function.



PDF

Stationary and Nonstationary Processes

- A random process whose statistical properties are invariant with time is classified as a ***stationary random process***. Accordingly, a shift of time origin will be impossible to notice because the process will appear to be the same statistically. Hence, if $t_2 = t_1 + \tau$,
$$f_X(x; t_1) = f_X(x; t_2)$$
- This is possible only if $f_X(x; t)$ is independent of t. Thus, the first-order density function of a stationary random process can be expressed as
$$f_X(x; t) = f_X(x) \tag{108}$$

- Similarly, the autocorrelation function must depend only on t_1 and t_2 only through the difference $t_2 - t_1$. Hence, for a real-valued stationary process,

$$R_X(t_1, t_2) = R_X(t_2 - t_1) = E\{X(t_1)X(t_1 + \tau)\} \tag{109}$$

- For a stationary random process, the joint PDF for x_1 and x_2 at t_1 and t_2 must also depend only on $t_2 - t_1$. Similarly, higher order PDFs, such as $f_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n)$, are all independent of the choice of origin.
- The random process $X(t)$ of slide #188 representing the temperature of a city is an example of nonstationary random process, because the temperature statistics (mean value, for example) depend on the time of the day.
- The noise process of slide #195 is stationary because the its statistics do not change with time.

Exercise

A random process has sample functions of the form

$$x(t) = \sum_{n=-\infty}^{\infty} A_n g(t - nT)$$

where T is the pulse duration, A_n's are independent random variables that are uniformly distributed from 0 to 10, and

$$g(t) = \begin{cases} 1, & 0 \le t \le T/2 \\ 0, & otherwise \end{cases}$$

a) *Find the mean value of the random variable $X\left(\frac{T}{4}\right)$.*

b) *Find the mean value of the random variable $X\left(\frac{3T}{4}\right)$.*

c) *Is the process stationary?*

<u>Answer</u>: (a) 5; (b) 0; (c) No

Wide-Sense (or Weakly) Stationary Processes

- A random process may not be stationary in the ***strict sense***. It may have a mean value and an autocorrelation function that are independent of the shift of time origin. That is,

$$E\{X(t)\} = \text{constant} \tag{110}$$

and

$$R_X(t_1, t_2) = R_X(t_2 - t_1) = R_X(\tau) \tag{110}$$

- Such a process is known as a ***wide-sense stationary process*** or ***weakly stationary process***.
- Note that stationarity is a much stronger condition than wide-sense stationarity. All stationary processes are wide-sense stationary, but the converse is not true (except Gaussian random processes).

Example 38

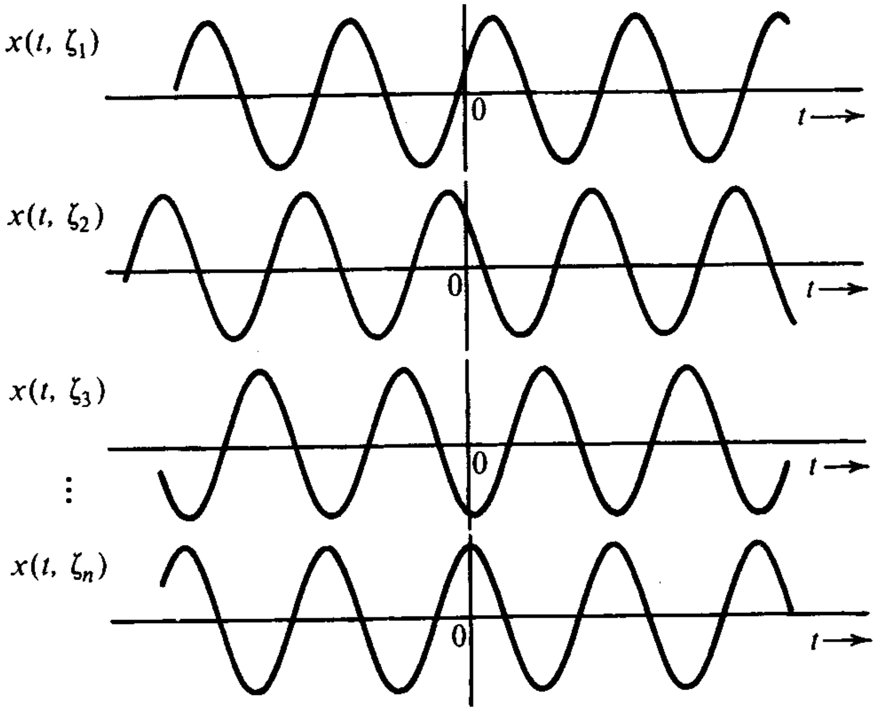
Show that the random process

$$X(t) = A\,cos(\omega_c t + \Theta)$$

where Θ is a random variable uniformly distributed over (0, 2π), is a wide-sense stationary process.

Answer

The ensemble consists of sinusoids of constant amplitude A and constant angular frequency ω_c, but the phase Θ is random.



Ensemble for the random process $A \cos(\omega_c t + \Theta)$.

$$E\{X(t)\} = E\{A\cos(\omega_c t + \Theta)\} = AE\{\cos(\omega_c t + \Theta)\}$$

Because $\cos(\omega_c t + \Theta)$ is a function of Θ, we have

$$E\{\cos(\omega_c t + \Theta)\} = \int_0^{2\pi} \cos(\omega_c t + \theta)\, f_\Theta(\theta)\, d\theta$$

Because $f_\Theta(\theta) = 1/(2\pi)$ over (0, 2π) and 0 outside the range,

$$E\{\cos(\omega_c t + \Theta)\} = \frac{1}{2\pi}\int_0^{2\pi} \cos(\omega_c t + \theta)\, d\theta = 0$$

Hence,

$$E\{X(t)\} = 0$$

$$R_X(t_1, t_2) = E\{A^2 \cos(\omega_c t_1 + \Theta)\cos(\omega_c t_2 + \Theta)\}$$
$$= \frac{A^2}{2} E\{\cos(\omega_c(t_2 - t_1)) + \cos(\omega_c(t_2 + t_1) + 2\Theta)\}$$

The first term on the RHS is not random. The second term is a function of Θ, and its expectation is

$$E\{\cos(\omega_c(t_2 + t_1) + 2\Theta)\} = \frac{1}{2\pi}\int_0^{2\pi} \cos(\omega_c(t_2 + t_1) + 2\theta)\ d\theta = 0$$

Hence,

$$R_X(t_1, t_2) = \frac{A^2}{2}\cos[\omega_c(t_2 - t_1)]$$

or

$$R_X(\tau) = \frac{A^2}{2}\cos(\omega_c \tau)$$

Hence, $X(t)$ is a wide-sense stationary process.

Ergodic Processes

- When we compute the mean and the autocorrelation function of a random process, they are ensemble average.
- We can also define ***time average*** for each sample function as

$$\langle x(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)dt \qquad (111)$$

- Similarly, the ***time-autocorrelation function*** $\Re_X(\tau)$ is defined as

$$\Re_X(\tau) = \langle x(t)x(t+\tau) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)\, dt \qquad (112)$$

- For ***ergodic processes***, ensemble averages are equal to the time averages of any sample function. That is, using any sample function,

$$E\{X(t)\} = \langle x(t) \rangle \quad \text{and} \quad R_X(\tau) = \Re_X(\tau) \qquad (113)$$

- Because a time average cannot be a function of time, it is evident that an ergodic process is necessarily a stationary process, but the converse is not true.
- Ergodic random processes possess the property that almost every sample function of the ensemble exhibits the same statistical behaviour as the whole ensemble. This feature makes it possible to determine the statistics by examining only one realization (sample function). This is the convenience of ergodic processes.
- For ergodic processes, the mean values and moments can also be determined by time averages as well as by ensemble averages:

$$E\{X^n\} = \int_{-\infty}^{\infty} x^n f_X(x) dx = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^n(t)\, dt$$

Exercise

A random process has sample functions of the form

$$X(t) = Y$$

where Y is a zero-mean Gaussian random variable having a variance of 4.

a) *Find the expected value and the correlation function of $X(t)$.*

b) *Is this process wide-sense stationary?*

c) *Is this process ergodic?*

Answer: (a) 0, 4; (b) Yes; (c) No

Exercise

A random waveform has sample functions of the form

$$X(t) = \sum_{n=-\infty}^{\infty} Af(t - nT - t_0)$$

where A is a constant, T is the pulse duration, and t_0 is a random variable uniformly distributed over (0, T). The function $f(t)$ is defined by

$$f(t) = \begin{cases} 1, & 0 \le t \le T/2 \\ 0, & otherwise \end{cases}$$

a) Find $E\{X(t)\}, E\{X^2(t)\}, \langle x \rangle$ and $\langle x^2 \rangle$.

b) Can the process be stationary?

c) Can the process be ergodic?

Answer: (a) $\frac{A}{2}, \frac{A^2}{2}, \frac{A}{2}, \frac{A^2}{2}$; (b) stationary; (c) ergodic

Why Correlation Functions?

- A probabilistic description (in terms of joint PDFs or joint CDFs of any order) of a random process is the most complete one because it incorporates all the knowledge that is available about the process. However, there are many engineering situations in which this degree of completeness is neither needed nor possible.
- In many cases, we are interested in an average power of a random process or the way that average power is distributed with respect to frequency. The entire probability model is actually not needed. A partial statistical description, in terms of certain average values, may provide an acceptable substitute for the probabilistic description.

Autocorrelation Function

- Recall that if two random variables come from the same random process, this correlation function is known as the ***autocorrelation function***. If they come from different random processes, it will be called the ***cross-correlation function***.
- If the random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ are obtained from a random process $X(t)$, then the autocorrelation function is given by

$$R_X(t_1, t_2) = E[X_1 X_2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2)\, dx_1\, dx_2 \qquad (114)$$

Wide-Sense Stationary Processes

- For a ***wide-sense stationary (WSS) process***,

$$R_X(t_1, t_2) = R_X(t_2 - t_1) \tag{115}$$

- Let $\tau = t_2 - t_1$. It is a common practice to write it as

$$R_X(\tau) = E[X(t)X(t+\tau)] \tag{116}$$

- Whenever correlation functions relate to nonstationary processes, they must be written as $R_X(t_1, t_2)$. It is because they are dependent on the particular time instant at which the ensemble average is taken as well as on the time difference between samples,
- It is assumed that all subsequent discussions, unless specifically stated otherwise, are related to WSS random processes.

Physical Meaning of $R_X(\tau)$

- When $\tau = 0$, $R_X(0) = E[X(t)X(t)]$ is equal to the mean-square value or total power of the process. For other values of τ, $R_X(\tau)$ can be thought of as measure of the ***similarity*** of the waveform $X(t)$ and the waveform $X(t + \tau)$.
- In order to illustrate this point, let us consider a sample function $X(t)$ from a zero-mean stationary random process. We define a new process

$$\epsilon(t) = X(t) - \rho X(t + \tau) \quad (117)$$

- We would like to determine the value of ρ such that the variance of $\epsilon(t)$ is minimized.

- In this way, ρ indicates how much of the waveform $X(t+\tau)$ is contained in the waveform $X(t)$.
- The determination of ρ is made by computing the variance of $\epsilon(t)$,

$$\begin{aligned}\sigma_\epsilon^2 &= E[\epsilon^2(t)] - E^2[\epsilon(t)] \\ &= E\{[X(t) - \rho X(t+\tau)]^2\} - E^2[X(t) - \rho X(t+\tau)] \\ &= E[X^2(t) - 2\rho X(t)X(t+\tau) + \rho^2 X^2(t+\tau)] \\ &= \sigma_X^2 - 2\rho R_X(\tau) + \rho^2\,\sigma_X^2\end{aligned}$$

- Setting the derivative with respect to ρ equal to zero yields

$$\frac{d\sigma_\epsilon^2}{d\rho} = -2R_X(\tau) + 2\rho\,\sigma_X^2 = 0$$

- Solving for ρ, we have

$$\rho = \frac{R_X(\tau)}{\sigma_X^2} \tag{118}$$

- Hence, ρ is exactly the ***correlation coefficient***, which can be thought of as the fraction of the waveform of $X(t)$ remaining after τ seconds have elapsed.
- The correlation coefficient ρ can vary from -1 to $+1$. For a value of $\rho = +1$, the waveforms would be identical. That is, they are completely correlated. For $\rho = 0$, the waveforms would be completely uncorrelated. That is, no part of $X(t + \tau)$ would be contained in $X(t)$. For $\rho = -1$, the waveforms would be identical except for opposite sign, i.e., the waveform $X(t + \tau)$ would be the negative of $X(t)$.

Exercise

A random process has sample functions of the form

$$X(t) = A \quad 0 \le t \le 1$$
$$= 0 \quad \text{otherwise}$$

where A is a random variable that is uniformly distributed from -12 to 12. Using the basic definition of the autocorrelation function, find the autocorrelation function of this process.

<u>Answer</u>

$$R_X(t_1, t_2) = 48 \quad 0 \le t_1, t_2 \le 1$$
$$= 0 \quad \text{otherwise}$$

Exercise

Define a random process $Z(t)$ *as*

$$Z(t) = X(t) + X(t + \tau_1)$$

where $X(t)$ *is a sample function from a stationary random process whose autocorrelation function is*

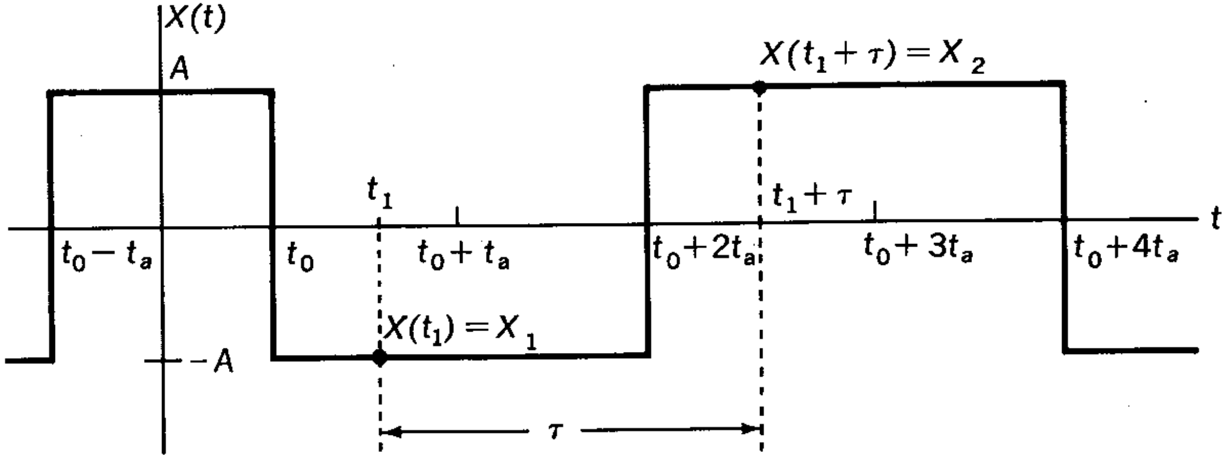
$$R_X(\tau) = exp(-\tau^2)$$

Find the autocorrelation function of the random process $Z(t)$ *.*

<u>Answer</u>

$$R_Z(\tau) = 2\exp(-\tau^2) + \exp[-(\tau - \tau_1)^2]$$
$$+ \exp[-(\tau + \tau_1)^2]$$

Example 39 - $R_X(\tau)$ of a Binary Process



Consider a sample function from a discrete, stationary, zero-mean random process $X(t)$ in which two values, $+A$ and $-A$, are possible. The sample function either can switch from one value to the other value every t_a seconds or remain the same, with equal probability. The time t_0 is a random variable with respect to the ensemble of possible time functions and is uniformly distributed over an interval of length t_a.

As far as the ensemble is concerned, switches can occur at any time with equal probability. It is also assumed that the value of $X(t)$ in any one interval is statistically independent of its value in any other interval.

a) The autocorrelation function is an even function. Hence, we only need to consider positive values of τ. $R_X(\tau)$ with negative values of τ can be obtained by the even symmetry.

b) When $\tau > t_a$, the two times instants, t_1 and $t_2 = t_1 + \tau$, cannot lie in the same t_a interval. Hence, $X_1 = X(t_1)$ and $X_2 = X(t_2)$ are statistically independent. Since X_1 and X_2 have zero mean, $R_X(\tau) = E[X_1 X_2] = E[X_1]E[X_2] = 0$.

c) When $0 \leq \tau \leq t_a$, t_1 and t_2 may or may not lie in the same t_a interval, depending on the value of t_0. Since t_0 can be anywhere, with equal probability, the probability that they do lie in the same interval is proportional to the difference between t_a and τ. Hence,

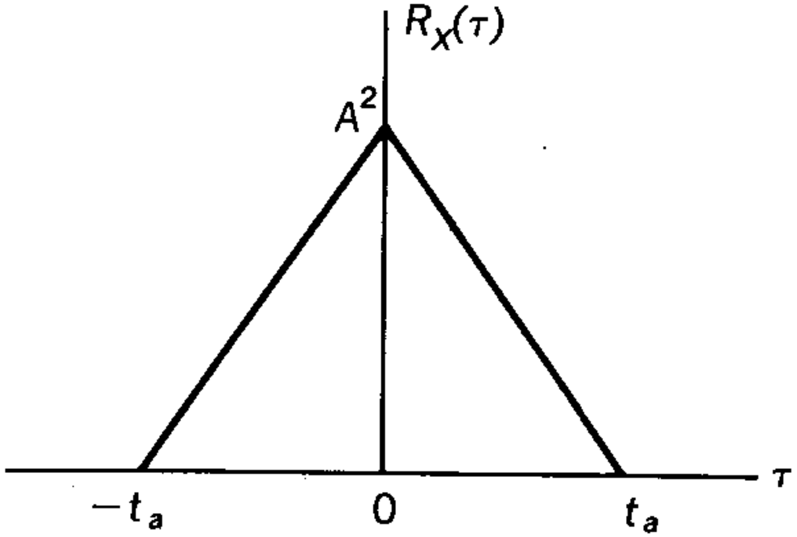
$$\Pr(t_1 \text{ and } t_2 \text{ are in the same interval}) = \frac{t_a - \tau}{t_a}$$

In general, with positive and negative values of τ,

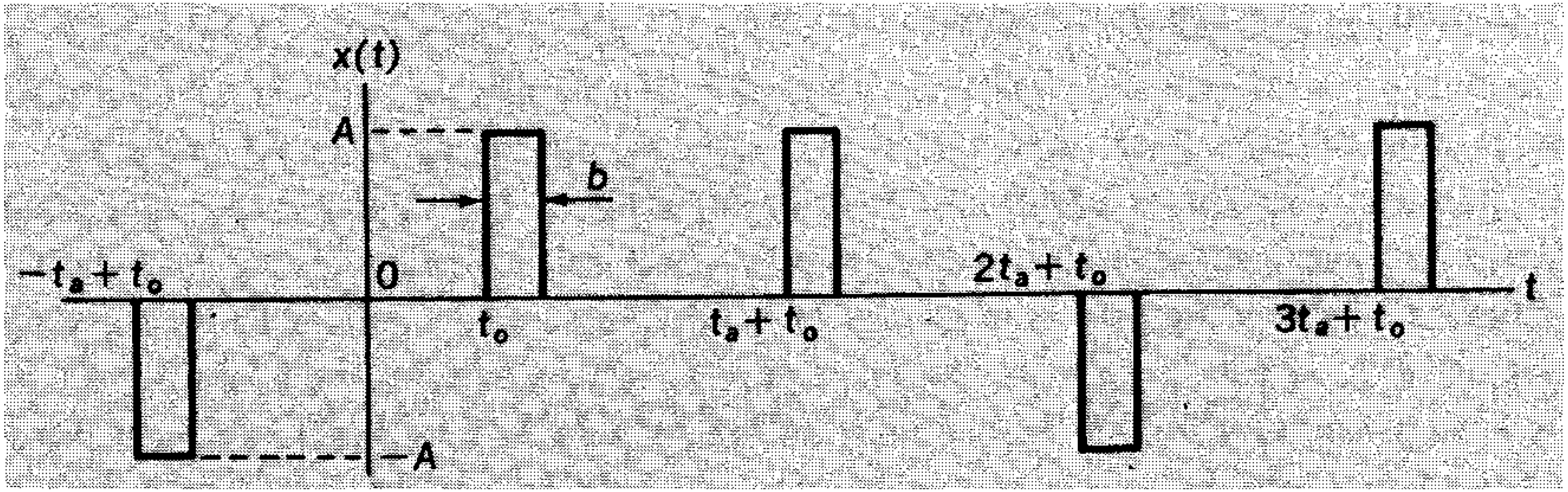
$$\Pr(t_1 \text{ and } t_2 \text{ are in the same interval}) = \frac{t_a - |\tau|}{t_a}$$

Hence,

$$R_X(\tau) = A^2 \left[\frac{t_a - |\tau|}{t_a}\right], \qquad 0 \le |\tau| \le t_a$$
$$= 0, \qquad \text{otherwise}$$



Exercise



A sample function from a stationary random process is shown above, The quantity t_0 is a random variable uniformly distributed from 0 to t_a. The pulse amplitudes are $+A$ and $-A$ with equal probability, which are independent from pulse to pulse. Find the autocorrelation function of this process.

Answer:

$$R_X(\tau) = A^2 \frac{b}{t_a}\left[1 - \frac{|\tau|}{b}\right], \qquad |\tau| \le b$$
$$= 0, \qquad \text{otherwise}$$

Properties of $R_X(\tau)$

1) $R_X(0) = E[X^2(t)]$. Hence, the mean-square value of the random process can always be computed by setting $\tau = 0$.

2) $R_X(\tau) = R_X(-\tau)$. The autocorrelation function is an ***even*** function of τ. The proof is quite straightforward:
$$R_X(\tau) = E[X(t)X(t+\tau)] = E[X(t+\tau)X(t)] = R_X(-\tau).$$

3) $|R_X(\tau)| \leq R_X(0)$. The largest value of the autocorrelation function always occurs at $\tau = 0$. There may be other values of τ for which it is just as large, but it cannot be bigger.

Exercise

Consider the inequality $E[(X_1 \pm X_2)^2] \geq 0$, where $X_1 = X(t_1)$ and $X(t_2)$ of a WSS process. Expand the expectation and show that

$$R_X(0) \geq |R_X(\tau)|, \qquad \tau = t_2 - t_1$$

Properties of $R_X(\tau)$ …

4) If $X(t)$ has a dc component or mean value, then $R_X(\tau)$ will have a constant component.

  For ergodic processes, the magnitude of the mean value of the process can be determined by examining the autocorrelation function as τ goes to infinity, provided that any periodic components in the autocorrelation function are ignored in the limit $(\tau \to \infty)$. Since the ***square*** of the mean value is obtained from this calculation, it is not possible to determined the sign of the mean value.

Exercise

If $X(t)$ *has a mean value* A *and a zero-mean component* $N(t)$ *so that*

$$X(t) = A + N(t)$$

Both $X(t)$ *and* $N(t)$ *are WSS. Find the autocorrelation function of* $X(t)$.

Answer

$$R_X(\tau) = A^2 + R_N(\tau)$$

Properties of $R_X(\tau)$ …

5) If $X(t)$ has a periodic component, then the autocorrelation function $R_X(\tau)$ will also have a periodic component with the same period.

   This property can be extended to any number of periodic components. If the random variables associated with the periodic components are statistically independent, then the autocorrelation function of the sum of the periodic components is simply the sum of the corresponding autocorrelation functions.

Exercise

Let $X(t) = Acos(\omega_c t + \Theta) + N(t)$*, where* A *and* ω_c *are constants and* Θ *is a random variable uniformly distribution over the range* $(0, 2\pi)$ *and 0 otherwise. The process* $N(t)$ *is WSS with zero mean. Note that* Θ *and* $N(t)$ *are statistically independent for all* t*. Show that*

$$R_X(\tau) = \frac{A^2}{2}\cos(\omega_c \tau) + R_N(\tau)$$

Hence, the autocorrelation function still contains a periodic component $\cos(\omega_c t)$ with the same period as $X(t)$.

Properties of $R_X(\tau)$ …

6) If the random process $\{X(t)\}$ is ergodic with zero mean, and has no periodic components, then

$$\lim_{|\tau|\to\infty} R_X(\tau) = 0 \tag{119}$$

For large values of τ, since the effect of the past values tends to die out as the time progress, the random variables $X(t)$ and $X(t+\tau)$ tend to be statistically independent.

7) Autocorrelation functions cannot have an arbitrary shape. This fact is more apparent when the power spectral density is introduced later. As $R_X(\tau) \Longleftrightarrow S_X(f)$ are a Fourier-transform pair, $\text{the restriction } S_X(f) \geq 0$ must be true for all f.

$R_X(\tau) \neq$ Joint PDF

- Although a knowledge of the joint PDF of the random process is sufficient to obtain a unique autocorrelation function, the converse is not true.
- There may be many different random processes that can yield the same autocorrelation function. That is, the autocorrelation function cannot unique determine the joint PDF.
- Hence, the specification of the correlation function of a random process is not equivalent to the specification of the joint PDF. In fact, $R_X(\tau)$ represents a considerably smaller amount of information.

Example 40

Given the autocorrelation function

$$R_X(\tau) = 25 + \frac{4}{1 + 6\tau^2}$$

find the mean value and variance of the process $X(t)$.

<u>Answers</u>

The mean value is $E[X(t)] = \sqrt{25} = \pm 5$, and the mean-square value is $R_X(0) = 25 + 4 = 29$. Hence, the variance of the process is

$$\sigma_X^2 = 29 - (\pm 5)^2 = 4$$

Exercise

a) An ergodic random process has an autocorrelation function of the form

$$R_X(\tau) = 25\, e^{-4|\tau|} + 16\cos(20\tau) + 36$$

Find the mean value, mean-square value, and variance of the process.

b) An ergodic random process has an autocorrelation function of the form

$$R_X(\tau) = \frac{25\tau^2 + 36}{6.25\tau^2 + 4}$$

Find the mean value, mean-square value, and variance of the process.

Answers: (a) $\pm 6, 77, 41$ (b) $\pm 2, 9, 5$

Cross-correlation Functions

- The cross-correlation function of two random processes $X(t)$ and $Y(t)$ is given by

$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)] \tag{120}$$

- If $X(t)$ and $Y(t)$ are at least jointly WSS, $R_{XY}(t, t+\tau)$ is independent of the time origin and can be written as

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] \tag{121}$$

- If $R_{XY}(t, t+\tau) = 0$, then $X(t)$ and $Y(t)$ are called ***orthogonal processes***.

Cross-correlation Functions …

- If $X(t)$ and $Y(t)$ are independent, then

$$R_{XY}(t, t+\tau) = E[X(t)]E[Y(t+\tau)] \tag{121}$$

- If, in addition to being independent, $X(t)$ and $Y(t)$ are at least WSS, then (121) can be written as

$$R_{XY}(\tau) = m_X m_Y = \text{constant} \tag{122}$$

where m_X and m_Y are the mean values of $X(t)$ and $Y(t)$, respectively.

Cross-correlation Functions …

- Similarly, we can define

$$R_{YX}(\tau) = E[Y(t)X(t+\tau)] \tag{123}$$

- Note that because both random processes are assumed to be ***jointly*** stationary, these cross-correlation functions depend only upon the time difference.
- It is important that the processes be joint stationary and not individually stationary. It is also possible to have two individually stationary random processes that are not joint stationary. In such a case, the cross-correlation function depends upon time t, as well as the time difference τ.

Cross-correlation Functions …

- The time cross-correlation functions are defined as

$$\Re_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)y(t+\tau)\, dt \tag{124}$$

and

$$\Re_{yx}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(t)x(t+\tau)\, dt \tag{125}$$

- If the random processes are joint ergodic, then ensemble averages and time averages are interchangeable as follows:

$$R_{XY}(\tau) = \Re_{xy}(\tau) \text{ and } R_{YX}(\tau) = \Re_{yx}(\tau) \tag{126}$$

Properties of Cross-Correlation Functions

In general, the physical interpretation of cross-correlation functions is no more concrete than that of autocorrelation functions. It is simply a measure of how much these two random variables depend upon one another.

In the later study of system analysis, however, the specific cross-correlation function between input and output will take on a very important physical significance.

The general properties of all cross-correlation functions are quite different from those of autocorrelation function. They are summarized in the next few slides:

Properties of Cross-Correlation Functions …

1) The quantities $R_{XY}(0)$ and $R_{YX}(0)$ have no particular physical meaning and do ***not*** represent mean-square values. However,

$$R_{XY}(0) = R_{YX}(0) \tag{127}$$

2) Unlike autocorrelation functions, cross-correlation functions are ***not*** generally even functions of τ. However,

$$R_{XY}(\tau) = R_{YX}(-\tau) \tag{128}$$

This can easily be observed from the derivation:

$$\begin{aligned} R_{XY}(\tau) &= E[X(t)Y(t+\tau)] \\ &= E[Y(\lambda)X(\lambda-\tau)] \\ &= R_{YX}(-\tau) \end{aligned}$$

Properties of Cross-Correlation Functions …

3) A cross-correlation function does not necessarily have its maximum value at $\tau = 0$. However,

$$|R_{XY}(\tau)| \leq \sqrt{R_X(0)R_Y(0)} \qquad (129)$$

with a similar relationship for $R_{YX}(\tau)$. The maximum of the cross-correlation function can occur anywhere, but it cannot exceed the above value. Furthermore, it may not achieve this value anywhere.

4) If two random processes are statistically independent, then

$$R_{XY}(\tau) = m_X m_Y = R_{YX}(\tau) \qquad (128)$$

If either process has zero mean, then $R_{XY}(\tau) = 0 = R_{YX}(\tau)$ for all τ. Hence, independence implies uncorrelation. However, the converse is not necessarily true, except for jointly Gaussian random processes.

Measurement of Autocorrelation Functions

- In general, autocorrelation functions cannot be calculated from the joint PDFs as these PDFs are seldom available. On the other hand, we cannot obtain the ensemble average because there is usually only one sample function available from the ensemble. Under these restrictions, the only available procedure is to compute a time autocorrelation function for a finite time interval, under the assumption that the random process is ergodic.
- The estimated correlation function can be written as

$$\hat{R}_X(\tau) = \frac{1}{T-\tau}\int_0^{T-\tau} x(t)\, x(t+\tau)\, dt, \quad 0 \le \tau \ll T \tag{129}$$

Note that the averaging time is $T - \tau$ rather than T because this is the only portion of the observed data in which both $x(t)$ and $x(t+\tau)$ are available.

Measurement of Autocorrelation Functions …

- In many cases it is not possible to carry out the integration in (129) because the mathematical expression for $x(t)$ is not available. To move forward, we can approximate the integral in (129) by summation as follows:

$$\hat{R}_X(n\Delta t) = \frac{1}{N-n+1} \sum_{k=0}^{N-n} x_k x_{k+n}, \qquad n = 0,1,2, \dots, M, \quad M \ll N \tag{130}$$

where $T = N\Delta t$, the samples of a particular sample function are taken at time instants of $\{k\Delta t\}_{k=0,1,\dots,N}$, and the corresponding values of $x(t)$ are $x_0, x_1, x_2, \dots, x_N$.

Introduction to Fourier Analysis

- The use of Fourier transform in the analysis of linear systems is widespread and frequently leads to much saving in labor. The principal reason for this simplification is that the convolution integral in the time domain is replaced by simple multiplication in the frequency domain.
- When the inputs to the system are random, frequency-domain methods are still useful but that some modifications are required. However, when properly used, frequency-domain methods offer essentially the same advantages in dealing with random signals as they do with non-random signals.

Fourier Transform of Non-Random Signals

- The Fourier transform of some ***non-random*** time function is

$$F_x(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt \tag{131}$$

- Hence, the Fourier-transform pair is denoted by

$$x(t) \longleftrightarrow F_x(\omega) \tag{132}$$

- It might seem reasonable to use exactly the same procedure in dealing with ***random*** signals. However, it is not possible to do this for at least two reasons. First, the Fourier transform will be a random variable over the ensemble (for each ω) since it will have a different value for each member of the ensemble of possible sample functions. Second, for WSS processes, Fourier transform will not exist because it is ***not absolutely integrable***, i.e., failing to satisfy

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty \tag{133}$$

Modifications

- In order to use the Fourier transform technique, it is necessary to modify the sample functions of a WSS process in such a way that the transform of each sample function exists. Let

$$X_T(t) = X(t) \times \mathrm{rect}\left(\frac{t}{2T}\right) \tag{134}$$

The function $\mathrm{rect}(\cdot)$ restricts the range to $|t| \leq T$.

- Note that the truncated time function $X_T(t)$ will satisfy the condition in (133) as long as T remains finite, provided that the WSS process from which it is taken has a finite mean-square value. Hence, $X_T(t)$ will be Fourier transformable. In fact, it will satisfy the more stringent requirement

$$\int_{-\infty}^{\infty} |X_T(t)|^2 dt < \infty \tag{135}$$

Fourier Transform of Random Signals

- Since $X_T(t)$ is Fourier transformable, its transform is

$$F_X(\omega) = \int_{-\infty}^{\infty} X_T(t) e^{-j\omega t}\, dt, \quad T < \infty \tag{136}$$

- Hence, the Fourier-transform pair is given by

$$X_T(t) \longleftrightarrow F_X(\omega) \tag{137}$$

- The spectral density of the random process $X(t)$ is defined as

$$S_X(\omega) = \lim_{T \to \infty} \frac{E[|F_X(\omega)|^2]}{2T} = \lim_{T \to \infty} \frac{E[F_X^*(\omega) F_X(\omega)]}{2T} \tag{138}$$

- It must be remembered that it is not possible to let $T \to \infty$ ***before*** taking the expectation. The spectral density above is sometimes referred to as the ***two-sided spectral density*** as it exists for both positive and negative values of ω.

Relation of $S_X(\omega)$ to $R_X(\tau)$

- For simplicity, we focus on WSS processes. Since $X_T(t) \longleftrightarrow F_X(\omega)$, we have, from (136),

$$F_X(\omega) = \int_{-T}^{T} X(t)e^{-j\omega t}\, dt, \;\; T < \infty \tag{139}$$

- Substituting (139) into (138) yields

$$S_X(\omega) = \lim_{T\to\infty} \frac{1}{2T}\, E\left[\int_{-T}^{T} X(t_1)e^{j\omega t_1}\, dt_1 \int_{-T}^{T} X(t_2)e^{-j\omega t_2}\, dt_2\right] \tag{140}$$

- The subscripts on t_1 and t_2 have been introduced for subsequent operations. We can rewrite (140) as

$$S_X(\omega) = \lim_{T\to\infty} \frac{1}{2T}\, E\left[\int_{-T}^{T} \int_{-T}^{T} X(t_1)X(t_2)\, e^{-j\omega(t_2-t_1)} dt_1\, dt_2\right]$$

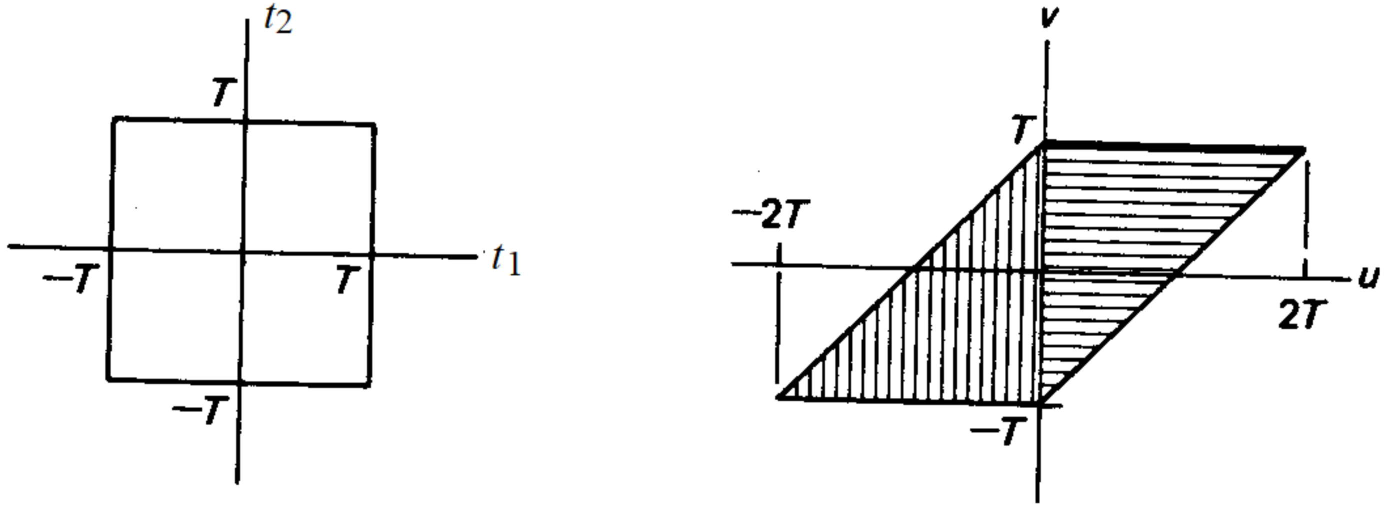
- Taking expectation inside the double integral can be shown to be valid in this case, but the details are not discussed here. Hence,

$$S_X(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} E[X(t_1)X(t_2)]\, e^{-j\omega(t_2-t_1)} dt_1\, dt_2$$

- The expectation in the integrand is recognized as the autocorrelation function of the truncated process.

$$S_X(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_X(t_2 - t_1)\, e^{-j\omega(t_2-t_1)} dt_1\, dt_2 \qquad (141)$$

- The change of variables $u = t_2 - t_1$ and $v = t_2$ is now made with the aid of below figure. In the uv-plane, we integrate over v first and then over u by breaking the integration into two integrals (+ve and -ve of u).



$$S_X(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{0} R_X(u) e^{-j\omega u} \left(\int_{-T}^{u+T} dv \right) du$$
$$+ \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{2T} R_X(u) e^{-j\omega u} \left(\int_{u-T}^{T} dv \right) du \qquad (142)$$

- Evaluating the inner integrals and combining the two terms yield

$$S_X(\omega) = \lim_{T\to\infty} \int_{-2T}^{2T} \left(1 - \frac{|u|}{2T}\right) R_X(u) e^{-j\omega u} du \qquad (143)$$

Hence,

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(u) e^{-j\omega u} du = \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau} d\tau \qquad (144)$$

or

$$R_X(\tau) \longleftrightarrow S_X(\omega) \qquad (145)$$

Wiener Khinchine Theorem

- The relationship in (145), which is known as the ***Wiener Khinchine theorem***, is of fundamental importance in analysing random signals because it provides the link between the time domain (correlation function) and the frequency domain (spectral density).
- The spectral density is also called the power spectral density (PSD).
- Because of the uniqueness of the Fourier transform, it follows that the autocorrelation function of a WSS random process is the inverse Fourier transform of the spectral density.

Example 41

Consider an autocorrelation function of the form

$$R_X(\tau) = Ae^{-\beta|\tau|}$$

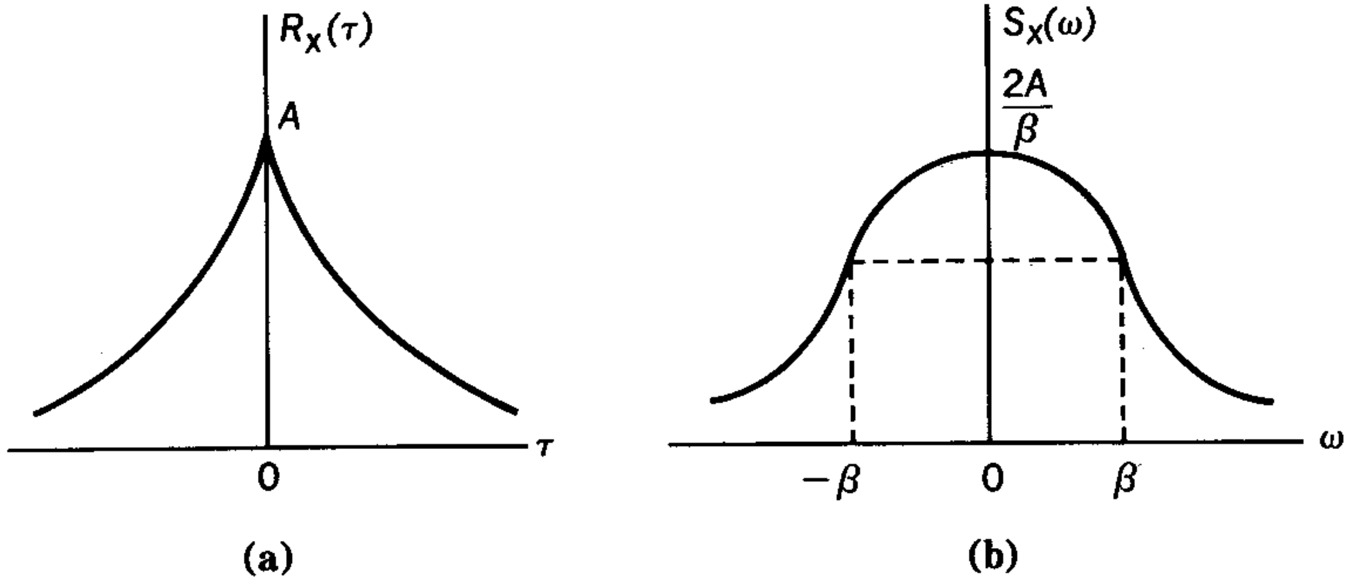
for which A *and* β *are positive constants. Find the PSD of* $X(t)$.

<u>Answer</u>

$$S_X(\omega) = \int_{-\infty}^{0} Ae^{\beta\tau}e^{-j\omega\tau}d\tau + \int_{0}^{\infty} Ae^{-\beta\tau}e^{-j\omega\tau}d\tau$$

$$= A\left[\frac{1}{\beta - j\omega} + \frac{1}{\beta + j\omega}\right] = \frac{2A\beta}{\omega^2 + \beta^2}$$

Example 41 …



Relation between (a) autocorrelation function and (b) spectral density.

Exercise

A WSS random process has an autocorrelation function of the form

$$R_X(\tau) = 16e^{-2|\tau|} - 8e^{-4|\tau|}$$

Find the PSD of the random process $X(t)$.

<u>Answer</u>

$$S_X(\omega) = \frac{768}{\omega^4 + 20\omega^2 + 64}$$

Exercise

A WSS random process has a PSD of the form

$$S_X(\omega) = \frac{16}{\omega^4 + 13\omega^2 + 36}$$

Find the autocorrelation function of the random process.

<u>Answer</u>

$$R_X(\tau) = \frac{4}{15}\left(3e^{-2|\tau|} - 2e^{-3|\tau|}\right)$$

Cross-Power Spectral Density

- The cross-correlation functions were previously defined as

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$
$$R_{YX}(\tau) = E[Y(t)X(t+\tau)]$$

- The cross-power spectral density functions of two jointly WSS processes are the Fourier transform of their cross-correlation functions:

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{E[F_X^*(\omega)F_Y(\omega)]}{2T} = \Im[R_{XY}(\tau)] \qquad (146)$$

$$S_{YX}(\omega) = \lim_{T\to\infty} \frac{E[F_Y^*(\omega)F_X(\omega)]}{2T} = \Im[R_{YX}(\tau)] \qquad (147)$$

The Power of Random Process

- The average power P_X of a WSS process $X(t)$ is the mean square value $E[X^2]$. Since $R_X(\tau) \longleftrightarrow S_X(\omega)$, we have

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(\omega) e^{j\omega\tau} \frac{d\omega}{2\pi} \tag{148}$$

- Hence,

$$P_X = R_X(0) = \int_{-\infty}^{\infty} S_X(\omega) \frac{d\omega}{2\pi} \tag{149}$$

- Since $S_X(\omega)$ is an even function of ω (because $R_X(\tau)$ is even), we have

$$P_X = E[X^2] = 2\int_{0}^{\infty} S_X(\omega) \frac{d\omega}{2\pi} \tag{150}$$

Example 42

Determine the autocorrelation function $R_X(\tau)$ and the power P_X of a low-pass white noise process.

<u>Answer</u>

We have

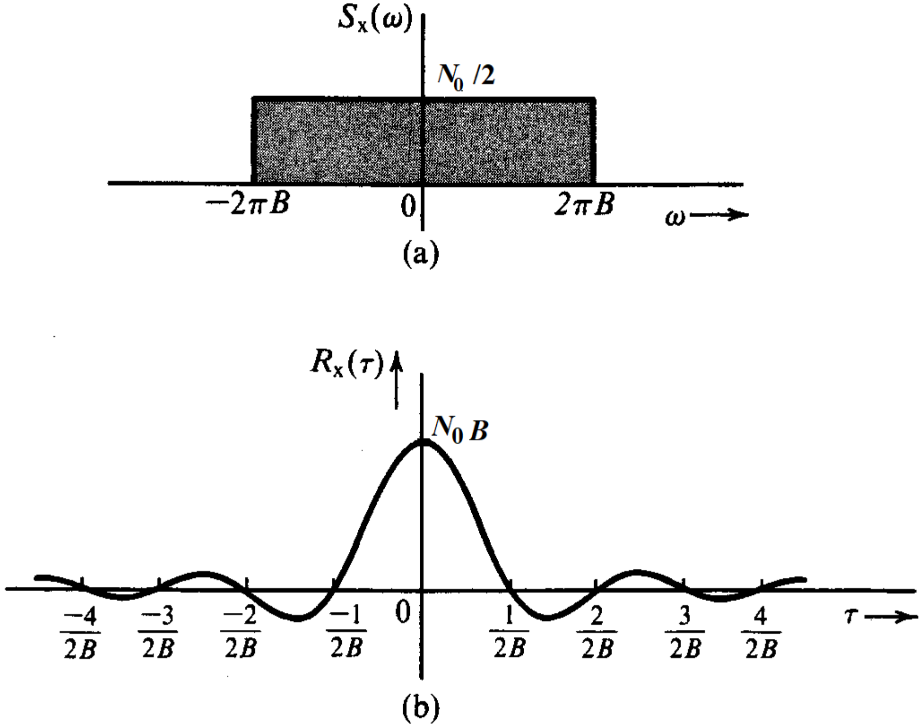
$$S_X(\omega) = \frac{N_0}{2}\ \mathrm{rect}\left(\frac{\omega}{4\pi B}\right)$$

Hence,

$$R_X(\tau) = N_0 B\ \mathrm{sinc}(2B\tau)$$

and

$$P_X = R_X(0) = N_0 B$$



Example 43

Determine the power spectral density and the mean-square value of

$$X(t) = Acos(\omega_c t + \Theta)$$

where Θ *is a random variable uniformly distributed over* $(0, 2\pi)$.

<u>Answer</u>

We have previously found

$$R_X(\tau) = \frac{A^2}{2} \cos \omega_c \tau$$

Hence,

$$S_X(\omega) = \frac{\pi A^2}{2} [\delta(\omega - \omega_c) + \delta(\omega + \omega_c)]$$

and

$$P_X = \int_{-\infty}^{\infty} S_X(\omega) \, \frac{d\omega}{2\pi} = \frac{A^2}{2}$$

Autocorrelation Function of System Output

If a random process $X(t)$ is applied at the input of a stable linear time-invariant (LTI) system with transfer function $H(\omega)$, we can determine the autocorrelation function and the PSD of the output process $Y(t)$.

First, we observe that

$$Y(t) = \int_{-\infty}^{\infty} X(t-\alpha)h(\alpha)\,d\alpha$$

$$Y(t+\tau) = \int_{-\infty}^{\infty} X(t+\tau-\alpha)h(\alpha)\,d\alpha$$

Hence,

$$\begin{aligned} R_Y(\tau) &= E[Y(t)Y(t+\tau)] \\ &= E\left[\int_{-\infty}^{\infty} X(t-\alpha)h(\alpha)\,d\alpha \int_{-\infty}^{\infty} X(t+\tau-\beta)h(\beta)\,d\beta\right] \end{aligned}$$

Linear Time-Invariant (LTI) Systems …

Moving the expectation inside the double integral, we obtain

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-\alpha)X(t+\tau-\beta)]h(\alpha)h(\beta)\, d\alpha\, d\beta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(\tau+\alpha-\beta)\, h(\alpha)\, h(\beta)\, d\alpha\, d\beta$$

- The double integral is precisely the double convolution

$$R_Y(\tau) = h(\tau) \otimes h(-\tau) \otimes R_X(\tau) \tag{150}$$

- Hence, for real-valued impulse response $h(t)$,

$$\begin{aligned} S_Y(\omega) &= H(\omega)H(-\omega)S_X(\omega) \\ &= H(\omega)H^*(\omega)\, S_X(\omega) \\ &= |H(\omega)|^2 S_X(\omega) \end{aligned} \tag{151}$$

$R_{XY}(\tau)$ and $S_{XY}(\omega)$

Similarly, we can find the cross-correlation function of $X(t)$ and $Y(t)$.

$$
\begin{aligned}
R_{XY}(\tau) &= E[X(t)Y(t+\tau)] \\
&= E\left[X(t)\int_{-\infty}^{\infty} X(t+\tau-\beta)\, h(\beta)\, d\beta\right] \\
&= \int_{-\infty}^{\infty} E[X(t)X(t+\tau-\beta)]\, h(\beta)\, d\beta \\
&= \int_{-\infty}^{\infty} R_X(\tau-\beta)\, h(\beta)\, d\beta \\
&= R_X(\tau) \otimes h(\tau)
\end{aligned}
\tag{152}
$$

The corresponding Spectral density is

$$S_{XY}(\omega) = H(\omega) S_X(\omega) \tag{153}$$

$R_{YX}(\tau)$ and $S_{YX}(\omega)$

A similar development is

$$\begin{aligned}
R_{YX}(\tau) &= E[Y(t)X(t+\tau)] \\
&= E\left[\int_{-\infty}^{\infty} X(t-\beta)\, h(\beta)\, d\beta \;\; X(t+\tau)\right] \\
&= \int_{-\infty}^{\infty} E[X(t-\beta)X(t+\tau)]\, h(\beta)\, d\beta \\
&= \int_{-\infty}^{\infty} R_X(\tau+\beta)\, h(\beta)\, d\beta \\
&= \int_{-\infty}^{\infty} R_X(\tau-\lambda)\, h(-\lambda)\, d\lambda = h(-\tau) \otimes R_X(\tau) \qquad (154)
\end{aligned}$$

The corresponding Spectral density is

$$S_{YX}(\omega) = H(-\omega)S_X(\omega) = H^*(\omega)S_X(\omega) \qquad (155)$$

Example 44

The input to a filter with impulse response $h(t)$ and transfer function $H(\omega)$ is a white-noise process with PSD

$$S_X(\omega) = \frac{N_0}{2}$$

for $-\infty < \omega < \infty$. The cross-power spectral density between input and output is

$$S_{XY}(\omega) = H(\omega)S_X(\omega) = \frac{N_0}{2}H(\omega)$$

and the cross-correlation function is

$$R_{XY}(\tau) = \frac{N_0}{2}\ h(\tau) \qquad \text{or} \qquad h(\tau) = \frac{2}{N_0}R_{XY}(\tau)$$

Hence, we can find the impulse response of a filter by driving it with white noise and measuring $R_{XY}(\tau)$ to obtain $h(\tau)$.

Sum of Random Processes

If two jointly WSS processes, $X(t)$ and $Y(t)$, are added to form a process $Z(t)$, the statistics of $Z(t)$ can be determined in terms of those of $X(t)$ and $Y(t)$. If

$$Z(t) = X(t) + Y(t) \tag{156}$$

then

$$\begin{aligned} R_Z(\tau) &= E[Z(t)Z(t+\tau)] \\ &= R_X(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{YX}(\tau) \end{aligned} \tag{157}$$

If $X(t)$ and $Y(t)$ are uncorrelated, then

$$R_{XY}(\tau) = E[X(t)]E[Y(t)] = m_X m_Y = R_{YX}(\tau)$$

and

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau) + 2m_X m_Y \tag{158}$$

Sum of Random Processes …

Most processes of interest in communication problems have zero means. If processes $X(t)$ and $Y(t)$ are uncorrelated with either m_X or m_Y equal to zero, then

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau) \tag{159}$$

and

$$S_Z(\omega) = S_X(\omega) + S_Y(\omega) \tag{160}$$

It also follows that

$$E[Z^2] = E[X^2] + E[Y^2] \tag{161}$$

Hence, the mean-square value of a sum of orthogonal processes is equal to the sum of the mean-square values of these processes.