

Hamilton cycles

COMS20017 (Algorithms and Data)

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Hamilton cycles

What if instead of using every edge once, we used every vertex once?

A **cycle** is a walk $W = w_0 \dots w_k$ with $w_0 = w_k$ and $k \geq 3$, in which every vertex appears at most once except for w_0 and w_k (which appear twice).

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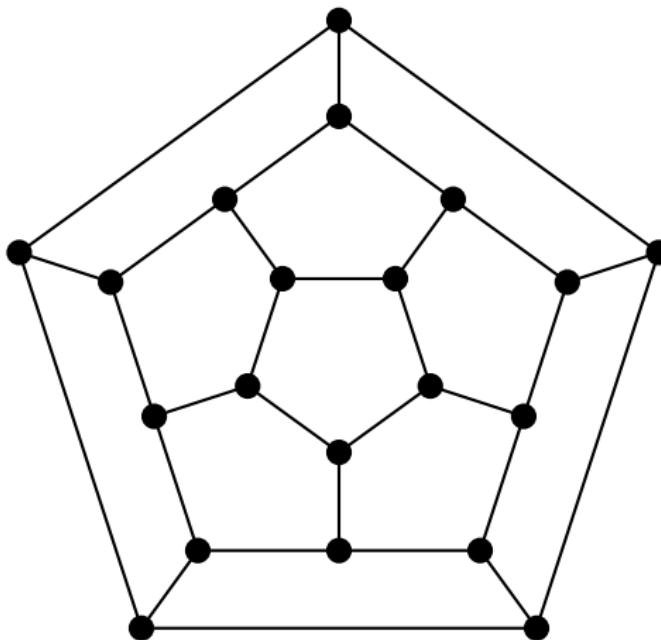


Hamilton actually made and sold a game based on trying to find Hamilton cycles in a dodecahedron!

Perhaps not surprisingly, it didn't sell very well.

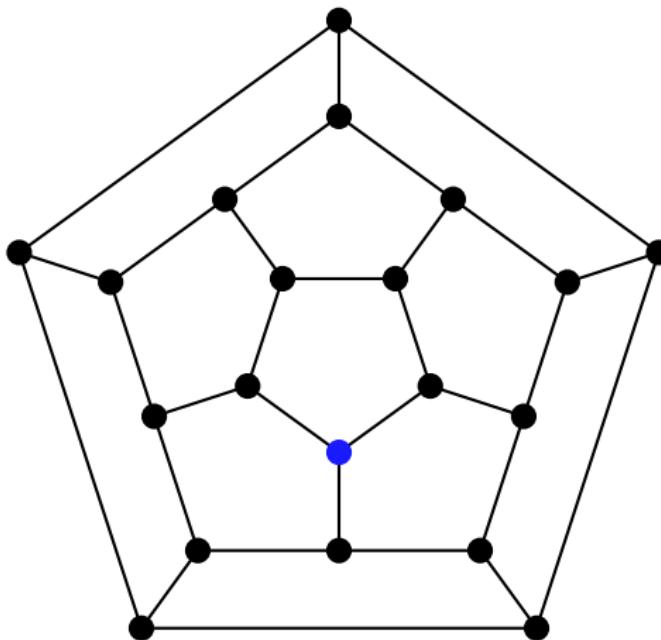
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In a particular (undirected) graph, Hamilton cycles can be easy to find:



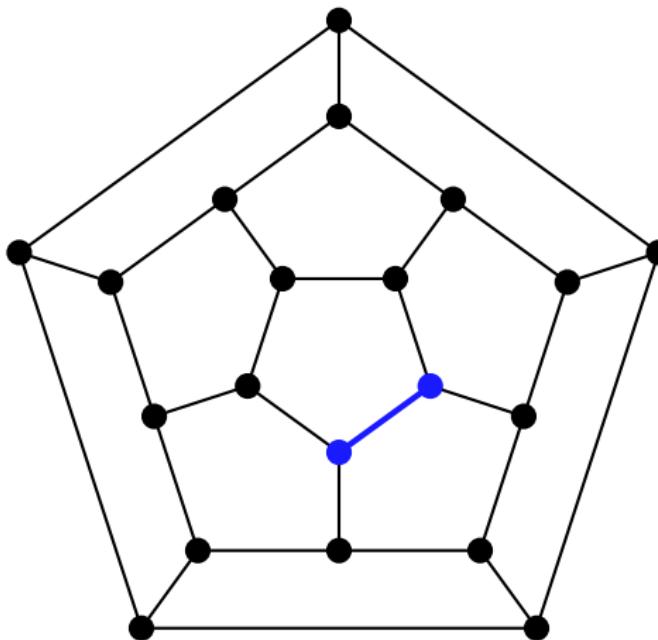
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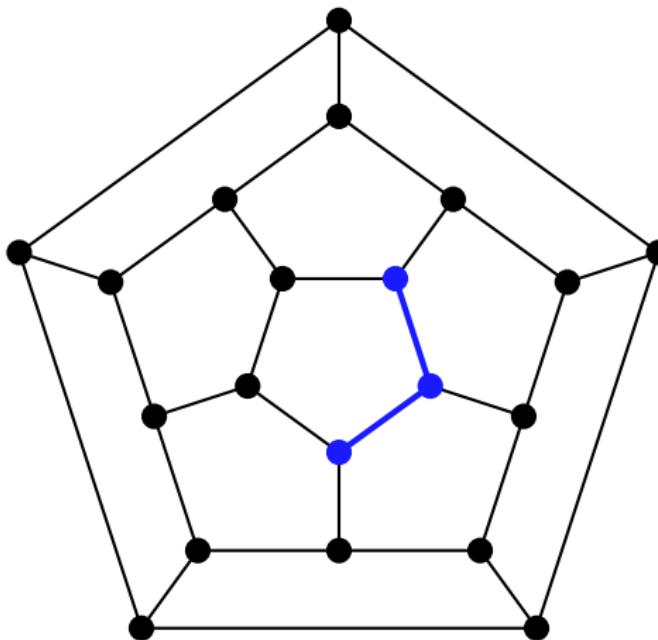
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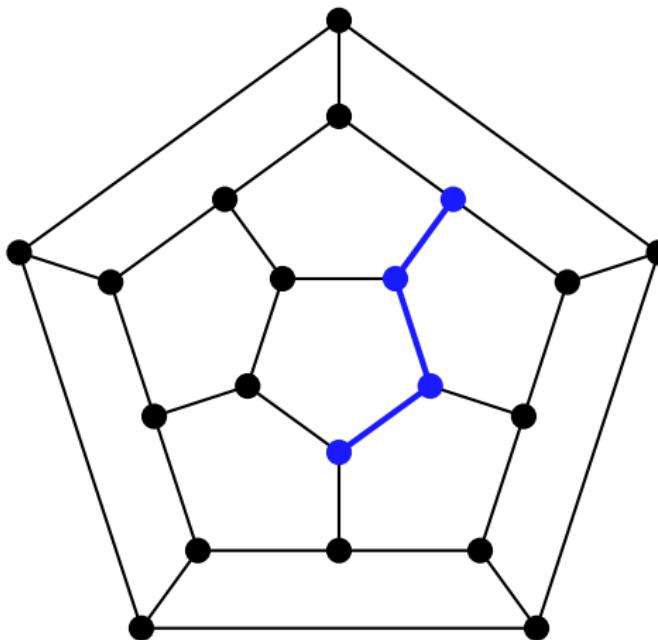
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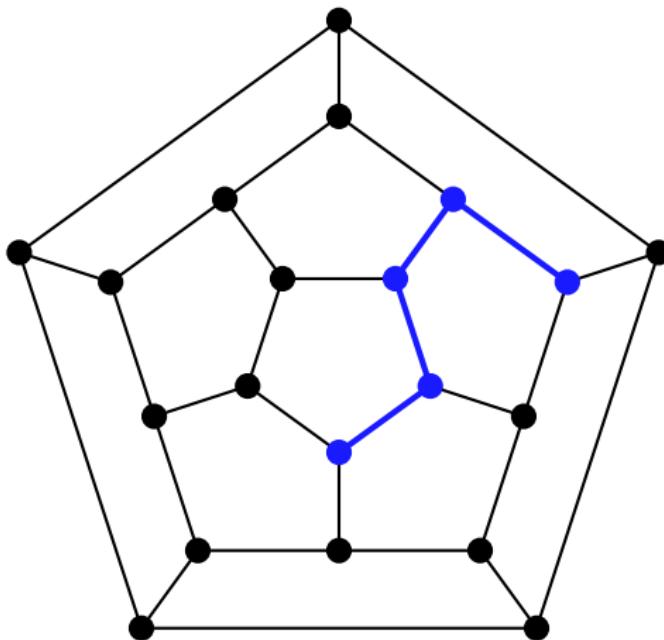
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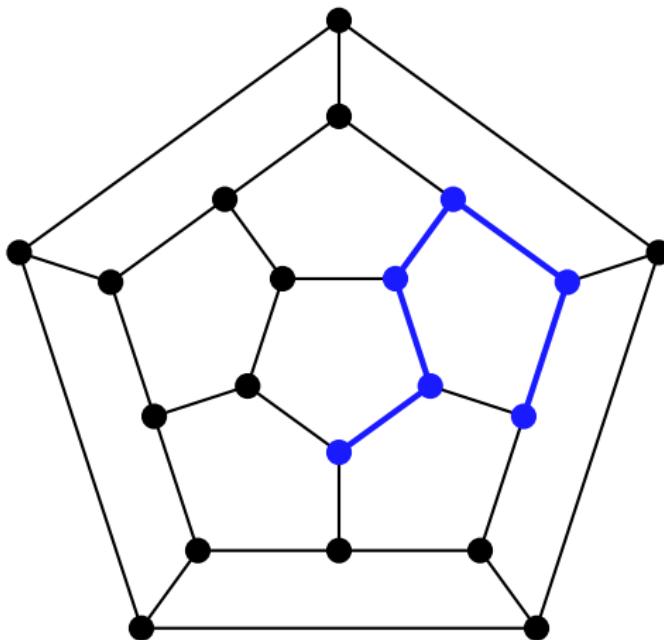
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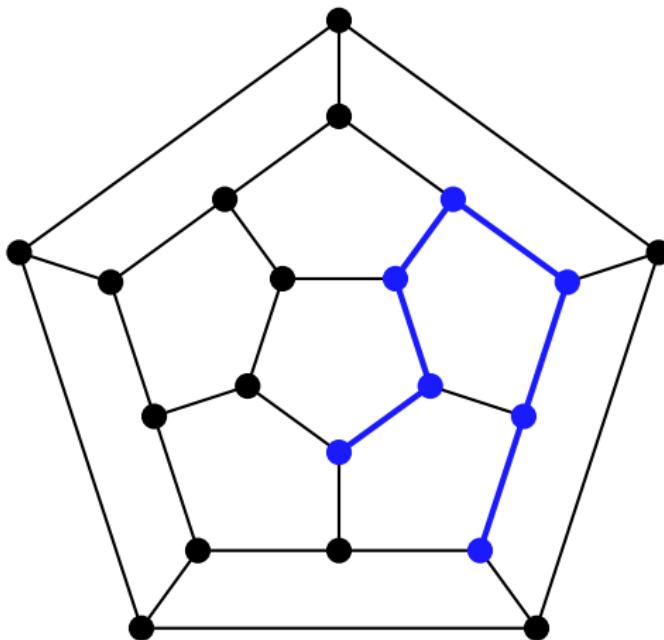
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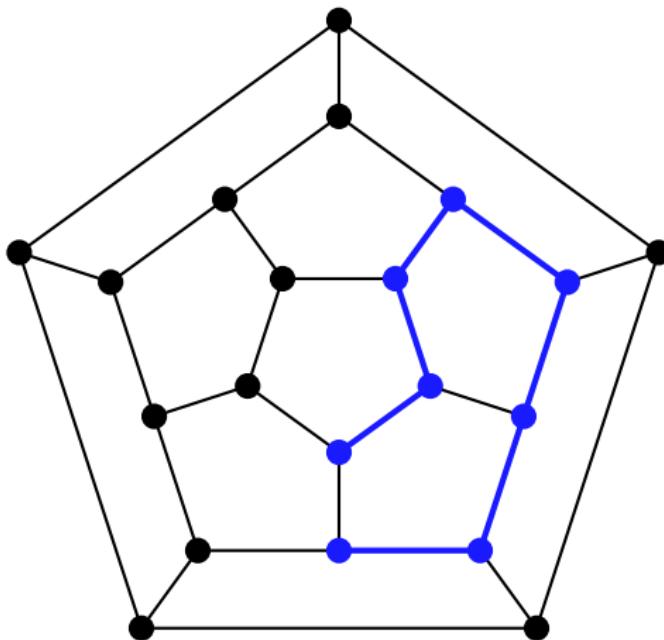
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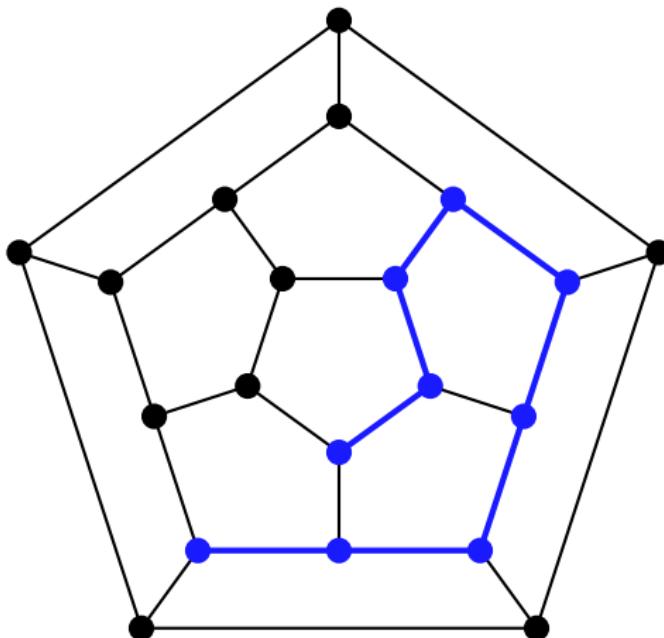
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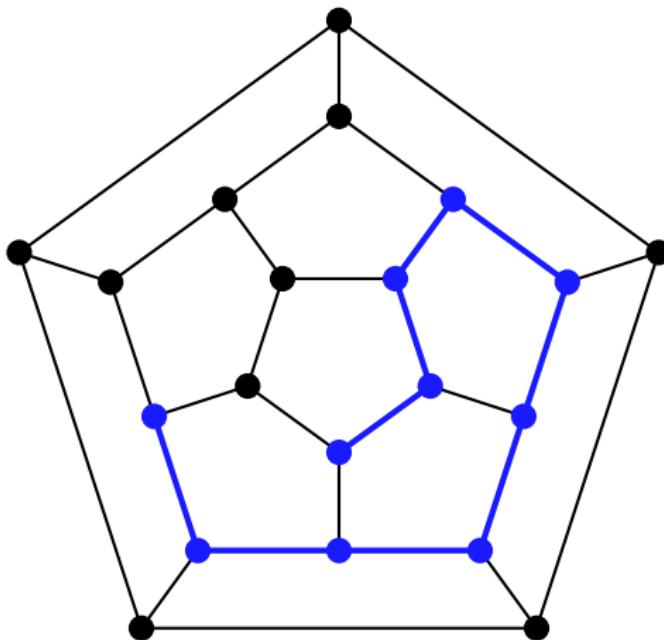
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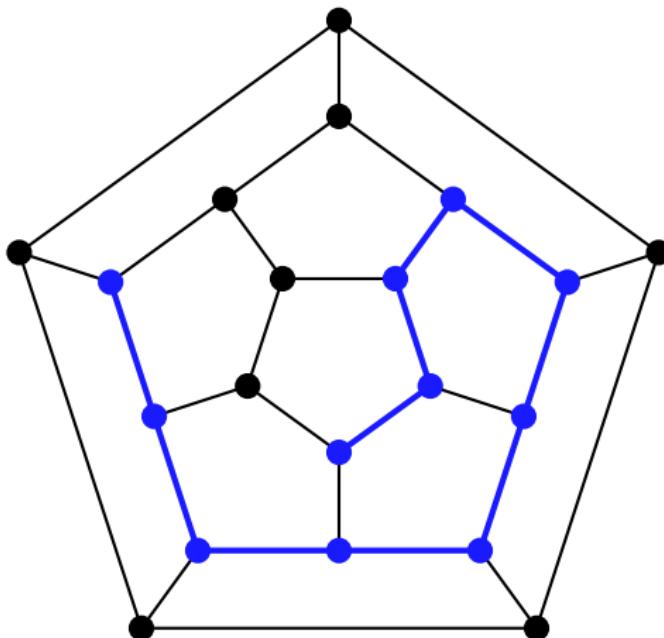
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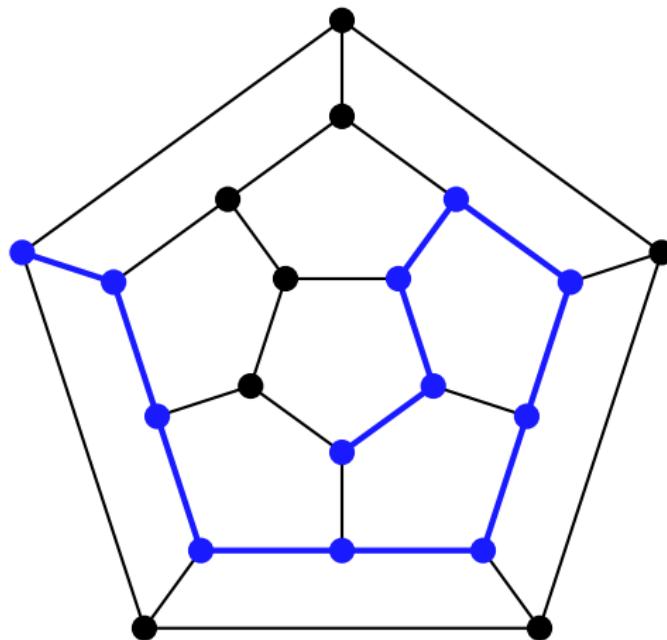
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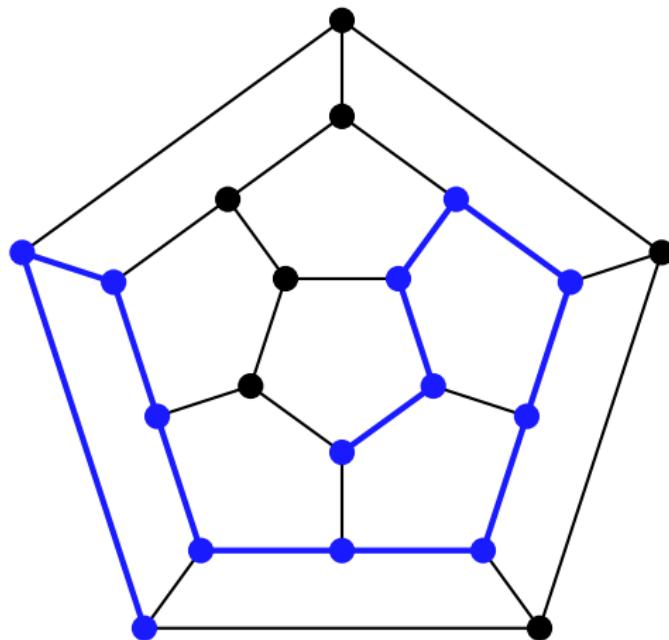
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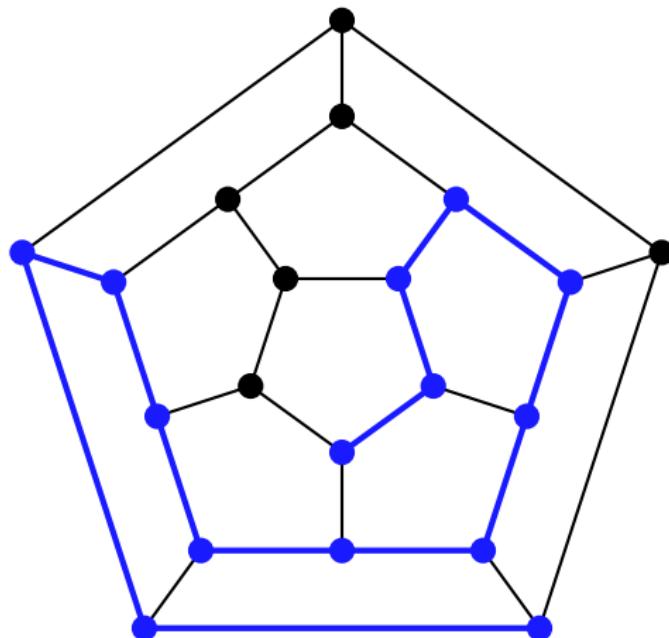
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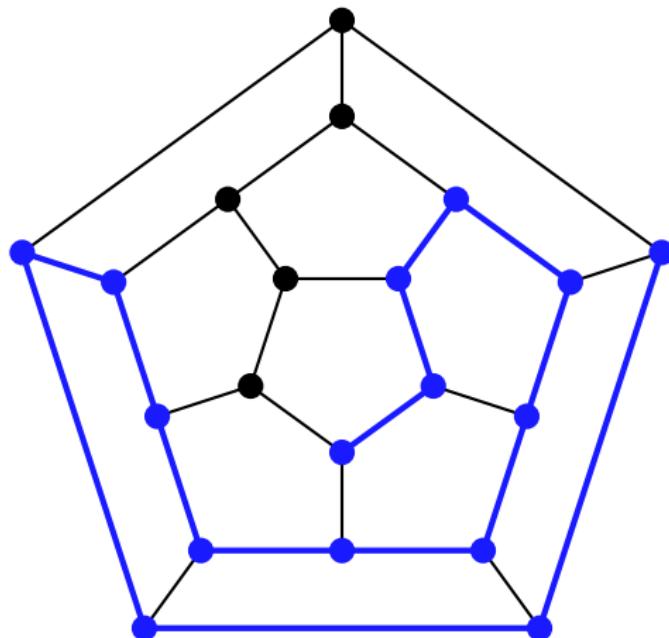
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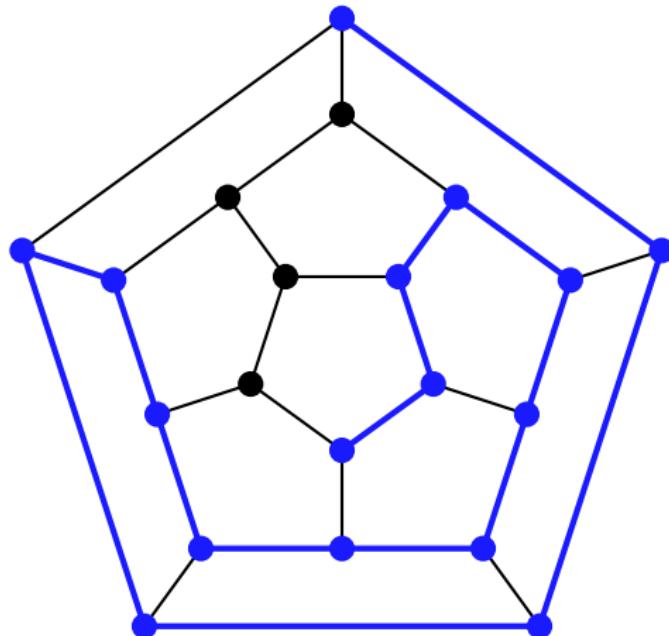
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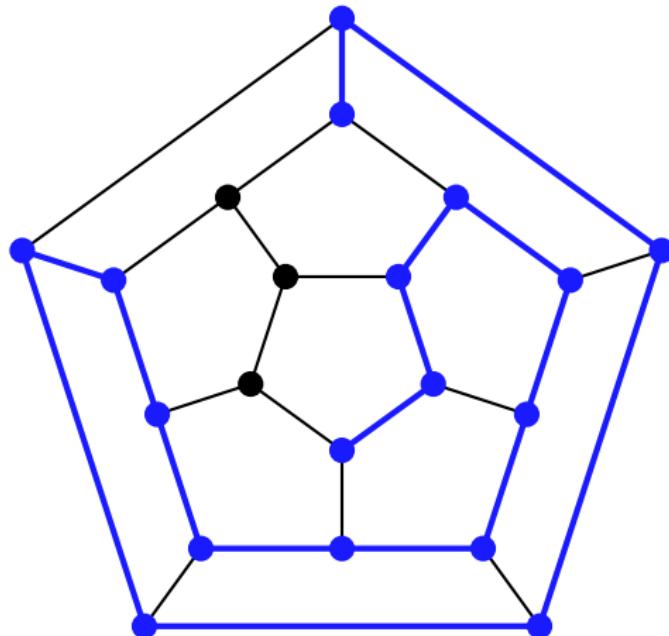
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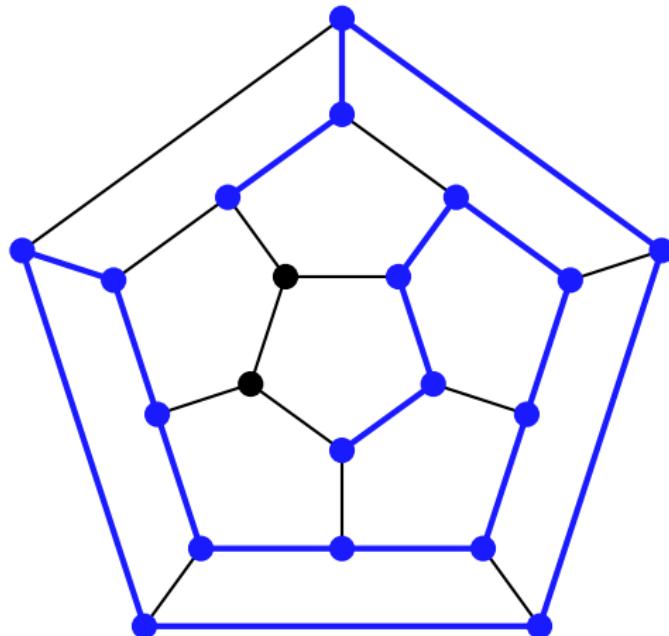
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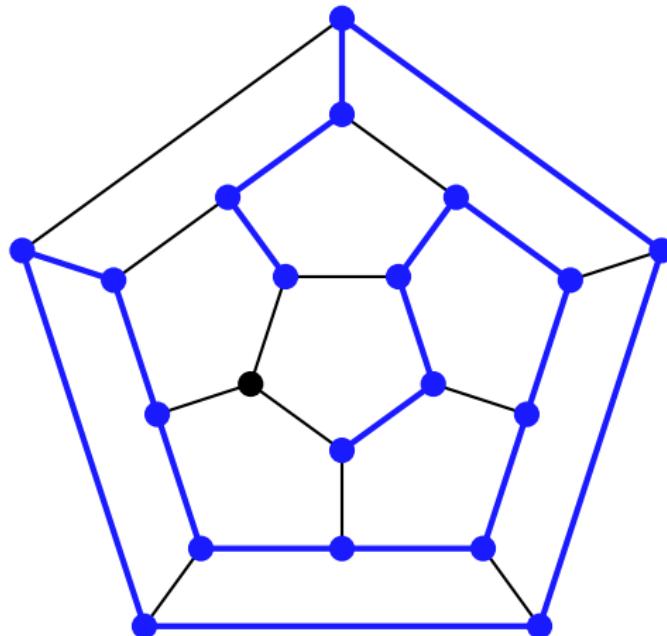
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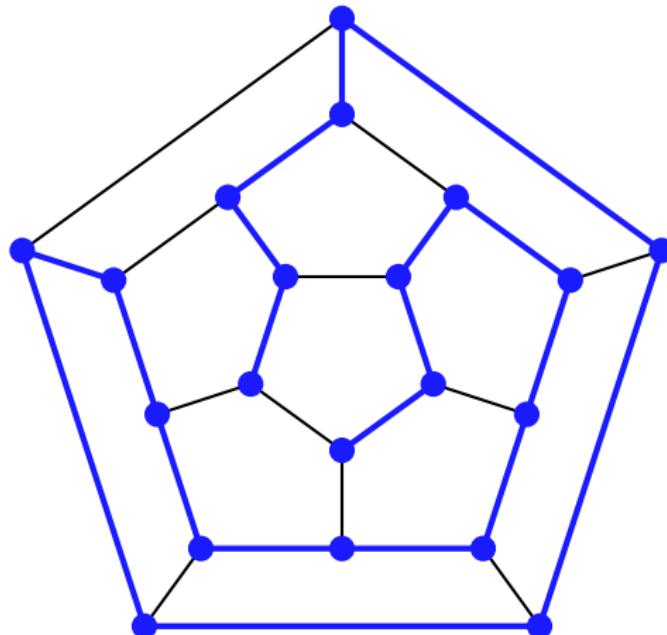
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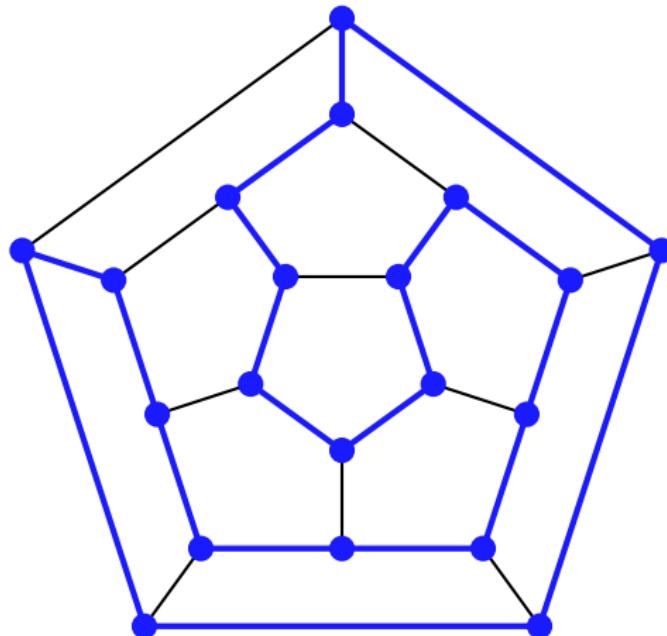
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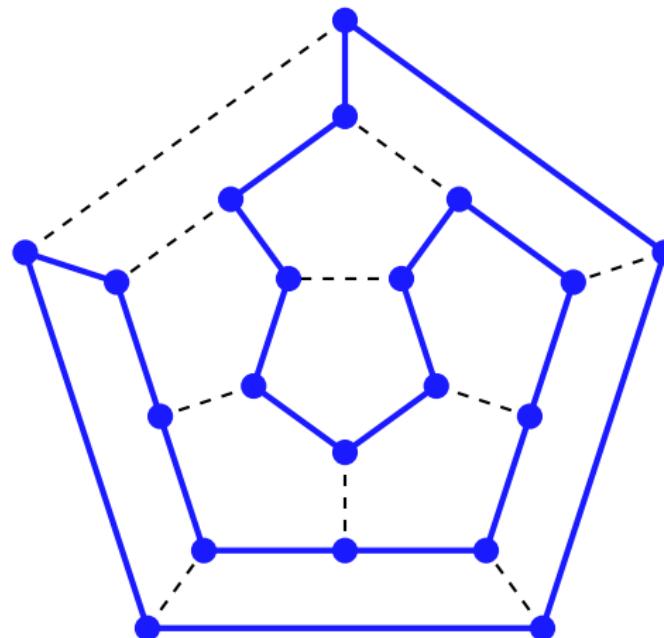
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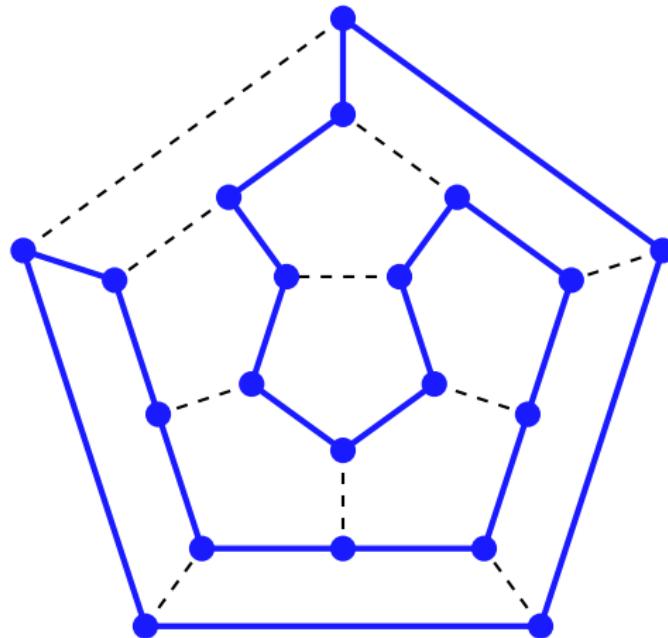
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But in general, they can be very hard to find. If you prove an easy-to-check condition like the one for Euler walks, you stand to win a million dollars!

Dirac's theorem

Just because we can't find Hamilton cycles in general doesn't mean we can't find them in special cases...

Dirac's Theorem: Let $n \geq 3$. Then any n -vertex graph G with minimum degree at least $n/2$ has a Hamilton cycle.

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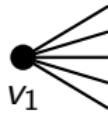
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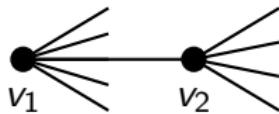
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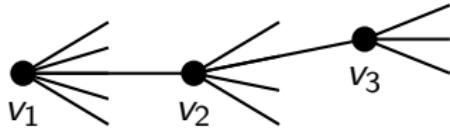
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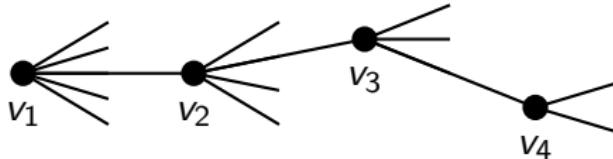
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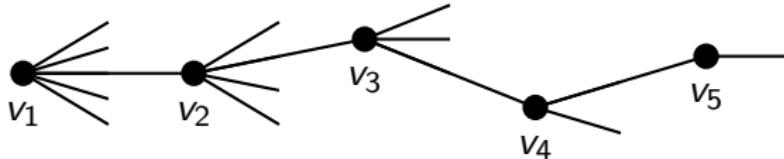
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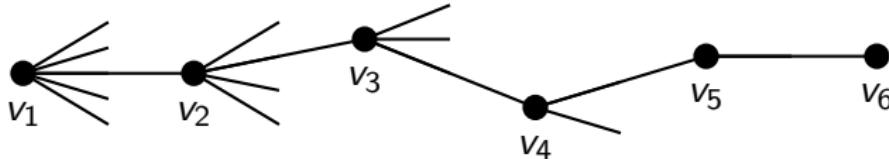
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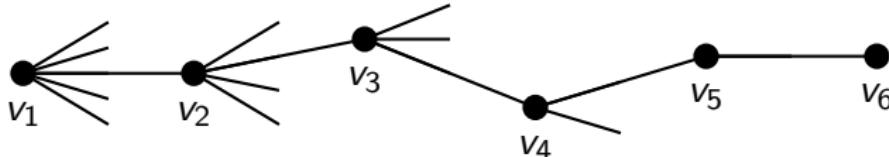
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In general, $d(v_k) \geq n/2 > |\{v_1, \dots, v_{k-1}\}|$, so there's a vertex v_{k+1} adjacent to v_k other than v_1, \dots, v_{k-1} . Then $v_1 \dots v_{k+1}$ is a path of length $k + 1$.



Dirac's Theorem: Any n -vertex graph G with minimum degree at least $n/2$ has a Hamilton cycle.

Idea: Repeatedly extend a k -vertex path in G .

Lemma 1: If G contains a k -vertex path with $1 \leq k \leq n/2$, then G contains a $(k + 1)$ -vertex path. ✓

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Case 2c: Both $N(v_1) \subseteq \{v_2, \dots, v_k\}$ and $N(v_k) \subseteq \{v_1, \dots, v_{k-1}\}$.
In this case, we *use* the fact that greedy extension fails to extend the path in another way.

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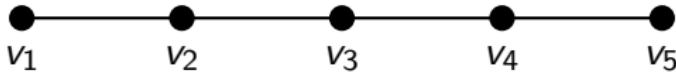
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We are done unless $N(v_1) \subseteq \{v_2, \dots, v_k\}$ and $N(v_k) \subseteq \{v_1, \dots, v_{k-1}\}$.

Never think about graphs without a picture. What does this **look like?**

Say just for $n = 8$, $k = \frac{1}{2}n + 1 = 5$?



So it looks like we should be able to turn our path into a cycle...

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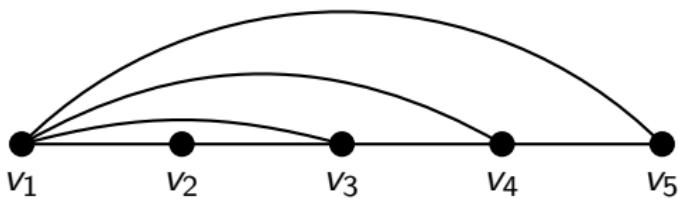
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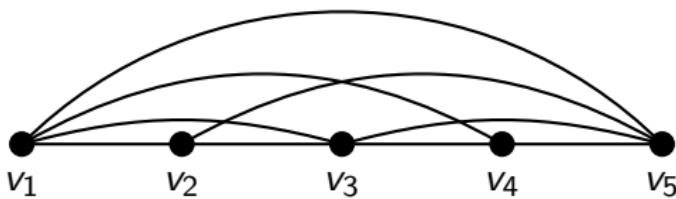
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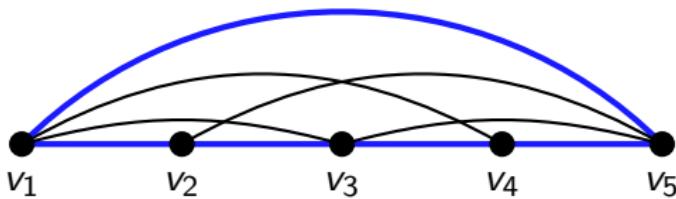
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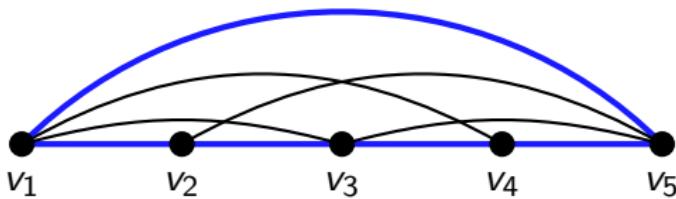
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But there are lots of other cycles available.

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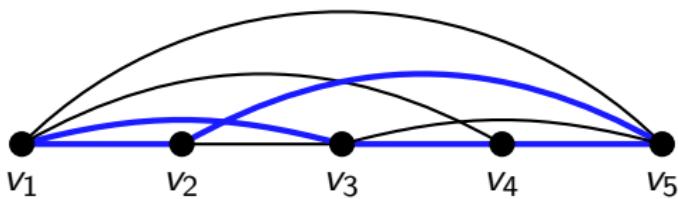
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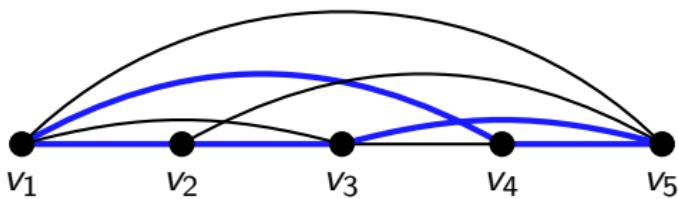
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Never think about graphs without a picture. What does this **look like**?

Say just for $n = 8$, $k = \frac{1}{2}n + 1 = 5$



So it looks like we should be able to turn our path into a cycle...

Of course, in general $\{v_1, v_k\}$ might not be an edge!

But there are lots of other cycles available.

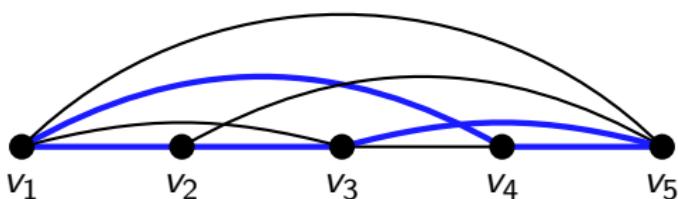
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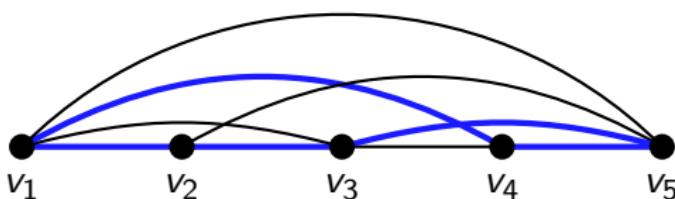
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Both sets are contained in $\{v_1, \dots, v_{k-1}\}$, which has size at most $n - 1$. So they must intersect.

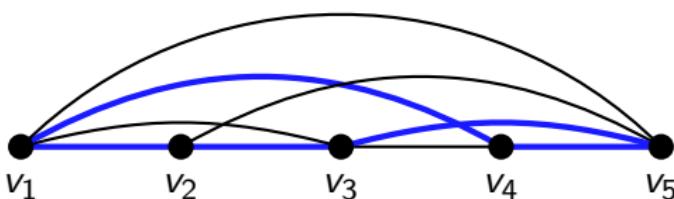
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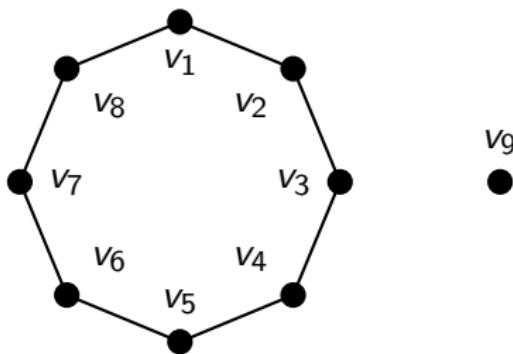
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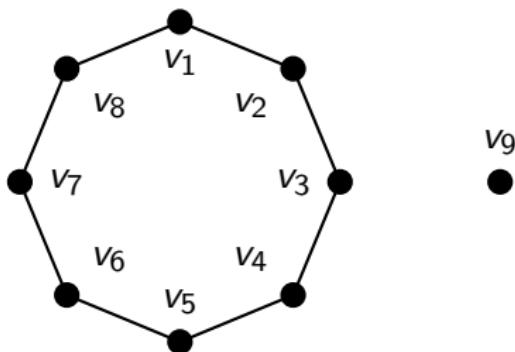


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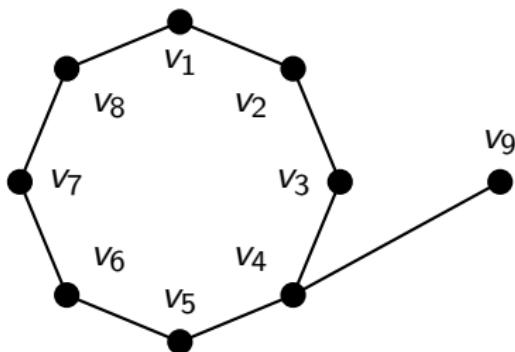
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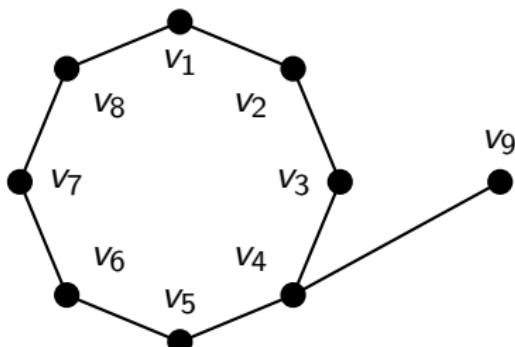
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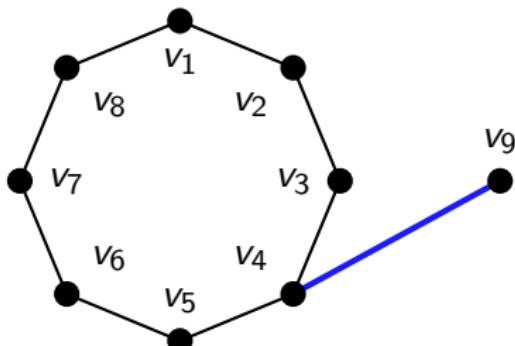
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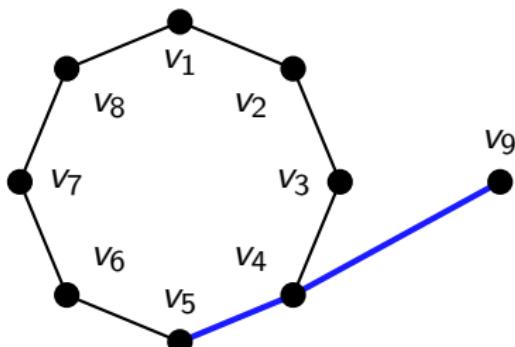
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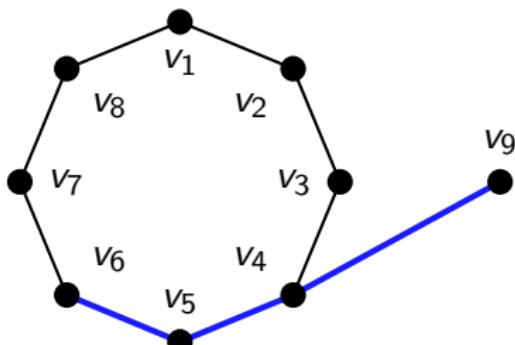
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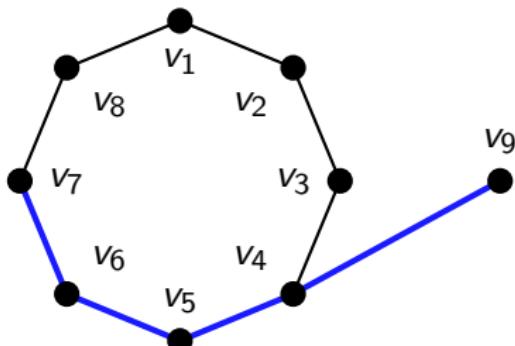
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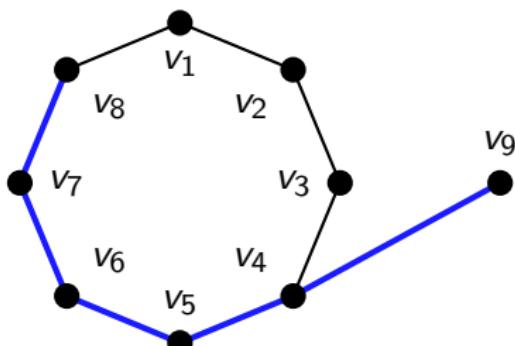
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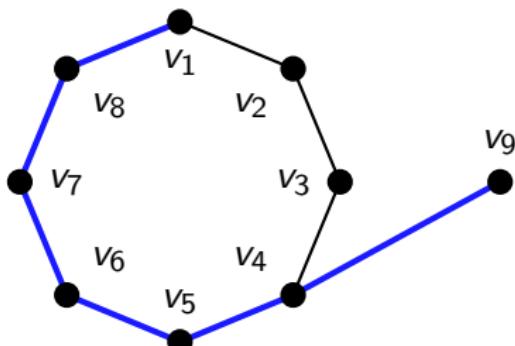
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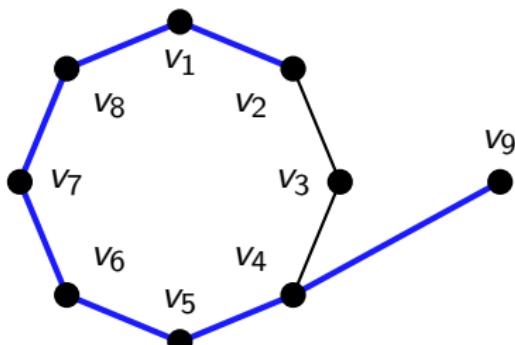
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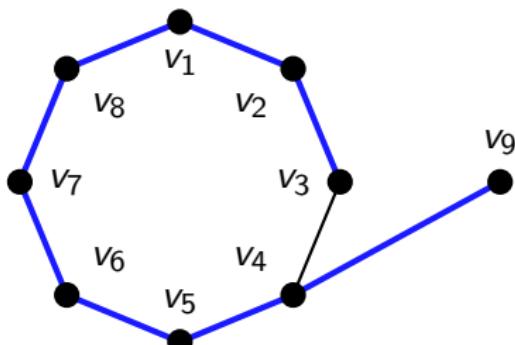
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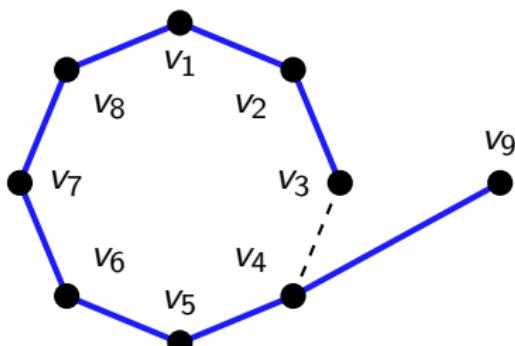
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So now by starting with a single-vertex path and repeatedly applying our three Lemmas, we reach an n -vertex path.

Then Lemma 2 turns this into a Hamilton cycle and we're done! □

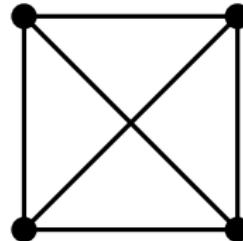
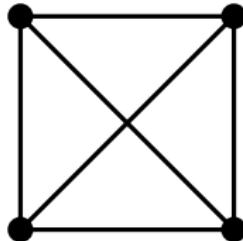
Note this proof gives us a (fairly fast) algorithm for finding a Hamilton cycle when Dirac's theorem applies. This often happens in graph theory!

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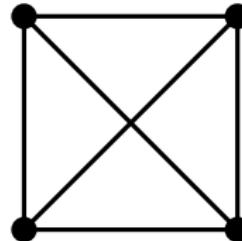
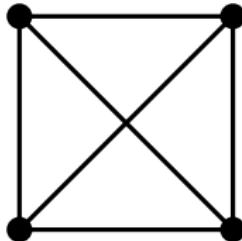
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But there are other ways to improve it. For example, when we do have minimum degree $n/2$, there's more than just one Hamilton cycle.

In fact, for large graphs, we can find $(n - 2)/8$ **disjoint** Hamilton cycles, decomposing almost half the graph!

(Proved in 2013–4 by Csaba, Kühn, Lapinskas, Lo, Osthus and Treglown.)