The union-find data structure
COMS20017 (Algorithms and Data)

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Last time...

A **union-find data structure** supports the following operations:

- `MakeUnionFind`(X): Makes a new union-find data structure containing a 1-element set $\{x\}$ for each element $x \in X$. Takes $O(|X|)$ time.
- `Union`(x, y): Merge the set containing x with the set containing y into a single set in the data structure. Takes $O(\log |X|)$ time.
- `FindSet`(x): Returns a unique identifier for the set containing x. Takes $O(\log |X|)$ time.

Set identifiers can be anything as long as they're unique.

If we implement the sets as linked lists, then `FindSet` is too slow. If we implement them as arrays, then `Union` is too slow.

We'll take the pointer structure of a linked list to make `Union` fast, but arrange it differently to make `FindSet` fast as well.

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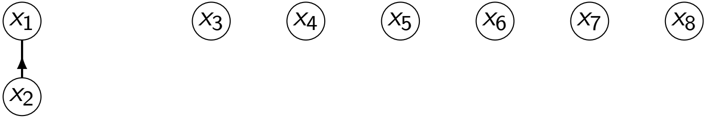

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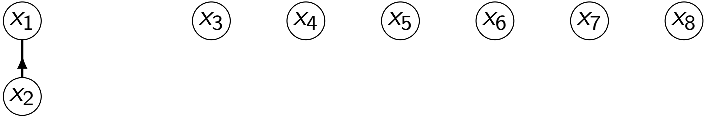

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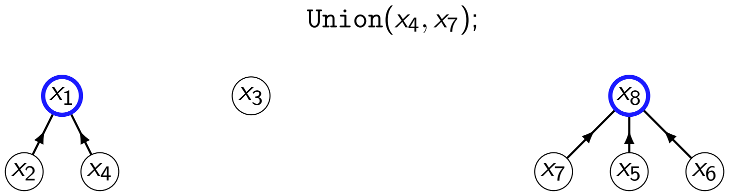

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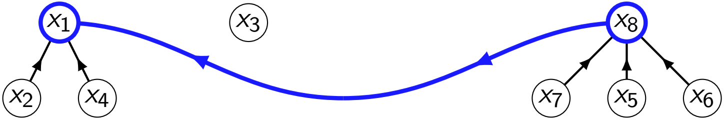

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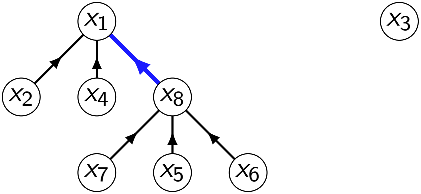

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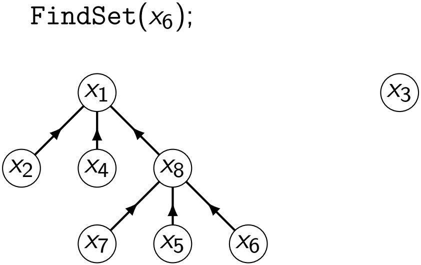

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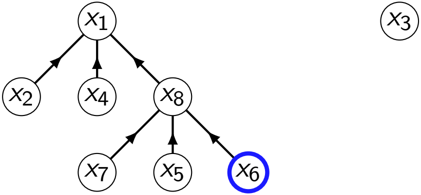

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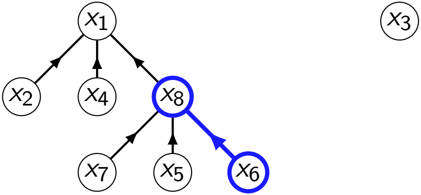

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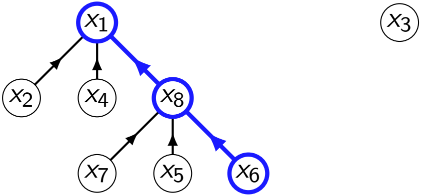

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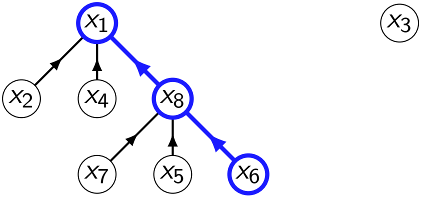

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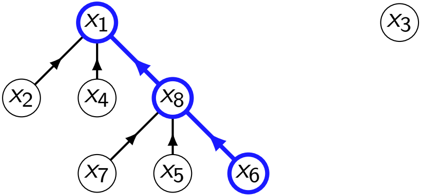

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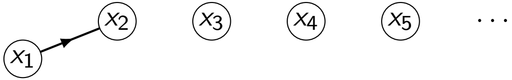

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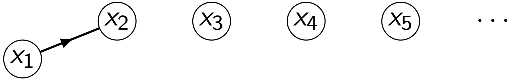

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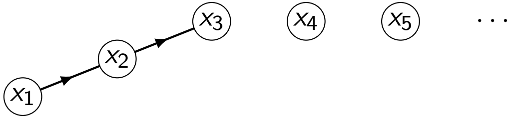

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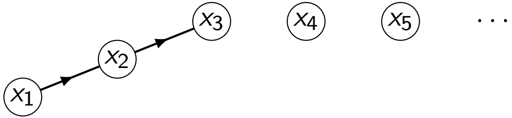

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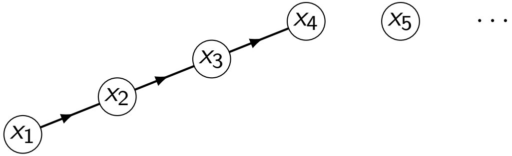

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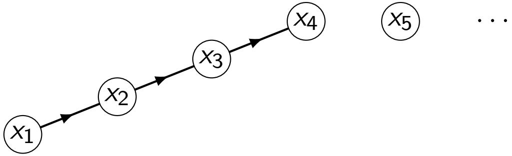

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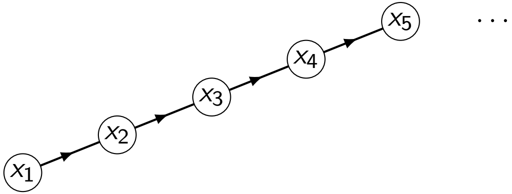

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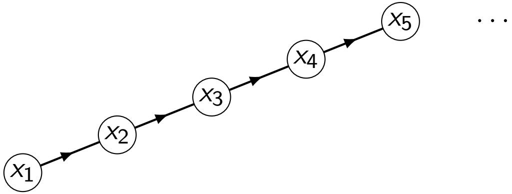

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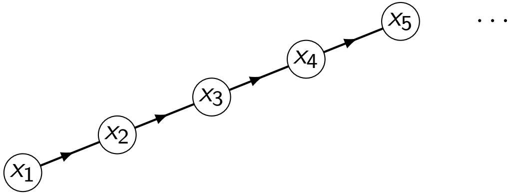

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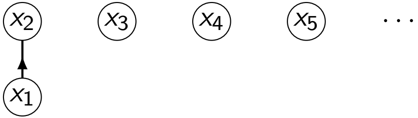

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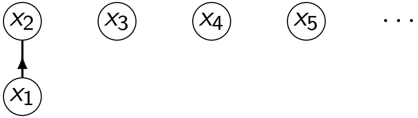

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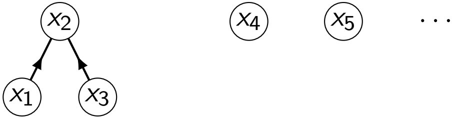

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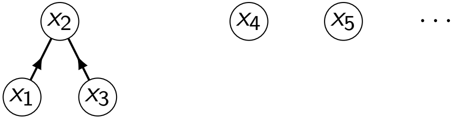

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Forests can degenerate into linked lists

Union and FindSet both take $\Theta(d)$ time, where d is the maximum depth of the tree components involved. How big can this be?

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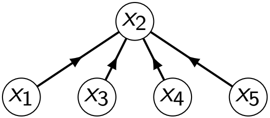

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This means any tree with depth greater than $\log |X|$ would contain more than $2^{\log |X|} = |X|$ vertices, which is impossible! So $d \leq \log |X|$.

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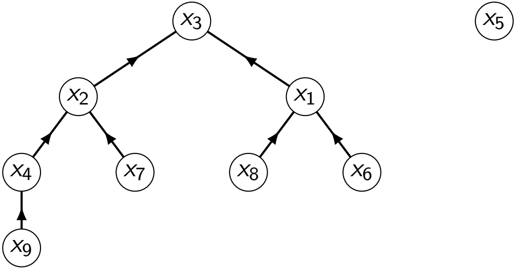
In particular, we can use this to implement Kruskal's algorithm and Borůvka's algorithm in $O(|E| \log |E|)$ time!

A possible improvement: Path compression

Right now, we are duplicating some work with root-finding.



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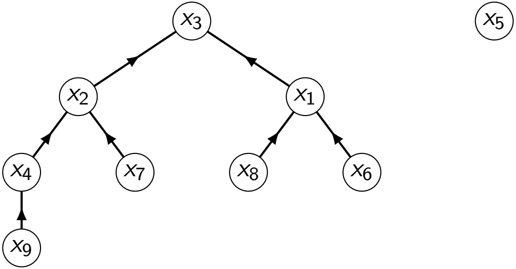

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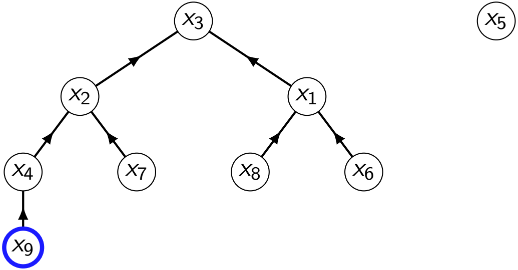

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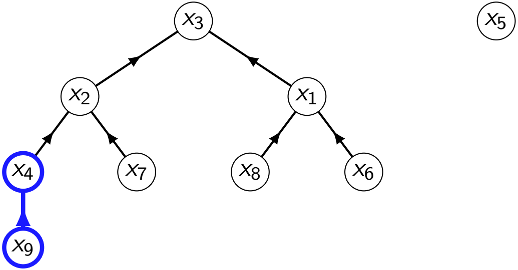

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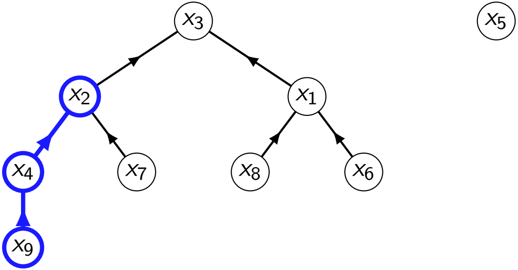

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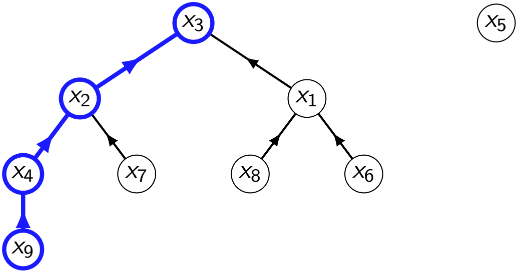

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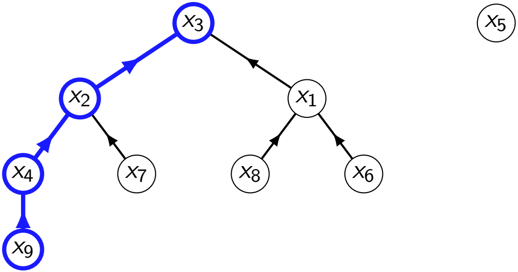

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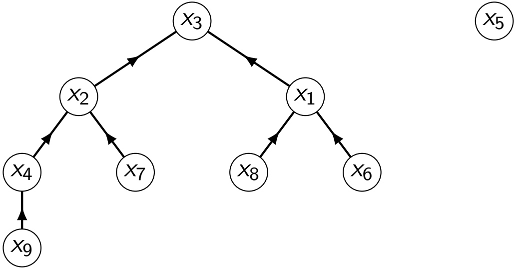

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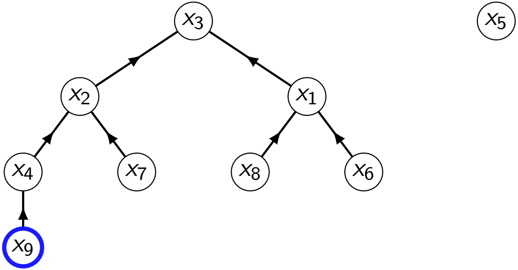

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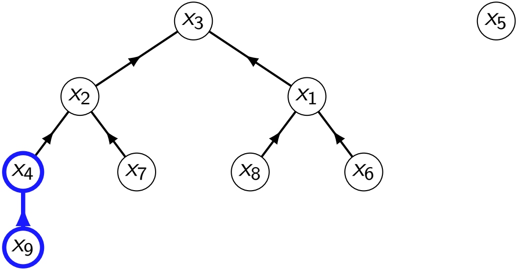

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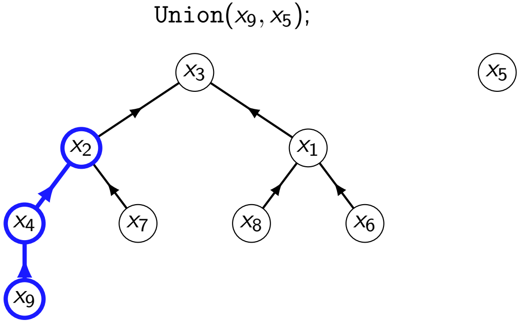

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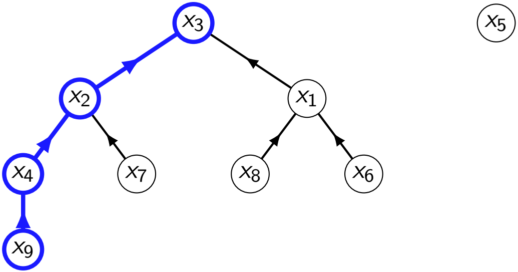

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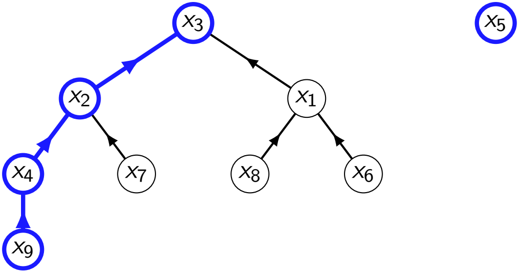

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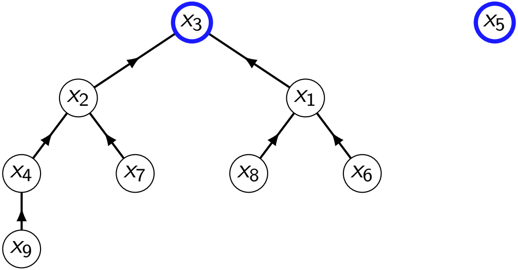

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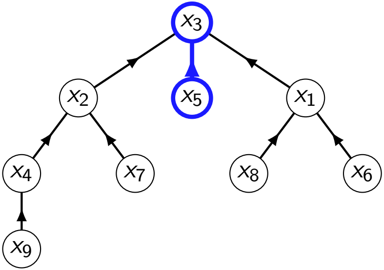

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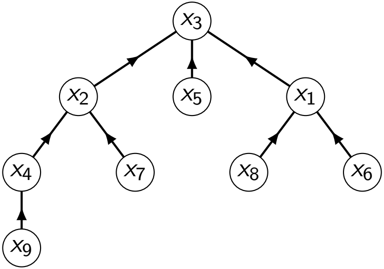
`FindSet`(x_4);



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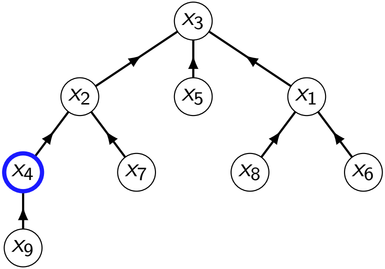

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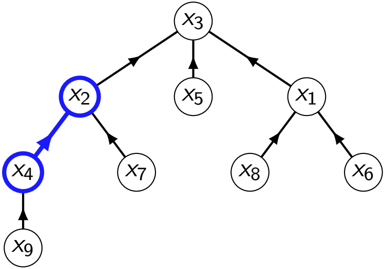

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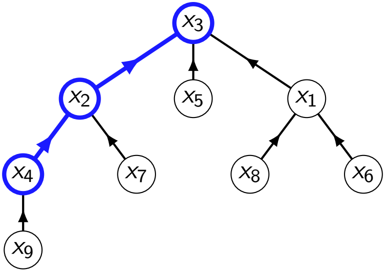

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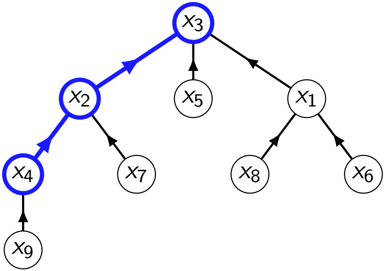

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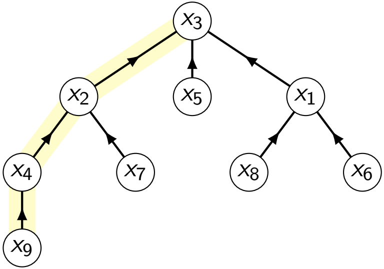
We traverse these edges several times!



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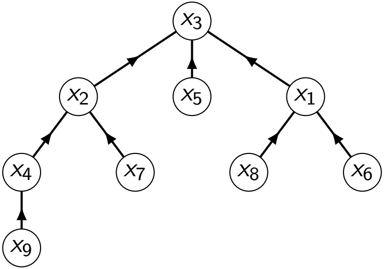

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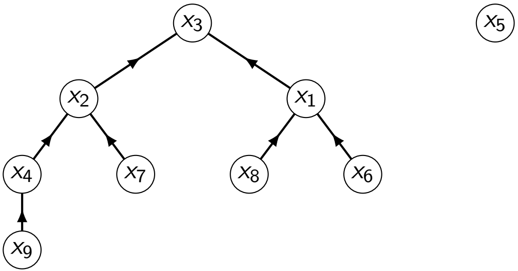

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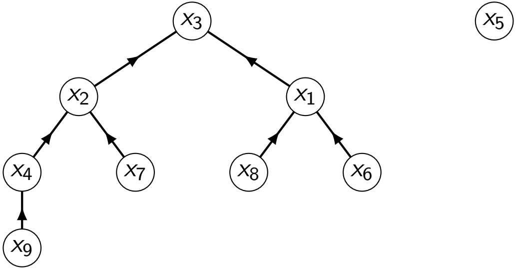

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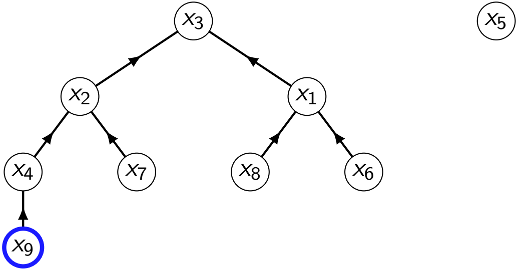

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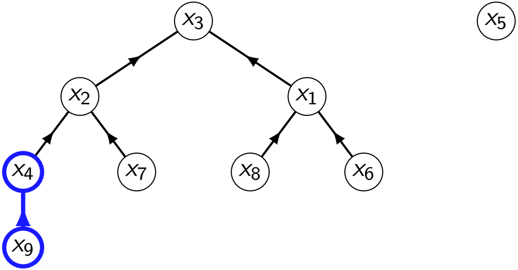

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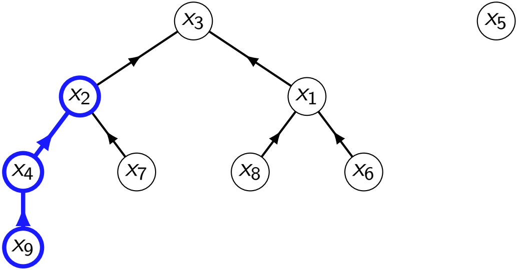

We could fix this by flattening our trees on each `Union` and `FindSet` operation, making every vertex we pass through a child of the root.

This technique is called **path compression**.

A possible improvement: Path compression

Right now, we are duplicating some work with root-finding.

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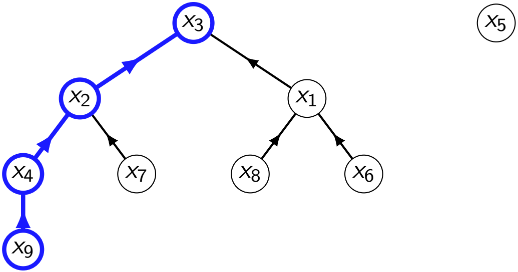

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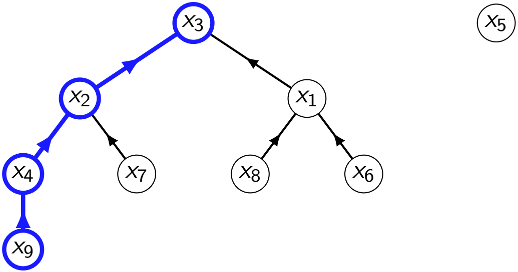

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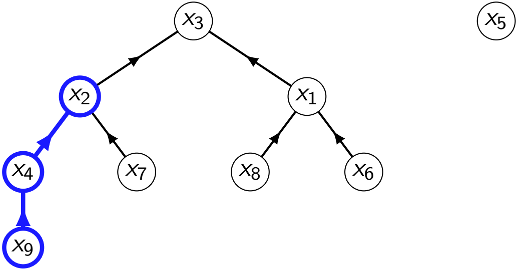

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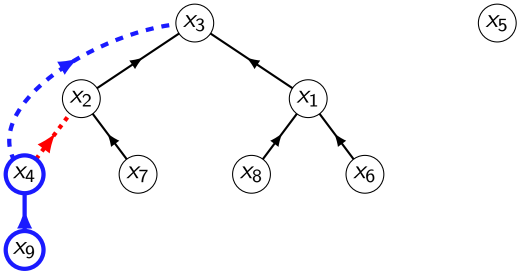

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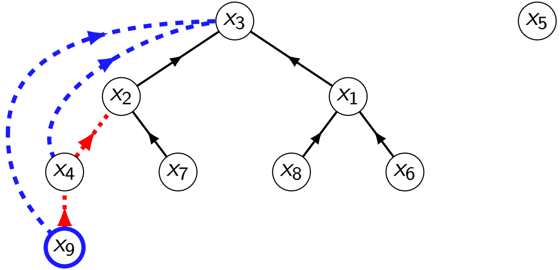

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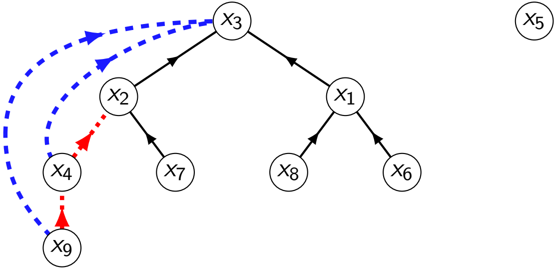

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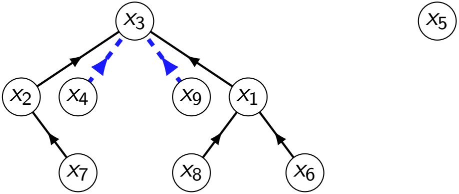

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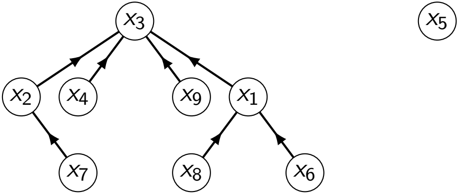

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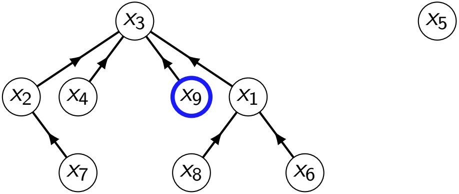

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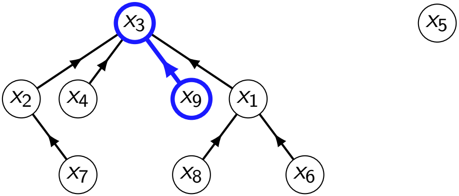

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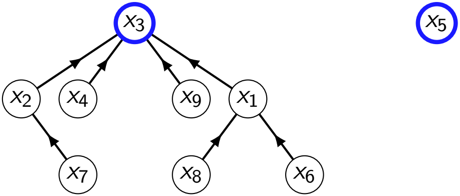

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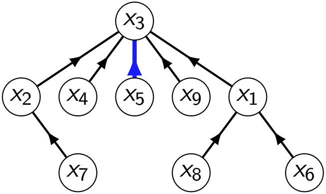

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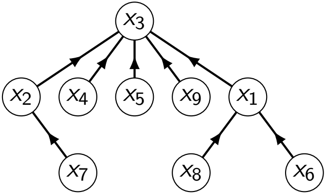

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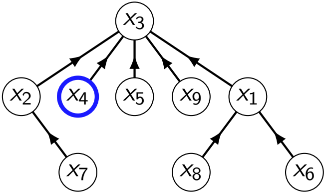

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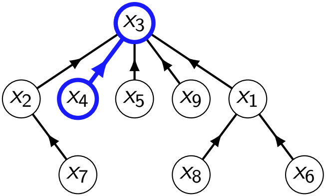

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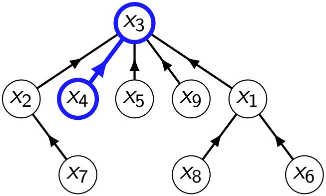

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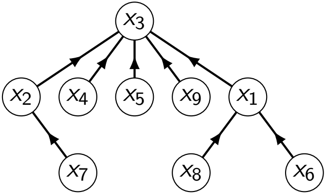

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This improves the running time of n operations from $O(n \log n)$ to $O(n\alpha(n))$, where $\alpha(n)$ is the **inverse Ackermann function**: $\alpha(n) = \min\{k \colon A(k,k) \geq n\}$.

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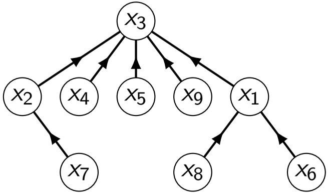

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