Trees
COMS20017 (Algorithms and Data)

John Lapinskas, University of Bristol

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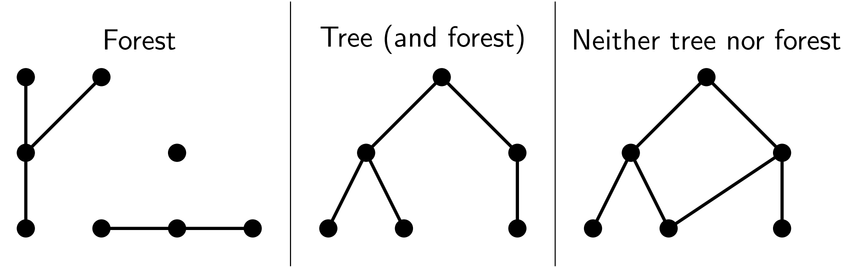

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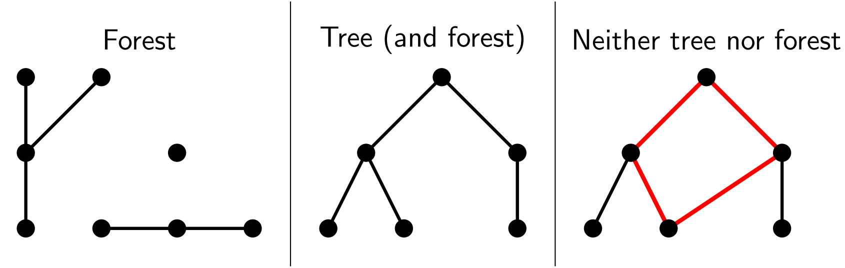

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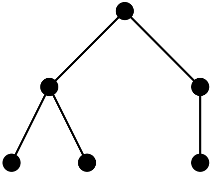

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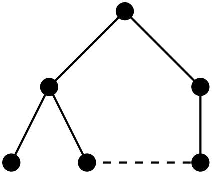

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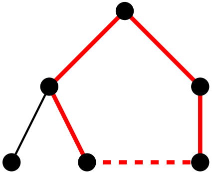

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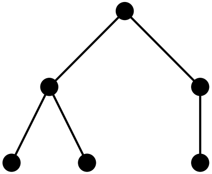

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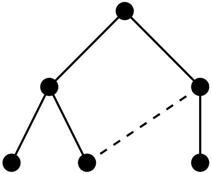

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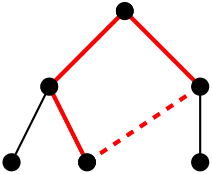

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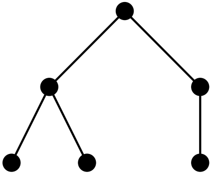

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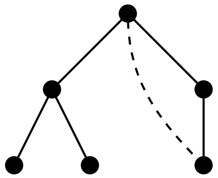

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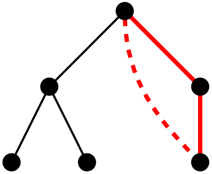

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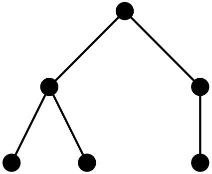

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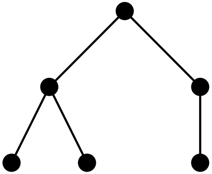

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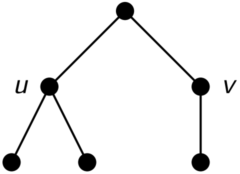

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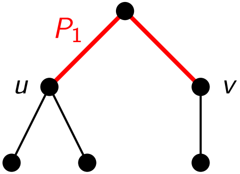

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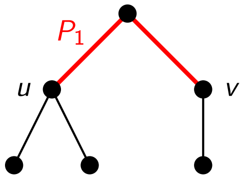

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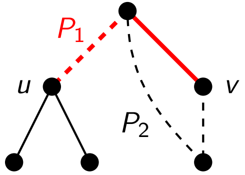

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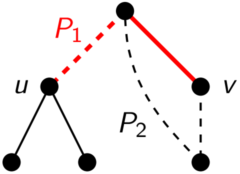

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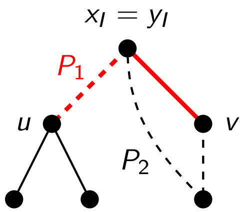

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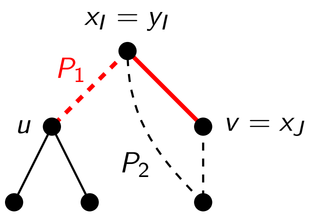

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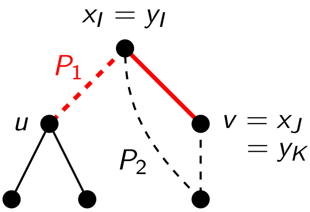

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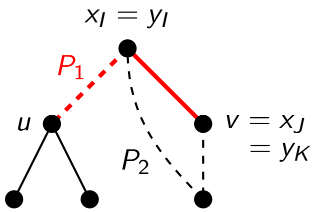

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Then $x_I x_{I+1} \dots x_J y_{K-1} y_{K-2} \dots y_I$ is a cycle, so T is not a tree. $\square$

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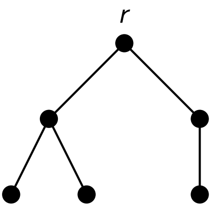

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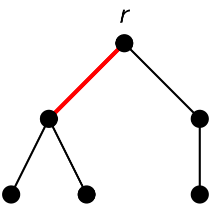

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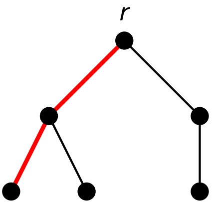

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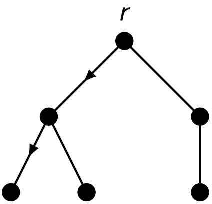

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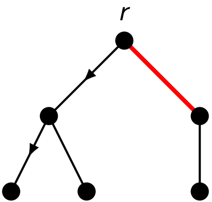

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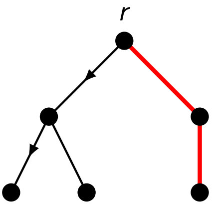

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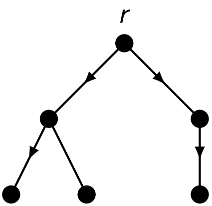

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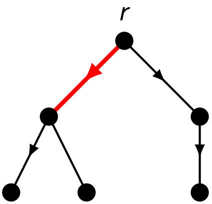

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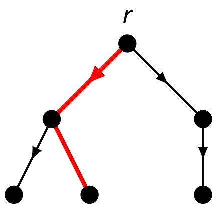

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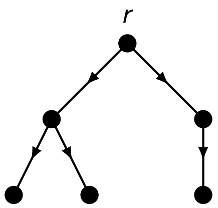

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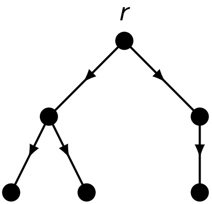

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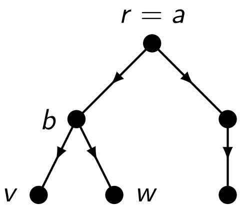

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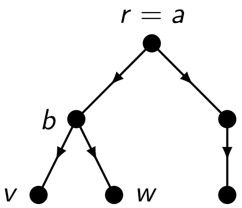

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Suppose some path P_v directs $a \to b$.
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Then both P_v and P_w must start with P_b, since P_b is the **unique** path from r to b.
So P_w also directs $a \to b$. ✓

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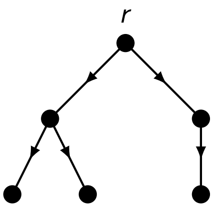
Proof idea: Take an arbitrary root $r \in V$. For all vertices v, let P_v be the unique path from r to v. Direct T's edges along these paths. ✓



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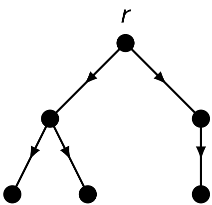

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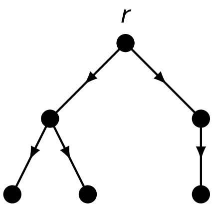

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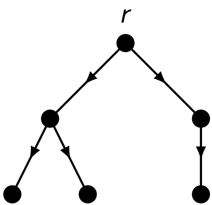

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Bonus: We also just defined rooted trees in terms of graphs.

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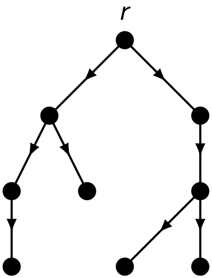

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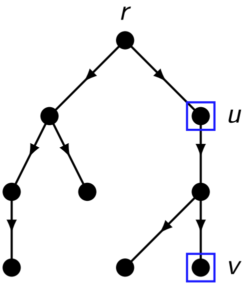

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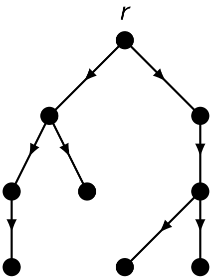

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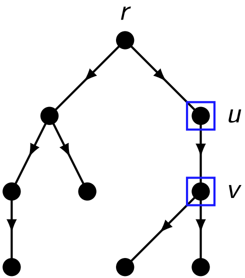

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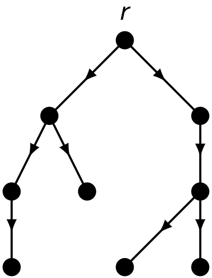

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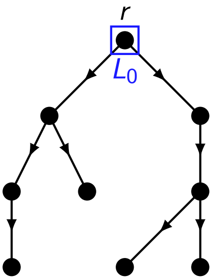

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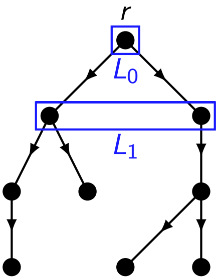

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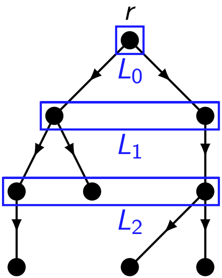

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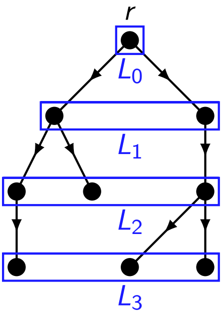

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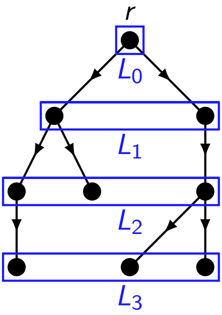

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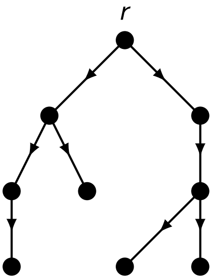

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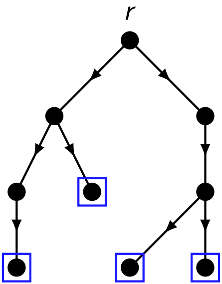

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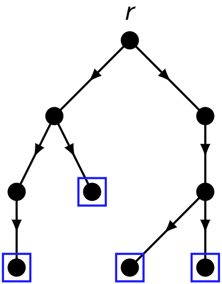

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These definitions agree except for a rooted tree whose root has one child.

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Solving for x gives $x \geq 2$, so we're done! □

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When you're actually working with trees, it's good to have one single result that tells you that all the "obvious" things are true. This is that result.

Lemma: The following are equivalent for an n-vertex graph $T = (V, E)$:

(A) T is connected and has no cycles, i.e. is a tree;

(B) T has $n-1$ edges and is connected;

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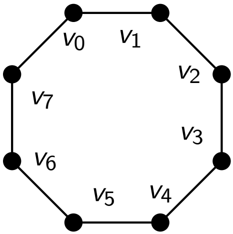
And on any cycle $v_0 \dots v_k$, there are two different paths from v_0 to v_k:



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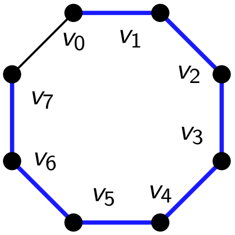

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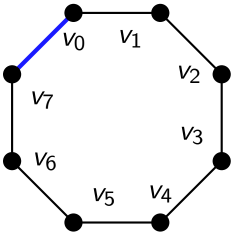

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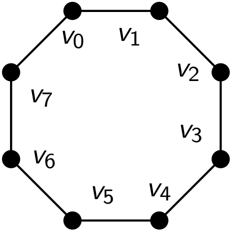

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Every edge of T is in some C_i, so $|E| = \sum_i (|V(C_i)| - 1) = n - r$. But we know $|E| = n - 1$, so we must have $r = 1$. ✓

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So T must have had **more than** $n - 1$ edges — a contradiction. $\square$

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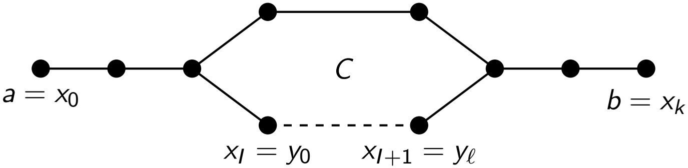

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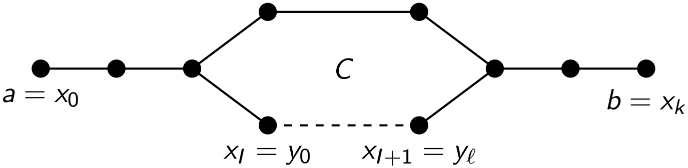

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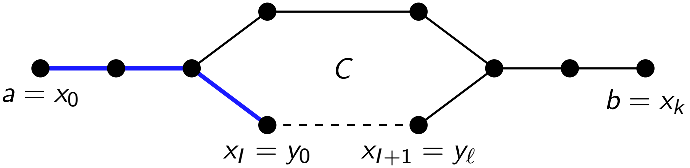

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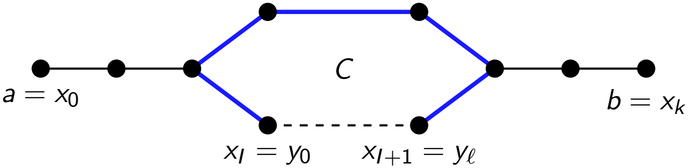

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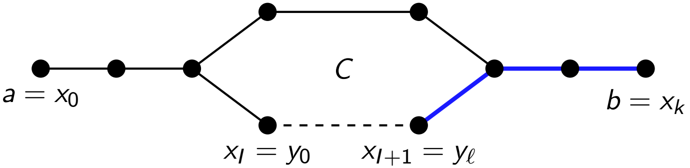

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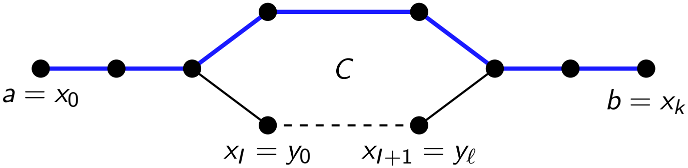

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And there was much rejoicing.