

# Hamilton cycles

## COMS20017 (Algorithms and Data)

John Lapinskas, University of Bristol

# Hamilton cycles

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A **cycle** is a walk  $W = w_0 \dots w_k$  with  $w_0 = w_k$  and  $k \geq 3$ , in which every vertex appears at most once except for  $w_0$  and  $w_k$  (which appear twice).

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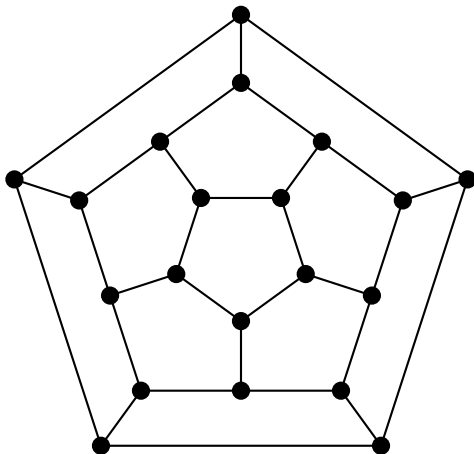


Hamilton actually made and sold a game based on trying to find Hamilton cycles in a dodecahedron!

Perhaps not surprisingly, it didn't sell very well.

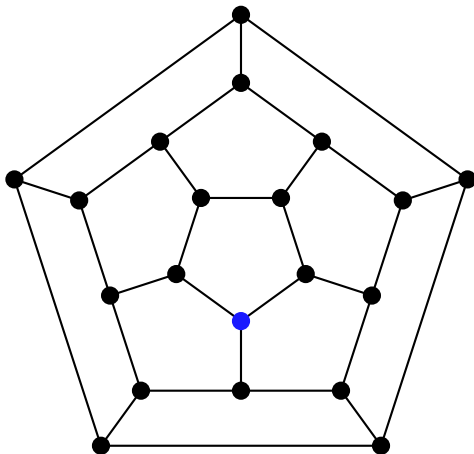
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In a particular (undirected) graph, Hamilton cycles can be easy to find:



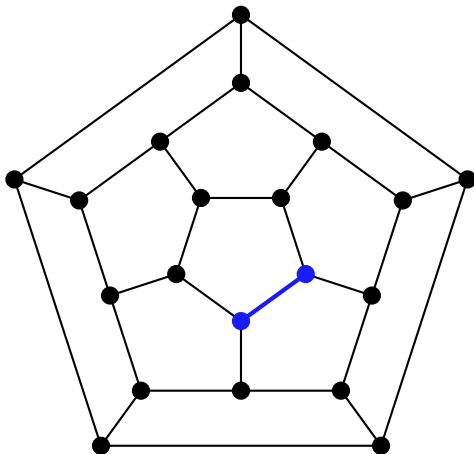
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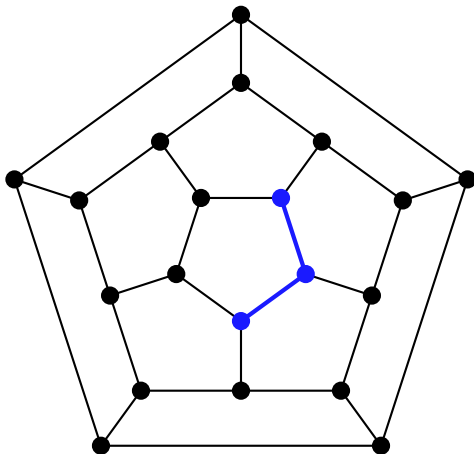
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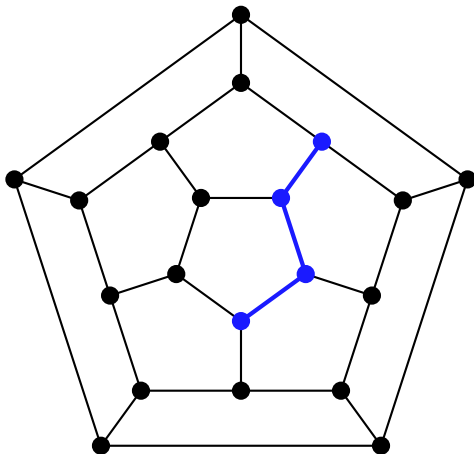
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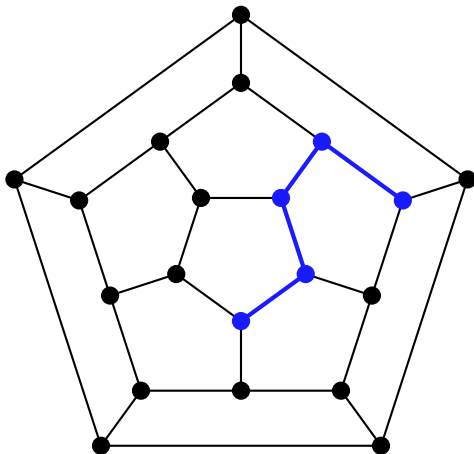
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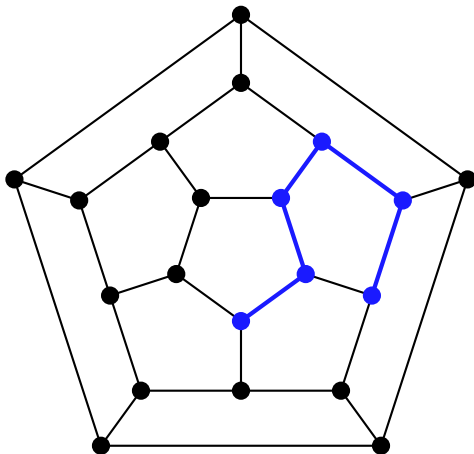
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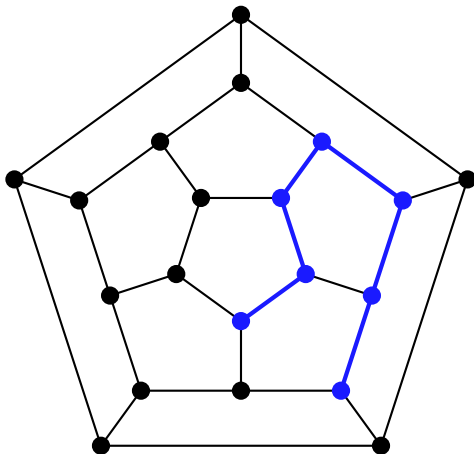
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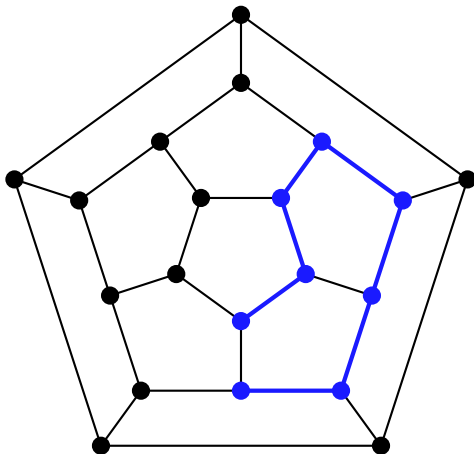
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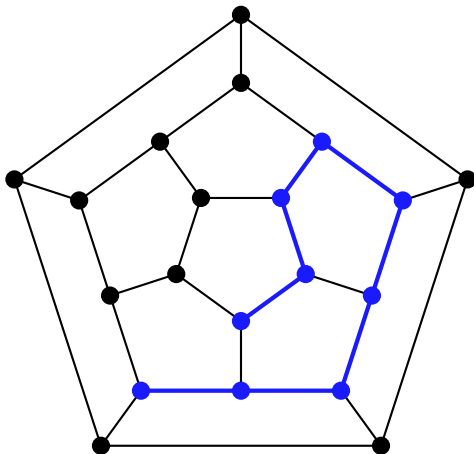
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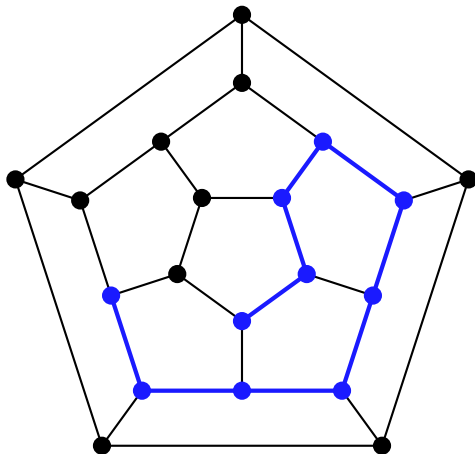
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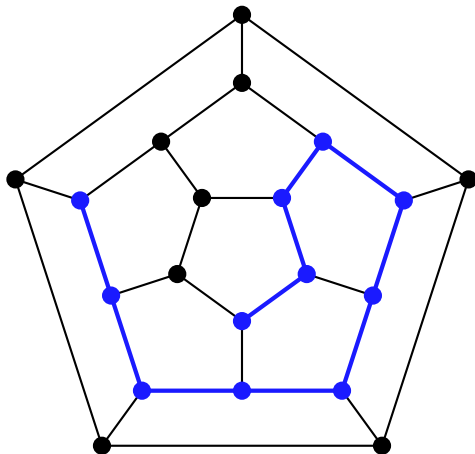
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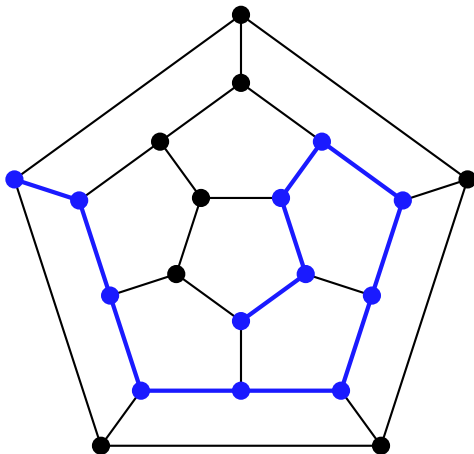
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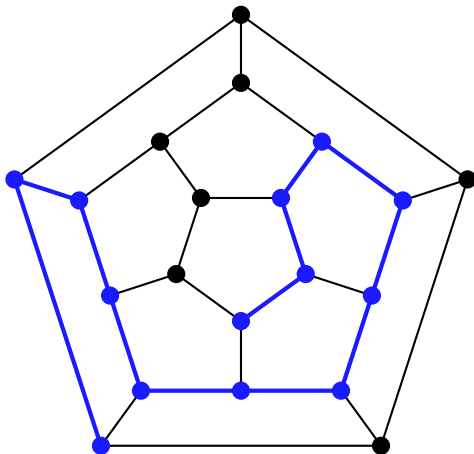
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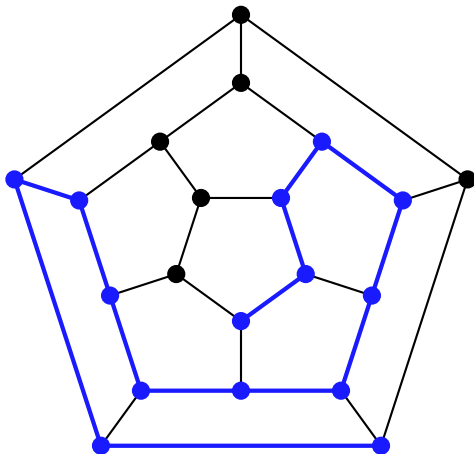
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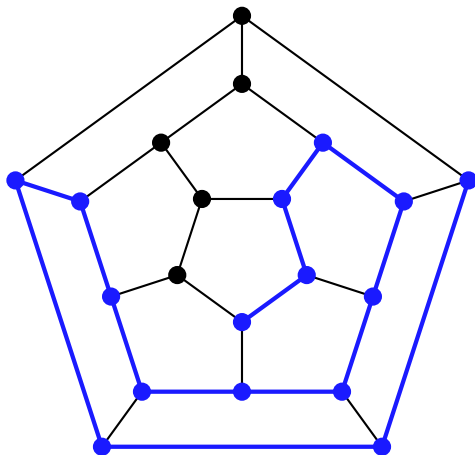
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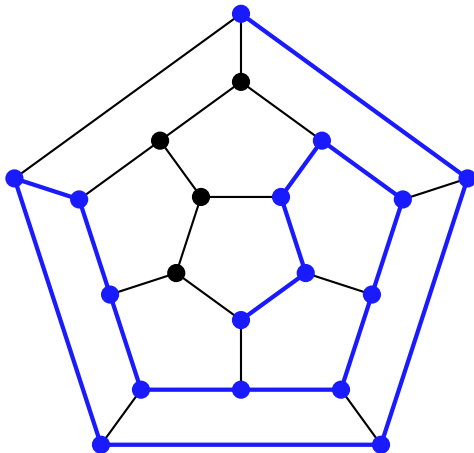
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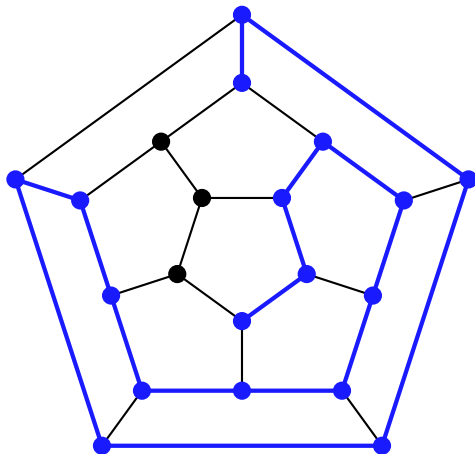
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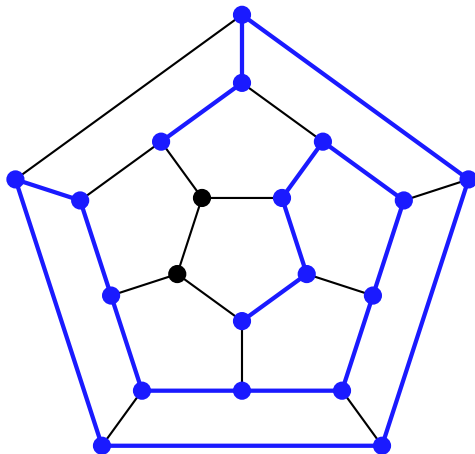
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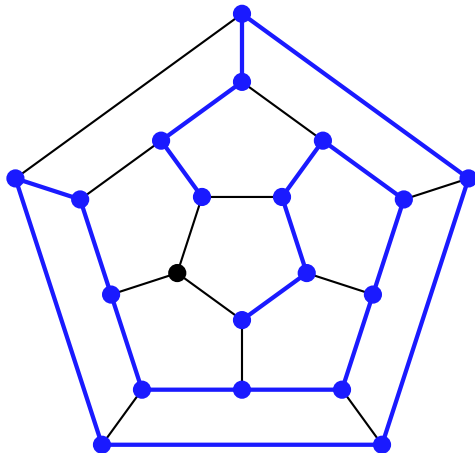
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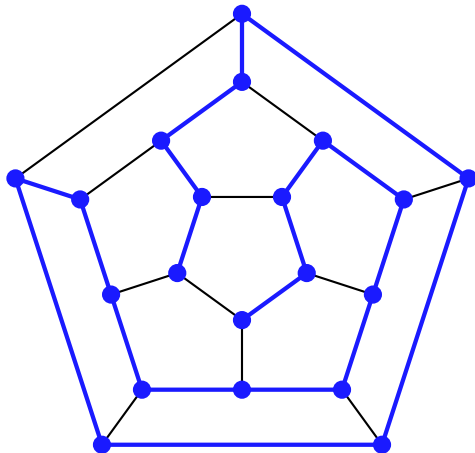
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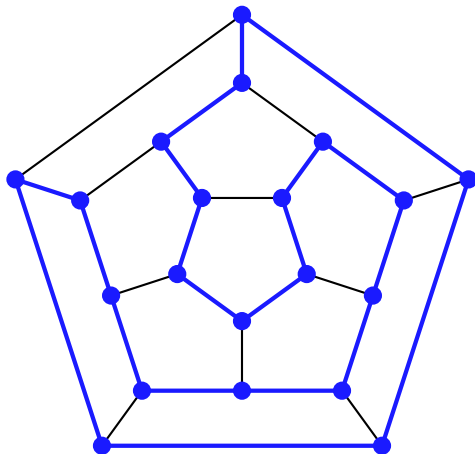
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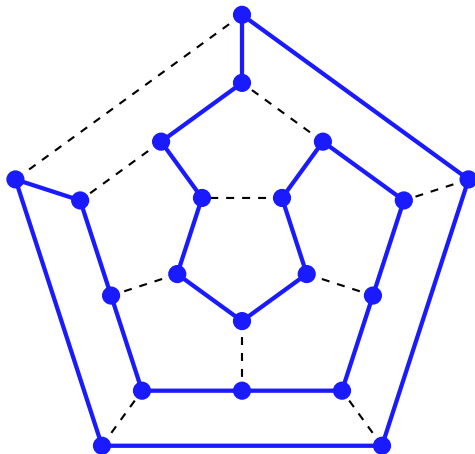
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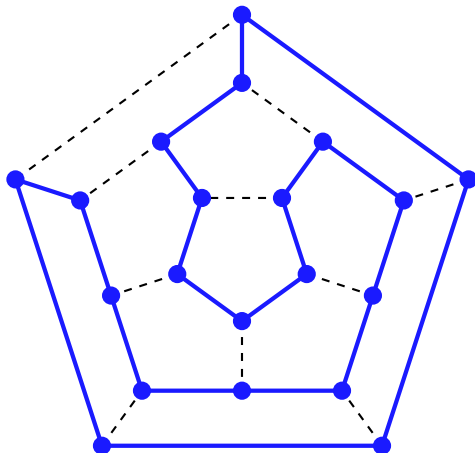
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But in general, they can be very hard to find. If you prove an easy-to-check condition like the one for Euler walks, you stand to win a million dollars!

# Dirac's theorem

Just because we can't find Hamilton cycles in general doesn't mean we can't find them in special cases...

**Dirac's Theorem:** Let  $n \geq 3$ . Then any  $n$ -vertex graph  $G$  with minimum degree at least  $n/2$  has a Hamilton cycle.

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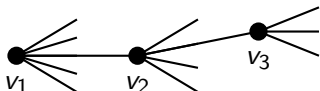
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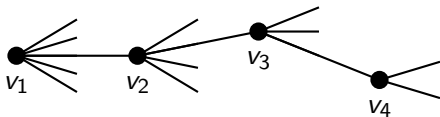
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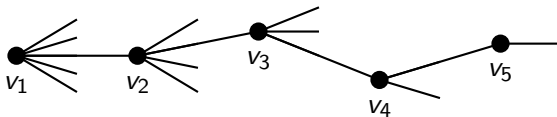
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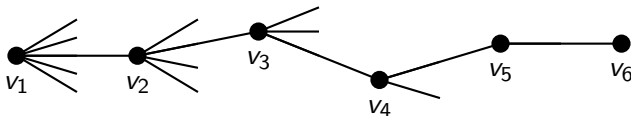
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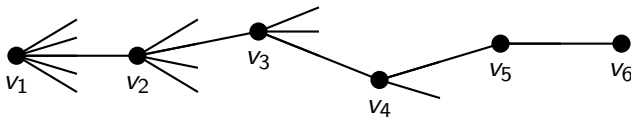
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In general,  $d(v_k) \geq n/2 > |\{v_1, \dots, v_{k-1}\}|$ , so there's a vertex  $v_{k+1}$  adjacent to  $v_k$  other than  $v_1, \dots, v_{k-1}$ . Then  $v_1 \dots v_{k+1}$  is a path of length  $k+1$ .



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**Case 2b: There exists a vertex  $v_0 \in N(v_1) \setminus \{v_2, \dots, v_k\}$ .**  
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**Case 2c: Both  $N(v_1) \subseteq \{v_2, \dots, v_k\}$  and  $N(v_k) \subseteq \{v_1, \dots, v_{k-1}\}$ .**  
In this case, we *use* the fact that greedy extension fails to extend the path in another way.

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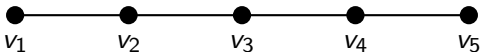
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We are done unless  $N(v_1) \subseteq \{v_2, \dots, v_k\}$  and  $N(v_k) \subseteq \{v_1, \dots, v_{k-1}\}$ .

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Never think about graphs without a picture. What does this **look like**?

Say just for  $n = 8$ ,  $k = \frac{1}{2}n + 1 = 5$ ?



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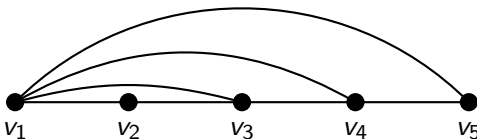
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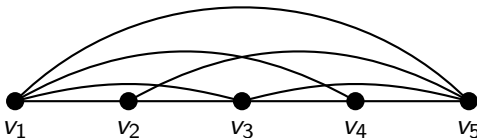
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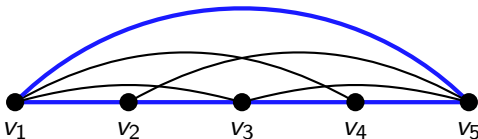
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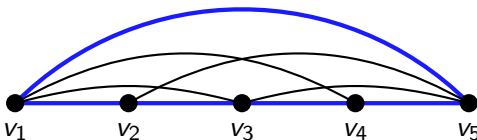
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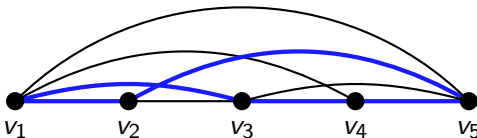
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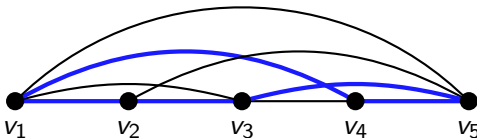
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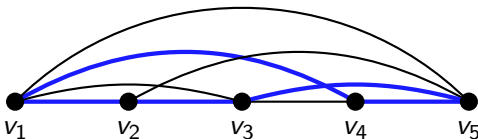
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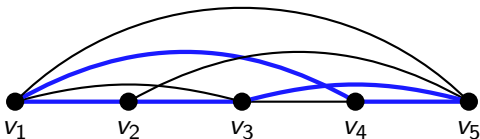
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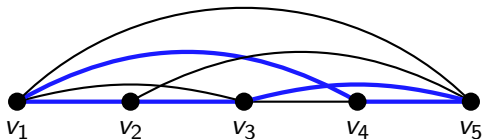
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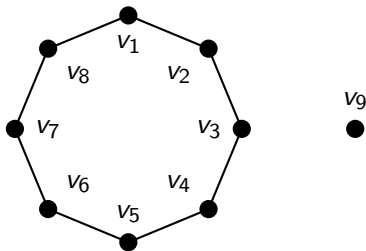
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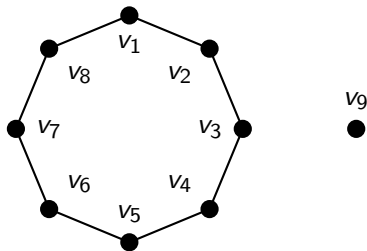
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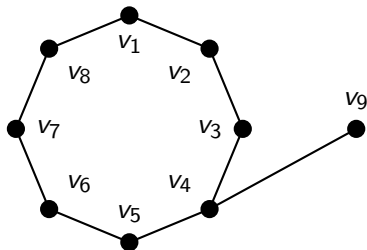
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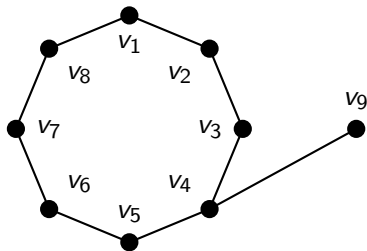
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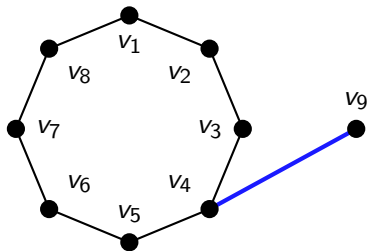
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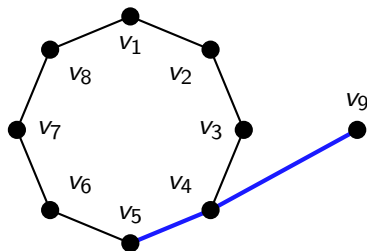
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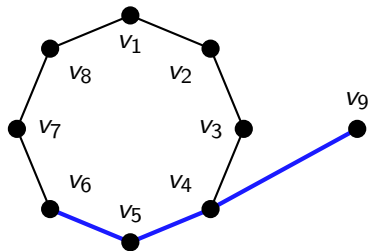
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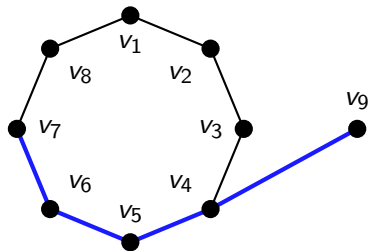
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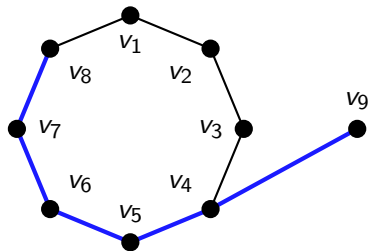
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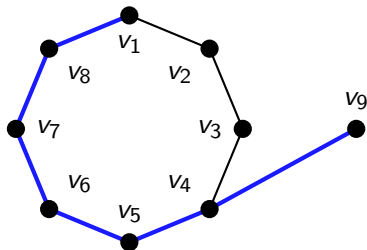
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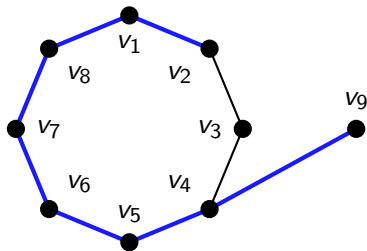
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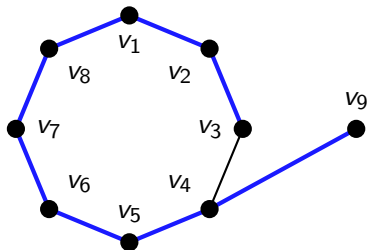
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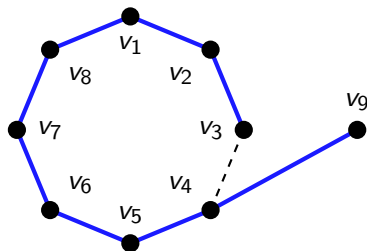


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So now by starting with a single-vertex path and repeatedly applying our three Lemmas, we reach an  $n$ -vertex path.

Then Lemma 2 turns this into a Hamilton cycle and we're done! □

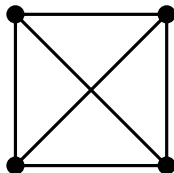
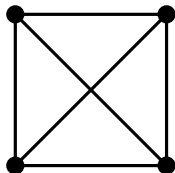
Note this proof gives us a (fairly fast) algorithm for finding a Hamilton cycle when Dirac's theorem applies. This often happens in graph theory!

# How good is Dirac's Theorem?

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But there are other ways to improve it. For example, when we do have minimum degree  $n/2$ , there's more than just one Hamilton cycle.

In fact, for large graphs, we can find  $(n - 2)/8$  **disjoint** Hamilton cycles, decomposing almost half the graph!

(Proved in 2013–4 by Csaba, Kühn, Lapinskas, Lo, Osthus and Treglown.)