Breadth-first search
COMS20017 (Algorithms and Data)

John Lapinskas, University of Bristol

Shortest path-finding

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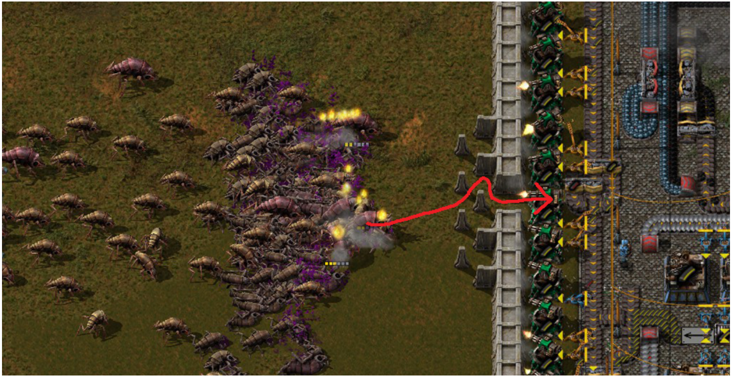

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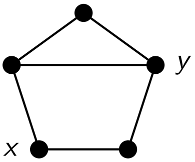

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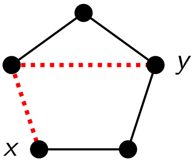

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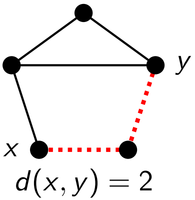

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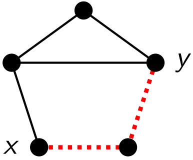

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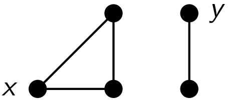

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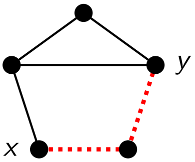

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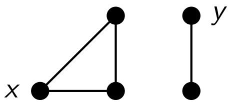

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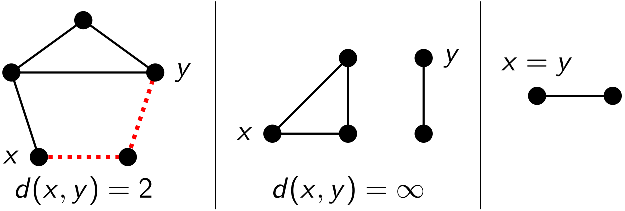

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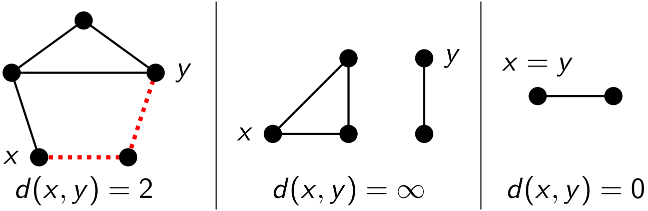

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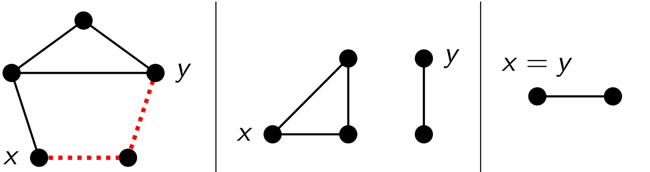

$d(x,y) = 2$ $\quad$ $d(x,y) = \infty$ $\quad$ $d(x,y) = 0$

In directed graphs, it's the same except that the path is **from** x **to** y.
So... we might not have $d(x,y) = d(y,x)$!

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Input: A graph G and two vertices x and y.
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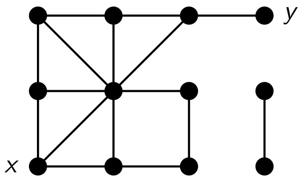

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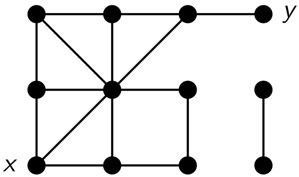

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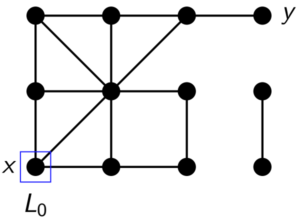

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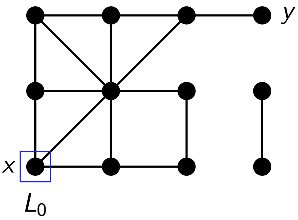

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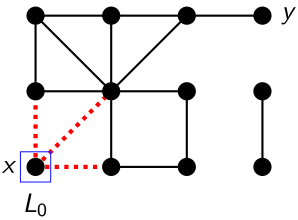

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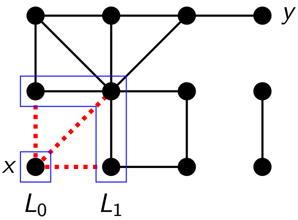

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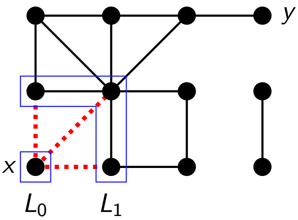

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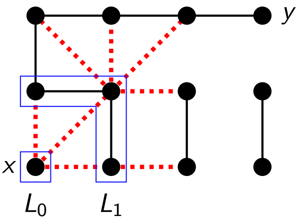

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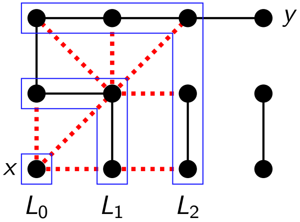

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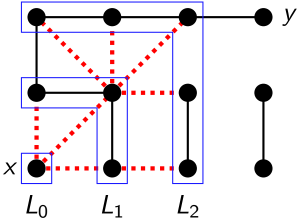

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In general, L_{i+1} is everything adjacent to L_i and not in $L_0 \cup \cdots \cup L_i$.

By continuing this until we find y, keeping track of which edges we use, we get a shortest path to y.

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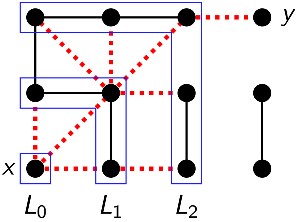

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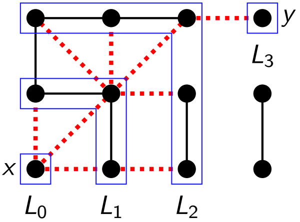

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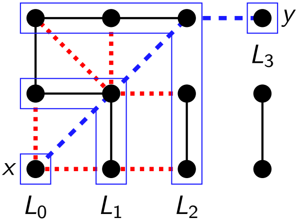

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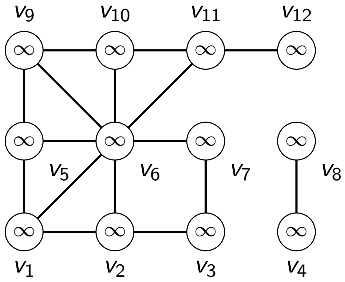

`queue`:

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
Output : $d(v, y)$ for all $y \in V$ and "a way of finding shortest paths".

Number the vertices of G as $v = v_1, \dots, v_n$.
Let $\mathtt{L}[i] \leftarrow \infty$ for all $i \in [n]$.
Let $\mathtt{L}[1] \leftarrow 0$, $\mathtt{pred}[1] \leftarrow \mathtt{None}$.
Let `queue` be a queue containing all tuples (v, v_j) with $\{v, v_j\} \in E$.
while `queue` *is not empty* **do**
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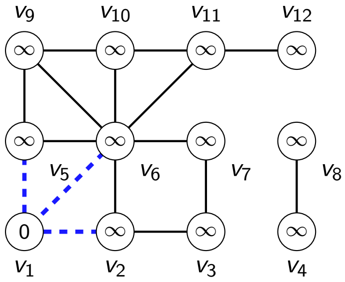

$\mathrm{queue}: (1,5), (1,6), (1,2)$

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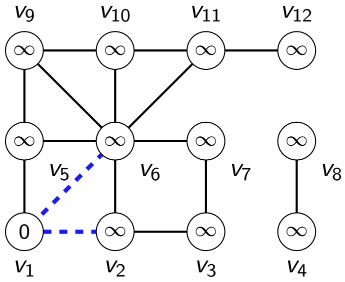

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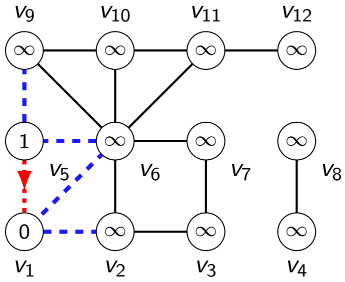

$\mathtt{queue}: (1,6), (1,2), (5,9), (5,6)$

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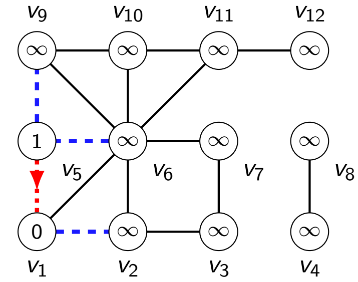

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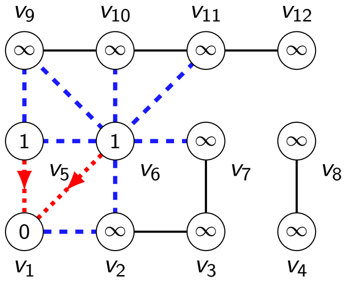

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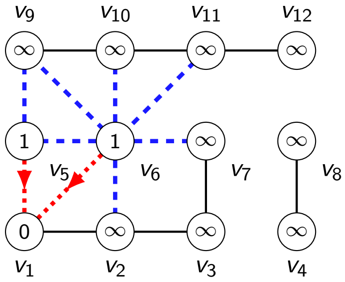

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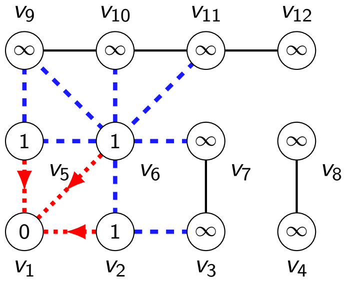

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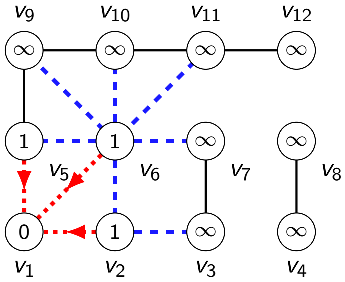
        Add (v_j, v_k) to $\texttt{queue}$ for all $\{v_j, v_k\} \in E$, $k \neq i$.
        Set $\texttt{L}[j] \leftarrow \texttt{L}[i] + 1$, $\texttt{pred}[j] = i$.
Return $\texttt{L}$ and $\texttt{pred}$.



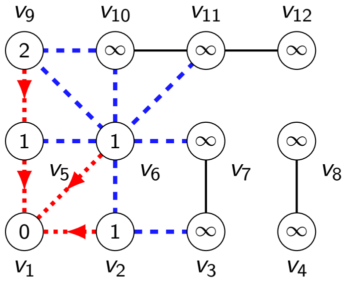
queue : $(5,6),(6,5),(6,9),(6,10),$
$(6,11),(6,7),(6,2),(2,6),$
$(2,3)$

Algorithm: BFS

Input : Graph $G=(V,E)$, vertex $v \in V$.
Output : $d(v,y)$ for all $y \in V$ and "a way of finding shortest paths".

Number the vertices of G as $v = v_1, \dots, v_n$.
Let $\mathtt{L}[i] \leftarrow \infty$ for all $i \in [n]$.
Let $\mathtt{L}[1] \leftarrow 0$, $\mathtt{pred}[1] \leftarrow$ `None`.
Let `queue` be a queue containing all tuples (v, v_j) with $\{v, v_j\} \in E$.
while `queue` *is not empty* **do**
  Remove front tuple (v_i, v_j) from `queue`.
  if $\mathtt{L}[j] = \infty$ **then**
    Add (v_j, v_k) to `queue` for all $\{v_j, v_k\} \in E$, $k \neq i$.
    Set $\mathtt{L}[j] \leftarrow \mathtt{L}[i] + 1$, $\mathtt{pred}[j] = i$.
Return `L` and `pred`.

Breadth-first search: Implementation

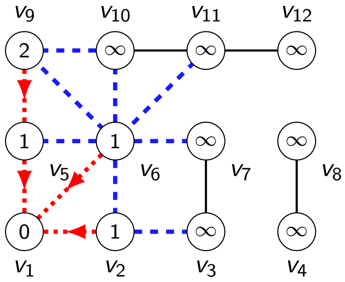


queue : $(5,6),(6,5),(6,9),(6,10),$
$(6,11),(6,7),(6,2),(2,6),$
$(2,3),(9,10),(9,6)$

Algorithm: BFS

Input : Graph $G=(V,E)$, vertex $v \in V$.
Output : $d(v,y)$ for all $y \in V$ and "a way of finding shortest paths".

Number the vertices of G as $v=v_1,\dots,v_n$.
Let $\texttt{L}[i] \leftarrow \infty$ for all $i \in [n]$.
Let $\texttt{L}[1] \leftarrow 0$, $\texttt{pred}[1] \leftarrow \texttt{None}$.
Let queue be a queue containing all tuples (v,v_j) with $\{v,v_j\} \in E$.
while queue *is not empty* **do**
  Remove front tuple (v_i,v_j) from queue.
  if $\texttt{L}[j]=\infty$ **then**
    Add (v_j,v_k) to queue for all $\{v_j,v_k\} \in E$, $k \neq i$.
    Set $\texttt{L}[j] \leftarrow \texttt{L}[i]+1$, $\texttt{pred}[j]=i$.
Return L and pred.



queue: $(6,5),(6,9),(6,10),(6,11),$
$(6,7),(6,2),(2,6),(2,3),$
$(9,10),(9,6)$

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
Output : $d(v, y)$ for all $y \in V$ and "a way of finding shortest paths".

Number the vertices of G as $v = v_1, \dots, v_n$.
Let $\mathtt{L}[i] \leftarrow \infty$ for all $i \in [n]$.
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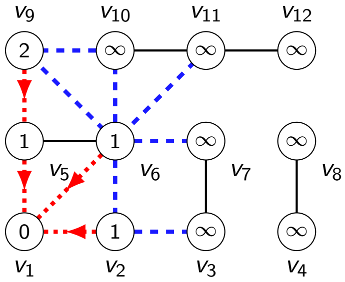
        Set $\mathtt{L}[j] \leftarrow \mathtt{L}[i] + 1$, $\mathtt{pred}[j] = i$.
Return L and pred.

Breadth-first search: Implementation

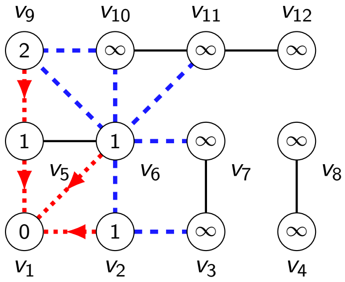


queue : $(6,9),(6,10),(6,11),(6,7),$
$(6,2),(2,6),(2,3),(9,10),$
$(9,6)$

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
Output : $d(v, y)$ for all $y \in V$ and "a way of finding shortest paths".

```
Number the vertices of G as v = v_1, ..., v_n.
Let L[i] ← ∞ for all i ∈ [n].
Let L[1] ← 0, pred[1] ← None.
Let queue be a queue containing all tuples
  (v, v_j) with {v, v_j} ∈ E.
while queue is not empty do
    Remove front tuple (v_i, v_j) from queue.
    if L[j] = ∞ then
        Add (v_j, v_k) to queue for all
          {v_j, v_k} ∈ E, k ≠ i.
        Set L[j] ← L[i] + 1, pred[j] = i.
Return L and pred.
```



$\texttt{queue}: (6,10), (6,11), (6,7), (6,2),$
$(2,6), (2,3), (9,10), (9,6)$

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
Output : $d(v, y)$ for all $y \in V$ and "a way of finding shortest paths".

Number the vertices of G as $v = v_1, \ldots, v_n$.
Let $\texttt{L}[i] \leftarrow \infty$ for all $i \in [n]$.
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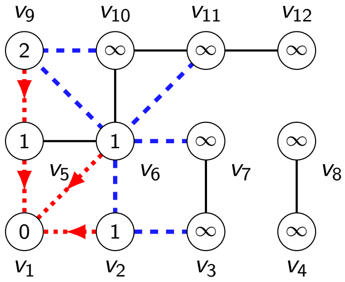
        Set $\texttt{L}[j] \leftarrow \texttt{L}[i] + 1$, $\texttt{pred}[j] = i$.
Return $\texttt{L}$ and $\texttt{pred}$.

Breadth-first search: Implementation



$\texttt{queue}: (6,11), (6,7), (6,2), (2,6),$
$(2,3), (9,10), (9,6)$

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
Output : $d(v, y)$ for all $y \in V$ and "a way of finding shortest paths".

Number the vertices of G as $v = v_1, \ldots, v_n$.
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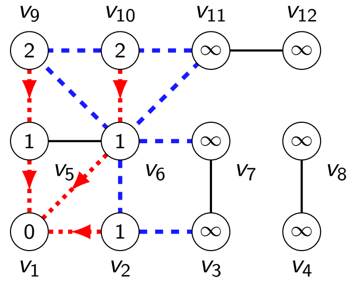
Return $\texttt{L}$ and $\texttt{pred}$.

Breadth-first search: Implementation



queue : $(6,11),(6,7),(6,2),(2,6),$
$(2,3),(9,10),(9,6),(10,11),$
$(10,9)$

Algorithm: BFS

Input : Graph $G=(V,E)$, vertex $v \in V$.
Output : $d(v,y)$ for all $y \in V$ and "a way of finding shortest paths".

```
Number the vertices of G as v = v_1, ..., v_n.
Let L[i] ← ∞ for all i ∈ [n].
Let L[1] ← 0, pred[1] ← None.
Let queue be a queue containing all tuples
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          {v_j, v_k} ∈ E, k ≠ i.
        Set L[j] ← L[i] + 1, pred[j] = i.
Return L and pred.
```

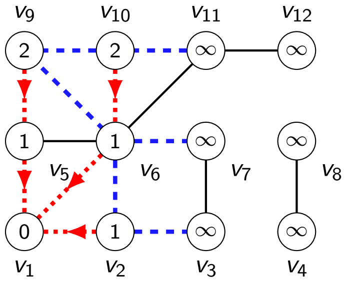

$\mathtt{queue}: (6,7), (6,2), (2,6), (2,3),$
$(9,10), (9,6), (10,11), (10,9)$

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
Output : $d(v, y)$ for all $y \in V$ and "a way of finding shortest paths".

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Number the vertices of G as v = v_1, ..., v_n.
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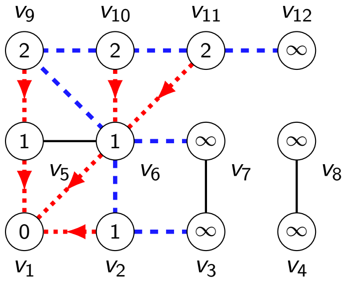
        Set L[j] ← L[i] + 1, pred[j] = i.
Return L and pred.
```



$\texttt{queue}: (6,7), (6,2), (2,6), (2,3),$
$(9,10), (9,6), (10,11), (10,9),$
$(11,10), (11,12)$

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
Output : $d(v, y)$ for all $y \in V$ and "a way of finding shortest paths".

```
Number the vertices of G as v = v_1, ..., v_n.
Let L[i] ← ∞ for all i ∈ [n].
Let L[1] ← 0, pred[1] ← None.
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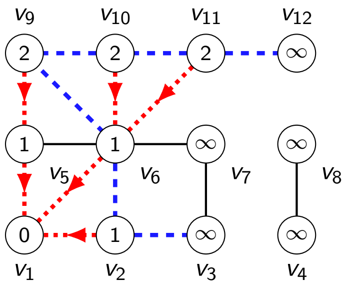
        Set L[j] ← L[i] + 1, pred[j] = i.
Return L and pred.
```



$\texttt{queue}: (6,2), (2,6), (2,3), (9,10),$
$(9,6), (10,11), (10,9), (11,10),$
$(11,12)$

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
Output : $d(v, y)$ for all $y \in V$ and "a way of finding shortest paths".

```
Number the vertices of G as v = v_1, ..., v_n.
Let L[i] ← ∞ for all i ∈ [n].
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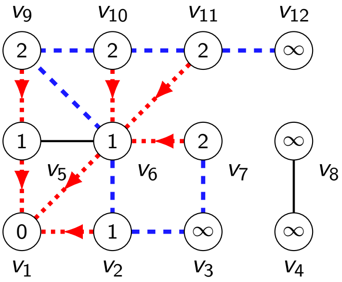
        Add (v_j, v_k) to queue for all
          {v_j, v_k} ∈ E, k ≠ i.
        Set L[j] ← L[i] + 1, pred[j] = i.
Return L and pred.
```

Breadth-first search: Implementation



queue : $(6,2),(2,6),(2,3),(9,10),$
$(9,6),(10,11),(10,9),(11,10),$
$(11,12),(7,3)$

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
Output : $d(v, y)$ for all $y \in V$ and "a way of finding shortest paths".

Number the vertices of G as $v = v_1, \dots, v_n$.
Let $\mathtt{L}[i] \leftarrow \infty$ for all $i \in [n]$.
Let $\mathtt{L}[1] \leftarrow 0$, $\mathtt{pred}[1] \leftarrow \mathtt{None}$.
Let `queue` be a queue containing all tuples (v, v_j) with $\{v, v_j\} \in E$.
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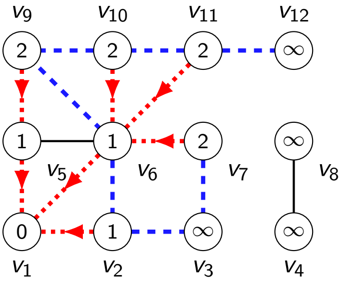
    Remove front tuple (v_i, v_j) from `queue`.
    if $\mathtt{L}[j] = \infty$ **then**
        Add (v_j, v_k) to `queue` for all $\{v_j, v_k\} \in E$, $k \neq i$.
        Set $\mathtt{L}[j] \leftarrow \mathtt{L}[i] + 1$, $\mathtt{pred}[j] = i$.
Return `L` and `pred`.

Breadth-first search: Implementation



queue : $(2,6),(2,3),(9,10),(9,6),$
$(10,11),(10,9),(11,10),(11,12),$
$(7,3)$

Algorithm: BFS

Input : Graph $G=(V,E)$, vertex $v \in V$.
Output : $d(v,y)$ for all $y \in V$ and "a way of finding shortest paths".

```
Number the vertices of G as v = v_1, ..., v_n.
Let L[i] ← ∞ for all i ∈ [n].
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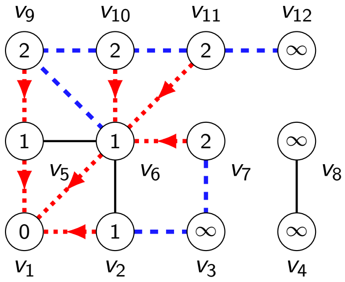
          {v_j, v_k} ∈ E, k ≠ i.
        Set L[j] ← L[i] + 1, pred[j] = i.
Return L and pred.
```



queue: $(2,3), (9,10), (9,6), (10,11),$
$(10,9), (11,10), (11,12), (7,3)$

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
Output : $d(v, y)$ for all $y \in V$ and "a way of finding shortest paths".

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Number the vertices of G as v = v_1, ..., v_n.
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        Set L[j] ← L[i] + 1, pred[j] = i.
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```
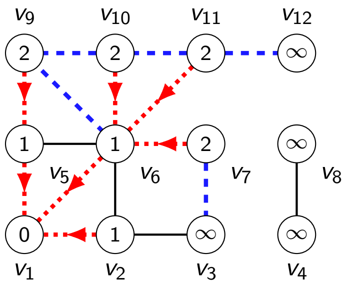

Breadth-first search: Implementation



$\text{queue}: (9,10),(9,6),(10,11),(10,9),$
$(11,10),(11,12),(7,3)$

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
Output : $d(v, y)$ for all $y \in V$ and "a way of finding shortest paths".

Number the vertices of G as $v = v_1, \dots, v_n$.
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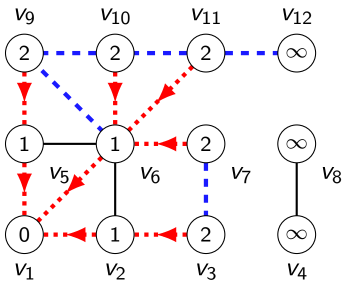
    Set $\mathtt{L}[j] \leftarrow \mathtt{L}[i] + 1$, $\mathtt{pred}[j] = i$.
Return `L` and `pred`.

Breadth-first search: Implementation



$\texttt{queue}: (9,10),(9,6),(10,11),(10,9),$
$(11,10),(11,12),(7,3),(3,7)$

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
Output : $d(v, y)$ for all $y \in V$ and "a way of finding shortest paths".

Number the vertices of G as $v = v_1, \dots, v_n$.
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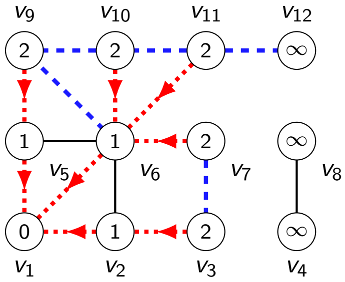
        Set $\texttt{L}[j] \leftarrow \texttt{L}[i] + 1$, $\texttt{pred}[j] = i$.
Return $\texttt{L}$ and $\texttt{pred}$.



$\text{queue}: (9,6),(10,11),(10,9),(11,10),$
$(11,12),(7,3),(3,7)$

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
Output : $d(v, y)$ for all $y \in V$ and "a way of finding shortest paths".

```
Number the vertices of G as v = v_1, ..., v_n.
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Let L[1] ← 0, pred[1] ← None.
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        Set L[j] ← L[i] + 1, pred[j] = i.
Return L and pred.
```

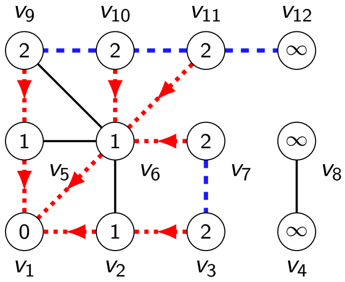

$\texttt{queue}: (10, 11), (10, 9), (11, 10), (11, 12),$
$(7, 3), (3, 7)$

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
Output : $d(v, y)$ for all $y \in V$ and "a way of finding shortest paths".

Number the vertices of G as $v = v_1, \dots, v_n$.
Let $\texttt{L}[i] \leftarrow \infty$ for all $i \in [n]$.
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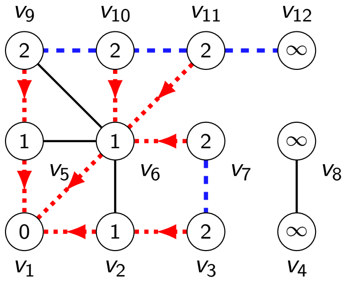
Return $\texttt{L}$ and $\texttt{pred}$.



queue : $(10, 9), (11, 10), (11, 12), (7, 3),$
$(3, 7)$

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
Output : $d(v, y)$ for all $y \in V$ and "a way of finding shortest paths".

Number the vertices of G as $v = v_1, \dots, v_n$.
Let $\mathtt{L}[i] \leftarrow \infty$ for all $i \in [n]$.
Let $\mathtt{L}[1] \leftarrow 0$, $\mathtt{pred}[1] \leftarrow \mathtt{None}$.
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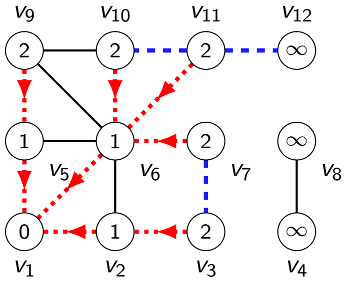
    Set $\mathtt{L}[j] \leftarrow \mathtt{L}[i] + 1$, $\mathtt{pred}[j] = i$.
Return $\mathtt{L}$ and $\mathtt{pred}$.

Breadth-first search: Implementation



queue : $(11, 10), (11, 12), (7, 3), (3, 7)$

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
Output : $d(v, y)$ for all $y \in V$ and "a way of finding shortest paths".

Number the vertices of G as $v = v_1, \dots, v_n$.
Let $\texttt{L}[i] \leftarrow \infty$ for all $i \in [n]$.
Let $\texttt{L}[1] \leftarrow 0$, $\texttt{pred}[1] \leftarrow \texttt{None}$.
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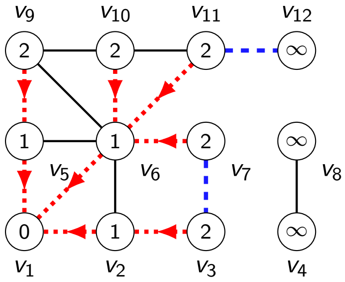
Return L and pred.



queue : $(11, 12), (7, 3), (3, 7)$

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
Output : $d(v, y)$ for all $y \in V$ and "a way of finding shortest paths".

```
Number the vertices of G as v = v_1, ..., v_n.
Let L[i] ← ∞ for all i ∈ [n].
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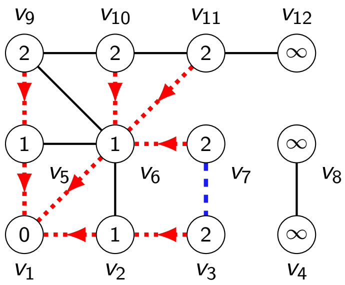
Breadth-first search: Implementation



$\texttt{queue}: (7,3), (3,7)$

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
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Number the vertices of G as $v = v_1, \dots, v_n$.
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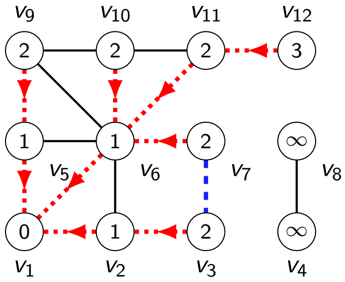

queue : $(7,3),(3,7)$

Algorithm: BFS

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```

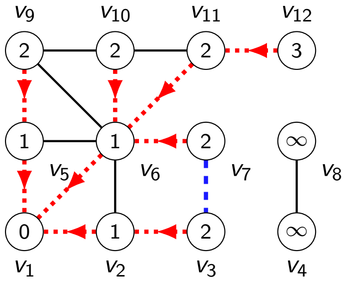

queue: $(3, 7)$

Algorithm: BFS

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Number the vertices of G as $v = v_1, \dots, v_n$.
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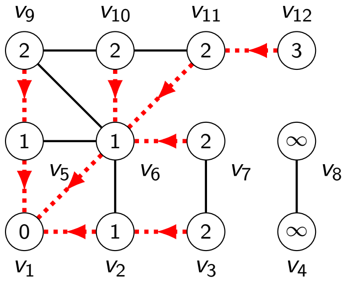
Breadth-first search: Implementation



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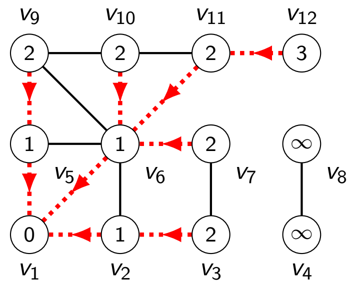

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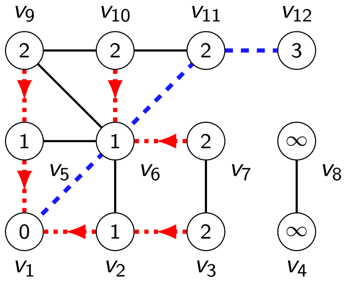
Return $\mathtt{L}$ and $\mathtt{pred}$.

In the output, $\mathtt{L}[i] = d(v, v_i)$. By following edges back from v_i via $\mathtt{pred}$, we can also quickly reconstruct a shortest path from v to v_i.

Breadth-first search: Implementation



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Let $\mathtt{L}[i] \leftarrow \infty$ for all $i \in [n]$.
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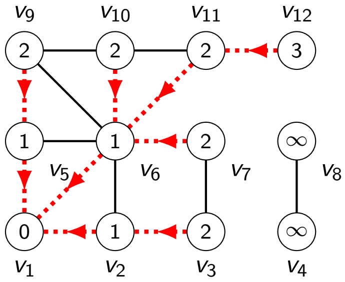
In the output, $\mathtt{L}[i] = d(v, v_i)$. By following edges back from v_i via $\mathtt{pred}$, we can also quickly reconstruct a shortest path from v to v_i.

E.g. $v_1 v_6 v_{11} v_{12}$ is a shortest path from v_1 to v_{12}.

Breadth-first search: Implementation



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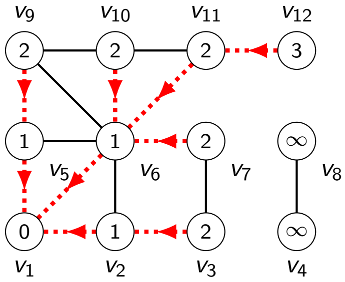
Return `L` and `pred`.

Time analysis: If G is in adjacency list form, each edge is added to `queue` at most twice, incurring $O(1)$ overhead each time, so the running time is $O(|V| + |E|)$.

Breadth-first search: Implementation



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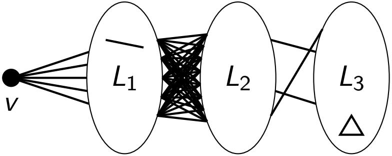
Return L and pred.

Important: There is a significant **space** inefficiency in this version of breadth-first search! See example sheet.

BFS trees

Definition: A **BFS tree** T of G is a rooted tree (call its root x) with:

1. $V(T)$ is the vertex set of a component of G;
2. The i'th layer of T is $\{x\colon d_G(x,v) = i\}$;
3. If $\{x,y\} \in E(G)$, then $|d_G(v,x) - d_G(v,y)| \leq 1$, i.e. x and y must be in the same or adjacent layers of T.



Theorem: The tree of edges from `pred` is always a BFS tree.

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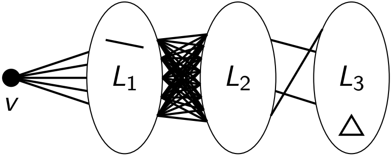

Theorem: The tree of edges from `pred` is always a BFS tree.

Proof: We already proved (1) and (2), so suppose $\{x,y\} \in E(G)$.

If P is a shortest path from v to x, then Pxy is a path from v to y, so $d(v,y) \leq d(v,x) + 1$. Likewise $d(v,x) \leq d(v,y) + 1$. ✓