result ensures that an equivalent of the octal game conjecture is not available for pure breaking games. Indeed, the following lemma establishes that if an even cut number belongs to L, the Grundy values are not bounded.

Lemma 3. *Let* $\mathrm{PB}(L)$ *be a pure breaking game, where* L *contains at least one even integer. Let* m *be the smallest even integer in* L.

For every pair $(x_1, x_2), x_1 \neq x_2$ *such that* $\mathcal{G}(x_1) = \mathcal{G}(x_2)$*, we have* $x_1 \not\equiv x_2 \bmod m$.

Proof. We reason by contradiction. Let x_1 and x_2 be two different integers such that $\mathcal{G}(x_1) = \mathcal{G}(x_2)$ and $x_1 \equiv x_2 \bmod m$. We have $x_1 = a_1 m + b$ and $x_2 = a_2 m + b$ for some $0 \leq b \leq m-1$. We assume without loss of generality that $x_1 > x_2$.

From a heap of x_1 counters, one can play to the option $O_{x_1} = (x_2, a_1 - a_2, \ldots, a_1 - a_2)$ obtained by an m-cut. Since m is even, $\mathcal{G}(O_{x_1}) = \mathcal{G}(x_2)$, and thus $\mathcal{G}(x_1) \neq \mathcal{G}(x_2)$, which contradicts our hypothesis. □

This result means that pure breaking games are somehow closer to hexadecimal games than octal games. One can then wonder whether the complexity of the $\mathcal{G}$-sequence increases with $\max(L)$. It does not seem to be the case, as we will show that in almost all cases, the $\mathcal{G}$-sequence is either periodic or arithmetic periodic. In Section 2, we consider several families of pure breaking games (e.g. those where $1 \notin L$, or those with only odd values in L) and prove their periodicity or arithmetic periodicity. For the remaining families, many games seem to have an arithmetic periodic behavior. To deal with them, we provide in Section 3 a set of testing conditions that are sufficient to show that a game is arithmetic periodic, and apply them to particular instances. Finally, in Section 4 we list the remaining sets L for which the regularity of the $\mathcal{G}$-sequence of $\mathrm{PB}(L)$ remains open.

2 Solving particular families of pure breaking games

In this section, we study specific families of pure breaking games. All the following results will be proved by contradiction. In each case, we will suppose that there exists an integer n for which the Grundy value is different from what was expected. By decomposing n into specific options, we will exhibit a contradiction. All the families will be proved to have arithmetic periodic sequences. We are going to use the following notation: $(m_1, \ldots, m_p)$ $(+s)$, which describes the arithmetic periodic sequence of period p and saltus s for which the first p values are $m_1, \ldots, m_p$. If a subsequence $(m_i, \ldots, m_j)$ is repeated q times, we will write $(m_i, \ldots, m_j)^q$. Thus, for example, the notation $(0,1,2)^2$ $(+3)$ denotes the arithmetic periodic sequence of period 6, saltus 3, and with first six values 0,1,2,0,1,2. We also use the notation $[\![a, b]\!]$ (with $a \leq b$) to describe the set of all the integers from a to b.

First, we study the games in which 1 is not an allowed cut number. In this case, optimal play is reduced to using only ℓ_1, and the Grundy sequence is arithmetic periodic with period ℓ_1 and saltus 1.

Proposition 4. *Let* $L = \{\ell_1, \ldots, \ell_k\}$ *be a set of cut numbers such that* $\ell_1 \geq 2$*. Then,* $\mathrm{PB}(L)$ *has a Grundy sequence of* $(0)^{\ell_1}$ $(+1)$.

Proof. We prove this result by contradiction. If n is a positive integer, then there exists a unique couple of nonnegative integers (a, b) such that : $0 \leq b \leq \ell_1 - 1$ and $n = a\ell_1 + b + 1$. We want to prove that for every positive integer n, $\mathcal{G}(n) = a$.
Assume that n is the smallest positive integer such that $\mathcal{G}(n) \neq a$.

Let $m \in L$. Suppose $\mathcal{G}(n) > a$. Then there exists $O_n = (a_0\ell_1 + b_0 + 1, \ldots, a_m\ell_1 + b_m + 1)$ an m-cut of n such that $\mathcal{G}(O_n) = a$. By minimality of n, $\mathcal{G}(O_n) = a_0 \oplus \ldots \oplus a_m = a$. Moreover, since O_n is an option of n, we have:

$$\sum_{i=0}^{m} (a_i\ell_1 + b_i + 1) = a\ell_1 + b + 1.$$

In particular, as $b < \ell_1$ we have $\sum_{i=0}^{m} a_i \leq a$. However, since $a = \bigoplus_{i=0}^{m} a_i \leq \sum_{i=0}^{m} a_i$ we have $\sum_{i=0}^{m} a_i = a$.

This implies that $1 + b = \sum_{i=0}^{m}(1 + b_i) = m + 1 + \sum_{i=0}^{m} b_i$.

This is a contradiction since $m \geq \ell_1$ which implies $b \geq \ell_1$.