But, by the sine formula for spherical triangles:

$$\frac{\sin(\beta_1)}{\sin(d_{03})} = \frac{\sin(\alpha_0)}{\sin(t)} ,$$

so that:

$$d\beta_1(U) = \frac{\cos(d_{03})\cos(t) - \cos(d_{01})}{\sin(d_{01})\sin(d_{03})\sin(t)\sin(\alpha_0)} .$$

The same computation (or a symmetry argument) shows that:

$$d\gamma_1(U) = \frac{\cos(d_{23})\cos(t) - \cos(d_{12})}{\sin(d_{12})\sin(d_{23})\sin(t)\sin(\alpha_2)} ,$$

and, taking the sum, we obtain that:

$$d\alpha_1(U) = \frac{\cos(d_{03})\cos(t) - \cos(d_{01})}{\sin(d_{01})\sin(d_{03})\sin(t)\sin(\alpha_0)} + \frac{\cos(d_{23})\cos(t) - \cos(d_{12})}{\sin(d_{12})\sin(d_{23})\sin(t)\sin(\alpha_2)} . \tag{4}$$

By the same computation (or a symmetry argument):

$$d\alpha_3(U) = \frac{\cos(d_{01})\cos(t) - \cos(d_{03})}{\sin(d_{01})\sin(d_{03})\sin(t)\sin(\alpha_0)} + \frac{\cos(d_{12})\cos(t) - \cos(d_{23})}{\sin(d_{12})\sin(d_{23})\sin(t)\sin(\alpha_2)} . \tag{5}$$

Proof of Theorem $\mathbf{B}_S$. Let $p = (v_1, \cdots, v_n)$ be a strictly convex spherical polygon. Recall that, by Definition 1.2, for any first-order deformation U of p, defined up to the Killing fields, we have:

$$b(U) = \sum_{i=0}^{n} d\alpha_i(U')dv_i(U') ,$$

where U' is any representative of U, i.e. any first-order deformation of p corresponding to U under the quotient by the trivial deformations.

So b is a quadratic form on $T_p\mathcal{P}_S(l)$, where $l = (l_1, \cdots, l_n)$ is the n-uple of the edge lengths of p. It is natural to define a bilinear form associated to b, which we call b_2; it is defined as follows: if U'_1 and U'_2 are two first-order deformations of p, then:

$$b_2(U'_1, U'_2) := \frac{1}{2}\left(\sum_{i=1}^{n} d\alpha_i(U'_1)dv_i(U'_2) + \sum_{i=1}^{n} d\alpha_i(U'_2)dv_i(U'_1)\right) .$$

As for b, an important point is that if one adds a Killing field to either U'_1 or U'_2, the result does not change. Moreover, it will be useful below to note that each of the two sums in the definition of b_2 is invariant under this transformation. Indeed, if $Y_1, Y_2 \in \mathbb{R}^3$ are two vectors, let V_1, V_2 be the trivial deformations defined by $dv_i(V_1) = Y_1 \times v_i, dv_i(V_2) = Y_2 \times v_i$; then, for each $i \in \{1, \cdots, n\}$, $d\alpha_i(U'_1 + V_1) = d\alpha_i(U'_i)$, and:

$$\begin{aligned}
\sum_{i=1}^{n} d\alpha_i(U'_1 + V_1)dv_i(U'_2 + V_2) &= \sum_{i=1}^{n} d\alpha_i(U'_1)(dv_i(U'_2) + Y_2 \times v_i) \\
&= \sum_{i=1}^{n} d\alpha_i(U'_1)dv_i(U'_2) + Y_2 \times \sum_{i=1}^{n} d\alpha_i(U'_1)v_i \\
&= \sum_{i=1}^{n} d\alpha_i(U'_1)dv_i(U'_2) ,
\end{aligned}$$

and the same computation can be applied to the second sum in the definition of b_2.