

# Properties of O-notation

## COMS20017 (Algorithms and Data)

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# The key to working with O-notation

Last time about comparing functions using the definitions of O-notation.  
You should almost never actually do this!

Your life will be much happier if you work mostly based on **intuition**.

**Usually** (not always!) if something is true for  $\leq$ , it is true for  $O$ .

For example, if  $x \leq y$  and  $y \leq z$  then  $x \leq z$ ;

likewise, if  $f(n) \in O(g(n))$  and  $g(n) \in O(h(n))$  then  $f(n) \in O(h(n))$ .

The same goes for  $\geq$  and  $\Omega$ ,  $=$  and  $\Theta$ ,  $<$  and  $o$ , and  $>$  and  $\omega$ .

For example, if  $x \leq y$  and  $x \geq y$  then  $x = y$ ;

likewise, if  $f(n) \in O(g(n))$  and  $f(n) \in \Omega(g(n))$ , then  $f(n) \in \Theta(g(n))$ .

This, combined with the following rough hierarchy, will let you solve most problems without thinking about  $C$ 's or  $n_0$ 's:

$$n! \in \omega(3^n) \subseteq \omega(2^n) \subseteq \omega(n^2) \subseteq \omega(n) \subseteq \omega(\log^2 n) \subseteq \omega(\log n) \subseteq \omega(1).$$

# When you *should* work formally

The time to fall back to definitions is when you need to confirm your intuition — when you're not sure if a general principle holds or not.

**Example:** Is it true that if  $f(n) \in \Omega(g(n))$ , then  $f(n)^2 \in \Omega(g(n)^2)$ ?

Think back to the definitions.

**We have:** There exist  $c, n_0 > 0$  such that  $f(n) \geq cg(n)$  for all  $n \geq n_0$ .

**We want:** There exist  $c', n'_0 > 0$  such that  $f(n)^2 \geq c'g(n)^2$  for all  $n \geq n'_0$ .

So we can just take  $c' = c^2$  and  $n'_0 = n_0$  to prove  $f(n)^2 \in \Omega(g(n)^2)$ . ✓

**Example:** Is it true that if  $f(n) < g(n)$  for all  $n$ , then  $f(n) \in o(g(n))$ ?

**We want:** For all  $C > 0$ , there exists  $n_0$  such that  $f(n) < Cg(n)$  for all  $n \geq n_0$ .

Since we only have  $f(n) < g(n)$ , this looks dubious when  $C \ll 1$ ...

One counterexample is  $f(n) = n/2, g(n) = n$  (taking  $C = 1/4$ ). ✓

# L'Hôpital's rule

This is like a more powerful form of the racetrack principle from last year.

**L'Hôpital's rule:** Suppose  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  are differentiable and that  $f(n), g(n) \in \omega(1)$ . Then:

- $f(n) \in \omega(g(n))$  if and only if  $f'(n) \in \omega(g'(n))$ ; and
- $f(n) \in o(g(n))$  if and only if  $f'(n) \in o(g'(n))$ .

**Intuitively:** This makes sense since  $f'$  and  $g'$  are the *rates of change* of  $f$  and  $g$  — if  $f$  grows much faster than  $g$ , then  $f'$  should grow much faster than  $g'$ , and vice versa.

I won't prove it, though! (It's also a weaker form of the “real” result.)

**Example:** Prove that  $n \in o(b^n)$  for all constants  $b > 1$ .

By L'Hôpital's rule, this holds if and only if  $1 \in o(b^n \ln b) = o(b^n)$ .

For any  $C > 0$ , we have  $1 \leq C \cdot b^n$  for all  $n \geq \log_b(1/C)$ , so this is true.

## Example: Proving that exponential beats polynomial

**Theorem:** For all polynomial functions  $f(n) = \sum_i a_i n^{x_i}$  and all  $y > 1$ , we have  $f(n) \in o(y^n)$ .

**Proof:** By the hierarchy, we have  $n^{x_i} \in o(n^{x_j})$  whenever  $x_i < x_j$ .

**Fact:** If  $g(n) \in o(f(n))$ , then  $f(n) + g(n) \in \Theta(f(n))$ . (Why?)

Hence  $f(n) \in \Theta(n^x)$  for some  $x > 0$ , and we must show  $n^x = o(y^n)$ .

We have that  $f(n)^x \in o(g(n)^x)$  if and only if  $f(n) \in o(g(n))$ , so it is enough to show  $n \in o(y^{n/x}) = o((y^{1/x})^n)$ .

We already saw this is true via L'Hôpital, so we're done. □

Notice the overall process here: rather than working with definitions directly, we reduce the question to one we know how to solve.

## Example: Dealing with unpleasant exponentials

**Example:** Prove that  $2^{(\log \log n)^2} \in o(n)$  and  $2^{(\log \log n)^2} \in \omega(\log n)$ .

Problems like this are much easier if you give the two things you're trying to compare a common base.

Here, we have  $n = 2^{\log n}$  and  $\log n = 2^{\log \log n}$ .

We have  $(\log \log n)^2 \in o(\log n)$  and  $(\log \log n)^2 = \omega(\log \log n)$ , so “clearly”  $2^{(\log \log n)^2} \in o(n)$  and  $2^{(\log \log n)^2} \in \omega(\log n)$ .

All we need is that if  $f(n) = o(g(n))$ , then  $2^{f(n)} \in o(2^{g(n)})$ , which is true... as long as  $g(n) \in \omega(1)$ . (Exercise!) ✓

(In practice, if you see a running time like this, you should be very careful even though it's theoretically fast — the constants are probably massive...)