

Proof by induction (recap)

COMS20017 (Algorithms and Data)

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Write $S(n)$ for the statement that $\sum_{i=0}^n x^i = \frac{x^{n+1} - 1}{x - 1}$ for all $x \neq 1$.

So for example, $S(0)$ says that $x^0 = \frac{x^{0+1} - 1}{x - 1} = 1$ for all x , which is true.

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Rather than proving $S(n)$ for all n , we write an algorithm which, given n as an input, outputs a proof of $S(n)$. So since we can prove $S(n)$ for all n , it must hold for all n !

Let $S(n)$ be the statement that $\sum_{i=0}^n x^i = \frac{x^{n+1}-1}{x-1}$ for all $x \neq 1$.

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The “program” has two subroutines:

- `BaseCase()` outputs a proof of $S(0)$;
- `InductiveStep(k)` outputs a proof of $S(k + 1)$ from $S(0), \dots, S(k)$.

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And the pseudocode reads:

Input: An integer $n \geq 0$.

Output: A proof of $S(n)$.

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1 begin
2   Output BaseCase().
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But this program is the same for every induction proof, so we only have to specify `BaseCase` and `InductiveStep`.

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And so we're done!



Strong induction

Let $S(n)$ be the statement that $\sum_{i=0}^n x^i = \frac{x^{n+1}-1}{x-1}$ for all $x \neq 1$.

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For example, this proof of the inductive step is also valid:

$$\sum_{i=0}^{k+1} x^i = \sum_{i=0}^{\lfloor k/2 \rfloor} x^i + \sum_{i=\lfloor k/2 \rfloor + 1}^{k+1} x^i$$

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This technique is sometimes called **strong induction**.

Another example

Consider the following (awful) pseudocode for a function $\text{Increment}(y)$:

Input: An integer $y > 0$.

Output: $y + 1$.

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1 begin
2   if  $y = 0$  then
3     Return 1.
4   else if  $y \pmod 2 = 0$  then
5     Return  $y + 1$ .
6   else
7     Return  $2 \cdot \text{Increment}(\lfloor y/2 \rfloor)$ .
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Base case: If $y = 0$, we return $1 = y + 1$. ✓

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Inductive step: If y is even, we return $y + 1$. ✓

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Inductive step: If y is even, we return $y + 1$. ✓

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Inductive step: If y is even, we return $y + 1$. ✓

Otherwise, by induction, we return $2(\lfloor y/2 \rfloor + 1)$.

Writing $y = 2z + 1$, we have $\lfloor y/2 \rfloor = z$, so this is $2(z + 1) = y + 1$. □

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Input: Integers $m, n \geq 0$.

Output: A proof of $S(m, n)$.

```
1 begin
2   if  $m = 0$  or  $n = 0$  then
3     Output BaseCase().
4   else
5     Output Proof( $m - 1, n - 1$ ).
6     Output InductiveStep( $m, n$ ).
7   Halt.
```

Other induction schemes

This corresponds to an induction proof of:

- **Base case:** Prove $S(m, 0)$ for all m and $S(0, n)$ for all n ;
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This is valid! There are lots of other induction schemes, e.g.:

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 - $S(m, n - 1)$ when $m = 0$ and $n \geq 1$;
 - $S(m - 1, n)$ when $m \geq 1$ and $n = 0$.
- For a one-variable statement $S(n)$: The base cases are $S(0)$ and $S(1)$, and the inductive step proves $S(n)$ from $S(n - 2)$ for all $n \geq 2$.

Other induction schemes

Without getting into foundational stuff: if you can put your induction proof in the form of an “induction program” that will output a (finite!) proof for any parameter choice, then you have a valid proof by induction.

This is useful in dealing with functions on multiple variables, or complicated structures that can be broken down into simpler parts.

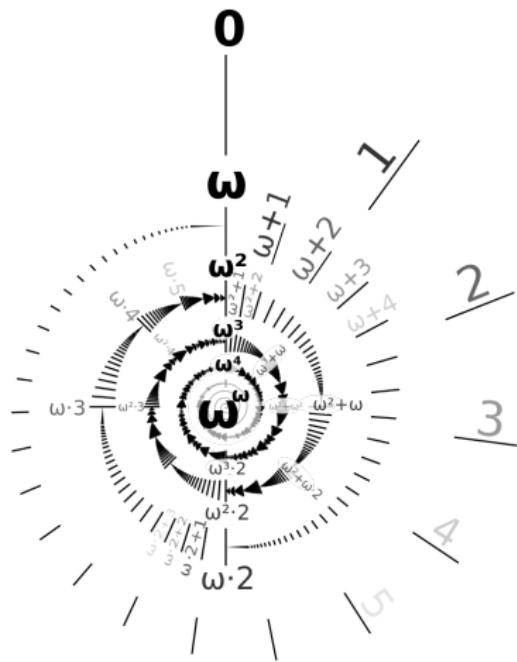
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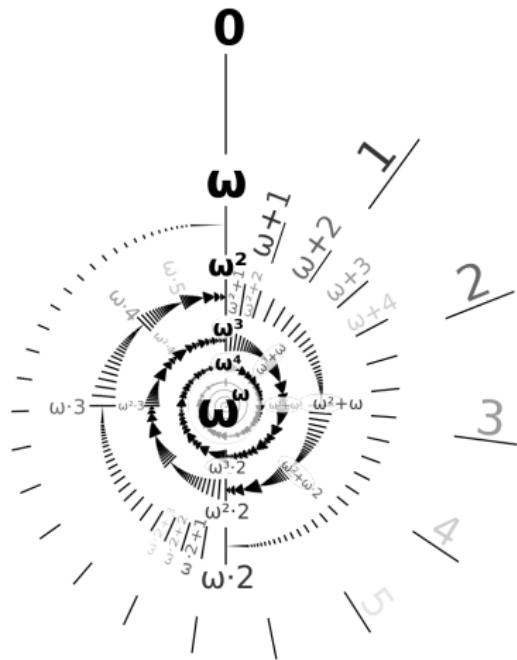
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But beware subtle errors! (See the quiz...)

Non-examinable: transfinite induction



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Just say no.