

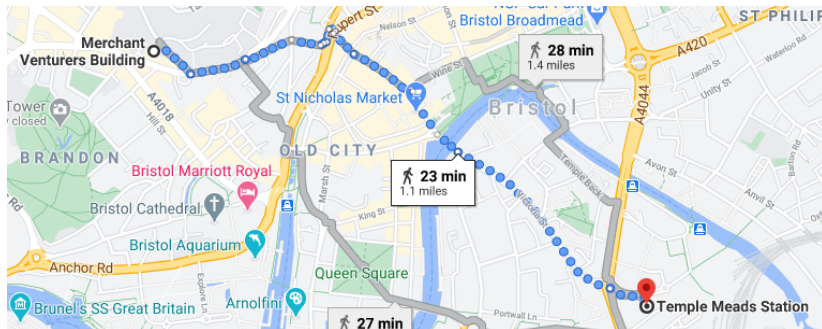
# Dijkstra's algorithm

## COMS20017 (Algorithms and Data)

John Lapinskas, University of Bristol

# Distances in real networks are weighted!

We often model road networks as graphs: junctions and destinations are vertices, roads are edges, one-way roads are directed edges.

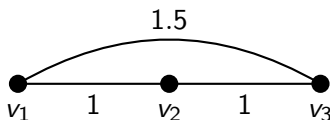


But when we want to find a “shortest path” in this graph, we don’t care about the number of edges, we care about the **physical distance**.

(We may also want to weight by e.g. elevation changes or current traffic.)

# Weighted graphs

A **weighted graph** is a pair  $(G, w)$ , where  $G$  is a graph and  $w: E(G) \rightarrow \mathbb{R}$  is a **weight function**. This could represent distances, costs, times, etc.



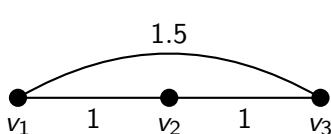
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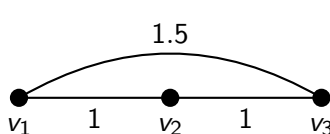
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The **length** of a path/walk  $P = x_1 \dots x_t$  is the total weight of  $P$ 's edges:

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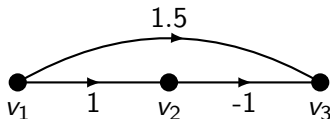
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The **distance** from  $x$  to  $y$  is the shortest length of any path/walk from  $x$  to  $y$ , or  $\infty$  if they are in different components. E.g.  $d(v_1, v_3) = 1.5$ .

# Negative-weight edges

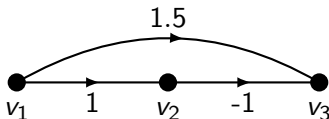
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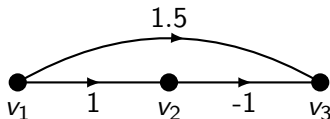
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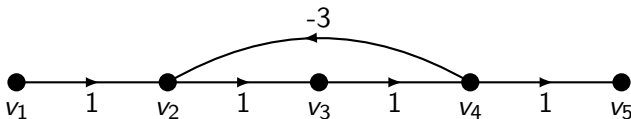
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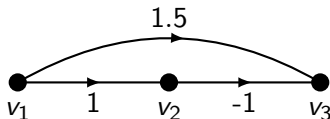


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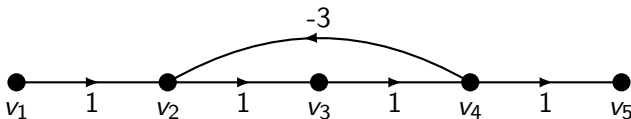


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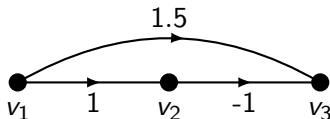
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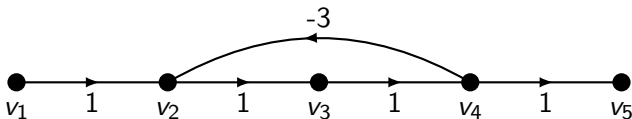
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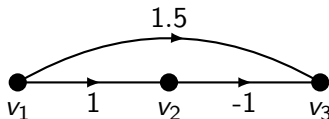
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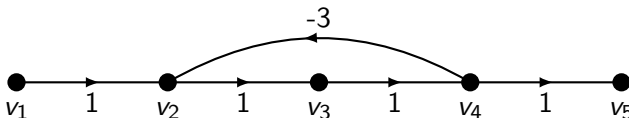
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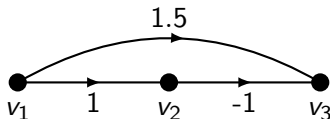
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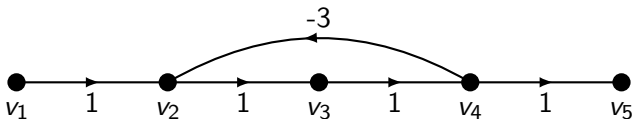
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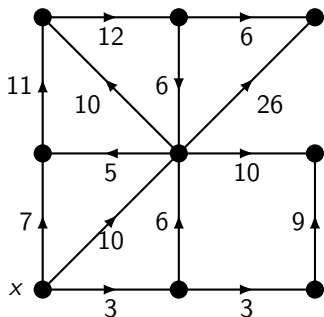
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This lecture, we ignore negative weights. (This is also faster!)

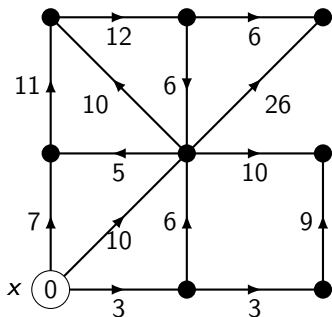
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Think of breadth-first search as water flooding a set of pipes, starting from  $x$ ... and now allow the pipes to have **different lengths**.

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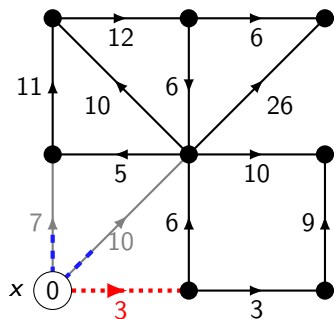
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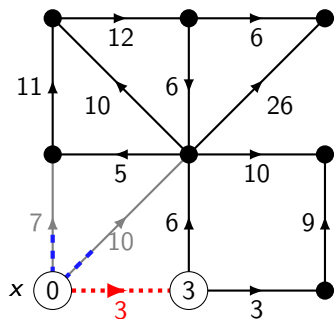
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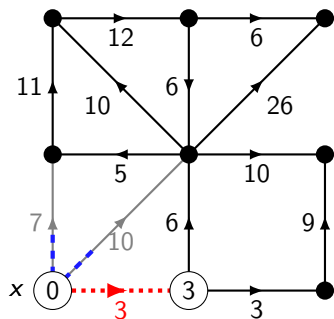
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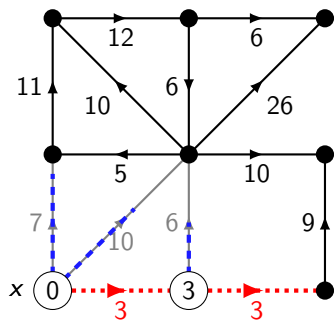
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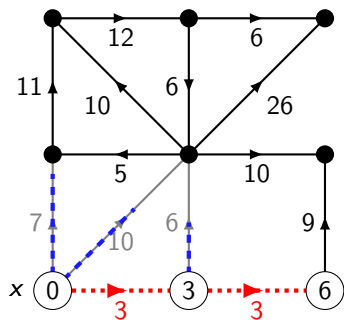
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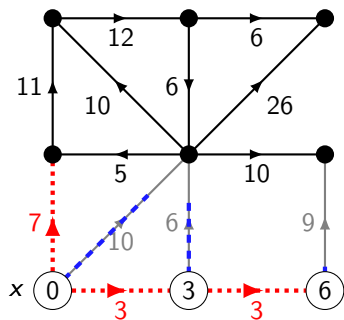
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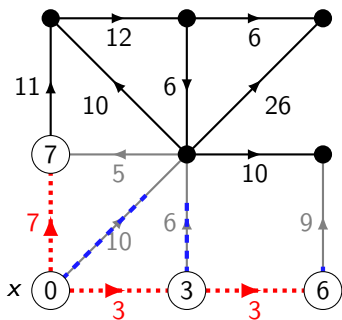
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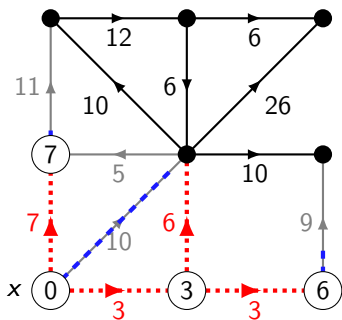
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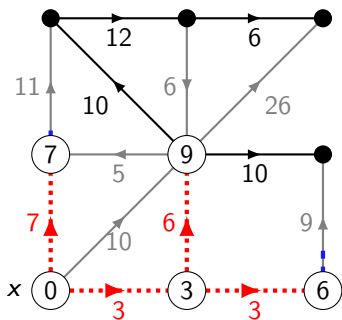
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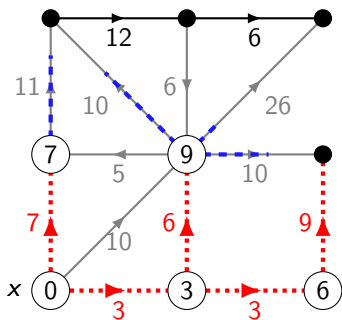
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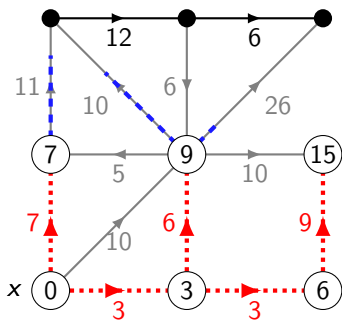
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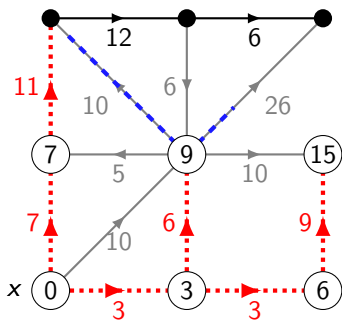
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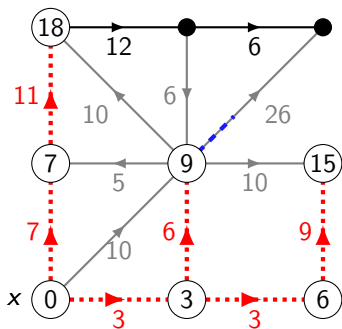
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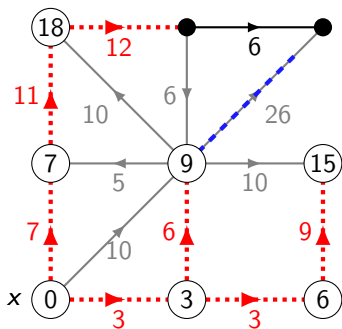
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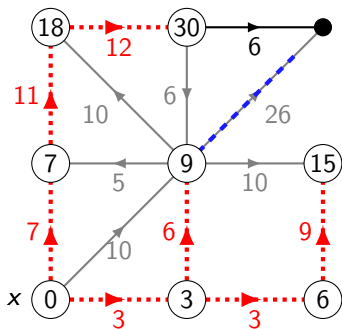
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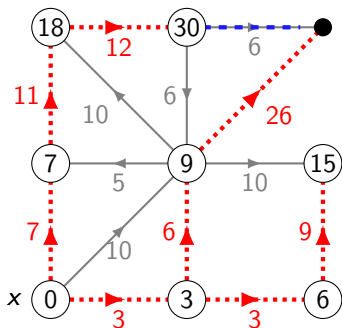
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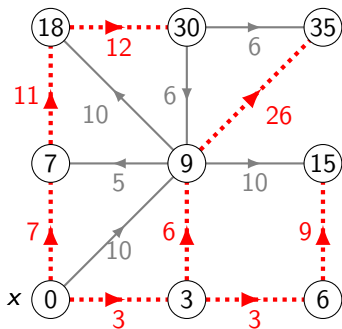
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**Claim:** Dijkstra's algorithm calculates distances correctly.

**Proof:** By induction on the number of vertices  $u$  with  $d(x, u)$  set.

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**For input graph  $G$ , vertex  $x$ :** At each stage, pick an edge  $(u, v)$  with  $d(x, u)$  known and  $d(x, v)$  unknown that minimises  $d(x, u) + \text{length}(u, v)$ . Then set  $d(x, v) = d(x, u) + \text{length}(u, v)$ .

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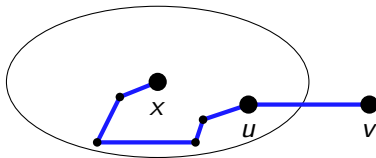
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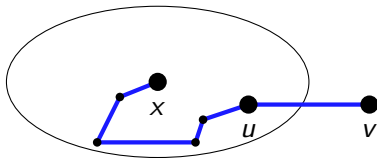
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We can append  $(u, v)$  to any path from  $x$  to  $u$ , so we have

$$d(x, v) \leq d(x, u) + \text{length}(u, v).$$

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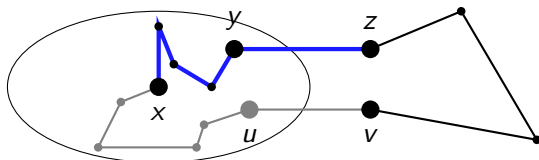
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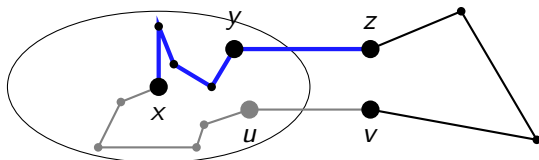
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Also, any path  $P$  from  $x$  to  $v$  has to leave  $X$  on some edge  $(y, z)$ . Hence  $P$  has length at least  $d(x, y) + \text{length}(y, z)$ . So from the way we picked  $(u, v)$ , we have  $d(x, v) \geq d(x, u) + \text{length}(u, v)$ . □

# Priority queues

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```
StartQueue(5);
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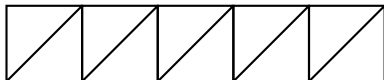
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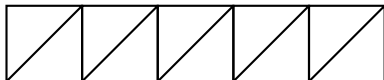
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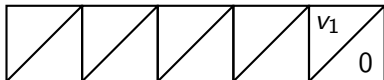
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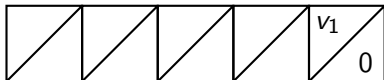
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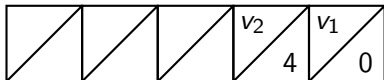
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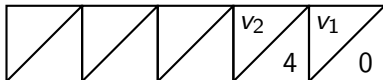
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$\text{Insert}(v_3, 2);$

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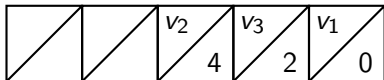
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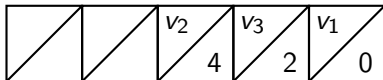
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$\text{Insert}(v_3, 2); \text{Insert}(v_5, 3);$

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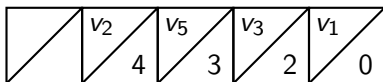
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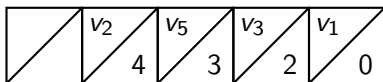
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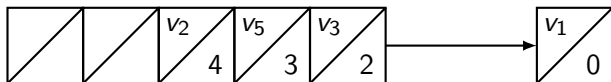
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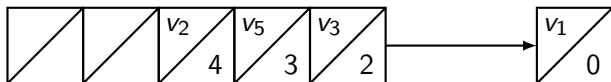
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$\text{Insert}(v_4, 20);$

# Priority queues

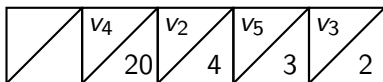
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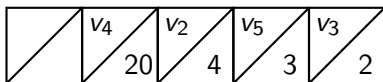
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$\text{Insert}(v_4, 20); \text{ChangeKey}(v_4, 1);$

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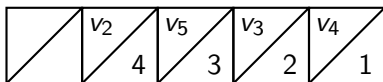
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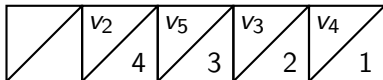
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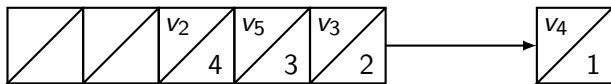
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# Dijkstra's algorithm: Implementation

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**Algorithm:** DIJKSTRA

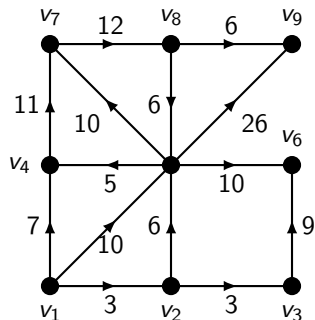
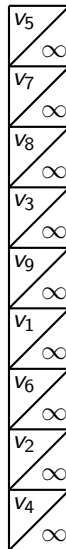
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**Input** : Weighted graph  $G = ((V, E), w)$ ,  $v \in V$ .

**Output** :  $d(v, y)$  for all  $y \in V$ .

```
1 Number the vertices of  $G$  as  $v = v_1, \dots, v_n$ .
2  $queue \leftarrow \text{StartQueue}(n)$ .
3 foreach  $i = 1$  to  $n$  do
4    $\text{dist}[i] \leftarrow \infty$  and call  $queue.\text{Insert}(v_i, \infty)$ .
5 Call  $queue.\text{ChangeKey}(v_1, 0)$ .
6 do
7    $v_i \leftarrow queue.\text{Extract}()$ .
8   foreach  $(v_i, v_j) \in E$  do
9      $\text{dist}[j] \leftarrow \min\{\text{dist}[j], \text{dist}[i] + w(i, j)\}$ .
10    Call  $queue.\text{ChangeKey}(v_j, \text{dist}[j])$ .
11 while  $queue$  is not empty
12 Return  $\text{dist}$ .
```

---



# Dijkstra's algorithm: Implementation

---

**Algorithm:** DIJKSTRA

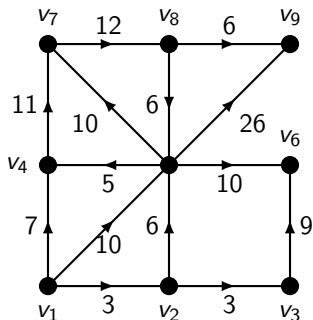
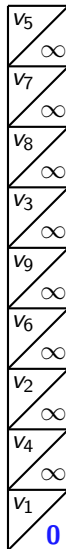
---

**Input** : Weighted graph  $G = ((V, E), w)$ ,  $v \in V$ .

**Output** :  $d(v, y)$  for all  $y \in V$ .

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```

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# Dijkstra's algorithm: Implementation

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**Algorithm:** DIJKSTRA

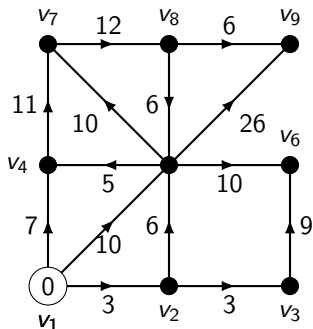
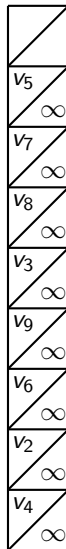
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# Dijkstra's algorithm: Implementation

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**Algorithm:** DIJKSTRA

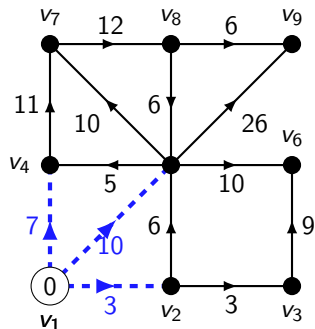
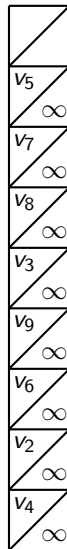
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# Dijkstra's algorithm: Implementation

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**Algorithm:** DIJKSTRA

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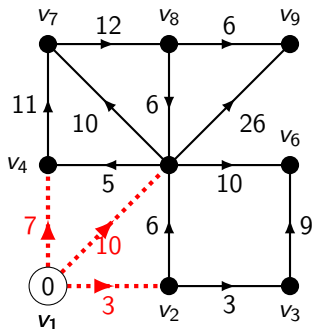
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```

---

$v_7$	$\infty$
$v_8$	$\infty$
$v_3$	$\infty$
$v_9$	$\infty$
$v_6$	$\infty$
$v_5$	10
$v_4$	7
$v_2$	3



# Dijkstra's algorithm: Implementation

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**Algorithm:** DIJKSTRA

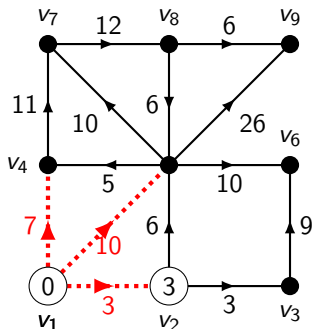
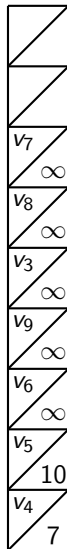
---

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# Dijkstra's algorithm: Implementation

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**Algorithm:** DIJKSTRA

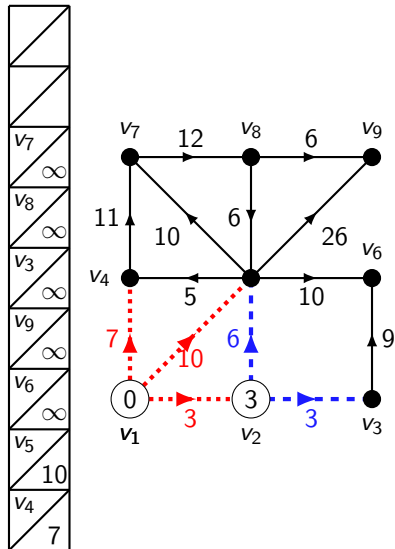
---

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# Dijkstra's algorithm: Implementation

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**Algorithm:** DIJKSTRA

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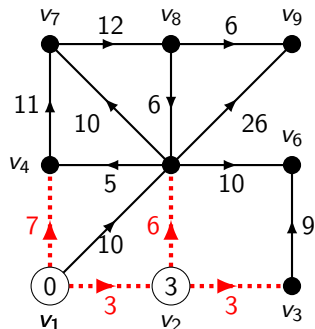
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---

		$v_7$					
			$v_8$				
				$v_9$			
	$v_7$						
		$v_8$					
			$v_9$				
				$v_6$			
	$v_4$						
		$v_6$					
			$v_5$				
				$v_4$			
					$v_3$		



# Dijkstra's algorithm: Implementation

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**Algorithm:** DIJKSTRA

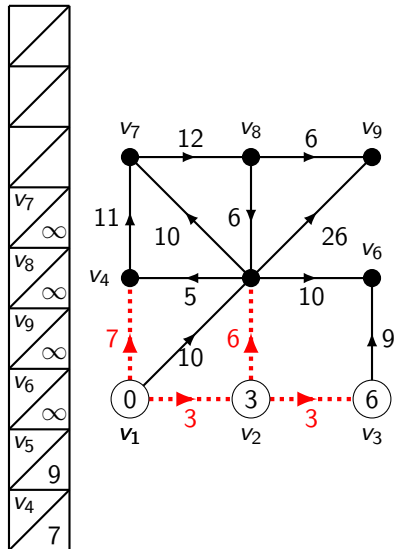
---

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# Dijkstra's algorithm: Implementation

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**Algorithm:** DIJKSTRA

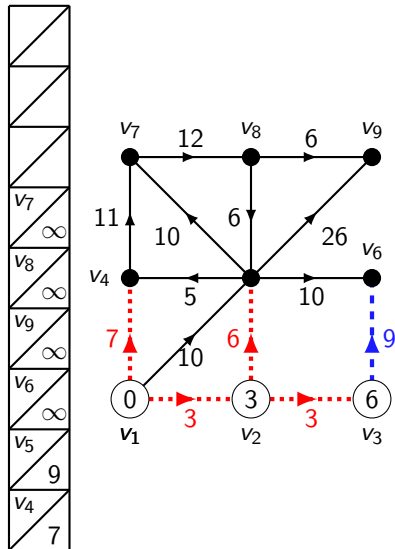
---

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# Dijkstra's algorithm: Implementation

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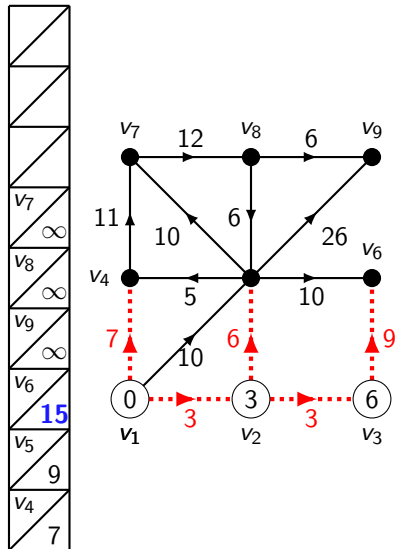
---

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# Dijkstra's algorithm: Implementation

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**Algorithm:** DIJKSTRA

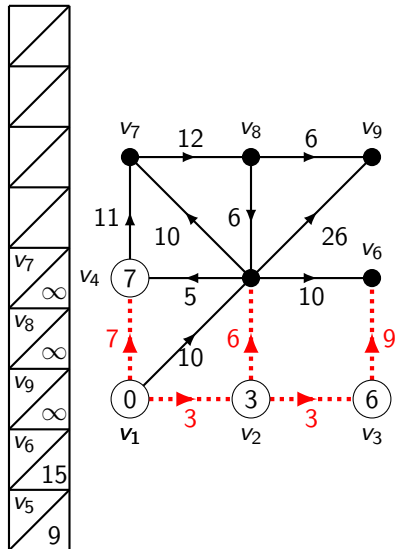
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# Dijkstra's algorithm: Implementation

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**Algorithm:** DIJKSTRA

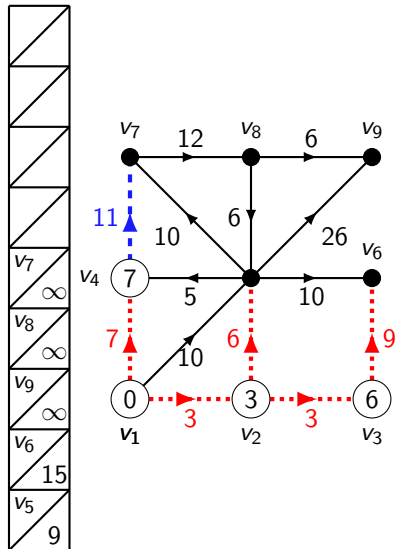
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# Dijkstra's algorithm: Implementation

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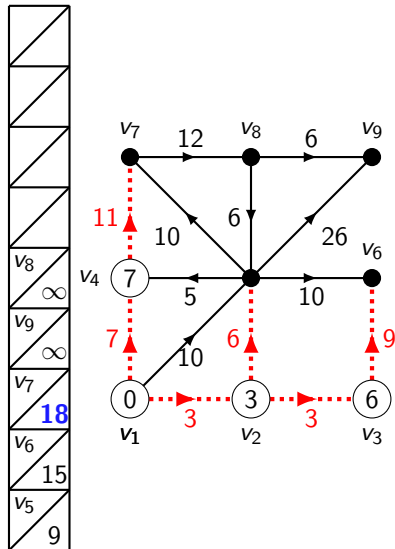
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# Dijkstra's algorithm: Implementation

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**Algorithm:** DIJKSTRA

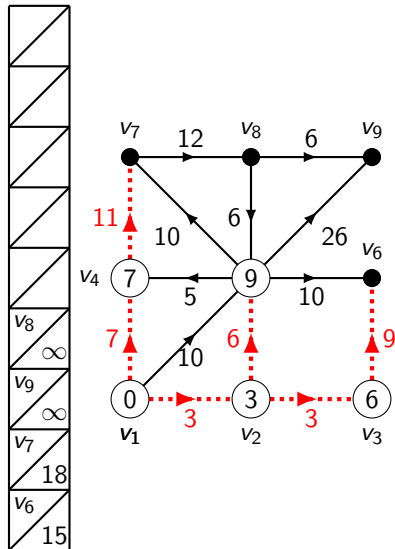
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# Dijkstra's algorithm: Implementation

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**Algorithm:** DIJKSTRA

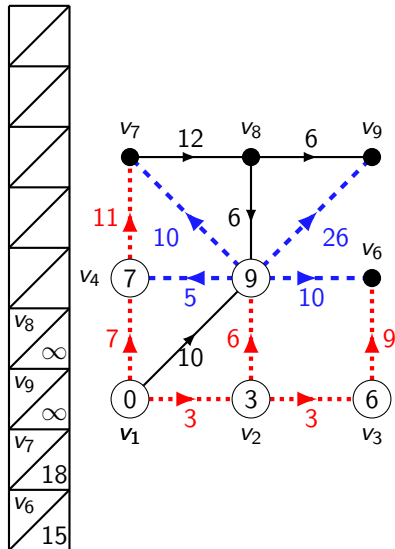
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# Dijkstra's algorithm: Implementation

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**Algorithm:** DIJKSTRA

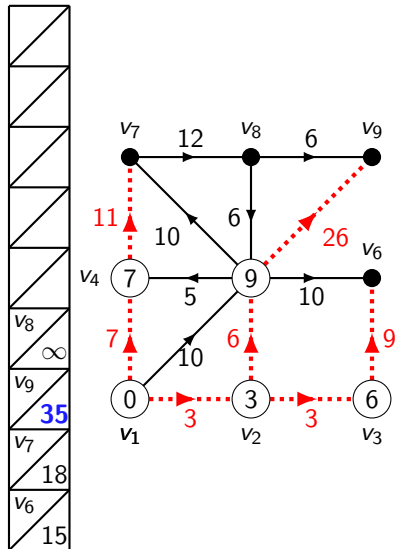
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**Algorithm:** DIJKSTRA

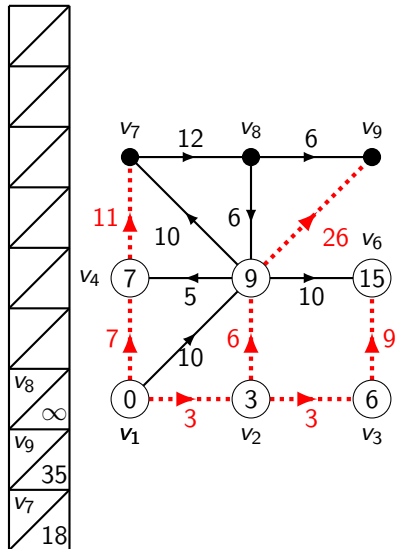
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# Dijkstra's algorithm: Implementation

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## Algorithm: DIJKSTRA

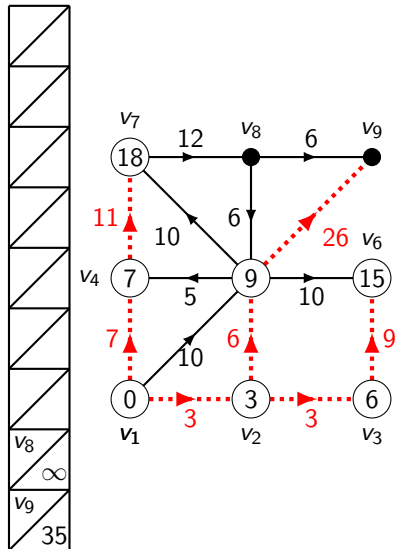
---

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# Dijkstra's algorithm: Implementation

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## Algorithm: DIJKSTRA

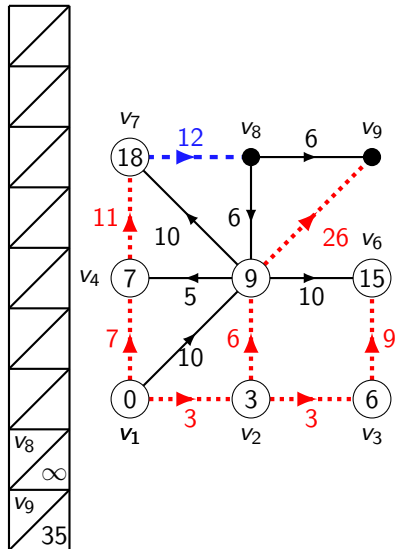
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**Input** : Weighted graph  $G = ((V, E), w)$ ,  $v \in V$ .

**Output** :  $d(v, y)$  for all  $y \in V$ .

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11 while  $queue$  is not empty
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# Dijkstra's algorithm: Implementation

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**Algorithm:** DIJKSTRA

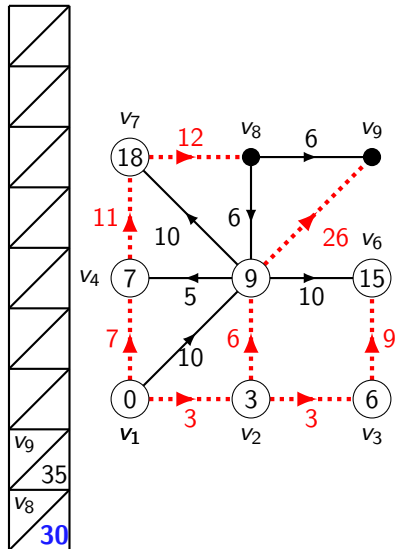
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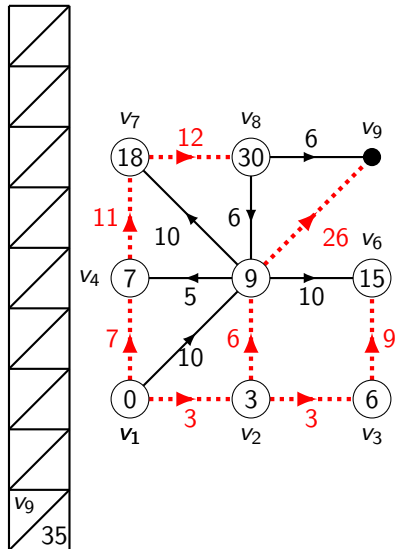
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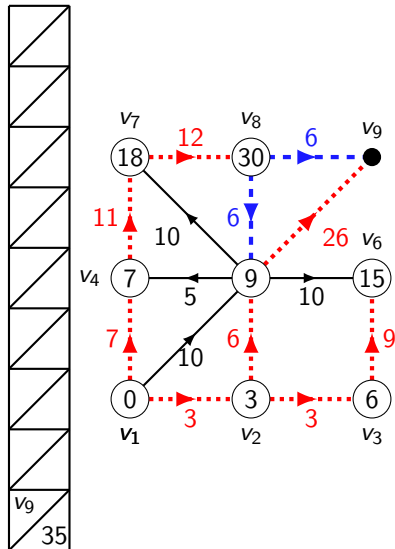
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# Dijkstra's algorithm: Implementation

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## Algorithm: DIJKSTRA

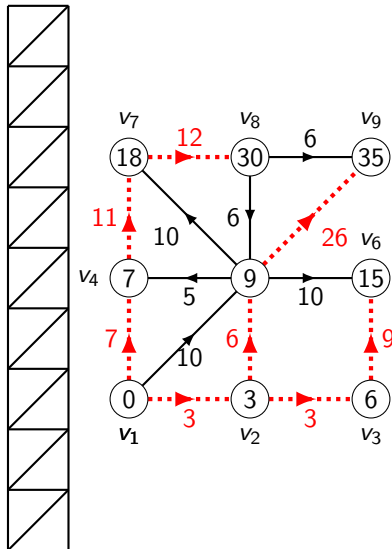
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# Dijkstra's algorithm: Implementation

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## Algorithm: DIJKSTRA

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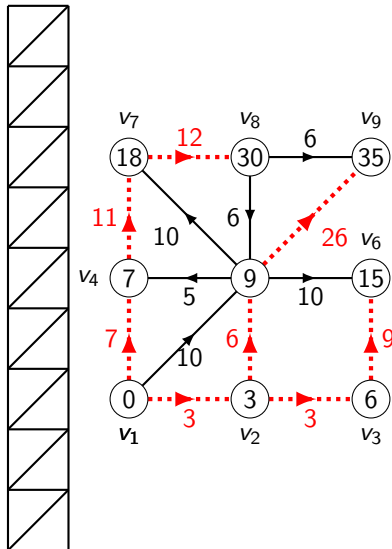
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---

**Invariant:**  $\text{dist}[j]$  is the minimum value of  $d(v_1, v_i) + w(v_i, v_j)$  over all  $v_i$ 's whose distances are finalised, as in mathematical version. ✓



## Dijkstra's algorithm: Implementation

### Algorithm: DIJKSTRA

**Input** : Weighted graph  $G = ((V, E), w), v \in V$ .

**Output** :  $d(v, y)$  for all  $y \in V$ .

1 Number the vertices of  $G$  as  $v = v_1, \dots, v_n$ .

```

2 queue  $\leftarrow$  StartQueue( $n$ ).

```

```
3  foreach  $i = 1$  to  $n$  do
```

```

4   |   dist[i] ← ∞ and call queue.Insert(vi, ∞).

```

5 Call `queue.ChangeKey( $v_1, 0$ )`.

**6 do**

```

7   |    $v_i \leftarrow \text{queue.Extract}()$ .

```

8	<b>foreach</b> $(v_i, v_j) \in E$ <b>do</b>
---	---

9      $\text{dist}[j] \leftarrow \min\{\text{dist}[j], \text{dist}[i] + w(i, j)\}.$

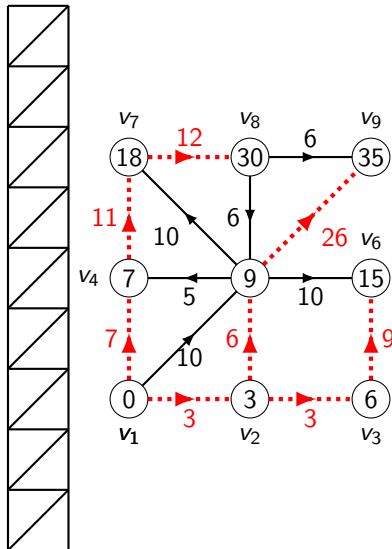
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----	---

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**Invariant:**  $\text{dist}[j]$  is the minimum value of  $d(v_1, v_i) + w(v_i, v_j)$  over all  $v_i$ 's whose distances are finalised, as in mathematical version. ✓

We can recover shortest paths by storing and returning the dotted red edges.



# Dijkstra's algorithm: Time analysis

---

**Algorithm:** DIJKSTRA

---

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We perform  $O(|V|)$  Insert operations and Extract operations, and  $O(|E|)$  ChangeKey operations, for a total of  $O((|V| + |E|) \log |V|)$  time when  $G$  is given in adjacency list form.

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We could drop this to  $O(|V| \log |V| + |E|)$  time by using a **Fibonacci heap** as a priority queue... But Fibonacci heaps have *awful* constants, and generally  $\log |V| \lesssim 50$ , so let's not!