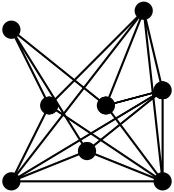
Shaking hands
COMS20017 (Algorithms and Data)

John Lapinskas, University of Bristol

The Handshake Lemma



Counting the edges of this graph seems unpleasant...
but adding up the vertex degrees would be much easier.

Lemma: For any graph $G = (V, E)$, $\sum_{v \in V} d(v) = 2|E|$.

Proof: All edges contain two vertices, and each vertex v is in $d(v)$ edges. Count the number of vertex-edge pairs: Let $X = \{(v, e) \in V \times E \colon v \in E\}$. Then $|X| = 2|E|$ and $|X| = \sum_{v \in V} d(v)$, so we're done. $\square$.

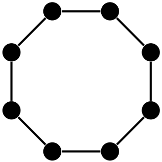
(This proof idea is called **double-counting**.)

Here, $\displaystyle\sum_v d(v) = 3 + 5 + 4 + 4 + 5 + 4 + 5 + 6 = 36,$ so 18 edges total.

Example applications

Handshake Lemma: For any graph $G = (V, E)$, $\sum_{v \in V} d(v) = 2|E|$.

Question: How many edges does an n-vertex cycle have?



Answer: Every vertex has degree 2, so

$$\#(\text{edges}) = \frac{1}{2} \sum_{v} d(v) = \frac{1}{2} \cdot n \cdot 2 = n.$$

A graph is ***k*-regular** if every vertex has degree k (so cycles are 2-regular).

Question: Are there 3-regular graphs $G = (V, E)$ with $|V|$ odd?

Answer: No, as then $\sum_{v \in V} d(v)$ would be $3|V|$ (which is odd).

$2|E|$ is even, so this can't happen.

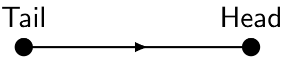
Handshake Lemma: For any graph $G = (V, E)$, $\sum_{v \in V} d(v) = 2|E|$.

In **directed** graphs, can we express the number of edges in terms of in- and out-degrees? Yes!

Directed Handshake Lemma:

For any digraph $G = (V, E)$, $\sum_{v \in V} d^+(v) = \sum_{v \in V} d^-(v) = |E|$.

Proof: Terminology: we call the first vertex in a directed edge the **tail**, and the second vertex the **head**. (Matching the direction of the arrow!)



Instead of counting vertex-edge pairs, we count tail-edge pairs.

So let $X = \{(v, e) \in V \times E \colon e = (v, w) \text{ for some } w\}$.
Each edge has one tail, so $|X| = |E|$.
And each vertex v is the tail of $d^+(v)$ edges, so $|X| = \sum_{v \in V} d^+(v)$.

Similarly, counting head-edge pairs gives $\sum_{v \in V} d^-(v) = |E|$. $\square$