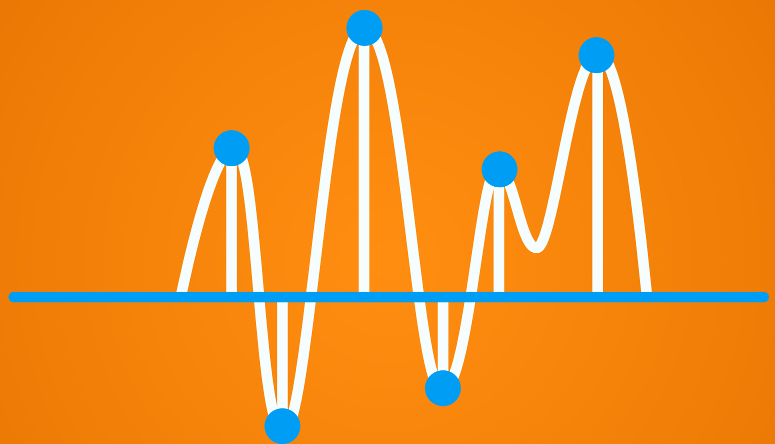
Chapter 1

Digital Signal Processing



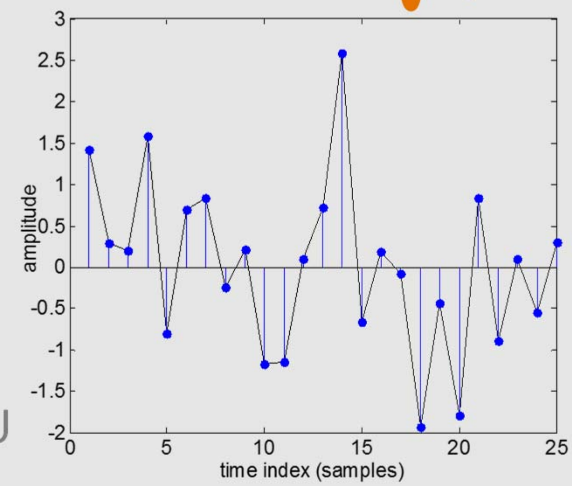
Dr. Andy W. H. Khong

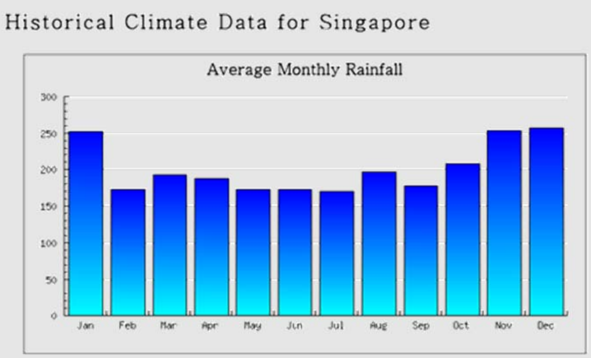
NANYANG TECHNOLOGICAL UNIVERSITY

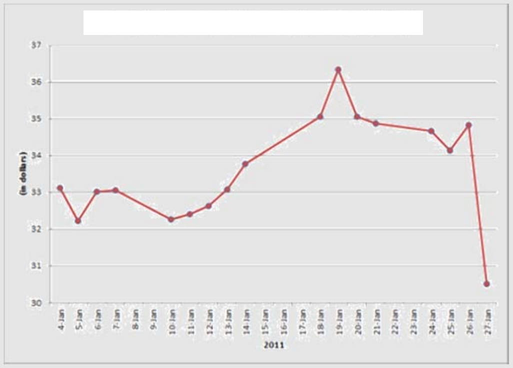
1.1 Definition

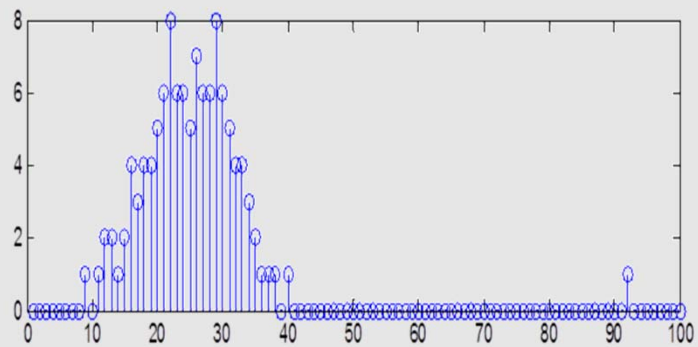
Digital signals

- discrete-time sequence
- sampling data at regular intervals
- *Examples:*
  - a) rainfall record for a given period
  - b) stock prices for a particular firm
  - c) grade distribution for a particular course in NTU









1.1 Definition

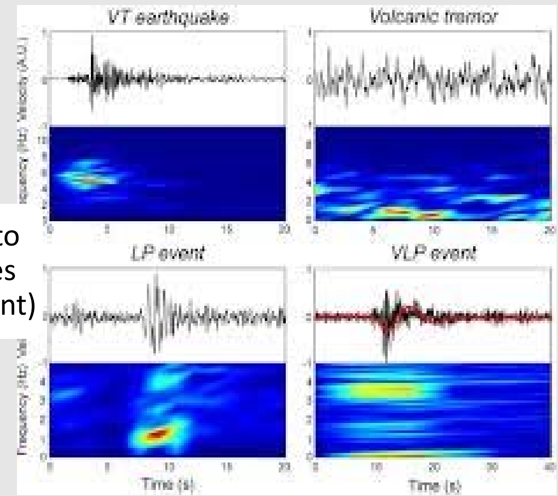
Processing

- Representation
- Transformation
- Manipulation

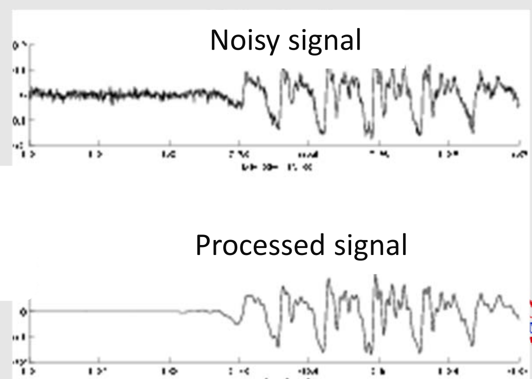
Analog Signals → Digital Signals → Representation / Transformation / Manipulation

Representation: Data compression

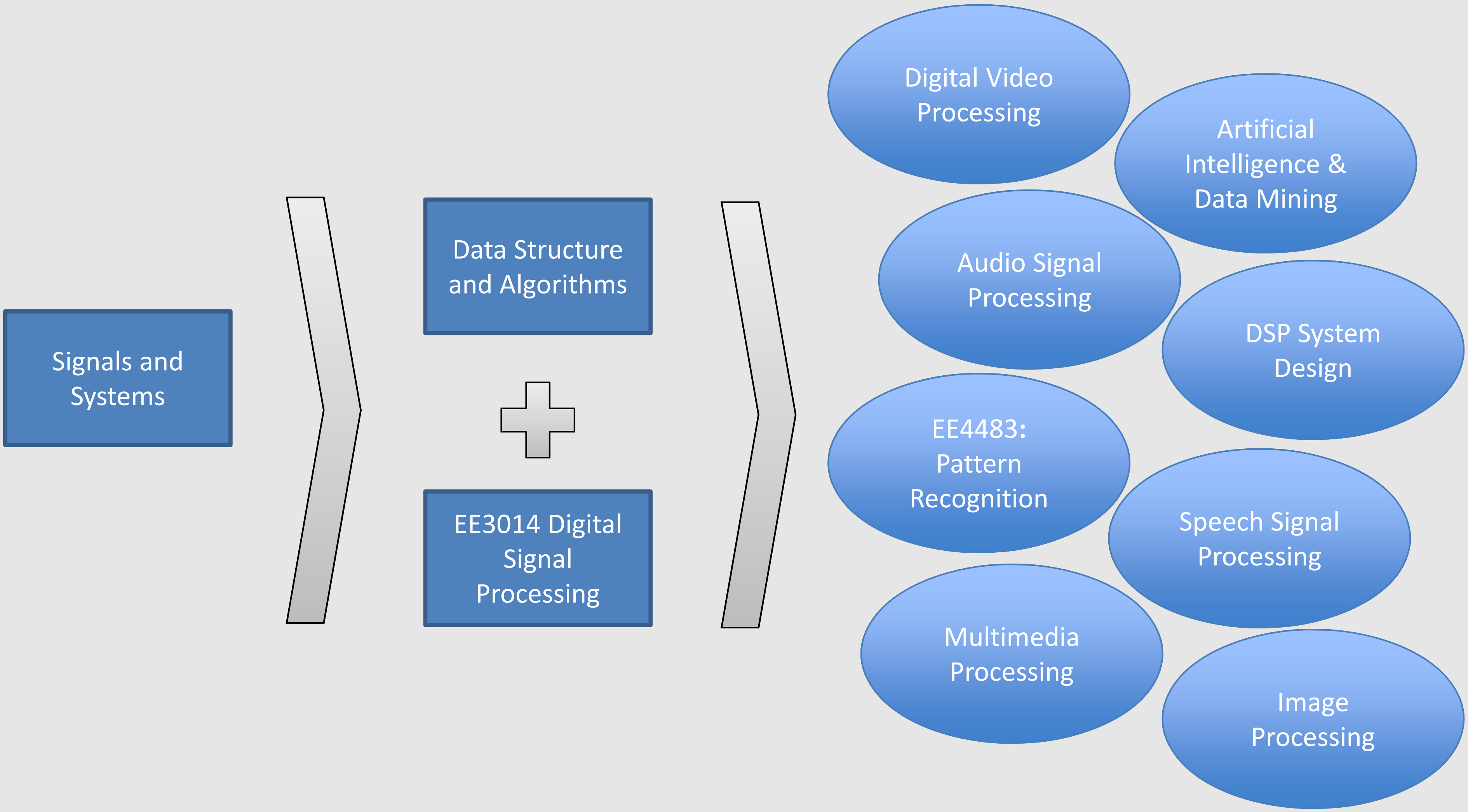
Transformation: Gain insights into signal properties (frequency content)



Manipulation: Reduce noise, sharpen image

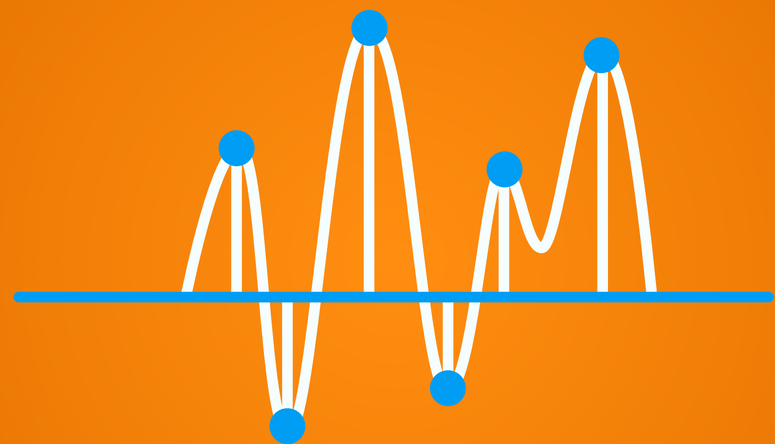


1.3 Related Courses



Chapter 2

Discrete-Time Signals



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Chapter Aims

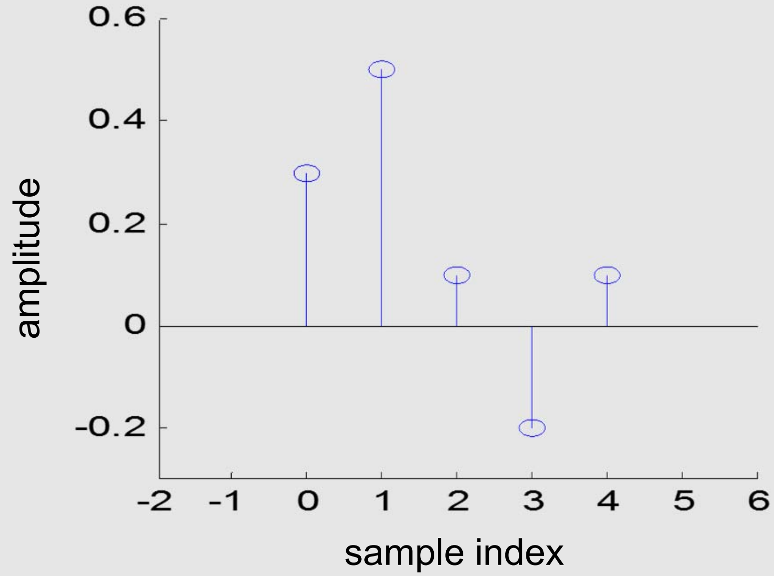
The aims of this chapter are to:

1. construct and compare the basic types of discrete signals
2. differentiate between different types of signal operations
3. formulate the process of sampling an analog signal
4. analyze and interpret properties of discrete signals

Discrete-time signals

- sequence of numbers
- normally represented in a vector form notation, e.g., $\mathbf{x}[n]$
- n is known as the sample index
- sometimes an arrow denotes the value when $n = 0$
- if there is no arrow, the first element is taken $n = 0$

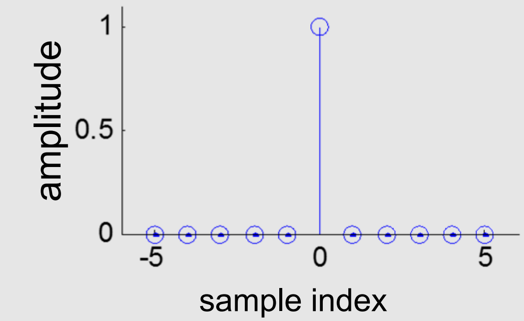
$$\mathbf{x}[n] = [\underset{\uparrow}{0.3} \quad 0.5 \quad 0.1 \quad -0.2 \quad 0.1]$$



Some basic signals

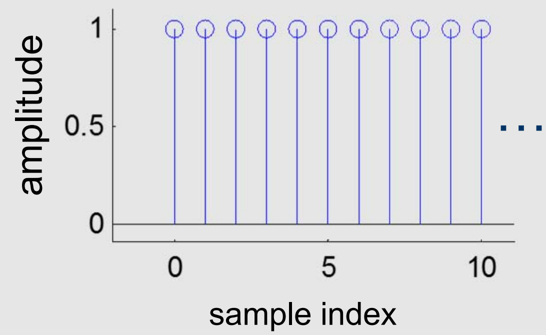
- Impulse

$$\delta[n] = \begin{cases} 0, & n \neq 0; \\ 1, & n = 0 \end{cases}$$



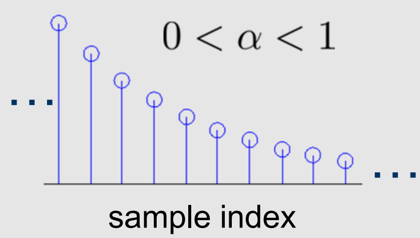
- Unit step

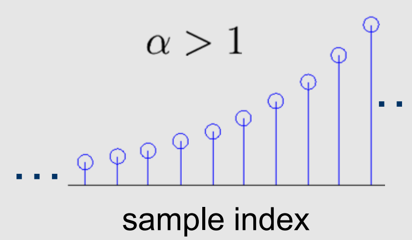
$$u[n] = \begin{cases} 1, & n \geq 0; \\ 0, & n < 0 \end{cases}$$



- Exponential

$$x[n] = A\alpha^n$$





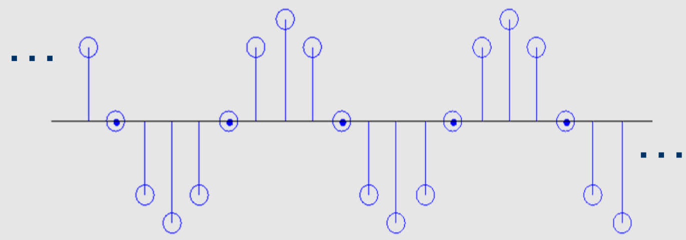
- Sinusoid

$$x[n] = A\cos(\omega_0 n + \phi)$$

A : amplitude

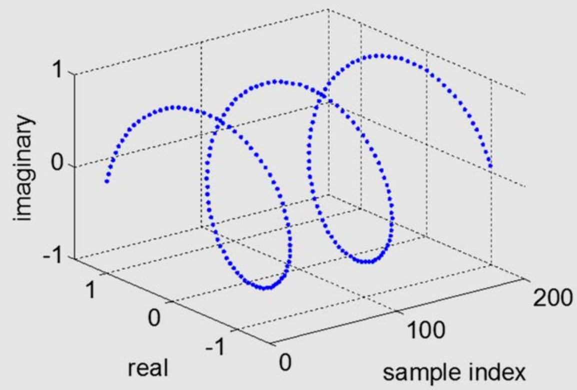
ω_0 : angular frequency (radian/sample)

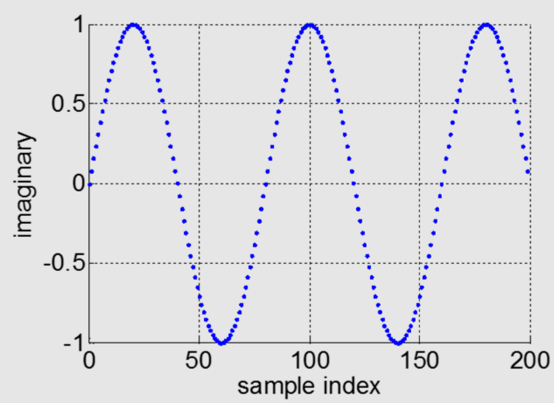
ϕ : phase

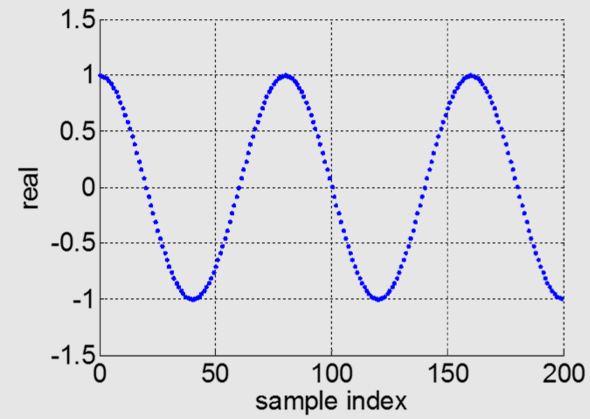


- Complex sinusoid

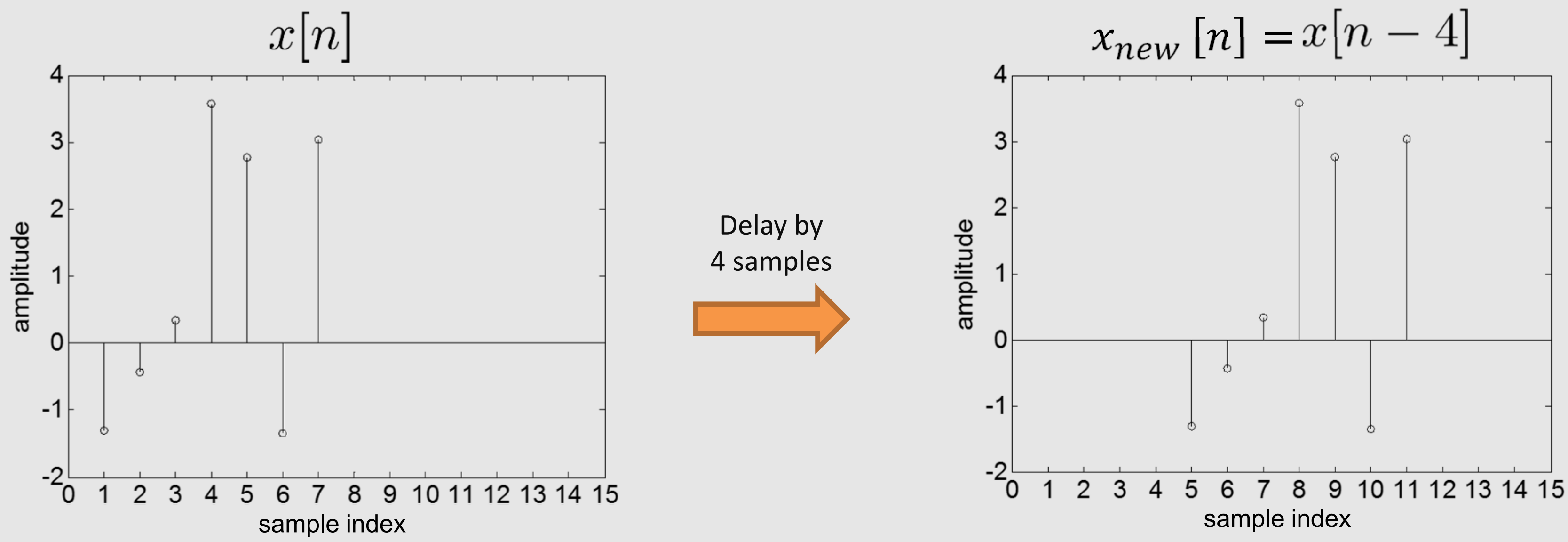
$$\begin{aligned} x[n] &= Ae^{\jmath(\omega_0 n + \phi)} \\ &= A\cos(\omega_0 n + \phi) + \jmath A\sin(\omega_0 n + \phi) \end{aligned}$$





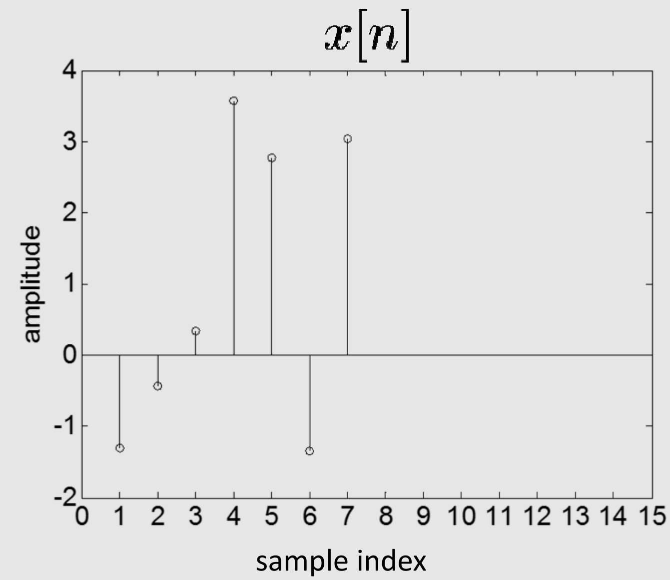


- Signal shift (delay)



2.3 Basic Operations of Signals

- Expressing a signal using the impulse function



We can express $x[n]$ by

$$x[n] = -1.3\delta[n-1] - 0.4\delta[n-2] + 0.3\delta[n-3] + \ldots + 3\delta[n-7]$$

More compactly, we can express a given signal as

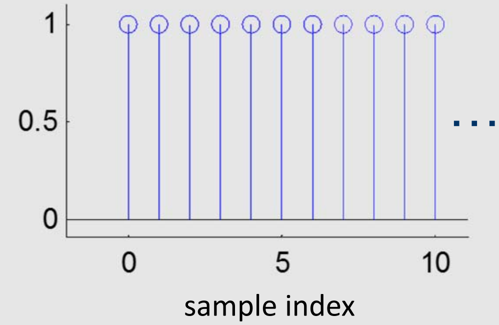
$$x[n] \quad = \quad \sum_{k=0}^{\infty} A_k \delta[n-k]$$

$$A_k \quad : \quad \text{coefficent of time index } k$$

Examples:

a) a unit step sequence can be expressed as

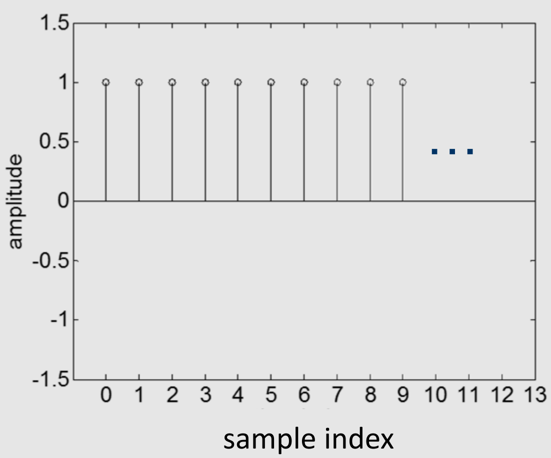
$$x[n] = \sum_{k=0}^{\infty} \delta[n-k]$$



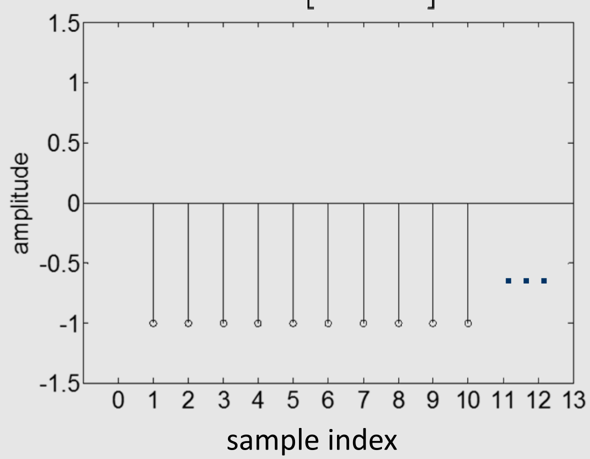
b) a unit impulse can be expressed as

$$\begin{aligned} \delta[n] &= u[n] - u[n-1] \\ &= \sum_{k=0}^{\infty} \delta[n-k] - \sum_{k=1}^{\infty} \delta[n-k] \end{aligned}$$

$u[n]$

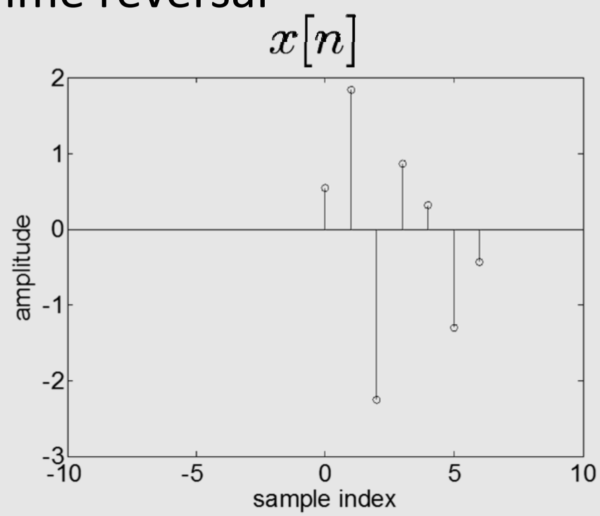


$-u[n-1]$

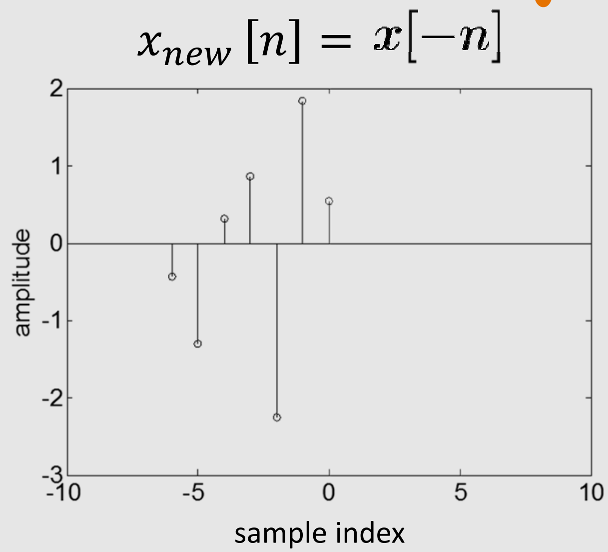


2.3 Basic Operations of Signals

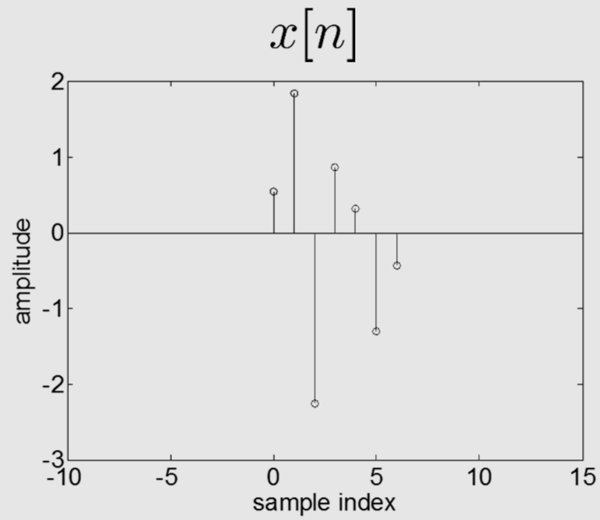
- Time reversal



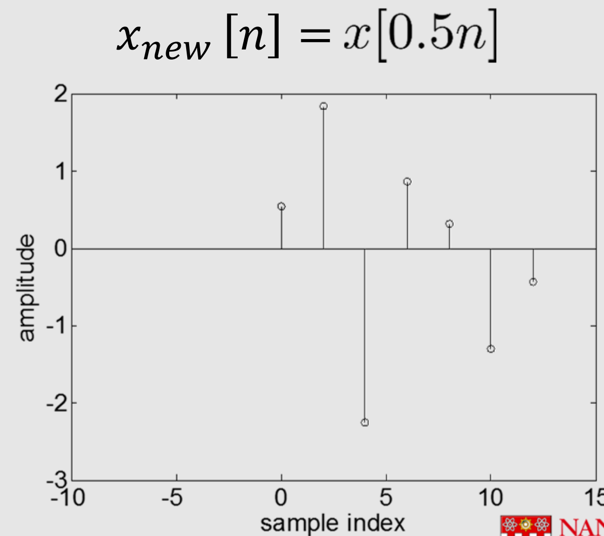
Time reversal



- Time scaling



Time scaling by a factor of 0.5



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2.4 Discrete-Time Signal from Continuous Time Signal

A) Definition

- An analog sinusoid is in the form of

$$\begin{aligned} x(t) &= A\cos(2\pi f t) \\ &= A\cos(\omega t) \end{aligned}$$

Therefore, by definition,

$$\begin{aligned} f &: \text{ analog frequency in cycles/sec (Hz)} \\ \omega &: \text{ angular (analog) frequency in rad/sec} \end{aligned}$$

- A digital sinusoid is in the form of

$$\begin{aligned} x[n] &= A\cos(2\pi f_0 n) \\ &= A\cos(\omega_0 n) \end{aligned}$$

Therefore, by definition,

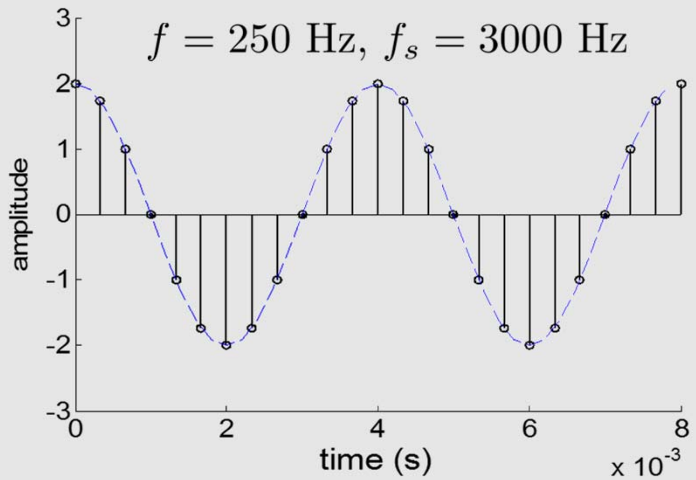
$$\begin{aligned} f_0 &: \text{ digital frequency in cycles/sample} \\ \omega_0 &: \text{ angular (digital) frequency in rad/sample} \end{aligned}$$

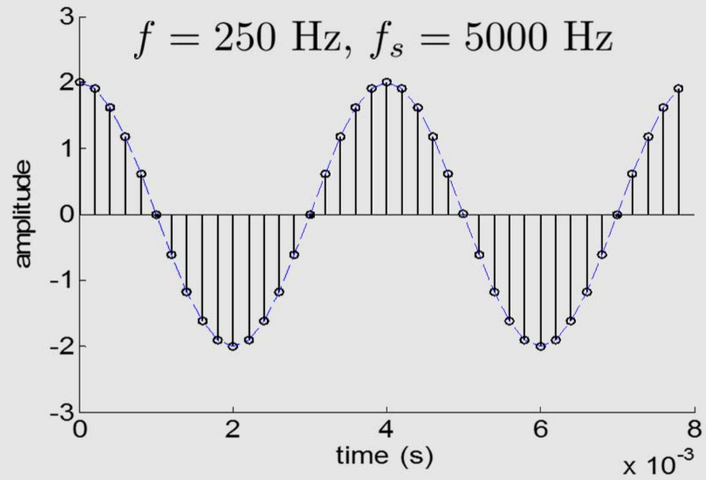
B) Sampling

- Sampling a continuous-time signal at regular interval results in a discrete-time signal.

- We often write
$$\begin{aligned} x[n] &= x_{\text{continuous}}(nT_s) \\ T_s &= 1/f_s \text{ is the sampling period in sec} \\ f_s &: \text{sampling frequency in Hz} \end{aligned}$$

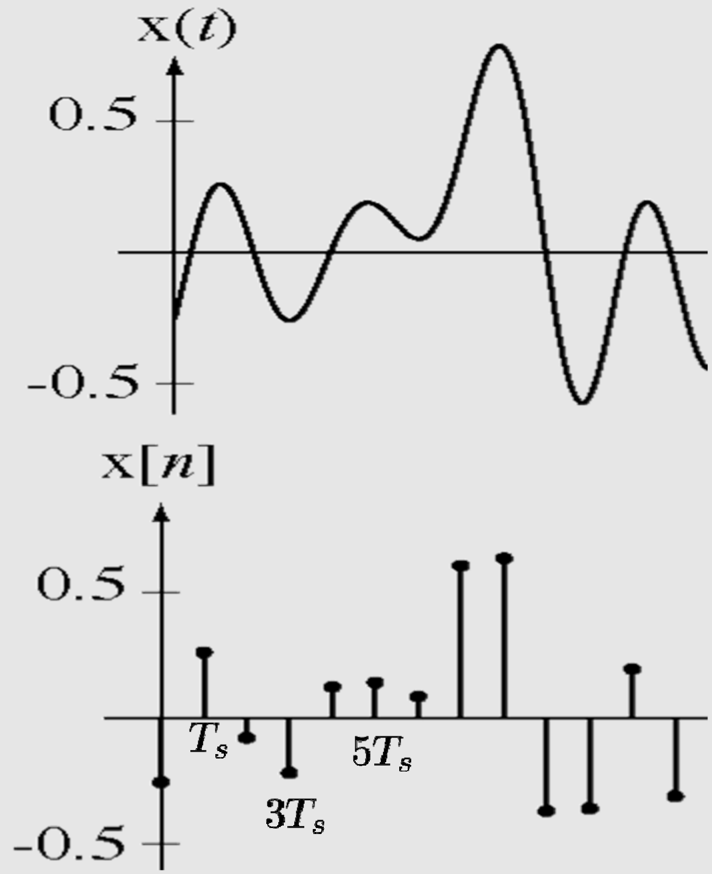
- A higher f_s implies that the analog signal is sampled more frequently.





2.4 Discrete-Time Signal from Continuous Time Signal

$$\begin{aligned} x[n] &= x_{\text{continuous}}(nT_s) \\ T_s &= 1/f_s \text{ is the sampling period} \\ f_s &: \text{ sampling frequency} \end{aligned}$$



- Therefore, samples are taken at regular time intervals

$$\ldots, 0, T_s, 2T_s, \ldots$$

- This is equivalent to replacing the variable t in $x(t)$ by nT_s where n is the sample index.

Example:

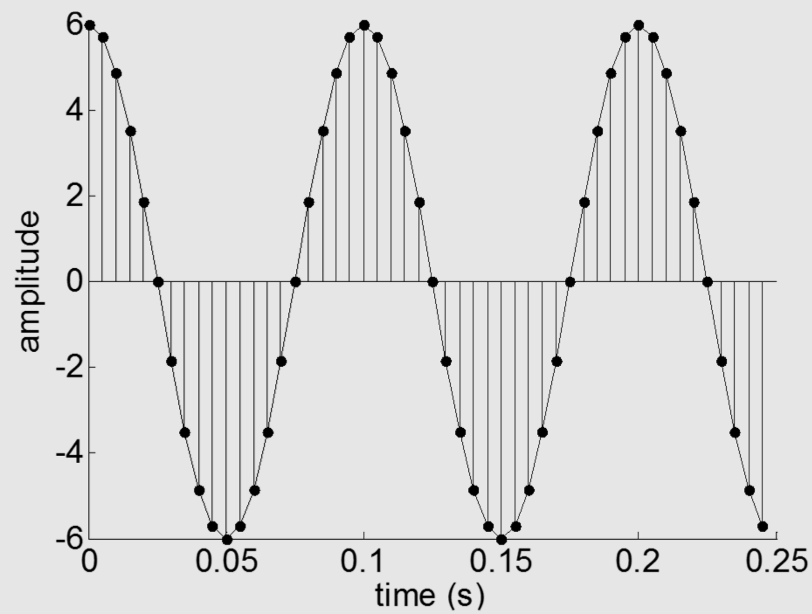
Consider an analog signal $x(t) = 6\cos(20\pi t)$. Given a sampling rate of $f_s = 200\ \text{Hz}$, find the discrete representation of the signal.

A sampling rate of $f_s = 200\ \text{Hz}$ corresponds to a sampling period of $T_s = 1/200 = 0.005\ \text{sec.}$

This implies that we have a digital signal at sample index n every 0.005 s.

Sampling $x(t)$ at this period will result in

$$\begin{aligned} x[n] &= 6\cos(20\pi \times nT_s) \\ &= 6\cos(0.1\pi n) \end{aligned}$$



2.4 Discrete-Time Signal from Continuous Time Signal

- In the above example, the analog signal is $x(t) = 6\cos(20\pi t)$ and therefore

$$
\begin{aligned}
f &= 10 \text{ Hz} \\
\omega &= 20\pi \text{ rad/s}
\end{aligned}
$$

Once sampled, the digital signal $x[n] = 6\cos(0.1\pi n)$ where

$$
\begin{aligned}
f_0 &= 0.05 \text{ cycles/sample} \\
\omega_0 &= 0.1\pi \text{ rad/sample}
\end{aligned}
$$

The variable f_0 is sometimes known as the *normalized frequency* since, from the above, we can show that

$$f_0 = \frac{f}{f_s} = \frac{10}{200} = 0.05$$

2.4 Discrete-Time Signal from Continuous Time Signal

- From the above, since

$$\omega_0 = 2\pi f_0$$

$$f_0 = f/f_s$$

we will have

$$\omega_0 = 2\pi f/f_s$$

This means that the *angular frequency* of the digital signal is equivalent to the *normalized frequency* multiplied by a factor of 2π.

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2.4 Discrete-Time Signal from Continuous Time Signal

C) Identical signals

- Two discrete-time sinusoids of different frequencies may be identical
- Consider two angular frequencies ω_0 and $\omega_0 + 2\pi$
- We can show that these two frequencies are identical by expressing

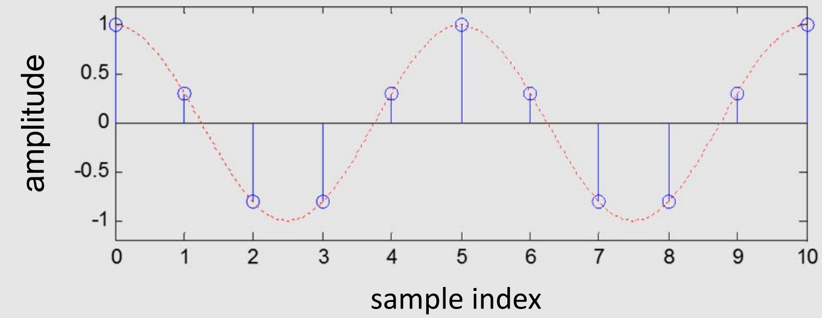
$$\begin{aligned}
A_0 \cos[(\omega_0 + 2\pi)n + \phi] &= A_0 \cos[(\omega_0 n + \phi) + 2\pi n] \\
&= A_0 \cos(\omega_0 n + \phi)\cos(2\pi n) - A_0 \sin(\omega_0 n + \phi)\sin(2\pi n) \\
&= A_0 \cos(\omega_0 n + \phi)
\end{aligned}$$

2.4 Discrete-Time Signal from Continuous Time Signal

- For example, consider the case where we have an analog signal with frequency $f = 200$ Hz. With a sampling rate of $f_s = 1000$ Hz,

$$\begin{aligned} \omega_0 &= 2\pi f / f_s \\ &= 0.4\pi \end{aligned}$$

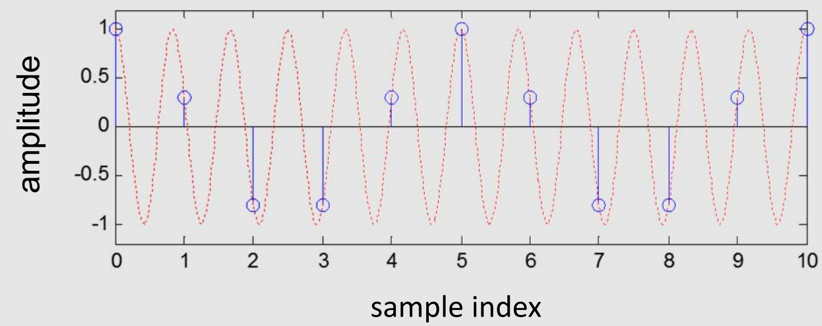
$$\cos(\omega_0 n) = \cos(0.4\pi n)$$



- Another analog signal of frequency $f = 1200$ Hz, with a sampling rate of $f_s = 1000$ Hz

$$\begin{aligned} \omega_0 &= 2\pi f / f_s \\ &= 2.4\pi \end{aligned}$$

$$\cos[(\omega_0 + 2\pi)n] = \cos(2.4\pi n)$$



- Therefore, analog signals of *different frequencies* can have the *same discrete signals*.

2.4 Discrete-Time Signal from Continuous Time Signal

D) Aliasing

- Consider an analog signal which contains 30 Hz and 170 Hz components, i.e.,

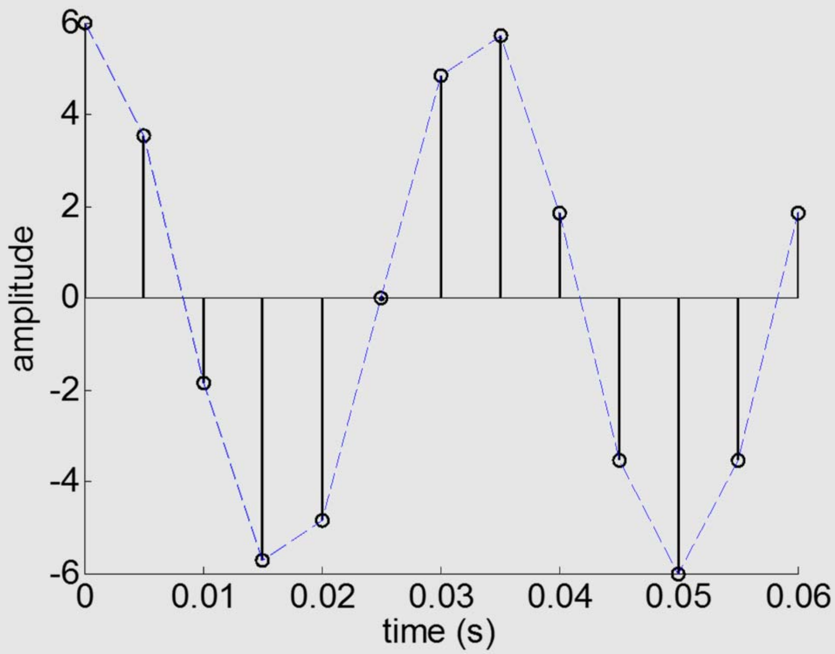
$$x(t) = 6\cos(2\times\pi\times 30t) + 6\cos(2\times\pi\times 170t)$$

Digitizing this signal with a sampling frequency of $f_s = 200\ \mathrm{Hz}$

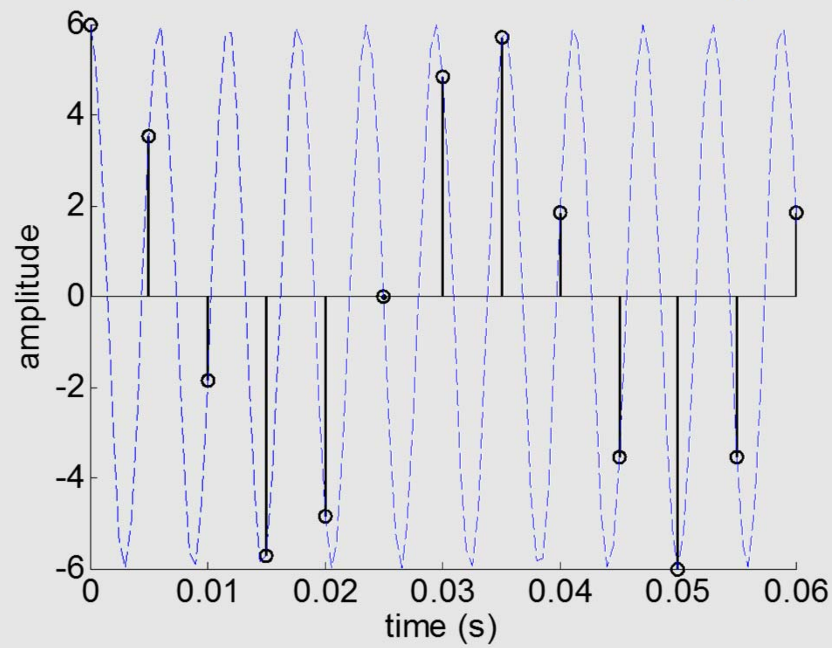
$$\begin{aligned} x[n] &= 6\cos(60\pi n T_s) + 6\cos(340\pi n T_s) \\ &= 6\cos(0.3\pi n) + 6\cos(1.7\pi n) \\ &= 6\cos(0.3\pi n) + 6\cos((2\pi - 0.3\pi)n) \\ &= 6\cos(0.3\pi n) + 6\cos(0.3\pi n) \\ &= 12\cos(0.3\pi n) \end{aligned}$$

The above implies that the digital signal (generated by 2 analog signals) only contains *one* frequency $f_0 = 0.15$ which is a false representation of the analog signal !

2.4 Discrete-Time Signal from Continuous Time Signal



30 Hz analog signal sampled at 200 Hz



170 Hz analog signal sampled at 200 Hz

$$
\begin{aligned}
x(t) &= 6\cos(2\times\pi\times 30t) + 6\cos(2\times\pi\times 170t)\\
x[n] &= 6\cos(60\pi n T_s) + 6\cos(340\pi n T_s)\\
&= 6\cos(0.3\pi n) + 6\cos(0.3\pi n)
\end{aligned}
$$

2.4 Discrete-Time Signal from Continuous Time Signal

$$
\begin{aligned}
x(t) &= 6\cos(2\times\pi\times 30t) + 6\cos(2\times\pi\times 170t)\\
x[n] &= 6\cos(60\pi n T_s) + 6\cos(340\pi n T_s)\\
&= 6\cos(0.3\pi n) + 6\cos(0.3\pi n)
\end{aligned}
$$

- The aliasing problem occurs because we are sampling at 200 Hz which is less than the Nyquist rate for the 170 Hz signal.

- If we are to sample at 500 Hz, i.e., more than twice the highest frequency, we can see that

$$
\begin{aligned}
x[n] &= 6\cos(60\pi n T_s) + 6\cos(340\pi n T_s)\\
&= 6\cos(0.12\pi n) + 6\cos(0.68\pi n)
\end{aligned}
$$

2.4 Discrete-Time Signal from Continuous Time Signal

- Note that when the analog signal is sampled without aliasing, i.e., the signal is sampled at least twice the frequency, we have

$$f_s \geq 2f$$

$$\Rightarrow f/f_s \leq 0.5$$

Since

$$\omega_0 = 2\pi f/f_s$$

we can therefore show that

$$\omega_0 \leq \pi$$

This implies that if the analog signal is to be faithfully represented, the *maximum* angular frequency of the digital signal is π.

2.4 Discrete-Time Signal from Continuous Time Signal

E) Periodicity of discrete sinusoids

- The period of discrete sinusoids is given by

$$N = 2\pi k f_s / \omega, \qquad k : \text{an integer}$$

The discrete signal will repeat itself after every $N = 2\pi k f_s / \omega$ samples.

- Rational: Consider an analog signal $x(t) = A\cos(2\pi f t)$. The digital signal is given by

$$x[n] = A\cos(2\pi f n T_s) = A\cos\left(2\pi \frac{f}{f_s} n\right) = A\cos\left(\frac{\omega}{f_s} n\right)$$

If $x[n]$ is to repeat itself, say every N samples, then $x[n] = x[n+N]$, i.e.,

$$x[n+N] = A\cos\left(\frac{\omega}{f_s}(n+N)\right) = A\cos\left(\frac{\omega}{f_s} n + \frac{\omega}{f_s} N\right)$$

Therefore $x[n] = x[n+N]$ can occur if $\omega N / f_s$ is a multiple of 2π, i.e.,

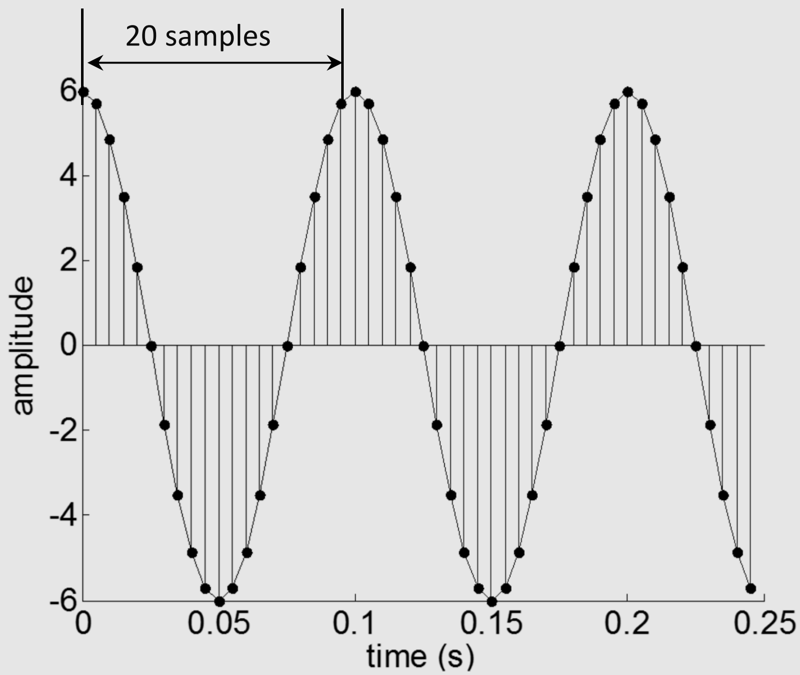
$$\frac{\omega N}{f_s} = 2\pi k$$

Example:

Consider an analog signal given by $x(t) = 6\cos(20\pi t)$ and sampled at a sampling rate of $f_s = 200\ \text{Hz}$.

For $k = 1$, the digital signal will repeat itself once every

$$\begin{aligned} N &= \frac{2\pi f_s}{\omega} \\ &= \frac{2\pi \times 200}{20\pi} \\ &= 20 \text{ samples} \end{aligned}$$



2.4 Discrete-Time Signal from Continuous Time Signal

F) High/low frequencies in digital signals

- The interpretation of high and low frequencies is somewhat different for continuous-time and discrete-time sinusoids.

- For continuous-time signal,

$$\begin{aligned} x(t) &= A\cos(2\pi f t) \\ &= A\cos(\omega t) \end{aligned}$$

higher value of ω translates to higher frequency.

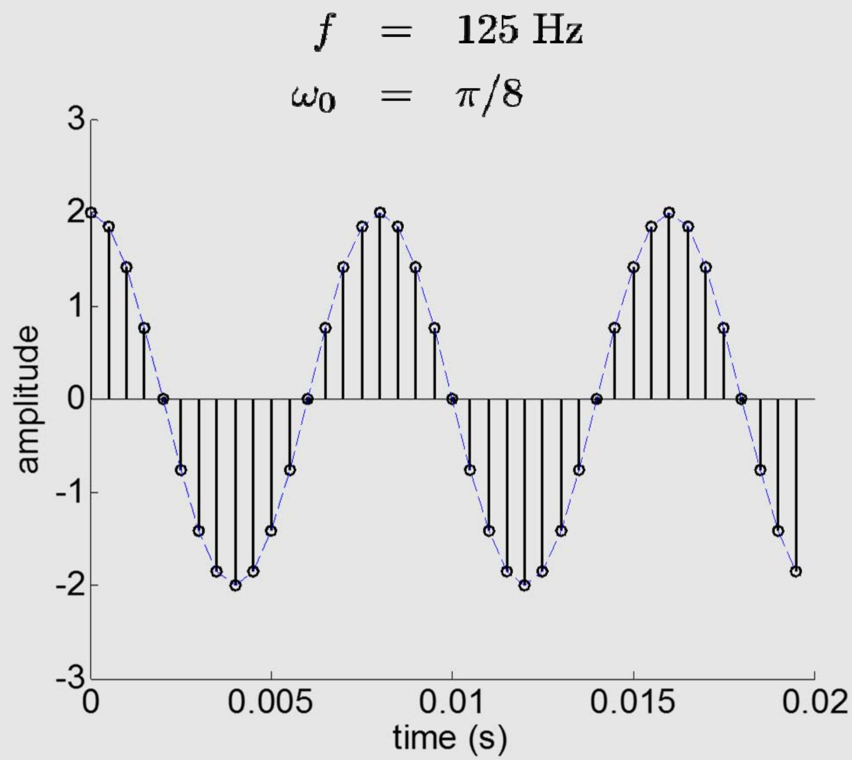
- For discrete-time signal,

$$\begin{aligned} x[n] &= A\cos(2\pi f_0 n) \\ &= A\cos(\omega_0 n) \end{aligned}$$

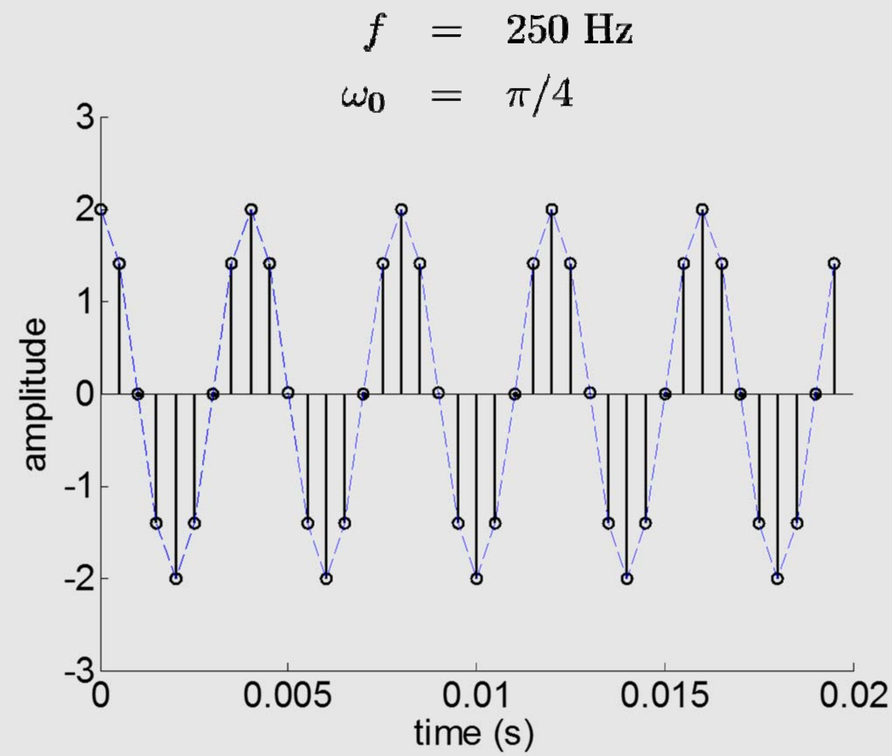
oscillation becomes *more* rapid for increasing ω_0 when $0 \leq \omega_0 \leq \pi$
oscillation becomes *less* rapid for increasing ω_0 when $\pi \leq \omega_0 \leq 2\pi$

2.4 Discrete-Time Signal from Continuous Time Signal

$$f_s = 2000 \text{ Hz}$$



2.5 oscillations per 0.02 s

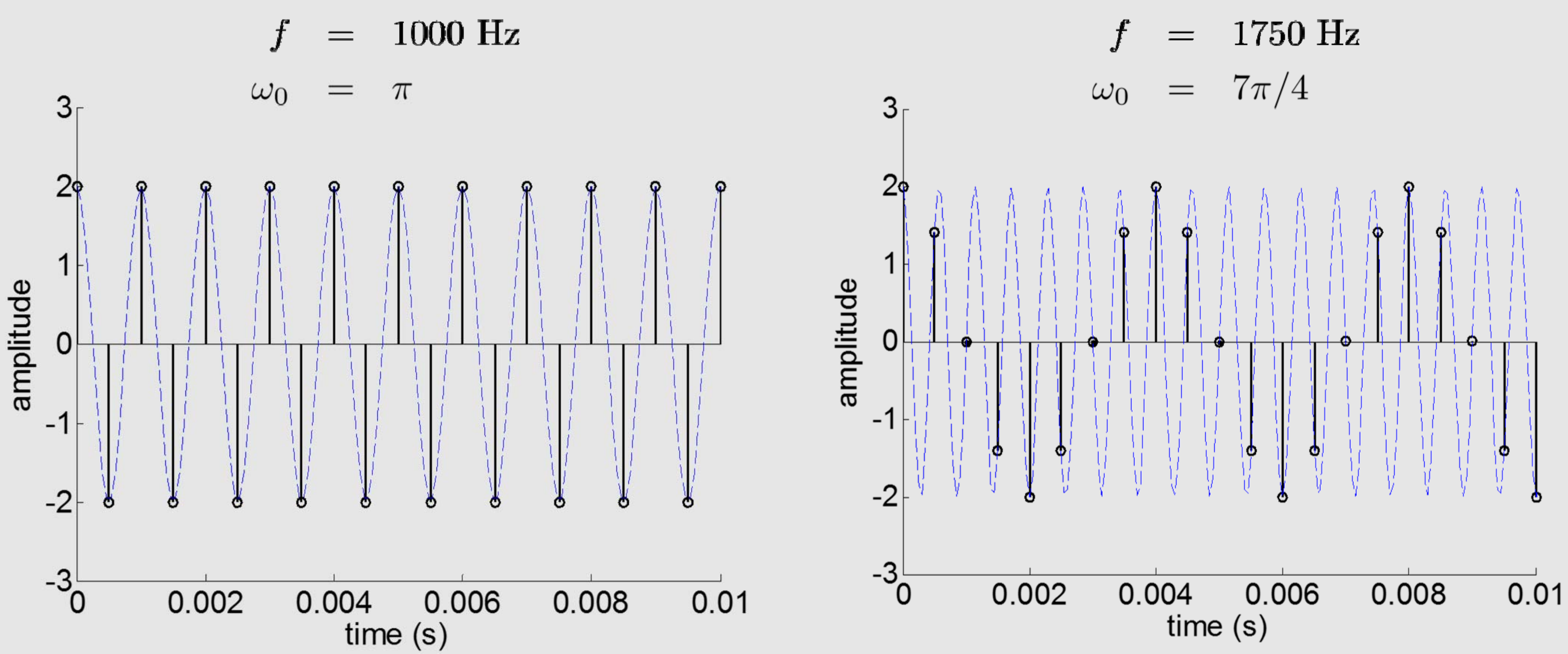


5 oscillations per 0.02 s

oscillation of discrete signal (black vertical lines) becomes *more* rapid for increasing ω_0 when $0 \leq \omega_0 \leq \pi$

2.4 Discrete-Time Signal from Continuous Time Signal

$$f_s = 2000 \text{ Hz}$$



10 oscillations per 0.01 s

2.5 oscillations per 0.01 s

oscillation of discrete signal (black vertical lines) becomes *<u>less</u>* rapid for increasing ω_0 when $\pi \le \omega_0 \le 2\pi$

2.5 Summary

- A discrete-time signal can be expressed from a continuous-time signal by

$$x[n] = x_{\text{continuous}}(nT_s)$$

- The normalized frequency in the digital domain is given by

$$f_0 = f/f_s = \omega_0/2\pi$$

and the maximum angular frequency of the digital signal corresponds to

$$\omega_0 \leq \pi$$

- The period of discrete sinusoid is given by $N = 2\pi k f_s/\omega$.

- For a discrete-time signal
oscillation becomes more rapid for increasing ω_0 when $0 \leq \omega_0 \leq \pi$.
oscillation becomes less rapid for increasing ω_0 when $\pi \leq \omega_0 \leq 2\pi$.

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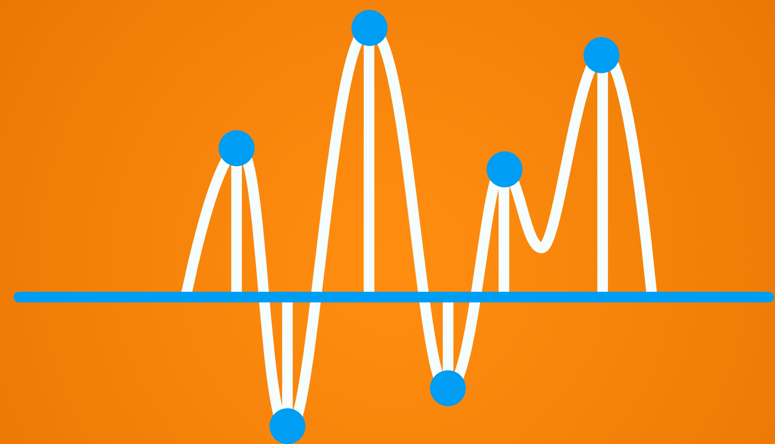
Chapter 3

Discrete-Time Systems



Dr. Andy W. H. Khong

Chapter Aims

At the end of this chapter, you will be expected to:

1. identify and determine characteristics of a discrete-time system
2. interpret what an impulse response is and how it is determined from a real system
3. analyze and formulate the process of convolution
4. establish the relationship between convolution and impulse response

3.1 Introduction

$$x[n] \longrightarrow \boxed{\text{system}} \longrightarrow y[n]$$

- A system is specified by an input-output relation. It can be a process developed by a DSP engineer or a natural occurring process.

- For example:

amplifying system $\quad y[n] = ax[n], \qquad \forall n$

delay system $\quad y[n] = x[n - n_{\mathrm{d}}], \qquad \forall n$

moving average system $\quad y[n] = \dfrac{1}{2M+1} \displaystyle\sum_{k=-M}^{M} x[n-k], \qquad \forall n$

$$x[n] = [0 \quad 0.3 \quad 0.5 \quad 0.1 \quad -0.2 \quad 0.1 \quad 0]$$

$$y[n] = [0.27 \quad 0.3 \quad 0.13 \quad 0 \quad -0.0333]$$

A) Memory/Memoryless systems

- A memoryless system is one where the output $y[n]$ depends only on the *present* input $x[n]$.

  For example: $y[n] = (x[n])^2$, $y[n] = 3x[n] + 8$

- If the output depends on the *past or future inputs*, then the system has memory.

  For example:

$$y[n] = x[n-2] + 8x[n-1]$$

$$y[n] = x[n-1] + x[n+1] + x[n+3]$$

B) Causal/Non-causal systems

- A causal system is one where the output $y[n]$ depends only on the *present and/or past input* $x[m]$ for $m \leq n$.

- For example:

$$y[n] = x[n-2] + 8x[n-1] + x[n] \qquad \text{causal}$$

$$y[n] = x[n-1] + x[n+1] + x[n+3] \qquad \text{non-causal}$$

- Causality is very important for realizability. A non-causal system needs *future input value(s)* to find the present output value, hence it cannot be realized in real-time.

3.2 Types of Systems

C) Stable/Unstable systems

$$x[n] \longrightarrow \boxed{\text{system}} \longrightarrow y[n]$$

- A system is stable in the bounded-input, bounded-output (BIBO) sense if and only if *__every bounded input__* sequence produces *__a bounded output__* sequence.

$$|x[n]| \leq B_x \longrightarrow \boxed{\text{system}} \longrightarrow |y[n]| \leq B_y$$

- For example if the input has a magnitude that is less than a fixed positive value B_x such that

$$|x[n]| \leq B_x < \infty$$

a BIBO stable system will have output that satisfies

$$|y[n]| \leq B_y < \infty$$

where B_y is a fixed positive finite value.

3.2 Types of Systems

Examples:

a) A system defined by $y[n] = (x[n])^2$ is stable because for $|x[n]| \leq B_x$, we have $y[n] = |x[n]|^2 \leq B_x^2$

$$x[n] \longrightarrow \boxed{(.)^2} \longrightarrow (x[n])^2$$

$$|x[n]| \leq B_x \longrightarrow \boxed{(.)^2} \longrightarrow |x[n]|^2 \leq B_x^2$$

Since $|x[n]| < \infty$, we therefore note that $|y[n]|$ is also bounded.

b) A system defined by $y[n] = \log|x[n]|$ is unstable since for a value of $x[n] = 0$, we have $y[n] = -\infty$ resulting in an infinite value of $|y[n]|$.

$$|x[n]| \longrightarrow \boxed{\log(.)} \longrightarrow \log(|x[n]|)$$

$$|x[n]| = 0 \longrightarrow \boxed{\log(.)} \longrightarrow \log(|0|) = -\infty$$

3.2 Types of Systems

D) Linear systems

- A *linear* system obeys the principle of *superposition*.

$$x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n]$$

$$x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_2[n]$$

$$x_1[n] + x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n] + y_2[n]$$

$$ax_1[n] + bx_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow ay_1[n] + by_2[n]$$

3.2 Types of Systems

- On the contrary, a non-linear system does not obey the principle of superposition.

Take for example a system described by $y[n] = \log|x[n]|$

This system is not linear since

$$x_1[n] \longrightarrow \boxed{\log(|\,.\,|)} \longrightarrow y_1[n] = \log|x_1[n]|$$

$$x_2[n] \longrightarrow \boxed{\log(|\,.\,|)} \longrightarrow y_2[n] = \log|x_2[n]|$$

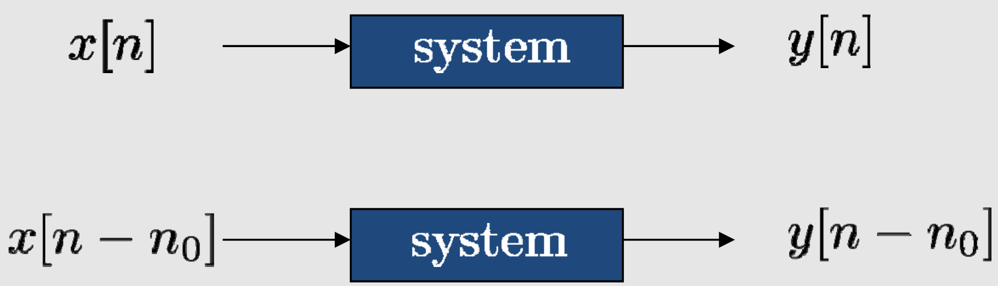
$$x_1[n] + x_2[n] \longrightarrow \boxed{\log(|\,.\,|)} \longrightarrow \log|x_1[n] + x_2[n]|$$

Not equivalent to the sum
of the outputs above

$$\log|x_1[n] + x_2[n]| \neq \log|x_1[n]| + \log|x_2[n]|$$

E) Time-invariant systems

- A time shift in the input of a time-invariant system causes a corresponding time shift in the output.



- Therefore, if a signal is delayed by n_0 number of samples, the output will also be delayed by the same number of samples.

Example:

Determine if the system $y[n] = \sum_{k=-\infty}^{n} x[k]$ is time invariant or not.

Let us delay the input signal by n_0 samples giving $x[n-n_0]$. The output is then given by

$$y_1[n] = \sum_{k=-\infty}^{n} x[k-n_0]$$

Substituting $k_1 = k - n_0$ we obtain

$$y_1[n] = \sum_{k_1=-\infty}^{n-n_0} x[k_1]$$

Now we need to show that $y_1[n]$ is a signal delayed by n_0 samples. From the system equation, we note that

$$y[n-n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$$

Since the last two equations are the same, the system is time-invariant.

The above can be expressed graphically by the following

$$x[n] \longrightarrow \boxed{\text{system}} \longrightarrow y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$x[n-n_0] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n] = \sum_{k_1=-\infty}^{n-n_0} x[k_1]$$

$$y[n-n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$$

These are the same

F) Summary of systems

property	test	effect
memoryless	output depends on present input	trivial system
causal	output depends on present/past inputs	realizable
stable	bounded input gives bounded output	realizable
linear	superposition holds	simplifies output computation
time-invariant	shifted input gives shifted output	simplifies output computation

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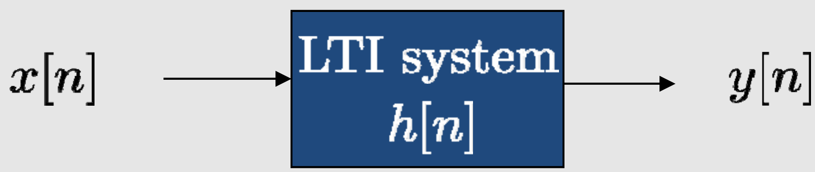
3.3 Impulse Response

- A system which obeys *superposition* and is *time-invariant* is known as a *linear time-invariant (LTI)* system.

- To find the *characteristic* (or everything we want to know) about an LTI system, we have to determine the *impulse response* of the system.

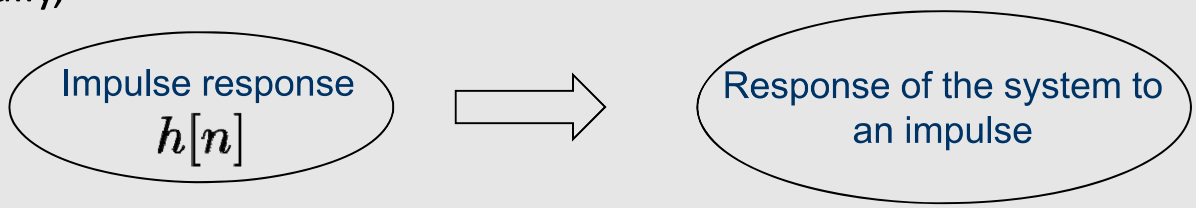
- Therefore, the impulse response of a system describes the behaviour or characteristic of the system.

- This impulse response consists of a sequence of numbers denoted by $h[n]$.

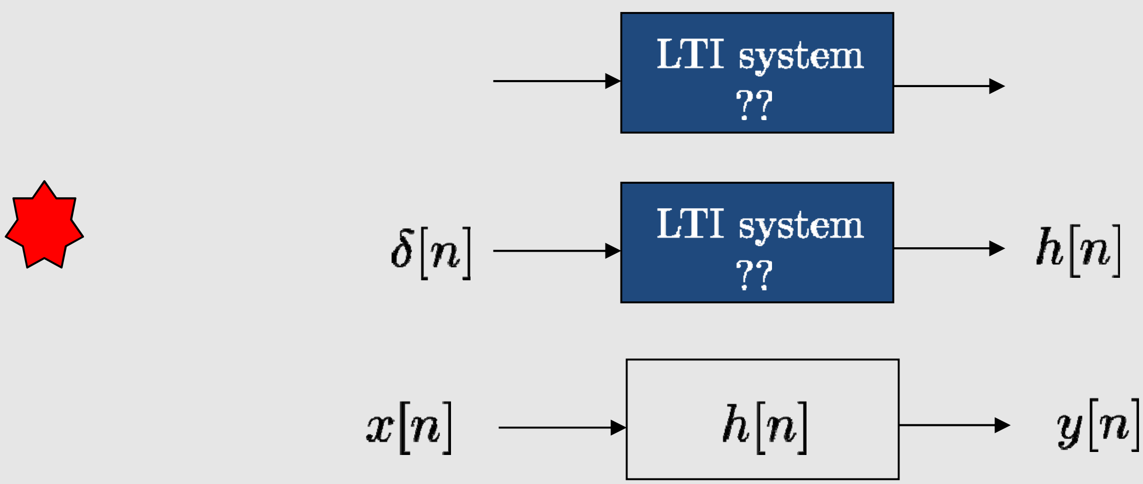


3.3 Impulse Response

- More formally,



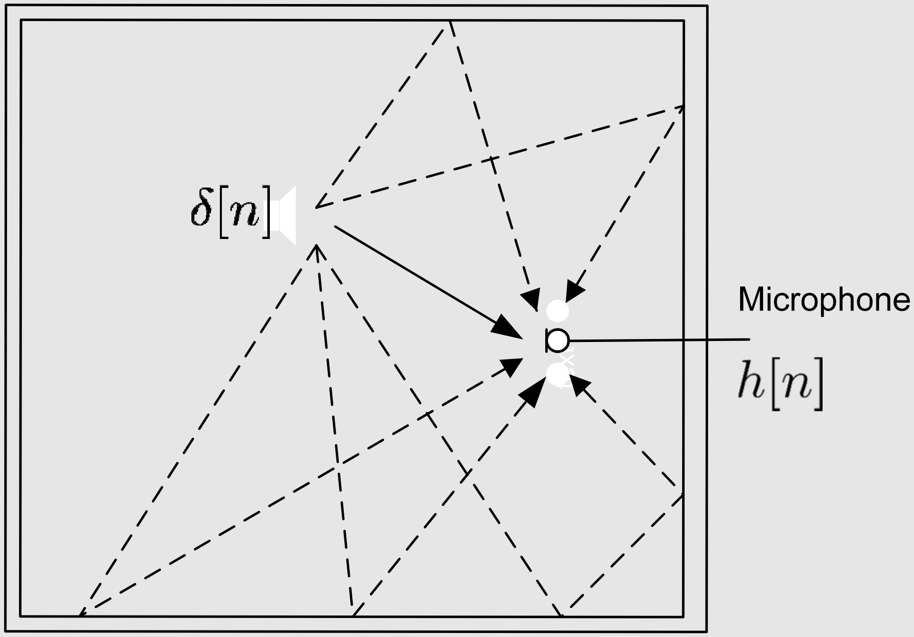
- The above technical definition of an impulse response gives us an idea of how to extract the impulse response of a system.

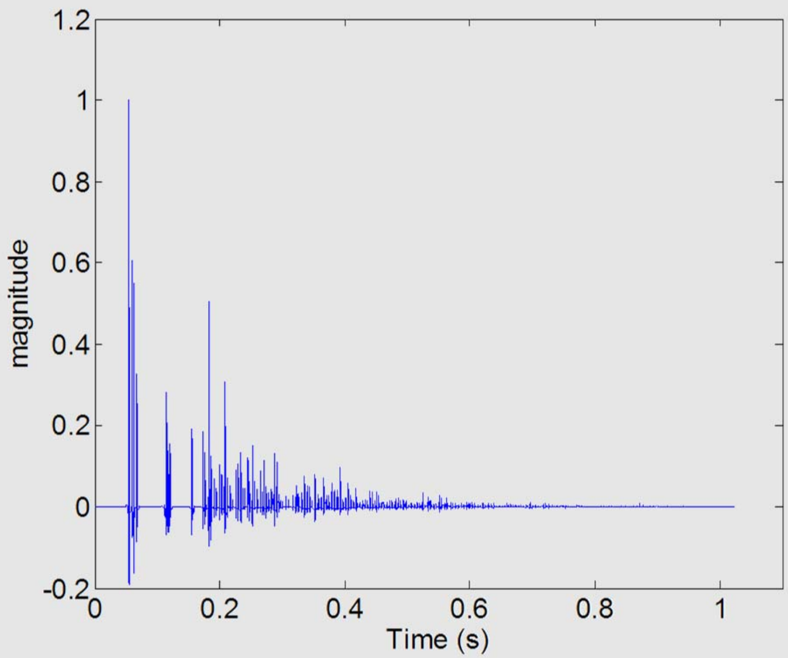


3.3 Impulse Response

- A real-life example of how to measure an acoustic impulse response

$$\delta[n] \longrightarrow \boxed{\begin{array}{c}\text{LTI system}\\ ??\end{array}} \longrightarrow h[n]$$





3.3 Impulse Response

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

- Given the impulse response of a system, we want to find the relationship between input $x[n]$, output $y[n]$ and impulse response $h[n]$.

- To do so, we first note that for an LTI system,

$$\delta[n] \longrightarrow \boxed{\text{LTI system}} \longrightarrow h[n]$$

Time-invariant $\quad \delta[n-k] \longrightarrow \boxed{\text{LTI system}} \longrightarrow h[n-k]$

Linearity

$$\ldots + a[0]\delta[n] + a[1]\delta[n-1] + \ldots \longrightarrow \boxed{\text{LTI system}} \longrightarrow \ldots + a[0]h[n] + a[1]h[n-1] + \ldots$$

$$\sum_{k=-\infty}^{\infty} a[k]\delta[n-k] \longrightarrow \boxed{\text{LTI system}} \longrightarrow \sum_{k=-\infty}^{\infty} a[k]h[n-k]$$

3.3 Impulse Response

$$\sum_{k=-\infty}^{\infty} a[k]\delta[n-k] \longrightarrow \boxed{\text{LTI system}} \longrightarrow \sum_{k=-\infty}^{\infty} a[k]h[n-k]$$

- In addition, from Section 2.3, we know that a discrete signal can be represented by a series of impulses, i.e.,

$$x[n] \quad = \quad \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- Therefore the above block diagram can be replaced by

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{\text{LTI system}} \longrightarrow \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

- Hence we note the important relationship

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

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3.4 Convolution

A) Definition

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

- The term *__linear convolution__*[Δ] is used to describe the process of achieving the output of an LTI system.

- We often use the notation $*$ to denote linear convolution.

- We say that the output $y[n]$ is obtained by convolving the input $x[n]$ with the impulse response of the system $h[n]$, i.e.,

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \end{aligned}$$

Δ Linear convolution is sometimes simply referred to as “convolution.” We are not dealing with circular convolution yet, and in this chapter, the word “convolution” will be used to mean linear convolution.

Example:

Find the linear convolution between $\delta[n]$ and $h[n]$.

Convolution between the two can be expressed as

$$y[n] = \delta[n] * h[n] = \sum_{k=-\infty}^{\infty} \delta[k]h[n-k]$$

We note that $\delta[n]$ is only non-zero when $n = 0$, and therefore we have

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} \delta[k]h[n-k] \\ &= \ldots + \delta[-1]h[n+1] + \delta[0]h[n] + \delta[1]h[n-1]\ldots \\ &= \delta[0]h[n] \end{aligned}$$

Since $\delta[0] = 1$, we have

$$\delta[n] * h[n] = h[n]$$

3.4 Convolution

$$\delta[n] * h[n] = h[n]$$

The above implies that an impulse function convolved with any signal will give the same signal itself.

This result has profound implication in how we extract the characteristic (impulse response) of a system.

To “peek” into the characteristic of a system, we only have to inject an impulse into the unknown system and the output will be the impulse response.

$$\longrightarrow \boxed{\text{LTI system ??}} \longrightarrow$$

$$\delta[n] \longrightarrow \boxed{\text{LTI system ??}} \longrightarrow h[n]$$

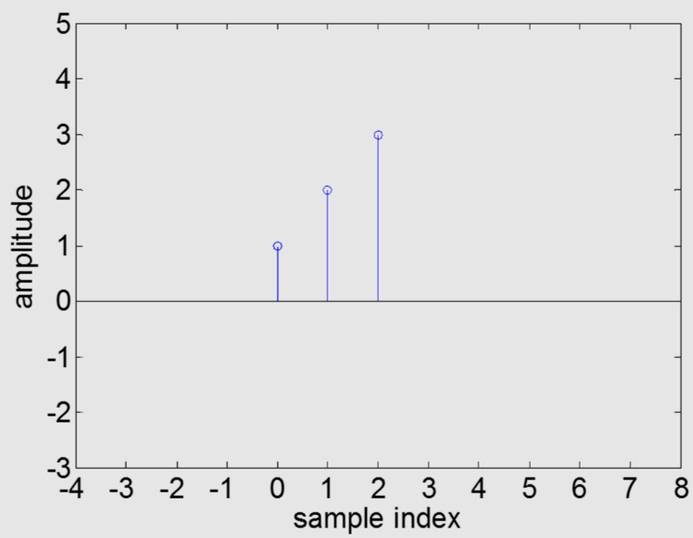
$$x(n) \longrightarrow \boxed{h[n]} \longrightarrow y(n) = x[n] * h[n]$$

B) "Flip-and-shift" method of convolution

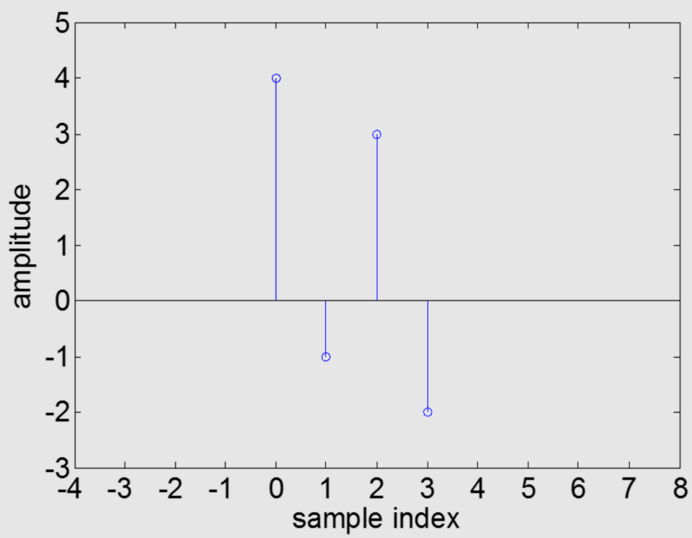
- The output sequence via a convolution process can be obtained using the "flip-and-shift" method.

- *Example:* Compute the convolution between $x[n]$ and $h[n]$ shown below.

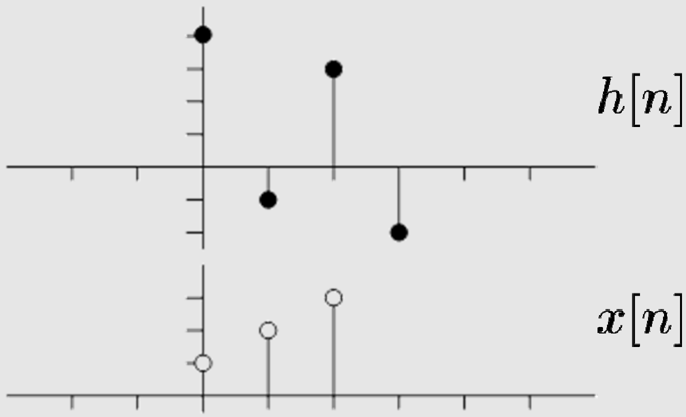
$$x[n] = [1\ 2\ 3]$$



$$h[n] = [4\ \ -1\ \ 3\ \ -2]$$



Step 1: draw one signal on top of the other



Subsequent steps: Shift the bottom signal by one sample to the right

C) Closed-form convolution

- Sometimes the sequence may be too long for one to use the "flip-and-shift" method.

- *Example:* Compute the convolution between $x[n] = [3\ \ 2]$ and impulse response $h[n] = 2^{-n}u[n]$ where $u[n]$ is a unit step function.

We can express

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} 2^{-k}u[k]x[n-k] \end{aligned}$$

Since a unit step function is defined as $u[n] = \begin{cases} 1, & n \geq 0; \\ 0, & n < 0 \end{cases}$

We can express
$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} 2^{-k} u[k] x[n-k] \\ &= \sum_{k=0}^{\infty} 2^{-k} x[n-k] \\ &= 2^{-0} x[n] + 2^{-1} x[n-1] + 2^{-2} x[n-2] + \ldots \end{aligned}$$

$$y[n] = \begin{cases} 0, & n < 0; \\ 2^{-n} x[0] = 3 \times 2^{-0} = 3, & n = 0 \\ 2^{-n+1} x[1] + 2^{-n} x[0] = (2^{-n+1})2 + (2^{-n})3 = 7(2^{-n}) & n \geq 1 \end{cases}$$

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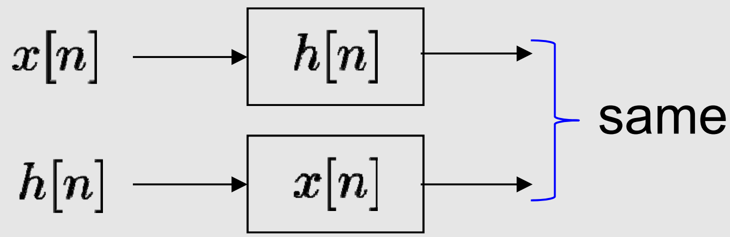
3.5 Properties of Convolution & LTI Systems

A) Convolution is commutative

$$x[n] * h[n] = h[n] * x[n]$$

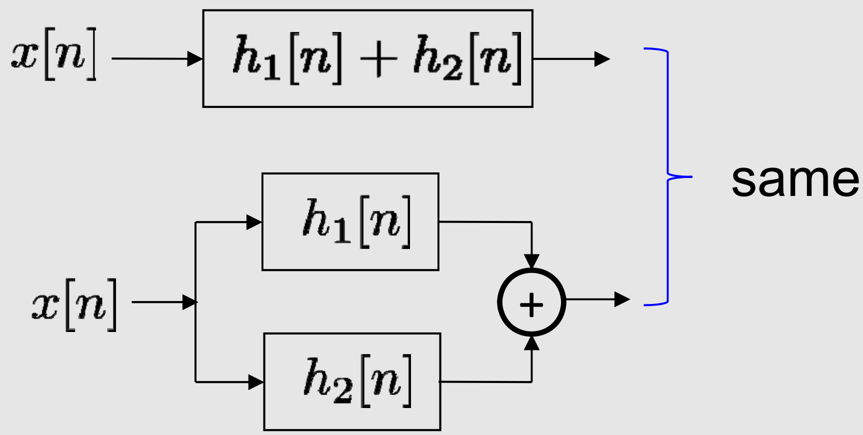


- The above also implies that when computing the convolution between two sequences using the "flip-and-shift" method (see Section 3.4-B), it does not matter which of the two signals are placed on top of the other.

3.5 Properties of Convolution & LTI Systems

B) Convolution is distributive over addition

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$



- The above process is also known as parallel combination.

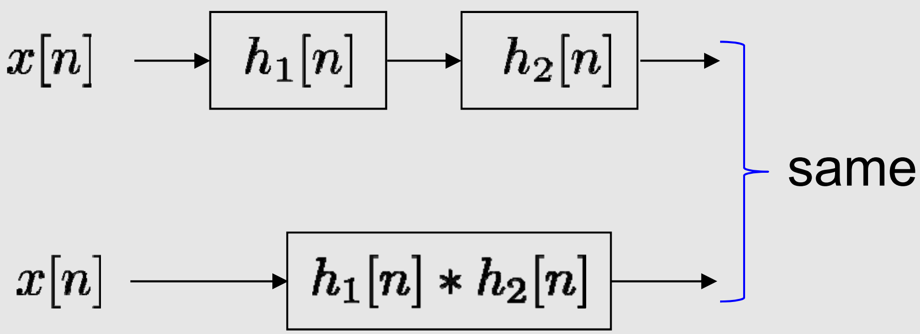
- In real application, the above implies that a system response can be *decomposed into two parts* before convolving these responses with the source and finally adding the output together.

- This decomposition may allow us to gain further insights into different parts of the overall system.

3.5 Properties of Convolution & LTI Systems

C) Convolution is associative

$$(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$$

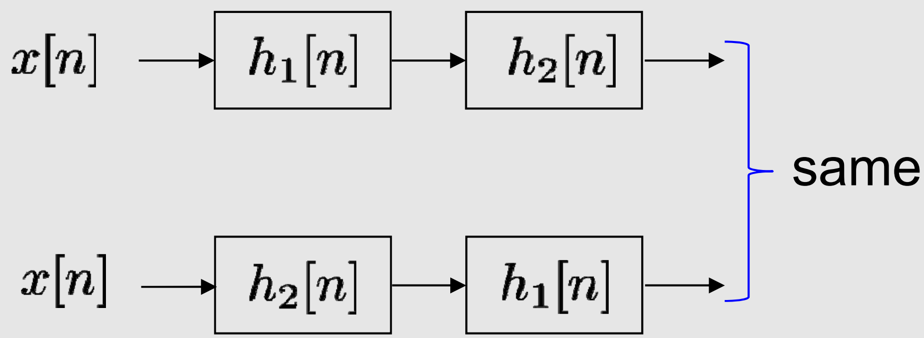


- The above process is also known as *cascade* combination.

- In real application, the above implies that if we want to generate certain audio effect, we can convolve the audio signal with an impulse response that is generated from the convolution of two impulse responses.

D) Commutative and associative property

$$(x[n] * h_1[n]) * h_2[n] = (x[n] * h_2[n]) * h_1[n]$$



- In real application, the above implies that it does not matter if the signal is convolved with the first or second impulse response.

E) LTI systems that are memoryless

- An LTI system is memoryless if and only if

$$h[n] = c\delta[n]$$

- For such a case, since

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

the output $y[n]$ depends only on $x[n]$ if $h[k] \neq 0$, or $y[n]$ is a scaled impulse.

- The above is consistent with the fact that an impulse convolve with a signal will result in the signal itself, i.e.,

$$\begin{aligned} y[n] &= \delta[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} \delta[k]x[n-k] \\ &= x[n] \end{aligned}$$

F) Causality of LTI systems

- An LTI system is causal if and only if

$$h[n] = 0, \quad n < 0$$

- For such a case, since

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

the output $y[n]$ depends only on $x[m]$ for $m \leq n$ if the LTI system is causal.

- The above implies that the output at time n depends only on the past value of the input sequence

3.5 Properties of Convolution & LTI Systems

G) Stability of LTI systems

- An LTI system is stable if and only if

$$\sum_{k=-\infty}^{\infty} |h[n]| < \infty$$

- For such a case, since

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

and that if $|x[n]| < B_x$, it follows that

$$|y[n]| \leq \sum_{k=-\infty}^{\infty} |h[k]|B_x$$

and hence, the output is bounded.

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3.6 Linear Constant-Coefficient Difference Equation

- Consider an accumulator system impulse response defined by

$$h[n] = \begin{cases} 1, & n \geq 0; \\ 0, & n < 0 \end{cases}$$

- Given $x[n]$ as the input signal, the output is given by

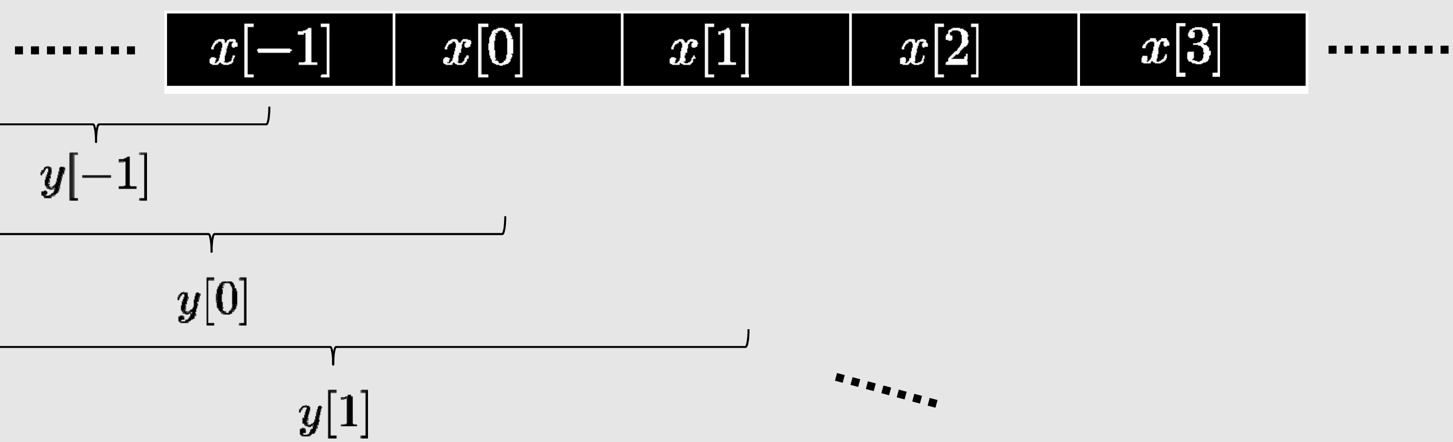
$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \end{aligned}$$

- Note that when $n - k < 0 \Rightarrow n < k$, we have $h[n-k] = 0$. Therefore, for each value of n, we can ignore higher order terms of $k > n$ resulting in

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] \quad = \quad \sum_{k=-\infty}^{n} x[k]$$

- Direct implementation of the above is infeasible since it involves infinite terms.



- The above implementation is not efficient at all as one will need infinite memory to store past values.

3.6 Linear Constant-Coefficient Difference Equation

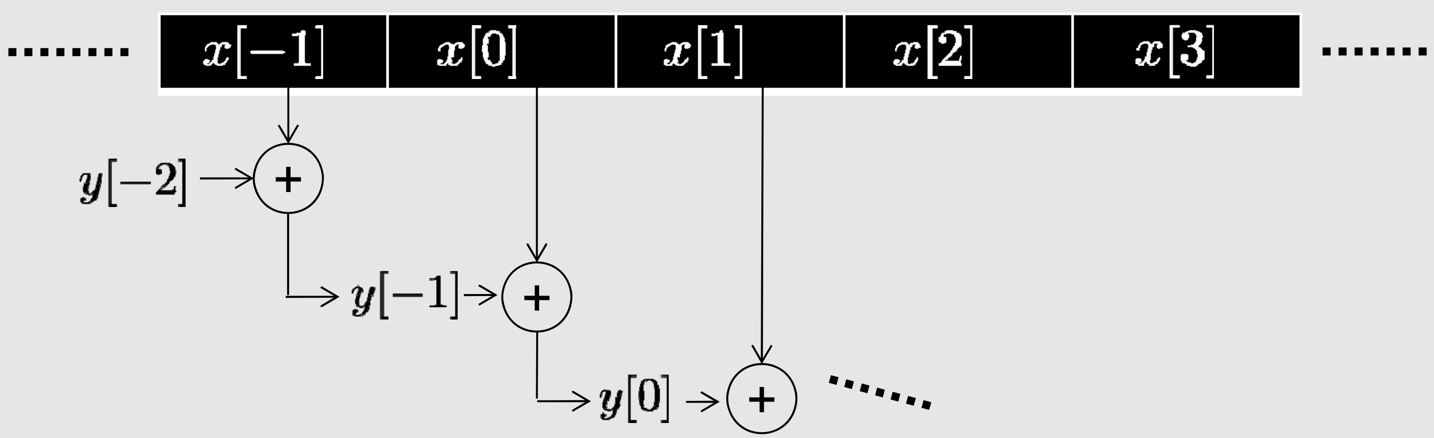
- We can express the above using finite terms by first noting that

$$y[n] = \sum_{k=-\infty}^{n} x[k] \qquad\qquad y[n-1] = \sum_{k=-\infty}^{n-1} x[k]$$

- Subtracting 2nd equation from the first, we obtain

$$\begin{aligned} y[n] - y[n-1] &= \sum_{k=-\infty}^{n} x[k] - \sum_{k=-\infty}^{n-1} x[k] \\ y[n] &= x[n] + y[n-1] \end{aligned}$$

- The above involves finite terms. The output of the system at time instant n is obtained by summing the current input sample with previous output sample.



3.6 Linear Constant-Coefficient Difference Equation

$$y[n] \quad = \quad x[n] + y[n-1]$$

- The above equation is an example of a linear constant-coefficient difference equation (CCDE).

- CCDEs are equations that show the relationship between *past inputs* and/or *outputs* and have a general form

$$\begin{aligned} a_0 y[n] + a_1 y[n-1] \quad &+ \quad \ldots + a_N y[n-N] \\ &= \quad b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M] \end{aligned}$$

- We can express the above more compactly using

$$\underbrace{a_0 y[n]}_{\text{Current output sample}} + \underbrace{\sum_{k=1}^{N} a_k y[n-k]}_{\text{Past output samples}} = \underbrace{\sum_{k=0}^{M} b_k x[n-k]}_{\text{Past and current input samples}}$$

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3.6 Linear Constant-Coefficient Difference Equation

- Using the CCDE, the output $y[n]$ at a given time n can be derived

$$a_0 y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k] - \sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k]$$

- Hence we note that auxiliary values $y[-1], y[-2], \ldots, y[-N]$ are required to compute later outputs $y[0], y[1], \ldots$

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3.7 Infinite and Finite Impulse Response Systems

A) Infinite Impulse Response (IIR)

- An infinite impulse response $h[n]$ is infinitely long.

- Convolution sum *cannot be used* to compute the output efficiently. We normally use CCDE in the form similar to

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k] - \sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k]$$

- When the CCDE involves past outputs, auxiliary values

$$y[-1], y[-2], \ldots, y[-N]$$

are needed.

3.7 Infinite and Finite Impulse Response Systems

Example: Consider the impulse response which we saw on Section 3.6

$$h[n] = \begin{cases} 1, & n \geq 0; \\ 0, & n < 0 \end{cases}$$

The convolution sum expression is given by $y[n] = \sum_{k=-\infty}^{n} x[k]$.

The CCDE is given by $y[n] = x[n] + y[n-1]$.

Therefore, for an input $x[n] = [1 \quad 2 \quad 4 \quad 8]$, given the auxiliary value $y[-1] = 0$.

$$\begin{aligned} y[0] &= x[0] + y[-1] &&= 1 \\ y[1] &= x[1] + y[0] &&= 3 \\ y[2] &= x[2] + y[1] &&= 7 \\ y[3] &= x[3] + y[2] &&= 15 \end{aligned}$$

3.7 Infinite and Finite Impulse Response Systems

B) Finite Impulse Response (FIR)

- A finite impulse response $h[n]$ has finite length.
- Convolution sum *may be used* to compute the output efficiently.
- The convolution sum is the same as CCDE for the case of FIR.
- The CCDE has no past output (i.e., no auxiliary values needed). It could be in the form of

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k]$$

- We can see that the coefficients b_k/a_0 are due to impulse response $h[n]$.

3.7 Infinite and Finite Impulse Response Systems

Example: Consider the impulse response $h[n] = [5 \;\; 2 \;\; 1 \;\; 3]$

The convolution sum expression is given by

$$
\begin{aligned}
y[n] &= \sum_{k=0}^{3} h[k]x[n-k] \\
&= h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] + h[3]x[n-3] \\
&= 5x[n] + 2x[n-1] + x[n-2] + 3x[n-3]
\end{aligned}
$$

The last equation is also the CCDE.

For an input $x[n] = [2 \quad \underset{\uparrow}{-1} \quad 1 \quad 2]$, the output is given by

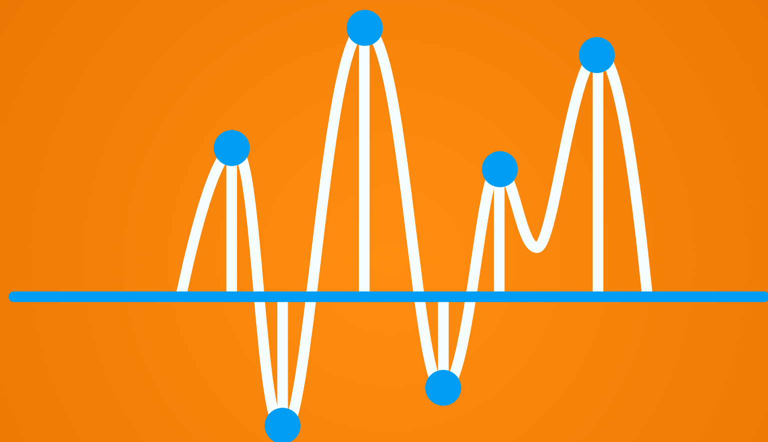
$$
\begin{aligned}
y[-1] &= 5x[-1] + 2x[-2] + x[-3] + 3x[-4] &&= 10 \\
y[0] &= 5x[0] + 2x[-1] + x[-2] + 3x[-3] &&= -1 \\
y[1] &= 5x[1] + 2x[0] + x[-1] + 3x[-2] &&= 5 \\
y[2] &= 5x[2] + 2x[1] + x[0] + 3x[-1] &&= -17
\end{aligned}
$$

3.8 Summary

- A system can be classified as one with memory/memoryless, causal/non-causal, stable/unstable, linear/non-linear, time-invariant or not.
- The impulse response of a system describes the characteristics of the system. It corresponds to the output of the system when the input is an impulse.
- The output of a system corresponds to the input convolved with the impulse response of the system.
- The CCDE describes the input and output relationship of the system.
- The output of an IIR system depends on both the previous outputs and past+present inputs.
- The output of an FIR system depends on the past+present input values.

Chapter 4

Discrete-Time Fourier Transform (DTFT)



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Chapter Aims

At the end of this chapter, you will be expected to:

1. understand the formulation of the DTFT
2. apply different properties of the DTFT
3. derive the DTFT for a given signal
4. understand what is the frequency response of a system and know how this is being derived

4.1 Definition of DTFT and IDTFT

- The DTFT is defined as a process of expressing a *time-domain discrete* signal $x[n]$ using a *continuous frequency domain* variable $X(e^{j\omega})$.

- The DTFT can be mathematically expressed using

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Frequency-domain continuous spectrum

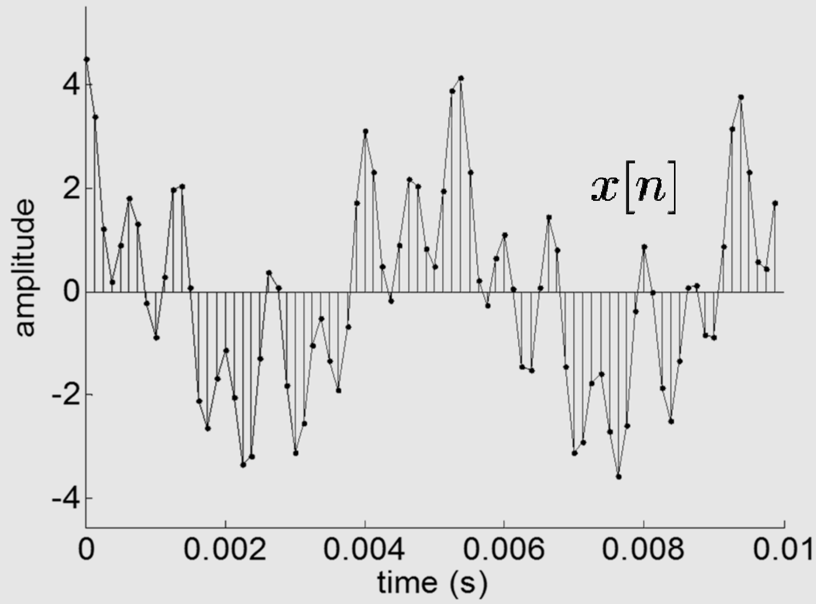
Time-domain discrete signal

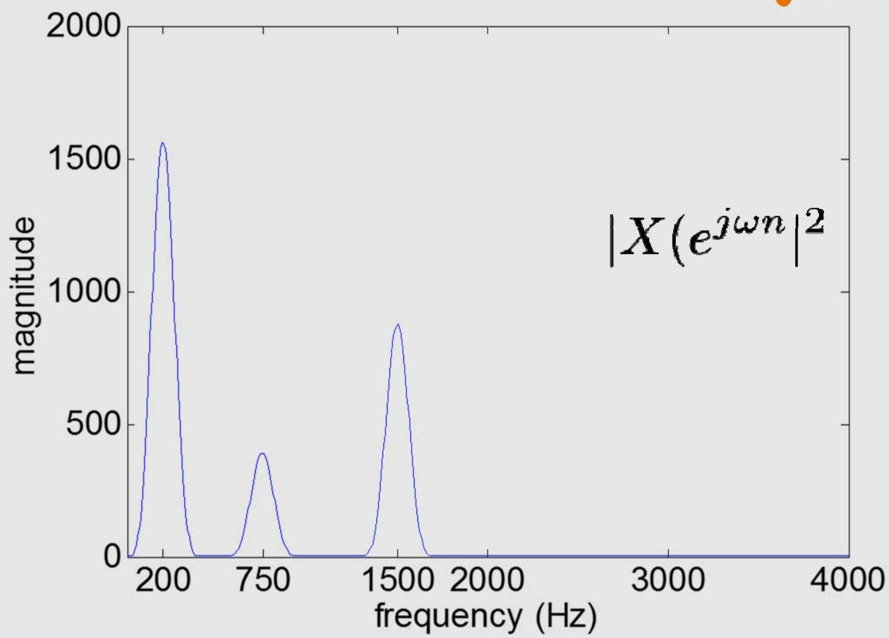
(basis function) complex sinusoid

- The inverse DTFT (IDTFT) can be expressed by

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

4.1 Definition of DTFT and IDTFT





- Frequency content of the discrete-time signal (left) can be achieved using the DTFT to generate the spectrum (right).

4.1 Definition of DTFT and IDTFT

- Recall that two discrete-time sinusoids of frequency ω_0 and $\omega_0 + 2m\pi$ are identical.

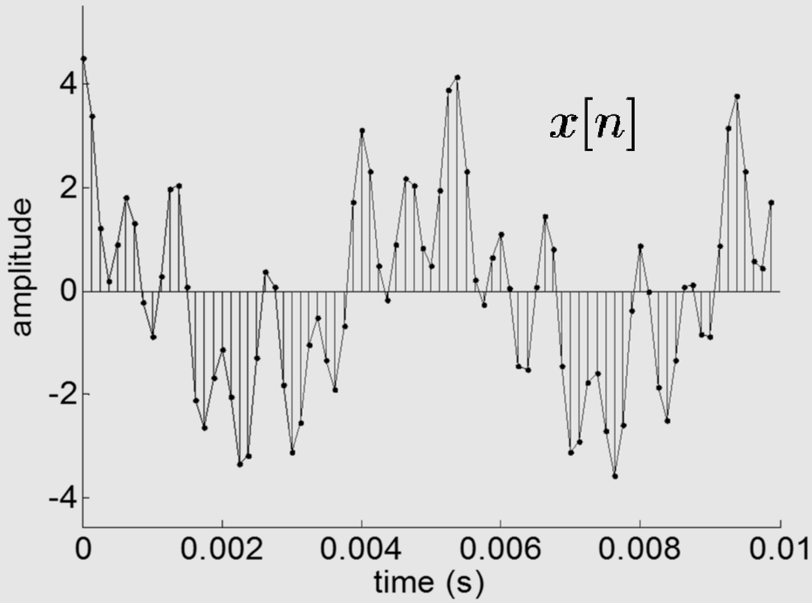
- In general, the spectrum is periodic with period 2π since

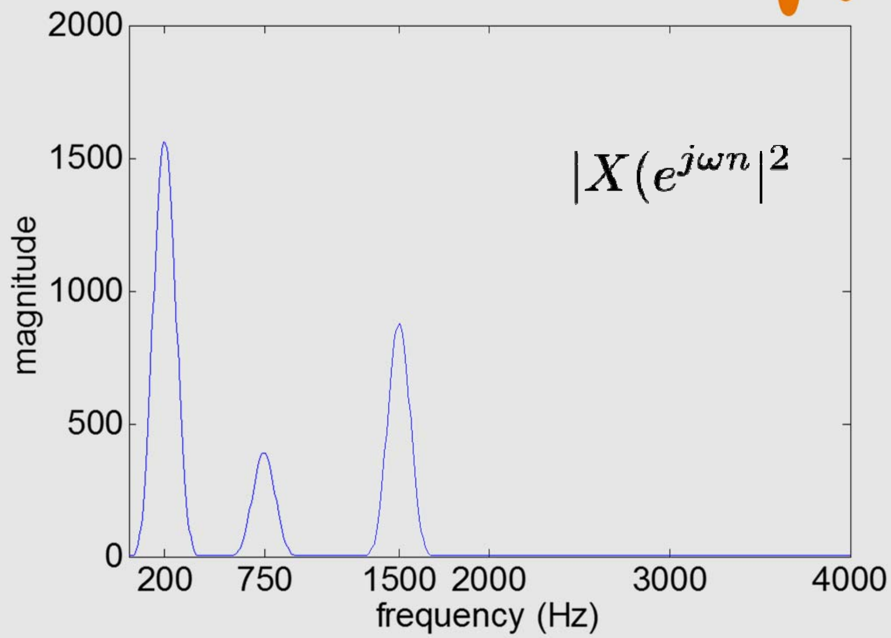
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\begin{aligned} X\left(e^{j(\omega+2m\pi)}\right) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2m\pi)n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}e^{-j2m\pi n} \\ &= X(e^{j\omega}) \end{aligned}$$

- Therefore the frequency range is limited by 2π from $-\pi$ to π.

4.1 Definition of DTFT and IDTFT





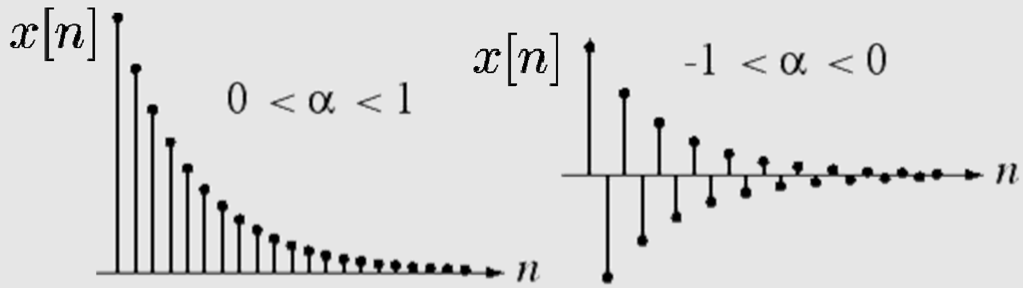
- In general, the spectrum is complex

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$= \underbrace{|X(e^{j\omega})|}_{\text{Magnitude spectrum}}\underbrace{e^{j\angle X(e^{j\omega})}}_{\text{Phase spectrum}}$$

4.1 Definition of DTFT and IDTFT

Example: Find the DTFT of $x[n] = \alpha^n u[n]$ where α is real.



$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \alpha^n u[n] e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n
\end{aligned}
$$

The sum diverges (becomes ∞) for $|\alpha| \geq 1$, so DTFT exists only for $|\alpha| < 1$.

4.1 Definition of DTFT and IDTFT

Since $X(e^{j\omega})$ is an infinite geometric series, its sum is given by

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n = \frac{1}{1-\alpha e^{-j\omega}}$$

$$= \frac{1}{1-\alpha\cos\omega + j\alpha\sin\omega}$$

The magnitude spectrum is given by

$$|X(e^{j\omega})| = \frac{1}{\sqrt{(1-\alpha\cos\omega)^2 + (\alpha\sin\omega)^2}}$$

$$= \frac{1}{\sqrt{1+\alpha^2\cos^2\omega + \alpha^2\sin^2\omega - 2\alpha\cos\omega}}$$

$$= \frac{1}{\sqrt{1+\alpha^2 - 2\alpha\cos\omega}}$$

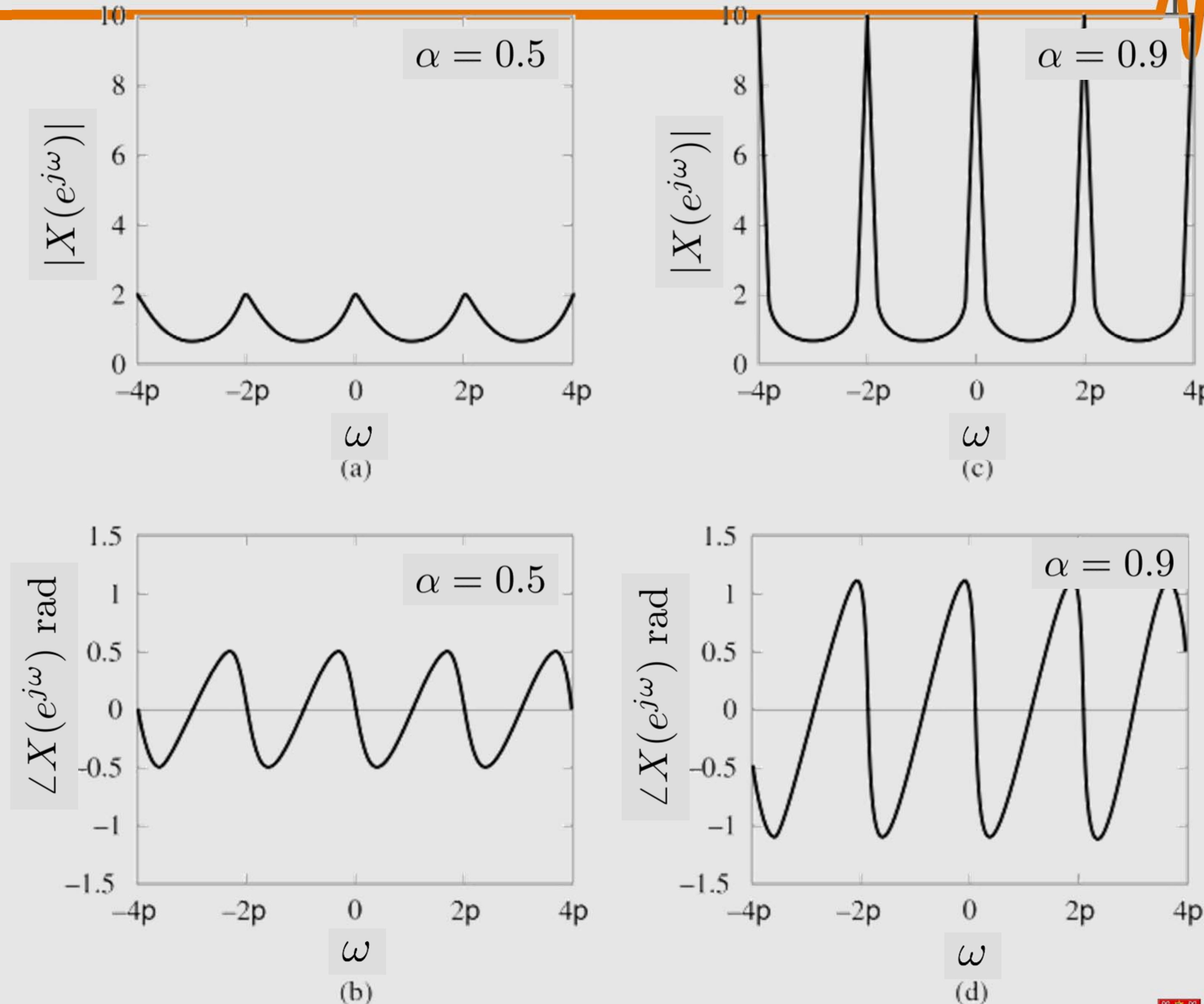
The phase spectrum is given by

$$\angle X(e^{j\omega}) = -\tan^{-1}\left(\frac{\alpha\sin\omega}{1-\alpha\cos\omega}\right)$$

4.1 Definition of DTFT and IDTFT

Magnitude spectrum

Phase spectrum



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- Symmetry

  If $x[n]$ is a real valued signal, then its spectrum has conjugate symmetry, i.e.,

$$x[n] \text{ real } \longleftrightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$$

To prove the above, we note that

$$\begin{aligned} X^*(e^{-j\omega}) = \left\{X(e^{j\omega})|_{\omega=-\omega}\right\}^* &= \left\{\sum x[n]e^{-j\omega n}|_{\omega=-\omega}\right\}^* \\ &= \left\{\sum x[n]e^{j\omega n}\right\}^* \\ &= \sum x[n]e^{-j\omega n} \\ &= X(e^{j\omega}) \end{aligned}$$

We note that the magnitude is symmetric (even function), i.e.,

$$|X(e^{j\omega})| = |X(e^{-j\omega})|$$

The phase is anti-symmetric (odd function), i.e.,

$$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$$

4.2 DTFT Properties

Linearity	$ax_1[n] + bx_2[n]$	$\longleftrightarrow$	$aX_1(e^{j\omega_1}) + bX_2(e^{j\omega_2})$
Time shift	$x[n - n_d]$	$\longleftrightarrow$	$e^{-j\omega n_d} X(e^{j\omega})$
Frequency shift	$e^{j\omega_0 n} x[n]$	$\longleftrightarrow$	$X(e^{j(\omega-\omega_0)})$
Time reversal	$x[-n]$	$\longleftrightarrow$	$X(e^{-j\omega})$
Differentiation in frequency	$nx[n]$	$\longleftrightarrow$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
Convolution	$x[n] * h[n]$	$\longleftrightarrow$	$X(e^{j\omega})H(e^{j\omega})$
Modulation	$x[n]w[n]$	$\longleftrightarrow$	$\dfrac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$

Periodic convolution

Example: Prove the differentiation property

$$nx[n] \quad \longleftrightarrow \quad j\frac{dX(e^{j\omega})}{d\omega}$$

We start with the definition of DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Differentiating both sides with respect to ω,

$$\begin{aligned} \frac{dX(e^{j\omega})}{d\omega} &= \frac{d}{d\omega}\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x[n](-jn)e^{-j\omega n} \\ \sum_{n=-\infty}^{\infty} nx[n]e^{-j\omega n} &= \frac{-1}{j}\frac{d}{d\omega}X(e^{j\omega}) \end{aligned}$$

We note that $-1/j = j$,

$$\Rightarrow nx[n] \quad \longleftrightarrow \quad j\frac{dX(e^{j\omega})}{d\omega}$$

4.3 DTFT Pairs

Impulse	δ	$\longleftrightarrow$	1	
Constant	1	$\longleftrightarrow$	$2\pi\delta(\omega)$ for one period	
Exponential	$a^n u[n]$	$\longleftrightarrow$	$\dfrac{1}{1 - ae^{-j\omega}}$	$\lvert a \rvert < 1$
Rectangular pulse	$x[n] = \begin{cases} 1, & -M \le n \le M; \\ 0, & \text{otherwise} \end{cases}$	$\longleftrightarrow$	$\dfrac{\sin[\omega(2M+1)/2]}{\sin(\omega/2)}$	

Example: Find the DTFT of the sinusoid

$$x[n] = A\cos(\omega_0 n + \phi)$$

We start by expressing

$$x[n] = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

Considering the first term,

$$1 \quad \longleftrightarrow \quad 2\pi\delta(\omega)$$

Exploiting frequency shift property,

$$e^{j\omega_0 n}x[n] \quad \longleftrightarrow \quad X(e^{j(\omega-\omega_0)})$$

$$e^{j\omega_0 n} \times 1 \quad \longleftrightarrow \quad 2\pi\delta(\omega - \omega_0)$$

Therefore, the DTFT of the sinusoid is given by

$$x[n] \longleftrightarrow A\pi e^{j\phi}\delta(\omega - \omega_0) + A\pi e^{-j\phi}\delta(\omega + \omega_0)$$

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4.4 Frequency Response

A. Definition

- The frequency response of an LTI system is defined as the DTFT of the impulse response.

impulse response

$$\longrightarrow \boxed{h[n]} \longrightarrow$$

$$\longrightarrow \boxed{H(e^{j\omega})} \longrightarrow$$

frequency response

- The frequency response comprises normally of two parts
  - ✓ *Magnitude response*
  - ✓ *Phase response*

$$\underbrace{H(e^{j\omega})}_{\text{Frequency response}} = \underbrace{|H(e^{j\omega})|}_{\text{Magnitude response}}\underbrace{e^{j\angle H(e^{j\omega})}}_{\text{Phase response}}$$

B. Input and output relationship with frequency response

- Knowing the frequency response, the output of an LTI system can be expressed in its frequency domain by

impulse response

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

$$X(e^{j\omega}) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

frequency response

- For example for a complex sinusoidal input,

$$e^{j\omega_0 n} \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow Y(e^{j\omega}) = e^{j\omega_0 n}H(e^{j\omega})$$

frequency response

- We can also find the frequency response of a CCDE. To do so, the CCDE can be expressed as (see Sec. 3.6)

$$\begin{aligned} a_0 y[n] + a_1 y[n-1] &+ \ldots + a_N y[n-N] \\ &= b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M] \end{aligned}$$

- We note that

$$y[n] \longleftrightarrow Y(e^{j\omega}) \qquad y[n-1] \longleftrightarrow e^{-j\omega} Y(e^{j\omega})$$

$$x[n] \longleftrightarrow X(e^{j\omega}) \qquad x[n-1] \longleftrightarrow e^{-j\omega} X(e^{j\omega})$$

- Taking the DTFT of the CCDE, we obtain

$$\left(a_0 + a_1 e^{-j\omega} + \ldots + a_N e^{-jN\omega}\right) Y(e^{j\omega}) = \left(b_0 + b_1 e^{-j\omega} + \ldots + b_M e^{-jM\omega}\right) X(e^{j\omega})$$

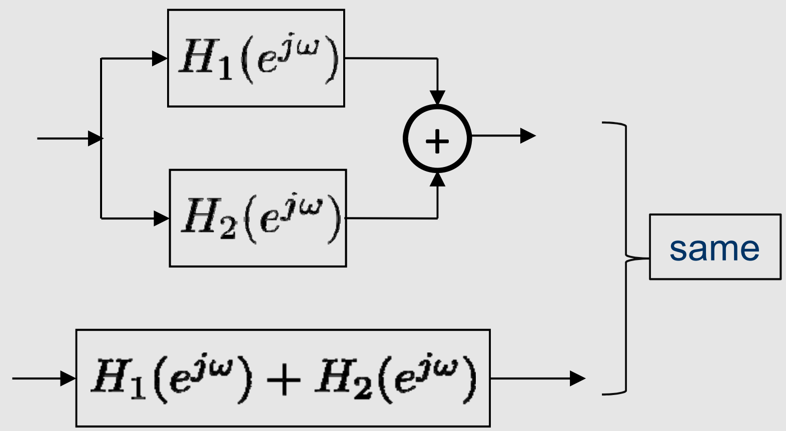
- Hence the frequency response of the LTI system (transfer function) is

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{b_0 + b_1 e^{-j\omega} + \ldots + b_M e^{-jM\omega}}{a_0 + a_1 e^{-j\omega} + \ldots + a_N e^{-jN\omega}}$$

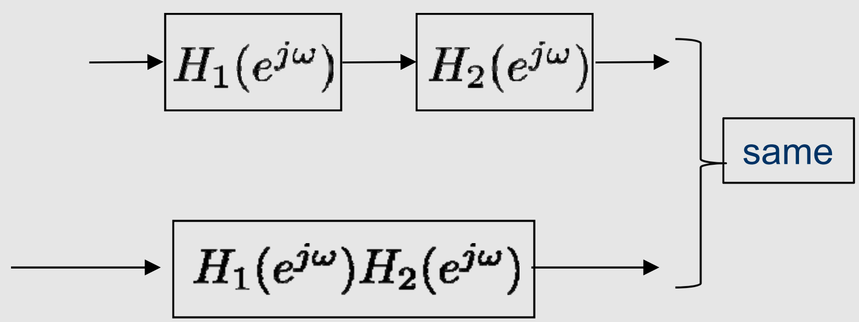
4.4 Frequency Response

C. Combination of LTI systems

Parallel combination



Cascade combination



4.5 Summary

- The DTFT is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

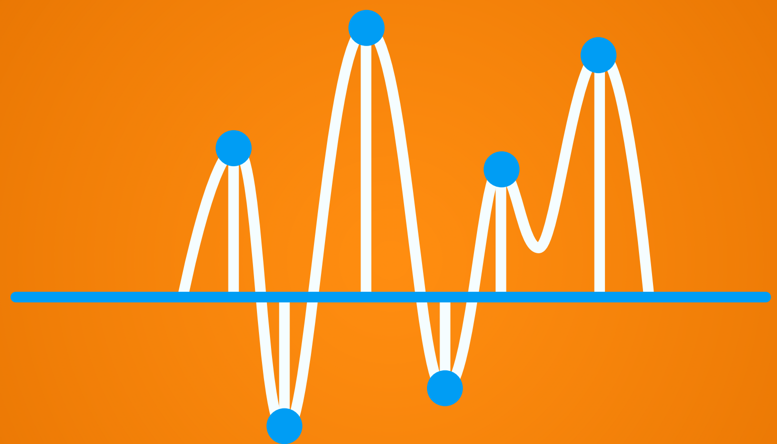
- The DTFT is a continuous complex variable.

- The DTFT of a real signal $x[n]$ is characterized by
  - ✓ magnitude spectrum which is symmetric about the vertical axis;
  - ✓ phase spectrum which is anti-symmetric about the vertical axis.

- The frequency response of a system is defined as the DTFT of the impulse response of the system.

Chapter 5

The z-Transform



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Chapter Aims

At the end of this chapter, you will be expected to:

1. understand the formulation of the z-transform
2. derive the z-transform for a given signal
3. apply different properties of the z-transform
4. derive the transfer function of a system and its poles and zeros
5. derive the inverse transfer function of a system

5.1 Definition

- There are many sequences for which the DTFT does not converge; e.g., as shown in Section 4.1, the DTFT of

$$x[n] = \alpha^n u[n], \qquad \alpha > 1$$

- The z-transform provides a *generalization* of the Fourier transform.

- The z-transform is often more convenient to use as it permits simple algebraic manipulations.

- The z-transform is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

5.1 Definition

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- The variable z is a *complex variable* of the form

$$z = re^{j\omega}$$

- The above is often referred to as the two-sided or bilateral z-transform.

- For signals which begins when $n \geq 0$, the *one-sided or unilateral* z-transform is given by

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

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Example: Find the z-transform of the discrete-time sequence

$$x[n] = [\underset{\uparrow}{1} \quad 0.2 \quad 3 \quad 0.4 \quad 5]$$

The z-transform is given by

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x[n] z^{-n} \\ &= x[0]z^{0} + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + x[4]z^{-4} \\ &= 1 + 0.2z^{-1} + 3z^{-2} + 0.4z^{-3} + 5z^{-4} \end{aligned}$$

5.1 Definition

- The z-transform has a close relationship with the DTFT. The DTFT is defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- The z- transform is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- Noting that $z = re^{j\omega}$, we can express the z-transform of $x[n]$ as

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x[n]\left(re^{j\omega}\right)^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left(x[n]r^{-n}\right)e^{-j\omega n} \end{aligned}$$

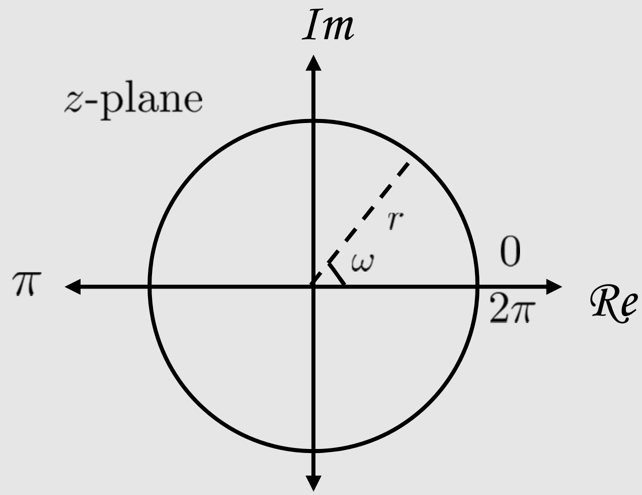
And therefore, the z-transform is equivalent to the DTFT when $r = 1$.

5.1 Definition

- The *complex variable* is of the form

$$z = re^{j\omega}$$

- It is convenient to describe and interpret it using the complex z-plane.



- Therefore, r defines the radius of the circle
  ω defines the angle w.r.t. the real axis
- We note that

$$z = 1 \qquad \Rightarrow \omega = 0, 2\pi$$

$$z = j \qquad \Rightarrow \omega = 0.5\pi$$

$$z = -1 \qquad \Rightarrow \omega = \pi, -\pi$$

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5.2 Poles and Zeroes

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z)$$

frequency response/system function

- Similar to the frequency response of an LTI system discussed in Section 4.4, the z- transform can be expressed as a ratio of polynomials

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}$$

- Expressing the above in the following form,

$$H(z) = z^{(N-M)} \times \frac{b_0 z^M + b_1 z^{M-1} + \ldots + b_M}{a_0 z^N + a_1 z^{N-1} + \ldots + a_N}$$

we can then factorize the numerator and denominator such that the above can be expressed in the form of

$$\begin{aligned} H(z) &= z^{(N-M)} \times \frac{b_0}{a_0} \times \frac{(z-\xi_1)(z-\xi_2)\ldots(z-\xi_M)}{(z-\lambda_1)(z-\lambda_2)\ldots(z-\lambda_N))} \\ &= z^{(N-M)} \times \frac{b_0}{a_0} \times \frac{\Pi_{l=1}^{M}(z-\xi_l)}{\Pi_{l=1}^{N}(z-\lambda_l)} \end{aligned}$$

5.2 Poles and Zeroes

$$H(z) = z^{(N-M)} \times \frac{b_0}{a_0} \times \frac{(z-\xi_1)(z-\xi_2)\ldots(z-\xi_M)}{(z-\lambda_1)(z-\lambda_2)\ldots(z-\lambda_N))}$$
$$= z^{(N-M)} \times \frac{b_0}{a_0} \times \frac{\Pi_{l=1}^{M}(z-\xi_l)}{\Pi_{l=1}^{N}(z-\lambda_l)}$$

- The factors

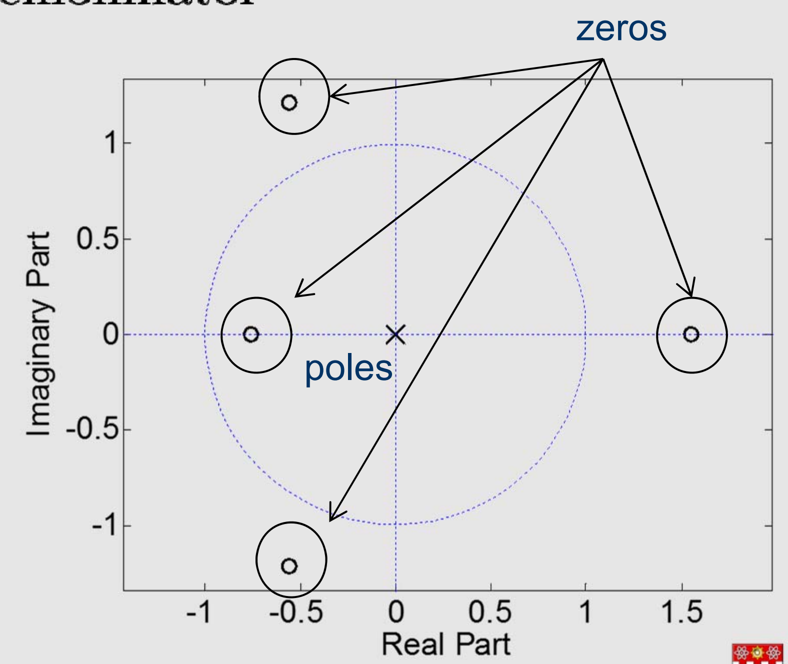
  $(z-\xi_l)$: roots of the numerator

  $(z-\lambda_l)$: roots of the demoninator

- The variables

  ξ_l: zeros of $H(z)$

  λ_l: poles of $H(z)$



Example: Consider the case where the input and output of a system is given by

$$\begin{aligned} x[n] &= [0.1 \quad 0.4 \quad 0.3 \quad 0.2 \quad 0.1] \\ y[n] &= [0.2 \quad 0.1 \quad 0.02 \quad 0.3] \end{aligned}$$

Find the transfer function of the system in the z-domain and state its poles and zeros.

The transfer function is given by

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{\sum_{n=0}^{3} y[n]z^{-n}}{\sum_{n=0}^{4} x[n]z^{-n}} \\ &= \frac{0.2 + 0.1z^{-1} + 0.02z^{-2} + 0.3z^{-3}}{0.1 + 0.4z^{-1} + 0.3z^{-2} + 0.2z^{-3} + 0.1z^{-4}} \end{aligned}$$

We may factorize the transfer function as follows

$$\begin{aligned}
H(z) &= \frac{0.2 + 0.1z^{-1} + 0.02z^{-2} + 0.3z^{-3}}{0.1 + 0.4z^{-1} + 0.3z^{-2} + 0.2z^{-3} + 0.1z^{-4}} \\
&= \frac{z^4}{z^3}\frac{0.2z^3 + 0.1z^2 + 0.02z + 0.3}{0.1z^4 + 0.4z^3 + 0.3z^2 + 0.2z + 0.1} \\
&= z\frac{0.2}{0.1}\frac{z^3 + 0.5z^2 + 0.1z + 1.5}{z^4 + 4z^3 + 3z^2 + 2z + 1} \\
&= 2z\frac{(z+1.3)\,(z-(0.4+j0.99))\,(z-(0.4-j0.99))}{(z+3.23)\,(z+0.67)\,(z+(0.05+j0.68))\,(z+(0.05-j0.68))}
\end{aligned}$$

The zeros of the system are:

$$\begin{aligned}
z &= 0 \\
z &= -1.3 \\
z &= 0.4 \pm j0.99
\end{aligned}$$

The poles of the system are:

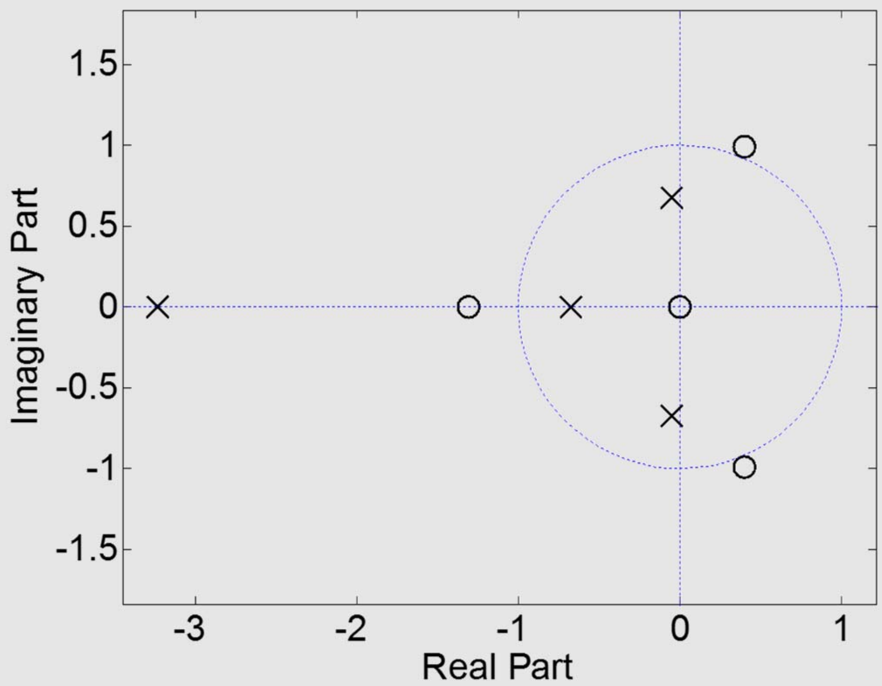
$$\begin{aligned}
z &= -3.23 \\
z &= -0.67 \\
z &= -0.05 \pm j0.68
\end{aligned}$$

The zeros of the system are:

$$
\begin{aligned}
z &= 0 \\
z &= -1.3 \\
z &= 0.4 \pm j0.99
\end{aligned}
$$

The poles of the system are:

$$
\begin{aligned}
z &= -3.23 \\
z &= -0.67 \\
z &= -0.05 \pm j0.68
\end{aligned}
$$



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5.3 Region of Convergence

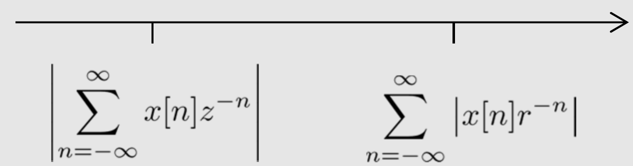
- As described in Sections 5.1 and 4.1, DTFT may not converge for some sequences. Similarly, the z-transform may not converge for all sequences or for all values of z.

- For any given sequence, the set of z for which the z-transform converges is called the *<u>region of convergence (ROC)</u>*.

- For the z-transform to converge, we require $|X(z)| = \left|\sum_{n=-\infty}^{\infty} x[n]z^{-n}\right| < \infty$.

- Since

$$\begin{aligned}\left|\sum_{n=-\infty}^{\infty} x[n]z^{-n}\right| &\le \sum_{n=-\infty}^{\infty} \left|x[n]z^{-n}\right| \\ &\le \sum_{n=-\infty}^{\infty} |x[n]| \ \left|r^{-n}\right| \\ &\le \sum_{n=-\infty}^{\infty} \left|x[n]r^{-n}\right|\end{aligned}$$



we note that the variable r regulates the convergence of the z-transform.

5.3 Region of Convergence

- From the above inequalities, we can therefore show the convergence of $X(z)$ by showing that

$$\sum_{n=-\infty}^{\infty} \left| x[n] r^{-n} \right| < \infty$$

$$\left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| \qquad \sum_{n=-\infty}^{\infty} \left| x[n] r^{-n} \right| \qquad \infty$$

since if $\sum_{n=-\infty}^{\infty} \left| x[n] r^{-n} \right| < \infty$, so is $|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| < \infty$.

- Because of the multiplication of $x[n]$ by r^{-n}, it is possible for the z-transform to converge even if the DTFT does not.

$$X(z) = \sum_{n=-\infty}^{\infty} \left(x[n] r^{-n} \right) e^{-j\omega n}$$

5.3 Region of Convergence

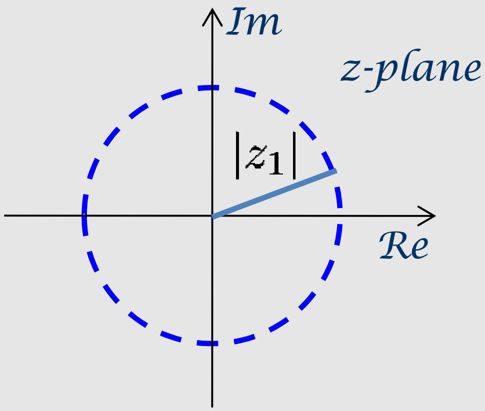
- For the z-transform to converge, we require

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| < \infty$$

- Since $\left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| \leq \sum_{n=-\infty}^{\infty} \left| x[n] z^{-n} \right|$

  $X(z)$ can converge if z is chosen such that

$$\sum_{n=-\infty}^{\infty} |x[n]| \, |z|^{-n} < \infty$$



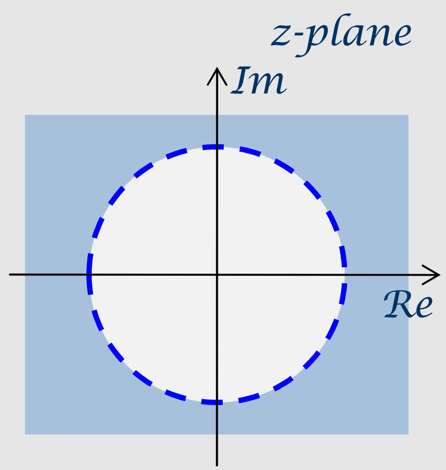
- Therefore if some value of $z = z_1$ is in the ROC, then all values of z on the circle defined by $|z| = |z_1|$ will also be in the ROC.

- The above also implies that for ROC, z is circular and the ROC will consist of a ring in the z-plane.

5.3 Region of Convergence

$$\sum_{n=-\infty}^{\infty} |x[n]|\,|z|^{-n} = \sum_{n=-\infty}^{\infty} |x[n]|\,\frac{1}{|z|^{n}}$$

- The ROC boundary is a circle and it may have three shapes



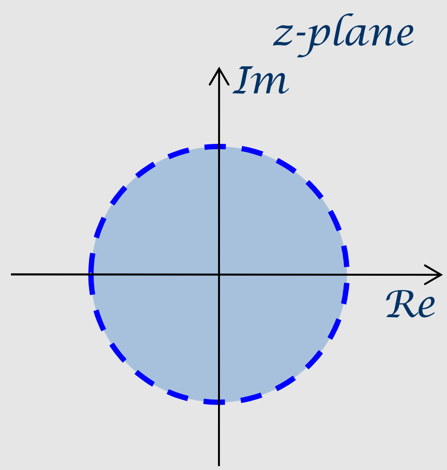
$r_R < |z|$

exterior of a circle

$x[n]$ non-zero at $n = \infty$

so $z \approx 0$ diverges

right-sided sequence



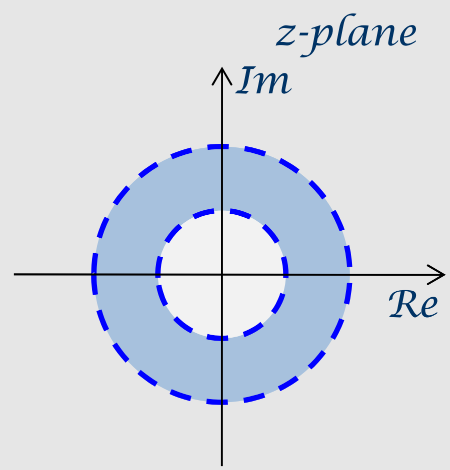
$r_L > |z|$

interior of a circle

$x[n]$ non-zero at $n = -\infty$

so $z \approx \infty$ diverges

left-sided sequence



$r_R < |z| < r_L$

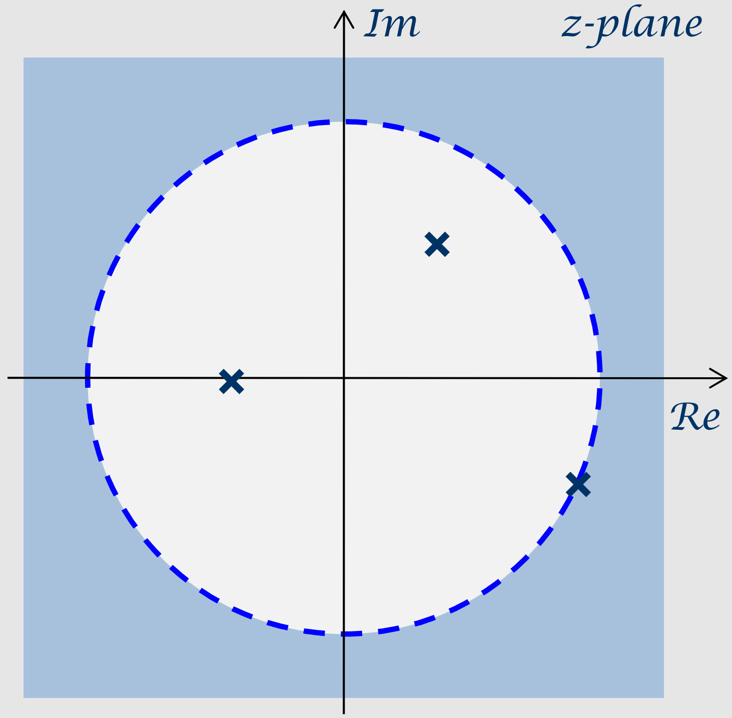
annular ring

$x[n]$ non-zero at both $n = \infty$ and $n = -\infty$

$z \approx 0$ and $z \approx \infty$ diverges *two-sided* sequence

- Furthermore, since $X(z)$ diverges at a pole, ROC does not contain any pole.
- The poles define the boundaries of an ROC.
- For example, for 3 poles shown below, there can be 4 possible ROCs:

left-sided
right-sided
two-sided

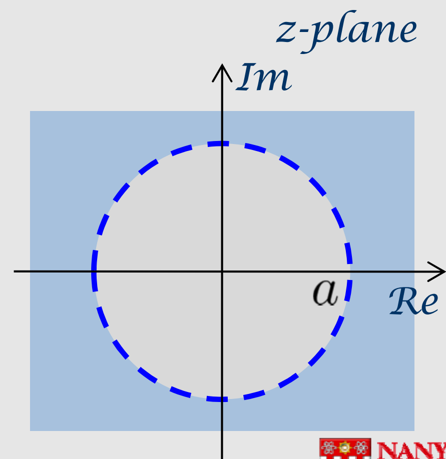


5.3 Region of Convergence

Example: Find the z-transform of $x[n] = a^n u[n]$ and its ROC.

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} \\
&= \sum_{n=0}^{\infty} a^n z^{-n} \\
&= \sum_{n=0}^{\infty} (az^{-1})^n
\end{aligned}
$$

For to converge, the “ratio” of the geometric series must be less than
Therefore, the ROC is defined by $|az^{-1}| < 1$ or $|z| > |a|$.



Under such condition, we have

$$
\begin{aligned}
X(z) &= \frac{1}{1 - az^{-1}} \\
&= \frac{z}{z-a}
\end{aligned}
$$

5.3 Region of Convergence

Example: Find the z-transform, poles and zeros of the left-sided exponential

$$x[n] = -a^n u[-n-1] = \begin{cases} -a^n, & n \leq -1; \\ 0, & n > -1 \end{cases}$$

The z-transform of the sequence is given by

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} \\ &= \sum_{n=-\infty}^{-1} -a^n z^{-n} \\ &= -\sum_{m=1}^{\infty} (a^{-1}z)^m \\ &= -\left[\sum_{m=0}^{\infty} (a^{-1}z)^m - (a^{-1}z)^0 \right] \\ &= 1 - \sum_{m=0}^{\infty} (a^{-1}z)^m \end{aligned}$$

dummy variable

$m = -n$

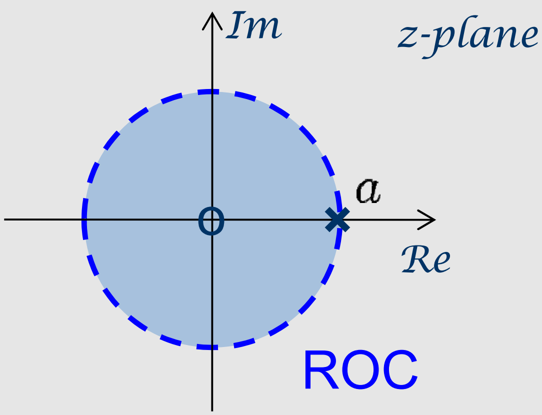
5.3 Region of Convergence

Therefore, for convergence,

$$|a^{-1}z| < 1 \Rightarrow |z| < |a|$$

and the series will converge to

$$\begin{aligned} X(z) &= 1 - \frac{1}{1 - a^{-1}z} \\ &= \frac{-a^{-1}z}{1 - a^{-1}z} \\ &= \frac{1}{1 - az^{-1}} \\ &= \frac{z}{z - a} \end{aligned}$$



Hence there is a zero at $z = 0$

there is a pole at $z = a$, assume a to be real positive.

Compared with the previous example, two sequences may have the same $X(z)$ but different ROC. Therefore, always specify the ROC when finding the z-transform.

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5.4 z-Transform Pairs

impulse $\delta[n] \overset{\mathcal{Z}}{\longleftrightarrow} 1$

exponential $a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \dfrac{1}{1-az^{-1}}, \quad |z| > |a|$

exponential $-a^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \dfrac{1}{1-az^{-1}}, \quad |z| < |a|$

truncated exponential $x[n] = \begin{cases} a^n, & 0 \le n \le N-1; \\ 0, & \text{otherwise} \end{cases} \overset{\mathcal{Z}}{\longleftrightarrow} \dfrac{1-a^N z^{-N}}{1-az^{-1}}, \quad |z| > 0$

5.5 Properties

Linearity	$ax_1[n] + bx_2[n]$	$\longleftrightarrow$	$aX_1(z) + bX_2(z),$	ROC :	$R_{x_1} \cap R_{x_2}$
Time-shift	$x[n - n_0]$	$\longleftrightarrow$	$z^{-n_0} X(z),$	ROC :	R_x
Multiplication with exponential	$z_0^n x[n]$	$\longleftrightarrow$	$X(z/z_0),$	ROC :	$\|z_0\| R_x$
Differentiation in z-domain	$nx[n]$	$\longleftrightarrow$	$-z\dfrac{dX(z)}{dz},$	ROC :	R_x
Conjugation	$x^*[n]$	$\longleftrightarrow$	$X^*(z^*),$	ROC :	R_x
Time-reversal	$x[-n]$	$\longleftrightarrow$	$X\left(\dfrac{1}{z}\right),$	ROC :	$\dfrac{1}{R_{x_1}}$
Convolution	$x_1[n] * x_2[n]$	$\longleftrightarrow$	$X_1(z)X_2(z),$	ROC :	$R_{x_1} \cap R_{x_2}$

A. Linearity

$$ax_1[n] + bx_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX_1(z) + bX_2(z), \qquad \text{ROC}: R_{x_1} \cap R_{x_2}$$

The z-transform of a signal that is composed of two signals is equivalent to the sum of their individual z-transforms.

B. Time-shift

$$x[n - n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} X(z), \qquad \text{ROC}: R_x$$

Proof: the z-transform is given by

$$\sum_{n=-\infty}^{\infty} x[n - n_0] z^{-n}$$

Substituting $m = n - n_0$, the z-transform becomes

$$\begin{aligned} \sum_{m=-\infty}^{\infty} x[m] z^{-m-n_0} &= z^{-n_0} \sum_{m=-\infty}^{\infty} x[m] z^{-m} \\ &= z^{-n_0} X(z) \end{aligned}$$

5.5 Properties

C. Multiplication by complex exponential

$$z_0^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z/z_0), \qquad \text{ROC} : |z_0| R_x$$

Proof: the z-transform is given by

$$\begin{aligned} \sum_{n=-\infty}^{\infty} z_0^n x[n] z^{-n} &= \sum_{n=-\infty}^{\infty} x[n](z/z_0)^{-n} \\ &= X(z/z_0) \end{aligned}$$

D. Differentiation in z

$$nx[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \qquad \text{ROC} : R_x$$

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad \Rightarrow \quad \begin{aligned} -z\frac{dX(z)}{dz} &= -z\frac{d}{dz}\sum_{n=-\infty}^{\infty} x[n] z^{-n} \\ &= -z\sum_{n=-\infty}^{\infty} x[n](-nz^{-n-1}) \\ &= \sum_{n=-\infty}^{\infty} nx[n] z^{-n} \end{aligned}$$

E. Time Reversal

$$x[-n] \longleftrightarrow X\left(\frac{1}{z}\right), \qquad \text{ROC}: \frac{1}{R_x}$$

Proof:

$$\begin{aligned} z - \text{transform of } x[-n] \quad &: \quad \sum_{n=-\infty}^{\infty} x[-n]z^{-n} \\ \text{Substitute } m = -n \quad &: \quad \sum_{m=\infty}^{-\infty} x[m]z^{m} \\ &= \quad \sum_{m=-\infty}^{\infty} x[m]\left(\frac{1}{z}\right)^{-m} \\ &= \quad X\left(\frac{1}{z}\right) \end{aligned}$$

F. Convolution

$$x_1[n] * x_2[n] \longleftrightarrow X_1(z)X_2(z), \qquad \mathrm{ROC}: R_{x_1} \cap R_{x_2}$$

Proof:

$$\begin{aligned}
x_1[n] * x_2[n] \quad &: \quad \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \\
z - \text{transform of } x_1[n] * x_2[n] \quad &: \quad \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \right\} z^{-n} \\
&= \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n} \\
\text{Substitute } m = n-k \quad &: \quad \sum_{k=-\infty}^{\infty} x_1[k] \left\{ \sum_{m=-\infty}^{\infty} x_2[m]z^{-m} \right\} z^{-k} \\
&= \sum_{k=-\infty}^{\infty} x_1[k]X_2(z)z^{-k} \\
&= X_2(z) \sum_{k=-\infty}^{\infty} x_1[k]z^{-k} \\
&= X_1(z)X_2(z)
\end{aligned}$$

Example: Find the z-transform of

$$x[n] = 0.5^{n-1}u[n] - u[n-3]$$

We first note that

$$0.5^{n-1}u[n] - u[n-3] = 0.5^{-1}\delta[n] + 0.5^{n-1}u[n-1] - u[n-3]$$

For the first term

$$2\delta[n] \longleftrightarrow 2, \qquad \text{ROC}: \text{ all } z$$

For the 2nd term

$$0.5^{n}u[n] \longleftrightarrow \frac{1}{1-0.5z^{-1}}, \qquad \text{ROC}: |z| > 0.5$$

$$0.5^{n-1}u[n-1] \longleftrightarrow z^{-1}\frac{1}{1-0.5z^{-1}}, \qquad \text{ROC}: |z| > 0.5$$

For the 3rd term

$$u[n] \longleftrightarrow \frac{1}{1-z^{-1}}, \qquad \text{ROC}: |z| > 1$$

$$u[n-3] \longleftrightarrow z^{-3}\frac{1}{1-z^{-1}}, \qquad \text{ROC}: |z| > 1$$

Combining the three terms, we have

$$0.5^{n-1}u[n] - u[n-3] = 0.5^{-1}\delta[n] + 0.5^{n-1}u[n-1] - u[n-3]$$

$$2 + \frac{z^{-1}}{1-0.5z^{-1}} - \frac{z^{-3}}{1-z^{-1}} = \frac{2-2z^{-1}-z^{-3}+0.5z^{-4}}{1-1.5z^{-1}+0.5z^{-2}}, \quad \text{ROC}: \ |z| > 1$$

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5.6 Inverse z-Transform

- We employ the method of partial fraction expansion to achieve inverse z-transform.

- Given $X(z) = \dfrac{P(z)}{Q(z)} = \dfrac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}$

  if the numerator order $M \geq$ denominator order N, then use long division to divide $P(z)$ by $Q(z)$.

- The above will result in $X(z)$ in the form of

$$X(z) = B_0 + B_1 z^{-1} + \ldots + B_{M-N} z^{-M+N} + \frac{c_0 + c_1 z^{-1} + \ldots + c_{N-1} z^{-N+1}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

  We then factorize $Q(z)$ by

$$a_0 + a_1 z^{-1} + \ldots + a_N z^{-N} = a_0 \prod_{k=1}^{N} (1 - d_k z^{-1})$$

$$X(z) \quad = \quad B_0 + B_1 z^{-1} + \ldots + B_{M-N} z^{-M+N} + \frac{c_0 + c_1 z^{-1} + \ldots + c_{N-1} z^{-N+1}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

where

$$a_0 + a_1 z^{-1} + \ldots + a_N z^{-N} = a_0 \prod_{k=1}^{N} (1 - d_k z^{-1})$$

- The last term of $X(z)$ can be expressed as

$$\frac{c_0 + c_1 z^{-1} + \ldots + c_{N-1} z^{-N+1}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{c_0 + c_1 z^{-1} + \ldots + c_{N-1} z^{-N+1}}{a_0 \prod_{k=1}^{N} (1 - d_k z^{-1})}$$

$$= \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

- The last objective is to find A_k using

$$A_k = (1 - d_k z^{-1}) W(z)|_{z=d_k}$$

Example: Find the inverse z-transform of

$$\frac{-3+12z^{-1}-7z^{-2}+2z^{-3}}{1-1.5z^{-1}+0.5z^{-2}}, \qquad 0.5<|z|<1$$

Since the order of numerator 3 is $\geq$ order of denominator 2, we use long division

$$\begin{array}{r|l}
 & \quad 4z^{-1} - 2 \\
\frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1 & \overline{\;2z^{-3}-7z^{-2}+12z^{-1}-3\;} \\
 & \;2z^{-3}-6z^{-2}+4z^{-1} \\
\hline
 & \qquad\quad -z^{-2}+8z^{-1}-3 \\
 & \qquad\quad \underline{-z^{-2}+3z^{-1}-2} \\
 & \qquad\qquad\qquad 5z^{-1}-1
\end{array}$$

The above implies that

$$\frac{-3+12z^{-1}-7z^{-2}+2z^{-3}}{1-1.5z^{-1}+0.5z^{-2}} = -2+4z^{-1}+\frac{-1+5z^{-1}}{1-1.5z^{-1}+0.5z^{-2}}$$

$$\frac{-3+12z^{-1}-7z^{-2}+2z^{-3}}{1-1.5z^{-1}+0.5z^{-2}} = -2+4z^{-1}+\frac{-1+5z^{-1}}{1-1.5z^{-1}+0.5z^{-2}}$$

Factorizing the denominator of the last term,

$$1-1.5z^{-1}+0.5z^{-2} = (1-0.5z^{-1})(1-z^{-1})$$

the partial fraction expansion of the last term can be expressed as

$$\frac{-1+5z^{-1}}{1-1.5z^{-1}+0.5z^{-2}} = \frac{-1+5z^{-1}}{(1-0.5z^{-1})(1-z^{-1})} = \frac{A_1}{1-0.5z^{-1}}+\frac{A_2}{1-z^{-1}}$$

from which

$$A_1 = \left[\frac{-1+5z^{-1}}{(1-0.5z^{-1})(1-z^{-1})}\times(1-0.5z^{-1})\right]_{z=0.5} = -9$$

$$A_2 = \left[\frac{-1+5z^{-1}}{(1-0.5z^{-1})(1-z^{-1})}\times(1-z^{-1})\right]_{z=1} = 8$$

Therefore

$$\frac{-1+5z^{-1}}{1-1.5z^{-1}+0.5z^{-2}} = \frac{-9}{1-0.5z^{-1}}+\frac{8}{1-z^{-1}}$$

5.6 Inverse z-Transform

$$\frac{-3+12z^{-1}-7z^{-2}+2z^{-3}}{1-1.5z^{-1}+0.5z^{-2}} = -2+4z^{-1}+\frac{-1+5z^{-1}}{1-1.5z^{-1}+0.5z^{-2}}$$
$$= -2+4z^{-1}+\frac{-9}{1-0.5z^{-1}}+\frac{8}{1-z^{-1}}$$

To obtain the inverse z-transform with ROC $0.5<|z|<1$,

$$-2 \longleftrightarrow -2\delta[n]$$
$$4z^{-1} \longleftrightarrow 4\delta[n-1]$$
$$\frac{-9}{1-0.5z^{-1}} \longleftrightarrow -9(0.5)^n u[n]$$
$$\frac{8}{1-z^{-1}} \longleftrightarrow -8\times 1^n u[-n-1]$$

Pole at 0.5, ROC is part of 0.5<|z| or exterior of circle. Therefore, right-sided sequence.

Pole at 1, ROC is part of 1>|z| or interior of circle. Therefore, left-sided sequence.

The inverse z-transform is finally given by

$$-2\delta[n]+4\delta[n-1]-9(0.5)^n u[n]-8u[-n-1]$$

5.6 Inverse z-Transform

- If the denominator $Q(z)$ has repeated factors, then partial fraction expansion is different. For example, if

$$Q(z) = a_0(1 - d_1 z^{-1})^s \prod_{k=2}^{N}(1 - d_k z^{-1})$$

then

$$W(z) = \sum_{m=1}^{s} \frac{C_m}{(1 - d_1 z^{-1})^m} + \sum_{k=2}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

- Coefficients A_k are found as before and C_m are found from

$$C_m = \frac{1}{(s-m)!(-d_1)^{s-m}} \left\{ \frac{d^{s-m}}{dx^{s-m}} \left[(1 - d_1 x)^s W(x^{-1}) \right] \right\}_{x=d_1^{-1}}$$

- Alternatively, the coefficients may be found by equating both sides of the partial fraction equation.

Example: Find the partial fraction expansion of

$$\frac{3z^{-1}}{(1-2z^{-1})^2}$$

Due to repeated factor $s = 2$, partial fraction expansion is

$$\begin{aligned} \frac{3z^{-1}}{(1-2z^{-1})^2} &= \frac{C_1}{1-2z^{-1}} + \frac{C_2}{(1-2z^{-1})^2} \\ &= \frac{(C_1+C_2) - 2C_1 z^{-1}}{(1-2z^{-1})^2} \end{aligned}$$

Equating both numerators, we have

$$\begin{aligned} C_1 + C_2 &= 0 \\ -2C_1 &= 3 \end{aligned}$$

from which we get $C_1 = -1.5,\ C_2 = 1.5$.

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5.7 System Function (Transfer Function)

- z-transform of an impulse response $\Longleftrightarrow$ system function of an LTI system

$$h[n] \longleftrightarrow H(z)$$

- Using the convolution property, the system function helps one to find the output

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

impulse response

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z) = X(z)H(z)$$

frequency response/
system function

5.7 System Function (Transfer Function)

- System function may also be obtained from the CCDE (see Section 3.6):

$$\begin{aligned} a_0 y[n] + a_1 y[n-1] &+ \ldots + a_N y[n-N] \\ &= b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M] \end{aligned}$$

- Noting that

$$y[n] \longleftrightarrow Y(z) \qquad\qquad y[n-1] \longleftrightarrow z^{-1}Y(z)$$

The CCDE in the z-domain becomes

$$\begin{aligned} a_0 Y(z) + a_1 z^{-1} Y(z) &+ \ldots + a_N z^{-N} Y(z) \\ &= b_0 X(z) + b_1 z^{-1} X(z) + \ldots + b_M z^{-M} X(z) \end{aligned}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}$$

5.7 System Function (Transfer Function)

- In the time domain, a system is stable if

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Since

$$\sum_{n=-\infty}^{\infty} h[n] \leq \sum_{n=-\infty}^{\infty} |h[n]|$$

We have

$$\sum_{n=-\infty}^{\infty} h[n] < \infty \quad \Rightarrow \quad \sum_{n=-\infty}^{\infty} \left(h[n]z^{-n}\right)_{z=1} < \infty$$

$$\Rightarrow \quad H(z)|_{z=1} < \infty$$

ROC includes the unit circle

- If the system is causal,

$$h[n] = 0 \text{ for } n < 0 \Rightarrow h[n] \text{ right} - \text{sided}$$

ROC is the exterior of a circle

5.7 System Function (Transfer Function)

We have for stability

$$\sum_{n=-\infty}^{\infty} h[n] < \infty \quad \Rightarrow \quad \sum_{n=-\infty}^{\infty} \left(h[n] z^{-n}\right)_{z=1}$$

$$\Rightarrow \quad H(z)|_{z=1} < \infty$$

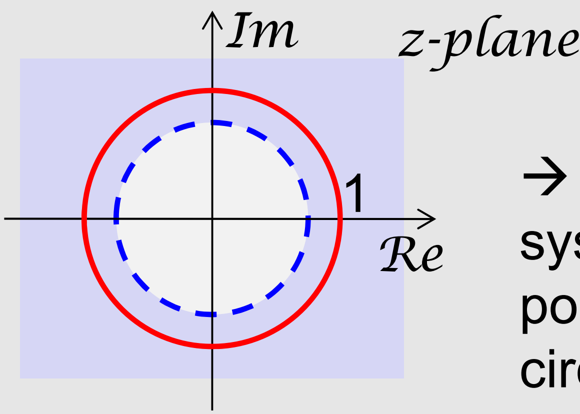
ROC includes the unit circle

If a system is causal,

$$h[n] = 0 \text{ for } n < 0 \Rightarrow h[n] \text{ right} - \text{sided}$$

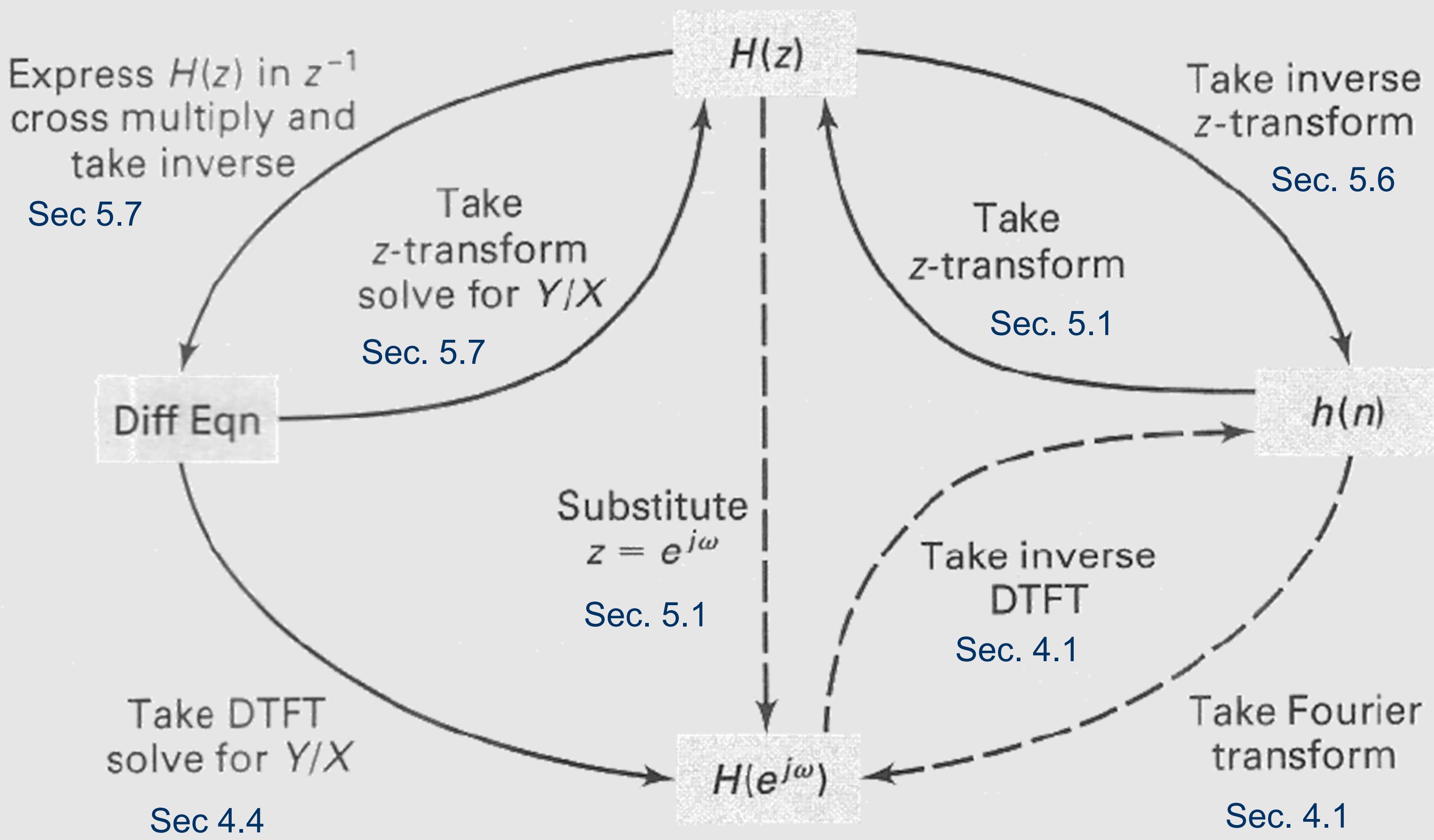
ROC is the exterior of a circle

Example: ROC of a causal stable system function



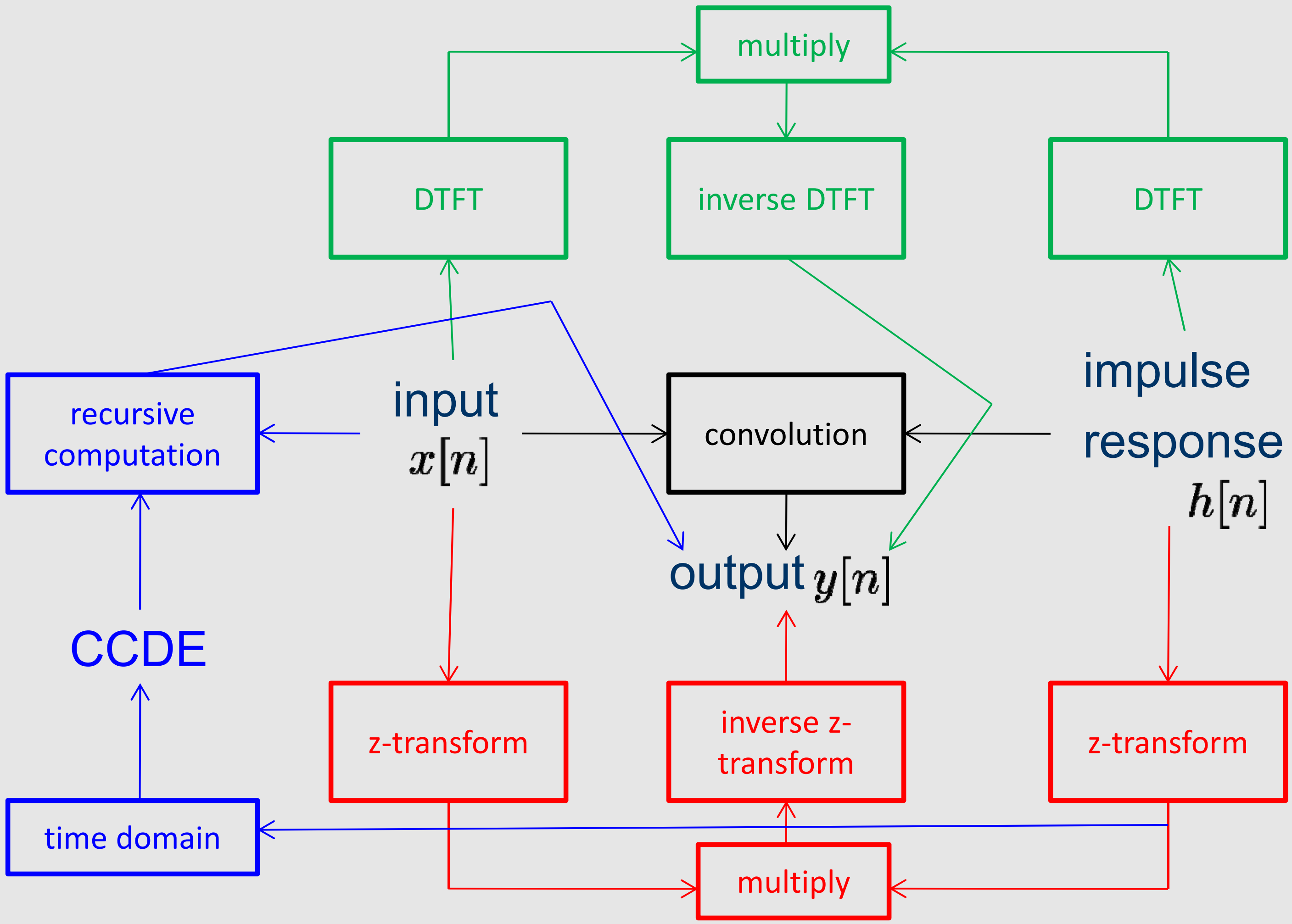
→ A causal stable system should have all poles inside the unit circle

5.7 System Function (Transfer Function)



5.7 System Function (Transfer Function)

Equivalent ways to compute an output



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5.7 System Function (Transfer Function)

Example: Find the system function, CCDE and frequency response of a system with impulse response $h[n] = [1 \;\; \underset{\uparrow}{2} \;\; 1]$.

To find the system function which is in the z-domain, we use

$$\begin{aligned} H(z) &= \sum_{n=-1}^{1} h[n]z^{-n} \\ &= z + 2 + z^{-1}, \qquad \text{ROC}: \text{ all } z \text{ except } 0, \infty \end{aligned}$$

Therefore, since $H(z) = Y(z)/X(z)$,

$$\frac{Y(z)}{X(z)} = z + 2 + z^{-1} \qquad \Rightarrow \qquad Y(z) = zX(z) + 2X(z) + z^{-1}X(z)$$

Bringing the above to the time domain, we have the CCD

$$y[n] = x[n+1] + 2x[n] + x[n-1]$$

To obtain the frequency response of the $h[n]$, we replace $z = e^{j\omega}$ in $H(z)$

$$\begin{aligned} H(e^{j\omega}) &= e^{j\omega} + 2 + e^{-j\omega} \\ &= 2(1 + \cos\omega) \end{aligned}$$

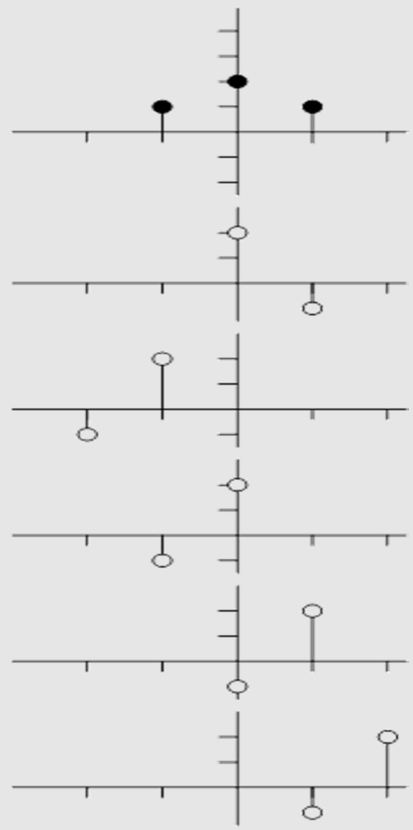
5.7 System Function (Transfer Function)

Example: Find the system $h[n] = [1 \quad \underset{\uparrow}{2} \quad 1]$, find the output using convolution, z-transform and DTFT for an input $x[n] = [2 \quad -1]$.

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

impulse response

Convolution using "flip-and-shift" method



$$h[n] = [1 \quad \underset{\uparrow}{2} \quad 1]$$

$$x[n] = [2 \quad -1]$$

$$y[-1] = 2(1) = 2$$

$$y[0] = -1(1) + 2(2) = 3$$

$$y[1] = -1(2) + 2(1) = 0$$

$$y[2] = -1(1) = -1$$

5.7 System Function (Transfer Function)

b) To compute the output using z-transform,

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z) = X(z)H(z)$$

frequency response/
system function

$$\begin{aligned} x[n] &= [2 \quad -1] & X(z) &\Rightarrow 2 - z^{-1} \\ h[n] &= [1 \quad \underset{\uparrow}{2} \quad 1] & H(z) &\Rightarrow z + 2 + z^{-1} \end{aligned}$$

Therefore the output is given by

$$\begin{aligned} Y(z) &= X(z)H(z) \\ &= (2 - z^{-1})(z + 2 + z^{-1}) \\ &= 2z + 3 - z^{-2} \end{aligned}$$

and hence

$$y[n] = [2 \quad \underset{\uparrow}{3} \quad 0 \quad -1]$$

5.7 System Function (Transfer Function)

c) To compute the output using DTFT,

$$X(e^{j\omega}) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

frequency response/
system function

$$\begin{aligned} x[n] &= [2 \quad -1] & \qquad X(e^{j\omega}) &\Rightarrow 2 - e^{-j\omega} \\ h[n] &= [1 \quad \underset{\uparrow}{2} \quad 1] & \qquad H(e^{j\omega}) &\Rightarrow e^{j\omega} + 2 + e^{-j\omega} \end{aligned}$$

Therefore the output is given by

$$\begin{aligned} Y(e^{j\omega}) &= X(e^{j\omega})H(e^{j\omega}) \\ &= (2 - e^{-j\omega})(e^{j\omega} + 2 + e^{-j\omega}) \\ &= 2e^{j\omega} + 3 - e^{-2j\omega} \end{aligned}$$

and hence

$$y[n] = [2 \quad \underset{\uparrow}{3} \quad 0 \quad -1]$$

5.7 System Function (Transfer Function)

Example: Find the impulse response, CCDE, frequency response, pole zero plot, stability and causality of a system with system function

$$H(z) = \frac{0.4}{1 - 0.9z^{-1}}, \qquad |z| > 0.9$$

a) To find the impulse response, we take the inverse z-transform giving

$$h[n] = 0.4(0.9)^n u[n]$$

b) To find the CCDE, we note that

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.4}{1 - 0.9z^{-1}}$$

$$Y(z) - 0.9z^{-1}Y(z) = 0.4X(z)$$

taking the inverse z-transform, we obtain the CCDE

$$y[n] = 0.4x[n] + 0.9y[n-1]$$

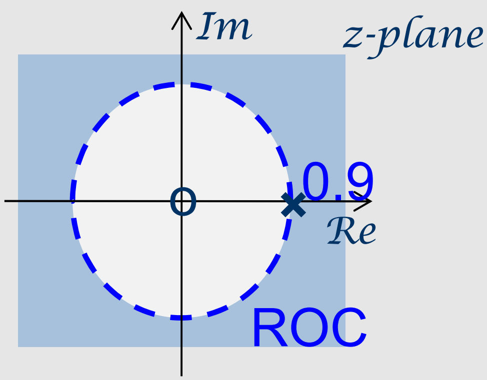
5.7 System Function (Transfer Function)

c) To find the frequency response, we simply replace z by $e^{j\omega}$

$$H(e^{j\omega}) = \frac{0.4}{1 - 0.9e^{-j\omega}}$$

d) To determine the pole and zeros, we refer to the z-transform

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.4}{1 - 0.9z^{-1}} = \frac{0.4z}{z - 0.9}$$



therefore, there is a pole at $z = 0.9$

and zero at $z = 0.$

c) For stability, unit circle $z = 1$ is included in the ROC → stable

$$h[n] = 0.4(0.9)^n u[n]$$

d) To check for causality: ROC is exterior of a circle → right-sided→ may be causal. Check with impulse response implies that $h[n] = 0$ for all $n < 0$→ causal.

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5.7 System Function (Transfer Function)

Example: For the system $h[n] = 0.4(0.9)^n u[n]$ above, find the output using convolution, z-transform, and CCDE for an input $x[n] = (0.5)^n u[n]$.

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

impulse response

a) To find the output using convolution, we note that we cannot use flip-and-shift method because of infinite sequences. Therefore

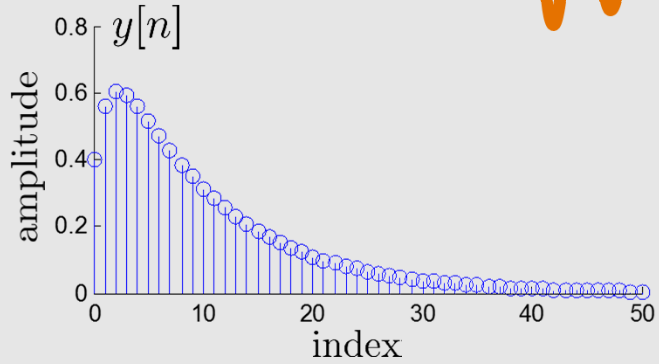
$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \quad = \quad \sum_{k=0}^{\infty} 0.4(0.9)^k x[n-k]$$

We note that for $k > n,\ n-k < 0$ hence $x[n-k] = 0$ giving

$$\begin{aligned}
y[n] = \sum_{k=0}^{n} 0.4(0.9)^k x[n-k] \quad &= \quad \sum_{k=0}^{n} 0.4(0.9)^k (0.5)^{n-k} \\
&= \quad 0.4(0.5)^n \sum_{k=0}^{n} (0.9/0.5)^k \\
&= \quad 0.4(0.5)^n \frac{1.8^{n+1} - 1}{1.8 - 1} \\
&= \quad 0.9^{n+1} - 0.5^{n+1}, \qquad n \geq 0
\end{aligned}$$

5.7 System Function (Transfer Function)

$$
\begin{aligned}
y[n] &= \left(0.9^{n+1} - 0.5^{n+1}\right) u[n] \\
&= [0 \;\; \underset{\uparrow}{0.4} \;\; 0.56 \;\; 0.604 \;\; 0.5936 \ldots]
\end{aligned}
$$



b) To find the output using z-transform,

$$
x[n] = (0.5)^n u[n] \qquad \Rightarrow \qquad X(z) = \frac{1}{1 - 0.5z^{-1}}, \quad |z| > 0.5
$$

therefore,

$$
\begin{aligned}
Y(z) &= X(z)H(z) \qquad\qquad \text{ROC}: R_x \cap R_h \Rightarrow |z| > 0.9 \\
&= \frac{0.4}{(1 - 0.5z^{-1})(1 - 0.9z^{-1})} \\
&= \frac{-0.5}{1 - 0.5z^{-1}} + \frac{0.9}{1 - 0.9z^{-1}}
\end{aligned}
$$

hence giving

$$
y[n] = -0.5(0.5)^n u[n] + 0.9(0.9)^n u[n]
$$

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5.7 System Function (Transfer Function)

c) To find the output using CCDE, we note that the CCDE is given (in the previous example) by

$$y[n] = 0.4x[n] + 0.9y[n-1]$$

Given that

$$x[n] = [0 \quad 1 \quad 0.5 \quad 0.25 \quad 0.125\ldots]$$

Substituting $x[n]$ into $y[n]$, and noting that $y[-1] = 0$ we obtain

$$\begin{aligned} y[0] &= 0.4x[0] + 0.9y[-1] &= 0.4 \\ y[1] &= 0.4x[1] + 0.9y[0] &= 0.56 \\ y[2] &= 0.4x[2] + 0.9y[1] &= 0.604 \\ y[3] &= 0.4x[3] + 0.9y[2] &= 0.5936 \end{aligned}$$

5.8 Summary

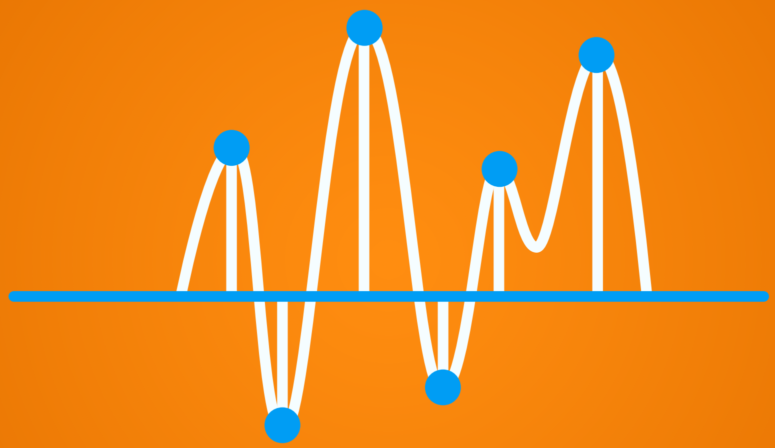
- The z-transform is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- The variable $X(z)$ is complex and can be plotted on a complex plane.
- The poles and zeros of a system is determined by the numerator and denominator of the system transfer function.
- The ROC defines all possible values of z such that $X(z)$ converges.
- The inverse z transform is obtained using the transform pair table.
- The system transfer function of a system is defined as the z transform of the impulse response.

Chapter 6

The Discrete Fourier Transform



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Chapter Aims

At the end of this chapter, you will be expected to:

1. understand the formulation of the DFT
2. highlight the similarity/difference between DTFT and DFT
3. establish the relationship between DFT, DTFT and z-transform
4. understand the properties of DFT

6.1 Definition

- The discrete Fourier transform (DFT) is one of the most important transform in DSP.

- Similar to the DTFT, the DFT allows one to know the *frequency components* of a given discrete signal.

- The DFT is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, \qquad 0 \le k \le N-1$$

- The inverse DFT (IDFT) is defined as

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}, \qquad 0 \le n \le N-1$$

6.1 Definition

- The DFT is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}, \qquad 0 \le k \le N-1$$

Frequency-domain discrete signal

Time-domain discrete signal

Length of time-domain signal

- Note that DFT coefficients $X[k]$ are complex in general
  - $|X[k]|$ gives the magnitude spectrum
  - $\angle X[k]$ gives the phase spectrum

- The variable k is an integer and known as the frequency-bin index.
  - ✓ A larger value of k corresponds to higher frequency.
  - ✓ Note that k is *not* frequency (Hz) but it is an index which is unit-less.

- The number of frequency bins corresponds to the number of samples N. Therefore, this is sometimes known as *N*-point DFT.

Example: Find the DFT of $x[n] = [2 \;\; 3 \;\; -3 \;\; -2]$.

To find the DFT, we note that $N = 4$ and

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{3} x[n] e^{-j2\pi kn/4} \\
&= 2 + 3e^{-j\pi k/2} - 3e^{-j\pi k} - 2e^{-j3\pi k/2} \\
&= e^{-j3\pi k/4}\left(2e^{j3\pi k/4} + 3e^{j\pi k/4} - 3e^{-j\pi k/4} - 2e^{-j3\pi k/4}\right) \\
&= e^{-j3\pi k/4}\left(4j\sin\frac{3\pi k}{4} + 6j\sin\frac{\pi k}{4}\right)
\end{aligned}
$$

$$X[k] = e^{-j3\pi k/4}\left(4j\sin\frac{3\pi k}{4} + 6j\sin\frac{\pi k}{4}\right)$$

We can now evaluate the frequency content for each frequency bin index

$$X[0] = \quad 1\,(4j\sin 0 + 6j\sin 0) \quad = 0$$

$$X[1] = \quad e^{-j3\pi/4}\,(4j\sin 3\pi/4 + 6j\sin \pi/4) \quad = 5\sqrt{2}e^{-j\pi/4}$$

$$X[2] = \quad e^{-j3\pi/2}\,(4j\sin 3\pi/2 + 6j\sin \pi/2) \quad = -2$$

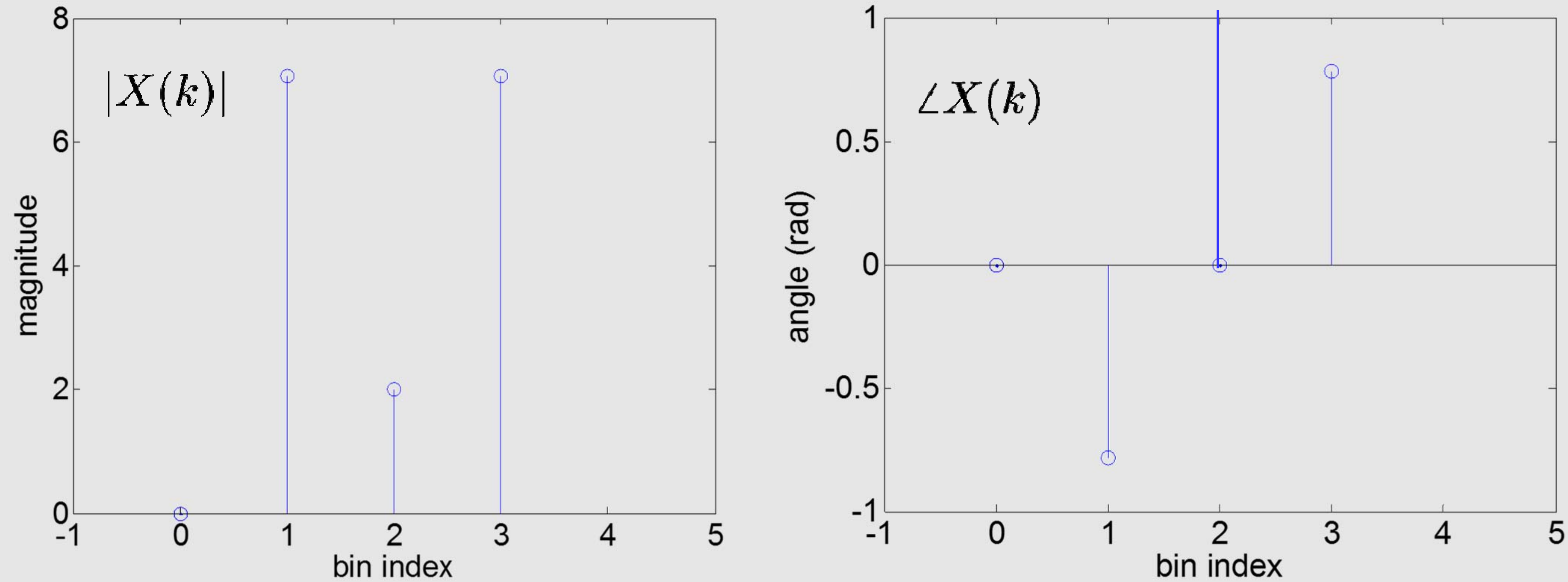
$$X[3] = \quad e^{-j9\pi/4}\,(4j\sin 9\pi/4 + 6j\sin 3\pi/4) \quad = 5\sqrt{2}e^{j\pi/4}$$

Note that there are 4 samples in $x[n]$ and $X[k]$.

$$X[k] = [0 \quad 5\sqrt{2}e^{-j\pi/4} \quad -2 \quad 5\sqrt{2}e^{j\pi/4}]$$

$$|X[k]| = [0 \quad 5\sqrt{2} \quad 2 \quad 5\sqrt{2}]$$

$$\angle X[k] = [0 \quad -\pi/4 \quad \pi \quad \pi/4]$$



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6.2 DFT, DTFT and the z-Transform

A) The DFT and DTFT

- The DTFT is defined as

continuous frequency

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fn}$$

continuous
(freq. domain)

discrete
(time domain)

- The DFT is defined as

discrete frequency

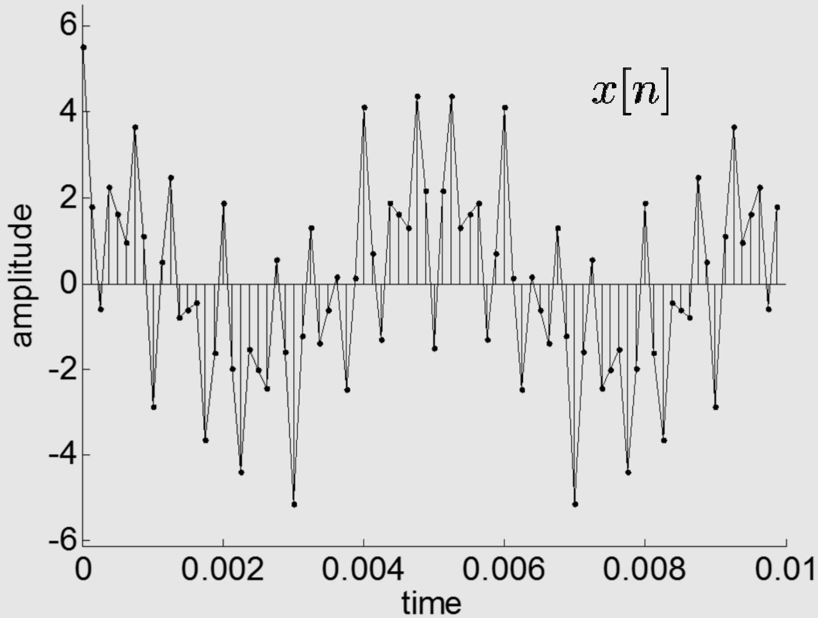
$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}, \qquad 0 \leq k \leq N-1$$

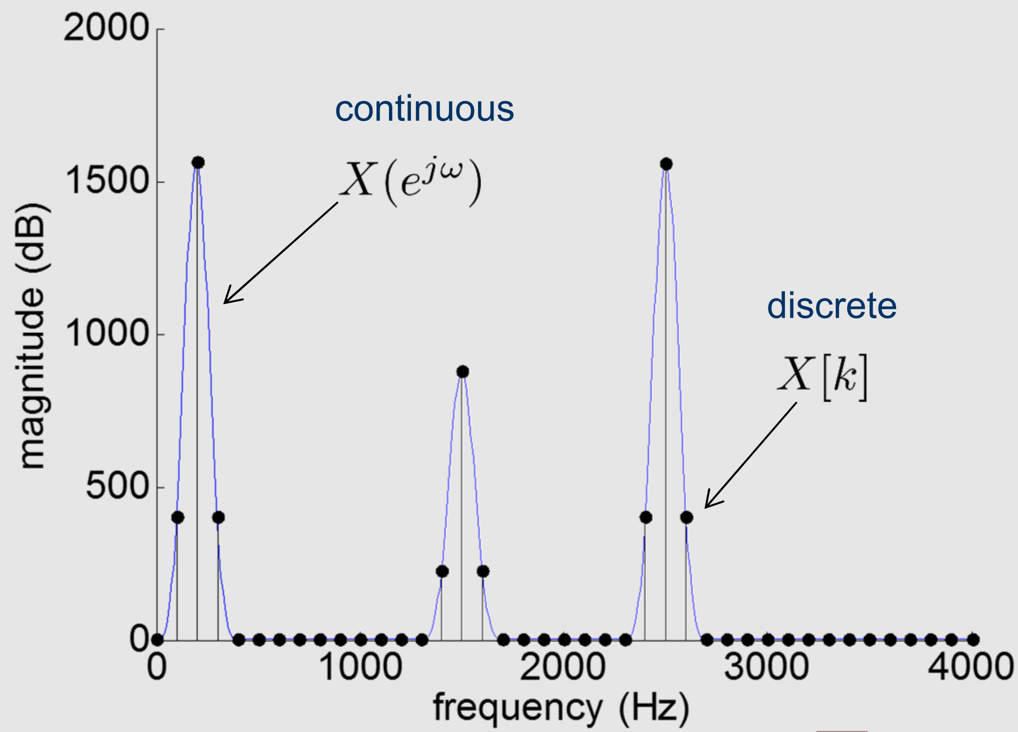
discrete
(freq. domain)

discrete
(time domain)

- The DFT corresponds to samples of the DTFT.

6.2 DFT, DTFT and the z-Transform





6.2 DFT, DTFT and the z-Transform

B) The DFT and the z-transform

- The z-transform is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- The DFT is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, \qquad 0 \leq k \leq N-1$$

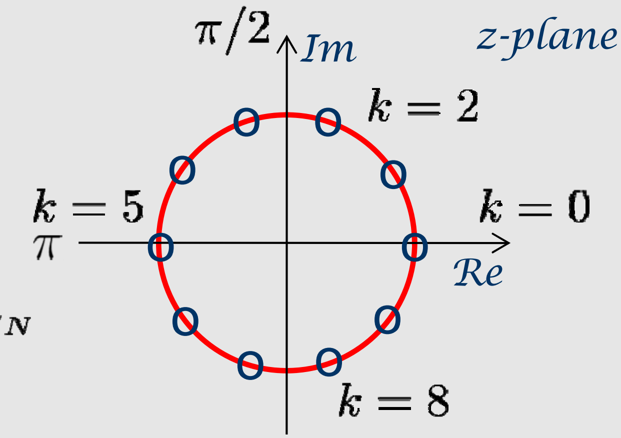
- We note that

$$X[k] \rightarrow X(z)\big|_{z=e^{j2\pi k/N}}$$

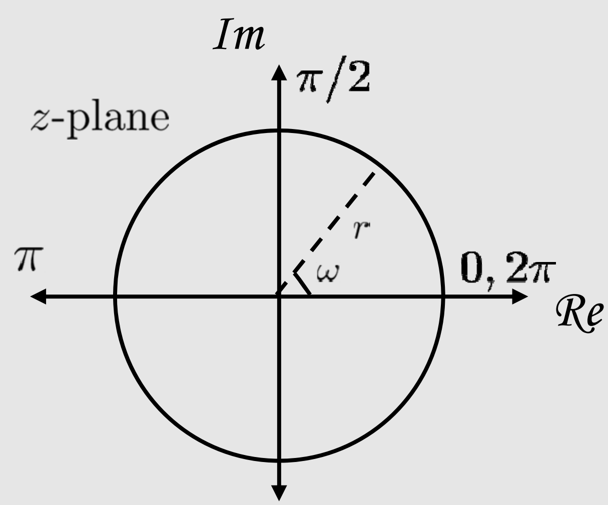
- This implies that the DFT corresponds to sampling N points uniformly around the unit circle.

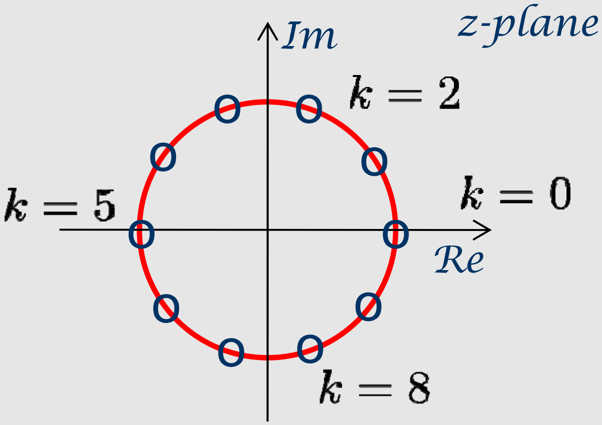
- For an example case of $N = 10$, we have $k = 0, 1, \ldots, 9$

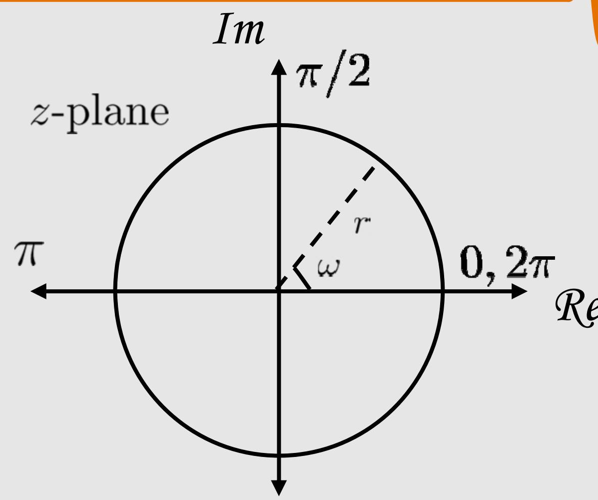
$$X[k] \rightarrow X(z)|_{z=e^{j2\pi k/N}}$$



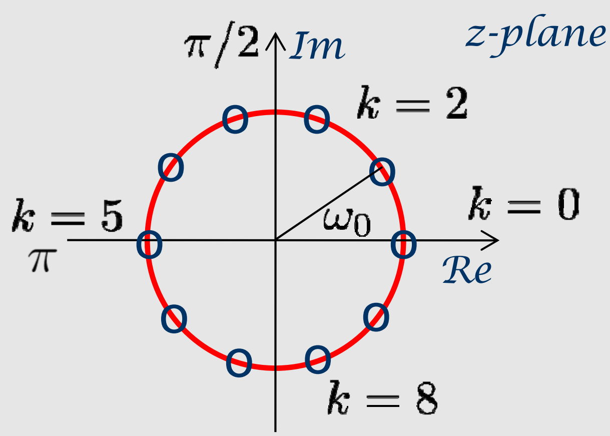
- From Section 5.1, we also know that



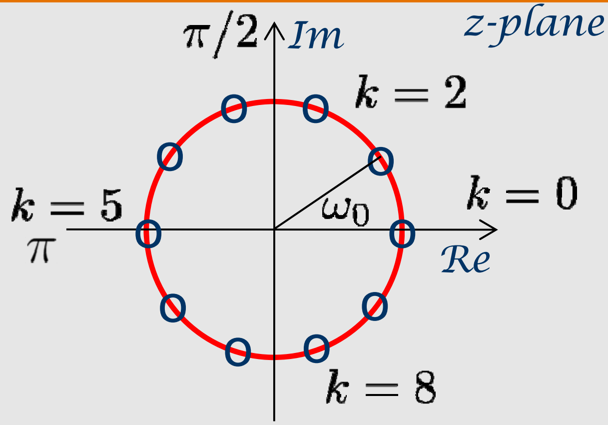




- Therefore when we sample the unit circle using N discrete points, we have discrete frequencies ω_0 around the unit circle.



6.2 DFT, DTFT and the z-Transform



- When $\omega_0 = \pi$, we have

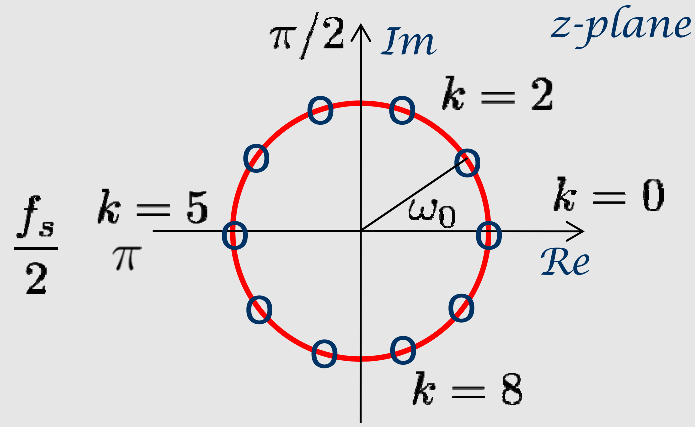
see Section 2.4

$$\begin{cases} \omega_0 = \pi \\ \dfrac{2\pi f}{f_s} = \pi \end{cases}$$

$$f = \frac{f_s}{2}$$

- The above implies that the frequency corresponds to half the sampling rate when $\omega_0 = \pi$.

6.2 DFT, DTFT and the z-Transform



- The above also implies that it is possible to convert frequency-bin index k to frequency f in Hz. This is achieved by

$$f = \frac{k}{N} \times f_s$$

- Therefore in the example above

$$N = 10, f_s = 1000$$

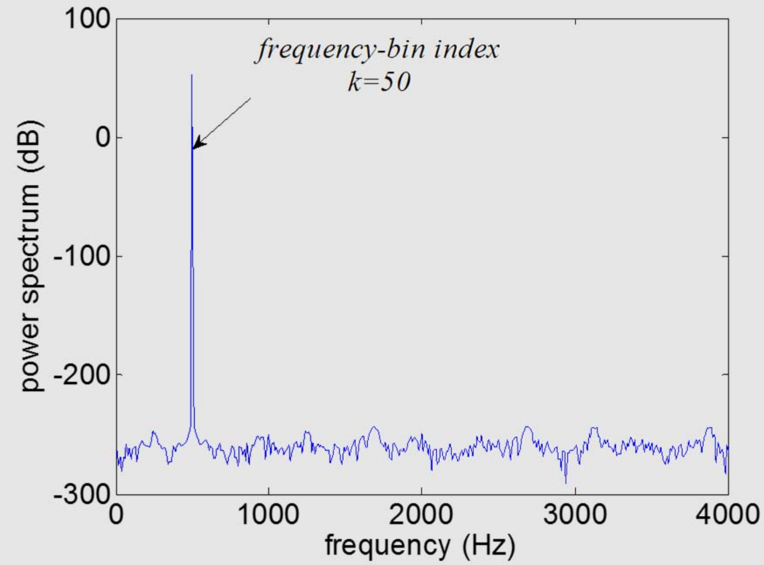
k	f (Hz)
0	0
1	100
5	500

6.2 DFT, DTFT and the z-Transform

Example: Consider an analog signal of $x(t) = \sin(1000\pi t)$. What is the frequency-bin index that will produce the highest energy if $N = 800$-point DFT is taken at a sampling rate of $f_s = 8000$?

We note that the analog signal has frequency of $f = 500\ \mathrm{Hz}$. Since this is a single-tone signal, we would expect that the peak of the magnitude spectrum corresponds to $f = 500\ \mathrm{Hz}$. The corresponding frequency-bin index would be

$$k = \frac{f}{f_s} \times N = 50$$



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6.3 Computation of DFT in matrix form

- Computation of the DFT can be done in matrix form.
- We first define $W_N = e^{-j2\pi/N}$, the DFT can be computed using

$$\underbrace{\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}}_{\mathbf{X}[k]} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}}_{\mathbf{D}_N} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\mathbf{x}[n]}$$

- Therefore the DFT of a time-domain signal is

$$\mathbf{X}[k] = \mathbf{D}_N \mathbf{x}[n]$$

6.3 Computation of DFT in matrix form

- The time-domain signal can be determined from the frequency-domain variable $\mathbf{X}[k]$ using

$$\mathbf{x}[n] = \mathbf{D}_N^{-1}\mathbf{X}[k]$$

where

$$\mathbf{D}_N^{-1} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)(N-1)} \end{bmatrix}$$

- Do note that

$$\mathbf{D}_N^{-1} = \frac{1}{N}\mathbf{D}_N^{*}$$

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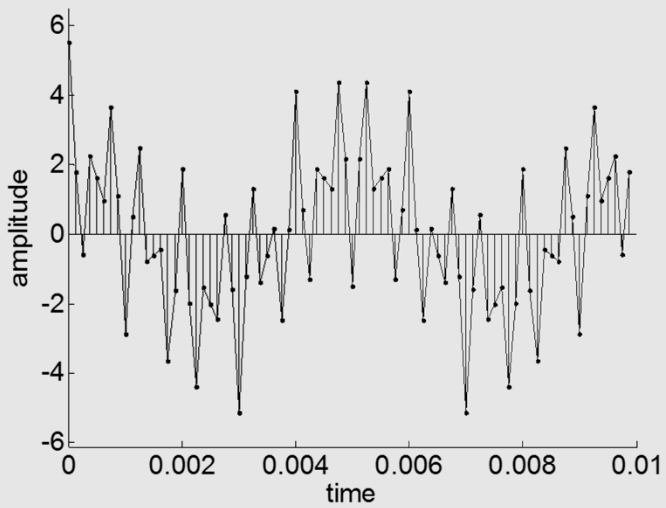
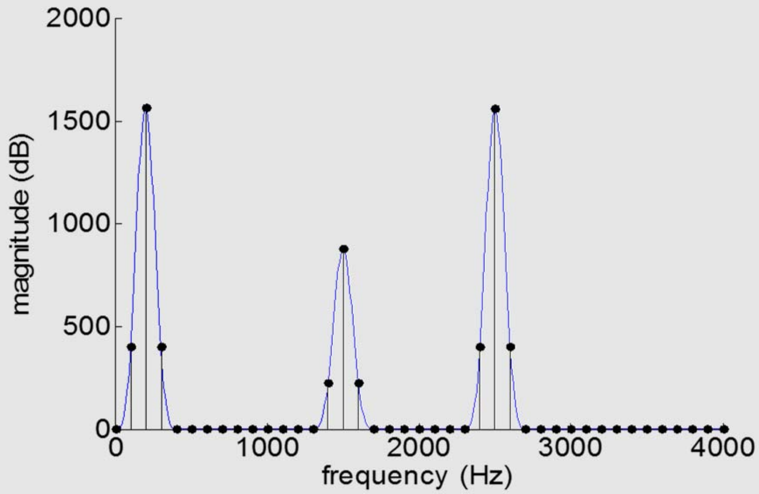
A) Linearity

For two equal length signals

$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1[k] + bX_2[k]$$

The above implies that the DFT of a sequence that is made up of two sequences $ax_1[n]$ and $bx_2[n]$, the resulting DFT will be the sum of the individual DFTs.





B) Circular Shift

We first define $((n))_N$ as $(n \text{ modulo } N)$ such that

$$\tilde{x}[n] = x[((n))_N]$$

denotes the periodic extension of $x[n]$. For example, if $N = 6$:

$$x[n] = [1 \quad 2 \quad 3 \quad 5 \quad 6 \quad 8]$$

$$\tilde{x}[n] = [\ldots 6 \quad 8 \quad \boxed{1 \quad 2 \quad 3 \quad 5 \quad 6 \quad 8} \quad 1 \quad 2 \quad 3 \quad 5 \quad 6 \quad 8 \quad 1 \quad 2 \ldots]$$

$$\tilde{x}[n-2] = [\ldots 3 \quad 5 \quad \boxed{6 \quad 8 \quad 1 \quad 2 \quad 3 \quad 5} \quad 6 \quad 8 \quad 1 \quad 2 \quad 3 \quad 5 \quad 6 \quad 8 \ldots]$$

$$\| $$

$$x[((n-2))_6]$$

The above implies that if a periodic sequence is shifted, it results in a circular shift and, similar to the DTFT properties (Section 4.2),

$$x[((n-m))_N] \longleftrightarrow e^{-j2\pi km/N} X[k]$$

C) Frequency Shift

$$e^{j2\pi nl/N}x[n] \longleftrightarrow X[((k-l))_N]$$

D) Duality

$$X[n] \longleftrightarrow Nx[((-k))_N]$$

E) Conjugation

$$x^*[n] \longleftrightarrow X^*[((-k))_N]$$

F) Time-reversal

$$x^*[((-n))_N] \longleftrightarrow X^*[k]$$

G) Real signals

$$x[n] \longleftrightarrow X[k] = X^*[((-k))_N]$$

H) Circular Convolution

The circular convolution is defined as

$$x_3[n] = x_1[n] \circledN x_2[n] = \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N], \qquad 0 \le n \le N-1$$

Example: The circular convolution between $x[n] = [\underset{\uparrow}{2} \ -1 \ 1 \ 2]$ and $h[n] = [\underset{\uparrow}{5} \ 2 \ 1 \ 3]$ is given by

				$\underset{\uparrow}{5}$	2	1	3	
Flip one signal similar to linear convolution	2	1	-1	$\underset{\uparrow}{2}$				
Shift the whole signal to the right				2	1	-1	$\underset{\uparrow}{2}$	
Align the time instances by rotating the elements, perform multiply-and-sum				$\underset{\uparrow}{2}$	2	1	-1	$y[0] = 12$
				-1	$\underset{\uparrow}{2}$	2	1	$y[1] = 4$
				1	-1	$\underset{\uparrow}{2}$	2	$y[2] = 11$
				2	1	-1	$\underset{\uparrow}{2}$	$y[3] = 17$

It is important to note that

$$x_1[n] \text{Ⓝ} x_2[n] \longleftrightarrow X_1[k]X_2[k]$$

$$x_1[n]x_2[n] \longleftrightarrow \frac{1}{N} X_1[k] \text{Ⓝ} X_2[k]$$

The first relationship implies that when two DFTs are multiplied together in the frequency domain, the resultant DFT corresponds to the DFT of the circular convolved signals.

Example: The DFT of $\cos(2\pi rn/N),\ 0 \leq n \leq N-1$ is given by first expressing

$$\cos(2\pi rn/N) = 0.5e^{j2\pi rn/N} + 0.5e^{-j2\pi rn/N}$$

and therefore

Using definition of DFT $$1 \longleftrightarrow \sum_{n=0}^{N-1} e^{-j2\pi kn/N} = \begin{cases} N, & k=0; \\ 0, & k \neq 0 \end{cases}$$

$$0.5 \longleftrightarrow 0.5 \sum_{n=0}^{N-1} e^{-j2\pi kn/N} = \begin{cases} N/2, & k=0; \\ 0, & k \neq 0 \end{cases}$$

$$1 \longleftrightarrow \sum_{n=0}^{N-1} e^{-j2\pi kn/N} = \begin{cases} N, & k=0; \\ 0, & k \neq 0 \end{cases}$$

$$0.5 \longleftrightarrow 0.5 \sum_{n=0}^{N-1} e^{-j2\pi kn/N} = \begin{cases} N/2, & k=0; \\ 0, & k \neq 0 \end{cases}$$

$$e^{j2\pi nr/N} x[n] \longleftrightarrow X[((k-r))_N]$$

$$0.5e^{j2\pi rn/N} + 0.5e^{-j2\pi rn/N} \longleftrightarrow \begin{cases} N/2, & k=r; \\ N/2, & k=N-r \\ 0, & \text{otherwise} \end{cases}$$

6.5 Summary

- The DFT is defined as

$$X[k] \;=\; \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}, \qquad 0 \leq k \leq N-1$$

- The DFT is in general complex.

- The DFT is a discretized version of the DTFT.

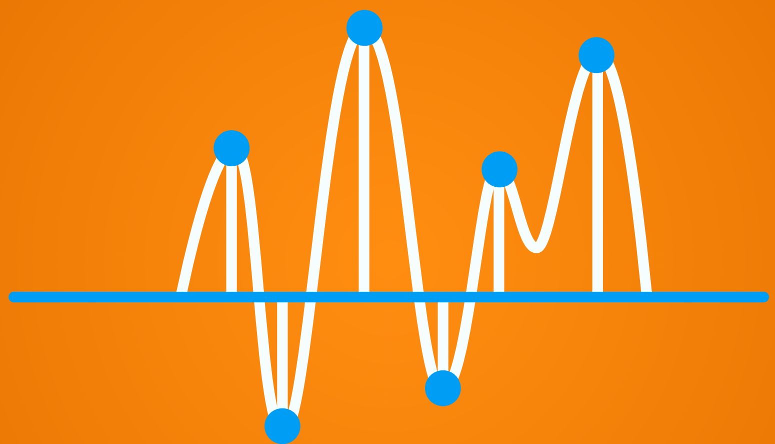
- The variable k is known as the frequency-bin index. It is not frequency in Hertz.

- The relationship between frequency (in Hertz) and frequency-bin index is given by

$$f = \frac{k}{N} \times f_s$$

Chapter 7

The fast-Fourier Transform



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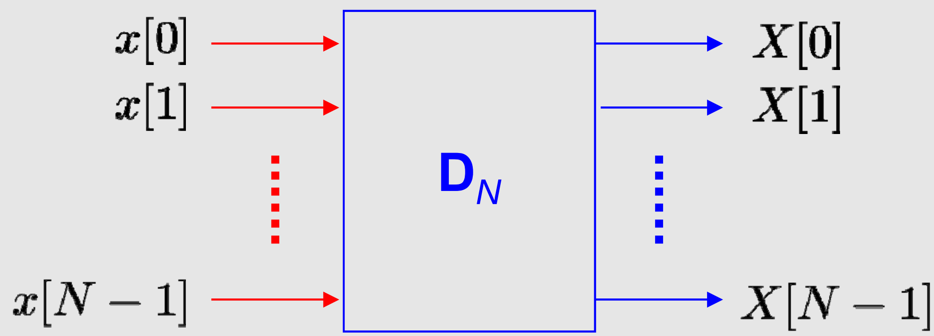
7.1 Motivation

- The DFT is used in a large number of applications such as MP3, MRI, Radar.

- Direct computation of the DFT is given, in Section 6.3, by

$$\mathbf{X}[k] = \mathbf{D}_N \mathbf{x}[n]$$

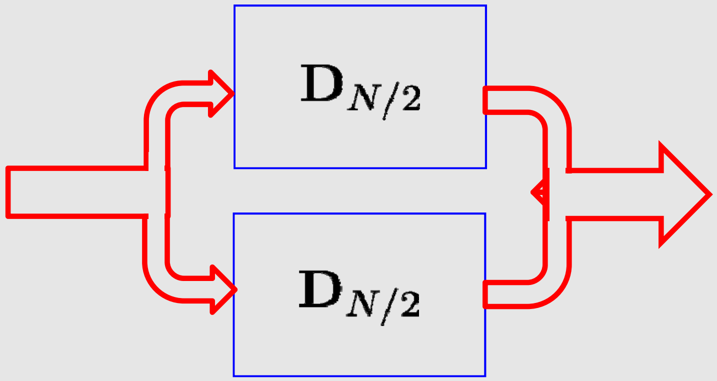


- Since $\mathbf{D}_N$ is an $N \times N$ matrix, direct implementation of the above requires N^2 complex multiplications and N^2 complex additions.

- The fast-Fourier transform (FFT) is a family of more efficient computations of DFT that requires less computation than the above.

- One simple way to reduce computation is to employ the divide-and-conquer approach.

- If N-point DFT may be computed using two $N/2$-point DFTs, then it $2(N/2)^2 = N^2/2$ requires multiplications/additions.

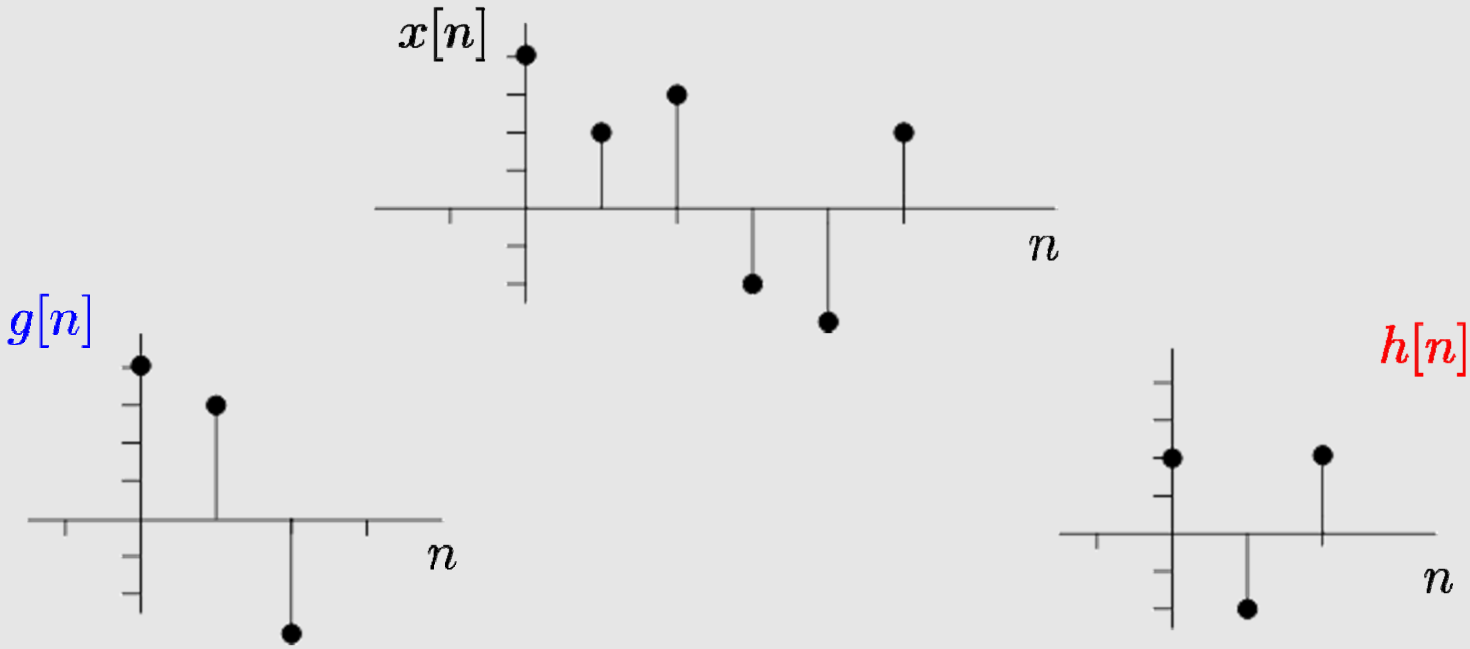


- In reality, as we shall see later, some more computations are required.

- The above process can be sub-divided further so that each $N/2$-point DFT can be computed using two $N/4$-point DFT, and so forth.

7.2 Decimation-in-time (DIT) FFT

- The decimation-in-time FFT begins by separating $x[n]$ into two $N/2$-point sequences:



$$g[n] = x[2n], \quad 0 \leq n \leq (N/2) - 1$$

$$h[n] = x[2n+1], \quad 0 \leq n \leq (N/2) - 1$$

7.2 Decimation-in-time (DIT) FFT

$$g[n] = x[2n], \quad 0 \le n \le (N/2) - 1$$

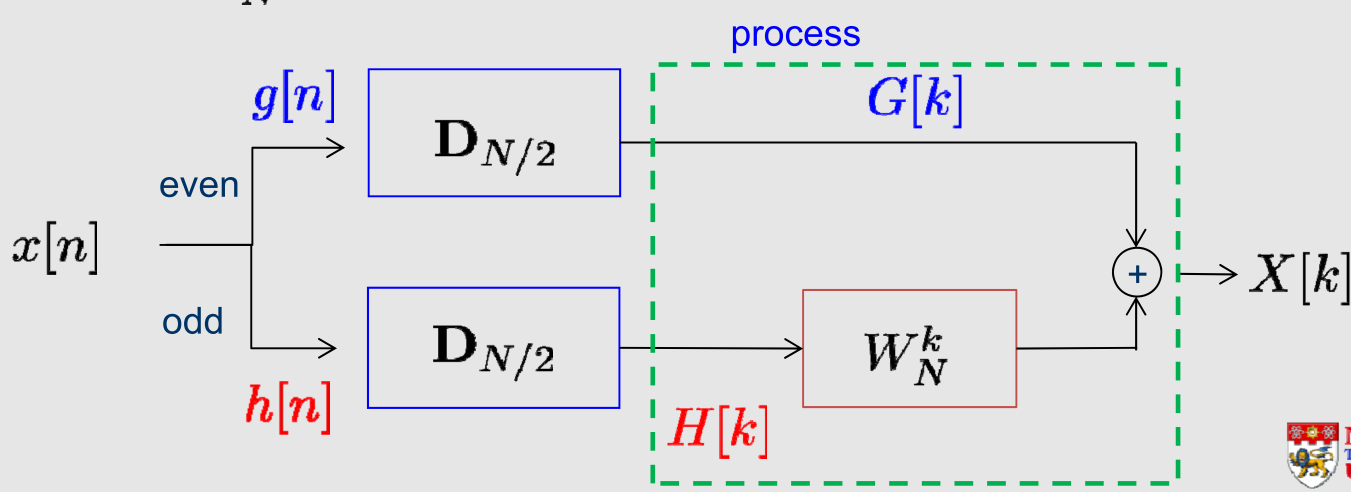
$$h[n] = x[2n+1], \quad 0 \le n \le (N/2) - 1$$

- Let $G[k]$ be the $N/2$-point DFT of $g[n]$ and assuming $G[k]$ to be periodic
  $H[k]$ be the $N/2$-point DFT of $h[n]$ and assuming $H[k]$ to be periodic

- The DFT of $x[n]$ is then given by

$$X[k] = G[k] + W_N^k H[k], \qquad 0 \le k \le N-1$$

$$W_N^k = e^{-j2\pi k/N}$$



7.2 Decimation-in-time (DIT) FFT

- Proof:

$$\begin{aligned}
G[k] + W_N^k H[k] &= \sum_{n=0}^{N/2-1} g[n] W_{N/2}^{nk} + W_N^k \sum_{n=0}^{N/2-1} h[n] W_{N/2}^{nk} \\
&= \sum_{n=0}^{N/2-1} x[2n] W_{N/2}^{nk} + \sum_{n=0}^{N/2-1} x[2n+1] W_N^k W_{N/2}^{nk}
\end{aligned}$$

We can show that

$$W_{N/2}^{nk} = e^{-j2\pi nk/(N/2)} = e^{-j2\pi(2n)k/N} = W_N^{2nk}$$

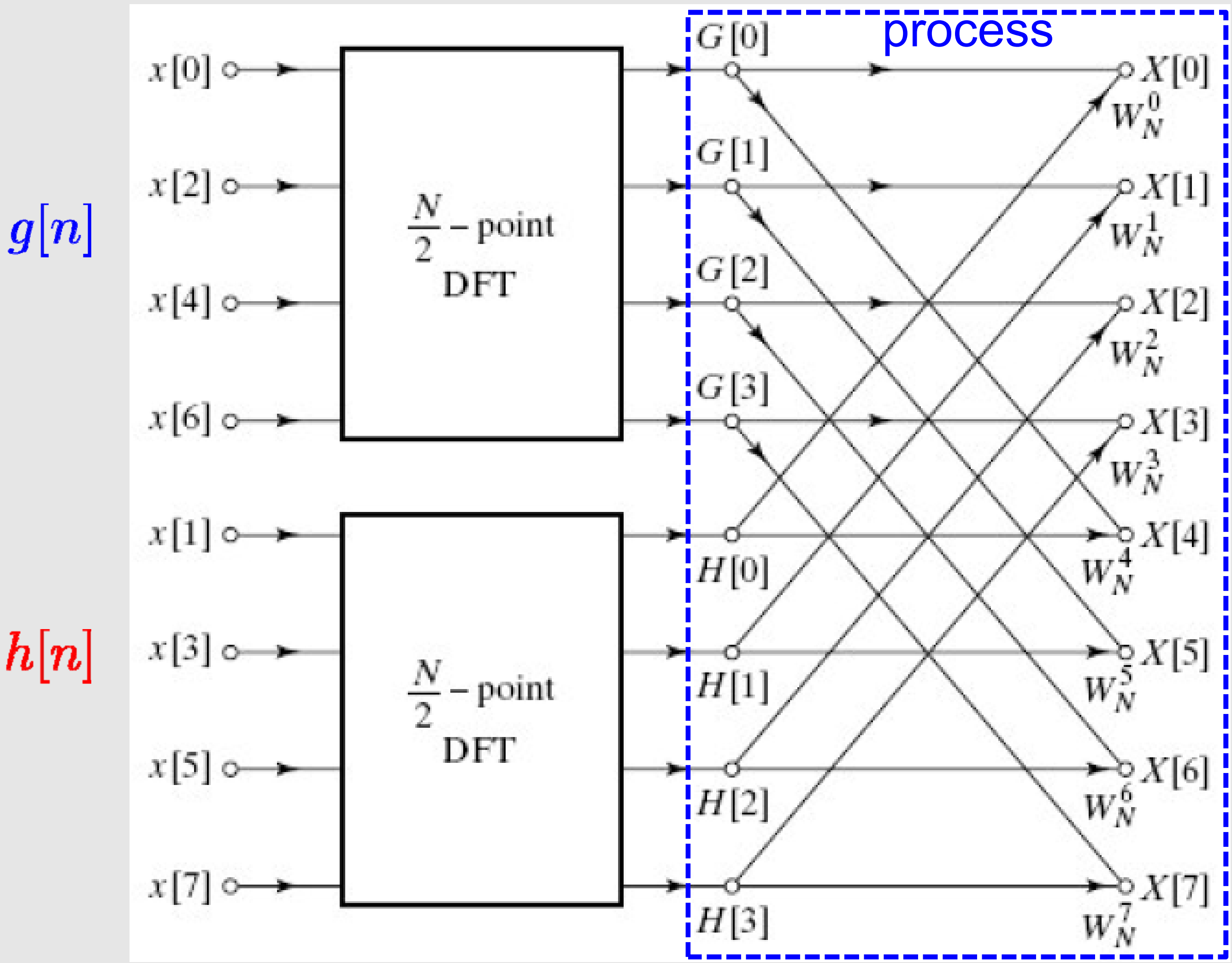
Therefore

$$\begin{aligned}
G[k] + W_N^k H[k] &= \sum_{n=0}^{N/2-1} x[2n] W_N^{2nk} + \sum_{n=0}^{N/2-1} x[2n+1] W_N^k W_N^{2nk} \\
&= \sum_{n=0}^{N/2-1} x[2n] W_N^{2nk} + \sum_{n=0}^{N/2-1} x[2n+1] W_N^{(2n+1)k} \\
&= \sum_{m=0}^{N-1} x[m] W_N^{mk} \\
&= X[k]
\end{aligned}$$

$\blacksquare$

7.2 Decimation-in-time (DIT) FFT

- Flow graph of 8-point DIT FFT (first)



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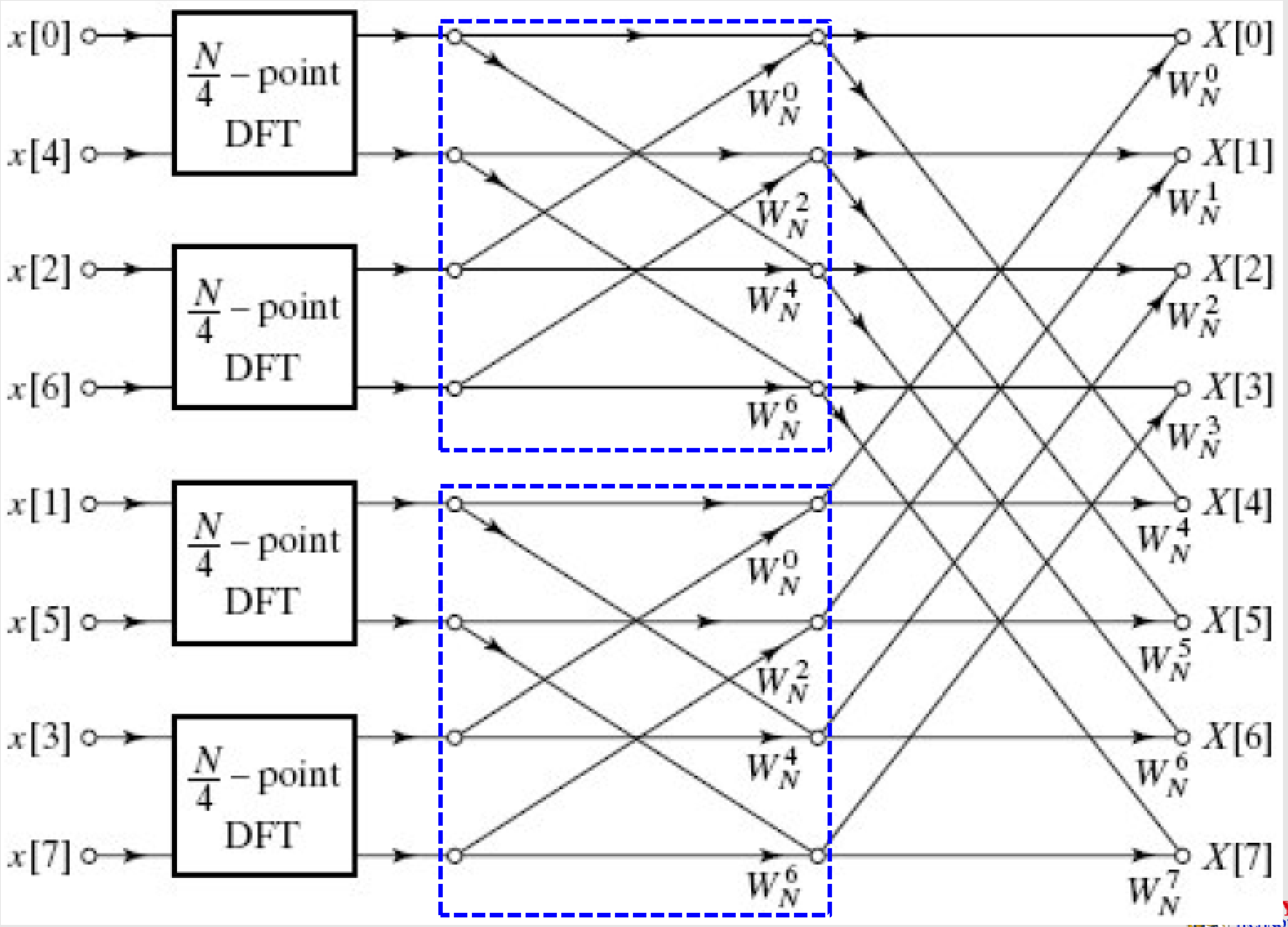
7.2 Decimation-in-time (DIT) FFT

- Flow graph of 8-point DIT FFT (second) $W_{N/2}^{k} = W_N^{2k}$

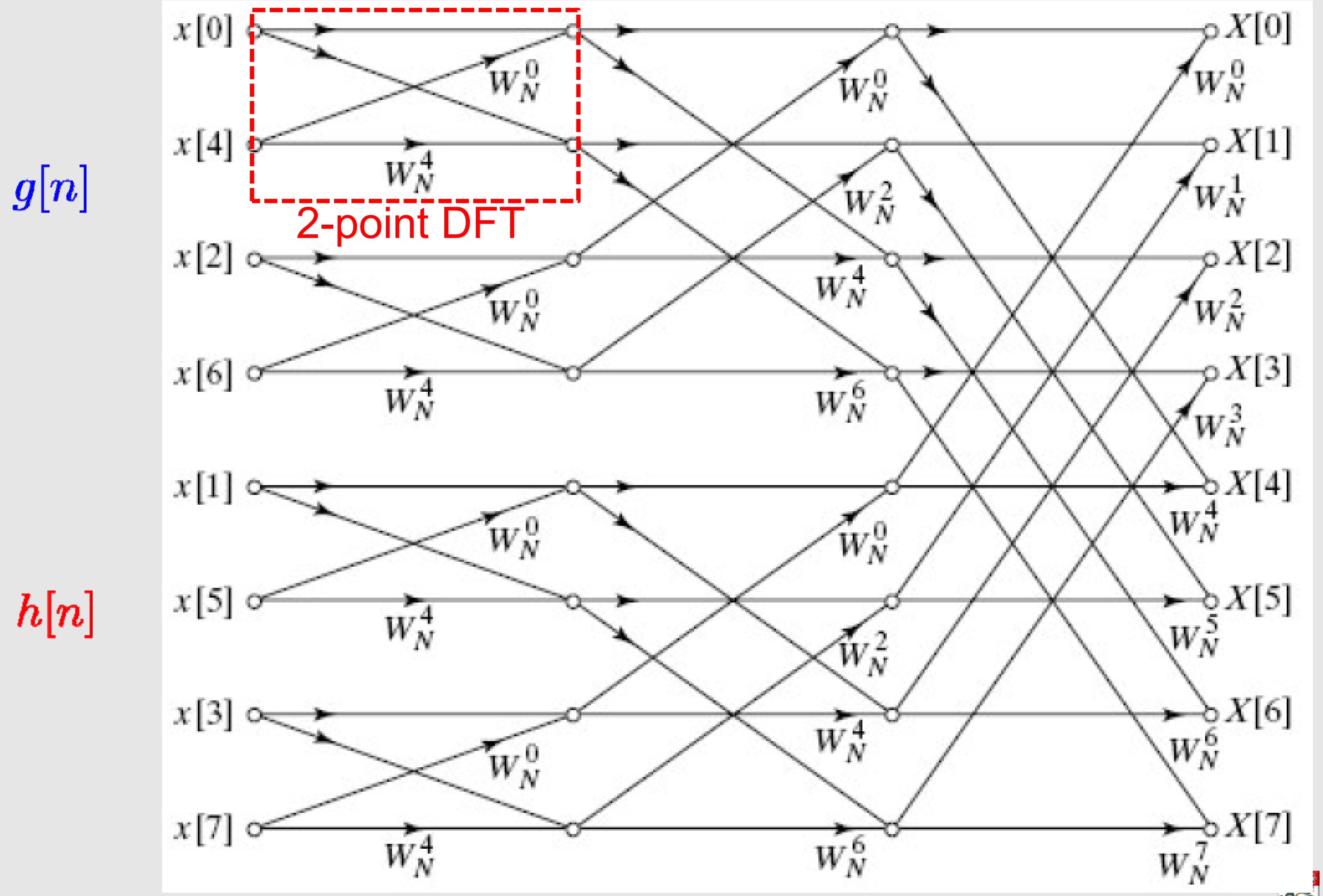
$g[n]$

$h[n]$



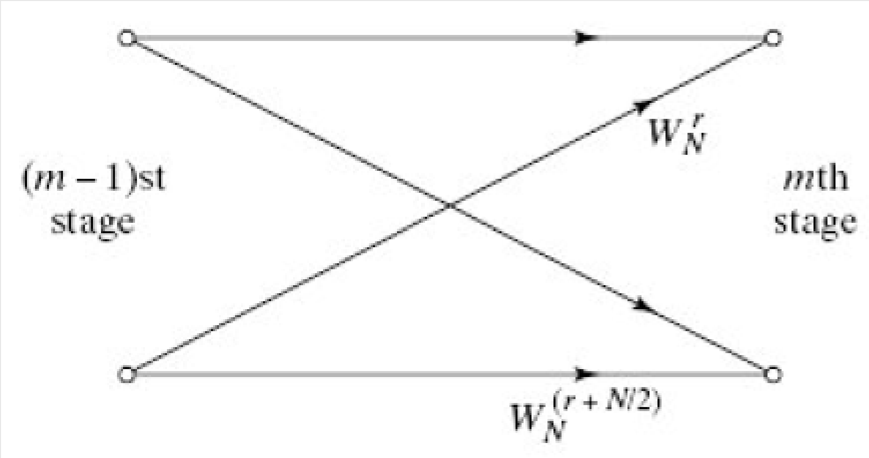
7.2 Decimation-in-time (DIT) FFT

- Flow graph of 8-point DIT FFT (third) $W_{N/4}^{k} = W_{N}^{4k}$

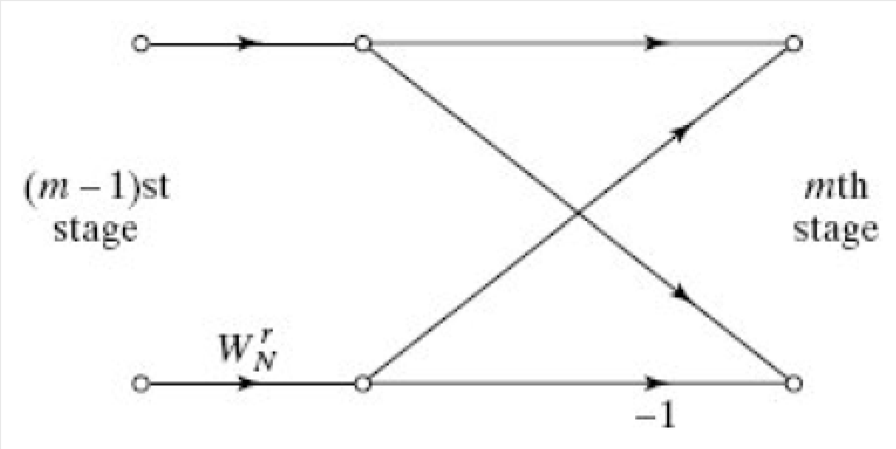


- We can further simplify the computation using the simplified butterfly computation by noting that

$$W_N^{r+N/2} = W_N^r W_N^{N/2} = -W_N^r$$



2 multiplications



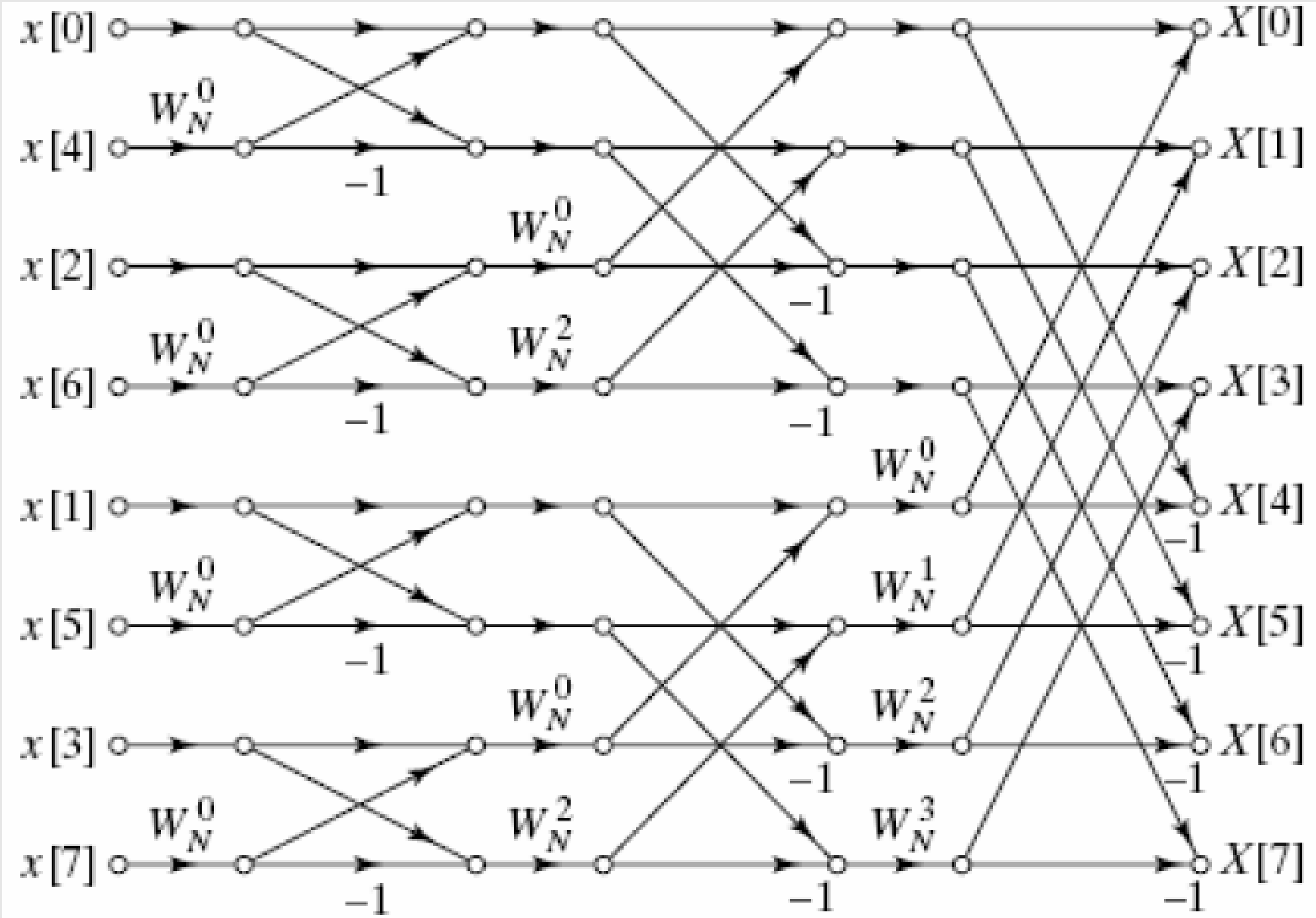
1 multiplication

7.2 Decimation-in-time (DIT) FFT

- Flow graph of 8-point DIT FFT (final)

$g[n]$

$h[n]$



7.2 Decimation-in-time (DIT) FFT

A short discussion of the computation:

- Number of stages: N-point to $N/2$-point, $N/2$-point to $N/4$-point
  There will be $\log_2 N$ stages

- Each stage requires a "process" block of $N/2$ complex multiplications and N complex additions.
  Total: $(N/2)\log_2 N$ multiplications, $N\log_2 N$ additions

- Bit-reverse ordering
  Input is presented as $x[0], x[4], x[2], x[6]$
  which, in binary is $x[000], x[100], x[010], x[110]$
  which correspond to the reverse natural ordering
  $x[000], x[001], x[010], x[011]$

- Only one N-length complex array storage is required, intermediate results are stored in the same array.

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7.3 Decimation-in-frequency (DIF) FFT

- The FFT coefficients $X[k]$ may be separated into two $N/2$-point sequences:

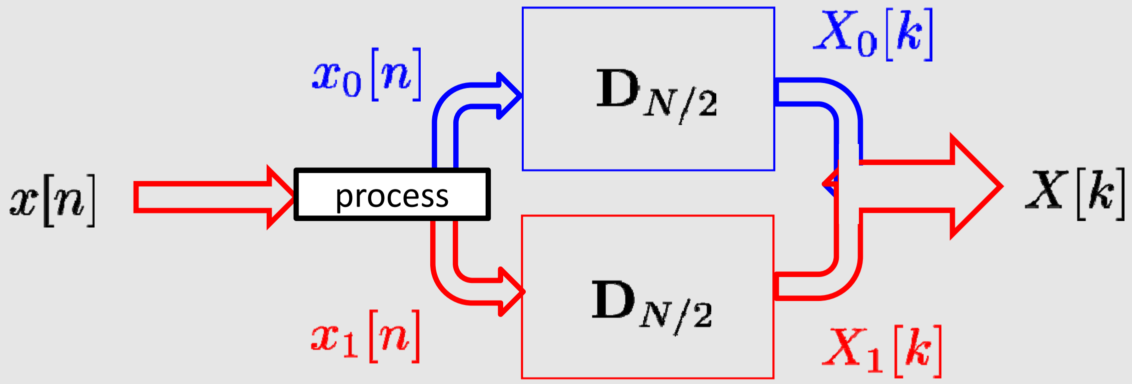
  even-numbered points $X_0[k] = X[2k], \quad 0 \le k \le (N/2) - 1$
  odd-numbered points $X_1[k] = X[2k+1], \quad 0 \le k \le (N/2) - 1$

- It may be shown that $X_0[k]$ is the $N/2$ -point DFT of $x_0[n]$ when

$$x_0[n] = x[n] + x[n + N/2], \quad 0 \le n \le N/2 - 1$$

- Similarly, $X_1[k]$ is the $N/2$-point DFT of $x_1[n]$ when

$$x_1[n] = (x[n] - x[n + N/2])W_N^n, \quad 0 \le n \le N/2 - 1$$



7.3 Decimation-in-frequency (DIF) FFT

- Proof- we will show that the first two equations on the previous page are valid.

- For $X_0[k]$,

$$
\begin{aligned}
X_0[k] = \sum_{n=0}^{N/2-1} x_0[n] W_{N/2}^{nk} &= \sum_{n=0}^{N/2-1} x[n] W_{N/2}^{nk} + \sum_{n=0}^{N/2-1} x[n+N/2] W_{N/2}^{nk} \\
\because W_{N/2}^{nk} = W_{N/2}^{(n+N/2)k} \quad &= \sum_{n=0}^{N/2-1} x[n] W_{N/2}^{nk} + \sum_{n=0}^{N/2-1} x[n+N/2] W_{N/2}^{(n+N/2)k} \\
&= \sum_{n=0}^{N-1} x[n] W_{N/2}^{nk} \\
\because W_{N/2}^{nk} = W_{N}^{2nk} \quad &= \sum_{n=0}^{N-1} x[n] W_{N}^{n(2k)} \\
&= X[2k] \qquad \blacksquare
\end{aligned}
$$

7.3 Decimation-in-frequency (DIF) FFT

- For $X_1[k]$,

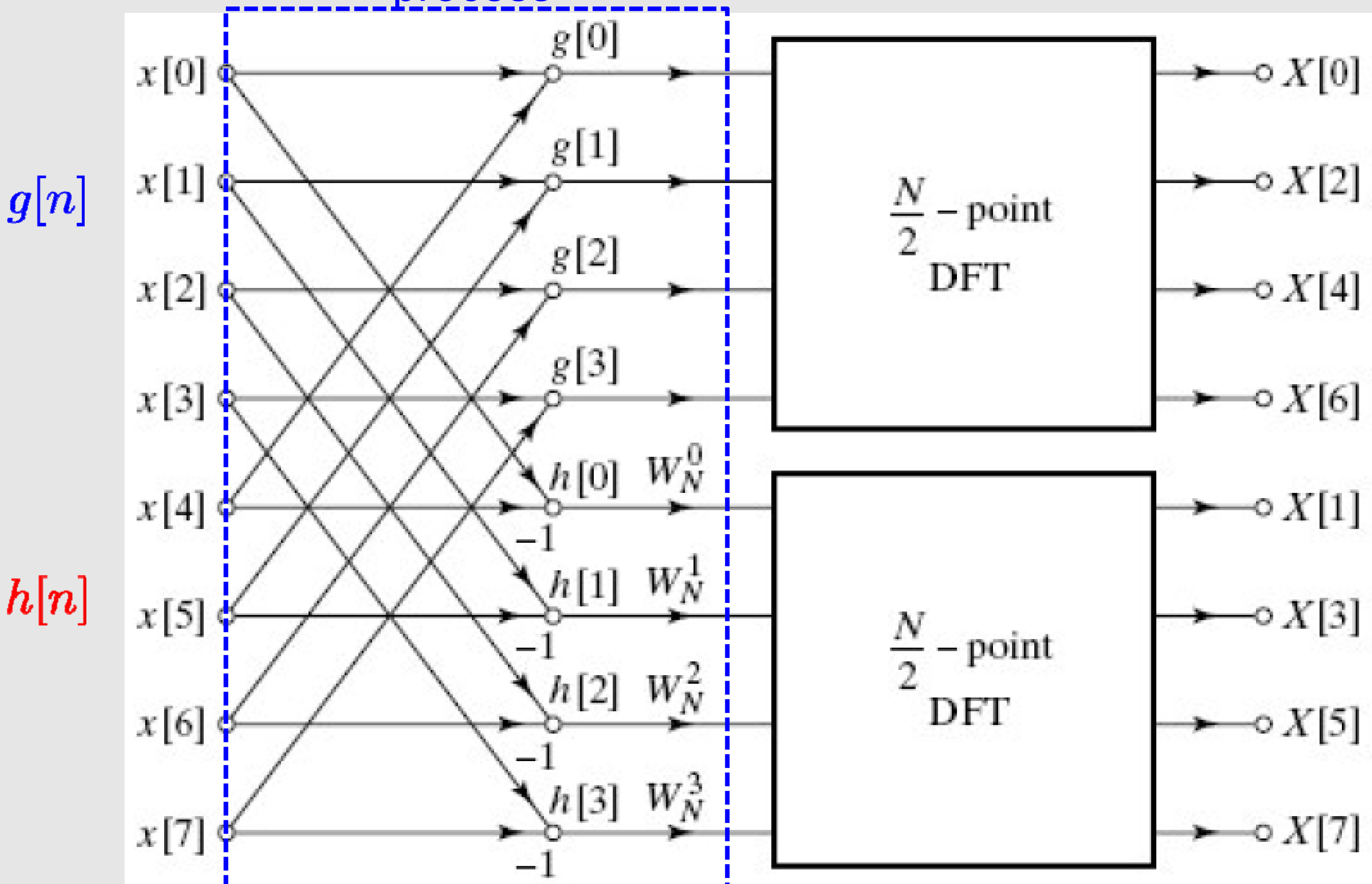
$$
\begin{aligned}
X_1[k] &= \sum_{n=0}^{N/2-1} x_1[n] W_{N/2}^{nk} \\
&= \sum_{n=0}^{N/2-1} x[n] W_N^n W_{N/2}^{nk} - \sum_{n=0}^{N/2-1} x[n+N/2] W_N^n W_{N/2}^{nk} \\
&= \sum_{n=0}^{N/2-1} x[n] W_N^n W_N^{n(2k)} + W_N^{(N/2)(2k+1)} \sum_{n=0}^{N/2-1} x[n+N/2] W_N^n W_N^{n(2k)} \\
&= \sum_{n=0}^{N/2-1} x[n] W_N^{n(2k+1)} + \sum_{n=0}^{N/2-1} x[n+N/2] W_N^{(n+N/2)(2k+1)} \\
&= \sum_{n=0}^{N-1} x[n] W_N^{n(2k+1)} \\
&= X[2k+1] \qquad \blacksquare
\end{aligned}
$$

$\because W_N^{N/2} = -1$

7.3 Decimation-in-frequency (DIF) FFT

- Flow graph of 8-point DIF FFT (first)

process

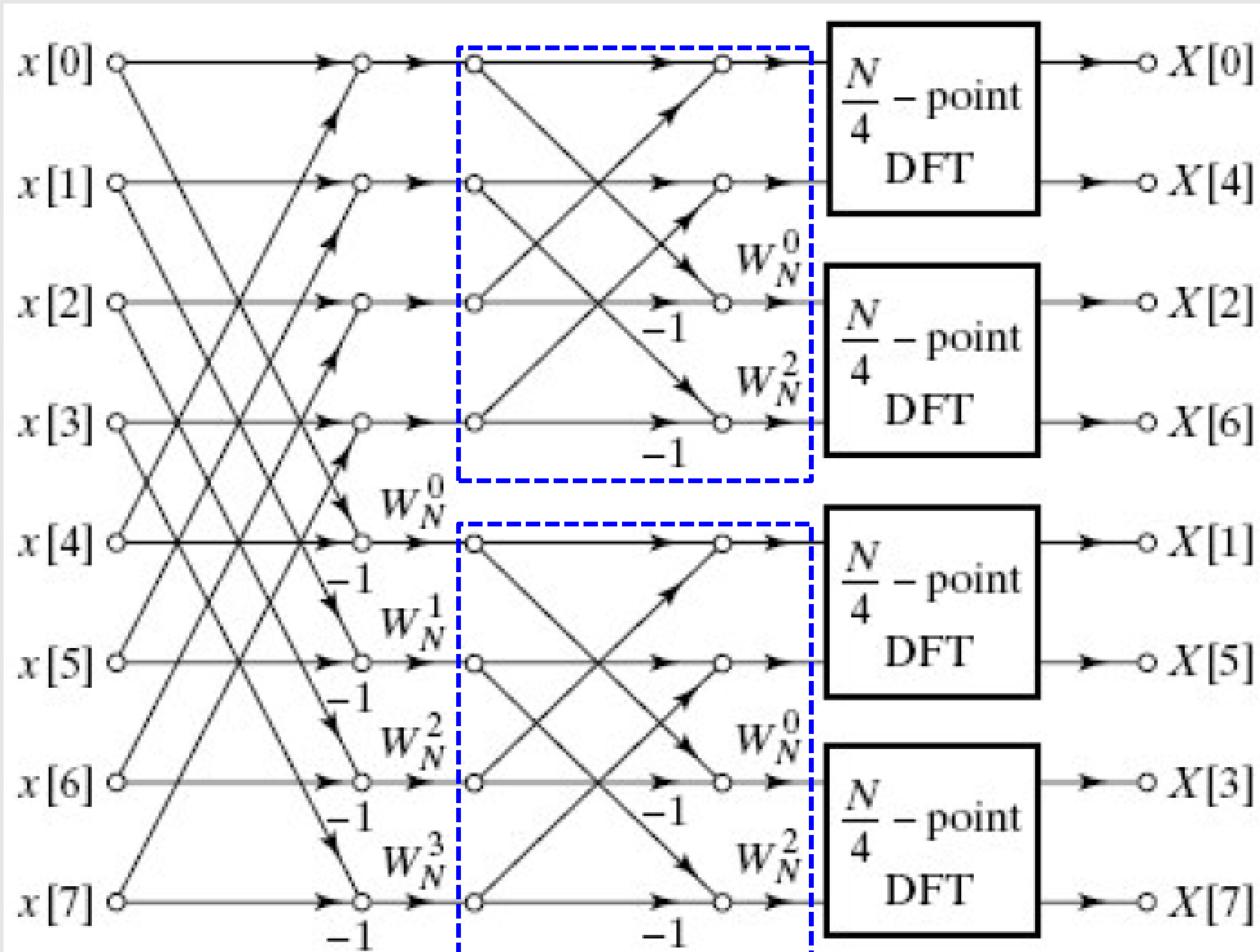


7.3 Decimation-in-frequency (DIF) FFT

- Flow graph of 8-point DIF FFT (second)

$g[n]$

$h[n]$

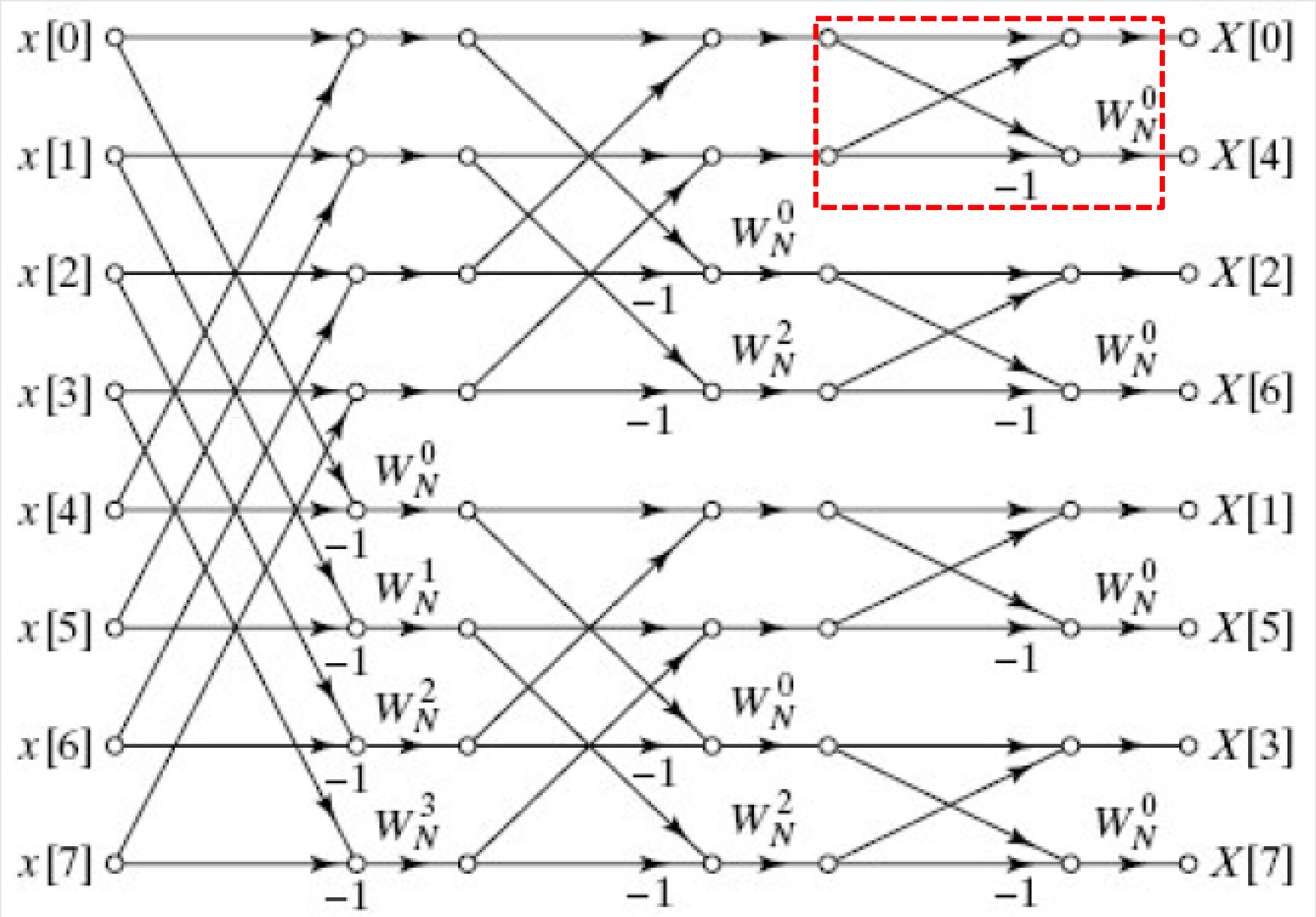


7.3 Decimation-in-frequency (DIF) FFT

- Flow graph of 8-point DIF FFT (third)

$g[n]$

$h[n]$



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7.3 Decimation-in-frequency (DIF) FFT

- DIF FFT is similar to DIT FFT but the “process” block is before.
- DIF FFT has the same required computation as DIT FFT.
- DIF FFT also uses bit-reversed ordering, but for the output.
- DIF FFT also uses in-place computation.

7.4 Summary

- The FFT is an efficient way to compute the DFT.

- There are two implementations of the FFT
  - Decimation in time
  - Decimation in frequency

- For decimation in time, the time-domain signal is split into even and odd samples (interleaved) before taking the FFT.

- For decimation in frequency, time-domain signal is processed first before taking the FFT. The resultant frequency bins are interleaved.