

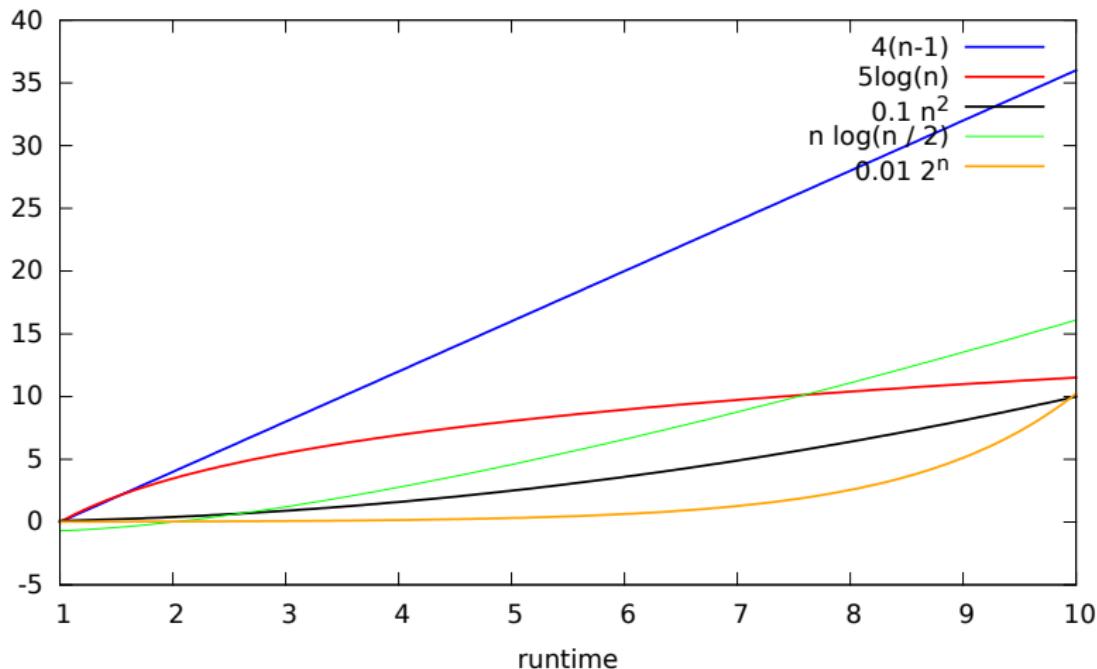
# Defining O-notation (recap)

## COMS20017 (Algorithms and Data)

John Lapinskas, University of Bristol

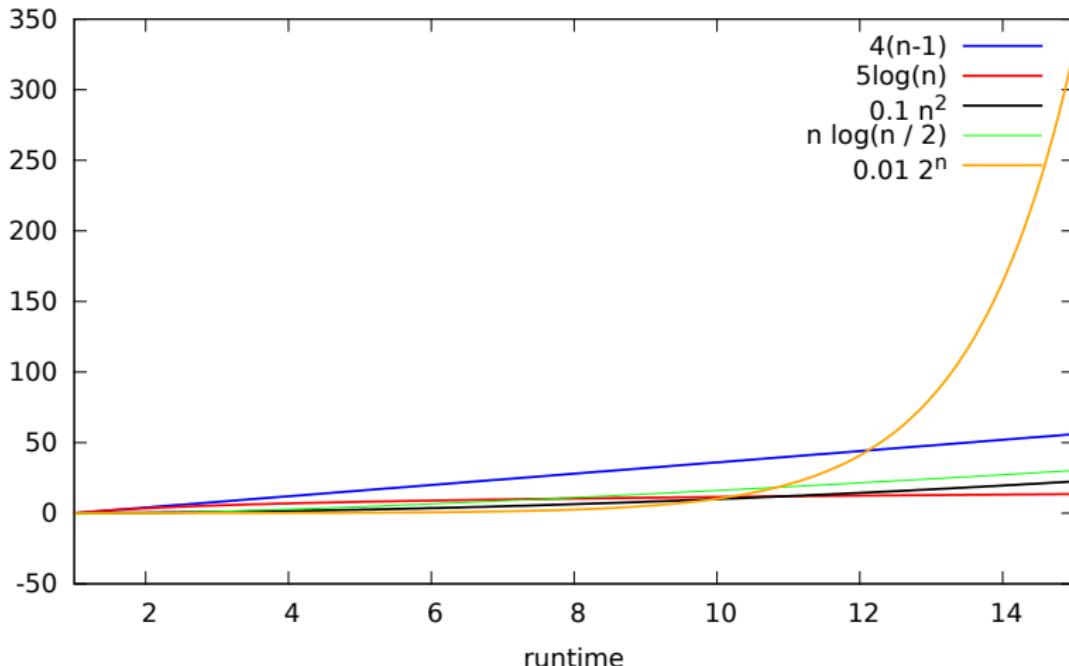
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**Intuition:** As input sizes get large, asymptotic growth rate matters more than constant factors. Also, constant factors are implementation-dependent. So we focus on growth rate.



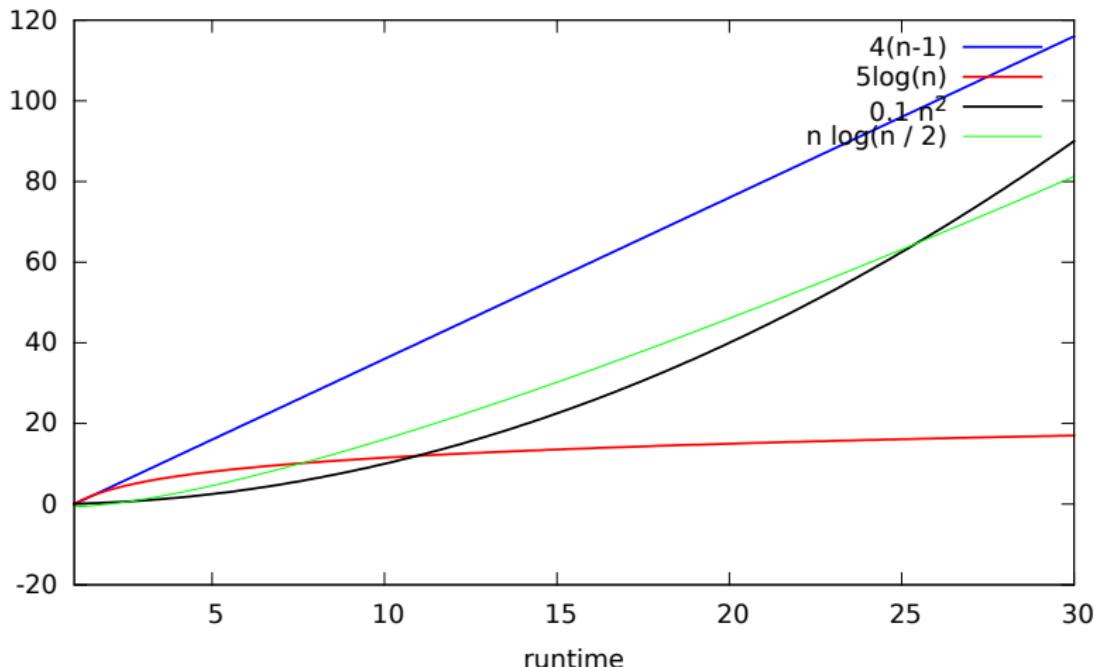
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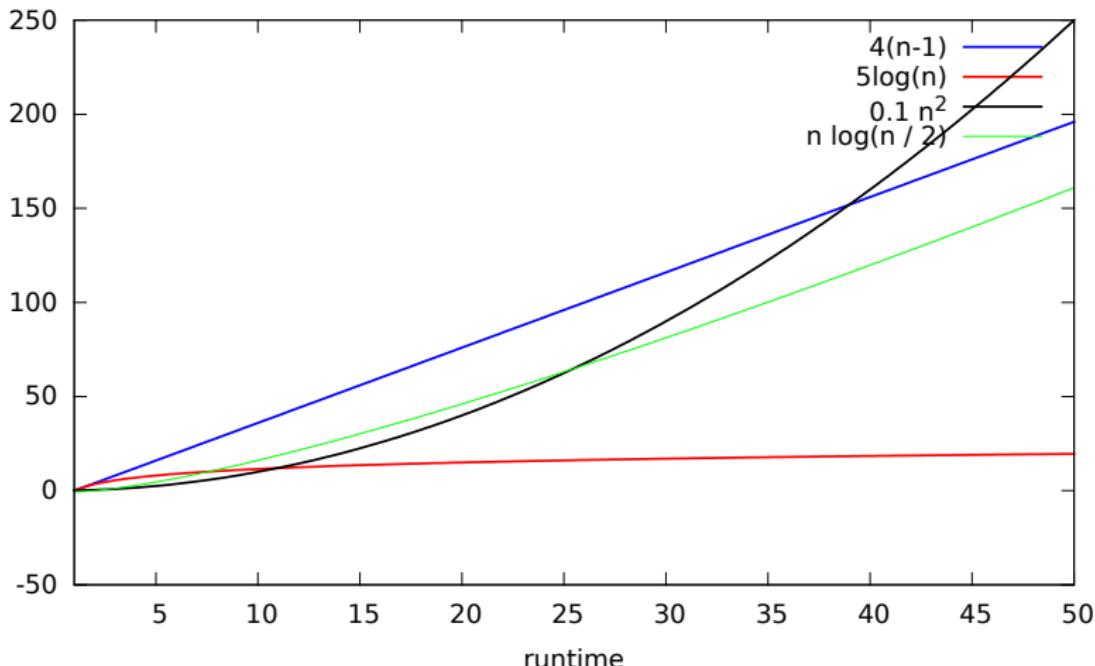
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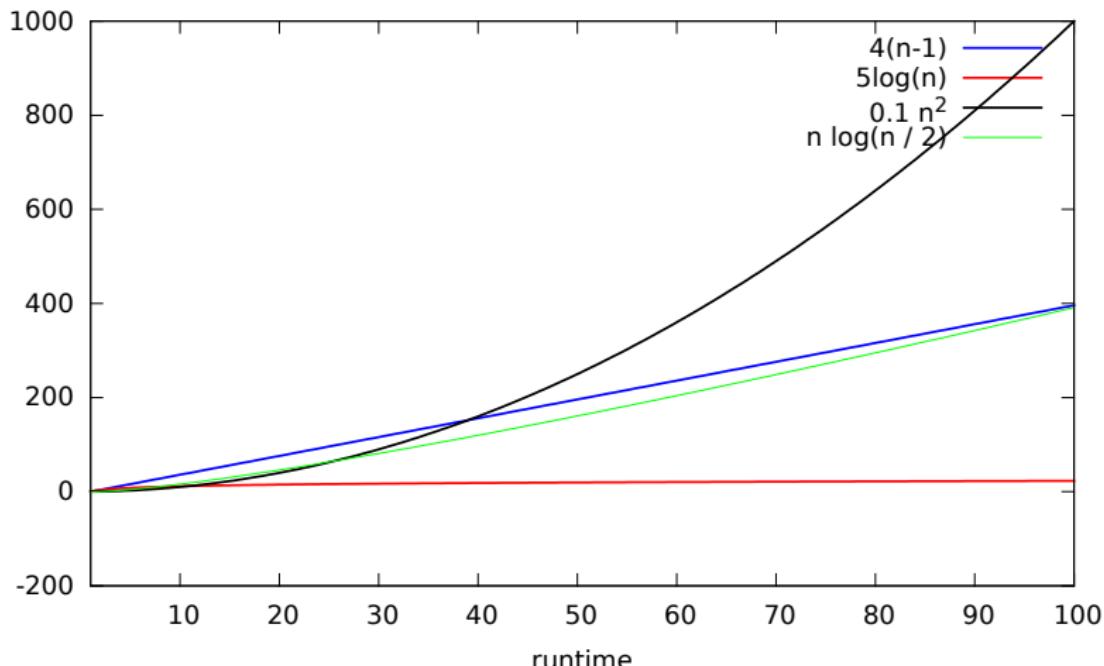
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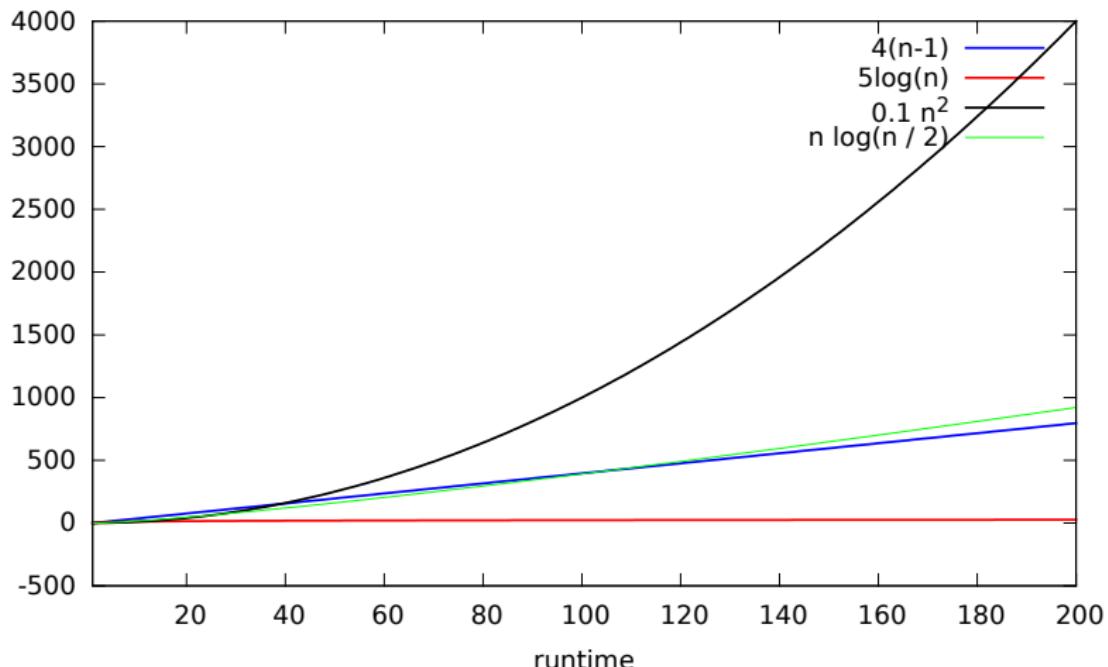
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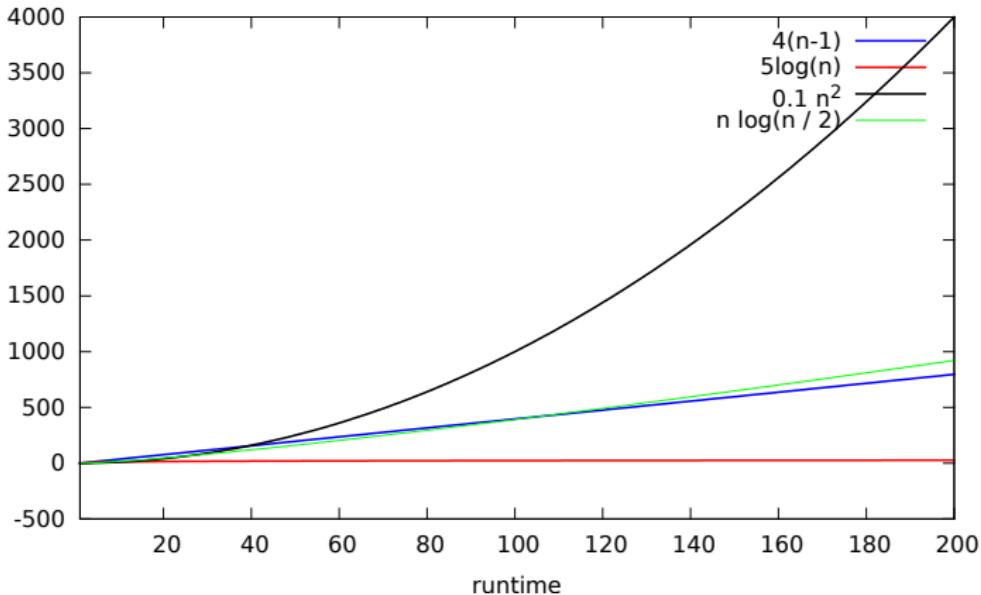
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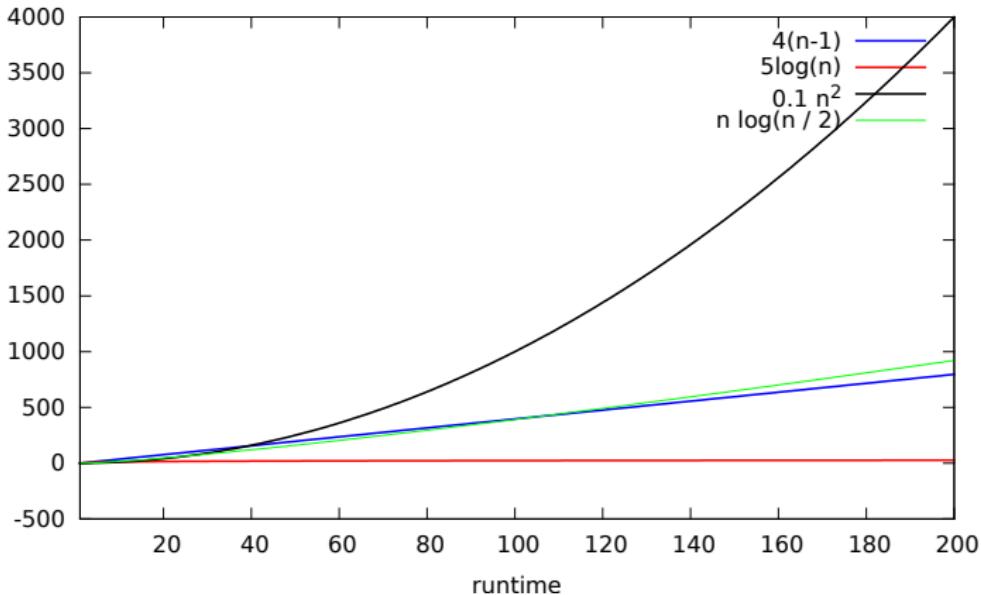
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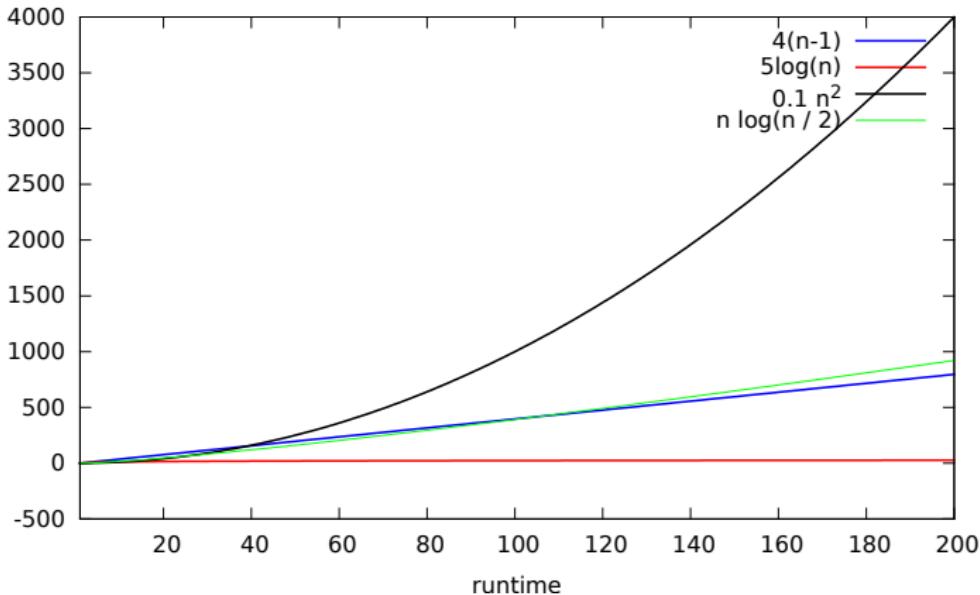
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- $f(n)$  “grows no faster than”  $g(n)$ , ignoring constant factors.



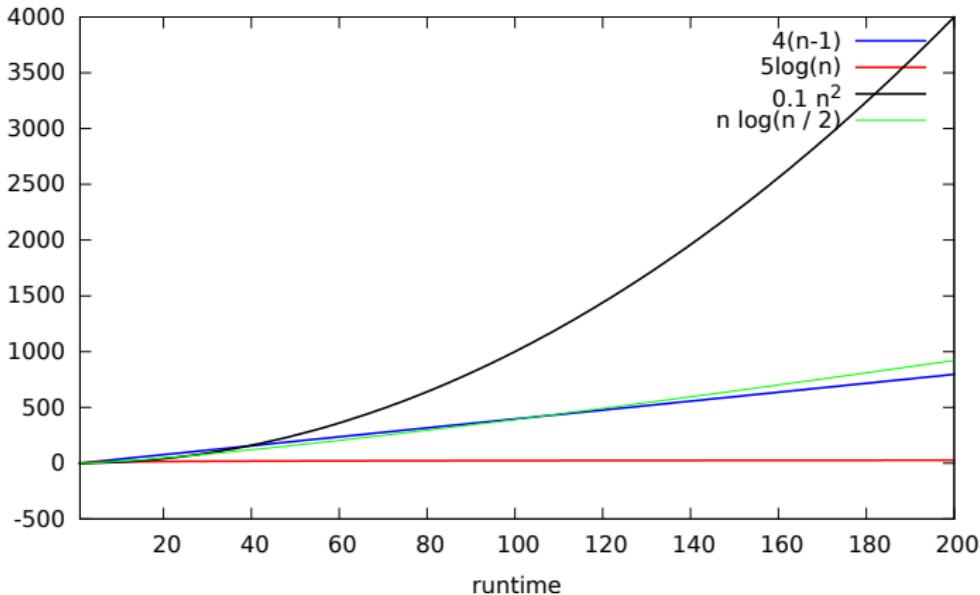
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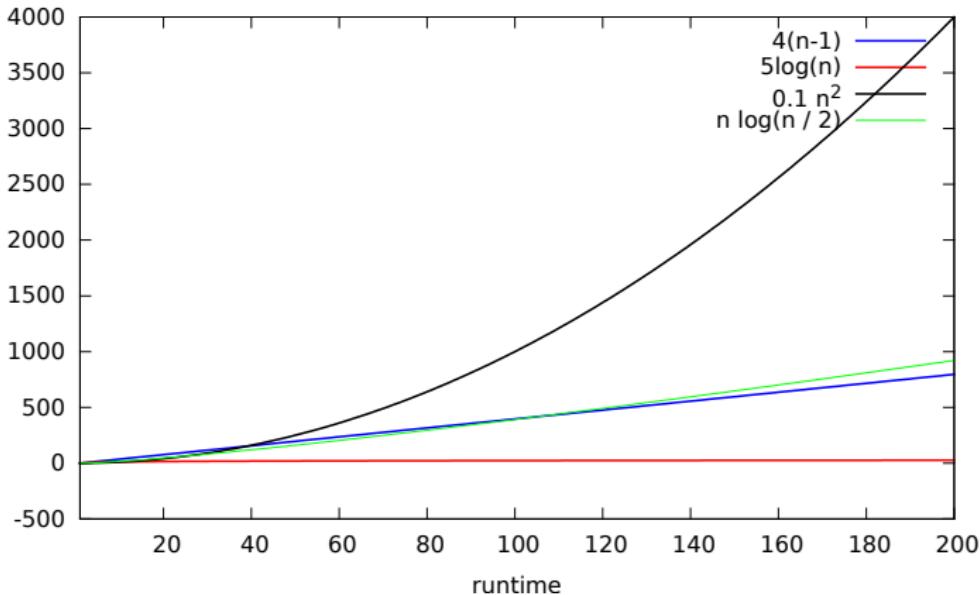
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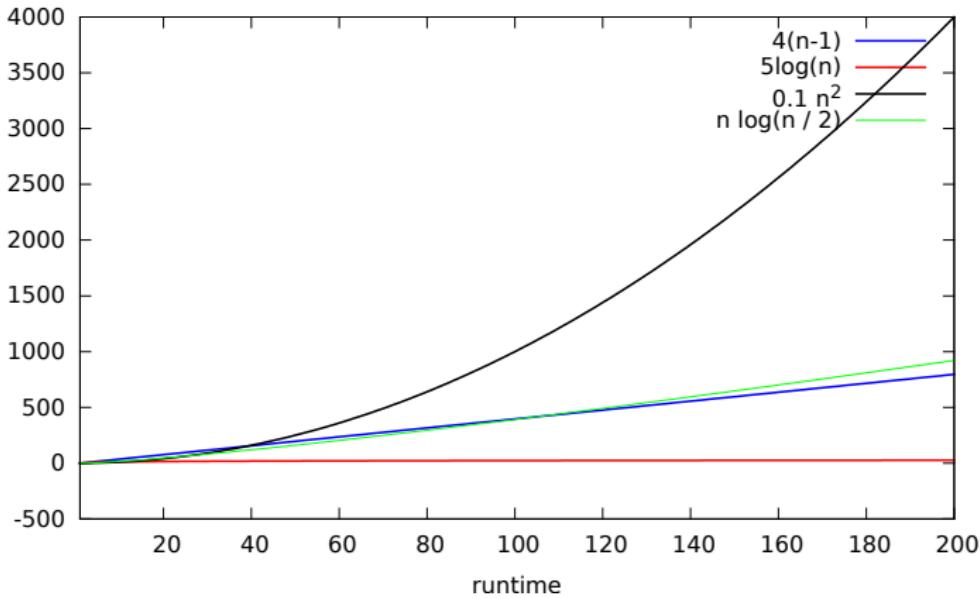
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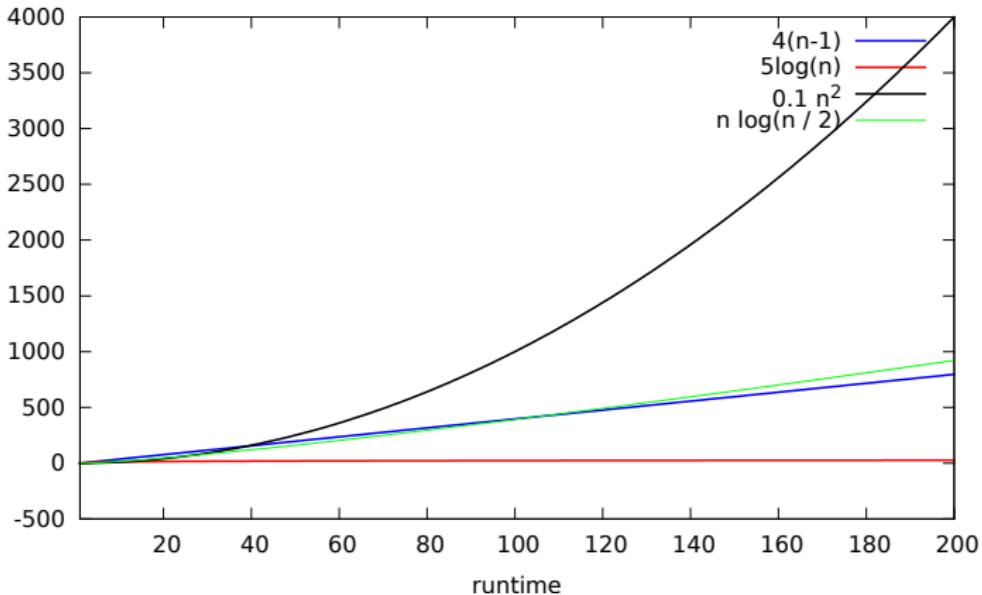
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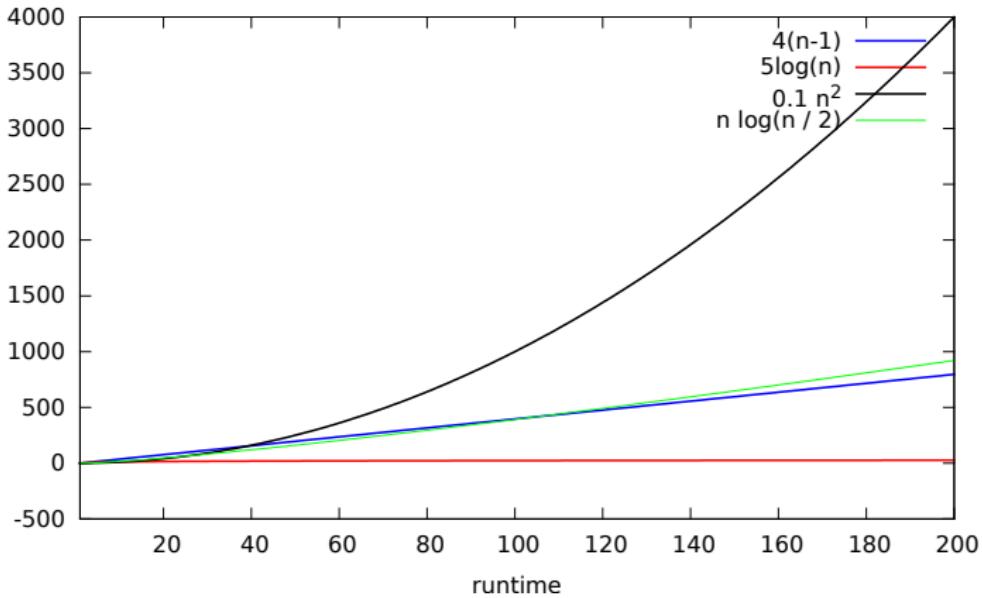
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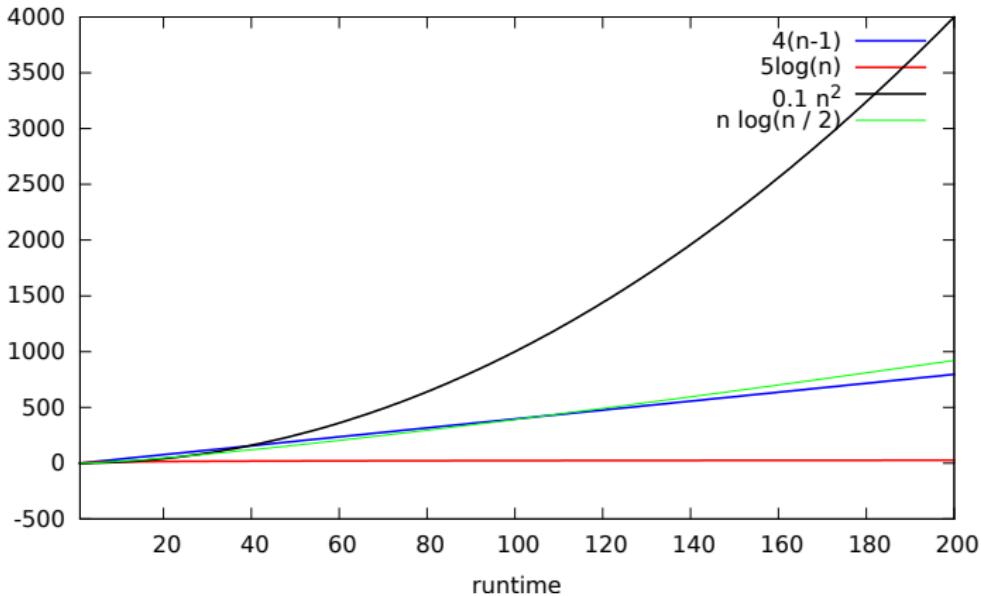
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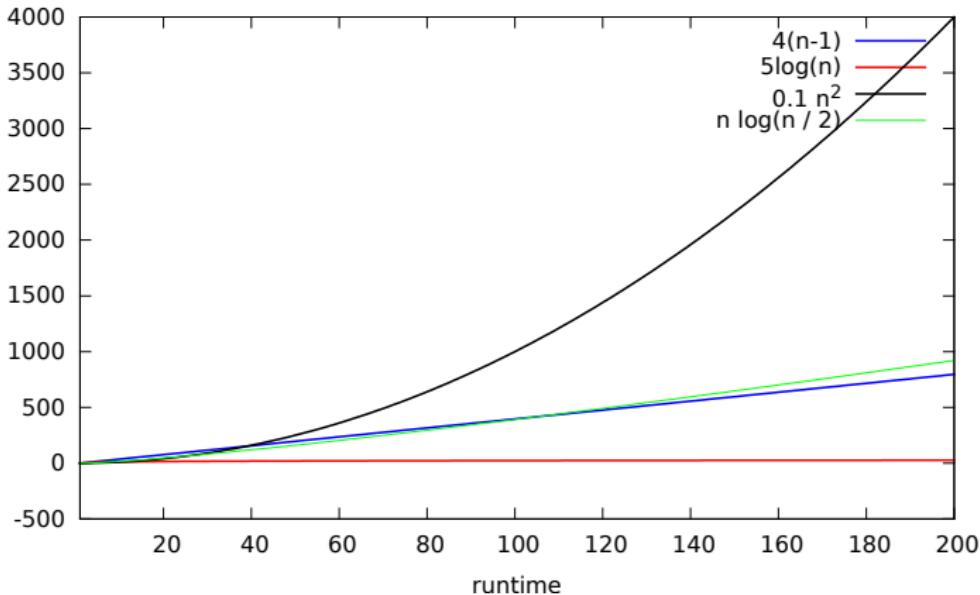
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There exist  $C, n_0 > 0$  such that  $f(n) \leq C \cdot g(n)$  whenever  $n \geq n_0$ . ✓

This rigorous definition is “just” a more precise version of our intuition.

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$f(n) \in O(g(n))$  is good notation for “ $f$  grows no faster than  $g$ , ignoring constants”. But what if we want to say “ $g$  grows no slower than  $f$ ”?

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Looking at it like this, it's much easier to see that

$$n^2 - 5n + 12 \leq 13n^2 \text{ for all } n \geq 1,$$

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So we prove  $n^2 - 5n + 12 \in \Theta(n^2)$  by taking  $c = \frac{1}{2}$ ,  $C = 13$ , and  $n_0 = 10$ .

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$$n! = \underbrace{n \cdot (n - 1) \cdots \cdots 1}_{n \text{ terms}}, \quad 2^n = \underbrace{2 \cdot 2 \cdots \cdots 2}_{n \text{ terms}}.$$

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# Multi-variable O-notation

We will often need O-notation for functions of more than one variable.

For example, an algorithm running on an  $n$ -vertex  $m$ -edge graph will often have running time depending on both  $m$  and  $n$ .

What does it mean to say that e.g.  $f(m, n) \in O(mn)$  or  $f(m, n) \in \Theta(m^2 \log n)$ ?

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For example,  $f(m, n) \in O(g(m, n))$  when there exist  $C$ ,  **$m_0$  and  $n_0$**  such that  $f(m, n) \leq C \cdot g(m, n)$  whenever  $m \geq m_0$  **and**  $n \geq n_0$ .

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All the useful properties of single-variable O-notation (see next video!) carry over to multi-variable O-notation, so e.g. if  $f(m, n) \in O(g(m, n))$  and  $f(m, n) \in \Omega(g(m, n))$  then we still have  $f(m, n) \in \Theta(g(m, n))$ .

## A pedantic clarification

O-notation can behave strangely with negative functions.

But we only care about O-notation for running times, which are positive!

So whenever you are asked to prove something general about O-notation in this course, you can assume the functions involved are non-negative.

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# A pedantic clarification

O-notation can behave strangely with negative functions.

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So the formal requirement is that the functions involved are **eventually non-negative** — that is, before we can say  $f(n) \in O(g(n))$  or similar, we require that  $f(n), g(n) \geq 0$  for all sufficiently large  $n$ .

Any fact that holds about O-notation for non-negative functions will also hold for eventually non-negative functions, by taking  $n_0$  large enough that “eventually non-negative” becomes “non-negative”.