

$$\pi(t_w) = \sum_{w \in \mathcal{J}} t_w w + \sum_{i+j \leq 2\bar{P}} (z_1^{-1} z_2^{-1} s, z^i z^j) - \sum_{w \in \mathcal{J} \cap \Xi} (i, j) t_w \xi_1^i \xi_2^j. \quad (12)$$

2) Consider the following sets in K :

$$\begin{aligned} T_q &= \{(t_w): (t_w) \in K^{\mathcal{J}}, \text{rank } \mathcal{H}_{\bar{P} \times \bar{P}}(\pi(t_w)) \leq q\} & q \leq m \\ V_q &= \{(t_w): (t_w) \in K^{\mathcal{J}}, \text{rank } \mathcal{H}_{\bar{P} \times (\bar{P}+1)}(\pi(t_w)) \leq q\} & q \leq m \\ U_q &= \{(t_w): (t_w) \in K^{\mathcal{J}}, \text{rank } \mathcal{H}_{(\bar{P}+1) \times \bar{P}}(\pi(t_w)) \leq q\} & q \leq m. \end{aligned} \quad (13)$$

Each condition on the rank of matrices in (13) is equivalent to a number of conditions on the minors of order $q+1$ and is expressed by a system of algebraic equations in the parameters $t_w, w \in \mathcal{J}$. Thus, T_q, V_q , and U_q are algebraic varieties in $K^{\mathcal{J}}$.

3) Evaluate the smallest value of the index q such that

$$\mathcal{W}_q \triangleq (T_q - T_{q-1}) \cap (V_q - V_{q-1}) \cap (U_q - U_{q-1}) \neq \emptyset. \quad (14)$$

Let \bar{m} denote this value. Then $(t_w) \in \mathcal{W}_{\bar{m}} = (V_{\bar{m}} \cap U_{\bar{m}}) \cap (T_{\bar{m}} - T_{\bar{m}-1})$ if and only if $\pi(t_w)$ belongs to \mathcal{P} and has minimal partial representation. In fact, \bar{m} is the smallest value of q for which the equations chain

$$\begin{aligned} \text{rank } \mathcal{H}_{\bar{P} \times \bar{P}}(\pi(t_w)) &= \text{rank } \mathcal{H}_{\bar{P} \times (\bar{P}+1)}(\pi(t_w)) \\ &= \text{rank } \mathcal{H}_{(\bar{P}+1) \times \bar{P}}(\pi(t_w)) = q \end{aligned}$$

admits a (t_w) solution.

As obvious consequence, minimal partial representation have dimension \bar{m} .

4) Use (12) for constructing polynomials in \mathcal{P} which have minimal partial representations.

Clearly, \mathcal{M} is constituted by all noncommutative rational power series of rank \bar{m} which extend the polynomials obtained by the above steps.

Once we obtained the set \mathcal{M} , we can use the generalized Ho algorithm for getting minimal partial representations. The set of these representations gives, modulo similarity transformations, all minimal realizations of \mathcal{S} .

Clearly, points 2) and 3) are the most difficult to be implemented because they involve the solution of several nonlinear algebraic equations. On the other hand, the necessity of introducing nonlinear algorithms is intrinsic to the problem as the dimension of minimal realizations depends on the ground field.

V. CONCLUSIONS

In this paper by pursuing the idea of introducing a state space model of two-dimensional filters, the realization problem has been further investigated along the directions outlined in previous works [1], [2].

The class of realizations introduced in this paper is characterized by a local state updating equation of the following form:

$$\begin{aligned} x(h+1, k+1) &= A_1 x(h+1, k) + A_2 x(h, k+1) \\ &\quad + B u(h, k). \end{aligned}$$

The minimality of the realizations is not guaranteed by reachability and observability. In general, the dimension of minimal realizations depends on the field K and does not coincide with the rank of $\mathcal{H}(s)$. Nevertheless, we can associate the commutative series s with noncommutative recognizable power series whose representations provide all the realizations of s . Hence, the problem of determining minimal realizations of s can be approached looking for minimal representations of noncommutative power series.

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The Approximation of Image Blur Restoration Filters by Finite Impulse Responses

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Abstract—Image blur can often be modeled by a linear spatially invariant, symmetric point spread function. For this class of functions, several restoration filters are known in the literature.

The approximation of their frequency transfer functions (ftf's) by the ftf's of small finite impulse response (FIR) filters has been studied. Accurate approximations will be possible by 9×9 FIR's with 8-bit elements if the approximation is done in a weighted MMSE sense, and if the truncation of the element values will be carried out such that the errors are small. A heuristic truncation algorithm MINIM will be described. An example of restoration by a 9×9 FIR will be shown.

Index Terms—Approximation, finite impulse response, finite register length, image blur, restoration filter.

I. INTRODUCTION

In the literature on image processing [1]–[3] a number of restoration filters are known, which can be applied in the case of spatially invariant image blurring. In general, these filters are implemented by means of the discrete Fourier transform (DFT), with help of the FFT algorithm. These filters, however, are operational only for static images as a consequence of the computational complexity.

Manuscript received March 22, 1978; revised June 20, 1979.

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