

each circle  $K_m$  is contained in the interior of  $R_v$ , provided  $v$  is sufficiently large, on the other hand, every  $R_v$  is bounded and is therefore contained in a circle  $K_M$  of sufficiently large radius  $M$ . Since the integrand  $e^{-x^2-y^2}$  is positive everywhere, it follows that

$$\iint_{K_m} e^{-x^2-y^2} dx dy \leq \iint_{R_v} e^{-x^2-y^2} dx dy \leq \iint_{K_M} e^{-x^2-y^2} dx dy.$$

As  $m$  and  $M$  increase, the integrals over  $K_m$  and  $K_M$  have the same limit  $\pi$ , so that the integral over  $R_v$  must have the same limit, this proves that the integral must converge to the limit  $\pi$ .

We obtain a particularly interesting result if for the regions  $R_v$  we choose the squares  $|x| \leq v, |y| \leq v$ . The integral  $\iint_{R_v} e^{-x^2-y^2} dx dy$  can then be reduced to two simple integrations (cf. section 2, p. 228):

$$\iint_{R_v} e^{-x^2-y^2} dx dy = \int_{-v}^v e^{-x^2} dx \int_{-v}^v e^{-y^2} dy = \left( \int_{-v}^v e^{-x^2} dx \right)^2 = \left( 2 \int_0^v e^{-x^2} dx \right)^2.$$

If we now let  $v$  tend to  $\infty$ , we must again obtain the same limit  $\pi$ . Hence

$$\left( 2 \int_0^{\infty} e^{-x^2} dx \right)^2 = \pi$$

or

$$\int_0^{\infty} e^{-x^2} dx = \frac{1}{2} \sqrt{\pi},$$

in agreement with Vol. I, p. 496.

## 5. Summary and Extensions

It is useful to consider the concepts of this section again from a single unifying point of view. Our extension of the concept of integral to cases in which the definitions in section 2 (p. 224) are not immediately applicable consists in regarding the value of the integral as the limiting value of a sequence of integrals over regions  $R_n$ , which approximate to the original region of integration  $R$  as  $n$  increases. For this purpose we regard the region  $R$  as *open* instead of closed, we assign all the points of discontinuity of the function  $f$  to the boundary and consider the boundary as not belonging to  $R$ . We then say that *the region is approximated to by a sequence of regions  $R_1, R_2, \dots, R_n, \dots$  if all the closed regions  $R_n$  lie in  $R$  and every arbitrarily chosen closed sub-region in the interior of  $R$  is also a sub-region of the region  $R_n$ , provided only that  $n$  is sufficiently large*. If in particular the sub-regions  $R_n$  are so chosen that each one contains the