1. (Setting up sample spaces) Assume two standard six-sided dice are rolled one after the other.

   i Restrict attention to the first roll, and list all possible outcomes of this random experiment.

   First roll can either be 1, 2, 3, 4, 5, or 6. $\left(S = \{x \in \mathbb{N} \mid 1 \leq x \leq 6\}\right)$
   We do not care about the outcomes of the second roll.

   ii Now consider both rolls, and list the outcomes of the most general sample space you can define.

   Each roll can either be 1, 2, 3, 4, 5, or 6. Therefore outcomes are: $\left(S = \{x, y \in \mathbb{N} \mid 1 \leq x, y \leq 6\}\right)$
   or
   {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
   (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
   (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
   (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
   (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),
   (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6) }

2. (Set theory) Consider two arbitrary events $A, B \in S$ and describe the following events using set operations.

   i Both events occur.

   $A \cap B$

   ii At least one event occurs.

   $A \cup B$

   iii Neither event occurs.

   $\neg (A \cup B)$

   iv Only event A but not event B occurs, i.e. $\{s \in S : s \in A \text{ and } s \notin B\}$.
   Note: this set operation is called *difference* and is denoted A – B (minus) or A\B (back-slash).

   $A \cap \neg B$

   v Exactly one event occurs.
   Note: this set operation is called *symmetric difference* and is denoted by $A \triangle B$. In logic it is also called *exclusive OR* (XOR).

   $(A \cap \neg B) \cup (\neg A \cap B)$

3. (Blitzstein: §1, Q42-43) For arbitrary events A, B use the probability axioms to show:
The definition of the difference ($-$) and symmetric difference ($\triangle$) set operations are given in problem 2.iv and 2.v.

i $P(A - B) = P(A) - P(A \cap B)$.

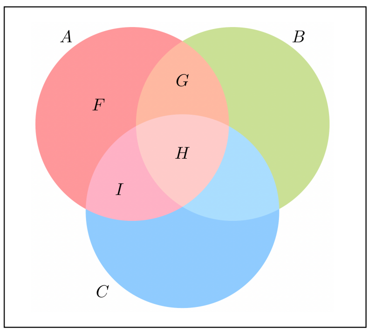
Proof. Recall: $A = (A \cap B) \cup (A \cap B^c) = (A \cap B) \cup (A - B)$, where $(A \cap B)$ and $(A - B)$ are disjoint.
$\therefore P(A) = \underbrace{P(A \cap B) + P(A - B)}_{\text{By the additivity axiom.}} \implies P(A - B) = P(A) - P(A \cap B)$ ■

ii $P(A \triangle B) = P(A) + P(B) - 2P(A \cap B)$.

Proof. Recall: $A - B$ and $B - A$ are disjoint.

$$\begin{aligned} P(A \triangle B) &= P((A - B) \cup (B - A)) \\ &= P(A - B) + P(B - A) && \text{By the additivity axiom.} \\ &= \big(P(A) - P(A \cap B)\big) + \big(P(B) - P(A \cap B)\big) && \text{By part i} \\ \therefore &= P(A) + P(B) - 2P(A \cap B) && \blacksquare \end{aligned}$$

4. Consider three arbitrary sets A, B, C represented by colored circles in the Venn diagram below.



i Describe the strictly pink area F using set operations on A, B, C.

$F = A \cap \neg(B \cup C)$

ii Describe the strictly brown area G using set operations on A, B, C.

$G = A \cap B \cap \neg C$

iii Show that $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$, i.e. prove the 3-set version of the inclusion-exclusion principle.

Proof. By using the 2-set version of the inclusion principle, we can prove

$$\begin{aligned}
P(A \cup B \cup C) &= P(A \cup (B \cup C)) && \text{Treating } B \cup C \text{ as one event.}\\
&= \underbrace{P(A) + P(B \cup C) - P(A \cap (B \cup C))}_{\text{By 2-set version.}}\\
&= P(A) + P(B \cup C) - P((A \cap B) \cup (A \cap C))\\
&= P(A) + \underbrace{P(B) + P(C) - P(B \cap C)}_{\text{By 2-set version.}} - P((A \cap B) \cup (A \cap C))\\
&= P(A) + P(B) + P(C) - P(B \cap C) - \underbrace{P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)}_{\text{By 2-set version.}}\\
&= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \blacksquare
\end{aligned}$$

5. (Inclusion-exclusion principle; SOA Exam P, May 2003) A survey of a groups viewing habits over the last year revealed the following information:
   - 28% watched gymnastics (G)
   - 19% watched soccer (S)
   - 12% watched baseball and soccer $(B \cap S)$
   - 8% watched all three sports. $(G \cap B \cap S)$
   - 29% watched baseball (B)
   - 14% watched gymnastics and baseball $(G \cap B)$
   - 10% watched gymnastics and soccer $(G \cap S)$

   Calculate the percentage of the group that:

   i Watched only gymnastics last year.

This is the same as watching gymnastics but not soccer or baseball. i.e. $(G - (S \cup B))$.

$$\begin{aligned}
(G - (S \cup B)) &= \underbrace{\Big(G - (G \cap (S \cup B))\Big)}_{\text{From 3i.}}\\
&= G - \big((G \cap S) \cup (G \cap B)\big)\\
&= G - \big((G \cap S) + (G \cap B) - (G \cap B \cap S)\big)\\
&= 0.28 - \big((0.10) + (0.14) - (0.08)\big)\\
&= \mathbf{0.12}
\end{aligned}$$

   ii Watched none of the three sports last year.

This is the same is $\neg(G \cup B \cup S)$

$$\begin{aligned}
\neg(G \cup B \cup S) &= 1 - (G \cup B \cup S)\\
&= 1 - \big(G + B + S - (G \cap B) - (G \cap S) - (B \cap S) + (G \cap B \cap S)\big) && \text{From 4iii.}\\
&= 1 - \big(0.28 + 0.29 + 0.19 - (0.14) - (0.10) - (0.12) + (0.08)\big)\\
&= 1 - 0.48\\
&= \mathbf{0.52}
\end{aligned}$$

6. (Betting) Probabilities are often expressed as fractional odds in betting situations. To fix ideas, consider a random experiment (e.g. basketball match) and a specific event (e.g. Raptors win by more than 10 points). When you bet money on the event, the amounts you win (W) or lose (L) are determined by the *odds*, which are usually quoted as the ratio W/L. The odds work as follows: if you wager L and the event occurs, then you win W on top of your original L. If you wager L and the event does not occur, then you lose the L you wagered. You can scale L up or down and W will change proportionately. Suppose the event at hand, call it A, has probability $P(A) = p$. If you gamble on it, on average you win p fraction of times and lose $1 - p$ fraction of times. So you average net gain is $pW - (1 - p)L$.