Minimum Spanning Trees I: Prim's algorithm
COMS20017 (Algorithms and Data)

John Lapinskas, University of Bristol

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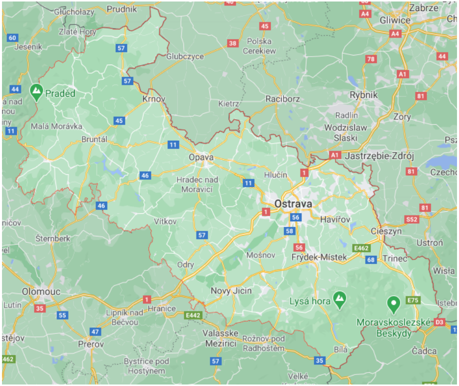

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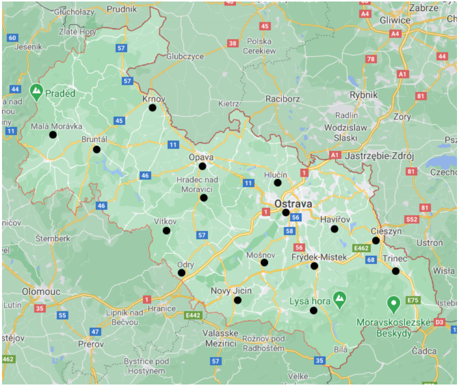

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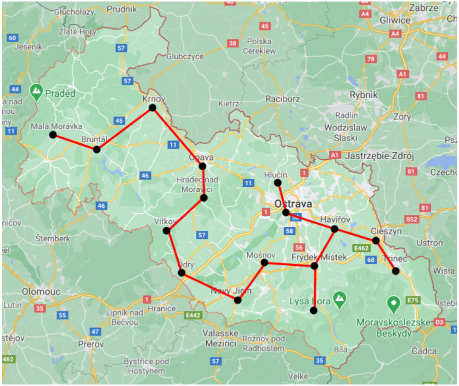

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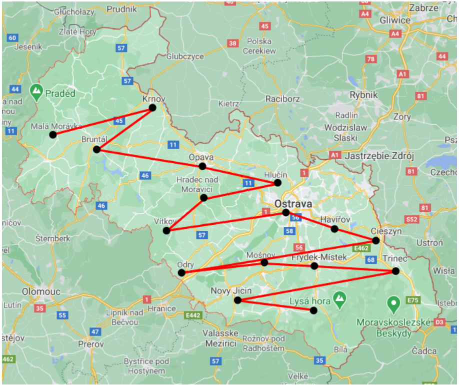
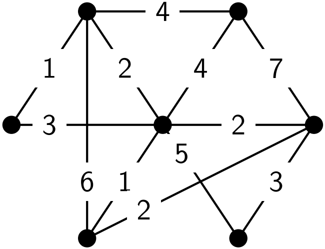


You need every town to be connected to every other town, and you want to spend as little as possible. So you want something like this, not like **this**.

Formal definition

We think of this situation as a connected weighted graph $G = ((V, E), w)$: the vertices are towns, and $w(x, y)$ is the cost of building a connection from x to y. (In this case, E would contain every possible edge.)



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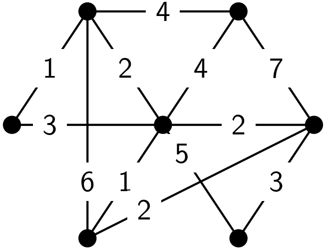

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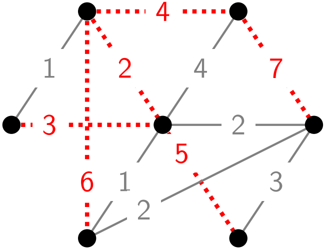

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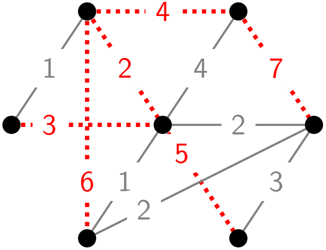

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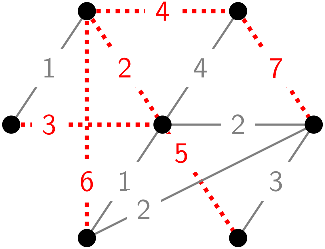

Total weight:
$4 + 2 + 7 + 3 + 6 + 5 = 27$

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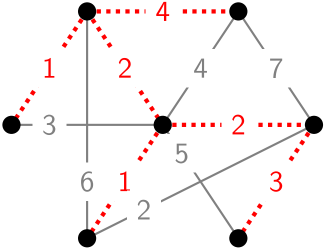

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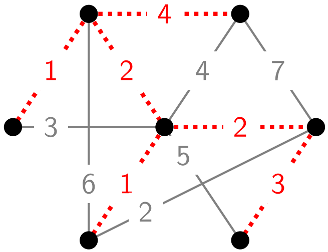

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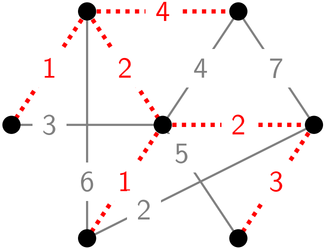

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This is called a **minimum spanning tree**.

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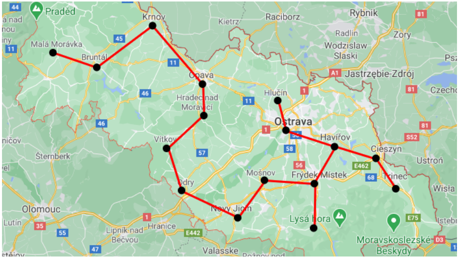

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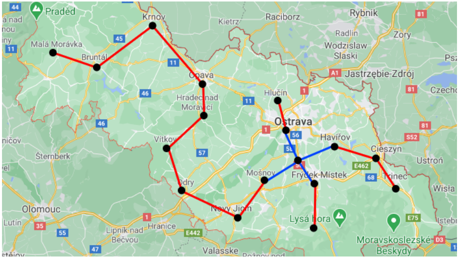

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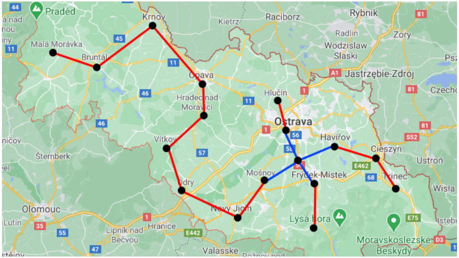

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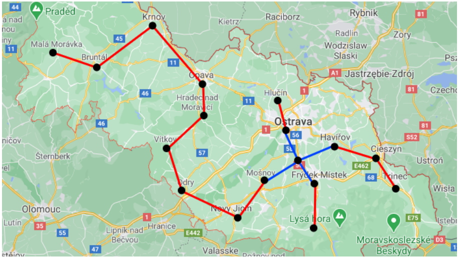

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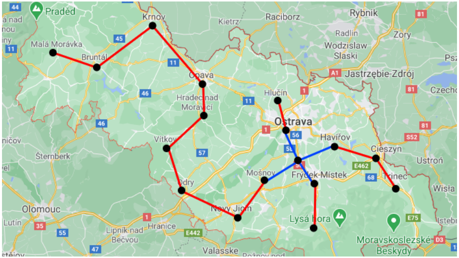

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- this is “NP-hard” (read: no polynomial-time algorithm);
- all the approximation algorithms are based on minimum spanning tree;
- using a minimum spanning tree is already “good enough” — at worst twice the weight of a minimum Steiner tree (see problem sheet).

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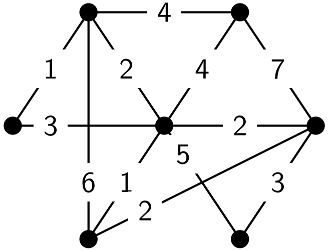

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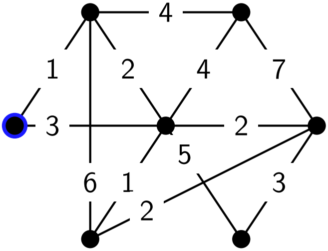

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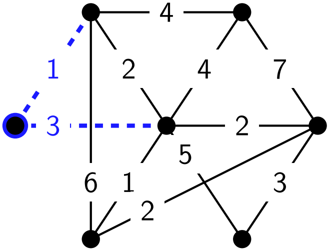

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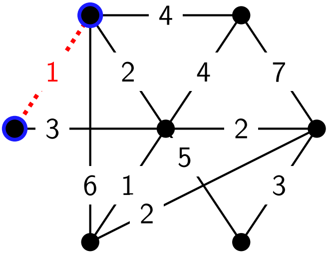

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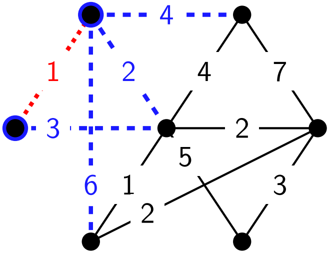

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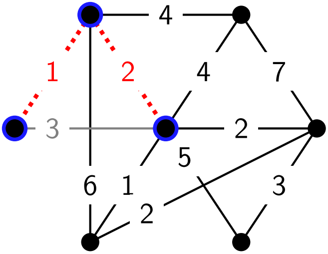

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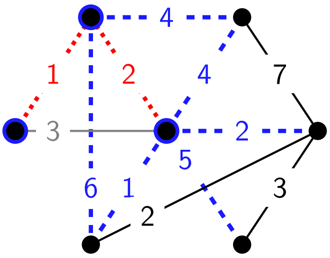

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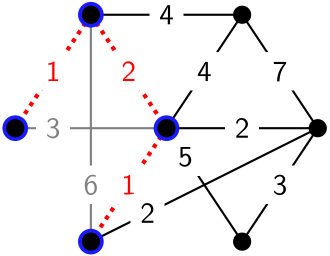

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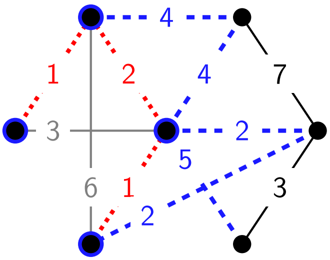

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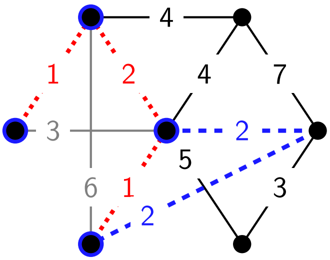

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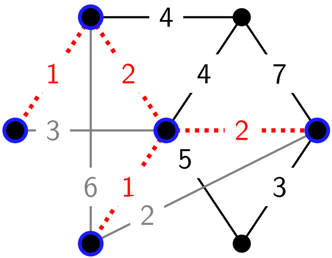

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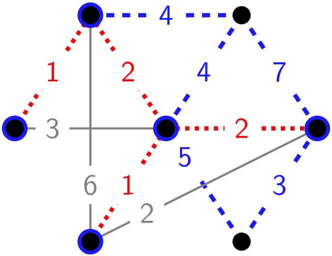

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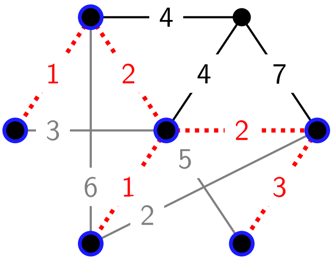

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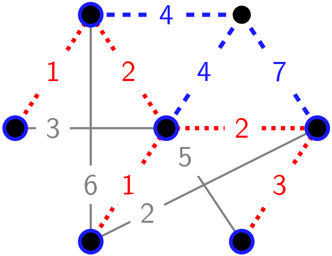

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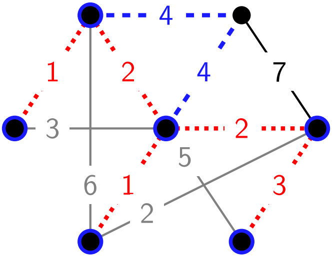

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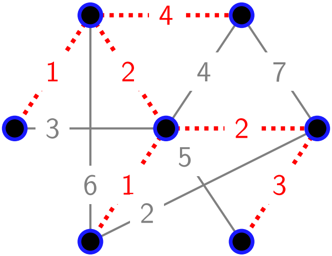

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Formally: Let $T_1 = (\{v\}, \emptyset)$ for some arbitrary $v \in V$.

Let E_i be the set of edges from $V(T_i)$ to $V \setminus V(T_i)$.

Form T_{i+1} by adding a lowest-weight edge $e_i \in E_i$ to T_i, so $V(T_{i+1}) = V(T_i) \cup e_i$ and $E(T_{i+1}) = E(T_i) \cup \{e_i\}$.

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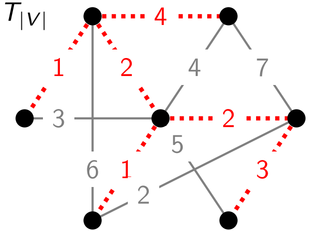
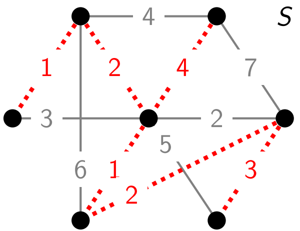
To prove it's a **minimum** spanning tree, we use an exchange argument.

That is, we show we can turn any minimum spanning tree into $T_{|V|}$ without increasing its weight (like with interval scheduling).

$T_{|V|}$ is minimum: Let S be a minimum spanning tree with $S \neq T_{|V|}$.





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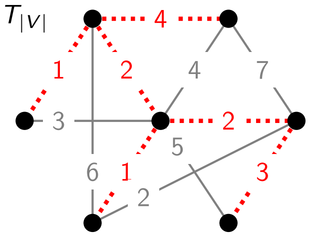

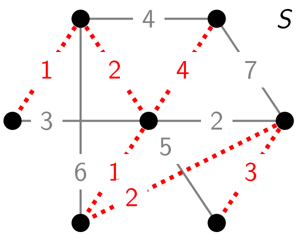

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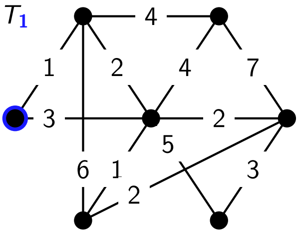

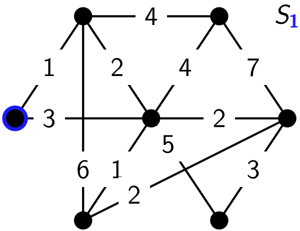

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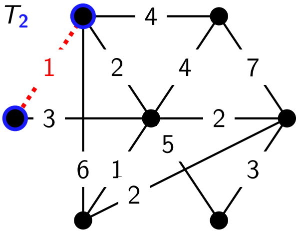

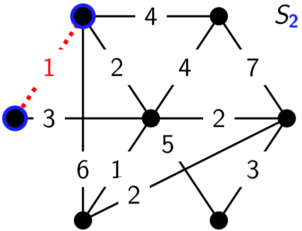

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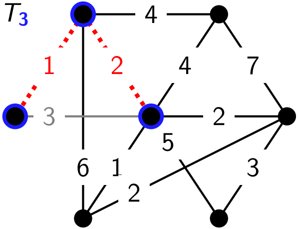

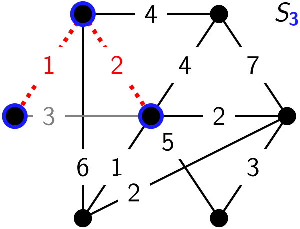

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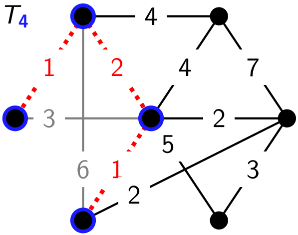

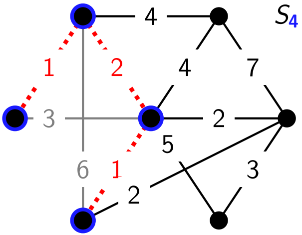

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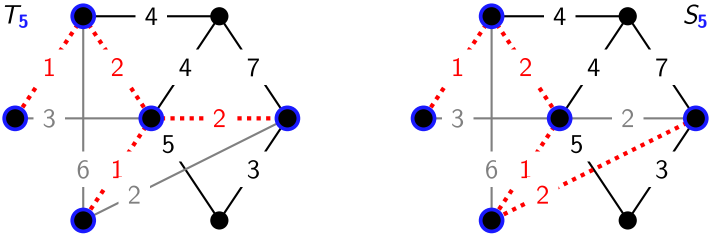

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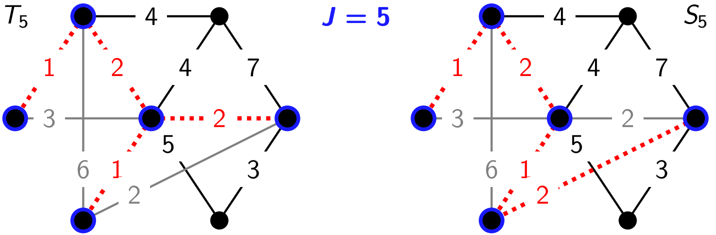

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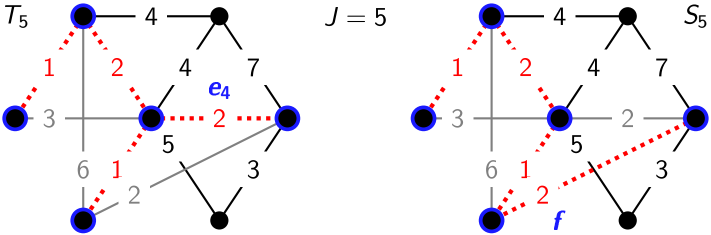

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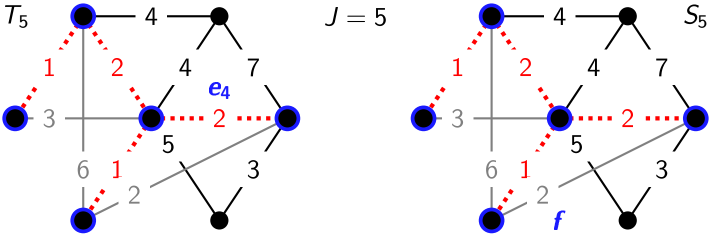

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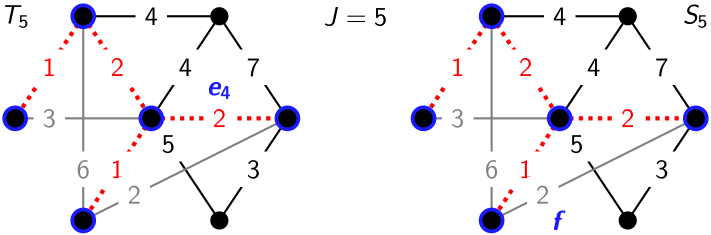

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Weight doesn't increase: True as both f and e_{J-1} join X to $V \setminus X$, so $w(e_{J-1}) \leq w(f)$ by Prim's choice of e_{J-1}. ✓

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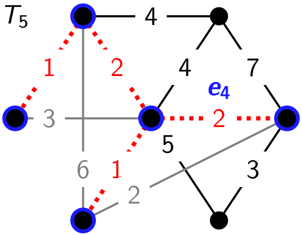
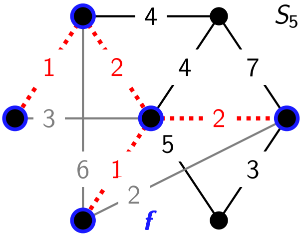


$J = 5$



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Weight doesn't increase: True as both f and e_{J-1} join X to $V \setminus X$, so $w(e_{J-1}) \leq w(f)$ by Prim's choice of e_{J-1}. ✓

Still a tree: Since there is only one edge f, $S[V \setminus X]$ is a tree as well (by the FLoT). Joining two disjoint trees by an edge gives another tree (by the FLoT). ✓

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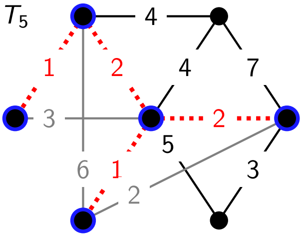

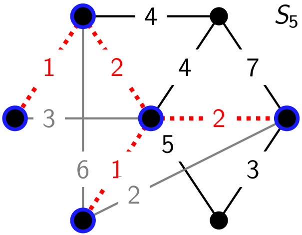

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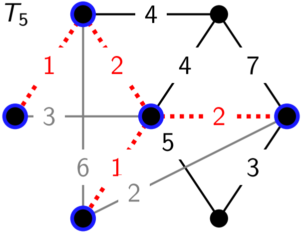
T_5



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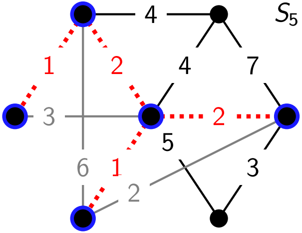

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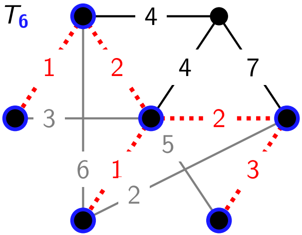

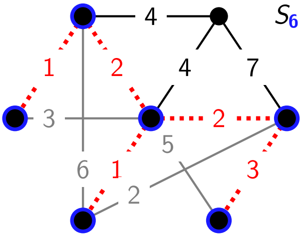

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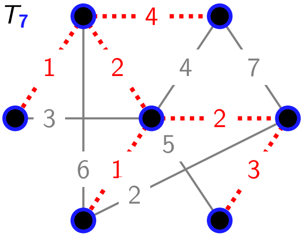

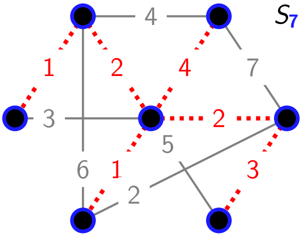

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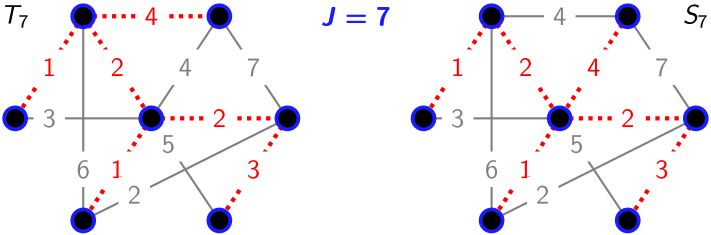

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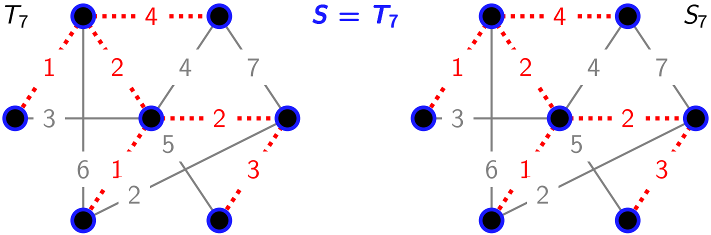

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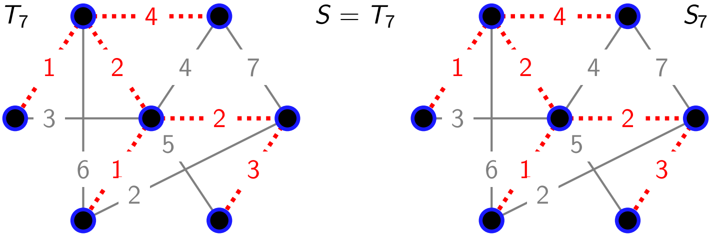

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By repeating the process, we can turn S **into** $T_{|V|}$ without increasing its weight. Hence $w(S) \geq w(T_{|V|})$. Since S was minimum, we're done! □

Prim's algorithm: Implementation

Literally just breadth-first search with a priority queue!

Algorithm: BFS

Input : **Connected weighted** graph $G = ((V, E), w)$.
Output : **A minimum spanning tree for G**.

Number the vertices of G **arbitrarily** as $v_1, \dots, v_n$.
Let `L`$[i] \leftarrow \infty$ for all $i \in [n]$.
Let `L`$[1] \leftarrow 0$, `pred`$[1] \leftarrow$ `None`.
Let `queue` be a **length-$|E|$ priority** queue containing all tuples (v_1, v_j) with $\{v_1, v_j\} \in E$,
  using their edge weights as priorities.
while `queue` *is not empty* **do**
  Remove front tuple (v_i, v_j) from `queue`.
  if `L`$[j] = \infty$ **then**
    Add (v_j, v_k) to `queue` for all $\{v_j, v_k\} \in E$, $k \neq i$.
    Set `L`$[j] \leftarrow$ `L`$[i] + 1$, `pred`$[j] = i$.
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Time analysis: As with breadth-first search, each edge is only processed twice. Processing each edge now takes $\Theta(\log |E|)$ worst-case time, so overall the algorithm runs in $O(|E| \log |E|)$ time. (Note $|E| \geq |V|$.)

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  Remove front tuple (v_i, v_j) from queue.
  if $\mathtt{L}[j] = \infty$ **then**
    Add (v_j, v_k) to queue for all $\{v_j, v_k\} \in E$, $k \neq i$.
    Set $\mathtt{L}[j] \leftarrow \mathtt{L}[i] + 1$, $\mathtt{pred}[j] = i$.
Return pred.

Like with Dijkstra, we could "improve" this to $O(|E| + |V| \log |V|)$ time (with a much worse constant) by using a Fibonacci heap in place of the priority queue.