Algorithm Foundations of Data Science and Engineering

Lecture 4: Computation of Eigenvalue and Eigenvector

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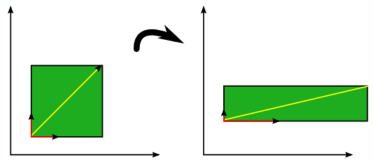
Outline

Introduction of Eigenvalue and Eigenvector

Calculating the Eigenvalue and Eigenvector
- Power Method
- Rayleigh Quotient Method
- Deflation Techniques

Transformation



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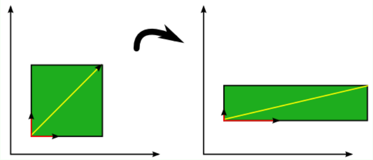

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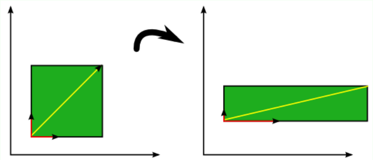

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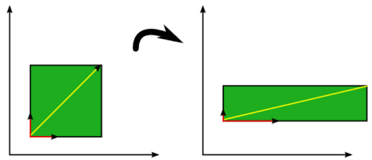

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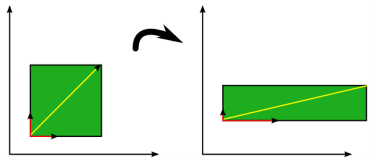

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- Eigenvectors (red) do not change direction when a linear transformation is applied to them. Other vectors (yellow) do.

Eigenvalue and eigenvector

Definition

Given a matrix $A \in \mathbb{R}^{n \times n}$, and non-zero column vector $\mathbf{v}$, if

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where λ is an eigenvalue of A and $\mathbf{v}$ is an eigenvector corresponding to eigenvalue λ.

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- $A\mathbf{v} = \lambda \mathbf{v}$ can be rewrote as

  $$A\mathbf{v} - \lambda \mathbf{v} = \mathbf{0}, \text{ i.e., } (A - \lambda I)\mathbf{v} = \mathbf{0},$$

  where I is the identity matrix of the same dimensions as A.

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How can we use computers to find eigenvalues and eigenvectors efficiently?

Directed approach

Given a matrix A

$$A = \left[\begin{array}{cc} 3 & -2 \\ 1 & 0 \end{array} \right]$$

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- For above example, we have

$$Det(A - \lambda I) = \lambda^2 - 3\lambda + 2 = 0.$$

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However, this approach is not scalable.

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- It can fail if there is not a single largest eigenvalue, i.e., $\lambda_1 = \lambda_2$.

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- It will find only one eigenvalue (the one with the greatest absolute value).

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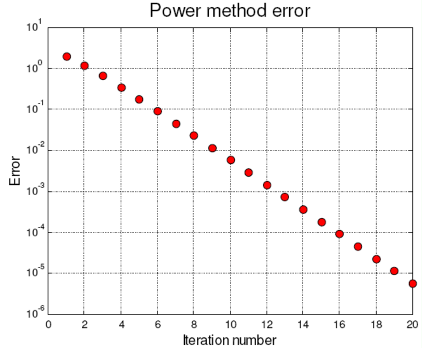

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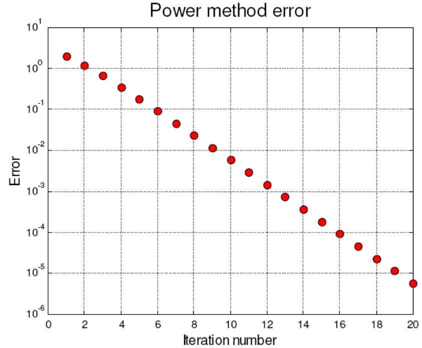

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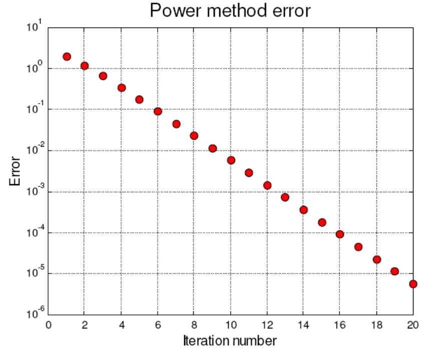

- The Power method can be used to find the dominant eigenvalue of a symmetric matrix.
- The method does work if the dominant eigenvalue has multiplicity r. The estimated eigenvector will then be a linear combination of the r eigenvectors.

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- Spectral shift: using the fact that the eigenvalues of $A - \alpha I$ are $\lambda_i - \alpha$. If we find the largest eigenvalue λ_1, we can find the largest in absolute value of $\lambda_i - \lambda_1$. However, it is not clear how it could be implemented in general to find all the eigenvalues of matrix A.

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1: Pick some μ close the desired eigenvalue;
2: Pick a starting vector $\mathbf{x}^{(0)} \in \mathbb{R}^n$, such that $\|\mathbf{x}^{(0)}\| = 1$;
3: For $k = 1, 2, \cdots$;
4: Solve $(A - \mu I)\mathbf{v} = \mathbf{x}^{(k-1)}$ for $\mathbf{v}$;
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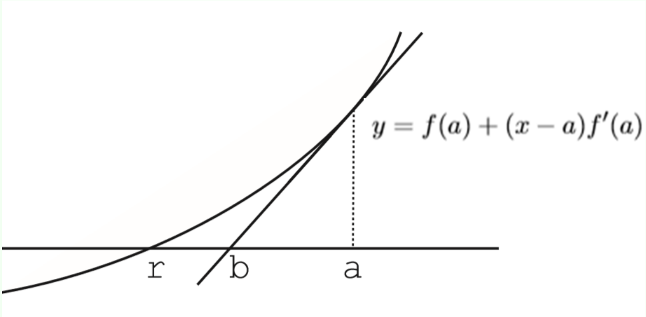
- Since the true root is r, and $h = r - x_0$, the number h measures how far the estimate x_0 is from the truth.
- Since h is 'small', we can use the linear (tangent line) approximation to conclude that

$$0 = f(r) = f(x_0+h) \approx f(x_0)+hf'(x_0) = f(x_0)+\frac{df(x)}{dx}|_{x=x_0}\triangle(x_0)$$

Newton iteration Cont'd



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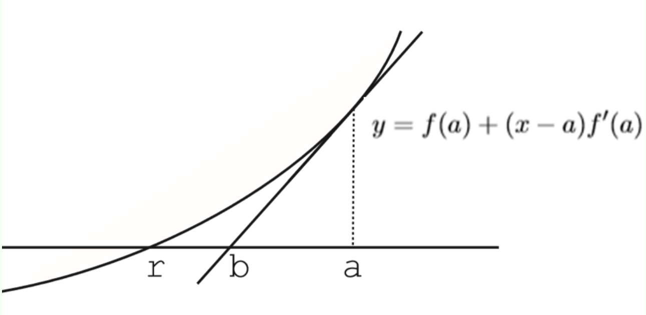
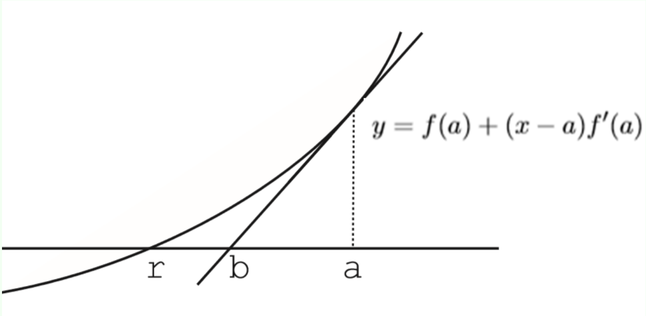


- Let a be the current estimate of r.

Newton iteration Cont'd



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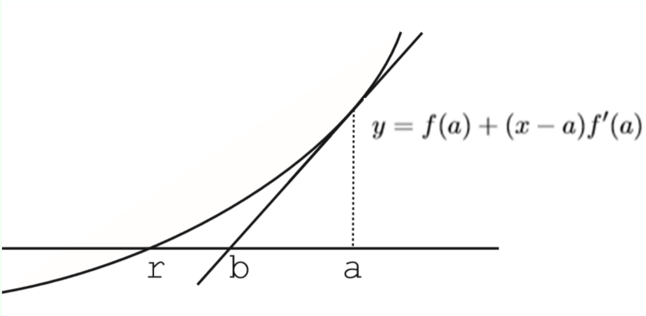

- Let a be the current estimate of r.
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- b is just the next Newton estimate of r.

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  2: Let $\lambda^{(0)} = r(\mathbf{x}^{(0)}) = (\mathbf{x}^{(0)})^T A\mathbf{x}^{(0)}$;
  3: For $k = 1, 2, \cdots$;
  4: Solve $(A - \lambda^{(k-1)} I)\mathbf{v} = \mathbf{x}^{(k-1)}$ for $\mathbf{v}$;
  5: Let $\mathbf{x}^{(k)} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$;
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- The method is only guaranteed to converge when the matrix A is both real and symmetric, and is known to fail in the cases where the matrix is not symmetric.

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- Newton iteration gives
$$\begin{aligned} 0 &= F(\mathbf{v}_k, \lambda_k) + \triangle F(\mathbf{v}_k, \lambda_k) \\ &= (A - \lambda_k I)(\mathbf{v}_k + \triangle\mathbf{v}_k) - (\triangle\lambda_k)\mathbf{v}_k \\ &= (A - \lambda_k I)\mathbf{v}_{k+1} - (\triangle\lambda_k)\mathbf{v}_k, \end{aligned}$$
which means that $\mathbf{v}_{k+1} = (\triangle\lambda_k)(A - \lambda_k I)^{-1}\mathbf{v}_k$.

How to compute the corresponding eigenvalue?

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For a real matrix A, the Rayleigh quotient of A, denoted as $R_A(\cdot)$ is a function from $R^n \backslash \{0\}$ to R, defined as follows

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- First note that if $\mathbf{v}_i$ is the eigenvector corresponding to the eigenvalue λ_i, then $R_A(\mathbf{v}_i) = \lambda_i$.

Analysis of Rayleigh Quotient method

Definition

Following the previous analysis, considering only the two largest eigenvalues,

$$R_A(\mathbf{x}^{(k)}) = \frac{{\mathbf{x}^{(k)}}^T A \mathbf{x}^{(k)}}{{\mathbf{x}^{(k)}}^T \mathbf{x}^{(k)}} \approx \frac{\left(c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2\right)^T\left(c_1\lambda_1^{k+1}\mathbf{v}_1 + c_2\lambda_2^{k+1}\mathbf{v}_2\right)}{\left(c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2\right)^T\left(c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2\right)}$$

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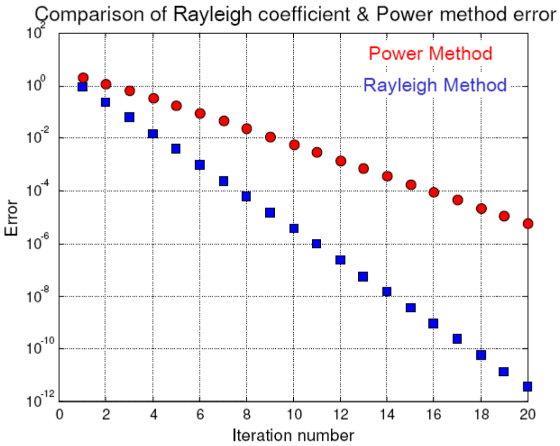
- It gives us a locally quadratically convergent algorithm, i.e. $|\frac{\lambda_2}{\lambda_1}|^2$.

Performance comparison



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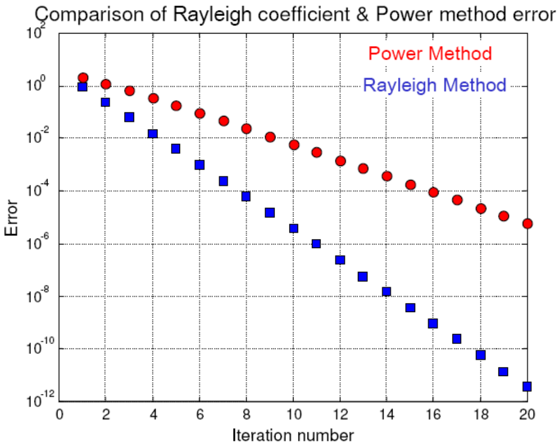

Since the reduction per iteration is quadratic, that is, quadratic or second order convergence and is much faster

Outline

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- We can find the similar conclusion for $A^{(i-1)} - \lambda_i \mathbf{v}_i \mathbf{v}_i^T$.
- Thus, we can use the power method with Rayleigh's coefficient to find the next biggest and so on.

Take-home messages

- Introduction to Eigenvalue and Eigenvector
- Calculating the Eigenvalue and Eigenvector
  - Power Method
  - Rayleigh Quotient Method
  - Deflation Technique