

Matchings I: Definitions

COMS20017 (Algorithms and Data)

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Definition by example

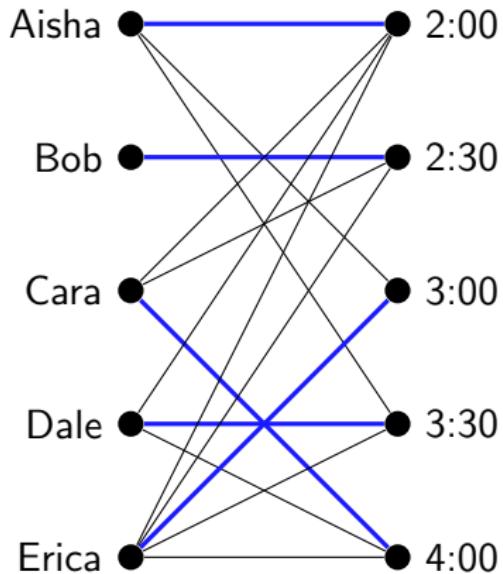
Suppose I'm trying to schedule a sequence of meetings with my tutees. I can only see one of them at once, and each of them can make a different set of times:

Tutee	Available times
Aisha	2:00, 3:00, 3:30
Bob	2:30
Cara	2:00, 2:30, 4:00
Dale	2:00, 3:30, 4:00
Erica	2:00, 2:30, 3:00, 3:30, 4:00

Who should I see when?

We can represent this with a graph:

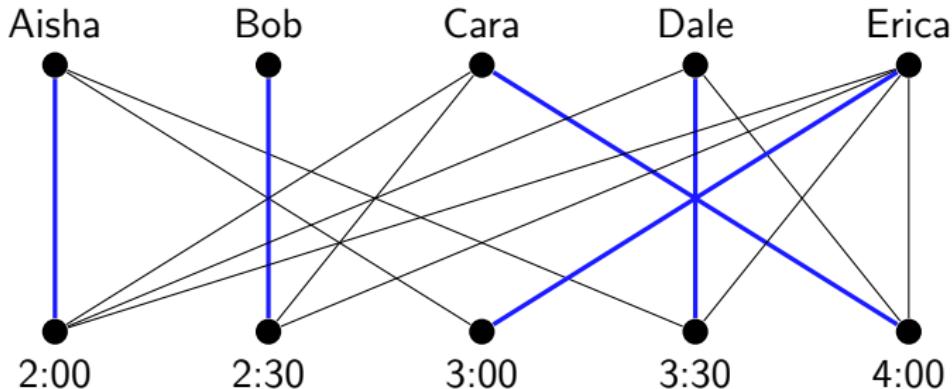
Tutee	Available times
Aisha	<u>2:00</u> , 3:00, 3:30
Bob	<u>2:30</u>
Cara	2:00, 2:30, <u>4:00</u>
Dale	2:00, <u>3:30</u> , 4:00
Erica	2:00, 2:30, <u>3:00</u> , 3:30, 4:00



We join a student to a time if they can make that time.

A valid choice of times corresponds to a **matching** in the graph.

Formal definitions



A **matching** in a graph is a collection of disjoint edges.

It is **perfect** if every vertex is contained in some matching edge.

A graph $G = (V, E)$ is **bipartite** if V can be partitioned into disjoint sets A and B which contain no edges; these are a **bipartition**. Here, we can take:

$$A = \{\text{Aisha, Bob, Cara, Dale, Erica}\}, \quad B = \{2:00, 2:30, 3:00, 3:30, 4:00\}.$$

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Note that bipartitions are not unique! E.g. we could also take:

$$A = \{2:00, 2:30, 3:00, 3:30, 4:00\}, \quad B = \{\text{Aisha, Bob, Cara, Dale, Erica}\}.$$

(**Exercise:** Are there ever more than two possibilities?)

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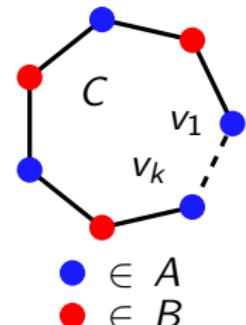
Lemma: G is bipartite if and only if it has no odd-length cycle.

Proof of “only if”: Suppose G has bipartition (A, B) and $C = v_1 \dots v_k$ is an odd cycle in G .

Wlog $v_1 \in A$. Since A has no edges, this means $v_2 \in B$.

Continuing the argument, $v_i \in A$ if i is odd, and $v_i \in B$ if i is even. In particular, $v_k \in A$.

But v_1 and v_k are adjacent, so this is a contradiction. ✓



“If”: See problem sheet.

General problem statement: Given a bipartite graph $G = (V, E)$, output a matching which is as large as possible.

Algorithms for this problem (or generalisations) can be applied to e.g.:

- Scheduling meetings.
- Matching lectures to available rooms.
- Matching people to tasks they're qualified for.
- Matching residents to hospitals. (Must output a “stable” matching.)
- Matching banner ads to viewers. (Must be an “online” algorithm.)
- Matching people as a dating site. (Must handle non-bipartite graphs!)