# Efficient Resource Allocation Contracts to Reduce Adverse Events

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Motivated by allocation of online visits to product/service/content suppliers in the platform economy, we consider a dynamic contract design problem in which a principal constantly determines the allocation of a resource to multiple agents. Although agents are capable of running the business, they introduce adverse events, the frequency of which depends on each agent's effort level. With an objective of maximizing social welfare, we study dynamic contracts that utilize resource allocation and monetary transfers to induce agents to exert effort and reduce the arrival rate of adverse events. In contrast to the single-agent case, in which efficiency is not achievable, we show that efficient and incentive-compatible contracts, which allocate all resources and induce agents to exert effort all the time, generally exist with two or more agents. We devise an iterative algorithm that characterizes and calculates such contracts. Furthermore, we provide a simple efficient and incentive-compatible dynamic contract that can be expressed in closed-form, and, therefore, is easy to understand and implement in practice.

*Key words*: dynamic contract design, moral hazard, self-generating set, platform economy, stochastic optimal control

## 1. Introduction

Some of the most valuable companies by market capitalization, such as Amazon, Apple, and Alibaba, and many of the \$1 billion unicorn startups, such as Airbnb, Bytedance, DiDi, and Uber, build online platforms to satisfy demand with supplies from individual players. These digital platforms facilitate and reshape a wide range of businesses and social activities, from selling products (e.g., Amazon, Apple, Alibaba) and providing services (e.g., DiDi, Uber, and Airbnb) to hosting news and information content (e.g., Bytedance). The success of platform economy depends crucially on moral suppliers providing high-quality services, products, or information content. Adverse events related to low-quality supply, however, hurt the reputation of platforms and the society at large. Recent examples of adverse events on platforms include the brushing and fake reviews from third-party sellers on Amazon in 2020 (Dai and Tang 2020), fraudulent drivers at DiDi in 2018 (Feng 2019), inappropriate content on Reddit in 2016 (Marantz 2018), and tampering with food at Uber Eats in 2018 (Edelstein 2018). In many situations, product/service/content suppliers on these platforms could exert effort to reduce the chance of adverse events. However, effort may be hard to verify, and adverse events may still occur despite the best effort.

In practice, some platforms create *resource allocation* incentives that adjust online visits allocated to different suppliers based on their performances. For example, Alibaba devised a point-based system on its retail platform. Each merchant is initially credited with a fixed amount of "points" that are deducted once the merchant breaches service quality promises or sells inferior products. Low-point merchants are penalized with low search visibility and limitations in marketing activities, or even termination (Chow 2010). Other platforms have implemented similar point-based systems, as summarized in Table 1. Nonetheless, some platforms still struggle with frequent occurrence of incidents (Mauldin 2019). From an operations point of view, it remains a challenge to properly design a resource allocation system to provide the right incentives for suppliers.

**Table 1** A summary of regulation systems for well-known platforms

| Platform | Category | Examples of adverse events | Possible ramification |
|---|---|---|---|
| JD.com<br>Taobao | Online retailing | • Counterfeit goods<br>• Breach of promises<br>• Fake transactions | • Demotion of search weight<br>• Limitation on promotion<br>• Termination |
| DiDi<br>Uber | Ride hailing | • Driving under influences<br>• Safety issues<br>• Fraud and theft | • Reduction in dispatch<br>• Suspension<br>• Termination |
| Ele.me<br>Uber Eats | Food delivery | • Spoiled food<br>• Delivery issues<br>• Tampering with food | • Demotion of search weight<br>• Reduction of subsidy<br>• Suspension<br>• Termination |
| ByteDance<br>Twitter<br>YouTube | User-generated content | • Low quality content<br>• Fake news | • Restrictions on posting<br>• Limitation on promotion |

In this paper, we study dynamic contracts that motivate agents to exert effort to reduce the frequency of adverse events in a continuous-time setting over an infinite time horizon. Specifically, a risk-neutral principal with commitment power to design long term contracts owns a fixed amount of resource to be allocated among multiple agents. Agents are able to use the resource to generate revenue, but can also generate adverse events costly to the society. Exerting effort allows agents to reduce the arrival rate of these adverse events. However, effort is costly to the agents and not observable to the principal. Following the usual limited liability assumption, the principal cannot align the incentive by making agents bear the cost of adverse events. Therefore, the principal must rely on dynamic resource allocation and payment decisions that depend on past arrival times to induce effort from agents.

We focus on the objective of maximizing social welfare. First, applications of this modeling framework are not limited to profit-maximizing firms. Second, maximizing social welfare may benefit even profit-maximizing platform companies due to the network effects that higher social welfare generates. Third, even traditional profit-driven companies are adopting social welfare objectives in recent years. For instance, on August 19, 2019, 181 top chief executive officers (CEOs) signed and released an open letter titled "Statement on the Purpose of a Corporation" via the association of Business Roundtable, whose members are CEOs of major U.S. companies. The statement explicitly negates the widespread view that "the social responsibility of business is to increase its profits." Instead, it proclaims the commitment of corporations to create values for their customers, employees, suppliers, and communities, in addition to their own shareholders, for the prosperity of the corporations, the communities and the country (Business Roundtable 2019).

Here is a summary of our main contributions. We extend continuous-time dynamic moral hazard models of Biais et al. (2010) and Myerson (2015) to a multi-agent setting, where a principal could leverage the allocation of a resource, besides payments, to incentivize effort from agents. In this context, we establish the existence of efficient (maximizing social welfare) and incentive-compatible (inducing effort) contracts, despite the fact that efficiency is not achievable in the corresponding single-agent setting. In particular, we describe the set of efficient and incentive-compatible (EIC) contracts by characterizing the *self-generating set* (Abreu et al. 1990) of agents' total future utilities (also called *promised utilities*) that can be achieved using these contracts. Following an idea from Balseiro et al. (2019) of using support functions to represent the convex set of achievable promised utilities, we propose an iterative algorithm based on solving a sequence of (infinite-dimensional) linear optimization models, which yields an EIC contract. The contract involves continuously adjusting all agents' resource allocations and promised utilities between arrivals, and letting them take discrete jumps upon adverse arrivals. In particular, whenever an agent experiences an adverse event, this agent's allocation and promised utility take downward jumps. At the same time, all other agents' allocations and promised utilities also take discrete jumps, in order to maintain efficiency and the incentive for agents to always exert effort. This insight allows us to further design a very simple dynamic contract in closed-form, which no longer requires solving linear optimization problems. Following this simple contract, each agent's income rate is proportional to the allocated resource. This feature is particularly relevant to online platform applications, where agents are individual players on the platform, whose income is often proportional to the amount of online visits they receive. Therefore, our simple contract allows online platforms to motivate players to maintain quality by adjusting online visit allocations in a straightforward manner. Overall, from a theoretical point of view, our paper contributes to the continuous-time dynamic contracting literature with Poisson arrivals of adverse events by considering a multi-agent setting. From a practical point of view, our results provide prescriptive guidance for practitioners to design easy-to-implement EIC contracts.

Now we review related literature, which explains methodological foundations and distinguishes contributions of our work. The study of moral hazard problems has gained growing attention since the early works of Holmström (1979) and Grossman and Hart (1983) on contract theory. In static settings, there has been a large literature on bilateral contracting problems (see, for example, Innes 1990, Baker 1992, Prendergast 2002). Early studies of dynamic moral hazard problems often consider discrete-time settings (see, for example, Rogerson 1985, Spear and Srivastava 1987, Gibbons and Murphy 1992, Holmström 1999). The main stochasticity in our setting is interarrival times, or equivalently, frequency of adverse events. Therefore, we consider continuous-time models. There has been a recent stream of literature on continuous-time moral hazard problems since Sannikov (2008), who uses a Brownian motion to capture uncertain outcomes following an agent's effort process. Relying on the Martingale Representation Theorem to represent the incentive-compatibility constraint, the methodology proposed in Sannikov (2008) paves the analytical foundation for this stream of research.

Biais et al. (2010) extend the continuous-time optimal contracting framework to study firm dynamics based on a Poisson process of adverse events, instead of a Brownian motion uncertainty process. Besides direct payments, an investor (the principal) can also change the size of a firm (the agent). In particular, they assume that there is an upper bound on the speed of scaling up the size of the firm, while there is no constraint on downsizing. Our modeling framework is closely related to Biais et al. (2010) with some important differences. The most obvious one is that we consider contracting with multiple agents, while the focus of Biais et al. (2010) is the single-agent case. With a single agent, no contract achieves efficiency in the setting with adverse events, and Biais et al. (2010) study optimal contracts. With at least two agents, on the other hand, efficient contracts do exist, which is the focus of our study. A more nuanced difference is that while it is quite reasonable to assume that the speed of scaling up a firm is upper-bounded in Biais et al. (2010), our principal can change the allocation to an agent at any time instantaneously to another level, with no speed limit. Our modeling assumption is justified in the context of platforms allocating online visits, and yields different analyses and results from Biais et al. (2010). Finally, we assume that there is a limited amount of resource to be allocated, while the firm's size in Biais et al. (2010) can grow without bounds.

From a modeling perspective, Myerson (2015) is closely related to Biais et al. (2010) and therefore also related to our paper. Myerson (2015) studies a problem in political economy, where the principal can dynamically pay and/or replace an agent in order to motivate effort. Instead of replacing the agent, Chen et al. (2019) uses costly monitoring to resolve information asymmetry, and dynamically schedules monitoring and payments to ensure an agent's effort. Both aforementioned papers study single-agent settings in which effort reduces the arrival rate of a Poisson process. Despite apparent connections, the specific dynamics of replacement (Myerson 2015), monitoring (Chen et al. 2019), and resource allocation (our paper) are different.

Analytically, we characterize efficient contracts by specifying the set of agents' promised utilities that are achievable by the contracts. Spear and Srivastava (1987) first proposed using the promised utility as a state variable to study infinite horizon contracts. Abreu et al. (1990) propose the concept of a *self-generating set* of promised utilities in the study of infinite horizon repeated games among multiple agents with imperfect monitoring. The concept of self-generating is crucial for Fudenberg et al. (1994) to establish a Folk Theorem, which further implies that efficient mechanisms and contracts exist in dynamic adverse selection and moral hazard problems, respectively, when agents are infinitely patient (the discount factor approaches one). However, it is hard to deduce the corresponding dynamic mechanisms or contracts for operational purposes from their existence proofs. Balseiro et al. (2019) consider dynamic mechanism design without money, and propose a mechanism that approaches efficiency as the discount factor approaches one. When there are only two agents, their mechanism's convergence rate to efficiency is optimal. An important analytical approach developed in Balseiro et al. (2019), which we also use in our paper, is to study the set of achievable promised utilities using its support functions in a recursive manner. However, Balseiro et al. (2019) study an adverse selection problem in a discrete-time setting, rather than a moral hazard problem in a continuous-time setting. Therefore, specifics of our self-generating set and support function analysis are quite different. For example, we need to develop a definition for the self-generating set for the continuous-time setting, which appears new. Furthermore, we directly obtain an efficient contract through iteratively solving linear programs to characterize support functions, an approach not pursued in Balseiro et al. (2019). In fact, an efficient mechanism does not exist without money unless time discount approaches one. In contrast, efficient contracts exist in our setting regardless of agents' patience level.

Poisson arrivals do not have to be adverse events. In other settings, a principal may want to motivate agents' effort to increase the arrival rate of "good" arrivals. Shan (2017), for example, considers a principal hiring two agents to carry out a multistage project, whose successful outcomes follow a Poisson process with rate jointly determined by the effort choices of both agents. The paper considers free riding issues when the total effort levels determine the arrival rate, an aspect that we do not consider. The main contractual lever of that paper is payment, and not resource allocation, which is another distinction with our paper. Other recent papers related to good arrivals include Green and Taylor (2016) and Sun and Tian (2018), which study optimal contract design for single-agent settings, and are more tangential to our paper.

The rest of this paper is organized as follows. We first describe the model in Section 2. Section 3 investigates efficient and incentive-compatible contracts. In particular, Subsection 3.1 introduces the key concepts of achievable utilities and self-generating set. Building upon these concepts, Subsection 3.2 proposes an iterative approach to construct the set of achievable promised utilities, and Subsection 3.3 describes an efficient contract that maximizes the principal's utility among all EIC contracts. Subsection 3.4 further illustrates insights of the EIC contracts through examples. Section 4 proposes an easy-to-implement contract with desirable properties making it practically relevant. Finally, Section 5 concludes the paper and discusses further insights and future research directions. All the proofs are presented in the Online Appendix.

## 2. The Model

We consider a principal-agent model in a continuous-time setting. The principal has a resource to run a business over time, which is normalized to one per unit of time. Consider the resource as the total visits to the platform. The principal can either run the business on her own, which generates a revenue normalized to zero, or allocate some of the resource among multiple symmetric agents. Agents are more efficient and can earn a positive revenue flow at rate $R$ per unit of resource per period of time. Both the principal and the agents are risk-neutral and discount future cash flows at the same rate $\rho$. Denote $\mathcal{I}=\{1,\dots,n\}$ to represent the set of agents. At each time epoch $t$, denote $X_{i,t}$ to be the resource that is allocated to agent $i$. Naturally, the allocation decisions satisfy the following condition, in which "FX" stands for "feasible $X$,"

$$X_{i,t}\geq 0, \text{ and } \sum_{i=1}^{n} X_{i,t}\leq 1, \ \forall t\geq 0, \ \forall i\in\mathcal{I}. \tag{FX}$$

Although more efficient, an agent's work may also generate adverse events following a Poisson process. An adverse event from agent $i$ arriving at time $t$ causes a cost $C\cdot X_{it}$, which is scaled with the resource allocated to the agent. It is helpful to use the following example to explain this modeling choice. Consider adverse arrivals as low quality or inappropriate content inadvertently posted by user-generated content providers (UGCs) on YouTube, Instagram or even Coursera. The more visits that the platform allocates to such a UGC, the larger the damage. While the expected cost from an adverse event is assumed to be proportional to the allocation, the arrival rate of adverse events is not. These modeling choices also capture adverse events in other settings too.

For example, outsourced parts without adequate quality control may trigger recalls; unmotivated representatives may cause customer complaints; and cutting corners in sanitation procedures may seed food poisoning episodes.

Each agent is able to reduce the arrival rate of the adverse events from $\bar{\lambda}$ to $\lambda = \bar{\lambda} - \Delta\lambda$ by exerting effort. Denote $\boldsymbol{\Lambda} = \{\boldsymbol{\lambda}_t\}_{t\geq 0} = \{(\lambda_{i,t})_{i\in\mathcal{I}}\}_{t\geq 0}$ to represent agents' effort processes, where $\lambda_{i,t} \in \{\lambda, \bar{\lambda}\}$. The positive revenue flow generated by agents satisfies

$$R > \lambda C. \tag{2.1}$$

That is, the societal benefit rate from allocating all resources to agents, $R$, is higher than the adverse event cost per unit of time, $\lambda \sum_{i\in\mathcal{I}} C \cdot X_{it} = \lambda C$, if agents exert effort. Therefore, it is socially efficient to hire diligent agents. However, the principal only observes the history of adverse events but not the effort processes, and, therefore, faces a dynamic moral hazard problem. Shirking brings an agent a benefit rate $b \cdot X_{it}$, also proportional to the resource allocated to the agent. Using the aforementioned UGC example, we may think of the moral hazard problem as UGC disguising advertisements as news reports, which may trigger complains towards the platform.

We assume that the principal has commitment power to issue a long-term contract with the agents, which is a contingency plan that both the principal and the agents are willing to follow through. It specifies both the payment the resource allocation policies over time. Denote an $n$-dimensional counting process $\{\boldsymbol{N}_t\}_{t\geq 0} = \{(N_{i,t})_{i\in\mathcal{I}}\}_{t\geq 0}$ to represent the number of adverse events induced by each agent up to time $t$. Define a filtration $\mathcal{F}^{\boldsymbol{N}} = \{\mathcal{F}_t^{\boldsymbol{N}}\}_{t\geq 0}$ generated by the counting process $\{\boldsymbol{N}_t\}_{t\geq 0}$, such that $\mathcal{F}_t^{\boldsymbol{N}}$ captures the entire information history up to time $t$ specified by the counting process $\{\boldsymbol{N}_t\}_{t\geq 0}$. The contract shall depend on the history of adverse events, that is, a contract consists of $\mathcal{F}^{\boldsymbol{N}}$-predictable payment and resource allocation processes.

We follow the standard assumption that the agents have limited liability and are cash constrained. That is, the principal can pay the agents, but cannot be paid by any of them at any point in time. Therefore, we denote an $n$-dimensional $\mathcal{F}^{\boldsymbol{N}}$-predictable process $\{\boldsymbol{L}_t\}_{t\geq 0} = \{(L_{i,t})_{i\in\mathcal{I}}\}_{t\geq 0}$ to represent the principal's cumulative payments to each agent $i \in \mathcal{I}$ up to time $t$. The limited liability and cash constrained condition is expressed as

$$\mathrm{d}L_{i,t} \geq 0, \quad \forall t \geq 0,\ \forall i \in \mathcal{I}. \tag{LL}$$

Moreover, we consider $\mathrm{d}L_{i,t} = I_{i,t} + l_{i,t}\mathrm{d}t$, where $I_{i,t}$ represents instantaneous payment, and $l_{i,t}$, flow payment, to agent $i$ at time $t$. Following the same conventions as for $\boldsymbol{\lambda}_t$, we use $n$-dimensional non-negative vectors $\boldsymbol{I}_t$ and $\boldsymbol{l}_t$ to represent vectors of instantaneous and flow payments, respectively.

Besides payments, the principal can also dynamically allocate the resource among the agents to induce effort. Let $\{\boldsymbol{X}_t\}_{t\geq 0} = \{(X_{i,t})_{i\in\mathcal{I}}\}_{t\geq 0}$ be the resource allocation rule, where $\boldsymbol{X}_t$ represents the resource allocation at time $t$. Vector $\boldsymbol{X}_t$ lies in the $n$-dimensional unit cube, that is, $\boldsymbol{X}_t \in [0,1]^n$ and $\sum_{i\in\mathcal{I}} X_{i,t} \le 1$. Formally, a contract $\Gamma$ consists of $\mathcal{F}_t^{\boldsymbol{N}}$-predictable payment and allocation processes, $\{\boldsymbol{L}_t\}_{t\ge 0}$ and $\{\boldsymbol{X}_t\}_{t\ge 0}$, respectively.

## 2.1. Agents' Utilities

Agents' utilities consist of discounted total payments and potential benefits form shirking. Given contract $\Gamma$ and agents' effort process $\boldsymbol{\Lambda} = \{\boldsymbol{\lambda}_t\}_{t\ge 0} = \{(\lambda_{i,t})_{i\in\mathcal{I}}\}_{t\ge 0}$, the *total expected utility* of agent $i$, denoted by $u_i(\Gamma, \boldsymbol{\Lambda})$, is defined as follows,

$$u_i(\Gamma, \boldsymbol{\Lambda}) := \mathbb{E}^{\boldsymbol{\Lambda}} \left[ \int_0^\infty e^{-\rho t} \left( \mathrm{d}L_{i,t} + bX_{i,t} \mathbb{1}_{\{\lambda_{i,t}=\bar{\lambda}\}} \mathrm{d}t \right) \right], \quad \forall i \in \mathcal{I}, \tag{2.2}$$

in which $\mathbb{E}^{\boldsymbol{\Lambda}}$ represents taking expectation with respect to the probability measure induced by the arrival rate process $\boldsymbol{\Lambda}$ of Poisson processes, and the symbol $\mathbb{1}$ represents the indicator function.

It is standard and convenient to work with the agents' *promised utility* (see, for example, Biais et al. 2010). The promised utility of agent $i$ at time $t$ is defined as follows:

$$W_{i,t}(\Gamma, \boldsymbol{\Lambda}) := \mathbb{E}^{\boldsymbol{\Lambda}} \left[ \int_t^\infty e^{-\rho(s-t)} \left( \mathrm{d}L_{i,s} + bX_{i,t} \mathbb{1}_{\{\lambda_{i,t}=\bar{\lambda}\}} \mathrm{d}s \right) | \mathcal{F}_t^{\boldsymbol{N}} \right], \quad \forall t \ge 0, \forall i \in \mathcal{I}. \tag{2.3}$$

The promised utility $W_{i,t}(\Gamma, \boldsymbol{\Lambda})$ is a right-continuous process capturing the total discounted utility of agent $i$ starting from time $t$. For convenience, we omit "$(\Gamma, \boldsymbol{\Lambda})$" when the context is clear and refer to $W_{i,t}(\Gamma, \boldsymbol{\Lambda})$ as $W_{i,t}$. Agents are free to walk away from the contract. Therefore, we require the following *individual rationality* (or IR) constraint to ensure agents' participation:

$$W_{i,t} \ge 0, \quad \forall t \ge 0, \forall i \in \mathcal{I}. \tag{IR}$$

In the next lemma, we characterize the dynamics of agents' promised utilities $\boldsymbol{W}_t = (W_{i,t})_{i\in\mathcal{I}}$ under an arbitrary contract in terms of the stochastic integral with respect to the right-continuous $\{W_{i,t}\}_{t\ge 0}$ process. For this purpose we define a left-continuous process $W_{i,t-} = \lim_{s\uparrow t} W_{i,s}$. We extend the definition of $W_{i,t}(\Gamma, \boldsymbol{\Lambda})$ such that $W_{i,0-}(\Gamma, \boldsymbol{\Lambda}) = u_i(\Gamma, \boldsymbol{\Lambda})$. Clearly, $W_{i,0}(\Gamma, \boldsymbol{\Lambda}) = W_{i,0-}(\Gamma, \boldsymbol{\Lambda}) = u_i(\Gamma, \boldsymbol{\Lambda})$.

Lemma 2.1. *For any contract $\Gamma$ and any effort process $\boldsymbol{\Lambda}$, there exist $\mathcal{F}_t^{\boldsymbol{N}}$-predictable processes $\{\boldsymbol{H}_t = (H_{ij,t})_{i,j\in\mathcal{I}}\}_{t\ge 0}$, such that for any $t_1$ and $t_2$ with $0 \le t_1 < t_2$, we have*

$$W_{i,t_2} = W_{i,t_1} + \int_{(t_1,t_2]} dW_{i,s}, \quad \forall i \in \mathcal{I}, \tag{2.4}$$

*in which*

$$dW_{i,t} = \left( \rho W_{i,t-} - bX_{i,t} \mathbb{1}_{\{\lambda_{i,t}=\bar{\lambda}\}} + \sum_{j\in\mathcal{I}} \lambda_{j,t} H_{ij,t} \right) dt - \sum_{j\in\mathcal{I}} H_{ij,t} dN_{j,t} - dL_{i,t}, \quad \forall t \ge 0, \forall i \in \mathcal{I}, \tag{PK}$$

*where the counting process $\{\boldsymbol{N}_t\}_{t\geq 0}$ is generated from the effort process $\boldsymbol{\Lambda}$. In addition, we need*

$$H_{ij,t} \leq W_{i,t-}, \quad \forall t \geq 0, \forall i,j \in \mathcal{I}, \tag{2.5}$$

*in order to satisfy* (IR).

The *promise keeping* condition (PK) generalizes Eq. (13) of Biais et al. (2010) to the multi-agent case, which ensures that $W_{i,t}$ is indeed the agent $i$'s continuation utility starting from time $t$. The term $H_{ij,t}$, if $H_{ij,t} > 0$, represents a downward jump in agent $i$'s promised utility when agent $j$ experiences an adverse event at time $t$ ($dN_{j,t} = 1$). If $H_{ij,t} < 0$, the jump is upward. It is intuitive that if agent $i$ experiences an adverse event, his own promised utility indeed takes a downward jump ($H_{ii,t} \geq 0$), because this decrease in agent $i$'s promised utility helps align the incentives between the principal and the agent.

Allowing $H_{ij,t} \neq 0$ for $i \neq j$, however, is an important modeling choice. A natural extension from the single-agent model may only include terms $H_{ii,t}$ without $H_{ij,t}$, and the corresponding dynamics of an agent's promised utilities would follow

$$\mathrm{d}W_{i,t} = \left(\rho W_{i,t-} - bX_{i,t}\mathbb{1}_{\{\lambda_{i,t}=\bar{\lambda}\}} + \lambda_{i,t}H_{ii,t}\right)\mathrm{d}t - H_{ii,t}\mathrm{d}N_{i,t} - \mathrm{d}L_{i,t}, \quad \forall t \geq 0, \forall i \in \mathcal{I},$$

which reduces to the single-agent promised keeping condition when $|\mathcal{I}| = 1$. As we will see later in the paper, allowing $H_{ij,t}$ terms to be non-zero not only provides more flexibility in contract design, it is essential for us to obtain efficient and incentive compatible contracts. Setting $H_{ij,t}$ to be positive or negative implies penalizing or rewarding agent $i$ when agent $j$ induces an adverse event. Generally speaking, the sign of $H_{ij,t}$ is unclear *a priori*, and we do not impose restrictions on it. Later in Section 3.4, we will explain in detail the economic implication of $H_{ij,t}$ for $i \neq j$. For the clarity of notation, denote $\boldsymbol{H}_{\cdot i,t} = \{H_{1i,t}, H_{2i,t}, \ldots, H_{ni,t}\}$ to represent the jumps in the promised utilities of agents due to an adverse arrival caused by agent $i$.

Finally, we need to introduce a parameter, $\bar{w}$, to upper bound the promised utilities of agents. This parameter reflects the "commitment power" of the principal, and is necessary to the model due to the "infinite back loading" problem, where the principal always prefers to delay paying agents while promising to pay the corresponding interest (Myerson 2015). Without $\bar{w}$, the principal can indefinitely delay the payments such that the promised utilities grow to infinity without agents ever being paid. Therefore, we need the following *Upper Bound* (or UB) constraints,

$$W_{i,t} \leq \bar{w}, \quad \forall t \geq 0, \forall i \in \mathcal{I}. \tag{UB}$$

### 2.2. Incentive Compatibility

In this paper, we focus on contracts that always induce effort from agents. We first denote $\beta$ to represent the ratio between the shirking benefit $b$ and the difference in arrival rates,

$$\beta := \frac{b}{\Delta\lambda}.$$

Similar to Biais et al. (2010), we assume that $\beta \leq C$, which ensures that it is worthwhile to induce full effort. A contract $\Gamma$ is called *incentive-compatible* (or IC), if it induces all agents to always exert effort and maintain a low arrival rate of adverse events. More precisely, this condition requires that compared with any other strategy, agent $i$ attains a higher total utility if always exerting effort, given all the other agents also always exert effort. That is,

$$u_i(\Gamma, \bar{\mathbf{\Lambda}}) \geq u_i(\Gamma, \tilde{\mathbf{\Lambda}}_i), \quad \forall i \in \mathcal{I},\ \forall \tilde{\mathbf{\Lambda}}_i, \tag{2.6}$$

where $\bar{\mathbf{\Lambda}} := \{\lambda_{i,t} = \lambda, \forall i \in \mathcal{I}, \forall t \geq 0\}$, and $\tilde{\mathbf{\Lambda}}_i := \{\lambda_{i,t} \text{ is } \mathcal{F}_t^{\boldsymbol{N}}\text{-predictable, and } \lambda_{j,t} = \lambda, \forall j \neq i, \forall t \geq 0\}$.

Intuitively, if the principal were to charge the agent $i$ an amount of $\beta X_{i,t}$ for each adverse arrival, then agent $i$ would be indifferent between exerting effort and shirking. Heuristically, in a small time interval $\delta$, agent $i$ enjoys a shirking benefit $bX_{i,t}\delta$, which needs to be offset by the higher penalty cost $\Delta\lambda\beta X_{i,t}$. Nonetheless, charging agents is not allowed in our setting, so the principal instead reduces agent $i$'s promised utility by at least $\beta X_{i,t}$ for each arrival to induce effort. In summary, the value $\beta X_{i,t}$ is the minimum penalty on agent $i$ if he induces an adverse event. We formalize this result in the following proposition, which extends Proposition 1 of Biais et al. (2010) to our multi-agent setting.

Proposition 2.1. *Contract $\Gamma$ satisfies the incentive-compatible condition (2.6) if and only if*

$$H_{ii,t} \geq \beta X_{i,t}, \quad \forall t \geq 0,\ \forall i \in \mathcal{I}. \tag{IC}$$

Proposition 2.1 and condition (PK) imply that if agent $i$ causes an adverse event, then his promised utility shall take a downward jump of $H_{ii,t}$, which is at least $\beta X_{i,t}$, in order to incentivize agent $i$ to exert effort. Note that $H_{ij,t}$ for $i \neq j$ is not involved in (IC) conditions. Intuitively, the principal does not need to penalize the agent for an adverse event that is not associated with this agent.

Finally, in order to satisfy the (IR) condition after downward jumps, we have the following bounds on allocation, besides constraints (FX),

$$X_{i,t} \leq \min\left\{\frac{W_{i,t-}}{\beta},\ 1\right\}, \quad \forall t \geq 0,\ \forall i \in \mathcal{I}. \tag{2.7}$$

Therefore, it is possible, although not efficient, that the total allocation is strictly less than one when the sum of agents' promised utilities is too low.

### 2.3. Social Welfare Objective

Under an incentive-compatible contract $\Gamma$, the total social welfare is defined as

$$S(\Gamma) := \mathbb{E}^{\bar{\mathbf{\Lambda}}}\left[\int_0^\infty e^{-\rho t}\sum_{i\in\mathcal{I}}(RX_{i,t}\mathrm{d}t - CX_{i,t}\mathrm{d}N_{i,t})\right], \tag{2.8}$$

which includes the revenue from allocating the resources to agents minus the cost of adverse events. Payments are internal transfers within the system and therefore do not appear in the social welfare definition (2.8).

In this paper, we focus on studying efficient contracts that maximize social welfare $S(\Gamma)$ among all contracts $\Gamma = \{\boldsymbol{L}_t, \boldsymbol{X}_t\}_{t\geq 0}$ that satisfy (FX), (IC), (IR), (LL), (PK), and (UB) starting from an initial vector of promised utilities. As mentioned in the introduction, social welfare maximization is not only the focus of non-for-profit organizations or governments, but has also become a key consideration of profit-driven companies.

It is easy to verify that

$$S(\Gamma) = (R - \lambda C)\mathbb{E}^{\bar{\mathbf{\Lambda}}}\left[\int_0^\infty e^{-\rho t}\sum_{i\in\mathcal{I}}X_{i,t}\mathrm{d}t\right]. \tag{2.9}$$

Together with the assumption (2.1), we know that the *efficient allocation*, that is, $\sum_i X_{i,t} = 1$ for all $t \geq 0$, maximizes the social welfare. The corresponding maximum social welfare is $(R - \lambda C)/\rho$. In the next section, we establish that incentive-compatible efficient allocation contracts indeed exist with two or more agents.

## 3. Incentive-compatible Contracts with Efficient Resource Allocation

As discussed in the previous section, efficient allocation maximizes the social welfare. However, with information asymmetry, it is unclear whether there exists an incentive-compatible contract that achieves efficient allocation. In fact, in the single-agent case, where effort reduces the arrival rate of adverse events, incentive-compatible contracts generally cannot achieve efficiency. Biais et al. (2010), for example, studies a similar single-agent problem, in which the agent is a firm who needs to exert effort to reduce the arrival rate of adverse events. The principal (a financier, or society at large) dynamically adjusts the firm size besides cash payments to induce effort. In their setting, the principal has to downsize the firm to yield credible threats when the agent's promised utility is lower than a certain threshold, which is economically inefficient.

The same logic behind inefficient allocation for a single-agent case works in our setting as well. Consider player $i$ as the only agent in town. No matter how high the promised utility has reached, as long as the arrival rate $\lambda > 0$, there is always a positive probability such that a sequence of frequent arrivals pushes the promised utility below $\beta$. At this point, the (IC) condition implies that we cannot maintain incentive compatibility with $X_{i,t} = 1$ while still satisfying (IR). In this case, we have to reduce $X_{i,t}$ to satisfy both (IC) and (IR), which is inefficient. Similar intuition behind efficiency and incentive compatibility is also discussed in Tian et al. (2019).

In contrast, and perhaps not *a priori* obvious, for the multi-agent setting of our paper, incentive-compatible contracts that achieve first-best efficient allocation do exist. The fundamental rationale is that when one agent has to be penalized by a reduction in allocated resource, the principal could transfer the revoked resources to other agent(s). This "pooling effect" in the multi-agent setting provides possibility for an efficient contract. We refer to incentive-compatible contracts that achieve efficiency as *efficient and incentive-compatible* (EIC) contracts.

Formally, we provide the following definition.

DEFINITION 3.1. We call a contract $\Gamma = \{\boldsymbol{L}_t, \boldsymbol{X}_t\}_{t\geq 0}$ *EIC contract*, if there exists a promised utility process $\{\boldsymbol{W}_t\}_{t\geq 0}$ with $W_{i,0} = u_i(\Gamma, \bar{\boldsymbol{\Lambda}})$, such that $\boldsymbol{L}_t$, $\boldsymbol{X}_t$, and $\boldsymbol{W}_t$ satisfy (IC), (IR), (PK), (LL), (UB), as well as the following "Efficient Allocation" condition,

$$\sum_{i\in\mathcal{I}} X_{i,t} = 1 \text{ and } X_{i,t} \geq 0, \quad \forall t \geq 0,\ \forall i \in \mathcal{I}. \tag{EA}$$

In other words, instead of satisfying (FX), the allocation decision in an EIC contract needs to always be efficient. It is worth noting that EIC contracts need to satisfy constraint (2.7), following (IC) and (IR).

### 3.1. Achievable Promised Utility and Self-generating Sets

In order to study EIC contracts, including their existence, it is helpful to consider the set of promised utilities that EIC contracts are able to achieve. Specifically, we define the following *achievable set* of promised utilities by EIC contracts as

$$\mathcal{U} := \left\{ \boldsymbol{w} = \{w_i\}_{i\in\mathcal{I}} \in [0, \bar{w}]^n \;\middle|\; \exists \text{ EIC contract } \Gamma, \text{ such that } w_i = u_i(\Gamma, \bar{\boldsymbol{\Lambda}}) \right\}. \tag{3.1}$$

We call elements of achievable set $\mathcal{U}$ *achievable promised utilities.* Therefore, the existence of an EIC contract is equivalent to the non-emptiness of the set of achievable utilities $\mathcal{U}$. Moreover, we claim, and later will show, that if an EIC contract $\Gamma$ exists and yields a promised utility process $\{\boldsymbol{W}_t\}_{t\geq 0}$, the definition of the achievable set $\mathcal{U}$ implies that for any time $t \geq 0$, we have $\boldsymbol{W}_t \in \mathcal{U}$. In other words, starting from an achievable set of promised utilities, all future promised utilities must also be achievable by some EIC contracts, and belong to this achievable set.

This evokes the concept of *self-generating set*, which was first introduced in the seminal paper Abreu et al. (1990) for repeated games, and is later used in Fudenberg et al. (1994) for imperfect public information repeated games, and Balseiro et al. (2019) for dynamic mechanism design problems. All these papers study games in discrete-time settings. In particular, Abreu et al. (1990) introduce a recursive approach to characterize a self-generating set. In all these papers, the fundamental definition of a self-generating set is that if a promised utility belongs to the set, the next period's promised utility, according to the promised keeping constraint, must also belong to this set. We cannot directly inherit the concept of "self-generation" from the aforementioned literature. This is because in our continuous-time setting, the notion of "next period" is not well defined. Therefore, we extend the definition of the self-generating set to the continuous-time setting as follows.

DEFINITION 3.2. A set $\mathcal{A} \subseteq [0, \bar{w}]^n$ is called a *self-generating* set if for any $\boldsymbol{W}_0 \in \mathcal{A}$, there exist $\mathcal{F}_t^{\boldsymbol{N}}$-predictable processes $\boldsymbol{H}_t$, $\boldsymbol{X}_t$, and $\boldsymbol{L}_t$, and an $\mathcal{F}_t^{\boldsymbol{N}}$-adapted process $\{\boldsymbol{W}_t\}_{t\geq 0}$ starting from $\boldsymbol{W}_0$, that satisfy (EA), (IC), (IR), (PK), (LL), and (UB), such that $\boldsymbol{W}_t \in \mathcal{A}$ for all $t \geq 0$.

Definition 3.2 implies that, should the agents start with promised utilities inside a self-generating set, their future promised utilities following an EIC contract would always stay in the same set.

Next, we draw the explicit connection between the self-generating set and the achievable set of promised utilities.

PROPOSITION 3.1. *If set $\mathcal{A}$ is a self-generating set, then $\mathcal{A} \subseteq \mathcal{U}$.*

Proposition 3.1 states that every self-generating set is a subset of the achievable set. It implies that for any self-generating set $\mathcal{A}$, and a vector of promised utilities $\boldsymbol{W}_0 \in \mathcal{A}$, we have $\boldsymbol{W}_t \in \mathcal{A} \subseteq \mathcal{U}$ for any time $t \geq 0$. That is, there must exist an EIC contract $\Gamma$, such that for any $i \in \mathcal{I}$, contract $\Gamma$ delivers utility $W_{i,0}$ to agent $i$.

The next proposition further tightens the relationship between self-generating sets and the set of achievable utilities.

PROPOSITION 3.2. *The achievable set $\mathcal{U}$ is a self-generating set.*

Propositions 3.1 and 3.2 imply that the achievable set $\mathcal{U}$ is the largest self-generating set. However, we have yet to show under what condition such a set is non-empty. Next, we propose an approach to characterize the achievable set of promised utilities, from which we further derive EIC contracts.

### 3.2. Iterative Approach for Achievable Set

In this subsection, we propose an iterative approach to characterize the achievable set of promised utilities, $\mathcal{U}$. In a nutshell, our approach is based on using "support functions" to describe the achievable set, which is convex. Balseiro et al. (2019) proposed the support function approach to study dynamic mechanism design without money and with multiple agents in a discrete-time setting. We extend the support function approach to our moral hazard problem in the continuous-time setting. Furthermore, our construction of support functions not only describes the achievable set, but also yields an EIC contract, as detailed in Subsection 3.3. This approach was not pursued in Balseiro et al. (2019) to obtain a mechanism.

The general idea of the iterative approach stems from the discussion in the previous subsection, that the achievable set $\mathcal{U}$ is the largest self-generating set. Specifically, in each iteration, based on a support function from the previous iteration, we solve a sequence of time-independent static optimization problems to obtain a new support function, which also defines a convex set. Starting from the set $[0,\bar{w}]^n$, support functions throughout the iterations gradually shrink the convex set until convergence. The final convex set, if not empty, is a self-generating set, and also the achievable set of promised utilities.

To this end, consider the following support function $\phi_{\mathcal{A}}:\mathbb{R}^n\to\mathbb{R}$ of any set $\mathcal{A}\subset\mathbb{R}^n$,

$$\phi_{\mathcal{A}}(\boldsymbol{\alpha}):=\inf_{\boldsymbol{w}\in\mathcal{A}}\boldsymbol{\alpha}^\top\boldsymbol{w},\quad \forall\boldsymbol{\alpha}\in\mathbb{R}^n_+ \text{ with } \|\boldsymbol{\alpha}\|_1=1.$$

The hyperplane $\{\boldsymbol{x}|\boldsymbol{\alpha}^\top\boldsymbol{x}=\phi_{\mathcal{A}}(\boldsymbol{\alpha})\}$ is a supporting hyperplane of set $\mathcal{A}$ with normal direction $\boldsymbol{\alpha}$. We focus on supporting hyperplanes with positive normal vectors $\boldsymbol{\alpha}\in\mathbb{R}^n_+$, because if a promised utility $\boldsymbol{w}$ is achievable by an EIC contract, any promised utility that is component-wise greater than or equal to $\boldsymbol{w}$ is also achievable, as shown in the technical Lemma EC.2.1 in the appendix. Moreover, from any support function $\phi$, we can define a closed convex set as the following,

$$\mathscr{G}(\phi):=\left\{\boldsymbol{w}\in[0,\bar{w}]^n \;\middle|\; \boldsymbol{\alpha}^\top\boldsymbol{w}\ge\phi(\boldsymbol{\alpha}),\ \forall\boldsymbol{\alpha}\in\mathbb{R}^n_+,\ \|\boldsymbol{\alpha}\|_1=1\right\}, \tag{3.2}$$

referred to as the set characterized by support function $\phi$. It is clear that for any set $\mathcal{A}\subseteq[0,\bar{w}]^n$, we must have

$$\mathcal{A}\subseteq\mathscr{G}(\phi_{\mathcal{A}}). \tag{3.3}$$

Next, we define an operator $T$, which maps from one support function to another, forming the foundation of our iterative approach. This operator is defined through the following linear program for any function $\phi:\mathbb{R}^n_+\to\mathbb{R}$ and vector $\boldsymbol{\alpha}\in\mathbb{R}^n_+$,

$$[T\phi](\boldsymbol{\alpha}):=\inf_{\boldsymbol{w},\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^n;\boldsymbol{H},\boldsymbol{Z}\in\mathbb{R}^{n\times n}}\boldsymbol{\alpha}^\top\boldsymbol{w}$$

$$\text{s.t.}\quad \sum_{i=1}^n x_i=1, x_i\ge 0,\quad \forall i\in\mathcal{I}, \tag{EA$_\text{s}$}$$

$$H_{ii}\ge\beta x_i,\quad \forall i\in\mathcal{I}, \tag{IC$_\text{s}$}$$

$$y_i=\rho w_i+\lambda\sum_{j\in\mathcal{I}}H_{ij},\quad \forall i\in\mathcal{I}, \tag{PK$_{\boldsymbol{y}}$}$$

$$Z_{ij} = w_i - H_{ij}, \quad \forall i, j \in \mathcal{I}, \tag{PK$_{\boldsymbol{Z}}$}$$

$$\hat{\boldsymbol{\alpha}}^\top \boldsymbol{w} \geq \phi(\hat{\boldsymbol{\alpha}}), \quad \forall \hat{\boldsymbol{\alpha}} \in \mathbb{R}^n_+, \ \|\hat{\boldsymbol{\alpha}}\|_1 = 1, \tag{SG$_{\boldsymbol{w}}$}$$

$$\boldsymbol{\alpha}^\top \boldsymbol{y} \geq 0, \tag{SG$_{\boldsymbol{y}}$}$$

$$\hat{\boldsymbol{\alpha}}^\top \boldsymbol{Z}_{\cdot j} \geq \phi(\hat{\boldsymbol{\alpha}}), \quad \forall j \in \mathcal{I}, \ \forall \hat{\boldsymbol{\alpha}} \in \mathbb{R}^n_+, \ \|\hat{\boldsymbol{\alpha}}\|_1 = 1, \tag{SG$_{\boldsymbol{Z}}$}$$

$$w_i \geq 0, \quad \forall i \in \mathcal{I}, \tag{IR$_\mathrm{s}$}$$

$$w_i \leq \bar{w}, \quad \forall i \in \mathcal{I}, \tag{UB$_\mathrm{s}$}$$

where notation $\boldsymbol{Z}_{\cdot j}$ represents the vector $(Z_{ij})_{i\in\mathcal{I}}$. For any normal vector $\boldsymbol{\alpha}$, the linear program $[T\phi](\boldsymbol{\alpha})$ returns a real value. Therefore, $T\phi$ is also a function that maps $\mathbb{R}^n_+$ to $\mathbb{R}$, the same as $\phi$.

In the optimization problem $[T\phi](\boldsymbol{\alpha})$, decision variables $\boldsymbol{x}$ and $\boldsymbol{w}$ correspond to resource allocation and the current promised utilities, respectively. Decision variable $H_{ij}$ in $\boldsymbol{H}$ corresponds to the jump to agent $i$'s promised utility upon the arrival of an adverse event at agent $j$. It is easy to see that constraints (EA$_\mathrm{s}$), (IC$_\mathrm{s}$), (IR$_\mathrm{s}$) and (UB$_\mathrm{s}$) resemble (EA), (IC), (IR) and (UB) for an EIC contract, respectively. Note that we do not include any payment decisions in this linear optimization model. In fact, we later show that it is sufficient to use this linear optimization to describe the achievable set, and adding decision variables representing payments does not help. As mentioned earlier, this linear optimization also helps us construct a particular EIC contract. In such a construction, we can identify payments from the optimal solution of this linear program.

In order to explain variables $\boldsymbol{y}$ and $\boldsymbol{Z}$, it is helpful to rewrite condition (PK) without payments as

$$\mathrm{d}W_{i,t} = \left(\rho W_{i,t-} + \lambda \sum_{j\in\mathcal{I}} H_{ij,t}\right)\mathrm{d}t - \sum_{j\in\mathcal{I}} H_{ij,t}\mathrm{d}N_{j,t}, \quad \forall t \geq 0, \ \forall i \in \mathcal{I}. \tag{3.4}$$

Therefore, constraint (PK$_{\boldsymbol{y}}$) defines variable $y_i$ as the smoothly changing term that multiplies $\mathrm{d}t$ in (3.4), and (PK$_{\boldsymbol{Z}}$) implies that $Z_{ij}$ represents the change in agent $i$'s promised utility when an arrival occurs at agent $j$ ($\mathrm{d}N_{j,t} = 1$ in (3.4)).

Finally, constraints (SG$_{\boldsymbol{w}}$), (SG$_{\boldsymbol{y}}$) and (SG$_{\boldsymbol{Z}}$) capture the self-generating property. In particular, constraint (SG$_{\boldsymbol{w}}$) ensures that the vector of current promised utilities, $\boldsymbol{w}$, is in the set $\mathscr{G}(\phi)$, while constraint (SG$_{\boldsymbol{Z}}$) ensures that promised utilities after an arrival of adverse events remain in the same set. Constraint (SG$_{\boldsymbol{y}}$) further makes sure that without an arrival, the vector of promised utilities moves towards the "interior" of the set $\mathscr{G}(\phi)$, a condition that is crucial when $\boldsymbol{w}$ is on the boundary of the convex set $\mathscr{G}(\phi)$. Constraint (SG$_{\boldsymbol{w}}$) implies that the optimal $\boldsymbol{w}$ to this linear program must satisfy $[T\phi](\boldsymbol{\alpha}) = \boldsymbol{\alpha}^\top \boldsymbol{w} \geq \phi(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ and $\|\boldsymbol{\alpha}\|_1 = 1$, which further implies that for any function $\phi$,

$$\mathscr{G}(T\phi) \subseteq \mathscr{G}(\phi). \tag{3.5}$$

The relationship (3.5) is at the foundation of our iterative approach. In particular, continuing applying operator $T$ to a support function $\phi$ generates a sequence of ever shrinking convex sets. The question is whether the set in the limit of the iterative process is desirable. The following result helps answer this question.

LEMMA 3.1. *If $\mathcal{A}$ is a self-generating set, then we have*

*1. $\mathcal{A} \subseteq \mathscr{G}(T\phi_{\mathcal{A}})$, and*

*2. $\mathscr{G}(\phi_{\mathcal{A}})$ is a self-generating set.*

*Conversely, if a convex set $\mathcal{A}$ satisfies $\mathcal{A} \subseteq \mathscr{G}(T\phi_{\mathcal{A}})$, then $\mathcal{A}$ is a self-generating set.*

Lemma 3.1 implies that for a convex set $\mathscr{G}(\phi)$ to be self-generating, a necessary and sufficient condition is $\mathscr{G}(\phi) \subseteq \mathscr{G}(T\phi)$. Also considering (3.5), we know that a set $\mathscr{G}(\phi)$ being self-generating is equivalent to

$$\mathscr{G}(\phi) = \mathscr{G}(T\phi).$$

Therefore, iteratively using operator $T$ on a support function, in the limit, we obtain the support function of a self-generating set. The following theorem further implies that if we start from the largest possible convex set $[0, \bar{w}]^n$ of promised utilities, this iterative process yields the largest possible self-generating set, the achievable set $\mathcal{U}$ defined in (3.1). (Recall the discussion right after Proposition 3.2.)

THEOREM 3.1. *Let $\mathcal{U}^0 = [0, \bar{w}]^n$, and define operator $T^k$ such that $T^k\phi = T(T^{k-1}\phi)$ for all $k > 1$. We have*

$$\lim_{k\to\infty} \mathscr{G}(T^k\phi_{\mathcal{U}^0}) = \mathcal{U}.$$

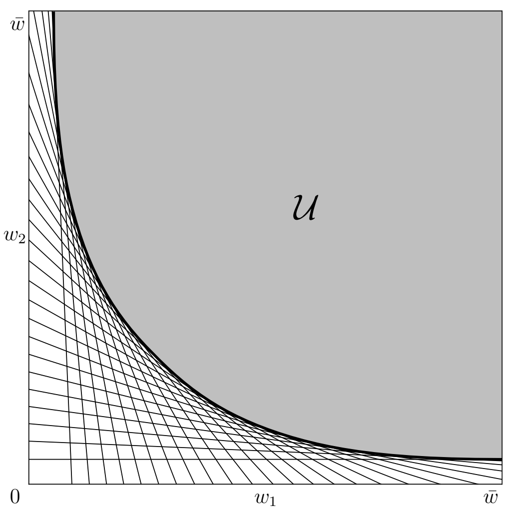
Theorem 3.1, together with Lemma 3.1 and Propositions 3.1 and 3.2, implies the following corollary.

COROLLARY 3.1. *The set of achievable utilities, $\mathcal{U}$, is closed and convex.*

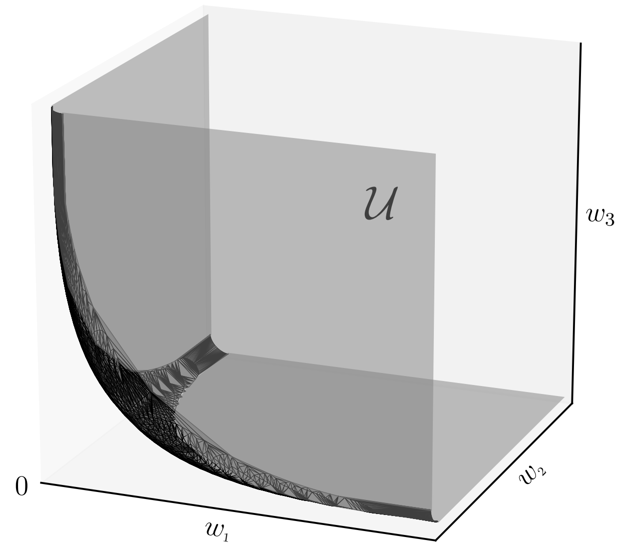
Next, in light of Theorem 3.1, we address the condition for the existence of the achievable set $\mathcal{U}$.

PROPOSITION 3.3. *There exists a threshold $\bar{\omega}$ that depends on model parameters $n$, $\rho$, $b$, $\lambda$ and $\bar{\lambda}$, such that the achievable set $\mathcal{U}$ is non-empty if and only if $\bar{w} \geq \bar{\omega}$.*

In Figure 1 we numerically illustrate the achievable sets found by the iterative approach for a two-agent and a three-agent case. Note that since the optimization problem $[T\phi](\boldsymbol{\alpha})$ is a semi-infinite linear program, we solve it approximately by considering a subset of constraints ($\mathrm{SG}_{\boldsymbol{w}}$) and ($\mathrm{SG}_{\boldsymbol{Z}}$) only for $\hat{\boldsymbol{\alpha}}$ on a grid, such that $\hat{\boldsymbol{\alpha}} \geq 0$ and $\|\hat{\boldsymbol{\alpha}}\|_1 = 1$. Echoing Corollary 3.1, both of the achievable sets are convex. Figure 1(a) also plots the supporting hyperplanes in solid lines, similar to Figure 1(a) in the Electronic Companion of Balseiro et al. (2019). It is noteworthy that with two agents, the set $\mathcal{U}$ does not intersect with the axes, indicating that the promised utilities of both agents inside $\mathcal{U}$ are strictly positive. Equivalently, an EIC contract never terminates either of the two agents.




(a) A two-agent case




(b) A three-agent case

**Figure 1** Illustration of achievable sets.

In Figure 1(b), however, the three hyperplanes defined by $w_1 = 0$, $w_2 = 0$, and $w_3 = 0$ do contribute to the boundary of the achievable set. That is, an EIC contract may terminate one of the agents. Furthermore, in this case, the intersection between the achievable set $\mathcal{U}$ and hyperplane $w_i = 0$ for $i = 1$, 2 or 3 is the achievable set in the two-agent case for the two remaining agents. In other words, an EIC contract for three agents may reduce to an EIC contract for a two-agent case upon terminating one agent. The reason for this observation is intuitive. If the current promised utilities for the three agents is $\boldsymbol{w} = (w_1, w_2, w_3 = 0)$, then any EIC contract cannot increase $w_3$ to be positive following (PK). Consequently, conditions $(w_1, w_2) \in \mathcal{U}(2)$ and $\boldsymbol{w} \in \mathcal{U}(3)$ are equivalent, where we use notation $\mathcal{U}(n)$ to highlight the achievable set in a setting with $n$ agents. In general, the aforementioned logic implies that on the boundary of the achievable set for $n$ agents, if we restrict one agent's promised utility to be zero, then we obtain the achievable set for $n-1$ agents.

Now that we know $\mathscr{G}(T\phi_{\mathcal{U}}) = \mathscr{G}(\phi_{\mathcal{U}}) = \mathcal{U}$, the linear program $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha})$ also directly implies the following sensitivity result on how model parameters affect the achievable set $\mathcal{U}$. Here we use notation $\mathcal{U}(\theta)$ to highlight model parameter $\theta$'s impact on the achievable set, in which $\theta$ could be $n$, $b$, $\rho$, $\Delta\lambda$, or $\bar{w}$.

PROPOSITION 3.4. *We have $\mathcal{U}(b_1) \supseteq \mathcal{U}(b_2)$ for $b_1 \leq b_2$ while keeping other model parameters the same. Similarly, we have $\mathcal{U}(\rho_1) \subseteq \mathcal{U}(\rho_2)$ for $\rho_1 \leq \rho_2$; $\mathcal{U}(\Delta\lambda_1) \subseteq \mathcal{U}(\Delta\lambda_2)$ for $\Delta\lambda_1 \leq \Delta\lambda_2$; and $\mathcal{U}(\bar{w}_1) \subseteq \mathcal{U}(\bar{w}_2)$ for $\bar{w}_1 \leq \bar{w}_2$.*

Finally, it is worth noting that allowing terms $H_{ij,t}$ for $i \neq j$ to be non-zero is an essential condition for the existence of EIC contracts. To see this, suppose that we fix $H_{ij,t} = 0$ for all $t$ and $i \neq j$. According to Definition 3.1, a contract $\Gamma = \{\boldsymbol{L}_t, \boldsymbol{X}_t\}_{t \geq 0}$ is an EIC contract if there exists a promised utility process $\{\boldsymbol{W}_t\}_{t \geq 0}$ with $W_{i,0} = u_i(\Gamma, \bar{\boldsymbol{\Lambda}})$, such that $\boldsymbol{L}_t$, $\boldsymbol{X}_t$, and $\boldsymbol{W}_t$ satisfy (EA), (IC), (IR), (PK), (LL), as well as (UB). Fixing $H_{ij,t} = 0$ for all $i \neq j$ effectively adds more constraints, and therefore shrinks the set of contracts under consideration. The following proposition states that with this restriction, EIC contracts no longer exist. This result highlights the importance of allowing the promised utilities of other agents to change when one agent experiences an adverse arrival.

PROPOSITION 3.5. *Introduce constraints $H_{ij} = 0$ for all $i \neq j$ to the linear programs $[T\phi](\boldsymbol{\alpha})$, and obtain a new linear program $[\tilde{T}\phi](\boldsymbol{\alpha})$. We have $\lim_{k \to \infty} \mathscr{G}(\tilde{T}^k \phi_{\mathcal{U}_0}) = \emptyset$, which implies that there does not exist an EIC contract with $H_{ij,t} = 0$ for all $t \geq 0$ and $i \neq j$.*

### 3.3. Boundary EIC Contract

The linear program $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha})$ not only provides the achievable set, but also yields an EIC contract. We call such contracts *boundary* EIC contracts because the promised utilities remain on the boundary of the achievable set. In this subsection, we formally define the boundary of the achievable set, and establish the existence of such a contract, along with its desirable properties.

First, we present the following technical result, which lays the foundation for the main results of this section.

PROPOSITION 3.6. *At any optimal solution to the linear program $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha})$, constraints ($\mathrm{IC_s}$) and ($\mathrm{SG}_{\boldsymbol{y}}$) hold as equalities, and for $\boldsymbol{\alpha}$ constraint ($\mathrm{SG}_{\boldsymbol{w}}$) holds as equality. Similarly, for each $j \in \mathcal{I}$, there exists an $\hat{\boldsymbol{\alpha}}$ such that the corresponding ($\mathrm{SG}_{\boldsymbol{Z}}$) constraint holds as equality.*

In order to define a boundary EIC contract, we formally define the *boundary* of an $n$-dimensional achievable set $\mathcal{U}(n) \subset [0, \bar{w}]^n$ to be

$$\mathrm{bd}(\mathcal{U}(n)) := \mathrm{cl}\left\{\boldsymbol{w} \in \mathcal{U} \text{ and } \boldsymbol{w} > 0 \mid \exists \boldsymbol{\alpha} \in \mathbb{R}^n_+ \text{ and } \|\boldsymbol{\alpha}\|_1 = 1 \text{ such that } \boldsymbol{\alpha}^\top \boldsymbol{w} = \phi_{\mathcal{U}}(\boldsymbol{\alpha})\right\}.$$

Therefore, it is the closure of the set of component-wise positive vectors on the boundary of the achievable set. When $n = 2$, any $\boldsymbol{w}$ on the boundary $\mathrm{bd}(\mathcal{U})$ must be component-wise positive. For $n > 2$ however, up to $n-2$ components of $\boldsymbol{w} \in \mathrm{bd}(\mathcal{U})$ may be zero. For instance, as shown in Figure 1(b), the boundary $\mathrm{bd}(\mathcal{U})$ corresponds to the dark meshed part of the set $\mathcal{U}$. The intersections of this boundary with the hyperplanes $w_i = 0$ for $i = 1,\ 2$, or 3 reduce to the boundary sets for the achievable sets of respective two-agent settings. In this case, we introduce a notation $\mathscr{C}(\boldsymbol{w})$ to represent collapsing vector $\boldsymbol{w}$ into one that only contains its positive elements, that is,

$$\mathscr{C}(\boldsymbol{w}) := (w_i)_{i:w_i>0}.$$

It is clear that if the dimension of $\mathscr{C}(\boldsymbol{w})$ is $m < n$ for a vector $\boldsymbol{w} \in \mathrm{bd}(\mathcal{U}(n))$, then we must have $\mathscr{C}(\boldsymbol{w}) \in \mathrm{bd}(\mathcal{U}(m))$.

Furthermore, for any $\boldsymbol{w} \in \mathbb{R}_+^n$, define $\Pi(\boldsymbol{w}) \in \mathrm{bd}(\mathcal{U})$ to be a projection of $\boldsymbol{w}$ onto the boundary $\mathrm{bd}(\mathcal{U})$, such that $\Pi(\boldsymbol{w})$ solves

$$\min_{\boldsymbol{\xi} \in \mathrm{bd}(\mathcal{U})} \|\boldsymbol{w} - \boldsymbol{\xi}\|_2. \tag{3.6}$$

Based on this projection, define a mapping $\check{\boldsymbol{\alpha}} : \mathbb{R}_+^n \to \mathbb{R}_+^n$, such that

$$\check{\boldsymbol{\alpha}}(\boldsymbol{w})^\top \Pi(\boldsymbol{w}) = \phi_{\mathcal{U}}\left(\check{\boldsymbol{\alpha}}(\boldsymbol{w})\right). \tag{3.7}$$

That is, for any $\boldsymbol{w} \in \mathcal{U}$, we have $\boldsymbol{w} - \Pi(\boldsymbol{w}) = c(\boldsymbol{w})\check{\boldsymbol{\alpha}}(\boldsymbol{w})$ for some scalar $c(\boldsymbol{w}) \in \mathbb{R}_+$ associated with $\boldsymbol{w}$. Following the EIC contract to be defined next, the vector of promised utility always stays on $\mathrm{bd}(\mathcal{U})$. Therefore, $\check{\boldsymbol{\alpha}}(\boldsymbol{w})$ is the normal vector of $\mathrm{bd}(\mathcal{U})$ at any point $\boldsymbol{w} \in \mathrm{bd}(\mathcal{U})$.

Starting from any vector of promised utilities $\boldsymbol{W}_0 \in \mathrm{bd}(\mathcal{U})$, we define a promised utility process $\{\boldsymbol{W}_t\}_{t\ge 0}$ together with an $\mathcal{F}^{\boldsymbol{N}}$-adapted process $\{\boldsymbol{\alpha}_t\}_{t\ge 0}$, a payment process $\{\boldsymbol{L}_t\}_{t\ge 0}$, an allocation process $\{\boldsymbol{X}_t\}_{t\ge 0}$, and a process of jumps $\{\boldsymbol{H}_t\}_{t\ge 0}$, such that we replace (2.4) with

$$\boldsymbol{W}_{t_2} = \mathscr{C}\left(\left(W_{i,t_1} + \int_{(t_1,t_2]} \mathrm{d}W_{i,s}\right)_{i:W_{i,t_1}>0}\right),\ \forall t_1 < t_2, \tag{3.8}$$

and define

$$\boldsymbol{\alpha}_t := \check{\boldsymbol{\alpha}}(\boldsymbol{W}_t), \tag{3.9}$$

$$\boldsymbol{X}_t := \boldsymbol{x}^*(\boldsymbol{\alpha}_{t-}), \tag{3.10}$$

$$\boldsymbol{H}_t := \boldsymbol{H}^*(\boldsymbol{\alpha}_{t-}), \tag{3.11}$$

$$\mathrm{d}L_{i,t} := (y_i^*(\boldsymbol{\alpha}_{t-}))^+ \mathbb{1}_{w_i^*(\boldsymbol{\alpha}_{t-}) = \bar{w}} \mathrm{d}t, \text{ and} \tag{3.12}$$

$$\mathrm{d}W_{i,t} := y_i^*(\boldsymbol{\alpha}_{t-})\mathrm{d}t + \sum_{j\in\mathcal{I}} \left(Z_{ij}^*(\boldsymbol{\alpha}_{t-}) - w_i^*(\boldsymbol{\alpha}_{t-})\right) \mathrm{d}N_{j,t} - \mathrm{d}L_{i,t}, \tag{3.13}$$

in which we use notations $\boldsymbol{H}^*(\boldsymbol{\alpha})$, $\boldsymbol{w}^*(\boldsymbol{\alpha})$, $\boldsymbol{x}^*(\boldsymbol{\alpha})$, $\boldsymbol{y}^*(\boldsymbol{\alpha})$ and $\boldsymbol{Z}^*(\boldsymbol{\alpha})$ to represent the optimal decision variables from $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha})$.

The next theorem indicates that there exists an EIC contract which yields a promised utility process following (3.8)-(3.13), and the promised utilities $\boldsymbol{W}_t$ always stay on the boundary $\mathrm{bd}(\mathcal{U})$.

THEOREM 3.2. *Starting from any $\boldsymbol{W}_0 \in \mathrm{bd}(\mathcal{U})$, processes $\{\boldsymbol{W}_t\}_{t\geq 0}$, $\{\boldsymbol{X}_t\}_{t\geq 0}$, $\{\boldsymbol{L}_t\}_{t\geq 0}$ and $\{\boldsymbol{H}_t\}_{t\geq 0}$ defined in (3.8)-(3.13) satisfy (EA), (IC), (IR), (PK), (LL), and (UB). Furthermore, $\boldsymbol{W}_t \in \mathrm{bd}(\mathcal{U})$ for all $t \geq 0$.*

It is worth explaining the payment process described in (3.12). According to this expression, the EIC contract as specified does not involve any instantaneous payment to an agent. This implies that upon an arrival at agent $j$, the upward jump $-H^*_{ij}$ to agent $i$'s promised utility is bounded above such that $w^*_i - H^*_{ij} \leq \bar{w}$. Otherwise agent $i$ would receive an instantaneous payment which equals to the difference, $w^*_i - H^*_{ij} - \bar{w}$, which would guarantee that agent $i$'s promised utility does not exceed the upper bound $\bar{w}$. In addition, the flow payment only occurs when the promised utility of an agent is hitting the upper bound $\bar{w}$. It is clear that the payment $\boldsymbol{L}_t$ according to (3.12) satisfies condition (LL). Therefore, although we do not explicitly include payment-related decision variables in the linear program $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha})$, the optimal solution of this linear program allows us to construct an EIC contract, including its payment process. It is worth noting that not all EIC contracts are defined according to Theorem 3.2. For instance, we cannot rule out the possibility that an EIC contract exists in which the payment is set as follows

$$\mathrm{d}L_{i,t} = \sum_j (w^*_i - H^*_{ij} - \bar{w})^+ \mathrm{d}N_{j,t} + (y^*_i)^+ \mathbb{1}_{w^*_i = \bar{w}} \mathrm{d}t,$$

where $x^+$ to represent $\max\{x, 0\}$, and the instantaneous payment $(w^*_i - H^*_{ij} - \bar{w})^+$ is positive.

The following result further states that following *any* EIC contract, if the vector of promised utilities is on the lower-left boundary of $\mathcal{U}$ at some point, they always stay on the boundary ever after.

COROLLARY 3.2. *For any EIC contract that yields a promised utility process $\{\boldsymbol{W}_t\}_{t\geq 0}$, if $\boldsymbol{W}_t \in \mathrm{bd}(\mathcal{U})$ for some time $t \geq 0$, then $\boldsymbol{W}_{t'} \in \mathrm{bd}(\mathcal{U})$ for all $t' \geq t$.*

Corollary 3.2 motivates us to study EIC contracts that keep the promised utility on the lower-left boundary of the achievable set $\mathcal{U}$ from the beginning and throughout the time horizon. We call these contracts *boundary EIC contracts*. In particular, the following corollary indicates that for any boundary EIC contract, the flow payment is zero unless one agent's promised utility is at $\bar{w}$.

COROLLARY 3.3. *For any boundary EIC contract, the flow payment $l_{i,t}$ is always zero except when agent $i$'s promised utility is at $\bar{w}$.*

Furthermore, we present the following result, which indicates that among all EIC contracts, a boundary EIC contract maximizes the principal's utility. Although our paper is focused on social welfare maximizing contracts, a principal may find this particular EIC contract desirable. Recall the definition of $u_i(\Gamma, \boldsymbol{\Lambda})$ from (2.2). With a slight abuse of notations, we use the following simplification to represent the agents' promised utilities that an EIC contract $\Gamma$ delivers under full efforts from agents,

$$\boldsymbol{u}(\Gamma) := \{u_i(\Gamma)\}_{i\in\mathcal{I}}, \quad \text{in which} \quad u_i(\Gamma) := u_i(\Gamma, \bar{\boldsymbol{\Lambda}}).$$

Clearly, for any EIC contract $\Gamma$, we have $\boldsymbol{u}(\Gamma) \in \mathcal{U}$. Further define the principal's total discounted utility under contract $\Gamma$ as

$$U(\Gamma) := S(\Gamma) - \sum_{i\in\mathcal{I}} u_i(\Gamma) = \mathbb{E}^{\bar{\boldsymbol{\Lambda}}}\left[\int_0^\infty e^{-\rho t}\sum_{i\in\mathcal{I}}(RX_{i,t}\mathrm{d}t - CX_{i,t}\mathrm{d}N_{i,t} - \mathrm{d}L_{i,t})\right],$$

where the social welfare from contract $\Gamma$ is defined in (2.8). That is, in order to obtain the principal's utility, we subtract from the social welfare the total agent utility, which is the total discounted payments that agents collect. The following proposition states that the EIC contract that maximizes the principal's utility is a boundary contract.

Proposition 3.7. *Consider the EIC contracts $\hat{\Gamma}$ such that $\boldsymbol{u}(\hat{\Gamma}) \in \mathrm{bd}(\mathcal{U})$ and $u_i(\hat{\Gamma}) = u_j(\hat{\Gamma})$ for all $i \neq j$. We have $U(\hat{\Gamma}) \geq U(\Gamma)$ for any EIC contract $\Gamma$.*

That is, the EIC contract that maximizes the principal's utility is the boundary contract that starts with all agents' promised utilities being the same to each other. In the next subsection, we illustrate details of boundary contracts, including $\hat{\Gamma}$, when the number of agents $n$ is two or three.

### 3.4. Sample Trajectories and Discussions

For the two-agent case, we first present a result, which partially describes boundary EIC contracts. We then use figures to illustrate the features and dynamics of boundary EIC contracts.

Proposition 3.8. *When $n = 2$, there exists an optimal solution to $[T\phi](\boldsymbol{\alpha})$, denoted by $(\boldsymbol{w}^*, \boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{Z}^*, \boldsymbol{H}^*)$, that satisfies the following conditions,*

*(i) if $w_1^* = \bar{w}$, then $w_2^* = \inf_{\boldsymbol{w}\in\mathcal{U}} w_2 > 0$, $x_1^* = 1$, $x_2^* = 0$, $y_1^* = \rho\bar{w} + \lambda\beta$, and $y_2^* = 0$;*

*(ii) if $\alpha_1 = \alpha_2 = 0.5$, then $w_1^* = w_2^*$, $x_1^* = x_2^* = 0.5$, and $y_1^* = y_2^* = 0$;*

Proposition 3.8(i) states that at this point, all of the resource is allocated to agent 1 while agent 2 receives no resource ($x_1^* = 1$ and $x_2^* = 0$). However, agent 2's promised utility is still positive, because the agent may still receive a positive resource and payments in the future due to adverse arrivals occurring to agent 1. Before such an adverse event occurs, however, both agents' promised utilities stay at this terminal point of the boundary ($y_1^* - l_{1,t} = y_2^* = 0$, Corollary 3.3 and equation (3.12)). Furthermore, Proposition 3.8(ii) illustrates that starting at the middle point of the boundary (contract $\hat{\Gamma}$), each of the two agents receives half of the resource, and both promised utilities stay the same before an adverse event occurs to one of them.

Although the latter point appears intuitive due to symmetry, it is worth mentioning that in single-agent settings with adverse events, the agent's promised utility generally does not stay the same. Instead, before reaching $\bar{w}$, it usually drifts upwards between arrivals, due to rent and accrued interests (see Biais et al. 2010, Myerson 2015, Chen et al. 2019, for several examples). As reflected in our (PK) condition with more than one agent, the term $H_{ij,t}$ not only captures the jump in agent $i$'s promised utility upon an arrival at agent $j$ (in $H_{ij,t}\mathrm{d}N_{j,t}$), but also modulates the drift of agent $i$'s promised utility when there is no arrival (in $\sum_j \lambda_{j,t}H_{j,t}\mathrm{d}t$). Effectively, when the two players exert full effort, the drift staying at zero implies that $-(\rho W_{1,t-} + \lambda H_{11,t}) = \lambda H_{12,t}$. That is, the term $\lambda H_{12,t}$ exactly offsets the rent from unobservable actions ($\lambda H_{11,t}$) and the interests from positive promised utilities ($\rho W_{i,t-}$). In other words, the fact that $H_{12,t} < 0$ implies that agent 1 receives a reward when the competitor, agent 2, experiences an adverse event ($-H_{12,t}\mathrm{d}N_{2,t}$ in (PK)). Consequently, the contract designer is able to take advantage of the competition among multiple agents to avoid offering interests or rent to either of them in this case. In the two-agent setting, $H_{12,t}$ and $H_{21,t}$ from our boundary EIC contract are always negative. That is, one agent is always better off if an adverse event occurs at his competitor. When there are more than two agents, $H_{ij,t}$ for all $i \neq j$ could also be positive, depending on the promised utilities of all agents at the time. This corresponds to the situation that one agent's adverse event may hurt a competitor's promised utility. We will illustrate this in a numerical example later. To summarize, introducing the terms $H_{ij,t}$ for all $i \neq j$ in the model allows the designer to better leverage competition in order to design dynamic contracts.

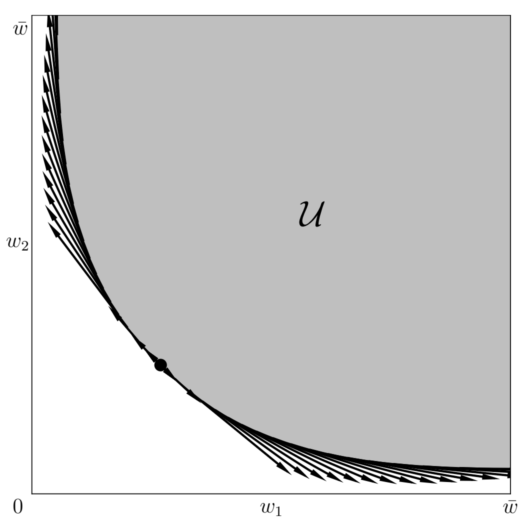
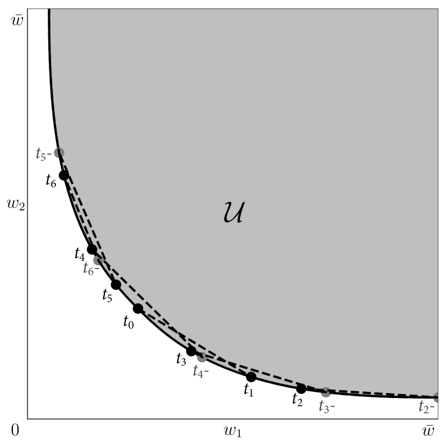
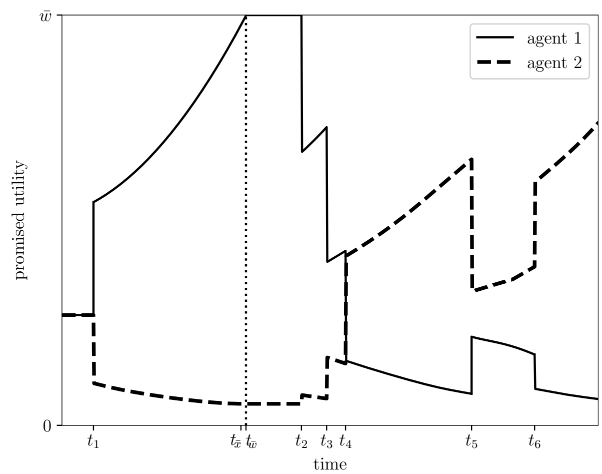



**Figure 2** Drift directions of promised utilities when no adverse event arrives for a two-agent case.

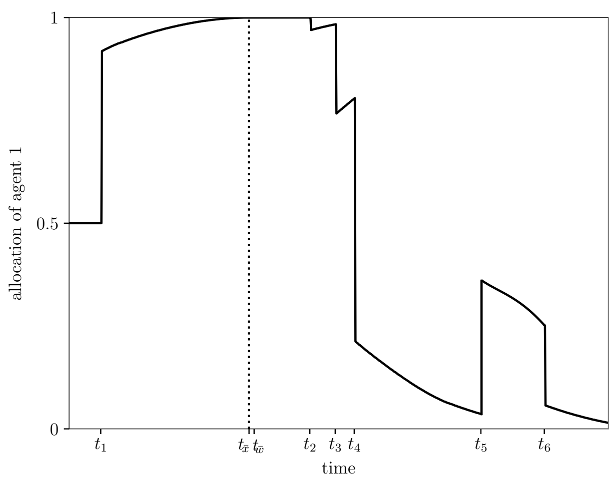
Figure 2 further depicts how the promised utilities move along the boundary beside the middle point and the two extreme points, absent of adverse events. In particular, each arrow represents a direction defined by $(y_1^*, y_2^*)$ starting from $(w_1^*, w_2^*)$. In general, we observe that when there is no arrival, the vector of promised utilities tends to move away from the center, and towards the nearest extreme point. For example, consider a point $(w_1^*, w_2^*) \in \mathrm{bd}(\mathcal{U})$ where $w_1^* > w_2^*$. We call agents 1 and 2 the better and worse performing agents, respectively. When there is no arrival, the promised utility gradually moves towards the bottom right extreme point of the boundary along the boundary, as shown in Figure 2. That is, the better (worse) agent gets even better (worse) absent of arrival.



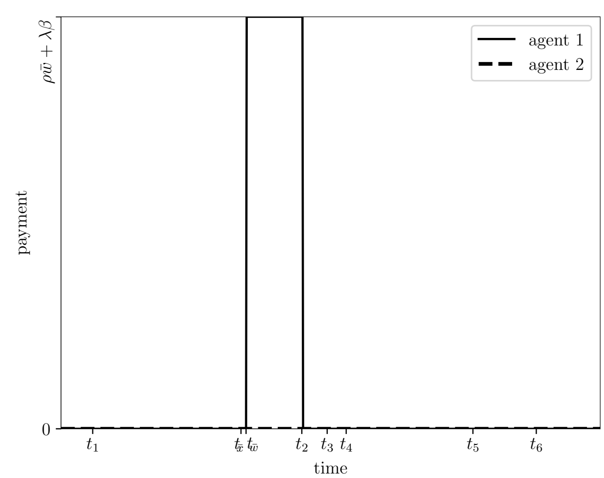

(a) Trajectory of promised utilities on $\mathrm{bd}(\mathcal{U})$




(b) Trajectories of promised utilities




(c) Trajectory of allocations



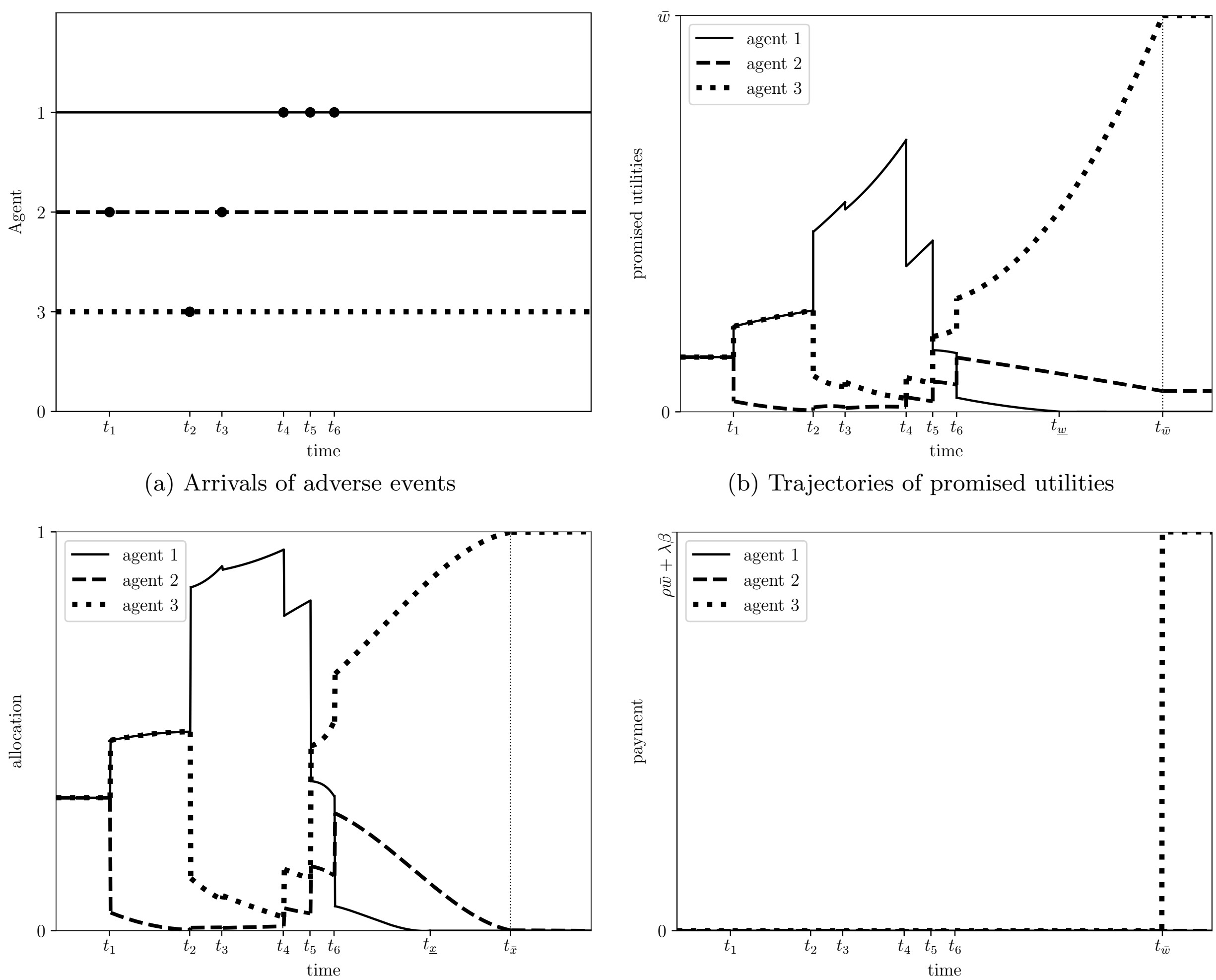

(d) Trajectories of payments

**Figure 3** A Sample trajectory under the boundary EIC contract for a two-agent case.

Figure 3 further illustrates a sample trajectory of the boundary EIC contract. In particular, the promised utilities start from the middle point, labeled as $t_0$ in Figure 3(a), and stay there until an arrival occurs to agent 2 at time $t_1$. The arrival triggers agent 2's promised utility to decrease and agent 1's to increase, as depicted by the jumps at time $t_1$ in Figure 3(b). The corresponding allocation also shifts from each agent having 0.5 unit of resource to agent 1 occupying the majority of the resource, as depicted by the jump at time $t_1$ in Figure 3(c). After time $t_1$, agent 1's promised utility keeps increasing while agent 2's decreasing, as we can see from both Figures 3(a) and 3(b). Correspondingly, the resource is shifted more towards agent 1. By time $t_{\bar{x}}$, all of the resource has been allocated to agent 1, although the promised utility of agent 1 has not reached $\bar{w}$ yet (see Figures 3(b) and 3(c)). Agent 1's promised utility finally reaches $\bar{w}$, and agent 2's reaches the lowest point, at time $t_{\bar{w}}$, as shown in Figure 3(b). Note that payment only starts at time $t_{\bar{w}}$, as shown in Figure 3(d), which also explains why agent 1's promised utility keeps increasing between $t_{\bar{x}}$ and $t_{\bar{w}}$. The next arrival occurs to agent 1, at time $t_2$. Therefore, the promised utility stays at the bottom right corner of the boundary until $t_2-$, as depicted in Figure 3(a), and jumps to the point labeled as $t_2$ right after the arrival. (Here we use notation $t-$ to represent the moment right before time $t$.) This arrival moves agent 1's promised utility below $\bar{w}$ (Figure 3(b)), which stops the payment (Figure 3(d)). Two additional arrivals occur to agent 1 at time epochs $t_3$ and $t_4$. The arrival at $t_4$ triggers the promised utility to jump from the bottom right portion of the boundary to the upper left (from $t_4-$ to $t_4$ on Figure 3(a)). At this point in time, agent 1's promised utility is lower than agent 2's (Figure 3(b)), and the corresponding allocation to agent 1 also decreases to be less than a half (Figure 3(c)). As a result, the promised utilities move towards the upper left of the boundary after time $t_4$. The next two arrivals, one to player 1 at time $t_5$, and the other to player 2 at $t_6$, are also depicted accordingly.

Figure 4 illustrates a sample trajectory of the boundary EIC contract for a three-agent case. In particular, Figure 4(a) represents the time epochs of arrivals to various agents. For example, agent 2 has an arrival at time $t_1$, followed by another arrival occurring to agent 3 at $t_2$, etc. Figures 4(b), 4(c), and 4(d), depict changes in the agents' promised utilities, allocations of the resource, and payments, respectively, over time. Given the similarities to Figure 3, we believe that Figure 4 is self-explanatory. However, there are two points worth mentioning. First, at time $t_3$, an arrival occurs to agent 2. As shown in Figure 4(b), not only does agent 2's promised utility take a downward jump due to this arrival, agent 1's promised utility also takes a downward jump. This corresponds to the case that $H_{12,t_3} > 0$, as mentioned earlier. Therefore, when there are multiple players, the sign of $H_{ij,t}$ may be either positive or negative, depending on the promised utilities of all players involved. Second, after a sequence of three arrivals occurring to agent 1 at time epochs $t_4$, $t_5$, and $t_6$, agent 1's promised utility drops to the lowest among the three agents. After a relatively long period with no additional arrival, agent 1's promised utility eventually decreases to 0 at time $t_{\underline{w}}$. As a result, agent 1 is terminated from the contract, and the system reduces to a two-agent case.

Therefore, an EIC contract exists as long as there are at least two agents in the system. In other words, as long as $\bar{w}$ is high enough, having two agents is sufficient to maintain competition




(a) Arrivals of adverse events

(b) Trajectories of promised utilities

(c) Trajectories of allocations

(d) Trajectories of payments

**Figure 4** A Sample trajectory under the boundary EIC contract for a three-agent case.

*Note.* In time period $[t_0, t_1]$, the promised utilities and allocations of the three agents coincide; and in time period $[t_1, t_2]$, the promised utilities and allocations of the agent 1 and agent 3 coincide.

between them such that efficient allocation is achieved. When there are more than two agents, with probability one, all but two will be eliminated eventually according to our boundary EIC. This observation also reminds us of Proposition 3.7, which states that the boundary contract $\hat{\Gamma}$, which starts from the same promised utility for all $n$ agents, maximizes the principal's utility among all EIC contracts. Following this boundary contract, the principal can threat agents with termination as long as the remaining number of agents is more than two.

Before we close this section, it is worth reviewing the main results. In this section, we first provide an iterative approach to obtain the achievable set of promised utilities in Theorem 3.1, such that any vector of promised utilities in this achievable set can be achieved by an EIC contract. Next, in Theorem 3.2 and Corollary 3.2, we demonstrate that the iterative approach not only yields an achievable set, but also a boundary EIC contract. It is worth noting that, different from the Folk Theorem of Fudenberg et al. (1994) and results of Balseiro et al. (2019), our contracts being efficient does no rely on agents being infinitely patient. One important insight from the boundary EIC contract is that, generally speaking, in order to achieve efficiency, whenever there is an adverse event arriving at one agent, all agents' allocations and promised utilities need to take discrete jumps. The boundary EIC contract has some nice features, for example, an agent is only paid a constant flow payment when the promised utility reaches the upper bound, and a boundary contract maximizes the principal's utility among all EIC contracts. Overall, however, the boundary contract can be quite complex to fully specify and accurately implement, because the promised utilities change all the time. This observation motivates us to propose a much simpler EIC contract that also possesses other useful properties.

## 4. Simple EIC Contract

Although the boundary EIC contract can be approximated through solving a sequence of linear optimization models $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha})$ with different $\boldsymbol{\alpha}$'s, such a procedure is complex from an operations point of view. In particular, at essentially any point in time, all agents' promised utilities keep moving. The movement, which is part of the solution to a linear program, is generally hard to characterize. Furthermore, in order to accurately calculate the boundary of the achievable set, the number of linear programs that we need to solve grows exponentially with the number of agents in the system. The size of each linear program $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha})$ also grows with the number of agents due to constraints ($\mathrm{SG}_{\boldsymbol{w}}$) and ($\mathrm{SG}_{\boldsymbol{Z}}$). In the spirit of finding solutions that are easy to calculate and implement, we propose a very simple EIC contract that possesses closed-form expressions in this section.

Our EIC contract of this section has the following desirable properties, in contrast to the boundary EIC contract. First, in our *simple EIC contract*, the agents' promised utilities and allocations do not change between arrivals. This property greatly simplifies implementation, because we only need to consider the jumps in promised utilities and allocations when there is an arrival. Second, according to our simple EIC contract, the payment to each agent at any point in time is always proportional to the resource allocation. This property is particularly desirable because, in an online platform setting, agents are often independent businesses running on the platform, and the resource is the total online visits to the platform. In such a setting, an agent's income is often proportional to the total volume of visits to this agent's web page, and a key lever of the platform is to decide which agent's web page to show to the next online visit. Therefore, modeling agents' income as proportional to allocation is practically relevant. Third, the simple EIC contract no longer relies on the exogenous parameter $\bar{w}$ as model input. In the last section, the entire position and shape of the boundary of the achievable set $\mathcal{U}$, and therefore the boundary EIC contract, critically depends on this $\bar{w}$ parameter. This upper bound captures the principal's commitment power in theory, but may be hard to estimate in practice. Instead, in our simple EIC contract, the entire set of promised utilities achievable by the simple EIC contract lies on a simplex, such that $\sum_i W_{i,t} = \hat{w}$ for some $\hat{w}$, which is the total discounted revenue that agents would share according to the EIC contract.

Now we present the simple EIC contract in more detail. At any point in time $t$, when an agent's allocation of the resource is $X_{i,t}$, assume the agent's income rate is $R^a X_{i,t}$, proportional to the agent's allocation $X_{i,t}$. Therefore, all the agents share a total income of $R^a$ per unit of time, while the principal collects the remaining $R - R^a$ revenue rate. Therefore, the total discounted revenue that agents receive is

$$\hat{w} := \frac{R^a}{\rho}. \tag{4.1}$$

Further denote $\check{w}$ to be the lowest promised utility of an agent, when the agent's allocation is zero. That is, the agents' promised utilities belong to the following set, which is a subset of an $(n-1)$-simplex,

$$\mathcal{U}_s := \left\{ \boldsymbol{w} \,\middle|\, \sum_{i \in \mathcal{I}} w_i = \hat{w},\ \check{w} \le w_i \le \hat{w} - (n-1)\check{w} \right\}.$$

If all $n-1$ agents' promised utilities are at $\check{w}$, then the only agent who receives all the resource must have a promised utility $\hat{w} - (n-1)\check{w}$. When this agent experiences an arrival, the promised utility needs to take a downward jump of at least $\beta \times 1$, following (EA) and (IC). Therefore, we need to require

$$\hat{w} - (n-1)\check{w} - \beta \ge \check{w},$$

which implies the following condition for $R^a$ in order to make our simple contract work,

$$R^a \ge \rho \left( n\check{w} + \beta \right). \tag{4.2}$$

With this setup, we formally define the *simple EIC contract.*

Definition 4.1. For any $R^a$ that satisfies conditions (4.2), and $\boldsymbol{w} \in \mathcal{U}_s$, in which $\hat{w}$ is defined in (4.1) and $\check{w}$ is defined as

$$\check{w} := \frac{\lambda \beta}{(n-1)\rho}, \tag{4.3}$$

define a *simple EIC contract* $\Gamma_s(\boldsymbol{w}; R^a)$ such that payments are

$$l_{i,t} = R^a X_{i,t} \text{ and } I_{i,t} = 0, \tag{4.4}$$

the allocations satisfy

$$X_{i,t} = \frac{W_{i,t-} - \check{w}}{\hat{w} - n\check{w}}, \tag{4.5}$$

and the promised utilities start from $\boldsymbol{W}_0 = \boldsymbol{w}$ and follow dynamic

$$dW_{i,t} = -\sum_{j \in \mathcal{I}} H_{ij,t} dN_{j,t}, \tag{4.6}$$

in which

$$H_{ij,t} = \begin{cases} X_{i,t}\beta, & \text{for } i = j, \\ -\dfrac{X_{j,t}\beta}{n-1}, & \text{for } i \neq j. \end{cases} \tag{4.7}$$

It is worth highlighting some intuitions behind various closed-form expressions and conditions related to Definition 4.1. First, according to (4.5), if an agent's promised utility is at the lower bound $\check{w}$, then this agent receives zero resource. If one agent's promised utility is at the upper bound, $\hat{w} - (n-1)\check{w}$, on the other hand, then all the resource must be allocated to this agent, and the definition of $\mathcal{U}_s$ implies that all other agents' promised utilities must be on the lower bound.

Next, the dynamic (4.6) implies that promised utilities only change upon arrivals. As mentioned earlier, this property significantly simplifies the contract implementation, especially compared with the boundary EIC contract. The change of promised utilities (4.7) further implies that when one agent experiences an adverse arrival, all other agents' promised utilities increase by evenly splitting the promised utility loss of the focal agent. Expression (4.7) also ensures that $\sum_i H_{ij,t} = 0$, which further guarantees that $\sum_i W_{i,t}$ remains a constant, and therefore the promised utilities reside on a simplex. Furthermore, in the proof of the following theorem, we illustrate that in order to construct the contract with desirable properties (4.4)-(4.7), we have to set $\check{w}$ according to (4.3).

Theorem 4.1. *Contract $\Gamma_s(\boldsymbol{w}; R^a)$ in Definition 4.1 satisfies* (EA)*,* (IC)*,* (IR)*,* (LL)*,* (PK)*, and* (UB) *as long as $\bar{w} \geq \hat{w} - (n-1)\check{w}$, therefore, is an EIC contract. Furthermore, $\boldsymbol{W}_t \in \mathcal{U}_s$ for any $t \geq 0$ starting from $\boldsymbol{W}_0 = \boldsymbol{w}$ following* (4.6)*. Therefore, $\mathcal{U}_s$ is a self-generating set.*

The proof of Theorem 4.1 critically depends on the lower bound condition (4.2) for the agents' total income rate $R^a$. Now that the value of $\check{w}$ is specified, it is worth discussing the intuitive reason why such a lower bound is needed and its practical relevance. The reason that we need a lower bound on $R^a$ is because in this setting we assume that agents only receive an income that is proportional to the resource allocation, as specified in (4.4). If $R^a$ is too low, the set of promised utilities would be too small to allow a downward jump of at least $\beta X_{i,t}$ within the set. Consequently, the contract would no longer be both efficient and incentive-compatible. In practice, $R^a$ could be exogenously determined as the profit rate an agent collects from running the business

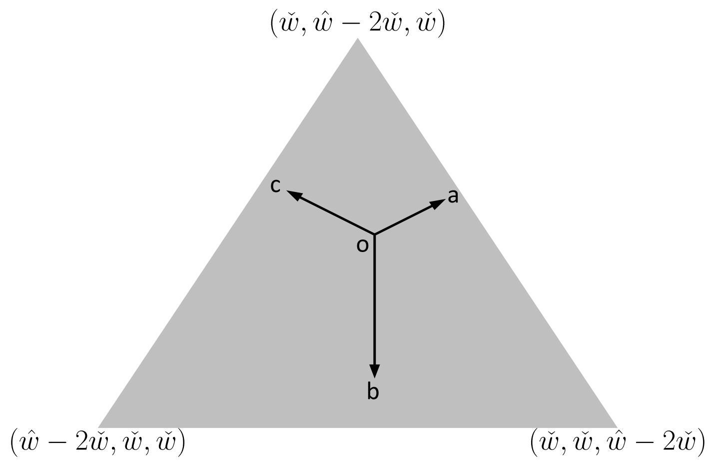



**Figure 5** Illustration of an achievable set under contract $\Gamma_s$ and the changes in promised utilities upon an adverse arrival for a three-agent case.

with one unit of resource. If such a profit rate is lower than the lower bound specified in (4.2), then the principal needs to subsidize by raising agents' income rate to at least this lower bound.

Before closing this section, we use Figure 5 to illustrate the self-generating set $\mathcal{U}_s$ in a three-agent case, as well as how promised utilities change upon an arrival. First, the triangular shape in the figure represents the set $\mathcal{U}_s$, which is similar to a 2-simplex. At each extreme point, two agents' promised utilities take value $\breve{w}$, while the other $\hat{w}-(n-1)\breve{w}$. In this figure, we consider a particular point `o`, representing the current promised utilities of the three agents. If an arrival occurs, the promised utility may jump to point `a`, `b`, or `c`, depending on which agent suffers from the arrival. Note that the jump from `o` to `a`, `b`, or `c` is always perpendicular to one of the facets of this self-generating set. This is because, as the point jumps towards one of the boundaries (the promised utility of the corresponding agent decreases), all other agents' promised utility increases equally, following (4.7). We show this result formally in Appendix EC.3.2. This three-agent example also highlights another key difference between the simple contract $\Gamma_s$ and boundary EIC contract. Recall that a boundary EIC contract eventually terminates all but two agents. In contrast, under the simple contract, an agent's promised utility is lower bounded by $\breve{w}$, and no agent is ever terminated. Even with zero resource and income at some points, an agent's promised utility remains positive, because the next arrival at another agent with positive allocation brings this agent $i$'s allocation and income flow back to be positive.

## 5. Concluding Remarks and Further Discussion

Motivated by applications of platforms allocating online visits among independent suppliers, this paper studies a dynamic moral hazard model, in which a principal motivates multiple symmetric agents to reduce the frequency of adverse events. We demonstrate, by construction, that there exist EIC contracts that always allocate all available resource to agents and ensure agents always exert effort to reduce the arrival rate of adverse events. It is worth noting that in the single-agent counterpart of our multi-agent setting, efficiency cannot be achieved by incentive-compatible contracts. Our construction involves an iterative approach that solves a sequence of semi-infinite linear programs. From a practical point of view, the contract from the linear programming-based approach can be cumbersome to characterize and implement. Therefore, we further provide a simple EIC contract in closed-form. The simple EIC contract possesses desirable properties that make it practically relevant. Therefore, our proposals provide prescriptive guidance for designing easy-to-implement EIC contracts that suit the need of practitioners.

In terms of methodology, our analyses are built upon the "self-generating set" idea first proposed in the discrete-time repeated games literature (Abreu et al. 1990). However, we have to extend the self-generating set concept to the continuous-time setting, which cannot rely on recursive expressions often used in discrete-time dynamical systems. In addition, our efficiency results do not rely on the assumption that agents are infinitely patient (which is required in the Folk Theorem of Fudenberg et al. 1994). Furthermore, our linear programming based iterative approach to characterize the self-generating set for a continuous-time moral hazard problem extends the support function approach proposed in Balseiro et al. (2019) for a discrete-time adverse selection problem.

Our analyses and results provide some interesting economic and managerial insights. For instance, in single-agent dynamic contracting settings similar to ours, one can perceive the decision of scaling firm size in Biais et al. (2010), or replacing governor in Myerson (2015), analogously to our resource allocation decisions. In those settings, incentive-compatible contracts cannot achieve efficiency, that is, the firm has to be downsized and the governor has to be replaced within finite time with probability one. This is because incentive compatibility requires the threat of reducing the agent's promised utility by a certain amount upon each arrival. In the multi-agent setting, however, when we penalize an agent for an adverse event by reducing his promised utility and allocation, we can simultaneously increase other agents' allocations to maintain efficiency. Consequently, all other agents' promised utilities are changed due to one agent's adverse event. This flexibility allows us to design efficient contracts that induce agents to constantly exert effort. In fact, without such an option of potentially rewarding other agents upon the arrival at one agent, we show that EIC contracts no longer exist.

Furthermore, EIC contracts exist as long as there are two agents. One can put it as "a little bit of competition goes a long way." This result also implies that the principal can threat agents with contract termination when the number of agents is more than two. In fact, we show that the EIC contract that maximizes the principal's utility is the boundary contract starting from the point where all agents' promised utilities are the same. This boundary contract has to terminate all but two agents within finite time.

Finally, it is worth concluding the paper with some thoughts on potential future research directions. First, from a theoretical point of view, it is interesting to investigate *optimal* dynamic contracts in our setting, that is, contracts that maximize the principal's utility. Generally speaking, one needs to formulate the optimal dynamic contract design problem as a continuous-time optimal control model. In our setting with Poisson arrivals of adverse events, the optimality conditions are in the form of a system of partial differential equations with delay. Solving such a system appears a technically challenging task. Second, in our model, we assume that agents do not collude. In particular, the EIC contracts in our setting leverage on the competition among agents. It is interesting to investigate whether agents can suppress competition by collusion in multilateral dynamic moral hazard problems, and, if so, how to mitigate collusion in dynamic contract design. Finally, in certain settings, rewarding one agent when there is an adverse event at another agent may create a perverse incentive for some agent to sabotage others. It is interesting to investigate designs to mitigate this effect if it is a real concern.

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# Proofs of Statements

All relevant proofs in this appendix involve the non-negative parameter $R^a$, which represents the agents' share of income rate per unit of resource. In particular, we assume that the total flow income rate that an agent $i$ receives at time $t$ is $R^a X_{i,t} + l_{i,t}$. Although the introduction of this parameter is postponed until Section 4 (where, for the clarity of exposition, by an abuse of notation we assume that the total flow income rate that an agent $i$ receives at time $t$ is $l_t = R^a X_{i,t}$, and we discuss the scenario where $R^a$ is too small and we need to supplement $l_t$), the corresponding statements in earlier sections do hold in the presence of $R^a$. In other words, the statements in the earlier sections, before Section 4, are special cases where $R^a$ is set to be zero. The proofs in this appendix prove the more general versions of the statements.

## EC.1. Proofs in Section 2

### EC.1.1. Proof of Lemma 2.1

LEMMA 2.1. *For any contract $\Gamma$ and any effort process $\mathbf{\Lambda}$, there exist $\mathcal{F}_t^{\boldsymbol{N}}$-predictable processes $\{\boldsymbol{H}_t = (H_{ij,t})_{i,j\in\mathcal{I}}\}_{t\geq 0}$, such that for any $t_1$ and $t_2$ with $0 \leq t_1 < t_2$, we have*

$$W_{i,t_2} = W_{i,t_1} + \int_{(t_1,t_2]} dW_{i,s}, \quad \forall i \in \mathcal{I}, \tag{2.4}$$

*in which*

$$dW_{i,t} = \left(\rho W_{i,t-} - bX_{i,t}\mathbb{1}_{\{\lambda_{i,t}=\bar{\lambda}\}} + \sum_{j\in\mathcal{I}} \lambda_{j,t} H_{ij,t}\right) dt - \sum_{j\in\mathcal{I}} H_{ij,t}\, dN_{j,t} - dL_{i,t}, \quad \forall t \geq 0,\ \forall i \in \mathcal{I}, \tag{PK}$$

*where the counting process $\{\boldsymbol{N}_t\}_{t\geq 0}$ is generated from the effort process $\mathbf{\Lambda}$. In addition, we need*

$$H_{ij,t} \leq W_{i,t-}, \quad \forall t \geq 0,\ \forall i,j \in \mathcal{I}, \tag{2.5}$$

*in order to satisfy (IR).*

*Proof:* For a generic contract $\Gamma$ and effort process $\mathbf{\Lambda}$, we introduce $u_{i,t}(\Gamma,\mathbf{\Lambda})$, which is the total expected utility of agent $i$ conditioning on the information available at time $t$ as

$$\begin{aligned} u_{i,t}(\Gamma,\mathbf{\Lambda}) &= \mathbb{E}\left[\int_0^\infty e^{-\rho s}(\mathrm{d}L_{i,s} + R^a X_{i,s}\mathrm{d}s + bX_{i,t}\mathbb{1}_{\{\lambda_{i,t}=\bar{\lambda}\}}\mathrm{d}s)\big|\mathcal{F}_t^{\boldsymbol{N}}\right] \\ &= \int_0^t e^{-\rho s}(\mathrm{d}L_{i,s} + R^a X_{i,s}\mathrm{d}s + bX_{i,s}\mathbb{1}_{\{\lambda_{i,t}=\bar{\lambda}\}}\mathrm{d}s) + e^{-\rho t} W_{i,t}(\Gamma,\mathbf{\Lambda}). \end{aligned} \tag{EC.1.1}$$

It is readily obtained from the definition of $u_{i,t}(\Gamma,\mathbf{\Lambda})$ that $u_{i,0}(\Gamma,\mathbf{\Lambda}) = u_i(\Gamma,\mathbf{\Lambda})$, which is defined by equation (2.2). Process $\{u_{i,t}\}_{t\geq 0}$ is an $\mathcal{F}_t^{\boldsymbol{N}}$-martingale, as it is constructed as a Doob martingale with respect to the filtration $\mathcal{F}_t^{\boldsymbol{N}}$.

Define process

$$M^{\mathbf{\Lambda}}_{j,t} = \int_0^t \lambda_{j,s} \mathrm{d}s - N_{j,t}.$$

Process $\{M^{\mathbf{\Lambda}}_{j,t}\}_{t\geq 0}$ is also an $\mathcal{F}^{\boldsymbol{N}}_t$-martingale. Consequently, following the Martingale Representation Theorem (see, for example, Theorem T9 of Brémaud 1981, page 64), there exists a unique $\mathcal{F}^{\boldsymbol{N}}_t$-predictable process $H^{\mathbf{\Lambda}}_{ij} = \{H^{\mathbf{\Lambda}}_{ij,t}\}_{t\geq 0}$ such that

$$u_{i,t}(\Gamma, \mathbf{\Lambda}) = u_{i,0}(\Gamma, \mathbf{\Lambda}) + \int_0^t e^{-\rho s} \sum_{j\in\mathcal{I}} H_{ij,s}(\Gamma, \mathbf{\Lambda}) \mathrm{d}M^{\mathbf{\Lambda}}_{j,s}, \quad \forall t \geq 0. \tag{EC.1.2}$$

Combining equations (EC.1.1) and (EC.1.2), we have

$$\mathrm{d}W_{i,t} = \left(\rho W_{i,t} - R^a X_{i,t} - b X_{i,t} \mathbb{1}_{\{\lambda_{i,t}=\bar{\lambda}\}} + \sum_{j\in\mathcal{I}} \lambda_{j,t} H_{ij,t}\right) \mathrm{d}t - \sum_{j\in\mathcal{I}} H_{ij,t} \mathrm{d}N_{j,t} - \mathrm{d}L_{i,t}, \quad \forall t\geq 0, \forall i \in \mathcal{I},$$

Following the convention to define stochastic integral with respect to the right-continuous $\{W_{i,t}\}_{t\geq 0}$ process, we have, for any $t_1 < t_2$, that

$$\begin{aligned}
W_{i,t_2} &= W_{i,t_1} + \int_{(t_1,t_2]} \mathrm{d}W_{i,s} \\
&= W_{i,t_1} + \int_{(t_1,t_2]} \left(\rho W_{i,s} - R^a X_{i,s} - b X_{i,s} \mathbb{1}_{\{\lambda_{i,s}=\bar{\lambda}\}} + \sum_{j\in\mathcal{I}} \lambda_{j,s} H_{ij,s}\right) \mathrm{d}s \\
&\qquad - \int_{(t_1,t_2]} \sum_{j\in\mathcal{I}} H_{ij,s} \mathrm{d}N_{j,s} - \int_{(t_1,t_2]} \mathrm{d}L_{i,s} \\
&= W_{i,t_1} + \int_{(t_1,t_2]} \left(\rho W_{i,s-} - R^a X_{i,s} - b X_{i,s} \mathbb{1}_{\{\lambda_{i,s}=\bar{\lambda}\}} + \sum_{j\in\mathcal{I}} \lambda_{j,s} H_{ij,s}\right) \mathrm{d}s \\
&\qquad - \int_{(t_1,t_2]} \sum_{j\in\mathcal{I}} H_{ij,s} \mathrm{d}N_{j,s} - \int_{(t_1,t_2]} \mathrm{d}L_{i,s},
\end{aligned}$$

where the second equality follows from the definition of Itô's stochastic integral and the definition of $W_{i,s-}$. Consequently, we obtain equation (PK).

At last, note that the promised utility after an adverse event still needs to satisfy (IR). Therefore, joining with equation (PK) we have equation (2.5). $\square$

### EC.1.2. Proof of Proposition 2.1

Proposition 2.1. *Contract* $\Gamma$ *satisfies the incentive-compatible condition* (2.6) *if and only if*

$$H_{ii,t} \geq \beta X_{i,t}, \quad \forall t \geq 0, \ \forall i \in \mathcal{I}. \tag{IC}$$

*Proof:* Fix any agent $i \in \{1,2,...,n\}$, and denote $\tilde{u}_{i,t}(\Gamma, \mathbf{\Lambda}'_i, \tilde{\mathbf{\Lambda}}_i)$ to represent an $\mathcal{F}_t^{\boldsymbol{N}}$-measurable random variable, representing the total expected utility of agent $i$ given that the effort process of all agents is $\mathbf{\Lambda}'_i$ before time $t$ and $\tilde{\mathbf{\Lambda}}_i$ after $t$. Furthermore, assume that $\mathbf{\Lambda}'_i$ and $\tilde{\mathbf{\Lambda}}_i$ are effort processes where agents other than $i$ always exert effort, that is, $\mathbf{\Lambda}'_i := \{\lambda_{i,\tau} \in \{\lambda, \bar{\lambda}\}, \lambda_{j,\tau} = \lambda \text{ for any } j \neq i\}_{\tau<t}$, and $\tilde{\mathbf{\Lambda}}_i := \{\lambda_{i,\tau} \in \{\lambda, \bar{\lambda}\}, \lambda_{j,\tau} = \lambda \text{ for any } j \neq i\}_{\tau \geq t}$. The expected utility $\tilde{u}_{i,t}(\Gamma, \mathbf{\Lambda}'_i, \tilde{\mathbf{\Lambda}}_i)$ can be written as follows:

$$\tilde{u}_{i,t}(\Gamma, \mathbf{\Lambda}'_i, \tilde{\mathbf{\Lambda}}_i) = \int_0^t e^{-\rho s}(\mathrm{d}L_{i,s} + R^a X_{i,s}\mathrm{d}s + bX_{i,s}\mathbb{1}_{\{\lambda'_{i,s}=\bar{\lambda}\}}\mathrm{d}s) + e^{-\rho t} W_{i,t}(\Gamma, \tilde{\mathbf{\Lambda}}_i). \tag{EC.1.3}$$

Therefore,

$$\tilde{u}_{i,0}(\Gamma, \mathbf{\Lambda}'_i, \tilde{\mathbf{\Lambda}}_i) = u_{i,0}(\Gamma, \tilde{\mathbf{\Lambda}}_i) = u_i(\Gamma, \tilde{\mathbf{\Lambda}}_i), \tag{EC.1.4}$$

$$\lim_{t\to\infty} \{\mathbb{E}[\tilde{u}_{i,t}(\Gamma, \mathbf{\Lambda}'_i, \tilde{\mathbf{\Lambda}}_i)|\mathcal{F}_0^{\boldsymbol{N}}]\} = u_i(\Gamma, \mathbf{\Lambda}'_i), \tag{EC.1.5}$$

For any given sample trajectory $\{N_{j,s}\}_{0\leq s\leq t}$ and effort process $\tilde{\mathbf{\Lambda}}_i$ and $\bar{\mathbf{\Lambda}}$,

$$\begin{aligned}
\tilde{u}_{i,t}(\Gamma, \tilde{\mathbf{\Lambda}}_i, \bar{\mathbf{\Lambda}}) = &\int_0^t e^{-\rho s}(\mathrm{d}L_{i,s} + R^a X_{i,s}\mathrm{d}s + bX_{i,s}\mathbb{1}_{\{\lambda_{i,s}=\bar{\lambda}\}}\mathrm{d}s) + e^{-\rho t} W_{i,t}(\Gamma, \bar{\mathbf{\Lambda}}) \\
=& u_{i,t}(\Gamma, \bar{\mathbf{\Lambda}}) + \int_0^t e^{-\rho s} bX_{i,s}\mathbb{1}_{\{\lambda_{i,s}=\bar{\lambda}\}}\mathrm{d}s \\
=& u_{i,0}(\Gamma, \bar{\mathbf{\Lambda}}) + \int_0^t e^{-\rho s}\sum_{j\in\mathcal{I}} H_{ij,s}(\Gamma, \bar{\mathbf{\Lambda}})\mathrm{d}M_{j,s}^{\bar{\mathbf{\Lambda}}} + \int_0^t e^{-\rho s} bX_{i,s}\mathbb{1}_{\{\lambda_{i,s}=\bar{\lambda}\}}\mathrm{d}s \\
=& u_{i,0}(\Gamma, \bar{\mathbf{\Lambda}}) + \int_0^t e^{-\rho s}\sum_{j\in\mathcal{I}} H_{ij,s}(\Gamma, \bar{\mathbf{\Lambda}})\mathrm{d}M_{j,s}^{\tilde{\mathbf{\Lambda}}_i} \\
&- \int_0^t e^{-\rho s} H_{ii,s}(\Gamma, \bar{\mathbf{\Lambda}})(\lambda_{i,s} - \lambda)\mathrm{d}s + \int_0^t e^{-\rho s} bX_{i,s}\mathbb{1}_{\{\lambda_{i,s}=\bar{\lambda}\}}\mathrm{d}s \\
=& u_{i,0}(\Gamma, \bar{\mathbf{\Lambda}}) + \int_0^t e^{-\rho s}\sum_{j\in\mathcal{I}} H_{ij,s}(\Gamma, \bar{\mathbf{\Lambda}})\mathrm{d}M_{j,s}^{\tilde{\mathbf{\Lambda}}_i} \\
&- \int_0^t e^{-\rho s}\Delta\lambda[H_{ii,s}(\Gamma, \bar{\mathbf{\Lambda}}) - \beta X_{i,s}]\mathbb{1}_{\{\lambda_{i,s}=\bar{\lambda}\}}\mathrm{d}s,
\end{aligned} \tag{EC.1.6}$$

where the first equality follows from equation (EC.1.3), the second equality from equation (EC.1.1), the third equality from equation (EC.1.2), the fourth equality from the definition of $M_{j,s}^{\tilde{\mathbf{\Lambda}}}$ and all agents other than $i$ exerting effort continuously, and the last equality from $\beta\Delta\lambda = b$.

Consider any two times $t' \leq t$,

$$\begin{aligned}
\mathbb{E}[\tilde{u}_{i,t}(\Gamma, \tilde{\mathbf{\Lambda}}_i, \bar{\mathbf{\Lambda}})|\mathcal{F}_{t'}^{\boldsymbol{N}}] =& u_{i,0}(\Gamma, \bar{\mathbf{\Lambda}}) + \int_0^{t'} e^{-\rho s}\sum_{j\in\mathcal{I}} H_{ij,s}(\Gamma, \bar{\mathbf{\Lambda}})\mathrm{d}M_{j,s}^{\tilde{\mathbf{\Lambda}}_i} \\
&- \int_0^{t'} e^{-\rho s}\Delta\lambda[H_{ii,s}(\Gamma, \bar{\mathbf{\Lambda}}) - \beta X_{i,s}]\mathbb{1}_{\{\lambda_{i,s}=\bar{\lambda}\}}\mathrm{d}s
\end{aligned}$$

$$\begin{aligned}
&-\mathbb{E}\left[\int_{t'}^{t} e^{-\rho s}\Delta\lambda[H_{ii,s}(\Gamma,\bar{\mathbf{\Lambda}})-\beta X_{i,s}]\mathbb{1}_{\{\lambda_{i,s}=\bar{\lambda}\}}\mathrm{d}s|\mathcal{F}_{t'}^{\boldsymbol{N}}\right]\\
=&\tilde{u}_{i,t'}(\Gamma,\tilde{\mathbf{\Lambda}}_i,\bar{\mathbf{\Lambda}})\\
&-\mathbb{E}\left[\int_{t'}^{t} e^{-\rho s}\Delta\lambda[H_{ii,s}(\Gamma,\bar{\mathbf{\Lambda}})-\beta X_{i,s}]\mathbb{1}_{\{\lambda_{i,s}=\bar{\lambda}\}}\mathrm{d}s|\mathcal{F}_{t'}^{\boldsymbol{N}}\right], \qquad \text{(EC.1.7)}
\end{aligned}$$

where the first equality follows from re-organizing the expected utility from $t'$ to $t$ under filtration $\mathcal{F}_{t'}^{\boldsymbol{N}}$, and the second equality from equation (EC.1.6).

We first prove that $H_{ii,s}\geq \beta X_{i,s}$ for all $s\geq 0$ is a sufficient condition for an incentive-compatible contract. Suppose $H_{ii,s}\geq \beta X_{i,s}$ for all $s\geq 0$. Under this condition, the integrand in the second term on the right-hand side of equation (EC.1.7) is always non-negative, which implies that

$$\mathbb{E}[\tilde{u}_{i,t}(\Gamma,\tilde{\mathbf{\Lambda}}_i,\bar{\mathbf{\Lambda}})|\mathcal{F}_{t'}^{\boldsymbol{N}}]\leq \tilde{u}_{i,t'}(\Gamma,\tilde{\mathbf{\Lambda}}_i,\bar{\mathbf{\Lambda}}),$$

that is, $\{\tilde{u}_{i,t}\}_{t\geq 0}$ is a super-martingale. Taking $t'=0$ and letting $t\to\infty$, we have

$$u_i(\Gamma,\bar{\mathbf{\Lambda}})=\tilde{u}_{i,0}(\Gamma,\tilde{\mathbf{\Lambda}}_i,\bar{\mathbf{\Lambda}})\geq \lim_{t\to\infty}\{\mathbb{E}[\tilde{u}_{i,t}(\Gamma,\tilde{\mathbf{\Lambda}}_i,\bar{\mathbf{\Lambda}})|\mathcal{F}_0^{\boldsymbol{N}}]\}=u_i(\Gamma,\tilde{\mathbf{\Lambda}}_i),$$

where the first equality follows from equation (EC.1.4), the second equality from equation (EC.1.5), the inequality from the Optional Stopping Theorem. In other words, agent $i$ prefers always exerting effort, given that the other agents are exerting effort. It then follows that, if a contract $\Gamma$ satisfies that for all $i$ and for all $s\geq 0$, $H_{ii,s}\geq \beta X_{i,s}$, then $\Gamma$ is incentive-compatible.

Next, we prove $H_{ii,s}\geq \beta X_{i,s}$ for all $s\geq 0$ is a necessary condition of a contract being incentive-compatible. Suppose, to the contrary, $H_{ii,s}<\beta X_{i,s}$ for $s$ belonging to some subset $\Omega\subset[0,t]$ with positive measure. Define an effort process $\tilde{\mathbf{\Lambda}}_i$ such that $\lambda_{i,s}=\lambda$ for all $s>t$, and for all $s\in[0,t]$:

$$\lambda_{i,s}=\begin{cases}\lambda, & \text{if } H_{ii,s}\geq \beta X_{i,s}\\ \bar{\lambda}, & \text{if } H_{ii,s}<\beta X_{i,s}\end{cases}.$$

Therefore, $\tilde{u}_{i,t}(\Gamma,\tilde{\mathbf{\Lambda}}_i,\bar{\mathbf{\Lambda}})=\tilde{u}_{i,t}(\Gamma,\tilde{\mathbf{\Lambda}}_i,\tilde{\mathbf{\Lambda}}_i)$ and

$$\mathbb{E}\left[\int_0^t e^{-\rho s}\Delta\lambda(H_{ii,s}-\beta X_{i,s})\mathbb{1}_{\{\lambda_{i,s}=\bar{\lambda}\}}\mathrm{d}s|\mathcal{F}_0^{\boldsymbol{N}}\right]<0.$$

Taking $t'=0$, equation (EC.1.7) then implies that $\mathbb{E}[\tilde{u}_{i,t}(\Gamma,\tilde{\mathbf{\Lambda}}_i,\bar{\mathbf{\Lambda}})|\mathcal{F}_0^{\boldsymbol{N}}]>\tilde{u}_{i,0}(\Gamma,\tilde{\mathbf{\Lambda}}_i,\bar{\mathbf{\Lambda}})$ for any $t>0$ such that set $\Omega$ has strictly positive measure. Letting $t\to\infty$, we obtain

$$u_i(\Gamma,\bar{\mathbf{\Lambda}})=\tilde{u}_{i,0}(\Gamma,\tilde{\mathbf{\Lambda}}_i,\bar{\mathbf{\Lambda}})<\lim_{t\to\infty}\{\mathbb{E}[\tilde{u}_{i,t}(\Gamma,\tilde{\mathbf{\Lambda}}_i,\bar{\mathbf{\Lambda}})|\mathcal{F}_0^{\boldsymbol{N}}]\}=u_i(\Gamma,\tilde{\mathbf{\Lambda}}_i).$$

In other words, agent $i$ prefers effort process $\tilde{\mathbf{\Lambda}}_i$ over $\bar{\mathbf{\Lambda}}$, which implies that if there exists some $s\in\Omega$ such that $H_{ii,s}<\beta X_{i,s}$, then $\Gamma$ is not incentive-compatible.

To sum up, $H_{ii,s}\geq \beta X_{i,s}$ for all $s\geq 0$ and for all $i$ is a necessary and sufficient condition for contract $\Gamma$ being incentive-compatible. $\square$

## EC.2. Proofs in Section 3

### EC.2.1. Proof of Proposition 3.1

PROPOSITION 3.1. *If set $\mathcal{A}$ is a self-generating set, then $\mathcal{A} \subseteq \mathcal{U}$.*

*Proof:* Suppose $\mathcal{A}$ is a self-generating set. By Definition 3.2, for every $\boldsymbol{v} \in \mathcal{A}$, then there exist corresponding $\mathcal{F}_t^{\boldsymbol{N}}$-predictable processes $\{\boldsymbol{H}_t\}_{t\geq 0}$, $\{\boldsymbol{X}_t\}_{t\geq 0}$, and $\{\boldsymbol{L}_t\}_{t\geq 0}$ that satisfy (EA), (IC), and (LL). In addition, the process $\{\boldsymbol{W}_t\}_{t\geq 0}$ with $\boldsymbol{W}_0 = \boldsymbol{v}$ that follows (IR), (PK), and (UB) satisfies $\boldsymbol{W}_t \in \mathcal{A}$ for all $t \geq 0$. To prove the proposition, we need to show that any element in the self-generating set $\mathcal{A}$ is achievable by an *Efficient and Incentive-Compatible* (EIC) contract, or equivalently, $\boldsymbol{W}_0 \in \mathcal{U}$. That is, we need to show that $W_{i,0}$ is equal to $u_i$ defined in equation (2.2) for all $i \in \mathcal{I}$.

Note that for any $t \geq 0$, we have

$$\begin{aligned}
e^{-\rho t}W_{i,t} =& e^{\rho 0}W_{i,0-} + \int_0^t e^{-\rho s}\mathrm{d}W_{i,s} + \int_0^t W_{i,s}\mathrm{d}e^{-\rho s} \\
=& W_{i,0-} + \int_0^t e^{-\rho s}\left[\left(\rho W_{i,s-} - R^a X_{i,s} + \lambda\sum_{j\in\mathcal{I}} H_{ij,s}\right)\mathrm{d}s - \sum_{j\in\mathcal{I}} H_{ij,s}\mathrm{d}N_{j,s} - \mathrm{d}L_{i,s}\right] \\
&- \int_0^t \rho e^{-\rho s}W_{i,s}\mathrm{d}s \\
=& W_{i,0-} - \int_0^t e^{-\rho s}\left(R^a X_{i,s}\mathrm{d}s + \mathrm{d}L_{i,s}\right) + \int_0^t e^{-\rho s}\sum_{j\in\mathcal{I}} H_{ij,s}(\lambda \mathrm{d}s - \mathrm{d}N_{j,s}),
\end{aligned} \tag{EC.2.1}$$

where the first equality follows from integration by parts, the second equality from (PK) and the third equality from canceling out the terms $\int_0^t \rho e^{-\rho s}W_{i,s}\mathrm{d}s$ and $\int_0^t \rho e^{-\rho s}W_{i,s-}\mathrm{d}s$.

Taking expectation on both sides of equation (EC.2.1), re-organizing the terms, and letting $t \to \infty$, we arrive at the following equation:

$$u_i = W_{i,0-} + \mathbb{E}\left[\int_0^t e^{-\rho s}\sum_{j\in\mathcal{I}} H_{ij,s}(\lambda\mathrm{d}s - \mathrm{d}N_{j,s})\right].$$

For a total number of $n$ agents, we have $\left|\sum_{j\in\mathcal{I}} H_{ij,s}\right| \leq n\bar{w}$, that is, the integrand is bounded. Then, by Fubini's theorem, we exchange the integration and expectation of the second term on the right-hand side of the above equation, and obtain

$$\mathbb{E}\left[\int_0^t e^{-\rho s}\sum_{j\in\mathcal{I}} H_{ij,s}(\lambda\mathrm{d}s - \mathrm{d}N_{j,s})\right] = \int_0^t e^{-\rho s}\sum_{j\in\mathcal{I}} \mathbb{E}[H_{ij,s}(\lambda\mathrm{d}s - \mathrm{d}N_{j,s})] = 0.$$

Therefore, $W_{i,0} = W_{i,0-} = u_i$. This completes the proof. $\square$

### EC.2.2. Proof of Proposition 3.2

PROPOSITION 3.2. *The achievable set $\mathcal{U}$ is a self-generating set.*

*Proof:* We need to show that if $\boldsymbol{W}_0 = \boldsymbol{v} \in \mathcal{U}$, then there exist $\mathcal{F}_t^{\boldsymbol{N}}$-predictable processes $\{\boldsymbol{X}_t\}_{t\geq 0}$ and $\{\boldsymbol{L}_t\}_{t\geq 0}$ that satisfy (EA), (IC), and (LL), such that the process $\{\boldsymbol{W}_t\}_{t\geq 0}$ that follows (IR), (PK), and (UB) satisfies $\boldsymbol{W}_t \in \mathcal{U}$ for all $t \geq 0$.

Assume to the contrary that there exists time $t$ such that $\boldsymbol{W}_t \notin \mathcal{U}$. We propose a contract $\hat{\Gamma}$ which implements at time 0 the EIC contract $\Gamma$ at time $t$ conditioning on the information available at time $t$, that is, we shift all allocation a period of time $t$. For any filtration $\hat{\mathcal{F}}_s^{\boldsymbol{N}}$ generated by a counting process $\{\hat{\boldsymbol{N}}_\tau\}_{0\leq\tau\leq s}$ under the shifted contract $\hat{\Gamma}$, consider the filtration $\mathcal{F}_{s+t}^{\boldsymbol{N}}$ under the original contract $\Gamma$, defined as $\sigma\big(\{\boldsymbol{N}_\tau\}_{0\leq\tau\leq t}, \{\boldsymbol{N}_t + \hat{\boldsymbol{N}}_{\tau-t}\}_{t<\tau\leq s+t}\big)$, where we follow the convention to use $\sigma(\mathcal{C})$ to represent the $\sigma$-field generated by $\mathcal{C}$. Since $\{\boldsymbol{H}_t\}_{t\geq 0}$, $\{\boldsymbol{X}_t\}_{t\geq 0}$, and $\{\boldsymbol{L}_t\}_{t\geq 0}$ are $\mathcal{F}_t^{\boldsymbol{N}}$-predictable, for clarity, we rewrite $\boldsymbol{H}_t$, $\boldsymbol{X}_t$, and $\boldsymbol{L}_t$ as $\boldsymbol{H}_t = \boldsymbol{H}_t(\boldsymbol{v}|\mathcal{F}_t^{\boldsymbol{N}})$, $\boldsymbol{X}_t = \boldsymbol{X}_t(\boldsymbol{v}|\mathcal{F}_t^{\boldsymbol{N}})$, and $\boldsymbol{L}_t = \boldsymbol{L}_t(\boldsymbol{v}|\mathcal{F}_t^{\boldsymbol{N}})$, respectively, where $\boldsymbol{v}$ represents the initial promised utility and $\mathcal{F}_t^{\boldsymbol{N}}$ denotes the information available at time $t$. For any filtration $\hat{\mathcal{F}}_s^{\boldsymbol{N}}$, we can construct a contract $\hat{\Gamma}$ that satisfies $\hat{\boldsymbol{H}}_s(\boldsymbol{v}|\hat{\mathcal{F}}_s^{\boldsymbol{N}}) = \boldsymbol{H}_{s+t}(\boldsymbol{v}|\mathcal{F}_{s+t}^{\boldsymbol{N}})$, $\hat{\boldsymbol{X}}_s(\boldsymbol{v}|\hat{\mathcal{F}}_s^{\boldsymbol{N}}) = \boldsymbol{X}_{s+t}(\boldsymbol{v}|\mathcal{F}_{s+t}^{\boldsymbol{N}})$, and $\hat{\boldsymbol{L}}_s(\boldsymbol{v}|\hat{\mathcal{F}}_s^{\boldsymbol{N}}) = \boldsymbol{L}_{s+t}(\boldsymbol{v}|\mathcal{F}_{s+t}^{\boldsymbol{N}})$.

Then, it follows that

$$\begin{aligned}
\boldsymbol{W}_t(\Gamma, \bar{\boldsymbol{\Lambda}}) &= \mathbb{E}^{\bar{\boldsymbol{\Lambda}}}\left[\int_t^\infty e^{-\rho(s-t)}\left(\mathrm{d}\boldsymbol{L}_s(\boldsymbol{v}|\mathcal{F}_s^{\boldsymbol{N}}) + R^a\boldsymbol{X}_s(\boldsymbol{v}|\mathcal{F}_s^{\boldsymbol{N}})\mathrm{d}s\right)|\mathcal{F}_t^{\boldsymbol{N}}\right]\\
&= \mathbb{E}^{\bar{\boldsymbol{\Lambda}}}\left[\int_0^\infty e^{-\rho s}\left(\mathrm{d}\boldsymbol{L}_{s+t}(\boldsymbol{v}|\mathcal{F}_{s+t}^{\boldsymbol{N}}) + R^a\boldsymbol{X}_{s+t}(\boldsymbol{v}|\mathcal{F}_{s+t}^{\boldsymbol{N}})\mathrm{d}s\right)|\mathcal{F}_t^{\boldsymbol{N}}\right]\\
&= \mathbb{E}^{\bar{\boldsymbol{\Lambda}}}\left[\int_0^\infty e^{-\rho s}\left(\mathrm{d}\hat{\boldsymbol{L}}_s(\boldsymbol{v}|\hat{\mathcal{F}}_s^{\boldsymbol{N}}) + R^a\hat{\boldsymbol{X}}_s(\boldsymbol{v}|\hat{\mathcal{F}}_s^{\boldsymbol{N}})\mathrm{d}s\right)|\hat{\mathcal{F}}_0^{\boldsymbol{N}}\right]\\
&= \boldsymbol{u}(\hat{\Gamma}, \bar{\boldsymbol{\Lambda}}),
\end{aligned}$$

where the first equality follows from the definition of $\boldsymbol{W_t}(\Gamma, \bar{\boldsymbol{\Lambda}})$, the second equality from shifting the index, the third equality from the definition of $\hat{H}_t$, $\hat{X}_t$, and $\hat{L}_t$, and the last equality from the definition of $\boldsymbol{u}(\hat{\Gamma}, \bar{\boldsymbol{\Lambda}})$. The above equation indicates that $\boldsymbol{W}_t$ is achievable by $\hat{\Gamma}$.

Next, we show that the shifted contract $\hat{\Gamma}$ is an EIC contract, that is, $\{\hat{\boldsymbol{H}}_t\}_{t\geq 0}$, $\{\hat{\boldsymbol{X}}_t\}_{t\geq 0}$, and $\{\hat{\boldsymbol{L}}_t\}_{t\geq 0}$ satisfy (EA), (IC), and (LL), and $\{\boldsymbol{W}_t\}_{t\geq 0}$ follows (IR), (PK), and (UB) under $\{\hat{\boldsymbol{H}}_t\}_{t\geq 0}$, $\{\hat{\boldsymbol{X}}_t\}_{t\geq 0}$, and $\{\hat{\boldsymbol{L}}_t\}_{t\geq 0}$. By the definition of $\hat{\boldsymbol{H}}_t$, $\hat{\boldsymbol{X}}_t$, and $\hat{\boldsymbol{L}}_t$, we know $\{\hat{\boldsymbol{H}}_t\}_{t\geq 0}$, $\{\hat{\boldsymbol{X}}_t\}_{t\geq 0}$, and $\{\hat{\boldsymbol{L}}_t\}_{t\geq 0}$ satisfy (EA), (IC), and (LL) because $\{\boldsymbol{H}_t\}_{t\geq 0}$, $\{\boldsymbol{X}_t\}_{t\geq 0}$, and $\{\boldsymbol{L}_t\}_{t\geq 0}$ satisfy (EA), (IC), and (LL). Moreover, for all $W_{i,s}$, we have

$$\begin{aligned}
W_{i,s}(\hat{\Gamma}, \bar{\boldsymbol{\Lambda}}|\hat{\mathcal{F}}_s^{\boldsymbol{N}}) &= \mathbb{E}^{\bar{\boldsymbol{\Lambda}}}\left[\int_s^\infty e^{-\rho(\tau-s)}\left(\mathrm{d}\hat{L}_{i,\tau} + R^a\hat{X}_{i,\tau}\mathrm{d}\tau\right)|\hat{\mathcal{F}}_s\right]\\
&= \mathbb{E}^{\bar{\boldsymbol{\Lambda}}}\left[\int_{s+t}^\infty e^{-\rho(\tau-s-t)}\left(\mathrm{d}L_{i,\tau} + R^aX_{i,\tau}\mathrm{d}\tau\right)|\mathcal{F}_{s+t}\right]\\
&= W_{i,s+t}(\Gamma, \bar{\boldsymbol{\Lambda}}|\mathcal{F}_{s+t}^{\boldsymbol{N}}),
\end{aligned}$$

where the first equality and third equality follow from the definition of $W_{i,s}$, and the second equality from shifting the index and the definition of $\hat{H}_t$, $\hat{X}_t$, and $\hat{L}_t$. This readily implies that $\{\boldsymbol{W}_t\}_{t\geq 0}$ follows (IR), (PK), and (UB) under $\{\hat{\boldsymbol{H}}_t\}_{t\geq 0}$, $\{\hat{\boldsymbol{X}}_t\}_{t\geq 0}$, and $\{\hat{\boldsymbol{L}}_t\}_{t\geq 0}$. Therefore, the shifted contract $\hat{\Gamma}$ is an EIC contract.

To sum up, $\boldsymbol{W}_t \in \mathcal{U}$ under another EIC contract $\hat{\Gamma}$, which contradicts with $\boldsymbol{W}_t \notin \mathcal{U}$. Therefore, $\boldsymbol{W}_t \in \mathcal{U}$ for all $t \geq 0$. To conclude, the achievable set $\mathcal{U}$ is a self-generating set. $\square$

### EC.2.3. Technical Lemma on the Sufficiency of the Supporting Function Representation

We provide the following technical lemma to show it is sufficient to use supporting function representation with positive $\boldsymbol{\alpha}$ to characterize the achievable set.

LEMMA EC.2.1. *The achievable set $\mathcal{U}$ satisfies $\mathcal{U} = [0,\bar{w}]^n \bigcap \mathbf{epi}(\mathcal{U})$, where $\mathbf{epi}(\mathcal{U}) = \{\boldsymbol{w} \in \mathbb{R}^n \mid \exists \underline{\boldsymbol{w}} \in \mathcal{U}$ s.t. $\boldsymbol{w} \geq \underline{\boldsymbol{w}}\}$.*

*Proof:* Note that $\mathcal{U}$ is a subset of $[0,\bar{w}]^n$ by definition. Then, since any set is in its epigraph, $\mathcal{U} \subseteq [0,\bar{w}]^n \bigcap \mathbf{epi}(\mathcal{U})$. To prove the converse, we need to show that if $\boldsymbol{w} \in \mathcal{U}$, then for any $\boldsymbol{w}' \in [0,\bar{w}]^n$ such that $\boldsymbol{w}' \geq \boldsymbol{w}$, $\boldsymbol{w}' \in \mathcal{U}$. Here, for any two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^n$, the inequality $\boldsymbol{x} \leq (\geq) \boldsymbol{y}$ represents that $x_i \leq (\geq) y_i$ for each component $i$. Since $\boldsymbol{w} \in \mathcal{U}$, by (3.1), we have that there exists an EIC contract $\Gamma = \{\boldsymbol{L}_t, \boldsymbol{X}_t\}_{t\geq 0}$ associated with process $\{\boldsymbol{W}_t\}_{t\geq 0}$ such that $\boldsymbol{W}_0 = \boldsymbol{w}$. For $\boldsymbol{W}_0' = \boldsymbol{w}'$, we can construct an EIC contract $\Gamma' = \{\boldsymbol{L}_t', \boldsymbol{X}_t\}_{t\geq 0}$ from $\Gamma$. In particular, let $\boldsymbol{I}_0' = \boldsymbol{I}_0 + \Delta \boldsymbol{I}$, where $\Delta I_i = w_i' - w_i$ for all $i$, and keep $\boldsymbol{I}_t'$ the same as $\boldsymbol{I}_t$ for all $t > 0$ and $\boldsymbol{l}_t'$ the same as $\boldsymbol{l}_t$ for all $t \geq 0$. It is readily obtained that $\boldsymbol{W}_0'$ is achievable, or equivalently, $\boldsymbol{W}_0' = \boldsymbol{w}' \in \mathcal{U}$. $\square$

The result suggests that it's sufficient to use supporting function representation with positive norm vectors $\boldsymbol{\alpha}$ to characterize the achievable set.

### EC.2.4. Proof of Lemma 3.1

LEMMA 3.1. *If $\mathcal{A}$ is a self-generating set, then we have*

1. *$\mathcal{A} \subseteq \mathscr{G}(T\phi_{\mathcal{A}})$, and*
2. *$\mathscr{G}(\phi_{\mathcal{A}})$ is a self-generating set.*

*Conversely, if a convex set $\mathcal{A}$ satisfies $\mathcal{A} \subseteq \mathscr{G}(T\phi_{\mathcal{A}})$, then $\mathcal{A}$ is a self-generating set.*

*Proof:* The proof consists of three parts. Specifically, we first prove that if $\mathcal{A}$ is a self-generating set, then $\mathcal{A} \subseteq \mathscr{G}(T\phi_{\mathcal{A}}))$. Next, we prove that if $\mathcal{A}$ is convex and $\mathcal{A} \subseteq \mathscr{G}(T\phi_{\mathcal{A}})$, then $\mathcal{A}$ is self-generating. At last, we use the above results to prove that if $\mathcal{A}$ is a self-generating set, then $\mathscr{G}(\phi_{\mathcal{A}})$ is a self-generating set.

**(Part I: If $\mathcal{A}$ is a self-generating set, then $\mathcal{A} \subseteq \mathscr{G}(T\phi_{\mathcal{A}})$)** We need to show that for every self-generating set $\mathcal{A}$, if $\boldsymbol{v} \in \mathcal{A}$, then $\boldsymbol{v} \in \mathscr{G}(T\phi_{\mathcal{A}})$. Recall that the set $\mathscr{G}(T\phi_{\mathcal{A}})$ is defined as $\{w \in [0,\bar{w}]^n | \boldsymbol{\alpha}^\top \boldsymbol{w} \geq T\phi_{\mathcal{A}}(\boldsymbol{\alpha}),\ \forall \boldsymbol{\alpha} \in \mathbb{R}_+^n,\ \|\boldsymbol{\alpha}\|_1 = 1\}$. Therefore, according to the definition of Problem $[T\phi](\boldsymbol{\alpha})$, it suffices to show that for all $\boldsymbol{v} \in \mathcal{A}$ where $\mathcal{A}$ is self-generating, $\boldsymbol{v}$ constitutes a feasible solution of Problem $[T\phi](\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \mathbb{R}_+^n$. Note that, as an evident property rooted in the Definition 3.2, if a set $\mathcal{A}$ is self-generating, then the closure of this set, denoted by $\mathrm{cl}(\mathcal{A})$, is also self-generating. Therefore, without loss of generality, we assume $\mathcal{A}$ to be closed sets. This result, together with the convexity of the feasible region of problem $T\phi_{\mathcal{A}}(\boldsymbol{\alpha})$, implies that it suffices to show the desired result hold for $\boldsymbol{v}$ such that both $\boldsymbol{v} \in \mathrm{bd}(\mathcal{A})$ and $\boldsymbol{v} \in \mathrm{bd}(\mathscr{G}(\phi_{\mathcal{A}}))$ (note that $\mathcal{A}$ and $\mathscr{G}(\phi_{\mathcal{A}})$ are not necessarily the same).

On the basis of Definition 3.2, the set $\mathcal{A}$ being self-generating is equivalent to the statement that for any $\boldsymbol{v} \in \mathcal{A}$, there exist processes $\{\boldsymbol{H}_t^{\boldsymbol{v}}\}_{t\geq 0}$, $\{\boldsymbol{X}_t^{\boldsymbol{v}}\}_{t\geq 0}$, $\{\boldsymbol{I}_t^{\boldsymbol{v}}\}_{t\geq 0}$, $\{\boldsymbol{l}_t^{\boldsymbol{v}}\}_{t\geq 0}$, and $\{\boldsymbol{W}_t^{\boldsymbol{v}}\}_{t\geq 0}$ with $\boldsymbol{W}_{0-} = \boldsymbol{W}_0 = \boldsymbol{v}$, such that for all $t \geq 0$, the processes satisfy (EA), (IC), (IR), (PK), (LL), and (UB). Meanwhile, for all $t \geq 0$, the discounted promised utility of agent $i$ at time $t$ can be re-written as follows:

$$\begin{aligned}
e^{-\rho t} W_{i,t}^{\boldsymbol{v}} =& e^{\rho 0} W_{i,0-}^{\boldsymbol{v}} + \int_0^t e^{-\rho s} \mathrm{d}W_{i,s}^{\boldsymbol{v}} + \int_0^t W_{i,s}^{\boldsymbol{v}} \mathrm{d}e^{-\rho s} \\
=& W_{i,0}^{\boldsymbol{v}} + \int_0^t e^{-\rho s} \left[ \left( \rho W_{i,s-}^{\boldsymbol{v}} - R^a X_{i,s}^{\boldsymbol{v}} + \lambda \sum_{j\in\mathcal{I}} H_{ij,s}^{\boldsymbol{v}} \right) \mathrm{d}s - \sum_{j\in\mathcal{I}} H_{ij,s}^{\boldsymbol{v}} \mathrm{d}N_{j,s} - l_{i,s}^{\boldsymbol{v}} \mathrm{d}s \right] \\
& - \sum_{s\in[0,t]: I_{i,s}^{\boldsymbol{v}} > 0} e^{-\rho s} I_{i,s}^{\boldsymbol{v}} - \int_0^t \rho e^{-\rho s} W_{i,s}^{\boldsymbol{v}} \mathrm{d}s \\
=& v_i - \int_0^t e^{-\rho s} \left( R^a X_{i,s}^{\boldsymbol{v}} + l_{i,s}^{\boldsymbol{v}} - \lambda \sum_{j\in\mathcal{I}} H_{ij,s}^{\boldsymbol{v}} \right) \mathrm{d}s - \sum_{s\in[0,t]: I_{i,s}^{\boldsymbol{v}} > 0} e^{-\rho s} I_{i,s}^{\boldsymbol{v}} \\
& - \int_0^t e^{-\rho s} \sum_{j\in\mathcal{I}} H_{ij,s}^{\boldsymbol{v}} \mathrm{d}N_{j,s},
\end{aligned}$$

where the first equality follows from integration by parts, the second equality from equation (PK), and the third equality from cancelling out the terms $\int_0^t \rho e^{-\rho s} W_{i,s}^{\boldsymbol{v}} \mathrm{d}s$ and $\int_0^t \rho e^{-\rho s} W_{i,s-}^{\boldsymbol{v}} \mathrm{d}s$. Since $\mathcal{A}$ is a self-generating set and $\boldsymbol{v} \in \mathcal{A}$, we have $\boldsymbol{W}_t^{\boldsymbol{v}} \in \mathcal{A}$, that is,

$$\begin{aligned}
\Bigg\{ e^{\rho t} \Bigg[ & v_i - \int_0^t e^{-\rho s} \left( R^a X_{i,s}^{\boldsymbol{v}} + l_{i,s}^{\boldsymbol{v}} - \lambda \sum_{j\in\mathcal{I}} H_{ij,s}^{\boldsymbol{v}} \right) \mathrm{d}s \\
& - \sum_{s\in[0,t]: I_{i,s}^{\boldsymbol{v}} > 0} e^{-\rho s} I_{i,s}^{\boldsymbol{v}} - \int_0^t e^{-\rho s} \sum_{j\in\mathcal{I}} H_{ij,s}^{\boldsymbol{v}} \mathrm{d}N_{j,s} \Bigg] \Bigg\}_{i\in[1,\ldots,n]} \in \mathcal{A}, \quad \forall \boldsymbol{\alpha} \in \mathbb{R}_+^n.
\end{aligned}$$

In addition, the above result indicates that, for all $t \geq 0$ and for any vector $\boldsymbol{\alpha} \in \mathbb{R}_+^n$ with $\|\boldsymbol{\alpha}\|_1 = 1$, $\boldsymbol{\alpha}^\top \boldsymbol{W}_t^{\boldsymbol{v}} \geq \phi_{\mathcal{A}}(\boldsymbol{\alpha})$, where $\phi_{\mathcal{A}}(\cdot)$ is the support function of set $\mathcal{A}$. Plugging in the previous expression for $\boldsymbol{W}_t^{\boldsymbol{v}}$, we have

$$\sum_{i\in\mathcal{I}} \alpha_i e^{\rho t} \Bigg[ v_i - \int_0^t e^{-\rho s} \left( R^a X_{i,s}^{\boldsymbol{v}} + l_{i,s}^{\boldsymbol{v}} - \lambda \sum_{j\in\mathcal{I}} H_{ij,s}^{\boldsymbol{v}} \right) \mathrm{d}s$$

$$- \sum_{s\in[0,t]:I^{\boldsymbol{v}}_{i,s}>0} e^{-\rho s} I^{\boldsymbol{v}}_{i,s} - \int_0^t e^{-\rho s} \sum_{j\in\mathcal{I}} H^{\boldsymbol{v}}_{ij,s} \mathrm{d}N_{j,s} \Bigg] \geq \phi_{\mathcal{A}}(\boldsymbol{\alpha}), \quad \forall \boldsymbol{\alpha} \in \mathbb{R}^n_+.$$

Then, taking the limit as $t$ approaches 0, we have

$$\sum_{i\in\mathcal{I}} \alpha_i \left[ v_i - I^{\boldsymbol{v}}_{i,0} - \sum_{j\in\mathcal{I}} H^{\boldsymbol{v}}_{ij,0} \mathrm{d}N_{j,0} \right] \geq \phi_{\mathcal{A}}(\boldsymbol{\alpha}), \quad \forall \boldsymbol{\alpha} \in \mathbb{R}^n_+. \tag{EC.2.2}$$

The above results shed light on a construction of a solution to Problem $[T\phi](\boldsymbol{\alpha})$. In particular, Let $\boldsymbol{x} = \boldsymbol{X}^{\boldsymbol{v}}_0$, $H_{ij} = H^{\boldsymbol{v}}_{ij,0}$, and set $\boldsymbol{w} = \boldsymbol{v}$, $y_i = \rho w_i - R^a x_i + \lambda \sum_{j\in\mathcal{I}} H_{ij}$, $Z_{ij} = w_i - H_{ij}$. It is readily obtained from this setting of $\boldsymbol{y}$ and $\boldsymbol{Z}$ that ($\mathrm{PK}_{\boldsymbol{y}}$) and ($\mathrm{PK}_{\boldsymbol{Z}}$) are satisfied. Then, following from the fact that processes $\{\boldsymbol{H}^{\boldsymbol{v}}_t\}_{t\geq 0}$, $\{\boldsymbol{X}^{\boldsymbol{v}}_t\}_{t\geq 0}$, $\{\boldsymbol{I}^{\boldsymbol{v}}_t\}_{t\geq 0}$, $\{\boldsymbol{l}^{\boldsymbol{v}}_t\}_{t\geq 0}$, and $\{\boldsymbol{W}^{\boldsymbol{v}}_t\}_{t\geq 0}$ with $\boldsymbol{W}_0 = \boldsymbol{v}$ satisfy (EA), (IC), (IR), (LL), and (UB) for all $t \geq 0$, constraints ($\mathrm{EA_s}$), ($\mathrm{IC_s}$), ($\mathrm{IR_s}$), and ($\mathrm{UB_s}$) are satisfied. Moreover, $\boldsymbol{v} \in \mathcal{A}$ implies that ($\mathrm{SG}_{\boldsymbol{w}}$) is satisfied. Furthermore, note that the Poisson arrival of adverse events implies that $\sum_{j\in\mathcal{I}} \mathrm{d}N_{j,t} \leq 1$ for all $t$. This, together with equation (EC.2.2) and the fact that $\boldsymbol{I}^{\boldsymbol{v}}_{i,0} \geq 0$, suggests that ($\mathrm{SG}_{\boldsymbol{Z}}$) is satisfied. At last, the remaining unverified constraint is constraint ($\mathrm{SG}_{\boldsymbol{y}}$). If the constructed solution also satisfies constraint ($\mathrm{SG}_{\boldsymbol{y}}$), then the solution is feasible to Problem $[T\phi](\boldsymbol{\alpha})$, that is, $\boldsymbol{v} \in \mathscr{G}(T\phi_{\mathcal{A}})$, which completes the proof of Part I.

We proceed to very that ($\mathrm{SG}_{\boldsymbol{y}}$) is also satisfied by the aforementioned constructed solution. Since $\boldsymbol{W}^{\boldsymbol{v}}_t \in \mathcal{A}$ for all $t \geq 0$, letting $t = \epsilon$, for all $\epsilon \in (0, \alpha]$ and $\alpha$ small enough, we have $\boldsymbol{W}^{\boldsymbol{v}}_\epsilon \in \mathcal{A}$, where

$$\begin{aligned}
W^{\boldsymbol{v}}_{i,\epsilon} =& e^{\rho\epsilon} v_i + \int_0^\epsilon e^{\rho(\epsilon - s)} \left( \rho W^{\boldsymbol{v}}_{i,s-} - R^a X^{\boldsymbol{v}}_{i,s} + \lambda \sum_{j\in\mathcal{I}} H^{\boldsymbol{v}}_{ij,s} - l^{\boldsymbol{v}}_{i,s} \right) \mathrm{d}s + \int_0^\epsilon \rho W^{\boldsymbol{v}}_{i,s} \mathrm{d}e^{\rho(\epsilon - s)} \\
& - \int_0^\epsilon e^{\rho(\epsilon - s)} \sum_{j\in\mathcal{I}} H^{\boldsymbol{v}}_{ij,s} \mathrm{d}N_{j,s} - \sum_{s\in[0,\epsilon]:I^{\boldsymbol{v}}_{i,s}>0} e^{\rho(\epsilon - s)} I^{\boldsymbol{v}}_{i,s} \\
=& e^{\rho\epsilon} v_i + \int_0^\epsilon e^{\rho(\epsilon - s)} \left( \rho W^{\boldsymbol{v}}_{i,s} - R^a X^{\boldsymbol{v}}_{i,s} + \lambda \sum_{j\in\mathcal{I}} H^{\boldsymbol{v}}_{ij,s} - l^{\boldsymbol{v}}_{i,s} \right) \mathrm{d}s + \int_0^\epsilon \rho W^{\boldsymbol{v}}_{i,s} \mathrm{d}e^{\rho(\epsilon - s)} \\
& - \int_0^\epsilon e^{\rho(\epsilon - s)} \sum_{j\in\mathcal{I}} H^{\boldsymbol{v}}_{ij,s} \mathrm{d}N_{j,s} - \sum_{s\in[0,\epsilon]:I^{\boldsymbol{v}}_{i,s}>0} e^{\rho(\epsilon - s)} I^{\boldsymbol{v}}_{i,s}.
\end{aligned}$$

$\boldsymbol{W}^{\boldsymbol{v}}_\epsilon \in \mathcal{A}$ implies for any given vector $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$,

$$\begin{aligned}
\sum_{i\in\mathcal{I}} \alpha_i \Bigg[ & e^{\rho\epsilon} v_i + \int_0^\epsilon e^{\rho(\epsilon - s)} \left( \rho W^{\boldsymbol{v}}_{i,s} - R^a X^{\boldsymbol{v}}_{i,s} + \lambda \sum_{j\in\mathcal{I}} H^{\boldsymbol{v}}_{ij,s} - l^{\boldsymbol{v}}_{i,s} \right) \mathrm{d}s + \int_0^\epsilon \rho W^{\boldsymbol{v}}_{i,s} \mathrm{d}e^{\rho(\epsilon - s)} \\
& - \int_0^\epsilon e^{\rho(\epsilon - s)} \sum_{j\in\mathcal{I}} H^{\boldsymbol{v}}_{ij,s} \mathrm{d}N_{j,s} - \sum_{s\in[0,\epsilon]:I^{\boldsymbol{v}}_{i,s}>0} e^{\rho(\epsilon - s)} I^{\boldsymbol{v}}_{i,s} \Bigg] \geq \phi_{\mathcal{A}}(\boldsymbol{\alpha}), \ \forall \boldsymbol{\alpha} \in \mathbb{R}^n_+.
\end{aligned}$$

This inequality can be further decomposed into the following two inequalities, depending on whether there exists an arrival of adverse events or instantaneous payment, as follows

$$\sum_{i\in\mathcal{I}} \alpha_i \Bigg[ e^{\rho\epsilon} v_i + \int_0^\epsilon e^{\rho(\epsilon - s)} \left( \rho W^{\boldsymbol{v}}_{i,s} - R^a X^{\boldsymbol{v}}_{i,s} + \lambda \sum_{j\in\mathcal{I}} H^{\boldsymbol{v}}_{ij,s} - l^{\boldsymbol{v}}_{i,s} \right) \mathrm{d}s$$

$$+\int_0^{\epsilon}\rho W^{\boldsymbol{v}}_{i,s}\mathrm{d}e^{\rho(\epsilon-s)}\Bigg]\geq\phi_{\mathcal{A}}(\boldsymbol{\alpha}),\quad\forall\boldsymbol{\alpha}\in\mathbb{R}^n_+,\qquad\text{(EC.2.3)}$$

and

$$\sum_{i\in\mathcal{I}}\alpha_i\left[v_i-I^{\boldsymbol{v}}_{i,0}-\sum_{j\in\mathcal{I}}H^{\boldsymbol{v}}_{ij,0}\mathrm{d}N_{j,0}\right]\geq\phi_{\mathcal{A}}(\boldsymbol{\alpha}),\quad\forall\boldsymbol{\alpha}\in\mathbb{R}^n_+.$$

Recall that $\boldsymbol{v}\in\mathrm{bd}(\mathscr{G}(\phi_{\mathcal{A}}))$, there exists some $\hat{\boldsymbol{\alpha}}\in\mathbb{R}^n_+$ with $\|\hat{\boldsymbol{\alpha}}\|_1=1$ such that $\hat{\boldsymbol{\alpha}}^\top\boldsymbol{v}=\phi_{\mathcal{A}}(\hat{\boldsymbol{\alpha}})$, and thus equation (EC.2.3) yields

$$\sum_{i\in\mathcal{I}}\hat{\alpha}_i\left(\rho W^{\boldsymbol{v}}_{i,0}-R^aX^{\boldsymbol{v}}_{i,0}+\lambda\sum_{j\in\mathcal{I}}H^{\boldsymbol{v}}_{ij,0}-l^{\boldsymbol{v}}_{i,0}\right)\geq 0,$$

which implies that constraint ($\mathrm{SG}_{\boldsymbol{y}}$) is satisfied for $\hat{\boldsymbol{\alpha}}$ (note that $\boldsymbol{l}^{\boldsymbol{v}}_0\geq 0$). Therefore, $\boldsymbol{v}$ is a feasible solution for problem $[T\phi](\boldsymbol{\alpha})$ with vector $\hat{\boldsymbol{\alpha}}$. This completes the proof of Part I.

**(Part II: If $\mathcal{A}$ is convex and $\mathcal{A}\subseteq\mathscr{G}(T\phi_{\mathcal{A}})$, then $\mathcal{A}$ is self-generating.)** It suffices to show that for any $\boldsymbol{W}_{0-}=\boldsymbol{W}_0=\boldsymbol{v}\in\mathcal{A}\subseteq\mathscr{G}(T\phi_{\mathcal{A}})$, there exist processes $\{\boldsymbol{H}^{\boldsymbol{v}}_t\}_{t\geq0}$, $\{\boldsymbol{X}^{\boldsymbol{v}}_t\}_{t\geq0}$, $\{\boldsymbol{I}^{\boldsymbol{v}}_t\}_{t\geq0}$, $\{\boldsymbol{l}^{\boldsymbol{v}}_t\}_{t\geq0}$, and $\{\boldsymbol{W}^{\boldsymbol{v}}_t\}_{t\geq0}$, that satisfy (EA), (IC), (IR), (PK), (LL), and (UB), and $\boldsymbol{W}^{\boldsymbol{v}}_t\in\mathcal{A}$ for all $t\geq0$. In the following text, we drop the superscript $\boldsymbol{v}$ when the context is clear.

We prove this result by induction. In particular, consider a partition of the time interval $[0,t]$ by a sequence of time points $\{t_k\}_{k=0}^{n+1}$, which satisfies $0=t_0<t_1<...<t_{n-1}<t_n\leq t_{n+1}=t$, where $n\geq0$ and $t_k$'s ($k\in\{1,2,...,n\}$) are time points at which adverse arrivals happen. Note that the Poisson assumption implies at most one adverse arrival at any time. Suppose $\boldsymbol{W}_{t_{k-1}}\in\mathcal{A}$, we claim that (i) there exist $\mathcal{F}^{\boldsymbol{N}}_t$-predictable processes $\{\boldsymbol{H}_t\}_{t\geq0}$, $\{\boldsymbol{X}_t\}_{t\geq0}$, and $\{\boldsymbol{L}_t\}_{t\geq0}$ that satisfy (EA), (IC), and (LL), and following (IR), (PK), and (UB) the process $\{\boldsymbol{W}_t\}_{t\geq0}$ satisfies $\boldsymbol{W}_t\in\mathcal{A}$ for any $t\in(t_{k-1},t_k)$; (ii) if $\boldsymbol{W}_t\in\mathcal{A}$ for all $t\in(t_{k-1},t_k)$, then following the $\mathcal{F}^{\boldsymbol{N}}_t$-predictable processes $\{\boldsymbol{H}_t\}_{t\geq0}$, $\{\boldsymbol{X}_t\}_{t\geq0}$, and $\{\boldsymbol{L}_t\}_{t\geq0}$ that satisfy (EA), (IC), and (LL), the processes of the promised utility of agents, $\{\boldsymbol{W}_t\}_{t\geq0}$, guided by conditions (IR), (PK), and (UB) satisfies $\boldsymbol{W}_{t_k}\in\mathcal{A}$. Combining (i) and (ii) together, we obtain that if $\boldsymbol{W}_{t_{k-1}}\in\mathcal{A}$, then there exist $\mathcal{F}^{\boldsymbol{N}}_t$-predictable processes $\{\boldsymbol{H}_t\}_{t\geq0}$, $\{\boldsymbol{X}_t\}_{t\geq0}$, and $\{\boldsymbol{L}_t\}_{t\geq0}$ that satisfy (EA), (IC), and (LL), abiding by which the process $\{\boldsymbol{W}_t\}_{t\geq0}$ following conditions (IR), (PK), and (UB) satisfies $\boldsymbol{W}_t\in\mathcal{A}$ for any $t\in(t_{k-1},t_k]$. At last, Part II follows by mathematical induction. Next, we prove the aforementioned two claims.

We first prove claim (i). Since $\boldsymbol{W}_{t_{k-1}}\in\mathcal{A}$, we have $\boldsymbol{\alpha}^\top\boldsymbol{W}_{t_{k-1}}\geq\phi_{\mathcal{A}}(\boldsymbol{\alpha})$, for all $\boldsymbol{\alpha}\in\mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1=1$. In addition, with $\mathcal{A}\subseteq\mathscr{G}(T\phi_{\mathcal{A}})$ as given, we have $\boldsymbol{W}_{t_{k-1}}\in\mathscr{G}(T\phi_{\mathcal{A}})$. Recall that, for any $t\in(t_{k-1},t_k)$, the definition of the promised utility of agents yields

$$e^{-\rho t}W^{\boldsymbol{v}}_{i,t}=e^{-\rho t_{k-1}}W^{\boldsymbol{v}}_{i,t_{k-1}}+\int_{(t_{k-1},t]}e^{-\rho s}\mathrm{d}W^{\boldsymbol{v}}_{i,s}+\int_{(t_{k-1},t]}W^{\boldsymbol{v}}_{i,s}\mathrm{d}e^{-\rho s}.$$

Multiplying both sides by $e^{\rho t}$, the above equation can be re-expressed as

$$
\begin{aligned}
W^{\boldsymbol{v}}_{i,t} =& e^{\rho(t-t_{k-1})}W^{\boldsymbol{v}}_{i,t_{k-1}} + \int_{(t_{k-1},t]} e^{\rho(t-s)}\mathrm{d}W^{\boldsymbol{v}}_{i,s} + \int_{(t_{k-1},t]} W^{\boldsymbol{v}}_{i,s}\mathrm{d}e^{\rho(t-s)} \\
=& e^{\rho(t-t_{k-1})}W^{\boldsymbol{v}}_{i,t_{k-1}} + \int_{(t_{k-1},t]} e^{\rho(t-s)}(\rho W^{\boldsymbol{v}}_{i,s-} - R^a X^{\boldsymbol{v}}_{i,s} + \lambda\sum_{j\in\mathcal{I}} H^{\boldsymbol{v}}_{ij,s} - l^{\boldsymbol{v}}_{i,s})\mathrm{d}s \\
& - \sum_{s\in(t_{k-1},t]:I^{\boldsymbol{v}}_{i,s}>0} e^{-\rho s}I^{\boldsymbol{v}}_{i,s} + \int_{(t_{k-1},t]} W^{\boldsymbol{v}}_{i,s}\mathrm{d}e^{\rho(t-s)} \\
=& e^{\rho(t-t_{k-1})}W^{\boldsymbol{v}}_{i,t_{k-1}} + \int_{(t_{k-1},t]} e^{\rho(t-s)}(\rho W^{\boldsymbol{v}}_{i,s} - R^a X^{\boldsymbol{v}}_{i,s} + \lambda\sum_{j\in\mathcal{I}} H^{\boldsymbol{v}}_{ij,s} - l^{\boldsymbol{v}}_{i,s})\mathrm{d}s \\
& - \sum_{s\in(t_{k-1},t]:I^{\boldsymbol{v}}_{i,s}>0} e^{-\rho s}I^{\boldsymbol{v}}_{i,s} + \int_{(t_{k-1},t]} W^{\boldsymbol{v}}_{i,s}\mathrm{d}e^{\rho(t-s)},
\end{aligned}
\tag{EC.2.4}
$$

where the second equality follows from the (PK) condition and the assumption that no adverse event arrives in $t\in(t_{k-1},t_k)$.

For any $\boldsymbol{w}\in\mathcal{A}\subseteq\mathscr{G}(T\phi_{\mathcal{A}})$, $\boldsymbol{w}$ constitutes a feasible solution to problem $[T\phi](\boldsymbol{\alpha})$, that is, there exist $\boldsymbol{x}$, $\boldsymbol{H}$, and the auxiliary variables $\boldsymbol{y}$ and $\boldsymbol{Z}$ satisfying ($\mathrm{IC_s}$), ($\mathrm{EA_s}$), ($\mathrm{PK}_{\boldsymbol{Z}}$), ($\mathrm{PK}_{\boldsymbol{y}}$), ($\mathrm{IR_s}$), ($\mathrm{UB_s}$), ($\mathrm{SG}_{\boldsymbol{w}}$), ($\mathrm{SG}_{\boldsymbol{y}}$), and ($\mathrm{SG}_{\boldsymbol{Z}}$). Each of such feasible solutions comprises a part of a contract that meets conditions (IC), (EA), (LL), (IR), and (UB) associated with the promised utility vector $\boldsymbol{w}$. Moreover, ($\mathrm{PK}_{\boldsymbol{y}}$) and ($\mathrm{PK}_{\boldsymbol{Z}}$) jointly ensure that condition (PK) is met. Recall that claim (i) states that if $\boldsymbol{W}_{t_{k-1}}\in\mathcal{A}$, then there exist $\mathcal{F}^{\boldsymbol{N}}_t$-predictable processes $\{\boldsymbol{H}_t\}_{t\geq0}$, $\{\boldsymbol{X}_t\}_{t\geq0}$, and $\{\boldsymbol{L}_t\}_{t\geq0}$ that satisfy (EA), (LL) and (IC) such that the $\mathcal{F}^{\boldsymbol{N}}_t$-adapted process $\{\boldsymbol{W}_t\}_{t\geq0}$ complying with (PK), (IR), (UB) must satisfies $\boldsymbol{W}_t\in\mathcal{A}$ for any $t\in(t_{k-1},t_k)$. We prove this by contradiction.

Suppose to the contrary that there exists $\hat{t}\in(t_{k-1},t_k)$ such that $\boldsymbol{W}_{\hat{t}}\notin\mathcal{A}$, then by the continuity of process $\{\boldsymbol{W}_t\}_{t\in(t_{k-1},t_k)}$ in the absence of adverse arrivals, there must exist $\tilde{t}\in(t_{k-1},\hat{t})$ such that $\boldsymbol{W}_{\tilde{t}}\in\mathrm{bd}(\mathcal{A})$, and that $\boldsymbol{W}_t\in\mathcal{A}$ for all $t\in(t_{k-1},\tilde{t}]$, whereas $\boldsymbol{W}_{\tilde{t}+\epsilon}\notin\mathrm{cl}(\mathcal{A})$ for all $\epsilon\in(0,\alpha]$ and $\alpha$ small enough, where $\mathrm{bd}(\mathcal{A})$ and $\mathrm{cl}(\mathcal{A})$ denote the boundary and closure of set $\mathcal{A}$, respectively. Next, we construct contracts in accordance with the above discussion. In particular, let $\boldsymbol{X}^{\boldsymbol{v}}_t=\boldsymbol{x}$ and $\boldsymbol{H}^{\boldsymbol{v}}_t=\boldsymbol{H}$, set $\boldsymbol{l}_t$ according to equation $l_{i,t}=(y_i)^+\mathbb{1}_{\{W^{\boldsymbol{v}}_{i,t-}=\bar{w}\}}$, and let $\boldsymbol{I}^{\boldsymbol{v}}_t=0$ for all $t\in(t_{k-1},\tilde{t}]$, where $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{H}$ are the feasible solutions to problem $[T\phi](\boldsymbol{\alpha})$ associated with $\boldsymbol{w}=\boldsymbol{W}^{\boldsymbol{v}}_{t-}$. Then, being feasible to constraints ($\mathrm{EA_s}$), ($\mathrm{IC_s}$), ($\mathrm{IR_s}$), and ($\mathrm{UB_s}$) and satisfying equation $l_{i,t}=(y_i)^+\mathbb{1}_{\{W^{\boldsymbol{v}}_{i,t-}=\bar{w}\}}$ implies that $\mathcal{F}^{\boldsymbol{N}}_t$-predictable processes $\{\boldsymbol{H}_t\}_{t\in(t_{k-1},\tilde{t}]}$, $\{\boldsymbol{X}_t\}_{t\in(t_{k-1},\tilde{t}]}$, and $\{\boldsymbol{L}_t\}_{t\in(t_{k-1},\tilde{t}]}$ satisfy conditions (EA), (IC), (IR), (LL), and (UB). Equation (EC.2.4) for $\boldsymbol{W}_t$ further implies that (PK) is satisfied. Moreover, since $\boldsymbol{I}^{\boldsymbol{v}}_t=0$ for all $t\in(t_{k-1},t_k)$, equation (EC.2.4) reduces to

$$
W^{\boldsymbol{v}}_{i,t} = e^{\rho(t-t_{k-1})}W^{\boldsymbol{v}}_{i,t_{k-1}} + \int_{(t_{k-1},t]} e^{\rho(t-s)}(\rho W^{\boldsymbol{v}}_{i,s} - R^a X^{\boldsymbol{v}}_{i,s} + \lambda\sum_{j\in\mathcal{I}} H^{\boldsymbol{v}}_{ij,s} - l^{\boldsymbol{v}}_{i,s})\mathrm{d}s + \int_{(t_{k-1},t]} W^{\boldsymbol{v}}_{i,s}\mathrm{d}e^{\rho(t-s)}.
\tag{EC.2.5}
$$

Next, since $\boldsymbol{W}_{\tilde{t}} \in \mathrm{bd}(\mathcal{A})$ and $\boldsymbol{W}_{\tilde{t}+\epsilon} \notin \mathrm{cl}(\mathcal{A})$, there exists $\tilde{\boldsymbol{\alpha}} \in \mathbb{R}^n_+$ and $\|\tilde{\boldsymbol{\alpha}}\|_1 = 1$, such that

$$\tilde{\boldsymbol{\alpha}} \boldsymbol{W}_{\tilde{t}} \geq \phi_{\mathcal{A}}(\tilde{\boldsymbol{\alpha}}) \text{ and } \tilde{\boldsymbol{\alpha}} \boldsymbol{W}_{\tilde{t}+\epsilon} < \phi_{\mathcal{A}}(\tilde{\boldsymbol{\alpha}}). \tag{EC.2.6}$$

Note that in a similar fashion to equation (EC.2.5), we have

$$W^{\boldsymbol{v}}_{i,\tilde{t}+\epsilon} = e^{\rho(\tilde{t}+\epsilon-\tilde{t})} W^{\boldsymbol{v}}_{i,\tilde{t}} + \int_{(\tilde{t},\tilde{t}+\epsilon]} e^{\rho(\tilde{t}+\epsilon-s)} (\rho W^{\boldsymbol{v}}_{i,s} - R^a X^{\boldsymbol{v}}_{i,s} + \lambda \sum_{j\in\mathcal{I}} H^{\boldsymbol{v}}_{ij,s} - l^{\boldsymbol{v}}_{i,s}) \mathrm{d}s + \int_{(\tilde{t},\tilde{t}+\epsilon]} W^{\boldsymbol{v}}_{i,s} \mathrm{d}e^{\rho(\tilde{t}+\epsilon-s)}.$$

Inequality (EC.2.6) implies that

$$\sum_{i\in\mathcal{I}} \tilde{\alpha}_i (\rho W_{i,\tilde{t}} - R^a X^{\boldsymbol{v}}_{i,\tilde{t}} + \lambda \sum_{j\in\mathcal{I}} H^{\boldsymbol{v}}_{ij,\tilde{t}} - l^{\boldsymbol{v}}_{i,\tilde{t}}) = \tilde{\boldsymbol{\alpha}}^\top \boldsymbol{y} < 0,$$

that is, $\boldsymbol{W}_{\tilde{t}}$ satisfies all the constraints in problem $[T\phi](\boldsymbol{\alpha})$, except for the constraint ($\mathrm{SG}_{\boldsymbol{y}}$).

However, since $\boldsymbol{W}_{\tilde{t}} \in \mathrm{bd}(\mathcal{A})$, $\tilde{\boldsymbol{\alpha}}$ defines a supporting hyperplane of set $\mathcal{A}$ at $\boldsymbol{W}_{\tilde{t}}$. In addition, since $\mathcal{A} \subseteq \mathscr{G}(T\phi_{\mathcal{A}})$, $\boldsymbol{W}_{\tilde{t}}$ is the optimal solution to problem $[T\phi](\tilde{\boldsymbol{\alpha}})$, that is, $[T\phi_{\mathcal{A}}](\tilde{\boldsymbol{\alpha}})$. Consequently, we have by construction

$$\sum_{i\in\mathcal{I}} \tilde{\alpha}_i (\rho W_{i,\tilde{t}} - R^a X^{\boldsymbol{v}}_{i,\tilde{t}} + \lambda \sum_{j\in\mathcal{I}} H^{\boldsymbol{v}}_{ij,\tilde{t}} - l^{\boldsymbol{v}}_{i,\tilde{t}}) \geq 0, \tag{EC.2.7}$$

which yields a contradiction. Therefore, $W^{\boldsymbol{v}}_{i,t} \in \mathcal{A}$ for any $t \in (t_{k-1}, t_k)$, that is, claim (i) holds.

Next, we show claim (ii). Since for $t < t_k$, by integration by parts, we have

$$e^{-\rho t_k} W^{\boldsymbol{v}}_{i,t_k} = e^{-\rho t} W^{\boldsymbol{v}}_{i,t-} + \int_t^{t_k} e^{-\rho s} \mathrm{d}W^{\boldsymbol{v}}_{i,s} + \int_t^{t_k} W^{\boldsymbol{v}}_{i,s} \mathrm{d}e^{-\rho s}.$$

Multiplying both sides by $e^{\rho t_k}$ the above equation becomes

$$\begin{aligned} W^{\boldsymbol{v}}_{i,t_k} =& e^{\rho(t_k-t)} W^{\boldsymbol{v}}_{i,t-} + \int_t^{t_k} e^{\rho(t_k-s)} \mathrm{d}W^{\boldsymbol{v}}_{i,s} + \int_t^{t_k} W^{\boldsymbol{v}}_{i,s} \mathrm{d}e^{\rho(t_k-s)} \\ =& e^{\rho(t_k-t)} W^{\boldsymbol{v}}_{i,t-} + \int_t^{t_k} e^{\rho(t_k-s)} \left( R^a X^{\boldsymbol{v}}_{i,s} + l^{\boldsymbol{v}}_{i,s} - \lambda \sum_{j\in\mathcal{I}} H^{\boldsymbol{v}}_{ij,s} \right) \mathrm{d}s \\ & - \sum_{s\in[t,t_k]: I^{\boldsymbol{v}}_{i,s}>0} e^{-\rho s} I^{\boldsymbol{v}}_{i,s} - \int_t^{t_k} e^{-\rho s} \sum_{j\in\mathcal{I}} H^{\boldsymbol{v}}_{ij,s} \mathrm{d}N_{j,s}. \end{aligned} \tag{EC.2.8}$$

Taking the limit for $t$ approaching $t_k$ from the left, we have

$$W^{\boldsymbol{v}}_{i,t_k} = W^{\boldsymbol{v}}_{i,t_k-} - I^{\boldsymbol{v}}_{i,t_k-} - \sum_{j\in\mathcal{I}} H^{\boldsymbol{v}}_{ij,t_k-} \mathrm{d}N_{j,t_k},$$

where $W^{\boldsymbol{v}}_{i,t_k-} = \lim_{t\uparrow t_k} W^{\boldsymbol{v}}_{i,t}$, and we define along the same vein that $H^{\boldsymbol{v}}_{ij,t_k-} := \lim_{t\uparrow t_k} H^{\boldsymbol{v}}_{ij,t}$. On the basis of claim (i), and $\boldsymbol{W}^{\boldsymbol{v}}_t$ being continuous in $(t_{k-1}, t_k)$, we have $\boldsymbol{W}_{t_k-} = \lim_{t\uparrow t_k} \boldsymbol{W}_t \in \mathscr{G}(T\phi_{\mathcal{A}})$ by the preservation of non-strict inequalities when taking limits. For $\boldsymbol{w} = \boldsymbol{W}^{\boldsymbol{v}}_{t_k-}$, since $\boldsymbol{W}_{t_k-} \in \mathscr{G}(T\phi_{\mathcal{A}})$, we can find $\boldsymbol{x}$ and $\boldsymbol{H}$ satisfying (EA$_\text{s}$), (IC$_\text{s}$), (IR$_\text{s}$), (PK$_{\boldsymbol{y}}$), (PK$_{\boldsymbol{Z}}$), (SG$_{\boldsymbol{w}}$), (SG$_{\boldsymbol{y}}$), (SG$_{\boldsymbol{Z}}$), and (UB$_\text{s}$).

Let $\boldsymbol{X}^{\boldsymbol{v}}_{t_k-}=\boldsymbol{x}$ and $\boldsymbol{H}^{\boldsymbol{v}}_{t_k-}=\boldsymbol{H}$. Since these controls are left-continuous adapted processes with right limits, it follows that $\boldsymbol{X}^{\boldsymbol{v}}_{t_k-}=\boldsymbol{X}^{\boldsymbol{v}}_{t_k}=\boldsymbol{x}$, $\boldsymbol{H}^{\boldsymbol{v}}_{t_k-}=\boldsymbol{H}^{\boldsymbol{v}}_{t_k}=\boldsymbol{H}$. In addition, set $\boldsymbol{I}^{\boldsymbol{v}}_{t_k}$ and $\boldsymbol{l}^{\boldsymbol{v}}_{t_k}$ according to equation (3.12) with $w^*_i=W^{\boldsymbol{v}}_{t_k-}$. Constraints (EA$_\text{s}$), (IC$_\text{s}$), (IR$_\text{s}$), (UB$_\text{s}$), and equation (3.12) imply that (EA), (IR), (IC), (LL), and (UB) are satisfied. Equation (EC.2.8) for $\boldsymbol{W}^{\boldsymbol{v}}_t$ implies that (PK) is satisfied. By constraints (PK$_{\boldsymbol{Z}}$) and (SG$_{\boldsymbol{Z}}$), for all $j\in\mathcal{I}$, we have

$$\sum_{i\in\mathcal{I}}\alpha_i(W^{\boldsymbol{v}}_{i,t_k-}-I^{\boldsymbol{v}}_{i,t_k-}-H^{\boldsymbol{v}}_{ij,t_k-})\geq\phi_{\mathcal{A}}(\boldsymbol{\alpha}),\quad\forall\boldsymbol{\alpha}\in\mathbb{R}^n_+\ \text{ with }\|\boldsymbol{\alpha}\|_1=1.$$

Since $\sum_{j\in\mathcal{I}}\mathrm{d}N_{j,t_k}=1$, then

$$\sum_{i\in\mathcal{I}}\alpha_i(w^{\boldsymbol{v}}_{i,t_k-}-I^{\boldsymbol{v}}_{i,t_k-}-\sum_{j\in\mathcal{I}}H^{\boldsymbol{v}}_{ij,t_k-}\mathrm{d}N_{j,t_k})\geq\phi_{\mathcal{A}}(\boldsymbol{\alpha}),\quad\forall\boldsymbol{\alpha}\in\mathbb{R}^n_+\ \text{ with }\|\boldsymbol{\alpha}\|_1=1.$$

Therefore, $\boldsymbol{W}^{\boldsymbol{v}}_{t_k}\in\mathcal{A}$ by the convexity of $\mathcal{A}$, which asserts claim (ii).

**(Part III: If $\mathcal{A}$ is a self-generating set, then $\mathscr{G}(\phi_{\mathcal{A}})$ is a self-generating set.)** First, for any $\boldsymbol{v}\in\mathscr{G}(\phi_{\mathcal{A}})$, $\boldsymbol{v}$ can be written as

$$\boldsymbol{v}=\lambda\boldsymbol{v}_1+(1-\lambda)\boldsymbol{v}_2,\quad\text{for some }\boldsymbol{v}_1,\boldsymbol{v}_2\in\ \mathbf{epi}(\mathcal{A})\cap[0,\bar{w}]^n.$$

This follows from the definition of $\mathscr{G}(\cdot)$. Then, since $\mathcal{A}$ is a self-generating set, from **Part I** we know that $\mathcal{A}\subseteq\mathscr{G}(T\phi_{\mathcal{A}})$. Similarly, by the definition of $\mathscr{G}(\cdot)$ and, we obtain that

$$\mathbf{epi}(\mathcal{A})\cap[0,\bar{w}]^n\subseteq\mathscr{G}(T\phi_{\mathcal{A}}).$$

This leads to $\boldsymbol{v}_1,\boldsymbol{v}_2\in\mathscr{G}(T\phi_{\mathcal{A}})$. Then, by the convexity of $\mathscr{G}(T\phi_{\mathcal{A}})$, $\boldsymbol{v}=\lambda\boldsymbol{v}_1+(1-\lambda)\boldsymbol{v}_2\in\mathscr{G}(T\phi_{\mathcal{A}})$. Since for all $\boldsymbol{v}\in\mathscr{G}(\phi_{\mathcal{A}})$ we have $\boldsymbol{v}\in\mathscr{G}(T\phi_{\mathcal{A}})$, $\mathscr{G}(\phi_{\mathcal{A}})\subseteq\mathscr{G}(T\phi_{\mathcal{A}})$. By **Part II** and the convexity of $\mathscr{G}(\phi_{\mathcal{A}})$, we obtain that $\mathscr{G}(\phi_{\mathcal{A}})$ is self-generating. □

### EC.2.5. Proof of Theorem 3.1

We need the following technical lemma to prove Theorem 3.1. First of all, let $\mathscr{M}(T\phi)$ be the set defined as follows:

$$\begin{aligned}\mathscr{M}(T\phi)=\{(\boldsymbol{H},\boldsymbol{x},\boldsymbol{y},\boldsymbol{Z})\mid\exists\boldsymbol{w}\in\mathscr{G}(T\phi),\ &\text{such that }(\boldsymbol{w},\boldsymbol{H},\boldsymbol{x},\boldsymbol{y},\boldsymbol{Z})\text{ is feasible to}\\ &\text{constraints (EA}_\text{s}),(\text{IC}_\text{s}),(\text{IR}_\text{s}),(\text{PK}_{\boldsymbol{y}}),(\text{PK}_{\boldsymbol{Z}}),(\text{SG}_{\boldsymbol{w}}),\\ &(\text{SG}_{\boldsymbol{Z}}),\ \text{and }(\text{UB}_\text{s}),\ \text{plus }(\text{SG}_{\boldsymbol{y}})\text{ if }\boldsymbol{w}\in\mathrm{bd}(\mathscr{G}(\phi))\}\end{aligned}$$

In other words, set $\mathscr{M}(T\phi)$ characterizes the collection of $(\boldsymbol{H},\boldsymbol{x},\boldsymbol{y},\boldsymbol{Z})$ for each of the $\boldsymbol{w}\in\mathscr{G}(T\phi)$ such that $(\boldsymbol{w},\boldsymbol{H},\boldsymbol{x},\boldsymbol{y},\boldsymbol{Z})$ is feasible to all constraints to problem $[T\phi](\boldsymbol{\alpha})$ except for constraint (SG$_{\boldsymbol{y}}$), which must also be satisfied when $\boldsymbol{w}\in\mathrm{bd}(\mathscr{G}(\phi))$.

Lemma EC.2.2. *Given any set $\mathcal{A}$ such that $\phi_{\mathcal{A}}(\boldsymbol{\alpha}) \geq 0$ for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$, we have the following properties:*

*(i) Convexity: $\mathscr{G}(T\phi_{\mathcal{A}})$ is convex and closed.*

*(ii) Compactness: both $\mathscr{G}(T\phi_{\mathcal{A}})$ and $\mathscr{M}(T\phi_{\mathcal{A}})$ are compact.*

*(iii) Monotonicity: given any set $\mathcal{B}$ such that $\phi_{\mathcal{B}}(\boldsymbol{\alpha}) \geq 0$ for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$, if $\boldsymbol{conv}(\mathcal{A}) \subseteq \boldsymbol{conv}(\mathcal{B})$, that is, $\phi_{\mathcal{A}}(\boldsymbol{\alpha}) \geq \phi_{\mathcal{B}}(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$, then $T\phi_{\mathcal{A}}(\boldsymbol{\alpha}) \geq T\phi_{\mathcal{B}}(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$. Consequently, $\mathscr{G}(T\phi_{\mathcal{A}}) \subseteq \mathscr{G}(T\phi_{\mathcal{B}})$ and $\mathscr{M}(T\phi_{\mathcal{A}}) \subseteq \mathscr{M}(T\phi_{\mathcal{B}})$.*

*Proof:* **(Proof of (i).)** Since $T\phi_{\mathcal{A}}$ maps $\mathbb{R}^n_+$ to $\mathbb{R}_+$ for all $\mathcal{A}$, $\mathscr{G}(T\phi_{\mathcal{A}})$ is the intersection of half spaces, and thus it is convex and closed. Moreover, denote $\mathbf{conv}(\mathcal{A})$ to represent the convex hull of set $\mathcal{A}$, we have $\mathbf{conv}(\mathscr{G}(T\phi_{\mathcal{A}})) = \mathscr{G}(T\phi_{\mathcal{A}})$. Note that this result does not require the set $\mathcal{A}$ to be convex and closed.

**(Proof of (ii).)** First, following from part (i), the set $\mathscr{G}(T\phi_{\mathcal{A}})$ is compact. Second, note that constraint ($\text{EA}_\text{s}$) implies that the allocation decisions have attainable lower and upper bounds. Therefore, the set of feasible allocations $\boldsymbol{x}$ is closed and bounded, and is a subset of the Euclidean space $\mathbb{R}^n$. Then, by the Heine-Borel theorem, the set of candidate allocation decisions forms a compact set. In a similar vein, the sets of candidate $\boldsymbol{H}$, $\boldsymbol{y}$, and $\boldsymbol{Z}$ all form respective compact sets. By the Tychonoff's theorem, the Cartesian product of these sets, $(\boldsymbol{H}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{Z})$, is also compact. At last, because $\boldsymbol{w} \in \mathscr{G}(T\phi_{\mathcal{A}})$ forms a compact set following from part (i), and because constraints ($\text{EA}_\text{s}$), ($\text{IC}_\text{s}$), ($\text{IR}_\text{s}$), ($\text{PK}_{\boldsymbol{y}}$), ($\text{PK}_{\boldsymbol{Z}}$), ($\text{SG}_{\boldsymbol{w}}$), ($\text{SG}_{\boldsymbol{y}}$), ($\text{SG}_{\boldsymbol{Z}}$) and ($\text{UB}_\text{s}$) are all equality constraints and non-strict inequality constraints that preserve compactness, the set $\mathscr{M}(T\phi_{\mathcal{A}})$ is compact for any set $\mathcal{A}$ such that $\phi_{\mathcal{A}} \geq 0$ for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$.

**(Proof of (iii).)** Since $\phi_{\mathcal{A}}(\boldsymbol{\alpha}) \geq \phi_{\mathcal{B}}(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$, the minimization problem $T\phi_{\mathcal{B}}(\boldsymbol{\alpha})$ has a larger feasible region than that of problem $T\phi_{\mathcal{A}}(\boldsymbol{\alpha})$, and thus a smaller optimal solution. In other words, $T\phi_{\mathcal{A}}(\boldsymbol{\alpha}) \geq T\phi_{\mathcal{B}}(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$. Then, by the definition of mapping $\mathscr{G}(\cdot)$, and the preservation of the ordering of sets under intersection, it is obtained that $\mathscr{G}(T\phi_{\mathcal{A}}) \subseteq \mathscr{G}(T\phi_{\mathcal{B}})$. At last, $\mathscr{M}(T\phi_{\mathcal{A}}) \subseteq \mathscr{M}(T\phi_{\mathcal{B}})$ follows from the definition of set $\mathscr{M}(T\phi)$, because $\mathscr{G}(T\phi_{\mathcal{A}}) \subseteq \mathscr{G}(T\phi_{\mathcal{B}})$. $\square$

Theorem 3.1. *Let $\mathcal{U}^0 = [0, \bar{w}]^n$, and define operator $T^k$ such that $T^k\phi = T(T^{k-1}\phi)$ for all $k > 1$. We have*

$$\lim_{k \to \infty} \mathscr{G}(T^k \phi_{\mathcal{U}^0}) = \mathcal{U}.$$

*Proof:* Let $\mathcal{U}^k = \mathbf{conv}(\mathscr{G}(T^k\phi_{\mathcal{U}^0}))$ for all $k \geq 1$. It then follows that $\mathcal{U}^k = \mathscr{G}(T^k\phi_{\mathcal{U}^0})$ by the convexity of the set $\mathscr{G}(T^k\phi_{\mathcal{U}^0})$ from part (i) of Lemma EC.2.2. It also follows that $T^k\phi_{\mathcal{U}^0}(\boldsymbol{\alpha})$ defines the support function representation of the convex set $\mathcal{U}^k$ for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$.

We first prove the existence of the limit $\mathcal{U}^\infty = \lim_{k\to\infty} \mathcal{U}^k$. Note that $\mathscr{G}(T\phi_{\mathcal{U}^0}) \subseteq \mathcal{U}^0$ by the definition of problem $[T\phi](\boldsymbol{\alpha})$ and $\mathscr{G}(T\phi_{\mathcal{A}})$, so $\mathcal{U}^1 = \mathscr{G}(T\phi_{\mathcal{U}^0}) \subseteq \mathcal{U}^0$, and then $T\phi_{\mathcal{U}^0}(\boldsymbol{\alpha}) \geq \phi_{\mathcal{U}^0}(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$. Thus, we have $\mathscr{G}(T^2\phi_{\mathcal{U}^0}) \subseteq \mathscr{G}(T\phi_{\mathcal{U}^0})$ by part (iii) of Lemma EC.2.2, which implies that $\mathcal{U}^2 \subseteq \mathcal{U}^1$. Along the same vein, we can prove that $\mathcal{U}^k \subseteq \mathcal{U}^{k-1}$ for all $k \geq 1$. At last, the monotone sequence of set $\mathcal{U}^k$ converges, that is, $\mathcal{U}^\infty = \lim_{k\to\infty} \mathcal{U}^k = \bigcap_{k=0}^\infty \mathcal{U}^k$ exists (the result follows from, for example, Proposition 1.4.1 in Resnick 2003, pages 8-9). Next, we first focus on the case where $\mathcal{U}$ is non-empty, and then return to the case where $\mathcal{U} = \emptyset$.

**(Part I: $\mathcal{U}$ is non-empty.)** The goal is to show that $\mathcal{U}^\infty = \mathscr{G}(T^\infty\phi_{\mathcal{U}^0}) = \mathcal{U}$. We first prove that the limit satisfies $\mathcal{U}^\infty \supseteq \mathcal{U}$. Then, we show that $\mathcal{U}^\infty \subseteq \mathcal{U}$. Merging these two results yields $\mathcal{U}^\infty = \mathcal{U}$.

**Step 1.** By the definition of $\mathcal{U}$ we know that $\mathcal{U} \subseteq \mathcal{U}^0$, and thus $\phi_{\mathcal{U}}(\boldsymbol{\alpha}) \geq \phi_{\mathcal{U}^0}(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$, then by Lemma EC.2.2, (iii), we have $\mathscr{G}(T\phi_{\mathcal{U}}) \subseteq \mathscr{G}(T\phi_{\mathcal{U}^0})$. Since $\mathcal{U}$ is a self-generating set by Proposition 3.2, we obtain by Lemma 3.1 that $\mathcal{U} \subseteq \mathscr{G}(T\phi_{\mathcal{U}})$. Thus, $\mathcal{U} \subseteq \mathscr{G}(T\phi_{\mathcal{U}}) \subseteq \mathscr{G}(T\phi_{\mathcal{U}^0}) = \mathcal{U}^1$. Next, $\mathcal{U} \subseteq \mathcal{U}^1$ implies that $\phi_{\mathcal{U}}(\boldsymbol{\alpha}) \geq \phi_{\mathcal{U}^1}(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$, then by part (iii) of Lemma EC.2.2 $\mathscr{G}(T\phi_{\mathcal{U}}) \subseteq \mathscr{G}(T\phi_{\mathcal{U}^1}) = \mathscr{G}(T^2\phi_{\mathcal{U}^0})$, and thus $\mathcal{U} \subseteq \mathscr{G}(T\phi_{\mathcal{U}}) \subseteq \mathscr{G}(T^2\phi_{\mathcal{U}^0}) = \mathcal{U}^2$. Along the same vein, we have $\mathcal{U} \subseteq \mathcal{U}^\infty$.

**Step 2.** We next prove that $\mathcal{U}^\infty \subseteq \mathcal{U}$. It suffices to show that $\mathcal{U}^\infty$ is self-generating, then by combining with Proposition 3.1, which states that every self-generating set is a subset of $\mathcal{U}$, the result follows. Note that $\mathcal{U}^k$ for all $k \geq 0$ is convex and compact by parts (i) and (ii) of Lemma EC.2.2. Therefore, $\mathcal{U}^\infty$ is convex and compact since the intersection of arbitrary convex and compact sets in $\mathbb{R}^n$ is still convex and compact. By Lemma 3.1, to prove $\mathcal{U}^\infty$ is self-generating, it is equivalent to prove $\mathcal{U}^\infty \subseteq \mathscr{G}(T\phi_{\mathcal{U}^\infty})$, that is, for any $\boldsymbol{w}' \in \mathcal{U}^\infty$, $\boldsymbol{w}' \in \mathscr{G}(T\phi_{\mathcal{U}^\infty})$. In particular, we need to show that for any fixed $\boldsymbol{w}' \in \mathcal{U}^\infty$, there exists $\boldsymbol{w} \leq \boldsymbol{w}'$ and $\boldsymbol{w} \in \mathrm{bd}(\mathcal{U}^\infty)$, together with $(\boldsymbol{H}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{Z})$ satisfying (EA$_\mathrm{s}$), (IC$_\mathrm{s}$), (IR$_\mathrm{s}$), (PK$_{\boldsymbol{y}}$), (PK$_{\boldsymbol{Z}}$), (SG$_{\boldsymbol{y}}$), and (UB$_\mathrm{s}$), for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$, $\boldsymbol{\alpha}^\top \boldsymbol{Z}_{\cdot j} \geq \phi_{\mathcal{U}^\infty}(\boldsymbol{\alpha})$ for all $j$, and $\boldsymbol{\alpha}^\top \boldsymbol{w} \geq \phi_{\mathcal{U}^\infty}(\boldsymbol{\alpha})$.

We first show that $\mathscr{M}(T\phi_{\mathcal{U}^k})$ is non-empty and compact for any $k \geq 0$. Since $\mathcal{U}$ is non-empty and self-generating by Proposition 3.2, $\mathscr{M}(T\phi_{\mathcal{U}})$ is non-empty. Furthermore, because $\mathcal{U} \subseteq \mathcal{U}^\infty \subseteq \mathcal{U}^k$ by Step 1, we have $\mathcal{U}^k$ is non-empty for any $k \geq 0$. Then, $\phi_{\mathcal{U}^k}(\boldsymbol{\alpha}) \leq \phi_{\mathcal{U}}(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$, and thus $\mathscr{M}(T\phi_{\mathcal{U}^k})$ is non-empty for any $k \geq 0$. Since $\mathcal{U}^\infty = \bigcap_{k=0}^\infty \mathcal{U}^k$, for any $\boldsymbol{w} \in \mathcal{U}^\infty$, $\boldsymbol{w} \in \mathcal{U}^k$ for all $k \geq 0$. Consequently, associated with $\boldsymbol{w}'$ there exists a sequence of $\boldsymbol{w}^k \leq \boldsymbol{w}'$ and $\boldsymbol{w}^k \in \mathrm{bd}(\mathcal{U}^k)$, together with a sequence of $(\boldsymbol{H}^k, \boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{Z}^k) \in \mathscr{M}(T\phi_{\mathcal{U}^k})$ that complies with (EA$_\mathrm{s}$), (IC$_\mathrm{s}$), (IR$_\mathrm{s}$), (PK$_{\boldsymbol{y}}$), (PK$_{\boldsymbol{Z}}$), (SG$_{\boldsymbol{y}}$), and (UB$_\mathrm{s}$), for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$, $\boldsymbol{\alpha}^\top \boldsymbol{Z}^k_{\cdot j} \geq \phi_{\mathcal{U}^k}(\boldsymbol{\alpha})$ for all $j$, and $\boldsymbol{\alpha}^\top \boldsymbol{w} \geq \phi_{\mathcal{U}^k}(\boldsymbol{\alpha})$. In other words, $\mathscr{M}(T\phi_{\mathcal{U}^k})$ is non-empty. In addition, for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$, $\boldsymbol{\alpha}^\top \boldsymbol{Z}^k_{\cdot j} \geq \phi_{\mathcal{U}^0}(\boldsymbol{\alpha})$ for all $j$, and $\boldsymbol{\alpha}^\top \boldsymbol{w} \geq \phi_{\mathcal{U}^0}(\boldsymbol{\alpha})$ because $\mathcal{U}^k \subseteq \mathcal{U}^0$. We obtain by part (ii) of Lemma EC.2.2 that $\mathscr{M}(T\phi_{\mathcal{U}^0})$ is compact, and the sequence of mechanism $(\boldsymbol{H}^k, \boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{Z}^k)$ lies in $\mathscr{M}(T\phi_{\mathcal{U}^0})$. Then, following the Bolzano–Weierstrass theorem, and by passing to a subsequence if necessary, we have that the sequence of $\boldsymbol{w}^k$ converges to $\boldsymbol{w}$ while the sequence of mechanism $(\boldsymbol{H}^k, \boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{Z}^k)$ converges to some $(\boldsymbol{H}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{Z}) \in \mathscr{M}(T\phi_{\mathcal{U}^0})$.

Next, we need to show that the above $(\boldsymbol{H}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{Z})$ satisfies ($\text{EA}_\text{s}$), ($\text{IC}_\text{s}$), ($\text{IR}_\text{s}$), ($\text{PK}_{\boldsymbol{y}}$), ($\text{PK}_{\boldsymbol{Z}}$), ($\text{SG}_{\boldsymbol{y}}$), and ($\text{UB}_\text{s}$), for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$, $\boldsymbol{\alpha}^\top \boldsymbol{Z}_{\cdot j} \geq \phi_{\mathcal{U}^\infty}(\boldsymbol{\alpha})$ for all $j$, and $\boldsymbol{\alpha}^\top \boldsymbol{w} \geq \phi_{\mathcal{U}^\infty}(\boldsymbol{\alpha})$. Since $(\boldsymbol{H}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{Z}) \in \mathscr{M}(T\phi_{\mathcal{U}^0})$, the decision variables satisfy ($\text{EA}_\text{s}$), ($\text{IC}_\text{s}$), ($\text{IR}_\text{s}$), ($\text{PK}_{\boldsymbol{y}}$), ($\text{PK}_{\boldsymbol{Z}}$), ($\text{SG}_{\boldsymbol{y}}$), and ($\text{UB}_\text{s}$). For any fixed $k$ and $m > k$, we have $\boldsymbol{\alpha}^\top \boldsymbol{Z}^m_{\cdot j} \geq \phi_{\mathcal{U}^k}(\boldsymbol{\alpha})$ for all $j$ and for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$, because the sequence of sets $\{\mathcal{U}^k\}$ is non-increasing. Since $\mathcal{U}^k$ is compact, we have $\boldsymbol{\alpha}^\top \boldsymbol{Z}_{\cdot j} \geq \phi_{\mathcal{U}^k}(\boldsymbol{\alpha})$ for all $j$ and for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$. Thus, $\boldsymbol{\alpha}^\top \boldsymbol{Z}_{\cdot j} \geq \phi_{\mathcal{U}^\infty}(\boldsymbol{\alpha})$ for all $j$ and $\boldsymbol{\alpha}$, because $\bigcap_{k=0}^\infty \mathcal{U}^k = \mathcal{U}^\infty$. Moreover, $\boldsymbol{w} \in \mathcal{U}^\infty$ implies $\boldsymbol{\alpha}^\top \boldsymbol{w} \geq \phi_{\mathcal{U}^\infty}(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$.

Combining the results in Step 1 and 2, we have $\mathcal{U}^\infty = \mathcal{U}$, which implies $\mathscr{G}(T^\infty \phi_{\mathcal{U}^0}) = \mathcal{U}$.

**(Part II: $\mathcal{U} = \emptyset$.)** When $\mathcal{U}$ is empty, suppose to the contrary that $\mathscr{G}(T^\infty \phi_{\mathcal{U}^0})$ is non-empty, that is, $\mathcal{U}^\infty$ is non-empty. Following the same argument, we have established that $\mathcal{U}^\infty = \bigcap_{k=0}^\infty \mathcal{U}^k$, which indicates that $\mathcal{U}^k$ is non-empty for any $k \geq 0$. Then, bypassing the same arguments as those in Step 2 of Part I, it follows that $\mathcal{U}^\infty$ is a self-generating set. Therefore, $\mathcal{U}^\infty \subseteq \mathcal{U}$ by Proposition 3.1, which contradicts $\mathcal{U}$ being an empty set. As a result, if $\mathcal{U} = \emptyset$, $\mathscr{G}(T^\infty \phi_{\mathcal{U}^0}) = \emptyset$, which again suggests that $\mathscr{G}(T^\infty \phi_{\mathcal{U}^0}) = \mathcal{U}$. This completes the proof. $\square$

### EC.2.6. Proof of Proposition 3.3

PROPOSITION 3.3. *There exists a threshold $\bar{\omega}$ that depends on model parameters $n$, $\rho$, $b$, $\lambda$ and $\bar{\lambda}$, such that the achievable set $\mathcal{U}$ is non-empty if and only if $\bar{w} \geq \bar{\omega}$.*

*Proof:* Theorem 3.1 establishes the equivalence between $\mathcal{U}$ and $\mathscr{G}(T^\infty \phi_{\mathcal{U}^0})$. In other words, the achievable set $\mathcal{U}$ is non-empty if problem $[T\phi](\boldsymbol{\alpha})$ remains feasible when we apply the operator $T$ infinite times. Problem $[T\phi](\boldsymbol{\alpha})$ is a linear semi-infinite programming problem with finite variables but an infinite number of constraints. To prove that problem $[T\phi](\boldsymbol{\alpha})$ is feasible, we need to show the equivalent statement that $(\mathbf{0}^n, 1) \notin \text{cl}(\mathcal{N})$, where $\mathbf{0}^n$ is an $n$-dimensional vector of all zeros, $\mathcal{N}$ is the second-moment cone associated with the set of constraints ($\text{EA}_\text{s}$), ($\text{IC}_\text{s}$), ($\text{IR}_\text{s}$), ($\text{PK}_{\boldsymbol{y}}$), ($\text{PK}_{\boldsymbol{Z}}$), ($\text{SG}_{\boldsymbol{w}}$), ($\text{SG}_{\boldsymbol{y}}$), ($\text{SG}_{\boldsymbol{Z}}$), ($\text{UB}_\text{s}$), and $\text{cl}(\mathcal{N})$ is the closure of set $\mathcal{N}$ (Fan 1968). A second-moment cone for a system of linear inequalities

$$\mathcal{S} = \{\boldsymbol{a}_i^\top \boldsymbol{x} \geq b_i, \ i \in \mathcal{M}\}$$

is defined as

$$\text{cone}\left(\left\{\begin{pmatrix}\boldsymbol{a}_i \\ b_i\end{pmatrix}, \ i \in \mathcal{M}\right\}\right).$$

Observing that the set of vectors $\{(\boldsymbol{\alpha}, \phi(\boldsymbol{\alpha}))|\boldsymbol{\alpha} \in \mathbb{R}_+^n,\ \|\boldsymbol{\alpha}\|_1 = 1\}$ forms a closed cone, the aforementioned equivalent statement can be proved by showing another equivalent statement, that every finite subsystem of problem $[T\phi](\boldsymbol{\alpha})$ is feasible (Goberna et al. 1995). In summary, to show that the achievable set $\mathcal{U}$ is non-empty, it suffices to show the feasibility of the following subproblem of problem $[T\phi](\boldsymbol{\alpha})$ with discretized $\hat{\boldsymbol{\alpha}}$:

$$
\begin{aligned}
\textbf{(Di-TS)} \qquad \min_{\boldsymbol{w},\boldsymbol{x},\boldsymbol{H}} \quad & 0 \\
\text{s.t.} \quad & \sum_{i\in\mathcal{I}} \hat{\alpha}_{ki}(w_i - H_{ij}) \ge \phi_{\mathcal{U}}(\hat{\boldsymbol{\alpha}}_k), \quad \forall j\in\mathcal{I},\ \forall k\in\mathcal{K}, && (m_{jk}) \\
& \sum_{i\in\mathcal{I}} \hat{\alpha}_{ki} w_i \ge \phi_{\mathcal{U}}(\hat{\boldsymbol{\alpha}}_k), \quad \forall k\in\mathcal{K}, && (o_k) \\
& \sum_{i} \alpha_i \left(\rho w_i - R^a x_i + \lambda \sum_{j\in\mathcal{I}} H_{ij}\right) \ge 0, && (\eta) \\
& H_{ii} - \beta x_i \ge 0, \quad \forall i\in\mathcal{I}, && (d_i) \\
& \sum_{i\in\mathcal{I}} x_i = 1, && (q) \\
& w_i \le \bar{w}, \quad \forall i\in\mathcal{I}, && (c_i) \\
& \boldsymbol{w}\ge 0, \quad \boldsymbol{x}\ge 0, \quad \boldsymbol{H} \text{ free},
\end{aligned}
$$

where $\{\hat{\boldsymbol{\alpha}}_k|k\in\mathcal{K}\}$ is a subset of $\{\tilde{\boldsymbol{\alpha}}|\tilde{\boldsymbol{\alpha}}\in\mathbb{R}_+^n,\ \|\tilde{\boldsymbol{\alpha}}\|_1 = 1\}$ and $|\mathcal{K}| < \infty$. Problem (**Di-TS**) is a linear programming problem with a constant objective function, and thus by the Farkas' lemma, the primal problem is feasible if and only if the dual problem is feasible and bounded. The dual problem of the problem (**Di-TS**) is

$$
\begin{aligned}
(\textbf{Du-TS}) \qquad \max \quad & \sum_{k\in\mathcal{K}} \left(\sum_{j\in\mathcal{I}} m_{jk} + o_k\right) \phi_{\mathcal{U}}(\hat{\boldsymbol{\alpha}}_k) + q - \bar{w}\sum_{i\in\mathcal{I}} c_i \\
\text{s.t.} \quad & \sum_{k\in\mathcal{K}} \hat{\alpha}_{ki} \left(\sum_{j\in\mathcal{I}} m_{jk} + o_k\right) + \rho\eta\alpha_i - c_i \le 0, \qquad \forall i\in\mathcal{I} && (w_i) \\
& -\sum_{k\in\mathcal{K}} \hat{\alpha}_{ki} m_{jk} + \lambda\eta\alpha_i + \mathbb{1}_{\{i=j\}} d_i = 0, \qquad \forall i,j\in\mathcal{I} && (H_{ij}) \\
& -R^a\eta\alpha_i - \beta d_i + q \le 0, \qquad \forall i\in\mathcal{I} && (x_i) \\
& \boldsymbol{m},\ \boldsymbol{o},\ \eta,\ \boldsymbol{d},\ \boldsymbol{c} \ge 0, \quad q \text{ free}.
\end{aligned}
$$

Note that $\boldsymbol{m} = \mathbf{0}, \boldsymbol{o} = \mathbf{0}, \eta = 0, \boldsymbol{d} = \mathbf{0}, \boldsymbol{c} = \mathbf{0}, q = 0$ is a feasible solution, we can readily obtain the feasibility of problem (**Du-TS**). In addition, summing over $i, j$ for the set of constraints ($H_{ij}$), we have

$$
\sum_{i\in\mathcal{I}} \eta\alpha_i = \eta = \Big(\sum_{j\in\mathcal{I},k\in\mathcal{K}} m_{jk} - \sum_{i\in\mathcal{I}} d_i\Big)/(n\lambda). \tag{EC.2.9}
$$

Then, summing the set of constraints ($x_i$) over $i$ yields

$$nq \leq R^a \sum_{i\in\mathcal{I}} \eta\alpha_i + \beta \sum_{i\in\mathcal{I}} d_i = \frac{R^a}{n\lambda} \sum_{j\in\mathcal{I},k\in\mathcal{K}} m_{jk} - (\frac{R^a}{n\lambda} - \beta) \sum_{i\in\mathcal{I}} d_i. \tag{EC.2.10}$$

By summing the set of constraints ($w_i$) over $i$ and substituting $\eta$ with equation (EC.2.9), we have

$$\sum_{k\in\mathcal{K}} \left( \sum_{j\in\mathcal{I}} m_{jk} + o_k \right) + \rho\eta - \sum_{i\in\mathcal{I}} c_i = (1 + \frac{\rho}{n\lambda}) \sum_{j\in\mathcal{I},k\in\mathcal{K}} m_{jk} + \sum_{k\in\mathcal{K}} o_k - \frac{\rho}{n\lambda} \sum_{i\in\mathcal{I}} d_i - \sum_{i\in\mathcal{I}} c_i \leq 0,$$

which implies

$$\sum_{i\in\mathcal{I}} c_i \geq -\frac{\rho}{n\lambda} \sum_{i\in\mathcal{I}} d_i + \frac{n\lambda + \rho}{n\lambda} \sum_{j\in\mathcal{I},k\in\mathcal{K}} m_{jk} + \sum_{k\in\mathcal{K}} o_k. \tag{EC.2.11}$$

Next, we prove that problem (**Du-TS**) is bounded. Suppose the dual problem is unbounded. Then, there exists an extreme ray, along which the dual objective is positive, that is,

$$\begin{aligned}
nq &> n\bar{w} \sum_{i\in\mathcal{I}} c_i - n \sum_{k\in\mathcal{K}} \left( \sum_{j\in\mathcal{I}} m_{jk} + o_k \right) \phi_{\mathcal{U}}(\hat{\alpha}_k) \\
&\geq \frac{n\lambda\bar{w} + \rho\bar{w}}{\lambda} \sum_{j\in\mathcal{I},k\in\mathcal{K}} m_{jk} - \frac{\rho\bar{w}}{\lambda} \sum_{i\in\mathcal{I}} d_i - n\underline{w} \sum_{j\in\mathcal{I},k\in\mathcal{K}} m_{jk} + n(\bar{w} - \underline{w}) \sum_{k\in\mathcal{K}} o_k \\
&= \left( \frac{\rho\bar{w}}{\lambda} + n(\bar{w} - \underline{w}) \right) \sum_{j\in\mathcal{I},k\in\mathcal{K}} m_{jk} + n(\bar{w} - \underline{w}) \sum_{k\in\mathcal{K}} o_k - \frac{\rho\bar{w}}{\lambda} \sum_{i\in\mathcal{I}} d_i,
\end{aligned} \tag{EC.2.12}$$

where $\underline{w} := \max_{\boldsymbol{\alpha}} \phi_{\mathcal{U}}(\boldsymbol{\alpha}) \leq \bar{w}$. The second inequality follows from inequalities (EC.2.11), and the last equality from re-organizing terms. It can be shown that for $\boldsymbol{\alpha} \in \mathbb{R}^n_+$, and $\|\boldsymbol{\alpha}\|_1 = 1$ that maximizes $\phi_{\mathcal{U}}(\boldsymbol{\alpha})$ satisfies $\alpha_i = 1/n$ for all $i \in \mathcal{I}$ (where $\mathcal{I}$ is the set of agents and $|\mathcal{I}| = n$), and

$$\underline{w} = \{w_i | \boldsymbol{w} \in \mathrm{bd}(\mathcal{U}), \ \text{ and } w_i = w_j, \ \forall i, j \in \mathcal{I}\}.$$

By comparing inequalities (EC.2.10) with (EC.2.12), we find that the dual problem (**Du-TS**) is bounded so long as we can show that these two inequalities cannot hold simultaneously. In other words, a sufficient condition for problem (**Di-TS**) to be feasible is specified by the following inequality

$$\left( \frac{\rho\bar{w}}{\lambda} - \frac{R^a}{n\lambda} + n(\bar{w} - \underline{w}) \right) \sum_{j\in\mathcal{I},k\in\mathcal{K}} m_{jk} + n(\bar{w} - \underline{w}) \sum_{k\in\mathcal{K}} o_k \geq (\frac{\rho\bar{w}}{\lambda} - \frac{R^a}{n\lambda} + \beta) \sum_{i\in\mathcal{I}} d_i. \tag{EC.2.13}$$

Next, we derive sufficient conditions for inequality (EC.2.13) to hold. Note that if $\bar{w} \geq \frac{R^a}{n\rho}$, both the left-hand side and right-hand side of the equation (EC.2.13) are non-negative. By equation (EC.2.9), the dual variables $m_{jk}$ and $d_i$ satisfy $\sum_{j\in\mathcal{I},k\in\mathcal{K}} m_{jk} \geq \sum_{i\in\mathcal{I}} d_i \geq 0$. Therefore, a sufficient condition for inequality (EC.2.13) to hold is

$$\frac{\rho\bar{w}}{\lambda} - \frac{R^a}{n\lambda} + n(\bar{w} - \underline{w}) \geq \frac{\rho\bar{w}}{\lambda} - \frac{R^a}{n\lambda} + \beta,$$

that is, $\bar{w} > \underline{w} + \frac{\beta}{n}$. In other words, if $\bar{w}$ satisfies

$$\bar{w} > \max\{\frac{R^a}{n\rho},\ \underline{w} + \frac{\beta}{n}\},$$

problem (**Di-TS**) is feasible, and thus the original problem $T\phi[\boldsymbol{\alpha}]$ is feasible. Note that a sufficient upper bound on $\underline{w}$ is $\frac{\bar{w}+\frac{\lambda}{\rho}\beta}{n}$, and thus a sufficient condition for inequality (EC.2.13) to hold is

$$\bar{w} > \bar{\omega} = \max\{\frac{R^a}{n\rho},\ \frac{\lambda+\rho}{\rho(n-1)}\beta\}.$$

An alternative construction of $\bar{w}$ is as follows. Suppose we relax the constraints $w_i \leq \bar{w}$ in the primal problem as well as the feasibility problem (**Di-TS**), then $c_i = 0$ must hold for the dual of the feasibility problem, (**Du-TS**). In addition, constraints ($w_i$) and ($H_{ij}$) imply that $m_{jk} = 0$ for all $j$ and $k$, $o_k = 0$ for all $k$, $\eta = 0$, $d_i = 0$ for all $i$. Consequently, the feasibility of problem (**Du-TS**) requires that $q \leq 0$ due to constraints ($x_i$). However, for the dual problem to be unbounded, we need $q > 0$ from inequality (EC.2.12), and thus the dual problem is feasible and bounded. By Farkas lemma, it follows that the relaxed problem (**Di-TS**) parameterized by any $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ with $\|\boldsymbol{\alpha}\|_1 = 1$ is always feasible and bounded. Denote $\tilde{\boldsymbol{w}}^*(\boldsymbol{\alpha})$ to represent an optimal vertex solution of the relaxed primal problem, and set $\bar{w} = \sup_{\boldsymbol{\alpha}\in\mathbb{R}^n_+, \|\boldsymbol{\alpha}\|_1=1} \tilde{\boldsymbol{w}}^*_i(\boldsymbol{\alpha})$, adding back the infinite-backlogging preventing constraints $w_i \leq \bar{w}$ for all $i$ will not affect the feasibility of the primal problem as well as the original problem (**Di-TS**). $\square$

### EC.2.7. Proof of Proposition 3.4

PROPOSITION 3.4. *We have $\mathcal{U}(b_1) \supseteq \mathcal{U}(b_2)$ for $b_1 \leq b_2$ while keeping other model parameters the same. Similarly, we have $\mathcal{U}(\rho_1) \subseteq \mathcal{U}(\rho_2)$ for $\rho_1 \leq \rho_2$; $\mathcal{U}(\Delta\lambda_1) \subseteq \mathcal{U}(\Delta\lambda_2)$ for $\Delta\lambda_1 \leq \Delta\lambda_2$; and $\mathcal{U}(\bar{w}_1) \subseteq \mathcal{U}(\bar{w}_2)$ for $\bar{w}_1 \leq \bar{w}_2$.*

*Proof:* Adjusting the model parameters $\rho$, $b$, $\Delta\lambda$, and $\bar{w}$ translates to modifying the right-hand side vector of the linear program $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha})$ for some $\boldsymbol{\alpha}$. Therefore, the results follow from the sensitivity properties of linear programming problems. $\square$

### EC.2.8. Proof of Proposition 3.5

PROPOSITION 3.5. *Introduce constraints $H_{ij} = 0$ for all $i \neq j$ to the linear programs $[T\phi](\boldsymbol{\alpha})$, and obtain a new linear program $[\tilde{T}\phi](\boldsymbol{\alpha})$. We have $\lim_{k\to\infty} \mathscr{G}(\tilde{T}^k \phi_{\mathcal{U}_0}) = \emptyset$, which implies that there does not exist an EIC contract with $H_{ij,t} = 0$ for all $t \geq 0$ and $i \neq j$.*

*Proof:* We prove this proposition by contradiction. Suppose that there exists an EIC contract such that $\sum_{i\neq j}|H_{ij,t}|=0$ for all $t\geq 0$, that is, the corresponding achievable set of promised utilities, $\mathcal{U}$, is non-empty. By a slightly different version of Theorem 3.1, we have $\mathcal{U}=\lim_{k\to\infty}\mathscr{G}(\tilde{T}^k\phi_{\mathcal{U}^0})$. The proof of this result follows the same line of logic as that of Theorem 3.1, hence is omitted for brevity. Then, it is readily obtained that $\mathscr{G}(\tilde{T}\phi_{\mathcal{U}})=\lim_{k\to\infty}\mathscr{G}(\tilde{T}^{k+1}\phi_{\mathcal{U}^0})=\mathcal{U}$.

Next, for a $\boldsymbol{w}\in \mathrm{bd}(\mathcal{U})$ satisfying $\hat{\boldsymbol{\alpha}}^\top\boldsymbol{w}=\phi(\hat{\boldsymbol{\alpha}})$, where $\hat{\alpha}_i>0$ for any $i$, by constraint ($\mathrm{EA_s}$) we know that there must exist $j$ such that $x_j>0$. Then, by constraint ($\mathrm{IC_s}$) we have $H_{jj}>0$. Therefore,

$$\hat{\boldsymbol{\alpha}}^\top\boldsymbol{Z}_{\cdot j}=\sum_i\hat{\alpha}_i(w_i-H_{ij})=\hat{\boldsymbol{\alpha}}^\top\boldsymbol{w}-\hat{\alpha}_jH_{jj}<\hat{\boldsymbol{\alpha}}^\top\boldsymbol{w}=\phi(\hat{\boldsymbol{\alpha}}), \tag{EC.2.14}$$

where the second equality follows from $H_{ij}=0$ for all $i\neq j$ and the inequality from $\hat{\alpha}_j>0$ and $H_{jj}>0$. Equation (EC.2.14) violates constraints ($\mathrm{SG}_{\boldsymbol{Z}}$), which implies that the feasible region must shrink after applying $\tilde{T}$ operator on the set $\mathcal{U}$, that is, $\mathscr{G}(\tilde{T}\phi_{\mathcal{U}})\subset\mathcal{U}$. However, this result contradicts with $\mathscr{G}(\tilde{T}\phi_{\mathcal{U}})=\mathcal{U}$. Therefore, the set $\mathcal{U}$ must be empty. Equivalently, $\lim_{k\to\infty}\mathscr{G}(\tilde{T}^k\phi_{\mathcal{U}_0})=\emptyset$, and EIC contracts such that $\sum_{i\neq j}|H_{ij,t}|=0$ for all $t\geq 0$ do not exist. $\square$

### EC.2.9. Proof of Proposition 3.6

PROPOSITION 3.6. *At any optimal solution to the linear program $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha})$, constraints ($\mathrm{IC_s}$) and ($\mathrm{SG}_{\boldsymbol{y}}$) hold as equalities, and for $\boldsymbol{\alpha}$ constraint ($\mathrm{SG}_{\boldsymbol{w}}$) holds as equality. Similarly, for each $j\in\mathcal{I}$, there exists an $\hat{\boldsymbol{\alpha}}$ such that the corresponding ($\mathrm{SG}_{\boldsymbol{Z}}$) constraint holds as equality.*

*Proof:* We prove that if $\boldsymbol{w}^*$ is an optimal solution to the linear program $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha})$, then not only $\boldsymbol{\alpha}^\top\boldsymbol{w}^*=\phi_{\mathcal{U}}(\boldsymbol{\alpha})$, but also the optimal solution $\boldsymbol{w}^*$, $\boldsymbol{x}^*$, $\boldsymbol{y}^*$, $\boldsymbol{H}^*$, and $\boldsymbol{Z}^*$ for problem $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha})$ satisfies: (i) for all $j$ there exists respective $\hat{\boldsymbol{\alpha}}\in\mathbb{R}^n_+$ and $\|\hat{\boldsymbol{\alpha}}\|_1=1$ such that $\hat{\boldsymbol{\alpha}}^\top\boldsymbol{Z}^*_{\cdot j}=\phi_{\mathcal{U}}(\hat{\boldsymbol{\alpha}})$, (ii) constraint ($\mathrm{IC_s}$) holds at equality, and (iii) $\boldsymbol{\alpha}^\top\boldsymbol{y}^*=0$. Whereas the first half of this statement (namely, $\boldsymbol{\alpha}^\top\boldsymbol{w}^*=\phi_{\mathcal{U}}(\boldsymbol{\alpha})$) directly follows from Lemma 3.1 and the fact that $\mathcal{U}$ is both convex (see Corollary 3.1) and self-generating (see Proposition 3.2), we prove the second half of this statement by contradiction.

We first show that, for all $j$, there exist respective $\hat{\boldsymbol{\alpha}}$'s, where $\hat{\boldsymbol{\alpha}}\in\mathbb{R}^n_+$ and $\|\hat{\boldsymbol{\alpha}}\|_1=1$, such that $\hat{\boldsymbol{\alpha}}^\top\boldsymbol{Z}^*_{\cdot j}=\phi_{\mathcal{U}}(\hat{\boldsymbol{\alpha}})$. Suppose there exists $\tilde{j}$ such that $\hat{\boldsymbol{\alpha}}^\top\boldsymbol{Z}^*_{\cdot\tilde{j}}=\hat{\boldsymbol{\alpha}}^\top(\boldsymbol{w}^*-\boldsymbol{H}^*_{\cdot\tilde{j}})>\phi_{\mathcal{U}}(\hat{\boldsymbol{\alpha}})$ for all $\hat{\boldsymbol{\alpha}}\in\mathbb{R}^n_+$ and $\|\hat{\boldsymbol{\alpha}}\|_1=1$. Then, there must exist a vector $\tilde{\boldsymbol{v}}$ satisfying $\tilde{\boldsymbol{\alpha}}^\top\tilde{\boldsymbol{v}}=\phi_{\mathcal{U}}(\tilde{\boldsymbol{\alpha}})$ for some $\tilde{\boldsymbol{\alpha}}\in\mathbb{R}^n_+$ and $\|\tilde{\boldsymbol{\alpha}}\|_1=1$, such that $\boldsymbol{Z}^*_{\cdot\tilde{j}}-\tilde{\boldsymbol{v}}\geq\boldsymbol{0}$, where the inequality holds component-wisely. We define vector $\boldsymbol{\epsilon}$ as follows

$$\boldsymbol{\epsilon}=\frac{\lambda(\boldsymbol{Z}^*_{\cdot\tilde{j}}-\tilde{\boldsymbol{v}})}{\lambda+\rho}\ ,$$

and construct $\tilde{\boldsymbol{w}}$ and $\tilde{\boldsymbol{H}}$, where

$$\tilde{w}_i = \begin{cases} w_i^* - \epsilon_i, & i \neq \tilde{j} \\ w_i^* - \dfrac{\lambda+\rho}{\lambda}\epsilon_i, & i = \tilde{j} \end{cases},$$

and

$$\tilde{H}_{ij} = \begin{cases} H_{ij}^* + \dfrac{\rho}{\lambda}\epsilon_i, & i \neq \tilde{j}, j = \tilde{j} \\ H_{ij}^* + \dfrac{\rho(\lambda+\rho)}{(n-1)\lambda^2}\epsilon_i, & i = \tilde{j}, j \neq \tilde{j} \\ H_{ij}^*, & \text{otherwise} \end{cases}.$$

By this construction, $\tilde{\boldsymbol{w}} < \boldsymbol{w}^*$, $\tilde{\boldsymbol{w}} - \tilde{\boldsymbol{H}}_{\cdot\tilde{j}} = \tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{y}} = \rho\tilde{\boldsymbol{w}} + \lambda\sum_j \tilde{\boldsymbol{H}}_{\cdot j} = \boldsymbol{y}^*$. Therefore, $\tilde{\boldsymbol{w}}$, $\boldsymbol{x}^*$, $\tilde{\boldsymbol{y}}$, $\tilde{\boldsymbol{H}}$, and $\tilde{\boldsymbol{Z}}$ consists of a feasible solution to problem $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha})$ except for constraint ($\mathrm{SG}_{\boldsymbol{w}}$). In other words, consider a set $\mathcal{U}' \supset \mathcal{U}$, with $\tilde{\boldsymbol{w}} \in \mathcal{U}'$, the constructed solution $\tilde{\boldsymbol{w}}$, $\boldsymbol{x}^*$, $\tilde{\boldsymbol{y}}$, $\tilde{\boldsymbol{H}}^*$, and $\tilde{\boldsymbol{Z}}$ is feasible to the new linear program $[T\phi_{\mathcal{U}'}](\boldsymbol{\alpha})$. This indicates that $\tilde{\boldsymbol{w}}$ is achievable under EIC contract. However, $\tilde{\boldsymbol{w}} < \boldsymbol{w}$, and $\boldsymbol{\alpha}^\top\boldsymbol{w} = \phi_{\mathcal{U}}(\boldsymbol{\alpha})$ leads to $\tilde{\boldsymbol{w}} \notin \mathcal{U}$, contradicting with $\mathcal{U}$ being the largest self-generating set. Therefore, constraints ($\mathrm{SG}_{\boldsymbol{Z}}$) must hold as equalities for each $j \in \mathcal{I}$ for some respective $\hat{\boldsymbol{\alpha}}$'s. Constraint ($\mathrm{IC_s}$) also holds as equality in the same vein.

Next, we show that $\boldsymbol{\alpha}^\top\boldsymbol{y}^* \geq 0$ must hold at equality. Suppose to the contrary that $\boldsymbol{\alpha}^\top\boldsymbol{y}^* > 0$. Let $\epsilon > 0$, pick $i \in \mathcal{I}$, and construct

$$\tilde{w}_j = \begin{cases} w_j^* - \epsilon, & j = i \\ w_i^*, & j \neq i \end{cases}.$$

Then, for each $j \in \mathcal{I}$, pick $k_j \neq j$, construct $\tilde{\boldsymbol{H}}_{\cdot j}$ as follows

$$\tilde{H}_{mj} = \begin{cases} H_{mj}^*, & m \neq k_j \\ H_{mj}^* - \dfrac{\hat{\alpha}_{ij}}{\hat{\alpha}_{mj}}\epsilon, & m = k_j \end{cases},$$

where $\hat{\boldsymbol{\alpha}}$ is selected such that $\hat{\boldsymbol{\alpha}}^\top\boldsymbol{Z}_{\cdot j}^* = \phi_{\mathcal{U}}(\hat{\boldsymbol{\alpha}})$. Note that $\tilde{H}_{jj} = H_{jj}^*$ for all $j$, therefore, constraint ($\mathrm{IC_s}$) remains binding. In addition, by this construction, we obtain $\tilde{\boldsymbol{Z}}$ from constraints ($\mathrm{PK}_{\boldsymbol{Z}}$), which satisfies

$$\begin{aligned} \hat{\boldsymbol{\alpha}}^\top\tilde{\boldsymbol{Z}}_{\cdot j} &= \sum_m \hat{\alpha}_{mj}(\tilde{w}_m - \tilde{H}_{mj}) \\ &= \sum_m \hat{\alpha}_{mj}(w_m^* - H_{mj}^*) - \hat{\alpha}_{ij}\epsilon + \hat{\alpha}_{k_j j}\cdot\frac{\hat{\alpha}_{ij}}{\hat{\alpha}_{k_j j}}\epsilon = \hat{\boldsymbol{\alpha}}^\top\boldsymbol{Z}_{\cdot j}^*, \end{aligned}$$

while for all other $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ and $\|\boldsymbol{\alpha}\|_1 = 1$, constraints ($\mathrm{SG}_{\boldsymbol{Z}}$) are satisfied for $\epsilon$ sufficiently small. Similarly, we obtain the following $\tilde{\boldsymbol{y}}$ from constraints ($\mathrm{PK}_{\boldsymbol{y}}$)

$$\tilde{y}_m = \begin{cases} \rho w^*_m + \lambda \sum_j H^*_{mj}, & m \neq i, m \neq k_j \\ \rho w^*_m - \epsilon + \lambda \sum_j H^*_{mj}, & m = i, m \neq k_j \\ \rho w^*_m + \lambda \sum_j H^*_{mj} - \frac{\hat{\alpha}_{ij}}{\hat{\alpha}_{k_j j}} \lambda \epsilon, & m \neq i, m = k_j \\ \rho w^*_m - \epsilon + \lambda \sum_j H^*_{mj} - \frac{\hat{\alpha}_{ij}}{\hat{\alpha}_{k_j j}} \lambda \epsilon, & m = i = k_j \end{cases} .$$

Since $\boldsymbol{\alpha}^\top \boldsymbol{y}^* > 0$, there exists $\epsilon > 0$ such that $\boldsymbol{\alpha}^\top \tilde{\boldsymbol{y}} \geq 0$. To sum up, we have constructed a solution $\tilde{\boldsymbol{w}}$, $\boldsymbol{x}^*$, $\tilde{\boldsymbol{y}}$, $\tilde{\boldsymbol{H}}$, and $\tilde{\boldsymbol{Z}}$, where $\tilde{\boldsymbol{w}} \leq \boldsymbol{w}$ and $\tilde{w}_i < w_i$. Therefore, by a similar argument as that for showing constraint ($\mathrm{SG}_{\boldsymbol{Z}}$) holds at equality for some respective $\hat{\boldsymbol{\alpha}}$, we obtain that constraint ($\mathrm{SG}_{\boldsymbol{y}}$) must hold at equality. This completes the proof. $\square$

### EC.2.10. Proof of Theorem 3.2

Lemma EC.2.3. *Suppose $\boldsymbol{w}^* \in \mathcal{U}$ satisfies that $w^*_1 = \bar{w}$. Then, there must exist an $\boldsymbol{\alpha}$ such that $\alpha_1 = 0$, and $\boldsymbol{w}^*$ is the optimal solution to $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha})$.*

*Proof:* We prove this result by contradiction. Suppose the statement does not hold. Equivalently, for any $\boldsymbol{\alpha}$ such that $\alpha_1 = 0$, we have

$$\boldsymbol{\alpha}^\top \boldsymbol{w}^* = \alpha_1 \bar{w} + \sum_{j \neq 1} \alpha_j w^*_j > \phi_{\mathcal{U}}(\boldsymbol{\alpha}). \tag{EC.2.15}$$

Since $\boldsymbol{w}^* \in \mathcal{U}$, there exists $\tilde{\boldsymbol{\alpha}}$ such that $\boldsymbol{w}^*$ is the optimal solution to $[T\phi_{\mathcal{U}}](\tilde{\boldsymbol{\alpha}})$. Note that following the assumption, we must have $\tilde{\alpha}_1 > 0$. Next, construct $\hat{\boldsymbol{\alpha}}$ as follows

$$\hat{\alpha}_j = \begin{cases} 0, & j = 1 \\ c \cdot \tilde{\alpha}_j, & j \neq 1 \end{cases} ,$$

where $c > 0$ is a scalar such that $\|\hat{\boldsymbol{\alpha}}\|_1 = 1$. From inequality (EC.2.15) we know that $\hat{\boldsymbol{\alpha}}^\top \boldsymbol{w}^* > \phi_{\mathcal{U}}(\hat{\boldsymbol{\alpha}})$. Suppose $\hat{\boldsymbol{w}}$ is the optimal solution to $[T\phi_{\mathcal{U}}](\hat{\boldsymbol{\alpha}})$, that is,

$$\hat{\boldsymbol{\alpha}}^\top \boldsymbol{w}^* = \sum_{j \neq 1} c\tilde{\alpha}_j w^*_j > \phi_{\mathcal{U}}(\hat{\boldsymbol{\alpha}}) = \hat{\boldsymbol{\alpha}}^\top \hat{\boldsymbol{w}} = \sum_{j \neq 1} c\tilde{\alpha}_j \hat{w}_j \quad . \tag{EC.2.16}$$

Then, putting the above inequalities together, we have

$$\begin{aligned} \tilde{\boldsymbol{\alpha}}^\top \boldsymbol{w}^* = \phi_{\mathcal{U}}(\tilde{\boldsymbol{\alpha}}) &= \tilde{\alpha}_1 \bar{w} + \sum_{j \neq 1} \tilde{\alpha}_j w^*_j \\ &> \tilde{\alpha}_1 \hat{w}_1 + \sum_{j \neq 1} \tilde{\alpha}_j \hat{w}_j = \tilde{\boldsymbol{\alpha}}^\top \hat{\boldsymbol{w}} \ , \end{aligned}$$

where the inequality follows from $\hat{w}_1 \le \bar{w}$ and the strict inequality (EC.2.16). This contradicts with $\hat{\boldsymbol{w}}$ being feasible to constraint ($\mathrm{SG}_{\boldsymbol{y}}$) of $[T\phi_{\mathcal{U}}](\hat{\boldsymbol{\alpha}})$, and completes the proof. $\square$

THEOREM 3.2. *Starting from any $\boldsymbol{W}_0 \in \mathrm{bd}(\mathcal{U})$, processes $\{\boldsymbol{W}_t\}_{t\ge 0}$, $\{\boldsymbol{X}_t\}_{t\ge 0}$, $\{\boldsymbol{L}_t\}_{t\ge 0}$ and $\{\boldsymbol{H}_t\}_{t\ge 0}$ defined in (3.8)-(3.13) satisfy (EA), (IC), (IR), (PK), (LL), and (UB). Furthermore, $\boldsymbol{W}_t \in \mathrm{bd}(\mathcal{U})$ for all $t \ge 0$.*

*Proof:* We prove the result in four steps.

**Step 1.** According to the theorem's statement, we construct the processes $\{\boldsymbol{W}_t\}_{t\ge 0}$, $\{\boldsymbol{X}_t\}_{t\ge 0}$, $\{\boldsymbol{L}_t\}_{t\ge 0}$ and $\{\boldsymbol{H}_t\}_{t\ge 0}$ from the optimal solution of problem $T\phi_{\mathcal{U}}$ by following (3.8)-(3.13). Suppose $\boldsymbol{W}_\tau \in \mathrm{bd}(\mathcal{U})$ for $\tau \in [0,t)$, and thus by Proposition 3.6 we have $\boldsymbol{W}_{t-} = \boldsymbol{w}^*(\boldsymbol{\alpha}_{t-}) \in \mathrm{bd}(\mathcal{U})$, where $\boldsymbol{\alpha}_{t-} = \check{\boldsymbol{\alpha}}(\boldsymbol{W}_{t-})$, and $\boldsymbol{w}^*(\boldsymbol{\alpha}_{t-})$ is the optimal solution to $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha}_{t-})$, that is, $\boldsymbol{\alpha}_{t-}^\top \boldsymbol{w}^*(\boldsymbol{\alpha}_{t-}) = \phi_{\mathcal{U}}(\boldsymbol{\alpha}_{t-})$. Then, the feasibility of the solution $\boldsymbol{x}^*(\boldsymbol{\alpha}_{t-})$, $\boldsymbol{y}^*(\boldsymbol{\alpha}_{t-})$, $\boldsymbol{H}^*(\boldsymbol{\alpha}_{t-})$, and $\boldsymbol{Z}^*(\boldsymbol{\alpha}_{t-})$ justifies that $\boldsymbol{X}_t$, $\boldsymbol{L}_t$ and $\boldsymbol{H}_t$ as constructed satisfy conditions (EA), (IC), (IR), and (LL).

**Step 2.** Next, on the basis of equation (3.12) and (3.13), we obtain the following expression describing the dynamics of $\{\boldsymbol{W}_t\}_{t\ge 0}$

$$\begin{aligned}
\mathrm{d}W_{i,t} = &\left(\rho w_i^*(\boldsymbol{\alpha}_{t-}) + \sum_{j\in\mathcal{I}} \lambda_{j,t} H_{ij}^*(\boldsymbol{\alpha}_{t-})\right)\mathrm{d}t - \sum_{j\in\mathcal{I}} H_{ij}^*(\boldsymbol{\alpha}_{t-})\mathrm{d}N_{j,t} \\
&- \left(\rho w_i^*(\boldsymbol{\alpha}_{t-}) + \sum_{j\in\mathcal{I}} \lambda_{j,t} H_{ij}^*(\boldsymbol{\alpha}_{t-})\right)\mathbb{1}_{w_i^*(\boldsymbol{\alpha}_{t-})=\bar{w}}\mathrm{d}t, \quad \forall i \in \mathcal{I},
\end{aligned}$$

where the equality follows from constraints ($\mathrm{PK}_{\boldsymbol{y}}$), ($\mathrm{PK}_{\boldsymbol{Z}}$). We need to show that $\{\boldsymbol{W}_t\}_{t\ge 0}$ follows the (UB) and (PK) conditions.

We first show that the construction of $\mathrm{d}L_{i,t}$ according to equation (3.12) guarantees that condition (UB) is satisfied. In particular, flow payment occurs only whenever $w_i^*(\boldsymbol{\alpha}_{t-}) = \bar{w}$, and the payment offsets $y_i^*(\boldsymbol{\alpha}_{t-})\mathrm{d}t$ such that $\mathrm{d}W_{i,t} = 0$, or equivalently, $W_{i,t} \le \bar{w}$. Next, we need to show that for any $i$ and $W_{i,t-} = w_i^*$ (we drop $(\boldsymbol{\alpha}_{t-})$ for clarity),

$$w_i^* - H_{ij}^* \le \bar{w} \quad \forall i,j \in \{\mathcal{I}\} .$$

Suppose for some $\boldsymbol{\alpha} \in \mathbb{R}_+^n$, $\|\boldsymbol{\alpha}\|_1 = 1$, there exist $\tilde{i} \in \{\mathcal{I}\}$ and $\tilde{j} \in \{\mathcal{I}\}$ such that $w_{\tilde{i}}^* - H_{\tilde{i}\tilde{j}}^* > \bar{w}$. Then, it is readily obtained that $\tilde{i} \ne \tilde{j}$, $\boldsymbol{Z}_{\cdot\tilde{j}}^* \notin \mathcal{U}$, and $Z_{ij}^* \ge \inf_{\boldsymbol{w}\in\mathcal{U}} w_i$ for all $i$ and $j$. Moreover, we must have that $H_{\tilde{i}\tilde{j}}^* < 0$, and that there exists at least an $H_{\tilde{i}\tilde{j}}' \in (H_{\tilde{i}\tilde{j}}^*, 0)$, such that $\tilde{\boldsymbol{H}}$ defined as follows is feasible

$$\tilde{H}_{ij} = \begin{cases} H_{ij}', & i = \tilde{i},\ j = \tilde{j} \\ H_{ij}^*, & \text{otherwise} \end{cases} .$$

In particular, denote $(\boldsymbol{w}^*, \boldsymbol{x}^*, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{H}}, \tilde{\boldsymbol{Z}})$ to represent the modified solution by substituting $\boldsymbol{H}^*$ with $\tilde{\boldsymbol{H}}$, and setting $\tilde{y}_i = \rho w_i^* + \lambda \sum_{j\in\mathcal{I}} \tilde{H}_{ij}$ and $\tilde{Z}_{ij} = w_i^* - \tilde{H}_{ij}$ for all $i, j \in \mathcal{I}$. Then, noting that $\boldsymbol{Z}^*_{\cdot\tilde{j}} \notin \mathcal{U}$, it can be verified that there exists $H'_{\tilde{i}\tilde{j}}$ such that the constructed solution is feasible to problem $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha})$. In addition, for the modified solution, we have

$$\tilde{\boldsymbol{\alpha}}^\top \boldsymbol{Z}^*_{\cdot\tilde{j}} > \tilde{\boldsymbol{\alpha}}^\top \tilde{\boldsymbol{Z}}_{\cdot\tilde{j}} \geq \phi_{\mathcal{U}}(\tilde{\boldsymbol{\alpha}}),\ \forall \tilde{\alpha}_{\tilde{i}} \in (0,1],\ \tilde{\boldsymbol{\alpha}} \in \mathbb{R}^n_+,\ \|\tilde{\boldsymbol{\alpha}}\|_1 = 1\ ,$$

where the first inequality follows from the construction of $\tilde{\boldsymbol{Z}}$, and the second from constraint ($\mathrm{SG}_{\boldsymbol{Z}}$). Without loss of generality, we can choose $H'_{\tilde{i}\tilde{j}}$ such that $\tilde{\boldsymbol{Z}}_{\cdot\tilde{j}} \in \mathrm{bd}(\mathcal{U})$. Since $\boldsymbol{w}^*$ remains the same, however, we know from Proposition 3.6 that some of the constraints ($\mathrm{SG}_{\boldsymbol{Z}}$) must be binding for $\boldsymbol{H}^*$ as well. In other words, in order to have $Z^*_{\tilde{i}\tilde{j}} = w^*_{\tilde{i}} - H^*_{\tilde{i}\tilde{j}} > \bar{w}$, we must have $\tilde{\boldsymbol{\alpha}}^\top \tilde{\boldsymbol{Z}}_{\cdot\tilde{j}} = \tilde{\boldsymbol{\alpha}}^\top \boldsymbol{Z}^*_{\cdot\tilde{j}} = \phi_{\mathcal{U}}(\tilde{\boldsymbol{\alpha}})$ for $\boldsymbol{\alpha}$ such that $\tilde{\alpha}_{\tilde{i}} = 0$. Note that this implies the possible existence of multiple optimal solutions. Indeed, if the optimal solution is unique, then the feasibility of the constructed solution and the binding of constraint ($\mathrm{SG}_{\boldsymbol{y}}$) jointly indicate that $\mathcal{U}$ may not be the achievable set. To resolve the issue of multiple optimal solutions, we can always choose $\tilde{\boldsymbol{H}}$ such that $\tilde{\boldsymbol{Z}}_{\cdot\tilde{j}} \in \mathrm{bd}(\mathcal{U})$, that is, $H'_{\tilde{i}\tilde{j}}$ such that $w^*_{\tilde{i}} - H'_{\tilde{i}\tilde{j}} \leq \bar{w}$, without affecting the optimality of the solution. Therefore, instantaneous payments are not necessary to ensure (UB) is satisfied. At last, it is not hard to verify that the constructed $\mathrm{d}W_{i,t}$ satisfies the (PK) condition.

**Step 3.** Equation (3.8) in effect helps drop agents when their promised utilities reach 0. It ensures the constructed $\boldsymbol{\alpha}_t$ to properly specify optimization problems $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha}_t)$ for all $t \geq 0$, and hence is necessary. Dropping agents with 0 promised utilities corresponds to the observation that if the current promised utilities of an agents is 0, then according to the (PK) condition, any EIC contract cannot increase the promised utilities of this agent to be positive. To see this, suppose that the promised utility vector is $\boldsymbol{W}_t$ with $W_{i,t} = 0$. Then, $\check{\boldsymbol{\alpha}}(\boldsymbol{W}_t) = \tilde{\boldsymbol{\alpha}}$ must satisfy $\tilde{\alpha}_i = 1$, as $\tilde{\boldsymbol{\alpha}}$ enables constraints ($\mathrm{SG}_{\boldsymbol{w}}$) to be binding (otherwise by constraints ($\mathrm{SG}_{\boldsymbol{w}}$) we have $\tilde{\alpha}_i W_{i,t} = 0 > \phi_{\mathcal{U}}(\tilde{\boldsymbol{\alpha}})$, however, we know from the construction of $\mathcal{U}$ that $\phi_{\mathcal{U}}(\tilde{\boldsymbol{\alpha}}) \geq 0$). Plugging $\tilde{\boldsymbol{\alpha}}$ into constraint ($\mathrm{SG}_{\boldsymbol{y}}$) yields $y_i \geq 0$. On the one hand, constraint ($\mathrm{PK}_{\boldsymbol{y}}$) leads to the inequality that

$$\rho W_{i,t} + \lambda \sum_{j\in\mathcal{I}} H_{ij} = \lambda \sum_{j\in\mathcal{I}} H_{ij} \geq 0\ .$$

On the other hand, constraints ($\mathrm{PK}_{\boldsymbol{Z}}$) and ($\mathrm{SG}_{\boldsymbol{Z}}$) implies that

$$Z_{ij} = W_{i,t} - H_{ij} \geq 0\ , \forall j \in \mathcal{I}\ ,$$

and thus, $H_{ij} \leq 0$. Putting together we must have $H_{ij} = 0$ and $y_i = 0$. Then, condition (PK) asserts that agent $i$'s promised utility will stay at 0 for all $t' \geq t$, or equivalently, agent $i$ is terminated, and the continuation of the contract does not involve agent $i$. At last, collapsing the dimension of vector $\boldsymbol{W}_t$ and consequently that of vector $\check{\boldsymbol{\alpha}}(\boldsymbol{W}_t)$ becomes evidently necessary by noting that when $\tilde{\alpha}_i = 1$, $\tilde{\alpha}_j = 0$ for all $j \neq i$, however, in order to properly define optimization problem $[T\phi]_{\mathcal{U}}(\boldsymbol{\alpha})$ for agents $\mathcal{I}\backslash\{i\}$, we need $\sum_{j\in\mathcal{I}\backslash\{i\}} \alpha_j > 0$, a condition achieved by equation (3.8) and (3.9).

**Step 4.** Now that we have $\boldsymbol{W}_{t-} = \boldsymbol{w}^*(\boldsymbol{\alpha}_{t-}) \in \mathrm{bd}(\mathcal{U})$, the goal is to show that $\boldsymbol{W}_t$ following the constructed contract always stays on $\mathrm{bd}(\mathcal{U})$. We first show the desired result for the case where there exists an adverse arrival at time $t$, that is, $\mathrm{d}N_{j,t} = 1$ for some $j \in \mathcal{I}$. Following from the (PK) condition, $\mathrm{d}W_{i,t} = Z^*_{ij}(\boldsymbol{\alpha}_{t-}) - w^*_i(\boldsymbol{\alpha}_{t-})$ and $\boldsymbol{W}_t = \boldsymbol{Z}^*_{\cdot j}$. Then, by Proposition 3.6, there exists $\hat{\boldsymbol{\alpha}}_t$ such that constraint ($\mathrm{SG}_{\boldsymbol{Z}}$) holds as equality, that is, $\hat{\boldsymbol{\alpha}}_t^\top \boldsymbol{Z}^*_{\cdot j} = \hat{\boldsymbol{\alpha}}_t^\top \boldsymbol{W}_t = \phi_{\mathcal{U}}(\hat{\boldsymbol{\alpha}}_t) = (\check{\boldsymbol{\alpha}}(\boldsymbol{W}_t))^\top \boldsymbol{W}_t$. In other words, $\boldsymbol{W}_t \in \mathrm{bd}(\mathcal{U})$.

Next, we consider the case where there is no adverse arrival during a period of time $[t, t+\bar{\delta})$ for some $\bar{\delta} > 0$. In this case, $\mathrm{d}W_{i,t} = y^*_i(\boldsymbol{\alpha}_{t-})\mathrm{d}t - \mathrm{d}L_{i,t}$ following from the (PK) condition. We first consider the situation that $W_{i,t+\delta} < \bar{w}$ for all $i \in \mathcal{I}$ and $\delta \in [0, \bar{\delta}]$, and use the binding of constraint ($\mathrm{SG}_{\boldsymbol{y}}$) to show $\boldsymbol{W}_{t+\delta} \in \mathrm{bd}(\mathcal{U})$ as long as $\boldsymbol{W}_t \in \mathrm{bd}(\mathcal{U})$. In this case, $\mathrm{d}L_{i,t} = 0$. For any $\delta \in [0, \bar{\delta}]$, define

$$\zeta(\delta) := \|\boldsymbol{W}_{t+\delta} - \Pi(\boldsymbol{W}_{t+\delta})\|_2^2.$$

Because $\boldsymbol{W}_t \in \mathrm{bd}(\mathcal{U})$, we have $\zeta(0) = 0$. For $\delta \in [0, \bar{\delta}]$, we have

$$\begin{aligned}
\zeta'(\delta) &= \frac{\mathrm{d}}{\mathrm{d}\delta}\|\boldsymbol{W}_{t+\delta} - \Pi(\boldsymbol{W}_{t+\delta})\|_2^2 \\
&= \sum_i [W_{i,t+\delta} - \Pi_i(\boldsymbol{W}_{t+\delta})] \frac{\mathrm{d}W_{i,t+\delta}}{\mathrm{d}\delta} \\
&= c(\boldsymbol{W}_{t+\delta}) \sum_i \check{\alpha}_i(\boldsymbol{W}_{t+\delta}) \frac{\mathrm{d}W_{i,t+\delta}}{\mathrm{d}\delta} \\
&= c(\boldsymbol{W}_{t+\delta}) \sum_i \check{\alpha}_i(\boldsymbol{W}_{t+\delta}) y^*_i(\check{\boldsymbol{\alpha}}(\boldsymbol{W}_{t+\delta})) \\
&= 0,
\end{aligned}$$

in which the second equality follows from the chain rule of taking derivative and the Envelope Theorem (where $\Pi(\boldsymbol{W}_t)$ is the optimal solution to the optimization problem (3.6) parameterized by $\boldsymbol{W}_t$, and the boundary $\mathrm{bd}(\mathcal{U})$ constraining $\boldsymbol{\xi}$ in (3.6) is time-invariant). The third equality follows from (3.7), in that we have shown from (3.7) that for any $\boldsymbol{w} \in \mathcal{U}$, $\boldsymbol{w} - \Pi(\boldsymbol{w}) = c(\boldsymbol{w}) \cdot \check{\boldsymbol{\alpha}}(\boldsymbol{w})$ for some constant $c(\boldsymbol{w}) \in \mathbb{R}_+$. Furthermore, $c(\boldsymbol{w}) > 0$ if and only if $\boldsymbol{w} \in \mathrm{int}(\mathcal{U})$ (we use $\mathrm{int}(\mathcal{A})$ to denote the interior of set $\mathcal{A}$). The fourth equality follows from equation (3.8) and (3.13), and the last equality from that constraint ($\mathrm{SG}_{\boldsymbol{y}}$) holds as equality. Therefore, the $\zeta(\delta)$ function is a constant over $[0, \bar{\delta}]$. Because $\zeta(0) = 0$, we have $\zeta(\delta) = 0$, or, equivalently, $\boldsymbol{W}_{t+\delta} \in \mathrm{bd}(\mathcal{U})$, for all $\delta \in [0, \bar{\delta}]$.

Next, we consider the situation in which $W_{i,t+\delta} = \bar{w}$ for an agent $i$ and $\delta \in [0, \bar{\delta}]$. Note that the first three equalities in the above equation describing $\zeta'(\delta)$ still hold. Plugging (3.12) that characterizes payments into equation (3.13), we obtain that $\mathrm{d}W_{i,t+\delta}=0$. Consequently, again we have $\zeta'(\delta)=0$. Specifically, if $y_i^*(\check{\boldsymbol{\alpha}}(\boldsymbol{W}_{t+\delta}))>0$, then the flow payment takes positive value such that $\mathrm{d}W_{i,t+\delta}=0$, and $\zeta'(\delta)$ becomes

$$\zeta'(\delta)=c(\boldsymbol{W}_{t+\delta})\sum_{j\in\{k|W_{k,t+\delta}<\bar{w}\}}\check{\alpha}_j(\boldsymbol{W}_{t+\delta})y_j^*(\check{\boldsymbol{\alpha}}(\boldsymbol{W}_{t+\delta}))\ .$$

To see that $\zeta'(\delta)=0$ under this situation, note that Lemma EC.2.3 indicates that $W_{i,t+\delta}$ is the optimal solution to $[T\phi_{\mathcal{U}}](\check{\boldsymbol{\alpha}}(\boldsymbol{W}_{t+\delta}))$, where $\check{\boldsymbol{\alpha}}(\boldsymbol{W}_{t+\delta})$ satisfies that $\check{\alpha}_i(\boldsymbol{W}_{t+\delta})=0$. Then, by the binding of constraint (SG$_{\boldsymbol{y}}$), we obtain that

$$\sum_i\check{\alpha}_i(\boldsymbol{W}_{t+\delta})y_i^*(\check{\boldsymbol{\alpha}}(\boldsymbol{W}_{t+\delta}))=\sum_{j\in\{k|W_{k,t+\delta}<\bar{w}\}}\check{\alpha}_j(\boldsymbol{W}_{t+\delta})y_j^*(\check{\boldsymbol{\alpha}}(\boldsymbol{W}_{t+\delta}))=0\ ,$$

that is, $\zeta'(\delta)=0$. On the other hand, if $y_i^*(\check{\boldsymbol{\alpha}}(\boldsymbol{W}_{t+\delta}))\le 0$, then $\zeta'(\delta)=0$ follows directly from the same arguments as in the situation that $W_{i,t+\delta}<\bar{w}$ for all $i\in\mathcal{I}$. It then follows from an argument in the same vein as above that $\boldsymbol{W}_\tau\in\mathrm{bd}(\mathcal{U})$ for all $\tau\in[\delta,\bar{\delta}]$.

To summarize, we have shown that there exists an $\mathcal{F}^N$-adapted process $\{\boldsymbol{\alpha}_t\}_{t\ge0}$, such that starting from $\boldsymbol{W}_0\in\mathrm{bd}(\mathcal{U})$, equations (3.8)-(3.13) defines processes $\{\boldsymbol{X}_t\}_{t\ge0}$ and $\{\boldsymbol{L}_t\}_{t\ge0}$ that constitute an EIC contract, following which the process $\{\boldsymbol{W}_t\}_{t\ge0}\in\mathrm{bd}(\mathcal{U})$ for all $t\ge0$. This completes the proof. □

## EC.2.11. Proof of Corollary 3.2

COROLLARY 3.2. *For any EIC contract that yields a promised utility process $\{\boldsymbol{W}_t\}_{t\ge0}$, if $\boldsymbol{W}_t\in\mathrm{bd}(\mathcal{U})$ for some time $t\ge0$, then $\boldsymbol{W}_{t'}\in\mathrm{bd}(\mathcal{U})$ for all $t'\ge t$.*

*Proof:* For any $\boldsymbol{W}_t\in\mathrm{bd}(\mathcal{U})$, there must exist at least an $\boldsymbol{\alpha}$ such that $\boldsymbol{W}_t$ is the optimal solution to the linear program $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha})$. Recall that in the proof of Proposition 3.6 (and Theorem 3.2), we have already shown by construction from the optimal solution to a sequence of linear programs $[T\phi_{\mathcal{U}}](\cdot)$ that there exists an EIC contract $\{\boldsymbol{L}_t,\boldsymbol{X}_t\}_{t\ge0}$ following which the process $\{\boldsymbol{W}_t\}_{t\ge0}$ satisfies $\boldsymbol{W}_{t'}\in\mathrm{bd}(\mathcal{U})$ for all $t'>t$. Then, to prove Corollary 3.2, we only need to note that the aforementioned construction does not require the constructed contracts to be based on the optimal solutions to the linear programs, and thus the result follows. □

## EC.2.12. Proof of Corollary 3.3

COROLLARY 3.3. *For any boundary EIC contract, the flow payment $l_{i,t}$ is always zero except when agent $i$'s promised utility is at $\bar{w}$.*

*Proof:* The proof follows directly from Corollary 3.2. In particular, assume to the contrary that there exist flow payments to agent $i$ when $W_{i,t-} < \bar{w}$. Then, there exists $\boldsymbol{\alpha} \in \mathbb{R}_+^n$ with $\alpha_i > 0$ such that $\{\boldsymbol{w} | \boldsymbol{\alpha}^\top \boldsymbol{w} = \phi_{\mathcal{U}}(\boldsymbol{\alpha})\}$ defines a supporting hyperplane of the achievable set $\mathcal{U}$ at $\boldsymbol{W}_{t-} \in \mathrm{bd}(\mathcal{U})$. Next, constraint (PK), together with the observation that constraint ($\mathrm{SG}_{\boldsymbol{y}}$) binds leads to the contradictory result that $\{\boldsymbol{w} | \boldsymbol{\alpha}^\top \boldsymbol{w} = \phi_{\mathcal{U}}(\boldsymbol{\alpha})\}$ does not support $\mathcal{U}$ at $\boldsymbol{W}_{t-}$. □

### EC.2.13. Proof of Proposition 3.7

PROPOSITION 3.7. *Consider the EIC contracts $\hat{\Gamma}$ such that $\boldsymbol{u}(\hat{\Gamma}) \in \mathrm{bd}(\mathcal{U})$ and $u_i(\hat{\Gamma}) = u_j(\hat{\Gamma})$ for all $i \neq j$. We have $U(\hat{\Gamma}) \geq U(\Gamma)$ for any EIC contract $\Gamma$.*

*Proof:* From equation (2.9), we obtain that for any EIC contract $\Gamma$, $S(\Gamma) = (R - \lambda C)/\rho$. Since $S(\Gamma) = U(\Gamma) + \sum_{i \in \mathcal{I}} u_i(\Gamma)$, we have

$$U(\Gamma) = \frac{R - \lambda C}{\rho} - \sum_{i \in \mathcal{I}} u_i(\Gamma). \tag{EC.2.17}$$

Therefore, we only need to show that $\sum_{i \in \mathcal{I}} u_i(\hat{\Gamma})$ attains the minimum value among all $\boldsymbol{u} \in \mathcal{U}$. To see this, first note from $\boldsymbol{u}(\hat{\Gamma}) \in \mathrm{bd}(\mathcal{U})$ and Corollary 3.2, we have that $\hat{\Gamma}$ is a boundary contract. We then show the desired result by contradiction. Suppose $\sum_{i \in \mathcal{I}} u_i(\hat{\Gamma})$ as specified in the statement does not attain the minimum value among all vectors of promised utilities in the achievable set. Then, there exists at least another EIC contract, denoted by $\bar{\Gamma}$, such that

$$\sum_{i \in \mathcal{I}} u_i(\bar{\Gamma}) \leq \sum_{i \in \mathcal{I}} u_i(\hat{\Gamma}), \tag{EC.2.18}$$

and $\sum_{i \in \mathcal{I}} u_i(\bar{\Gamma})$ attains the minimum. It is readily obtained that $\boldsymbol{u}(\bar{\Gamma}) \in \mathrm{bd}(\mathcal{U})$, because otherwise there exists another boundary contract, $\tilde{\Gamma}$, such that the corresponding $\boldsymbol{u}(\tilde{\Gamma}) \in \mathrm{bd}(\mathcal{U})$ satisfies $\boldsymbol{u}(\tilde{\Gamma}) \leq \boldsymbol{u}(\bar{\Gamma})$ and attains a lower agents' total promised utility. Consequently, together with the assumption that $\sum_{i \in \mathcal{I}} u_i(\hat{\Gamma})$ as specified in Proposition 3.7 does not attain the minimum value among the vectors of promised utilities in the achievable set, we must have that $\boldsymbol{u}(\bar{\Gamma})$ does not satisfy the condition that $u_i(\bar{\Gamma}) = u_j(\bar{\Gamma})$ for all $i \neq j$. Without loss of generality, assume there exist $i \neq j$ such that $u_i(\bar{\Gamma}) < u_j(\bar{\Gamma})$. Then, by the symmetry among agents, we must have $\tilde{\boldsymbol{u}}(\bar{\Gamma}) \in \mathrm{bd}(\mathcal{U})$, where

$$\tilde{u}_k(\bar{\Gamma}) = \begin{cases} u_i(\bar{\Gamma}), & \text{if } k = j, \\ u_j(\bar{\Gamma}), & \text{if } k = i, \\ u_k(\bar{\Gamma}), & \text{otherwise.} \end{cases}$$

Then, by Corollary 3.1, we have $\frac{1}{2}\left(\boldsymbol{u}(\bar{\Gamma}) + \tilde{\boldsymbol{u}}(\bar{\Gamma})\right)$ resides in the achievable set and attains the same agents' total promised utility. In other words, there must exist $\tilde{\boldsymbol{u}}(\bar{\Gamma})$, satisfying $\tilde{u}_i(\bar{\Gamma}) = \tilde{u}_j(\bar{\Gamma})$ for all $i \neq j$, that attains the same agents' total promised utility. In addition, by Corollary 3.1, the $\boldsymbol{u}(\hat{\Gamma})$ as specified in Proposition 3.7 satisfies $\boldsymbol{u}(\hat{\Gamma}) \leq \tilde{\boldsymbol{u}}(\bar{\Gamma})$, contradicting with (EC.2.18).

Therefore, $\sum_{i\in\mathcal{I}} u_i(\hat{\Gamma})$ attains the minimum value among all $\boldsymbol{u} \in \mathcal{U}$. Then, by (EC.2.17), for any EIC contract $\Gamma$,

$$U(\hat{\Gamma}) \geq U(\Gamma).$$

This completes the proof. $\square$

### EC.2.14. Proof of Proposition 3.8

PROPOSITION 3.8. *When $n = 2$, there exists an optimal solution to $[T\phi](\boldsymbol{\alpha})$, denoted by $(\boldsymbol{w}^*, \boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{Z}^*, \boldsymbol{H}^*)$, that satisfies the following conditions,*

*(i) if $w_1^* = \bar{w}$, then $w_2^* = \inf_{\boldsymbol{w}\in\mathcal{U}} w_2 > 0$, $x_1^* = 1$, $x_2^* = 0$, $y_1^* = \rho\bar{w} + \lambda\beta$, and $y_2^* = 0$;*

*(ii) if $\alpha_1 = \alpha_2 = 0.5$, then $w_1^* = w_2^*$, $x_1^* = x_2^* = 0.5$, and $y_1^* = y_2^* = 0$;*

*Proof:* **(Proof of (i).)** If $w_1^* = \bar{w}$, since $\boldsymbol{w}^* \in \mathrm{bd}(\mathcal{U})$, then $w_2^* > 0$. This is because, suppose otherwise that $w_2^* = 0$, then agent 2 is terminated and the average allocation will no longer be efficient as explained at the beginning of Section 3, violating the condition that $\boldsymbol{w}^* \in \mathrm{bd}(\mathcal{U})$. To see $w_2^* = \inf_{\boldsymbol{w}\in\mathcal{U}} w_2$, it suffices to note that $\tilde{\boldsymbol{\alpha}} = (0,1)^\top$, $\boldsymbol{w}^*$ is the optimal solution of $[T\phi_{\mathcal{U}}](\tilde{\boldsymbol{\alpha}})$, and $\boldsymbol{w}^* \in \mathrm{bd}(\mathcal{U})$ (See Lemma EC.2.3).

Next, we show that $x_1^* = 1$ and $x_2^* = 0$. Suppose to the contrary that $x_2^* > 0$. By ($\mathrm{IC_s}$) and ($\mathrm{PK}_{\boldsymbol{Z}}$) we have

$$Z_{22}^* = w_2^* - \beta x_2^* < w_2^* \ .$$

Thus, for $\tilde{\boldsymbol{\alpha}} = (0,1)^\top$,

$$\tilde{\boldsymbol{\alpha}}^\top \boldsymbol{Z}_{\cdot 2}^* = Z_{22}^* < w_2^* = \tilde{\boldsymbol{\alpha}}^\top \boldsymbol{w}^* = \phi(\tilde{\boldsymbol{\alpha}}) \ ,$$

which contradicts with constraint ($\mathrm{SG}_{\boldsymbol{Z}}$). Therefore, $x_2^* = 0$. Consequently, $x_1^* = 1$ following from ($\mathrm{EA_s}$).

At last, we show $y_1^* = \rho\bar{w} + \lambda\beta$ and $y_2^* = 0$. Since $\boldsymbol{w}^*$ is part of the optimal solution to $[T\phi_{\mathcal{U}}](\tilde{\boldsymbol{\alpha}})$, by the binding of constraint ($\mathrm{SG}_{\boldsymbol{y}}$), we directly have $y_2^* = 0$. On the other hand, $y_1^* = \rho\bar{w} + \lambda\beta + \lambda H_{12}^*$. To ensure $y_1^* = \rho\bar{w} + \lambda\beta$, observe that since $x_2^* = 0$, $H_{22}^* = 0$, then $Z_{22}^* = w_2^*$. By Theorem 3.2, $\boldsymbol{Z}_{\cdot 2}^* \in \mathrm{bd}(\mathcal{U})$, then $Z_{12}^* = w_1^*$, and thus $H_{12}^* = 0$. Plugging into constraint ($\mathrm{PK}_{\boldsymbol{y}}$), we obtain that $y_1^* = \rho\bar{w} + \lambda\beta \geq 0$. This completes the proof for (i).

**(Proof of (ii).)** We first show $w_1^* = w_2^*$. Suppose $w_1^* \neq w_2^*$, since $\boldsymbol{\alpha}^\top(w_1^*, w_2^*) = \boldsymbol{\alpha}^\top(w_2^*, w_1^*)$, both $(w_1^*, w_2^*)$ and $(w_2^*, w_1^*)$ are optimal to $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha})$. Therefore, $(\frac{w_1^*+w_2^*}{2}, \frac{w_1^*+w_2^*}{2})$ is optimal to $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha})$ due to the convexity of $\mathcal{U}$. By resetting $w_1^* = \frac{w_1^*+w_2^*}{2}$ and $w_2^* = \frac{w_1^*+w_2^*}{2}$, we have $w_1^* = w_2^*$.

Next, we show $x_1^*=x_2^*=0.5$. Suppose $x_1^*\neq x_2^*$, since $w_1^*=w_2^*$, $(x_2^*,x_1^*)$ is also optimal to $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha})$. Therefore, $(\frac{x_1^*+x_2^*}{2},\frac{x_1^*+x_2^*}{2})$ is optimal to $[T\phi_{\mathcal{U}}](\boldsymbol{\alpha})$ due to the convexity of $\mathscr{M}(T\phi)$ (see Lemma EC.2.2). By resetting $x_1^*=\frac{x_1^*+x_2^*}{2}$ and $x_2^*=\frac{x_1^*+x_2^*}{2}$, we have $x_1^*=x_2^*=0.5$. Similarly, we can obtain $H_{12}^*=H_{21}^*$.

At last, we get $y_1^*=y_2^*$ by $w_1^*=w_2^*$, $x_1^*=x_2^*$, and $H_{12}^*=H_{21}^*$. Since $\boldsymbol{\alpha}^\top(y_1^*,y_2^*)=0$, it is straightforward that $y_1^*=y_2^*=0$. $\square$

## EC.3. Proofs in Section 4

### EC.3.1. Proof of Theorem 4.1

THEOREM 4.1. *Contract $\Gamma_s(\boldsymbol{w};R^a)$ in Definition 4.1 satisfies (EA), (IC), (IR), (LL), (PK), and (UB) as long as $\bar{w}\geq\hat{w}-(n-1)\breve{w}$, therefore, is an EIC contract. Furthermore, $\boldsymbol{W}_t\in\mathcal{U}_s$ for any $t\geq 0$ starting from $\boldsymbol{W}_0=\boldsymbol{w}$ following (4.6). Therefore, $\mathcal{U}_s$ is a self-generating set.*

*Proof:* For notation convenience, we abbreviate $\Gamma_s(\boldsymbol{w};R^a)$ to $\Gamma_s$ when the context is clear. Before presenting the proofs, we first show that for any $W_{i,t}\in\mathcal{U}_s$, condition $\mathrm{d}W_{i,t}=-\sum_{j\in\mathcal{I}}H_{ij,t}\mathrm{d}N_{j,t}$ implies that $\Gamma_s$ satisfies (PK). Next, we show that agents' promised utilities stay inside $\mathcal{U}_s$ under $\Gamma_s$. At last, we prove that $\Gamma_s$ is an EIC contract.

We first show show that for any $W_{i,t}\in\mathcal{U}_s$, $\mathrm{d}W_{i,t}=-\sum_{j\in\mathcal{I}}H_{ij,t}\mathrm{d}N_{j,t}$ implies that $\Gamma_s$ satisfies (PK). By equation (4.5) in Definition 4.1, we know that

$$\sum_{i\in\mathcal{I}}X_{i,t}=\sum_{i\in\mathcal{I}}\frac{W_{i,t-}-\breve{w}}{\hat{w}-n\breve{w}}=1, \tag{EC.3.1}$$

where the second equality follows from $W_{i,t}\in\mathcal{U}_s$, i.e., $\sum_{i\in\mathcal{I}}W_{i,t-}=\hat{w}$. Then we have

$$\begin{aligned}
\rho W_{i,t-}-R^aX_{i,t}+\lambda\sum_{j\in\mathcal{I}}H_{ij,t}=&\rho W_{i,t-}-R^aX_{i,t}+\lambda H_{ii,t}+\lambda\sum_{j\in\mathcal{I},j\neq i}H_{ij,t}\\
=&\rho W_{i,t-}-R^aX_{i,t}+\lambda\beta X_{i,t}-\lambda\beta\sum_{j\in\mathcal{I},j\neq i}\frac{X_{j,t}}{n-1}\\
=&\rho W_{i,t-}-\rho\hat{w}X_{i,t}+\rho(n-1)\breve{w}X_{i,t}-\rho\breve{w}\sum_{j\in\mathcal{I},j\neq i}X_{j,t}\\
=&\rho W_{i,t-}-\rho(\hat{w}-n\breve{w})X_{i,t}-\rho\breve{w}\sum_{i\in\mathcal{I}}X_{i,t}\\
=&\rho W_{i,t-}-\rho(\hat{w}-n\breve{w})X_{i,t}-\rho\breve{w}\\
=&0
\end{aligned}$$

where the second equality follows from equation (4.7), the third equality from equations (4.1) and (4.3) (i.e., the definition of $\breve{w}$), the fourth from re-organizing terms, the fifth equality from equation (EC.3.1), and the last equality from equation (4.5). Consequently, $\mathrm{d}W_{i,t}=-\sum_{j\in\mathcal{I}}H_{ij,t}\mathrm{d}N_{j,t}$ implies that $\Gamma_s$ satisfies (PK).

We next show that for any $\boldsymbol{W}_0\in\mathcal{U}_s$, $\boldsymbol{W}_t\in\mathcal{U}_s$ for any $t\geq 0$ under $\Gamma_s$. Consider a partition of the time interval $[0,t]$: $0=t_0<t_1<...<t_{m-1}<t_m\leq t_{m+1}=t$, where $n\geq 0$ and $t_k$ ($k\in\{1,2,...,m\}$) are time points at which adverse events happen. Note that the Poisson arrival process ensures one adverse event at a time. Suppose $\boldsymbol{W}_{t_{k-1}}\in\mathcal{U}_s$, we claim that (i) $\boldsymbol{W}_t\in\mathcal{U}_s$ for any $t\in(t_{k-1},t_k)$; (ii) if $\boldsymbol{W}_t\in\mathcal{U}_s$ for any $t\in(t_{k-1},t_k)$, then $\boldsymbol{W}_{t_k}\in\mathcal{U}_s$. Combining (i) and (ii) together, we obtain that if $\boldsymbol{W}_{t_{k-1}}\in\mathcal{U}_s$, then $\boldsymbol{W}_t\in\mathcal{U}_s$ for any $t\in(t_{k-1},t_k]$. Then, we prove the aforementioned two claims. We

first prove claim (i). Since $\boldsymbol{W}_{t_{k-1}} \in \mathcal{U}_s$, for any $t \in (t_{k-1}, t_k)$, we have $\mathrm{d}W_{i,t} = 0$ for any $t \in (t_{k-1}, t_k)$ by the definition of $\Gamma_s$, and thus $W_{i,t} = W_{i,t_{k-1}}$, which implies $\boldsymbol{W}_t \in \mathcal{U}_s$ any $t \in (t_{k-1}, t_k)$. Next, we show claim (ii). Suppose agent $j$ causes an adverse event at time $t = t_k$, for $t < t^k$, we have

$$W_{i,t_k} = W_{i,t-} + \int_t^{t^k} \mathrm{d}W_{i,s} = W_{i,t-} - \int_t^{t^k} \sum_{j\in\mathcal{I}} H_{ij,s}\mathrm{d}N_{j,s} = W_{i,t_{k-1}} - H_{ij,t_{k-1}},$$

where the second equality follows from $\mathrm{d}W_{i,t} = -\sum_{j\in\mathcal{I}} H_{ij,t}\mathrm{d}N_{j,t}$ by the definition of $\Gamma_s$, and the third equality follows from $W_{i,t-} = W_{i,t_{k-1}}$ and $H_{ij,t} = H_{ij,t_{k-1}}$ for any $t \in (t_{k-1}, t_k)$. Therefore,

$$W_{i,t_k} = \begin{cases} W_{i,t_{k-1}} - \dfrac{W_{i,t_{k-1}} - \check{w}}{\hat{w} - n\check{w}}\beta & \text{if } i = j, \\ W_{i,t_{k-1}} + \dfrac{W_{j,t_{k-1}} - \check{w}}{(n-1)(\hat{w} - n\check{w})}\beta & \text{if } i \neq j. \end{cases} \tag{EC.3.2}$$

Then, we show that $\boldsymbol{W}_{t_k} \in \mathcal{U}_s$. Note that

$$\begin{aligned} \sum_{i\in\mathcal{I}} W_{i,t_k} &= W_{j,t_k} + \sum_{i\in\mathcal{I}, i\neq j} W_{i,t_k} \\ &= W_{j,t_{k-1}} - \frac{W_{j,t_{k-1}} - \check{w}}{\hat{w} - n\check{w}}\beta + \sum_{i\in\mathcal{I}, i\neq j}\left[W_{i,t_{k-1}} + \frac{W_{j,t_{k-1}} - \check{w}}{(n-1)(\hat{w} - n\check{w})}\beta\right] \\ &= \sum_{i\in\mathcal{I}} W_{i,t_{k-1}} - \frac{W_{j,t_{k-1}} - \check{w}}{\hat{w} - n\check{w}}\beta + \frac{W_{j,t_{k-1}} - \check{w}}{\hat{w} - n\check{w}}\beta \\ &= \sum_{i\in\mathcal{I}} W_{i,t_{k-1}} = \hat{w}, \end{aligned}$$

where the last equality follows from $\boldsymbol{W}_{t_{k-1}} \in \mathcal{U}_s$. To prove $\boldsymbol{W}_{t_k} \in \mathcal{U}_s$, we need to further show that $W_{i,t_k} \in [\check{w}, \hat{w} - (n-1)\check{w}]$ for any $i$. On the one hand, from equation (EC.3.2) we know that $W_{i,t_k}$ is a linear function of $W_{i,t_{k-1}}$ if $i = j$, and thus we just need to verify the values of $W_{i,t_k}$ for both $W_{i,t_{k-1}} = \check{w}$ and $W_{i,t_{k-1}} = \hat{w} - (n-2)\check{w}$. If $i = j$, we have

$$W_{i,t_k}\big|_{W_{i,t_{k-1}} = \check{w}} = \check{w}, \quad W_{i,t_k}\big|_{W_{i,t_{k-1}} = \hat{w} - (n-1)\check{w}} = \hat{w} - (n-1)\check{w} - \beta.$$

It is straightforward that $\check{w} \in [\check{w}, \hat{w} - (n-1)\check{w}]$. From equations (4.1) and (4.2) we have

$$\hat{w} \geq n\check{w} + \beta, \tag{EC.3.3}$$

then $\hat{w} - (n-1)\check{w} - \beta \in [\check{w}, \hat{w} - (n-1)\check{w}]$. Note that we do not need equation (4.2) to be binding. Equivalently, the proof holds for all $R^a \geq \rho(n\check{w} + \beta)$. Consequently, the above result suggests that $W_{i,t_k} \in [\check{w}, \hat{w} - (n-1)\check{w}]$. On the other hand, if $i \neq j$, by equation (EC.3.2), we know that $W_{i,t_k}$ obtains its minimum at $W_{i,t_{k-1}} = W_{j,t_{k-1}} = \check{w}$,

$$W_{i,t_k}\big|_{W_{i,t_{k-1}} = W_{j,t_{k-1}} = \check{w}} = \check{w},$$

which is straightforward in $[\check{w}, \hat{w}-(n-1)\check{w}]$. By equation (4.5) we can rewrite $W_{i,t_k}$ as

$$W_{i,t_k} = (\hat{w}-n\check{w})X_{i,t_{k-1}} + \check{w} + \frac{X_{j,t_{k-1}}\beta}{n-1} \leq (\hat{w}-n\check{w})X_{i,t_{k-1}} + \check{w} + \frac{\beta}{n-1}(1-X_{i,t_{k-1}}),$$

and thus the maximum of $W_{i,t_k}$ is

$$g(X_{i,t_{k-1}}) := (\hat{w}-n\check{w})X_{i,t_{k-1}} + \check{w} + \frac{\beta}{n-1}(1-X_{i,t_{k-1}}).$$

Note that

$$g(0) = \check{w} + \frac{\beta}{n-1} < \check{w} + \beta \leq \hat{w}-(n-1)\check{w},$$

where the second inequality follows from (EC.3.3), and $g(1) = \hat{w}-(n-1)\check{w}$, we have

$$g(0) \in [\check{w}, \hat{w}-(n-1)\check{w}],\ g(1) \in [\check{w}, \hat{w}-(n-1)\check{w}].$$

Since $g(X_{i,t_{k-1}})$ is monotone in $X_{i,t_{k-1}}$, the maximum of $W_{i,t_k}$ lies in $[\check{w}, \hat{w}-(n-1)\check{w}]$. Therefore, $W_{i,t_k} \in [\check{w}, \hat{w}-(n-1)\check{w}]$ for any $i \in \mathcal{I}$. Combining with $\sum_{i\in\mathcal{I}} W_{i,t_k} = \hat{w}+\check{w}$, we know $\boldsymbol{W}_t \in \mathcal{U}_s$. Therefore, combining claim (i) and (ii), if $\boldsymbol{W}_{t_{k-1}} \in \mathcal{U}_s$, then $\boldsymbol{W}_t \in \mathcal{U}_s$ for any $t \in (t_{k-1}, t_k]$. Then, by mathematical induction, we obtain that for any $\boldsymbol{W}_0 \in \mathcal{U}_s$, $\boldsymbol{W}_t \in \mathcal{U}_s$ for any $t \geq 0$.

At last, we show that $\Gamma_s$ is an EIC contract. To prove $\Gamma_s$ is an EIC, we need to show $\Gamma_s$ satisfies (EA), (IC), (IR), (PK), (LL), and (UB). We have proved that $\mathrm{d}W_{i,t} = -\sum_{j\in\mathcal{I}} H_{ij,t}\mathrm{d}N_{j,t}$ implies that $\Gamma_s$ satisfies (PK) and equation (4.5) implies $H_{ii,t} = \beta X_{i,t}$, implying that $\Gamma_s$ satisfies (IC). Following equation (4.4) we know that $\Gamma_s$ satisfies (LL). Since we have proved that for any $\boldsymbol{W}_0 \in \mathcal{U}_s$, $\boldsymbol{W}_t \in \mathcal{U}_s$ for any $t \geq 0$, which implies $\check{w} \leq W_{i,t} \leq \hat{w}-(n-1)\check{w}$ and $\sum_{i\in\mathcal{I}} W_{i,t} = \hat{w}$. Therefore, by equation (4.5), we have

$$X_{i,t} \geq 0,$$

which combining with (EC.3.1) implies that $\Gamma_s$ satisfies (EA). Moreover, since $W_{i,t} \geq \check{w} \geq 0$, $\Gamma_s$ satisfies (IR). At last, agents' promised utilities is endogenously upper bounded by $\hat{w}-(n-1)\check{w}$, and hence if $\hat{w}-(n-1)\check{w} \leq \bar{w}$, agents' promised utilities will not exceed the principal's commitment power, or equivalently, constraint (UB) is satisfied. To conclude, $\Gamma_s$ is an EIC contract. □

### EC.3.2. Proposition EC.3.1

This subsection provides a result on the directions of jumps in promised utilities triggered by adverse arrivals under a simple EIC contract. The result states that, if an arrival occurs, the direction of jump is always perpendicular to one of the facets of the self-generating set associated with the simple EIC contract. Note that the result is established for the more general case with $n \geq 3$, compared to the numerical illustration as in Section 4.

Proposition EC.3.1. *Denote $\mathcal{U}_{s,i}$ to represent a facet of $\mathcal{U}_s$, and $\boldsymbol{w}_i$ the $i$th vertex of $\mathcal{U}_s$. Specifically,*

$$\mathcal{U}_{s,i} = \left\{ \sum_{j \neq i} \theta_j \boldsymbol{w}_j | \sum_{j \neq i} \theta_j = 1, \theta_j \geq 0, \quad \forall j \in \mathcal{I} \right\}$$

*Suppose at time point $t-$, the promised utilities of agents $\boldsymbol{W}_{t-}$ satisfies $W_{i,t-} > \check{w}$, and hence $X_{i,t}$, the allocation to agent $i$ is non-zero. If agent $i$ causes an adverse arrival at time $t$, then $\boldsymbol{H}_{\cdot i,t}$ is orthogonal to the facet $\mathcal{U}_{s,i}$.*

*Proof:* We drop the subscripts denoting time in this proof for notation convenience. Denote $\tilde{w} = \hat{w} - (n-1)\check{w}$ to represent the highest promised utility that any agent can get under the simple EIC contract $\Gamma_s$. Denote the vertices of simplex $\mathcal{U}_s$ by $\{\boldsymbol{w}_i\}_{i \in \mathcal{I}}$, where $\boldsymbol{w}_i = [\check{w}, \check{w}, \ldots, \tilde{w}, \ldots, \check{w}]$ is a vector such that all elements except for the $i$th one equal to $\check{w}$, and the $i$th element equals to $\tilde{w}$. That is, the simplex $\mathcal{U}_s$ under the simple EIC contract $\Gamma_s$ can be characterized as follows

$$U_s = \left\{ \sum_i \theta_i \boldsymbol{w}_i | \sum_i \theta_i = 1, \theta_i \geq 0, \quad \forall i \in \mathcal{I} \right\}.$$

Similarly, a facet of $\mathcal{U}_s$ excluding $\boldsymbol{w}_i$ is

$$U_{s,i} = \left\{ \sum_{j \neq i} \theta_j \boldsymbol{w}_j | \sum_{j \neq i} \theta_j = 1, \theta_j \geq 0, \quad \forall j \in \mathcal{I} \right\}.$$

Note that $U_{s,i}$ lies in an affine space generated by affine combinations of the vertices of $U_{s,i}$, namely, $\{\boldsymbol{w}_j\}_{j \in \mathcal{I}, j \neq i}$. The directions form a linear span $S_i$, defined as follows

$$S_i = \text{span}(\boldsymbol{w}_2 - \boldsymbol{w}_1, \boldsymbol{w}_3 - \boldsymbol{w}_1, ..., \boldsymbol{w}_{i-1} - \boldsymbol{w}_1, \boldsymbol{w}_{i+1} - \boldsymbol{w}_1, ..., \boldsymbol{w}_n - \boldsymbol{w}_1).$$

At last, when an adverse event associated with agent $i$ arrives, the promised utility vector takes a jump, whose direction is characterized by vector $\boldsymbol{H}_{\cdot i} = (H_{1i}, H_{2i}, \ldots, H_{ni})^\top$, where following from equation (4.7), $H_{ji}$ is

$$H_{ji} = \begin{cases} X_i \beta, & \text{for } j = i, \\ -\dfrac{X_i \beta}{n-1}, & \text{for } j \neq i. \end{cases}$$

Then, we have

$$\begin{aligned} \boldsymbol{H}_{\cdot i}^\top (\boldsymbol{w}_j - \boldsymbol{w}_1) &= -\frac{H_{ii}}{n-1} \times (1, 1, \ldots, \underbrace{-(n-1)}_{i\text{-th entry}}, \ldots, 1)^\top (\check{w} - \tilde{w}, 0, \ldots, 0, \underbrace{\tilde{w} - \check{w}}_{j\text{-th entry}}, 0, \ldots, 0) \\ &= 0, \end{aligned}$$

where $(\boldsymbol{w}_j - \boldsymbol{w}_1)$ takes non-zero values at the first and $j$th entries. The above equation implies that $\boldsymbol{H}_{\cdot i}$ is orthogonal to all directions in the linear span $S_i$, and thus $\boldsymbol{H}_{\cdot i}$ is orthogonal to the facet $\mathcal{U}_{s,i}$. $\square$