

complete symmetry. Lagrange's rule is usually expressed as follows:

To find the extreme values of the function $f(x, y)$ subject to the subsidiary condition $\phi(x, y) = 0$, we add to $f(x, y)$ the product of $\phi(x, y)$ and an unknown factor λ independent of x and y , and write down the known necessary conditions,

$$f_x + \lambda \phi_x = 0, \quad f_y + \lambda \phi_y = 0,$$

for an extreme value of $F = f + \lambda \phi$. In conjunction with the subsidiary condition $\phi = 0$ these serve to determine the co-ordinates of the extreme value and the constant of proportionality λ .

Before proceeding to prove the rule of undetermined multipliers rigorously we shall illustrate its use by means of a simple example. We wish to find the extreme values of the function

$$u = xy$$

on the circle with unit radius and centre the origin, that is, with the subsidiary condition

$$x^2 + y^2 - 1 = 0$$

According to our rule, by differentiating $xy + \lambda(x^2 + y^2 - 1)$ with respect to x and to y we find that at the stationary points the two equations

$$\begin{aligned} y + 2\lambda x &= 0 \\ x + 2\lambda y &= 0 \end{aligned}$$

have to be satisfied. In addition we have the subsidiary condition

$$x^2 + y^2 - 1 = 0.$$

On solving we obtain the four points

$$\begin{aligned} \xi &= \frac{1}{2}\sqrt{2}, & \eta &= \frac{1}{2}\sqrt{2}, \\ \xi &= -\frac{1}{2}\sqrt{2}, & \eta &= -\frac{1}{2}\sqrt{2}, \\ \xi &= \frac{1}{2}\sqrt{2}, & \eta &= -\frac{1}{2}\sqrt{2}, \\ \xi &= -\frac{1}{2}\sqrt{2}, & \eta &= \frac{1}{2}\sqrt{2}. \end{aligned}$$

The first two of these give a maximum value $u = \frac{1}{2}$, the second two a minimum value $u = -\frac{1}{2}$, of the function $u = xy$. That the first two do really give the greatest value and the second two the least value of the function u can be seen as follows: on the circumference the function must assume a greatest and a least value (cf. p. 97), and since the circumference has no boundary point, these points of greatest and least value must be stationary points for the function.