Directed Euler walks
COMS20017 (Algorithms and Data)

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Last week...

One piece of new notation: For all integers $n \geq 1$, $[n] := \{1, \dots, n\}$.

A **walk** from u to v in a graph $G = (V, E)$ is a sequence of vertices $w_0 \dots w_k$ with $w_0 = u$, $w_k = v$, and with $\{v_i, v_{i+1}\} \in E$ for all $i \leq k - 1$.

A **path** is a walk with no repeated vertices.

An **Euler walk** is a walk containing every edge in G exactly once.

A vertex's **degree** is the number of edges intersecting ("incident to") it.

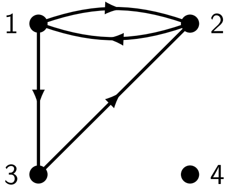
A graph is **connected** if any two vertices are joined by a path.

We showed that a connected graph has an Euler walk if and only if either all, or all but two, of its vertices have even degree.

Directed graphs

A **directed graph** (or **digraph**) is a pair $G = (V, E)$, where V is a set of vertices and E is a set of edges contained in $\{(u, v) \colon u, v \in V,\ u \neq v\}$.

E.g. $V = [4]$ and $E = \{(1,2), (2,1), (1,3), (3,2)\}$ looks like:



We use directed graphs when we want to model **asymmetric** relations.

For example, a software dependency graph: "vi depends on the kernel" shouldn't imply "the kernel depends on vi"!

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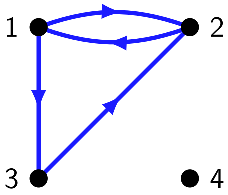
Most graph definitions (subgraphs etc.) carry over to digraphs unchanged.

To develop our intuition for those that don't, we now generalise our Euler walks result to digraphs.

A **walk** in a digraph G is defined in (almost) the same way as in a graph: a sequence of vertices $w_0 \dots w_k$ with $(w_i, w_{i+1}) \in E$ for all $i \leq k - 1$.

Again as in graphs, a **path** is a walk with no repeated vertices, and an **Euler walk** is a walk which uses every edge in the graph exactly once. E.g.:



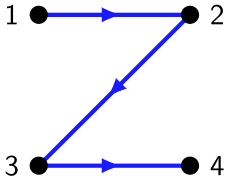
A **directed graph** (or **digraph**) is a pair $G = (V, E)$, where V is a set of vertices and E is a set of edges contained in $\{(u, v) : u, v \in V,\ u \neq v\}\}$.

A **walk** from u to v is a sequence of vertices $w_0 \dots w_k$ with $w_0 = u$, $w_v = v$ and $(w_i, w_{i+1}) \in E$ for all $i \leq k - 1$.

A **path** is a walk with no repeated vertices.

G is **strongly connected** if for all $u, v \in V$, there is a path from u to v **and** a path from v to u.

Warning: For undirected graphs, these two paths would be the same. But here, G can have a path from u to v, but no path from v to u!



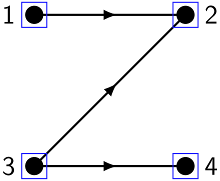
So this graph is **not** strongly connected, as there's no path from 4 to 1.

And yet it still has an Euler walk...

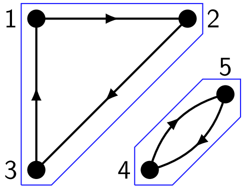
G is **strongly connected** if for all $u, v \in V$, there is a path from u to v.

A digraph can have an Euler walk despite not being strongly connected...

So we need another notion of connectivity too. G is **weakly connected** if for all $u, v \in V$, there is a path from u to v **ignoring edge directions**.



This graph is weakly connected...



But this graph is not (e.g. it has no undirected path from 2 to 4).

As with graphs, **strong components** (resp. **weak components**) of digraphs are maximal strongly (resp. weakly) connected induced subgraphs.

A digraph definitely can't have an Euler walk if it's not weakly connected!
And it can't have one **with equal endpoints** if it's not strongly connected.
(At least if there are no isolated vertices...)

G is **strongly connected** if for all $u, v \in V$, there is a path from u to v.

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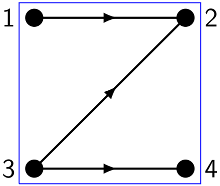
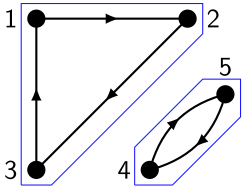


This graph is weakly connected...



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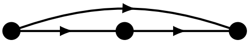
A digraph definitely can't have an Euler walk if it's not weakly connected!
And it can't have one **with equal endpoints** if it's not strongly connected.
(At least if there are no isolated vertices...)

Ignoring isolated vertices, an **undirected** graph has an Euler walk iff it is connected and all, or all but two, of its vertices have even degree.

For digraphs, we think "connected" will become "strongly connected" or "weakly connected". What about "even degree"?

As with undirected graphs, the **neighbourhood** $N(v)$ of a vertex v is the set of vertices it's adjacent to, and its **degree** $d(v)$ is the number of edges v is contained in. But...



Here every vertex has even degree (namely degree 2), but there's no Euler walk!

We define the **in-neighbourhood** $N^-(v)$ and the **out-neighbourhood** $N^+(v)$ of a vertex v by

$$N^-(v) = \{u \in V(G) \colon (u, v) \in E(G)\},$$
$$N^+(v) = \{w \in V(G) \colon (v, w) \in E(G)\}.$$

The **in-degree** $d^-(v)$ is $|N^-(v)|$, and the **out-degree** $d^+(v)$ is $|N^+(v)|$.

Note that $d(v) = d^-(v) + d^+(v)$.

Ignoring isolated vertices, an **undirected** graph has an Euler walk iff it is connected and all, or all but two, of its vertices have even degree.

For digraphs, we think "connected" will become "strongly connected" or "weakly connected". What about "even degree"?

The **in-degree** of v is the number of edges pointing towards v.
The **out-degree** of v is the number of edges pointing away from v.

Let $W = w_0 \ldots w_k$ be a walk in a digraph G. For any vertex x, W has:

- one edge out of x and one edge into x for each time x appears in $\{w_1, \ldots, w_{k-1}\}$;
- one extra edge out of x if $x = w_0$;
- one extra edge into x if $x = w_k$.

If W is Euler, it contains $d^-(x)$ edges into x and $d^+(x)$ edges out of x.

So if $x \notin \{w_0, w_k\}$, or $x = w_0 = w_k$, then $d^+(x) = d^-(x)$.
If $x = w_0 \neq w_k$, then $d^+(x) = d^-(x) + 1$.
And if $x = w_k \neq w_0$, then $d^-(x) = d^+(x) + 1$.

Ignoring isolated vertices, an **undirected** graph has an Euler walk iff it is connected and all, or all but two, of its vertices have even degree.

For digraphs, we think "connected" will become "strongly connected" or "weakly connected". What about "even degree"?

The **in-degree** of v is the number of edges pointing towards v.
The **out-degree** of v is the number of edges pointing away from v.

We have shown that if W is an Euler walk, for any vertex x, either:

- $x = w_0 = w_k$ and $d^+(x) = d^-(x)$; or
- $x \notin \{w_0, w_k\}$ and $d^+(x) = d^-(x)$; or
- $x = w_0 \neq w_k$ and $d^+(x) = d^-(x) + 1$; or
- $x = w_k \neq w_0$ and $d^-(x) = d^+(x) + 1$.

Theorem: Let G be a digraph with no isolated vertices containing an Euler walk $W = w_0 \dots w_k$. Then G is weakly connected and either:

- $d^+(v) = d^-(v)$ for all $v \in V$, and $w_0 = w_k$; or
- $d^-(v) = d^+(v)$ for all $v \notin \{w_0, w_k\}$, $d^+(w_0) = d^-(w_0) + 1$, and $d^-(w_k) = d^+(w_k) + 1$. (So also $w_0 \neq w_k$.) $\square$

Theorem: Let G be a digraph with no isolated vertices containing an Euler walk $W = w_0 \dots w_k$. Then G is weakly connected and either:

- G is strongly connected, $d^+(v) = d^-(v)$ for all $v \in V$, and $w_0 = w_k$; or
- $d^-(v) = d^+(v)$ for all $v \notin \{w_0, w_k\}$, $d^+(w_0) = d^-(w_0) + 1$, and $d^-(w_k) = d^+(w_k) + 1$. (So also $w_0 \neq w_k$.) $\square$

As with undirected graphs, this turns out to be sufficient!

Theorem: Let $G = (V, E)$ be a digraph with no isolated vertices, and let $u, v \in V$. Then G has an Euler walk from u to v if and only if G is **weakly** connected and either:

(i) $u = v$ and every vertex of G has equal in- and out-degrees; or

(ii) $u \neq v$, $d^+(u) = d^-(u) + 1$, $d^-(v) = d^+(v) + 1$, and every other vertex of G has equal in- and out-degrees. $\square$

It's surprising that weak connectedness turns out to be good enough!

It turns out that weak connectedness implies strong connectedness when every vertex has equal in- and out-degrees.