Minimum Spanning Trees II: Kruskal's algorithm
COMS20017 (Algorithms and Data)

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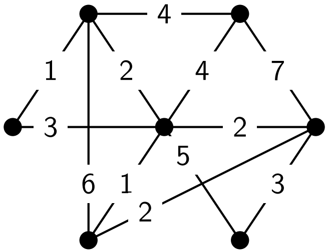

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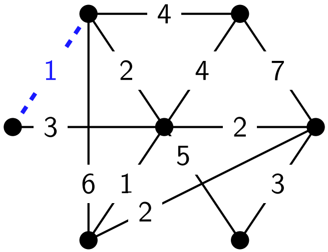

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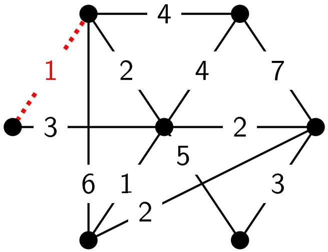

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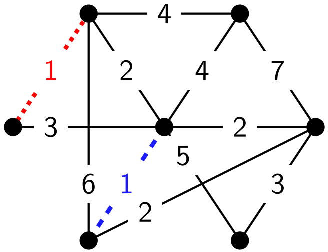

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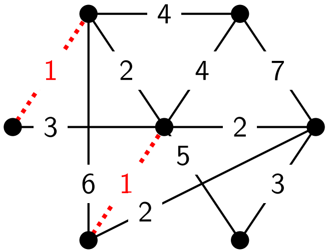

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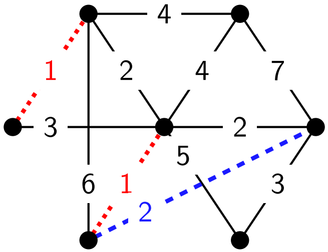

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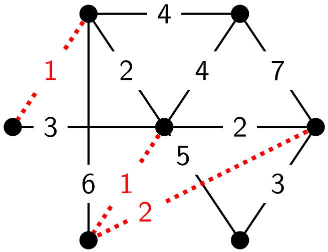

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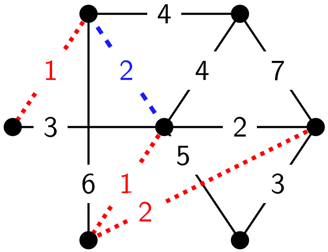

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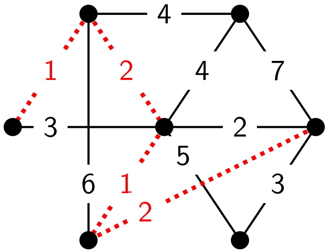

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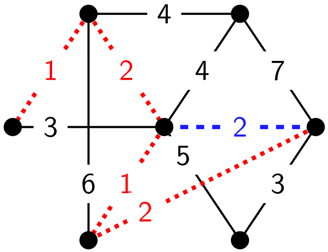

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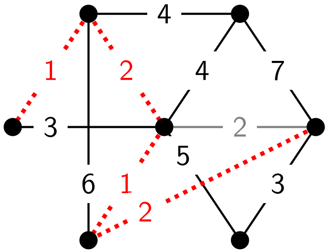

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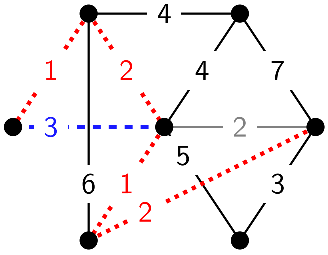

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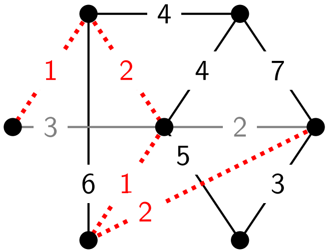

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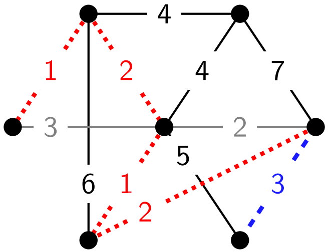

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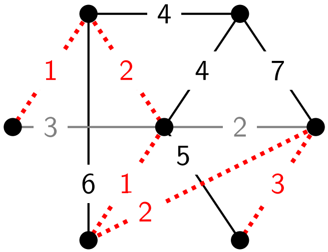

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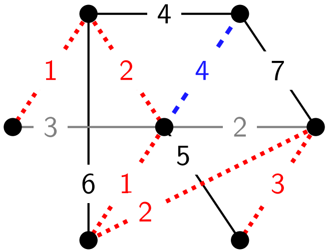

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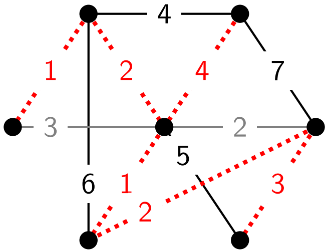

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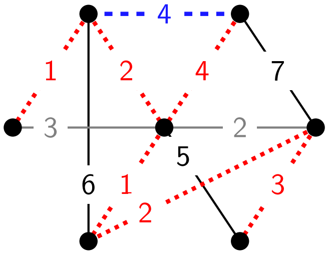

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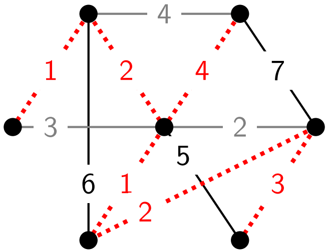

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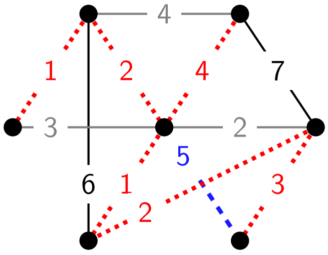

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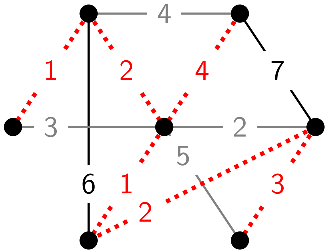

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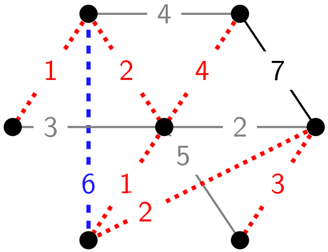

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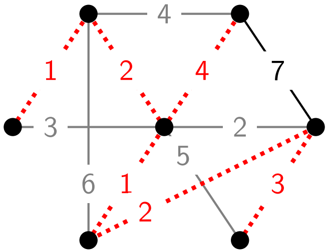

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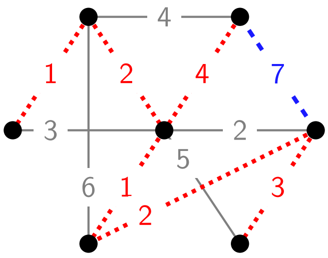

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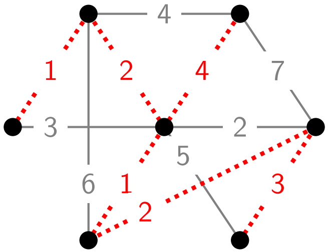

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Formally: Let $e_1, \ldots, e_m$ be the edges of G, with $w(e_1) \leq \cdots \leq w(e_m)$.

Let $T_0 = (V, \emptyset)$ be the empty graph on V.
Given T_i, let $T_{i+1} = T_i + e_{i+1}$ if this is a forest, or T_i otherwise.

Kruskal's algorithm is to calculate and return T_m. Why does this work?

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$\boldsymbol{T_m}$ **is a spanning tree:** Suppose not, for a contradiction.

By construction, T_m has no cycles and $V(T_m) = V$, so T_m must have at least two components C_1 and C_2 (both of which are trees).

Since G is connected, it must contain an edge e_i between C_1 and C_2.
$T_m + e_i$ contains no cycles since C_1 and C_2 are trees, so nor does $T_{i-1} + e_i$, so we should have $e_i \in E(T_i)$ — a contradiction. $\square$

Kruskal's algorithm: Correctness II

$\boldsymbol{T_m}$ **is minimum:** Again we will use an exchange argument.

Let S be a minimum spanning tree with $S \neq T_m$. We will turn S into a tree S^+ with one more edge in common with T_m, and $w(S^+) \leq w(S)$.

By repeating the process, we prove: $w(S) \geq w(S^+) \geq \cdots \geq w(T_m)$, and we're done.

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If we then remove any other edge from C, the result is a tree.
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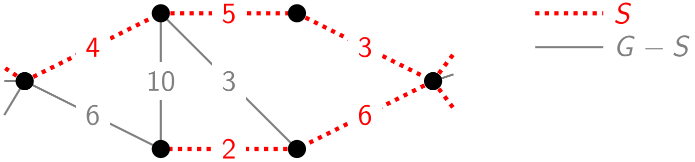

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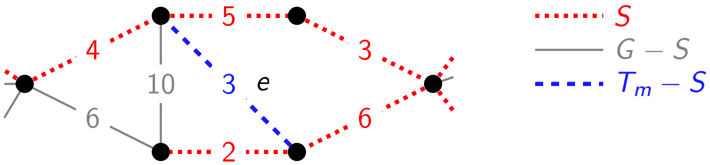

Kruskal's algorithm: Correctness II

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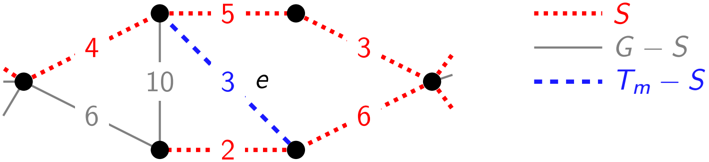

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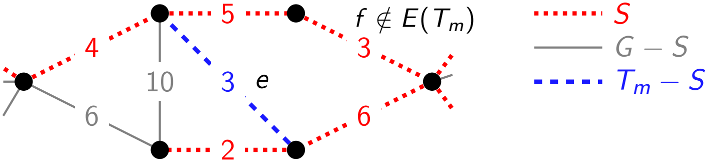

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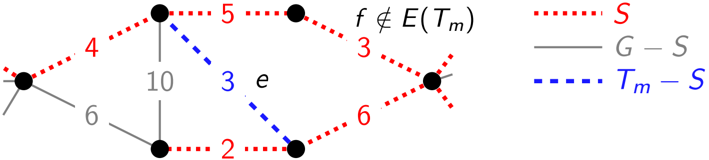

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Since Kruskal's algorithm added e instead of f, we have $w(e) \leq w(f)$. We therefore take $S^+ = S - f + e$.

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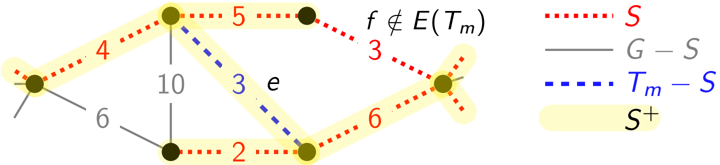

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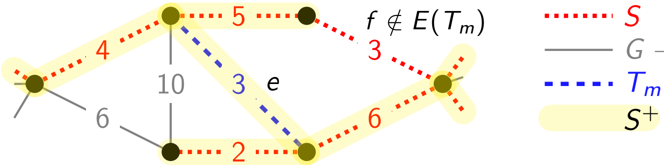

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