Algorithm Foundations of Data Science and Engineering

Lecture 2: Hashing Algorithms

YANHAO WANG

DaSE @ ECNU
(for course related communications)
yhwang@dase.ecnu.edu.cn

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Outline

Introduction

Searching

To find an entry (field of information) in the table, you only have to use the contents of one of the fields (say name in the case of the telephone book). You don't have to know the contents of all the fields. The field you use to find the contents of the other fields is called the **key**.

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	key	entry
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1	45	<data>
2	22	<data>
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4	17	<data>
⋮	*and so on*	

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A telephone book has fields name, address and phone number. When you want to find somebodys phone number, you search the book based on the name field.

Ideally, the key should uniquely identify the entry, i.e. if the key is the name then no two entries in the telephone book have the same name.

Hash function

Hash function h:

$$h : key \rightarrow \text{ value }, i.e., h(key) = value \in \mathcal{Z}^+,$$

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- **Resistant to collisions:** two keys of different inputs may not lead to a value of identical output. It should be noted that even a minimal variation between two keys of inputs can lead to two values of outputs completely different. The reverse is also true. The same value of entry may not lead to two different outputs.

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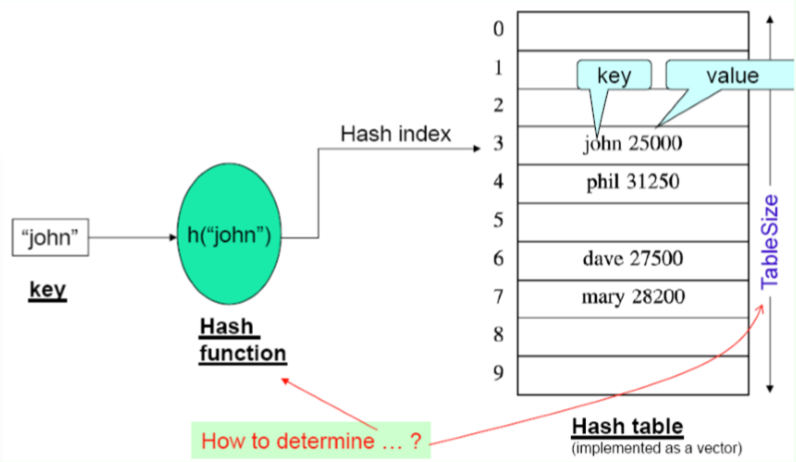
- **Pre-image resistance:** it is impossible to find an input key from the output. The only way to find the input value is to apply a method of "brute force".

Hash table



"john"
key
h("john")
Hash function
Hash index
0
1
2
3
4
5
6
7
8
9
key
value
john 25000
phil 31250
dave 27500
mary 28200
TableSize
Hash table
(implemented as a vector)
How to determine … ?

Complexity VS. Data structure

Operation	Unso. array	So. array	Linked list	Or. bin. Tree
Insert	$O(1)$	$O(n)$	$O(1)$ or $O(n)$	$O(\log n)$
Find	$O(n)$	$O(\log n)$	$O(n)$	$O(\log n)$
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- Table size: too large a table will cause a wastage of memory; too small a table will cause increased collisions.

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- **Duplicate checker**: it help Web crawler to determine whether an URL has already been crawled.

Problem setting

Let S be a set of n elements. We are given a set of k hash functions, saying $h_1, h_2 \cdots, h_k$, with range $\{1, 2, \cdots, m\}$ or $(\{0, 1, \cdots, m-1\})$. Initially, $m-$long array of bits are set to 0.

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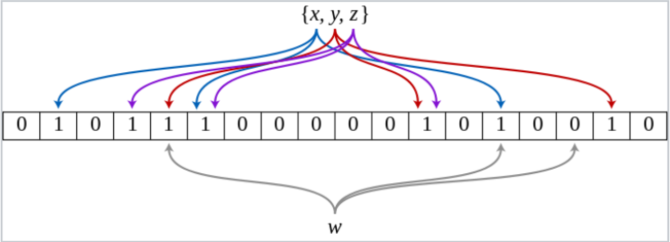

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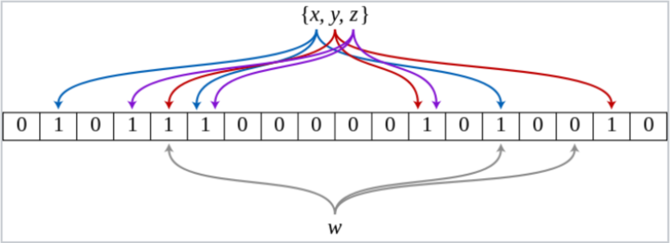

An example of a Bloom filter, representing the set $\{x, y, z\}$. The colored arrows show the positions in the bit array that each set element is mapped to. For this figure, $m = 18$ and $k = 3$.

Example

We insert and query on a Bloom filter of size $m = 10$ and number of hash functions $k = 3$. Let $H(x)$ denote the result of the three hash functions which we will write as a set of three values $\{h_1(x), h_2(x), h_3(x)\}$.

0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0

0	1	2	3	4	5	6	7	8	9
0	1	0	0	1	0	0	0	0	1

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Insert x_1 with $H(x_1) = \{4, 5, 8\}$.

Query on Bloom filter

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- Query y_1 with $H(y_1) = \{0, 4, 8\} \rightarrow$ NO ($\surd$);
- Query y_2 with $H(y_2) = \{1, 5, 8\} \rightarrow$ Yes (False Positive);

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And consequently the probability that the bit is one is

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$$f = \left(1 - (1 - \frac{1}{m})^{kn}\right)^k \approx (1 - e^{-kn/m})^k.$$

x	$\left(1 - \frac{1}{x}\right)^{-x}$
4	3.160494
16	2.808404
64	2.739827
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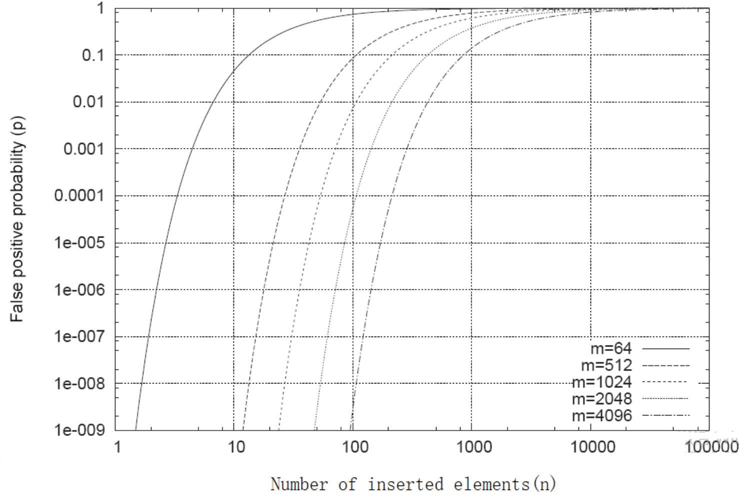
Now, we want to minimize the probability of false positives, by minimizing $(1 - e^{-kn/m})^k$ with respect to k.

False positive rate Cont'd



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Thus, we see that g is minimized when $p = \frac{1}{2}$.

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Thus, the optimal filter structure is $p = \frac{1}{2}$, which corresponds to a half-full Bloom filter array.

m, n, k Examples

False positive rates for choices of k given $\frac{m}{n}$

m/n	k	k=1	k=2	k=3	k=4	k=5
2	1.39	0.393	0.400			
3	2.08	0.283	0.237	0.253		
4	2.77	0.221	0.155	0.147	0.160	
5	3.46	0.181	0.109	0.092	0.092	0.101
6	4.16	0.154	0.0804	0.0609	0.0561	0.0578
7	4.85	0.133	0.0618	0.0423	0.0359	0.0347
8	5.55	0.118	0.0489	0.0306	0.024	0.0217

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- Today: Jaccard distance / similarity
  - The Jaccard similarity of two sets is the size of their intersection divided by the size of their union

$$sim(A, B) = \frac{|A \cap B|}{|A \cup B|}$$

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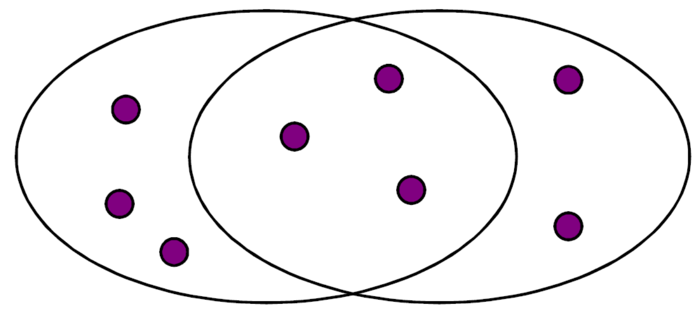
- For each application, we first need to define what “distance” means
- Today: Jaccard distance / similarity
  - The Jaccard similarity of two sets is the size of their intersection divided by the size of their union

$$sim(A, B) = \frac{|A \cap B|}{|A \cup B|}$$

  - Jaccard distance (non-negative, symmetric, triangle inequality):

$$d(A, B) = 1 - \frac{|A \cap B|}{|A \cup B|}$$

Example



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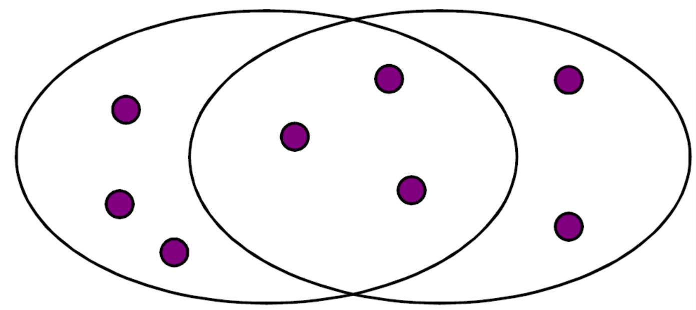

- 3 elements in intersection, 8 elements in union.

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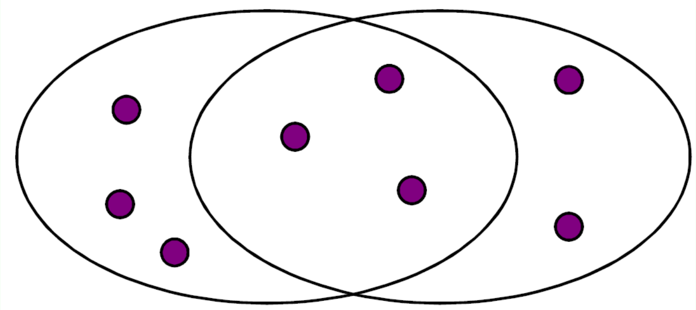

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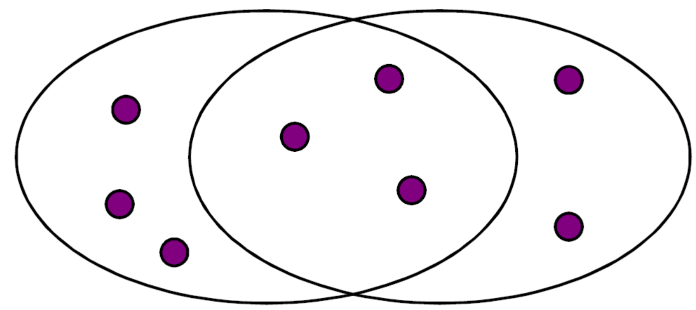

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  - Documents are so large or so many that they cannot fit in main memory.

Three essential steps for similar documents

- Shingling: Convert documents to sets

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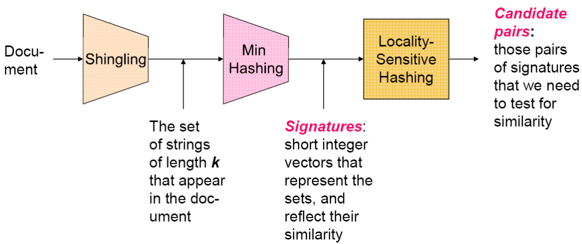

Shingling: Documents as high-dimensional data

Approach

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Need to account for ordering of words! A different way: Shingles!

Shingles

Definition

A k-shingle (or k-gram) for a document is a sequence of k tokens that appears in the document

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Option: Shingles as a bag (multiset), count ab twice

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Hash the singles: $h(D) = \{1, 5, 7\}$.

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- For $N = 10$ million, it takes more than a year...

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Documents

Shingles

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Hash docs into buckets. Expect that "most" pairs of near duplicate docs hash into the same bucket!

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- A good example of a random permutation is the shuffling of a deck of cards: this is ideally a random permutation of the 52 cards.

Min-hashing Cont'd

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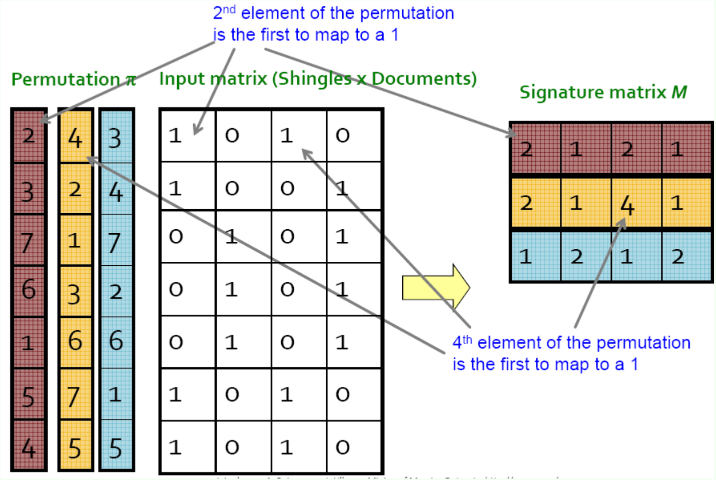
- Use several (e.g., 100) independent hash functions (that is, permutations) to create a signature of a column

Min-hashing example



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- Let y be s.t. $\pi(y) = \min(\pi(C_1 \cup C_2))$, then

$$\pi(y) = \min(\pi(C_1)) \text{ or } \pi(y) = \min(\pi(C_2))$$

The Min-hashing property

Given a random permutation π, we have

$$P\big(h_\pi(C_1) = h_\pi(C_2)\big) = sim(C_1, c_2)$$

- Let X be a doc (set of shingles), $y \in X$ is a shingle. Then

  $$P[\pi(y) = \min(\pi(X))] = 1/|X|,$$

  it is equally likely that any $y \in X$ is mapped to the *min* element.
- Let y be s.t. $\pi(y) = \min(\pi(C_1 \cup C_2))$, then

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- Thus, so the prob. that both are true is the prob. $y \in C_1 \cap C_2$. Final, we have

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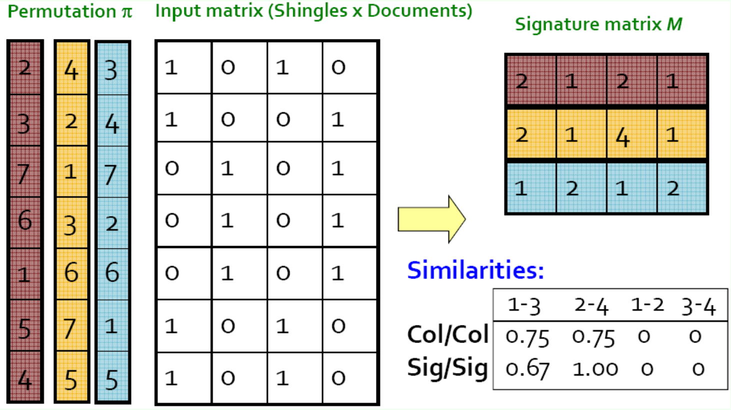
- Now generalize to multiple hash functions
- The similarity of two signatures is the fraction of the hash functions in which they agree
- Note: Because of the Min-Hash property, the similarity of columns is the same as the expected similarity of their signatures

Min-hashing-based similarity



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- We achieved our goal! We “compressed” long bit vectors into short signatures

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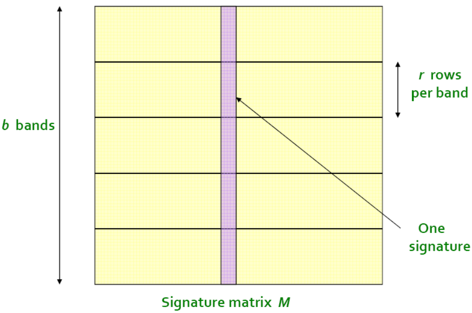
Grouping signatures



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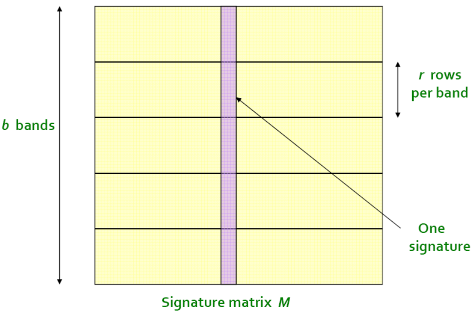

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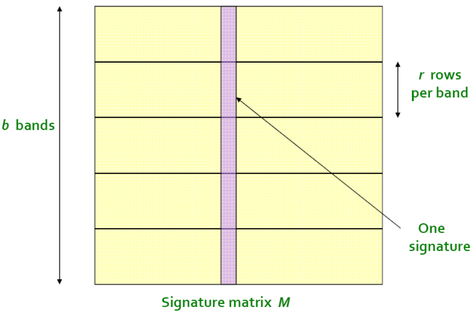

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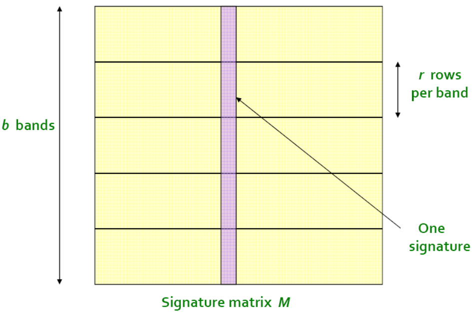

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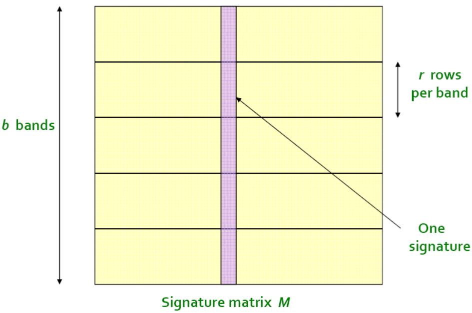

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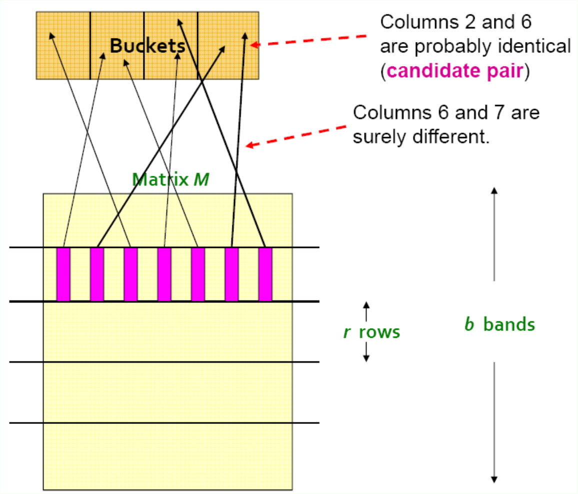
Tune b and r to catch most similar pairs, but few non-similar pairs

Hashing bands



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Assume the following case:

- Suppose 100,000 columns of M ($100k$ docs)
- Signatures of 100 integers (rows)
- Therefore, signatures take $40Mb$
- Choose $b = 20$ bands of $r = 5$ integers/band

Goal: Find pairs of documents that are at least $s = 0.8$

C_1, C_2 are 80% similar

Find pairs of $\geq s = 0.8$ similarity, set $b = 20, r = 5$

- Assume: $sim(C_1, C_2) = 0.8$. Since $sim(C_1, C_2) \geq s$, we want C_1, C_2 to be a candidate pair: We want them to hash to at least 1 common bucket (at least one band is identical)

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  - We would find 99.965% pairs of truly similar documents

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- Assume: $sim(C_1, C_2) = 0.3$. Since $sim(C_1, C_2) < s$, we want C_1, C_2 to hash to no common buckets (all bands should be different)

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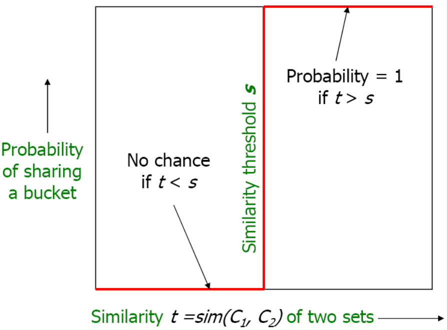
  - They are false positives since we will have to examine them (they are candidate pairs) but then it will turn out their similarity is below threshold s

LSH involves a tradeoff



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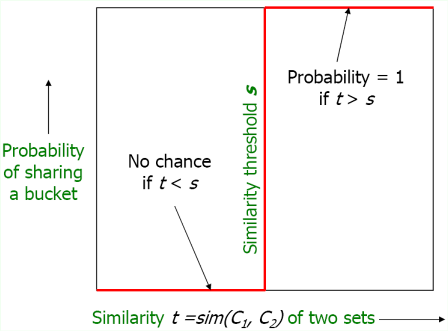

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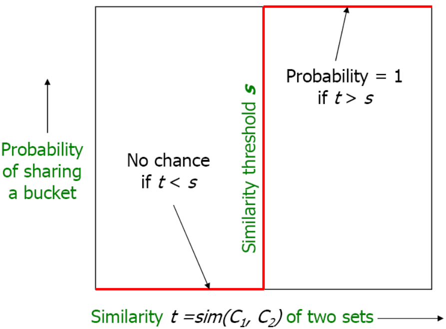

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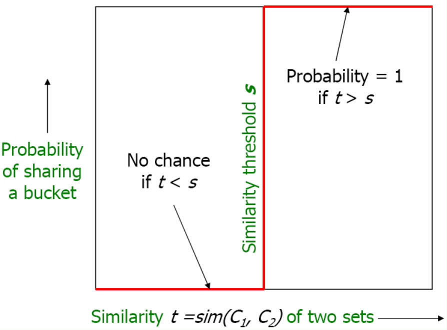

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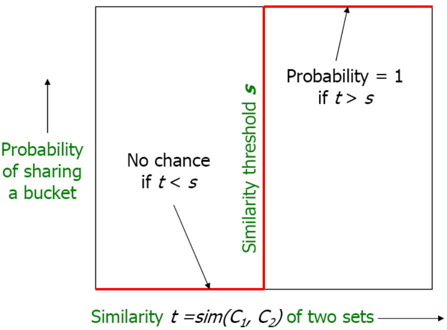

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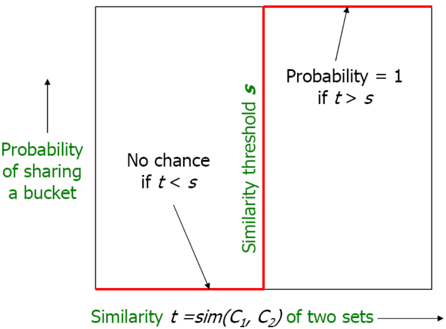

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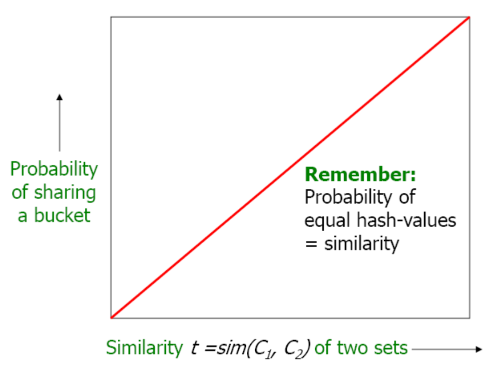
- Example: If we had only 15 bands of 5 rows, the number of false positives would go down, but the number of false negatives would go up

What 1 band of 1 row gives you



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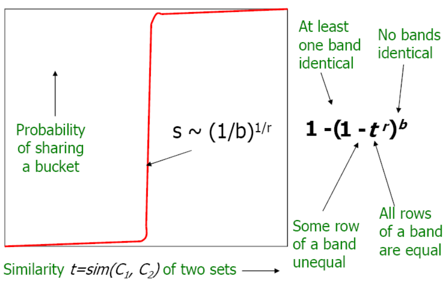

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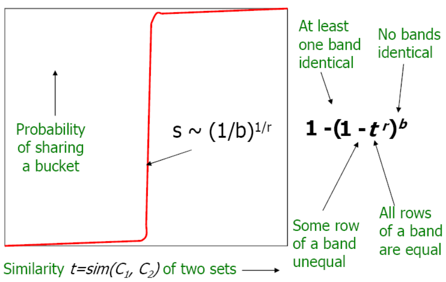
Columns C_1 and C_2 have similarity t



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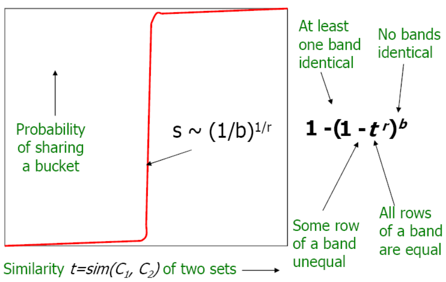

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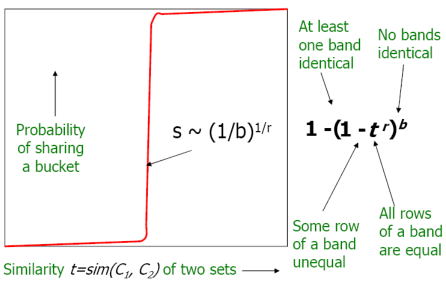

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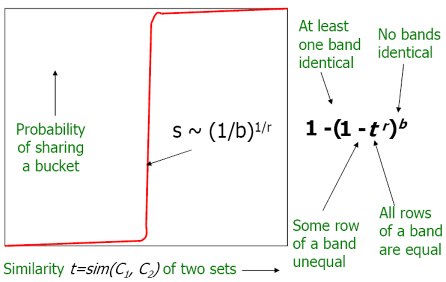

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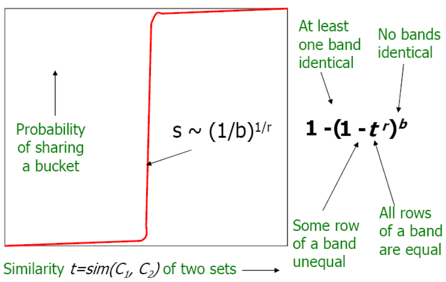

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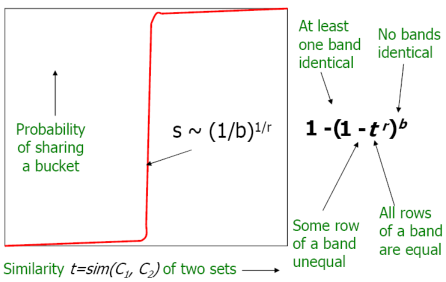

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- Prob. that at least 1 band identical $= = 1 - (1 - t^r)^b$

Analysis of LSH

s	$1\text{-}(1\text{-}s^r)^b$
.2	.006
.3	.047
.4	.186
.5	.470
.6	.802
.7	.975
.8	.9996

Prob. that at least 1 band is identical.

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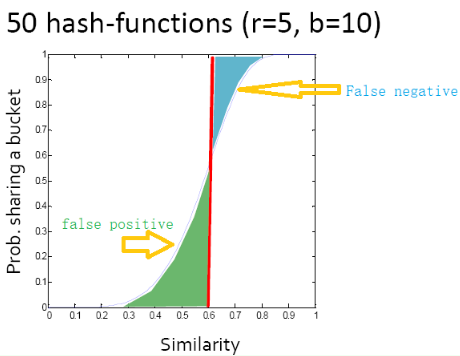

Picking r and b to get the best S-curve

LSH summary

- Tune M, b, r to get almost all pairs with similar signatures, but eliminate most pairs that do not have similar signatures

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- Check in main memory that candidate pairs really do have similar signatures
- Optional: In another pass through data, check that the remaining candidate pairs really represent similar documents

Take-aways

- Hash functions
- Bloom filtering
- Locality sensitive hashing
  - Shingling
  - Min-hashing
  - Locality sensitive hashing