Properties of O-notation
COMS20017 (Algorithms and Data)

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For example, if $x \leq y$ and $y \leq z$ then $x \leq z$;
likewise, if $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$ then $f(n) \in O(h(n))$.

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The same goes for $\geq$ and Ω, $=$ and Θ, $<$ and o, and $>$ and ω.
For example, if $x \leq y$ and $x \geq y$ then $x = y$;
likewise, if $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$, then $f(n) \in \Theta(g(n))$.

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This, combined with the following rough hierarchy, will let you solve most problems without thinking about C's or n_0's:

$$n! \in \omega(3^n) \subseteq \omega(2^n) \subseteq \omega(n^2) \subseteq \omega(n) \subseteq \omega(\log^2 n) \subseteq \omega(\log n) \subseteq \omega(1).$$

When you *should* work formally

The time to fall back to definitions is when you need to confirm your intuition — when you're not sure if a general principle holds or not.

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We have: There exist c, $n_0 > 0$ such that $f(n) \geq cg(n)$ for all $n \geq n_0$.
We want: There exist c', $n_0' > 0$ such that $f(n)^2 \geq c'g(n)^2$ for all $n \geq n_0'$.

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So we can just take $c' = c^2$ and $n_0' = n_0$ to prove $f(n)^2 \in \Omega(g(n)^2)$. ✓

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Since we only have $f(n) < g(n)$, this looks dubious when $C \ll 1$...
One counterexample is $f(n) = n/2$, $g(n) = n$ (taking $C = 1/4$). ✓

L'Hôpital's rule

This is like a more powerful form of the racetrack principle from last year.

L'Hôpital's rule: Suppose $f, g\colon \mathbb{R} \to \mathbb{R}$ are differentiable and that $f(n),\ g(n) \in \omega(1)$. Then:

- $f(n) \in \omega(g(n))$ if and only if $f'(n) \in \omega(g'(n))$; and
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Intuitively: This makes sense since f' and g' are the *rates of change* of f and g — if f grows much faster than g, then f' should grow much faster than g', and vice versa.

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Example: Prove that $n \in o(b^n)$ for all constants $b > 1$.

By L'Hôpital's rule, this holds if and only if $1 \in o(b^n \ln b) = o(b^n)$.
For any $C > 0$, we have $1 \leq C \cdot b^n$ for all $n \geq \log_b(1/C)$, so this is true.

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Fact: If $g(n) \in o(f(n))$, then $f(n) + g(n) \in \Theta(f(n))$. (Why?)

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Notice the overall process here: rather than working with definitions directly, we reduce the question to one we know how to solve.

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(In practice, if you see a running time like this, you should be very careful even though it's theoretically fast — the constants are probably massive...)