

Making Kruskal's algorithm fast

COMS20017 (Algorithms and Data)

John Lapinskas, University of Bristol

Implementing Kruskal's algorithm

Algorithm: KRUSKAL

Input : Connected weighted graph $G = ((V, E), w)$ in adjacency list form.

Output : A minimum spanning tree for G .

- 1 Sort the edges by weight as e_1, \dots, e_m , with $w(e_1) \leq \dots \leq w(e_m)$.
 - 2 Let $T \leftarrow (V, \emptyset)$ be the empty tree on V .
 - 3 **for** $i = 1$ to m **do**
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Lines 1, 2 and 6 take $O(|E| \log |E|)$ time, and lines 3–5 repeat $|E|$ times.

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Lines 1, 2 and 6 take $O(|E| \log |E|)$ time, and lines 3–5 repeat $|E|$ times.

We *could* implement line 4 with BFS... but this would take $\Theta(|E|)$ time, giving us a worst-case running time of $\Theta(|E|^2)$. That's bad.

Implementing Kruskal's algorithm: Take 2

Idea: Joining two tree components with an edge will never add a cycle, and adding an edge inside a tree component will always add one.

So when we consider an edge e_i to T , we just need to make sure both endpoints aren't in the same component — this implementation will work:

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A linked list for each component? Then merging will take $O(1)$ time, but finding C_1 and C_2 could take $\Omega(|V|)$ time, giving a runtime of $\Omega(|V||E|)$.

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A linked list for each component? Then merging will take $O(1)$ time, but finding C_1 and C_2 could take $\Omega(|V|)$ time, giving a runtime of $\Omega(|V||E|)$.

An array for each component? Then finding C_1 and C_2 will take $O(1)$ time, but merging will take $\Omega(|V|)$, so we still get $\Omega(|V||E|)$ overall...

The solution

We need to use a **union-find** data structure, also known as a **disjoint-set** or **merge-find** data structure. It supports the following operations:

- $\text{MakeUnionFind}(X)$: Makes a new union-find data structure containing a 1-element set $\{x\}$ for each element $x \in X$.
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$\text{MakeUnionFind}(v_1, v_2, v_3, v_4, v_5, v_6);$

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1	2	3	4	5	6
$\{v_1\}$	$\{v_2\}$	$\{v_3\}$	$\{v_4\}$	$\{v_5\}$	$\{v_6\}$

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$\text{FindSet}(v_5);$ Returns 5.

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42 $\{v_1, v_2, v_4\}$	7 $\{v_3, v_5\}$	6 $\{v_6\}$
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MakeUnionFind takes $O(|X|)$ time, and Union and FindSet take $O(\log |X|)$ time. (It is also possible to add elements dynamically, but we won't need to.) So if we use this for \mathcal{C} ...

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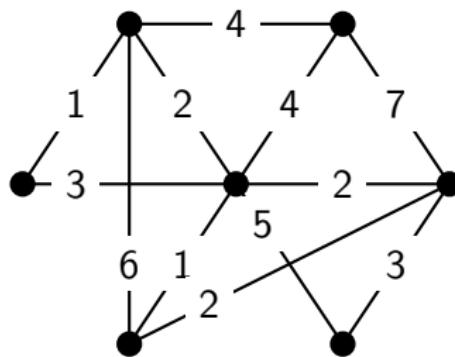
So overall, since G is connected and $|E| \geq |V| - 1$, the running time is $O(|E| \log |V|)$ — exactly what we got from Prim's algorithm!

Non-examinable: Borůvka's algorithm

Neither Kruskal's algorithm and Prim's algorithm parallelise effectively.

But Borůvka's original algorithm, from 40 years earlier, works nicely.

At each step, it **simultaneously** finds and adds the cheapest edge out of **each component** of the output tree T .

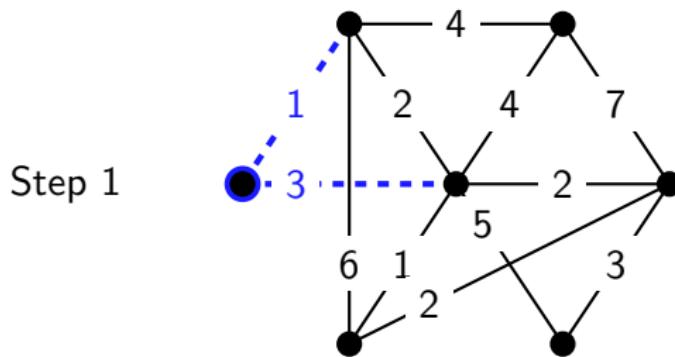


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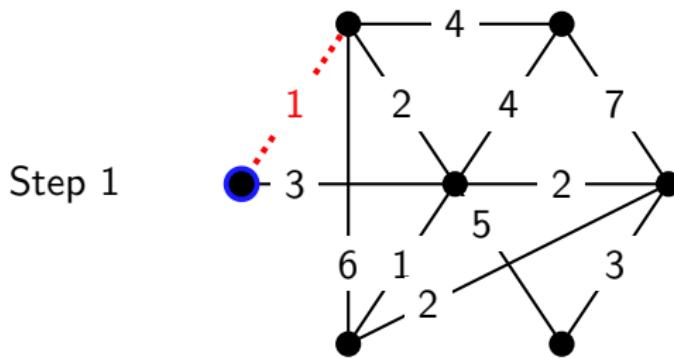


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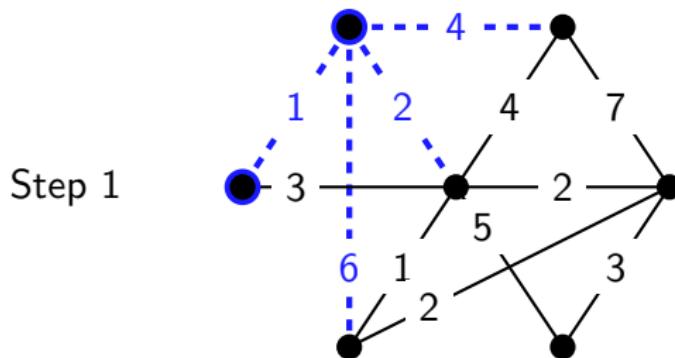


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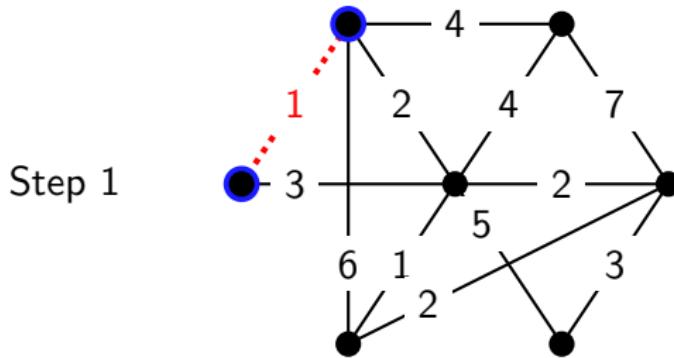


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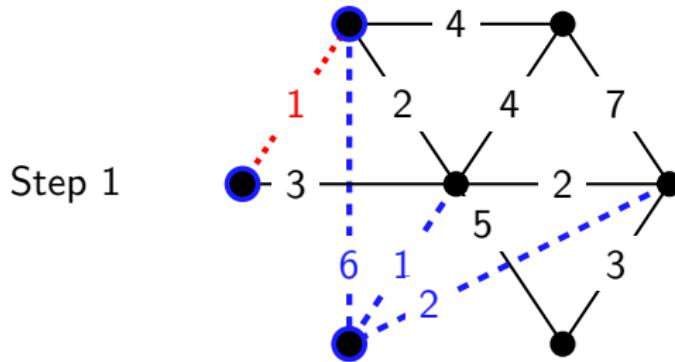


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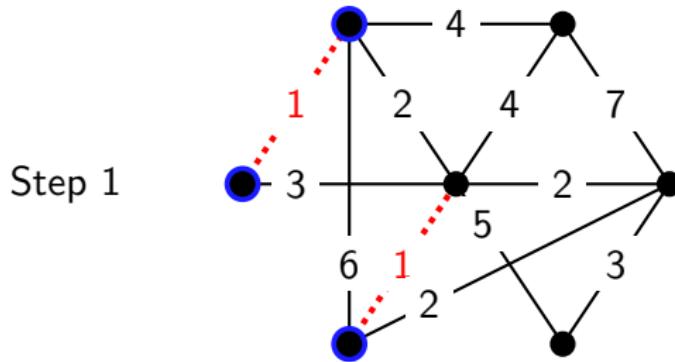


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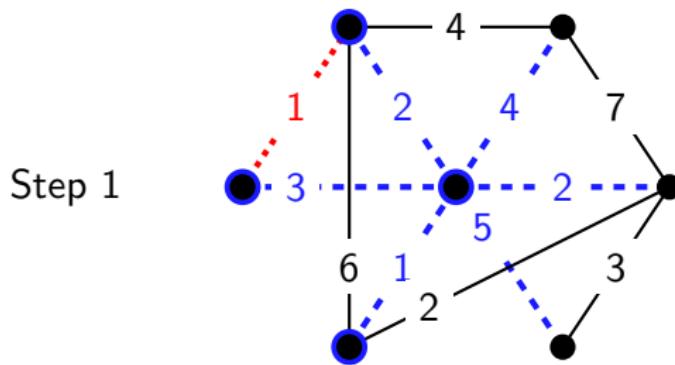


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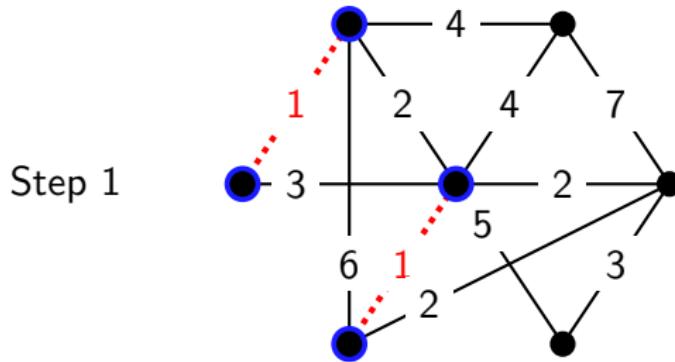


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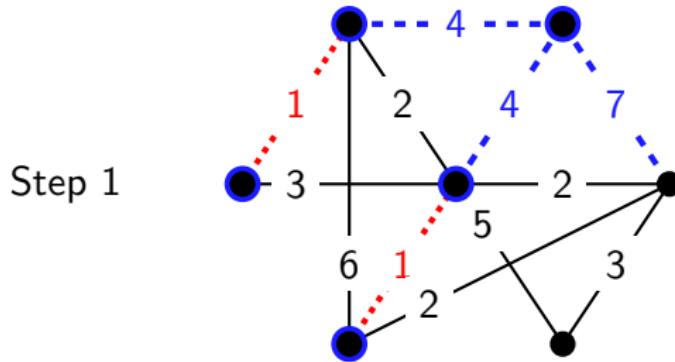


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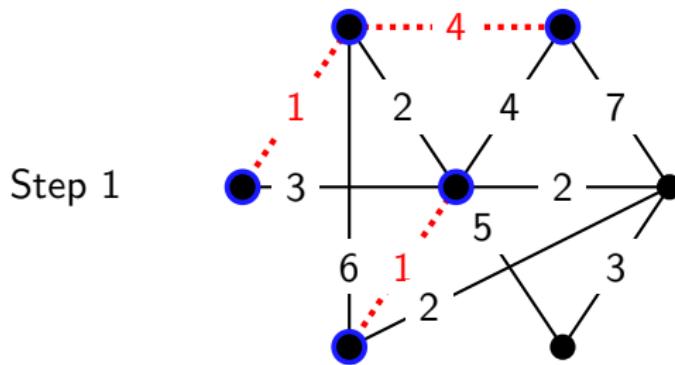


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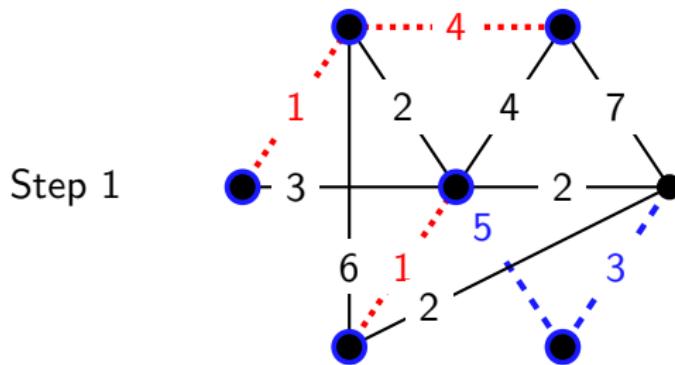


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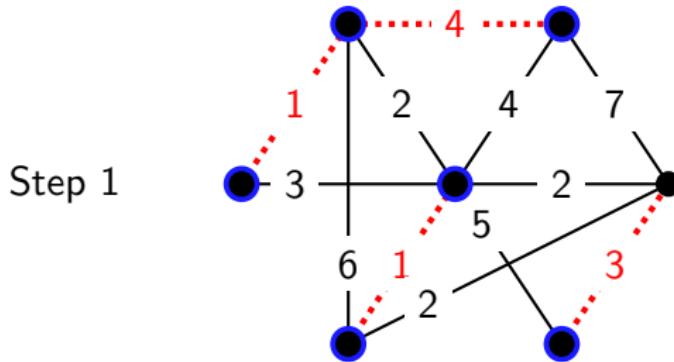


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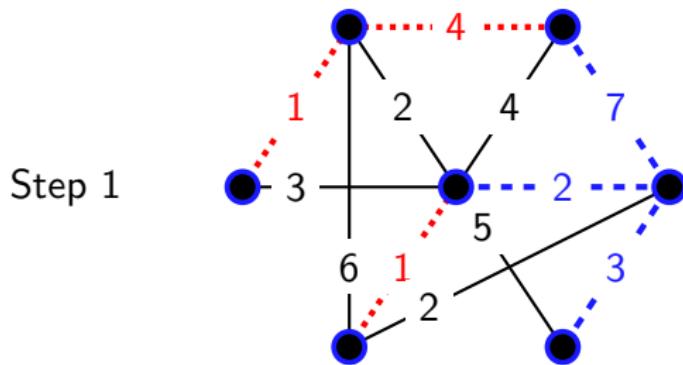


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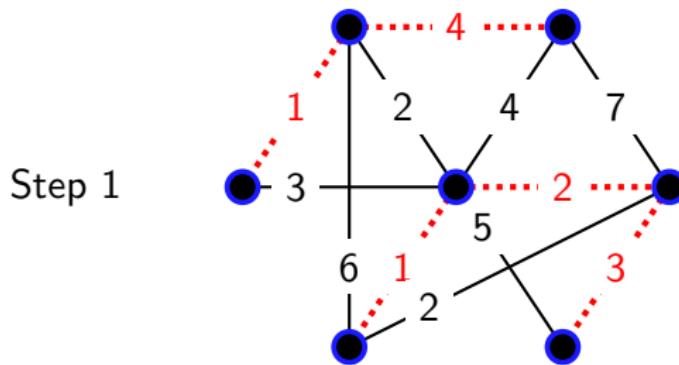


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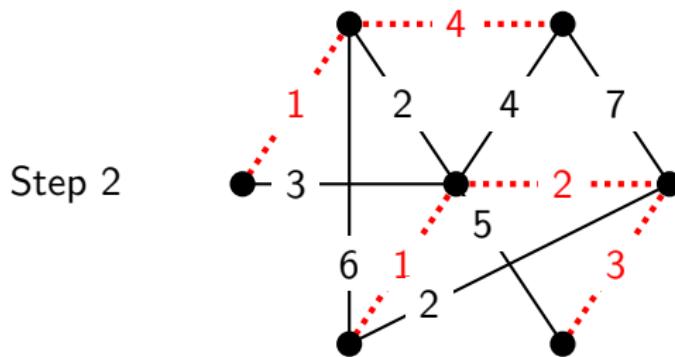


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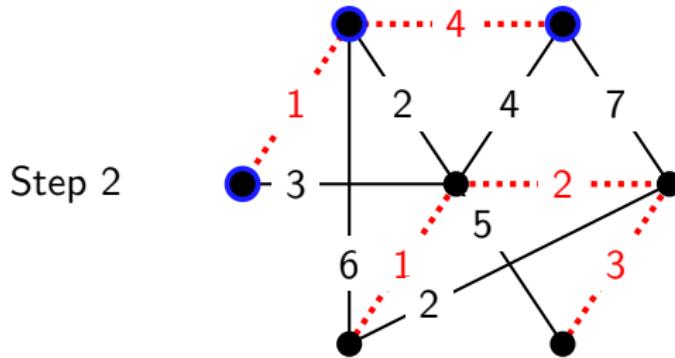


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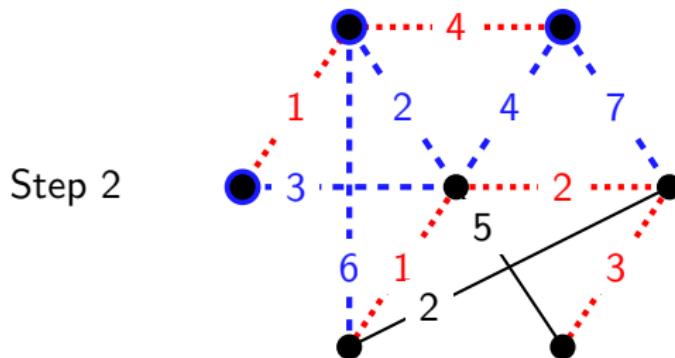


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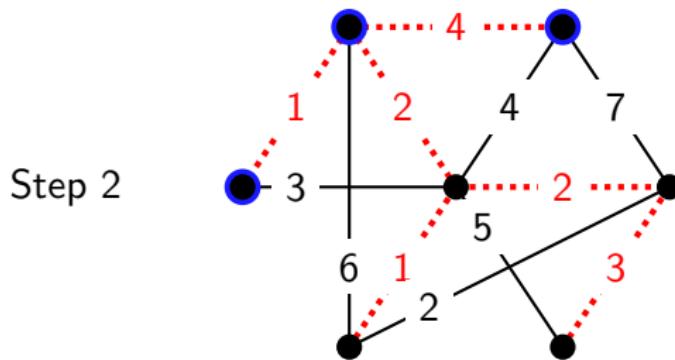


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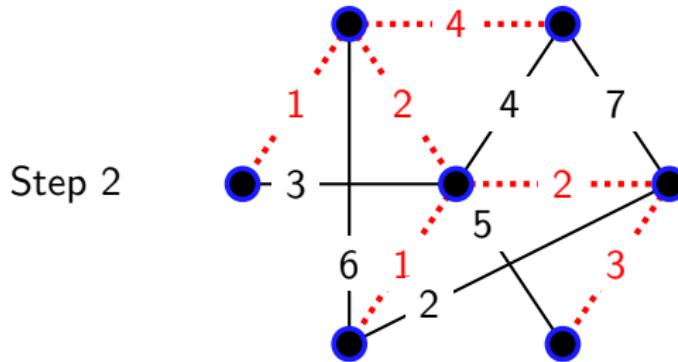


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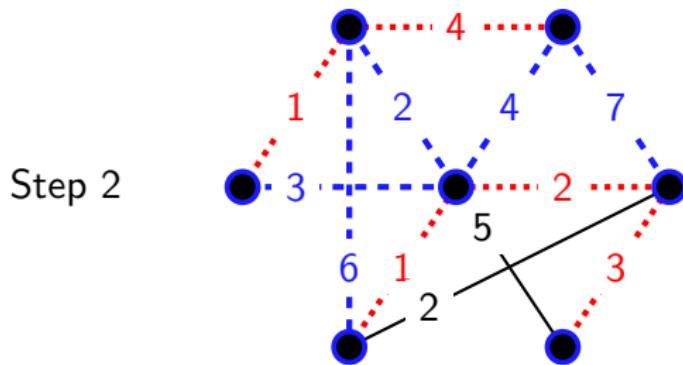


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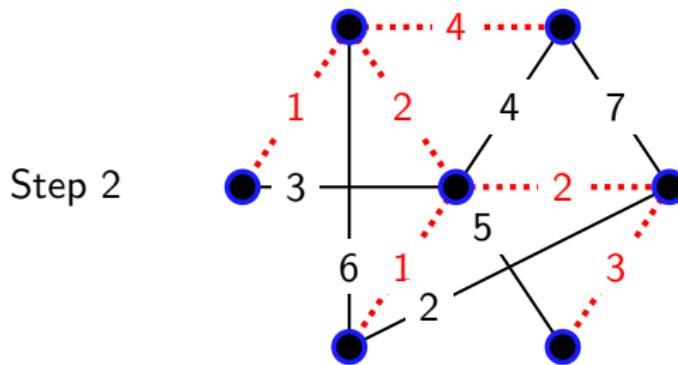


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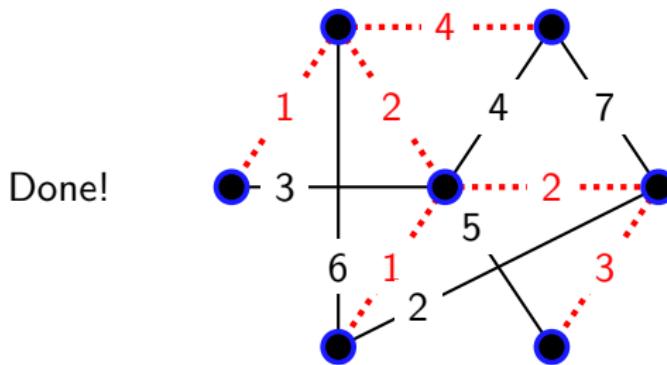


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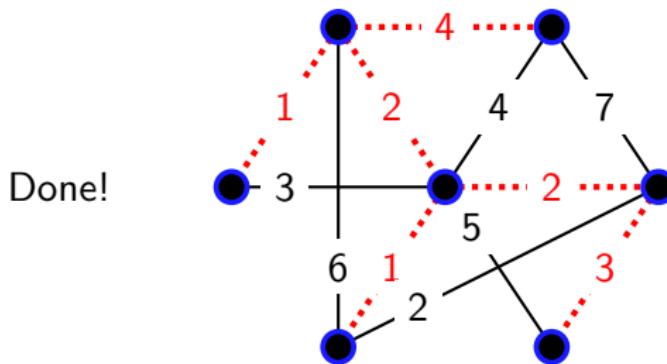


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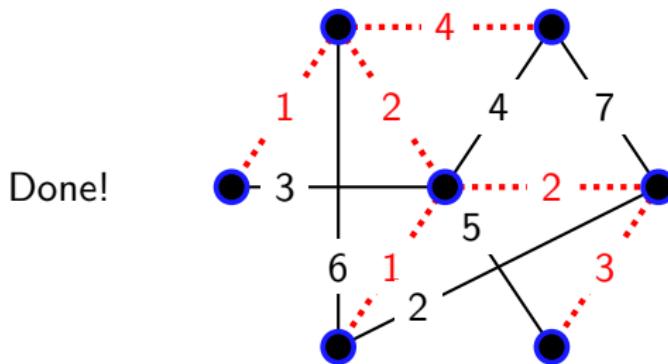
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Most modern algorithms for minimum spanning tree are variants of Borůvka's algorithm...and they use a union-find data structure to keep track of the components! So it is useful, after all.