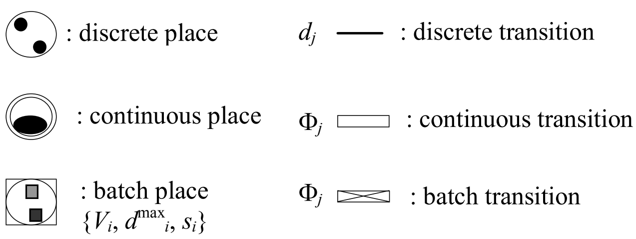

Fig. 1. Nodes of GBPN.

***Definition** 2.1.* A *Generalized Batches Petri net* (GBPN) is a 6-tuple $N = (P, T, Pre, Post, \gamma, Time)$ where:

- $P = P^D \cup P^C \cup P^B$ is finite set of places partitioned into the three classes of *discrete*, *continuous* and *batch* places.
- $T = T^D \cup T^C \cup T^B$ is finite set of transitions partitioned into the three classes of *discrete*, *continuous* and *batch* transitions.
- $Pre, Post : (P^D \times T \to \mathbb{N}) \cup ((P^C \cup P^B) \times T \to \mathbb{R}_{\geq 0})$ are, respectively, the pre-incidence and post-incidence matrixes, denoting the weigth of the arcs from places to transitions and transitions to places.
- $\gamma : P^B \to \mathbb{R}^3_{\geq 0}$ is the *batch place function.* It associates to each batch place $p_i \in P^B$ the triple $\gamma(p_i) = (V_i, d_i^{\max}, s_i)$ that represents, respectively, *speed*, *maximum density* and *length* of p_i.
- $Time : T \to \mathbb{R}_{\geq 0}$ associates a non negative number to every transition:
  - if $t_j \in T^D$, then $Time(t_j) = d_j$ denotes the firing delay associated to the discrete transition;
  - if $t_j \in T^C \cup T^B$, then $Time(t_j) = \Phi_j$ denotes the maximal firing flow associated to the continuous or batch transition. ■

To every continuous and batch transition, $t_j \in T^C \cup T^B$, is associated an instantaneous firing flow (IFF), noted $\varphi_j(\tau)$, representing the quantity of markings by time unit that fires transition t_j. Section 3 will discuss on the computation of this vector.

We denote the number of places and transitions, resp., $m = |P|$ and $n = |T|$. The *preset* and *postset* of transition t_j are: ${}^\bullet t_j = \{p_i \in P \mid Pre(p_i, t_j) > 0\}$ and $t_j^\bullet = \{p_i \in P \mid Post(p_i, t_j) > 0\}$. Similar notations may be used for pre and post transition sets of places and its restriction to discrete, continuous or batch transitions is denoted as ${}^{(d)}p_i = {}^\bullet p_i \cap T^D$, ${}^{(c)}p_i = {}^\bullet p_i \cap T^C$, and ${}^{(b)}p_i = {}^\bullet p_i \cap T^B$.

In this paper we will only consider well-formed nets, introduced in the next definition.

***Definition** 2.2.* A GBPN is said to be (*well-formed*) if the following conditions hold:

- discrete places can be connected to continuous and batch transitions only by self-loops, i.e., for all $p_i \in P^D$ and for all $t_j \in T^C \cup T^B$ it holds $Pre(p_i, t_j) = Post(p_i, t_j)$.
- the pre and post sets of batch places contain only batch transitions, i.e., for all $p_i \in P^B$ it holds ${}^\bullet p_i \cup p_i^\bullet \subset T^B$. ■

The first condition, that is also commonplace in the framework of hybrid nets, is required to ensure the marking of discrete place is not changed by the firing of continuous and batch transitions. The second condition is due to the rules concerning the creation and destruction of batches.

The *incidence matrix* of a net is $\mathbf{C} = Post - Pre$ and for a well-formed GBPN can be partitioned as follows.

$$\begin{array}{cc} & \begin{array}{ccc} T^D & T^C & T^B \end{array} \\ \boldsymbol{C} = & \begin{bmatrix} \boldsymbol{C}^{DD} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{C}^{CD} & \boldsymbol{C}^{CC} & \boldsymbol{C}^{CB} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{C}^{BB} \end{bmatrix} \begin{array}{c} P^D \\ P^C \\ P^B \end{array} \end{array}$$

The main extension of GBPN with respect to Hybrid Petri Nets (David and Alla, 2005) is related to the notions of batch, i.e., a group of discrete entities characterized by three continuous variables.

***Definition** 2.3.* A *batch* β_r at time τ, is defined by a triple, $\beta_r(\tau) = (l_r(\tau), d_r(\tau), x_r(\tau))$, where $l_r(\tau) \in \mathbb{R}_{\geq 0}$ is the *length*, $d_r(\tau) \in \mathbb{R}_{\geq 0}$ is the *density* and $x_r(\tau) \in \mathbb{R}_{\geq 0}$ is the *head position.* ■

A batch place contains a series of batches, ordered by their head positions and moving forward at the same speed V_i.

The state of a GBPN is represented by its marking.

***Definition** 2.4.* The *marking* of a GBPN at time τ is defined as $\boldsymbol{m}(\tau) = [m_1(\tau)...m_i(\tau)...m_n(\tau)]^T$, where:

- if $p_i \in P^D$ then $m_i \in \mathbb{N}$, i.e., the marking of a discrete place is a non negative integer.
- if $p_i \in P^C$ then $m_i \in \mathbb{R}_{\geq 0}$, i.e., the marking of a continuous place is a non negative real.
- if $p_i \in P^B$ then $m_i = \{\beta_h, ..., \beta_r\}$, i.e., the marking of a batch place is a series of batches.

The *quantity of marks* contained in batch place $p_i \in P^B$, with $m_i = \{\beta_h, ..., \beta_r\}$, is defined by:

$$q_i(\tau) = \sum_{\beta_j \in m_i} l_j(\tau) \cdot d_j(\tau),$$

and represents the sum of the quantities of the batches contained in the place. ■

We denote $\boldsymbol{m}_0 = \boldsymbol{m}(\tau_0)$ the initial marking. When time can be omitted, we denote the marking as $\boldsymbol{m}$.

***Definition** 2.5.* Let $\beta_r(\tau) = (l_r(\tau), d_r(\tau), x_r(\tau)) \in m_i(\tau)$ be a batch in place $p_i \in P^B$, with $\gamma(p_i) = (V_i, d_i^{\max}, s_i)$. β_r is called an *output batch* if its head position is equal to the length associated to the batch place, i.e., $x_r(\tau) = s_i$. A batch is said to be *dense* if its density is equal to the maximal density of batch place p_i, $d_r(\tau) = d_i^{\max}$. The *output density* d_i^{out} of batch place p_i is defined as follows. If at time τ, place p_i has an output batch $\beta_r(\tau)$, then $d_i^{\text{out}}(\tau) = d_r(\tau)$, else $d_i^{\text{out}}(\tau) = 0$. ■

Note that a place in GBPN can have at most one output batch. Due to the bounded characteristics of a batch place, some constraints on batches characteristics have to be respected: $0 \leq l_r \leq x_r \leq s_i$ (position and length constraints) and $0 \leq d_r \leq d_i^{\max}$ (density constraint).

The notion of batch place function associated to a batch place implicitly assumes that the place capacity is finite. This is formalised in the following definition.

***Definition** 2.6.* The *maximum capacity* of batch place $p_i \in P^B$, with $\gamma(p_i) = (V_i, d_i^{\max}, s_i)$, is $Q_i = s_i \cdot d_i^{\max}$.