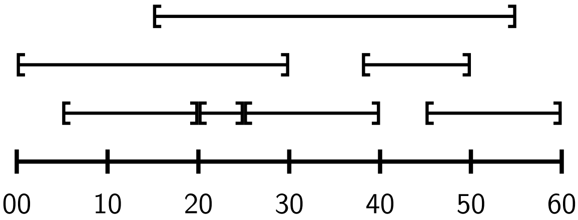
# Correctness proofs for interval scheduling
# COMS20017 (Algorithms and Data)

John Lapinskas, University of Bristol

# Recall from last time

To solve interval scheduling with input $\mathcal{R}$, we repeatedly choose the compatible interval with the earliest finish time and add it to the output.

**Goal:** Prove our pseudocode $\textsc{GreedySchedule}$ is correct.



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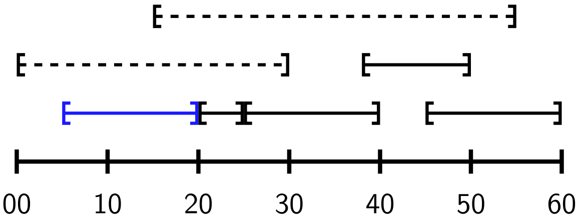

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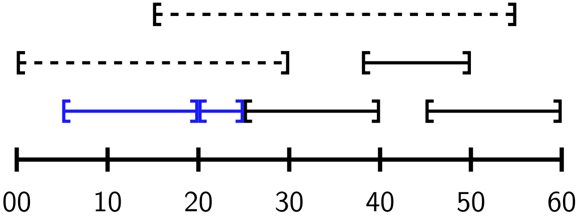

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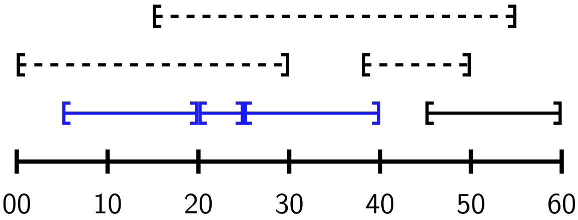

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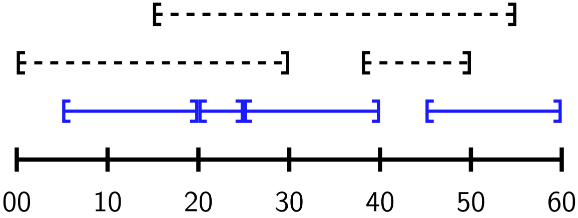

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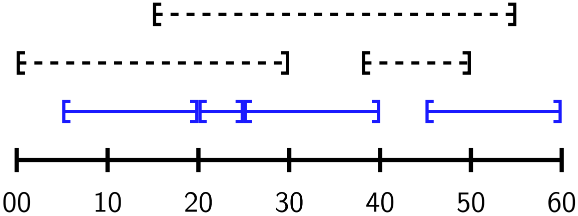


Let's define this formally: breaking ties arbitrarily, let

$$A^+ := \arg\min\{f : (s,f) \in \mathcal{R}, A \cup \{(s,f)\} \text{ is compatible}\} \text{ for all } A \subseteq \mathcal{R},$$

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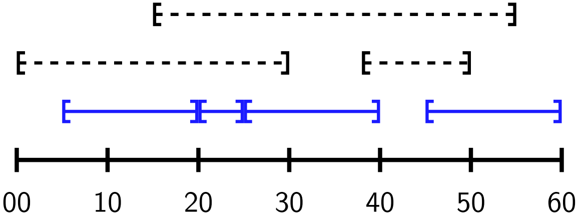


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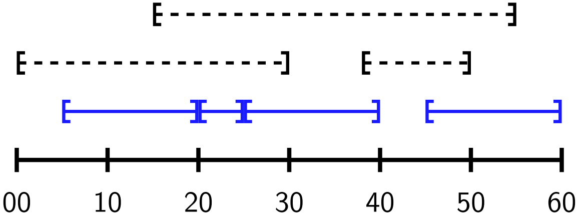


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So (we think) $\textsc{GreedySchedule}$ calculates $A_0, \dots, A_t$ and outputs $A_t$. Much easier to work from this than pseudocode when proving correctness!

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**Algorithm:** GreedySchedule

**Input** : An array $\mathcal{R}$ of $n$ requests.
**Output** : Maximum compatible subset of $\mathcal{R}$.

**begin**
- Sort $\mathcal{R}$'s entries so that $\mathcal{R} \leftarrow [(s_1, f_1), \dots, (s_n, f_n)]$ where $f_1 \leq \dots \leq f_n$.
- Initialise $A \leftarrow [\,]$, `lastf` $\leftarrow 0$.
- **foreach** $i \in \{1, \dots, n\}$ **do**
  - **if** $s_i \geq$ `lastf` **then**
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**Lemma:** GreedySchedule always outputs $A_t$.

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**Proof:** By induction from the following loop invariant. At the start of the $i$'th iteration of 4–7:

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**Proof:** Instant by induction; $A_0$ is compatible, and if $A_i$ is compatible then so is $A_{i+1} = A_i \cup \{A^+\}$ by the definition of $A_i^+$. □

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Sometimes, life is easy!

(Without the lemma, this would have needed a tedious loop invariant...)

# Step 3: Prove our algorithm outputs a maximum set

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More formally, let $B \subseteq \mathcal{R}$ be any other compatible set with $|B| \geq |A_t|$, and let $B_i$ consist of the $i$ fastest-finishing elements of $B$.

Then we will show by induction that for all $0 \leq i \leq t$, the last finish time of $B_i$ is no earlier than the last finish time of $A_i$.

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**Proof:** By induction on $i$.

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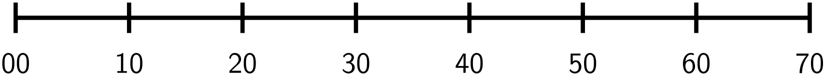

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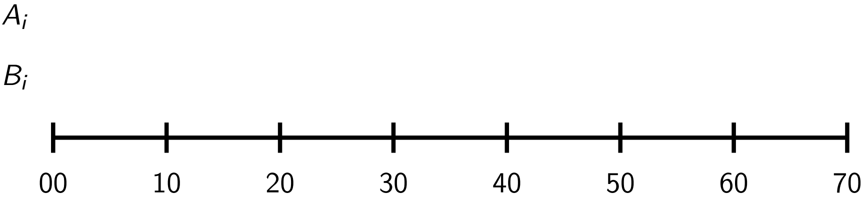


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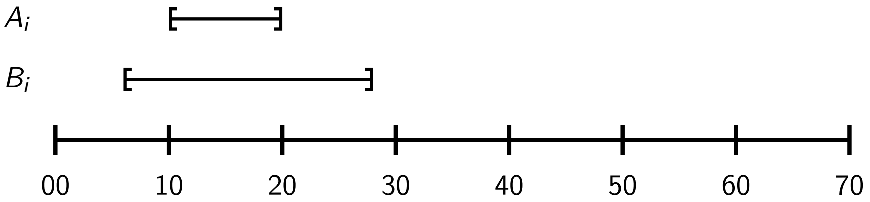


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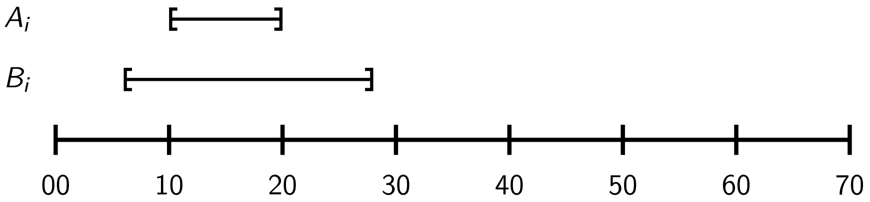

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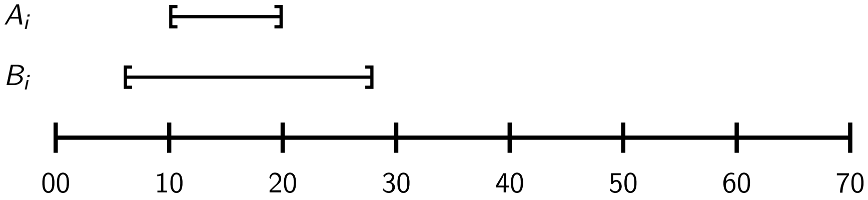


**Inductive step:** Suppose $A_i$ finishes faster than $B_i$.

Let $B_i^+$ be the $(i+1)'$st fastest-finishing element of $B$.
Since $A_i$ finishes faster than $B_i$, $A_i \cup \{B_i^+\}$ is compatible.
Hence by definition, $A_i^+$ exists and finishes no later than $B_i^+$. ✓

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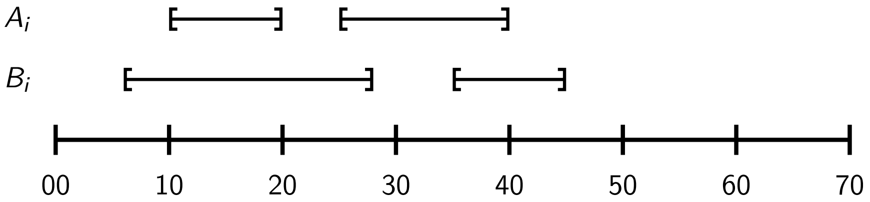


**Inductive step:** Suppose $A_i$ finishes faster than $B_i$.

Let $B_i^+$ be the $(i+1)$'st fastest-finishing element of $B$.
Since $A_i$ finishes faster than $B_i$, $A_i \cup \{B_i^+\}$ is compatible.
Hence by definition, $A_i^+$ exists and finishes no later than $B_i^+$. ✓

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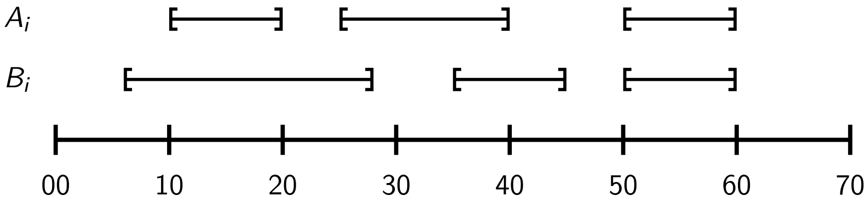


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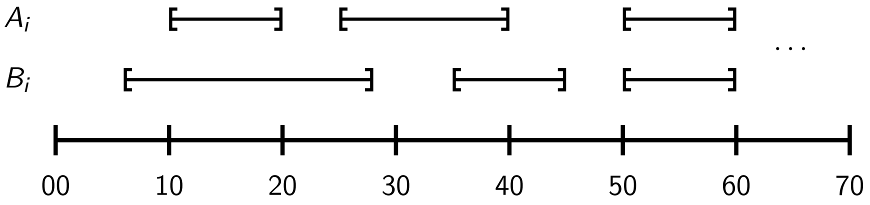


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Putting it all together, we obtain...

**Theorem:** GreedySchedule outputs $A_t$, which is a **maximum** compatible set. □

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This technique of proving that the greedy solution "stays ahead" of any other solution is very useful for other greedy algorithms as well!

# An alternative proof of optimality

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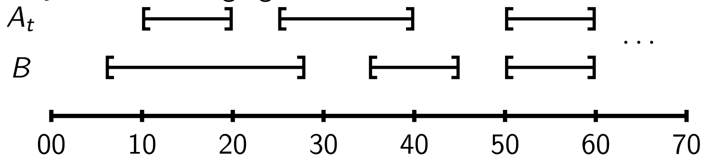

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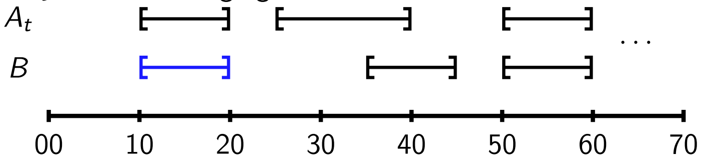

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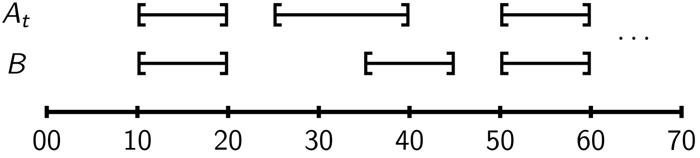

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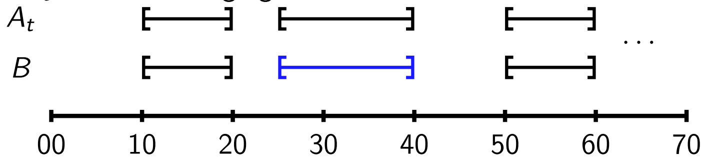

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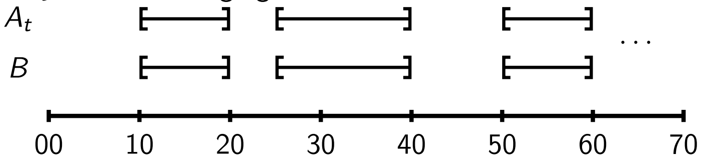

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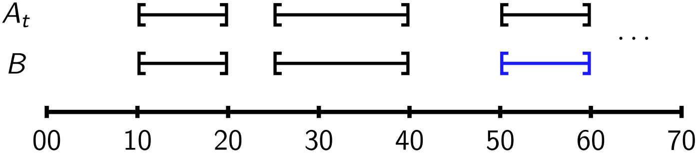

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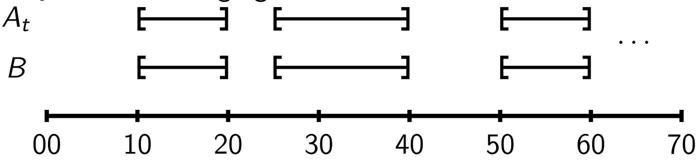

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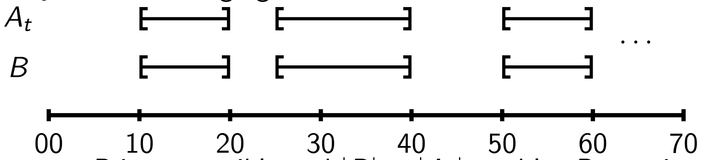


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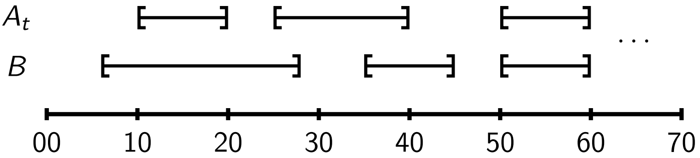


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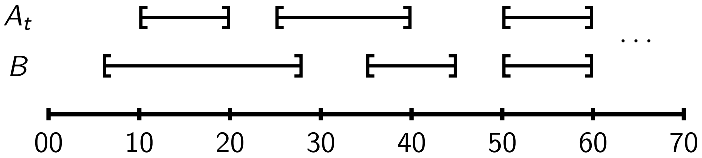


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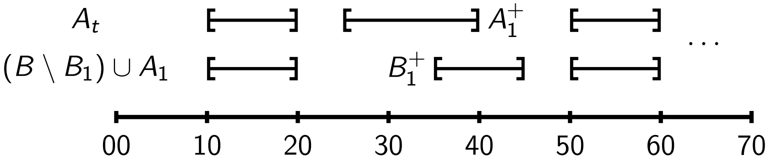


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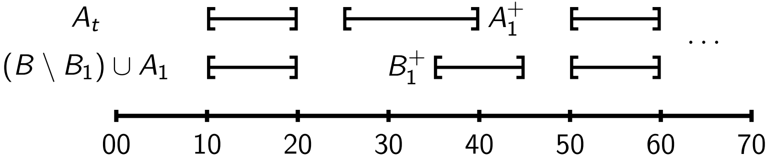


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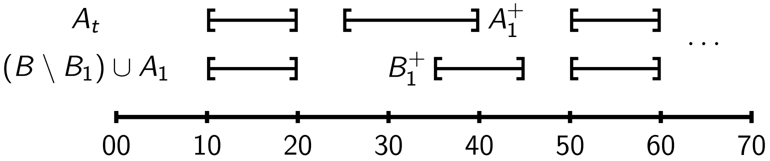


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**Theorem:** $A_t$ is a **maximum** compatible set.

On taking $i = t$, we see that $(B \setminus B_t) \cup A_t$ is compatible — i.e. we can remove the first $t$ intervals from $B$ and replace them with the whole of $A_t$.

Since $A_t$ is **maximal** — that is, since we can't add any intervals to $A_t$ and keep it compatible — it follows that $|B| = |A_t|$.

(**Exercise:** Prove that $A_t$ is maximal...)

# Choosing between the two methods

Both types of argument used this lecture, "**greedy stays ahead** proofs" and "**exchange** proofs", are powerful and widely-used.

Sometimes only one approach will work easily, but often (like here) the two approaches feel like they are doing the same thing under the surface. Use whichever one you find more natural — it's a matter of taste!

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