Algorithm Foundations of Data Science and Engineering

Lecture 11: Submodular and Its Applications

YANHAO WANG

DaSE @ ECNU
(for course related communications)
yhwang@dase.ecnu.edu.cn

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Outline

Motivation: set functions

Feature selection

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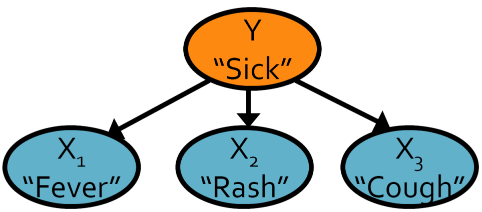
Sensor placement

- Given a water distribution network;
- Where should we place sensors to quickly detect contaminations?

Feature selection



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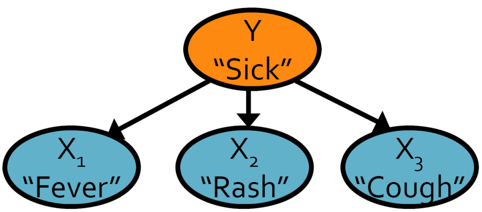

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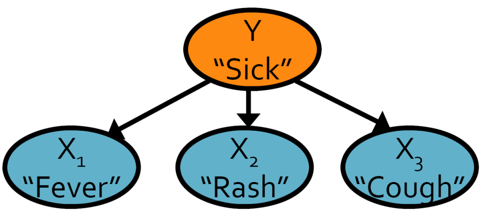

- Given r.v.s $Y, X_1, \cdots, X_n$, predict Y from a subset $A = (X_{i1}, \cdots, X_{ik})$;
- Information gain:

$$I(A; Y) = H(Y) - H(Y|A),$$

where $H(Y)$ is the conditional entropy, $I(A; Y)$ measures the difference of uncertainty before and after knowing A;

Cut function in a graph

Let $G = (V, E)$ be an undirected graph.

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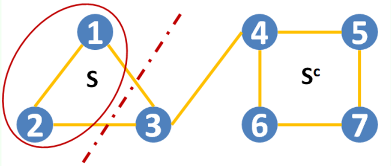

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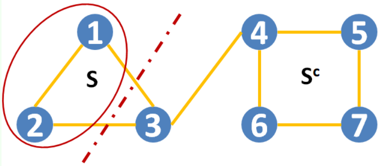

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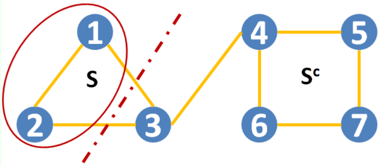

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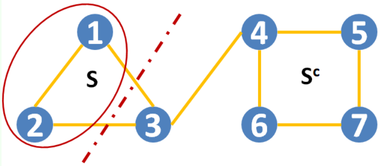

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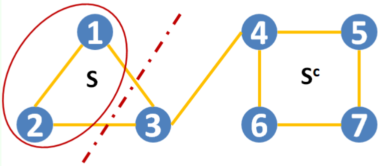

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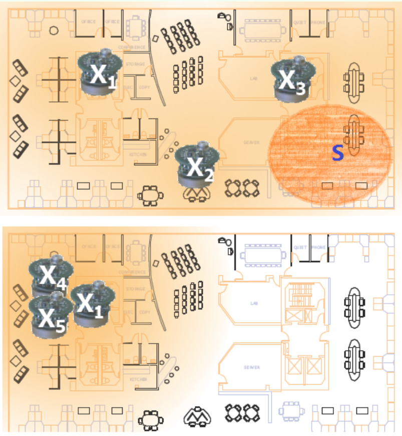
  - For $S = \{1, 2, 3\}$, $f(S) = 1$;
  - For $S = \{1, 2\}$, $f(S) = 2$;
- The graph cut is a set function.

Sensor placement



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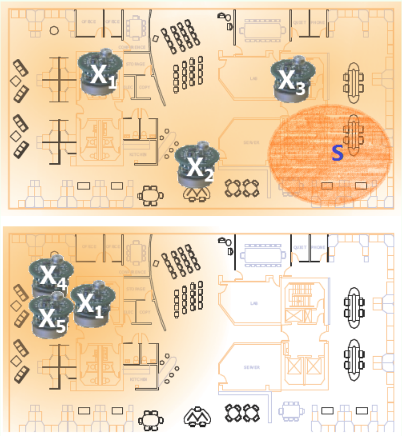

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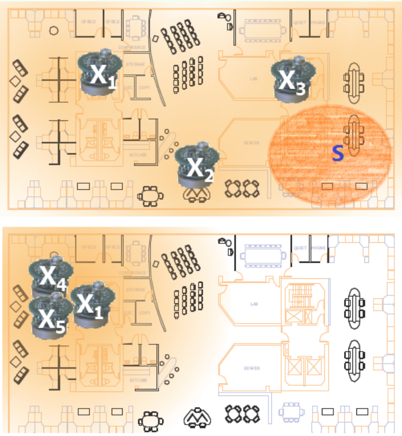

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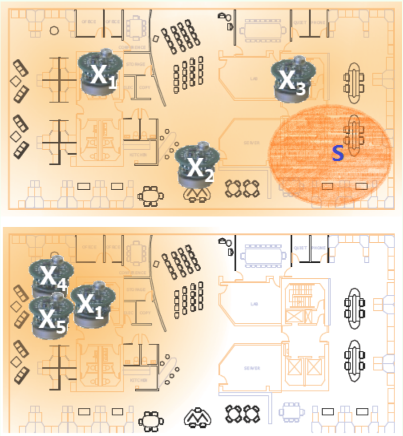

Function $f(A)$ is the value of utility of having sensors at subset A of all locations.

- $A = \{1, 2, 3\}$ very informative (high value of $f(A)$).
- $A = \{1, 4, 5\}$ redundant information (low value of $f(A)$).

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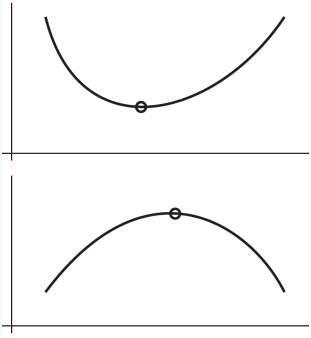
There are many set functions, such as information gain, graph cut, and sensor utility, etc.

Continuous optimization

What makes continuous optimization tractable?



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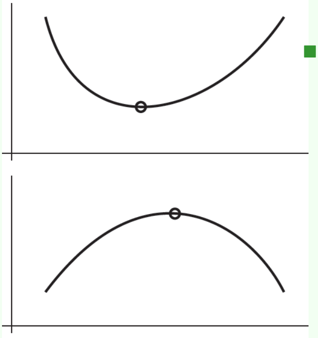

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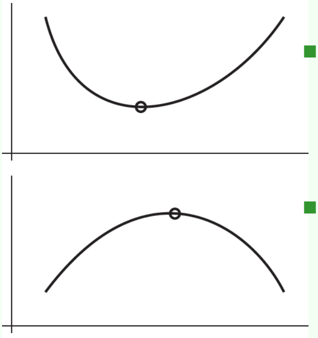

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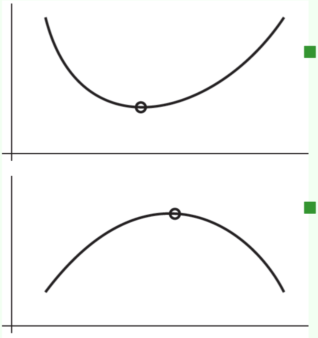

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Discrete analogy?

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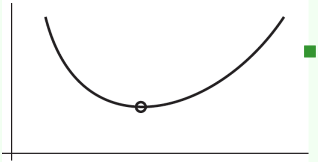

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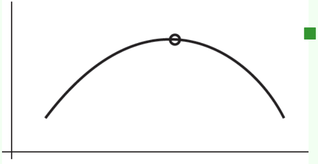

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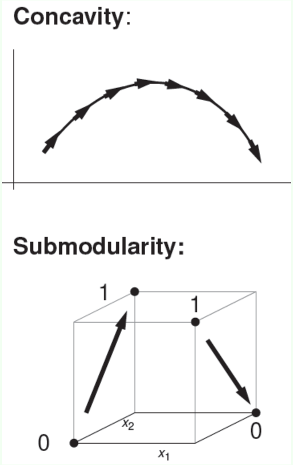
Discrete analogy?
f is now a set function, or equivalently $f : 2^V \rightarrow R$ or $f : \{0, 1\}^n \rightarrow R$.

From concavity to submodularity



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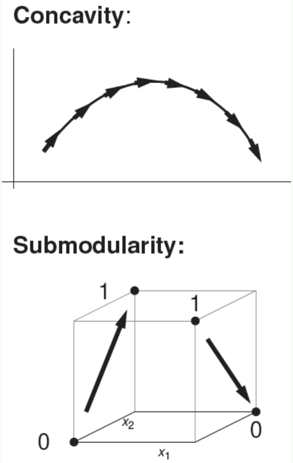

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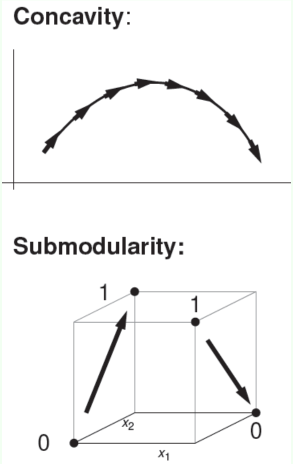

- $f : R \to R$ is concave, if the derivative $f'(x)$ is non-increasing in x.

- $f : \{0,1\}^n \to R$ is submodular, if $\forall i$, the discrete derivative

$$\partial_i f(x) = f(x + e_i) - f(x)$$

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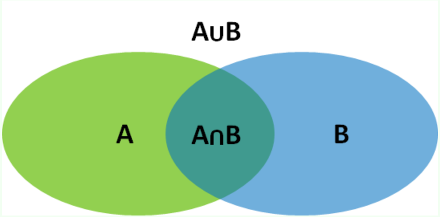

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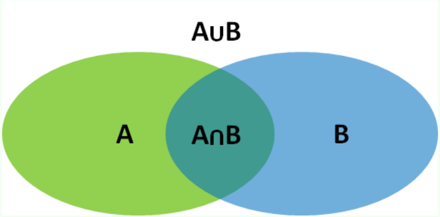

i.e.,

$$f(A) - f(A \cap B) \geq f(A \cup B) - f(B).$$

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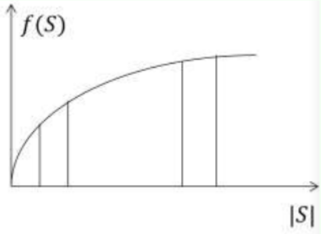
- Machine learning:
  Submodular functions often appear as objective functions of machine learning tasks such as sensor placement, document summarization or feature selection $\rightarrow$ simple algorithms such as Greedy or local search work well.

Submodularity: or equivalently

We have two equivalent definitions:



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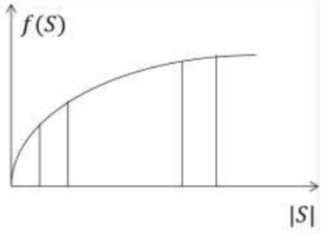

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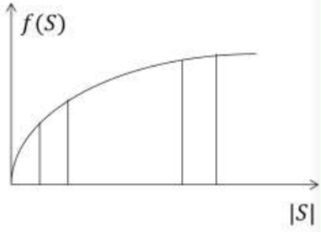

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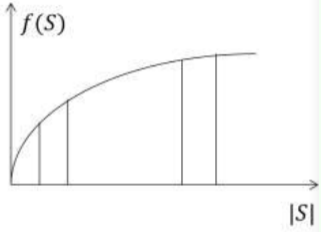

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Submodularity is the discrete analogue of concavity; in economics, known as diminishing returns.

Proof of equivalence

$f(S \cup T) + f(S \cap T) \leq f(S) + f(T) \Leftrightarrow f(S \cup \{v\}) - f(S) \geq f(T \cup \{v\}) - f(T)$.

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- $\Rightarrow$: let $S \subset T$, consider two sets $S \cup \{v\}$ and T, if $v \notin T$, then
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  Thus, we have $f(S \cup \{v\}) - f(S) \geq f(T \cup \{v\}) - f(T)$.
- $\Leftarrow$: let $T \backslash S = \{v_1, v_2, \cdots, v_k\}$, $T_j = \{v_1, v_2, \cdots, v_j\}$, $A_j = (S \cap T) \cup T_j$, and $B_j = S \cup T_j$, then we have $f(A_j \cup \{v_{j+1}\}) - f(A_j) \geq f(B_j \cup \{v_{j+1}\}) - f(B_j)$ for $j = 0, 1, 2, \cdots, k-1$.

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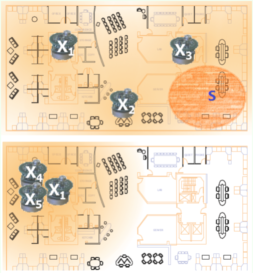
  Summing up all these equations, we have
  $f(S \cup T) + f(S \cap T) \leq f(S) + f(T)$.

Submodular example I: sensor placement



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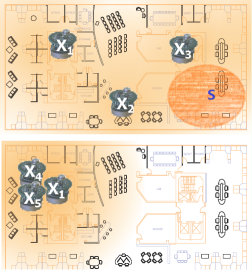

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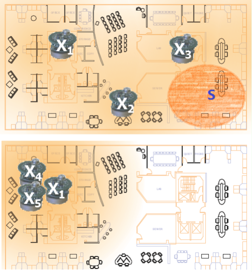

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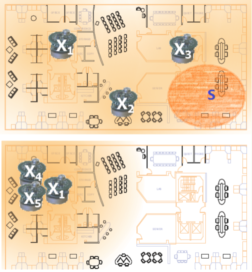

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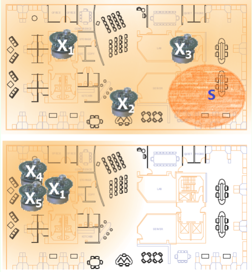

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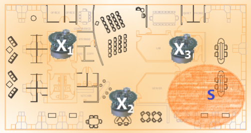

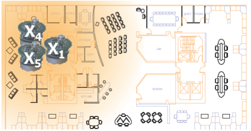

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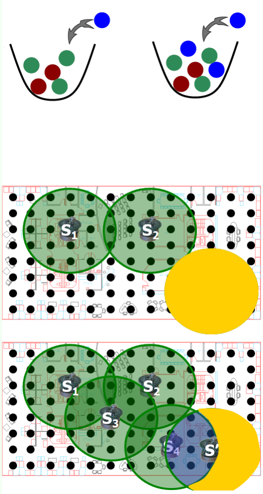
There are many similar applications, such as information cascade, document summarization, community detection, etc.

Submodular examples



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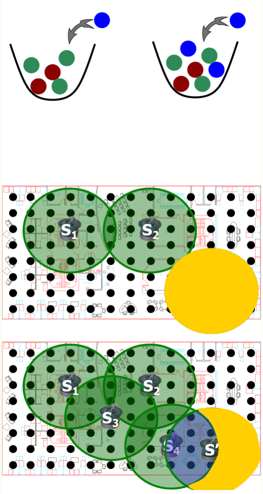

Count distinct colors

Given a set S of balls, $f(S)$ counts the number of distinct colors.

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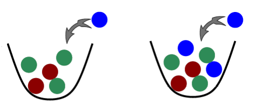

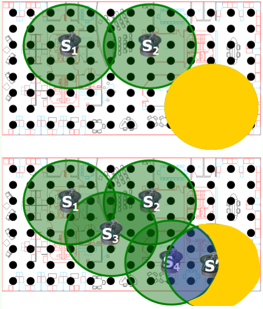

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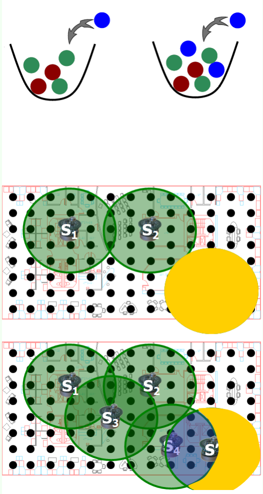

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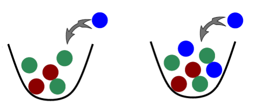

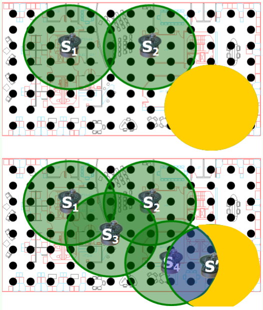

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Set covering

Assume that $A = \{S_1, S_2\}$ and $B = \{S_1, S_2, S_3, S_4\}$, then we have $f(A \cup \{S'\}) - f(A) \geq f(B \cup \{S'\}) - f(B)$.

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- $f_\theta(A)$ is a submodular $\Rightarrow \sum_\theta P(\theta) f_\theta(A)$ is a submodular;
- Multicriterion optimization: $f_1, \cdots, f_m$ are submodulars, and $\lambda_i > 0 \Rightarrow \sum_i \lambda_i f_i(A)$ is a submodular;

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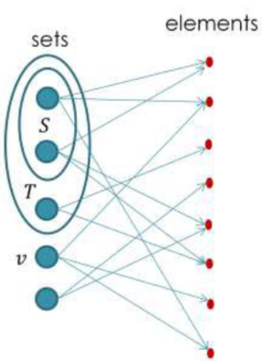
- For set covering problem:

$$\begin{aligned}
&\text{minimize} \sum_{i=1}^{|S|} c_i x_i \\
&\text{s.t.} \sum_{i=1}^{|S|} x_i S_{ij} > 0, \text{ for } j = 1, 2, \cdots, |U| \\
&x_i \in \{0, 1\}
\end{aligned}$$

Outline

Set cover problem



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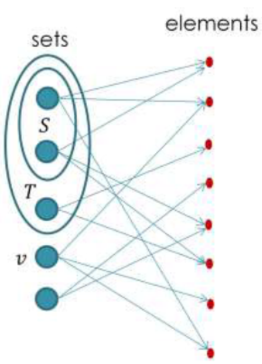

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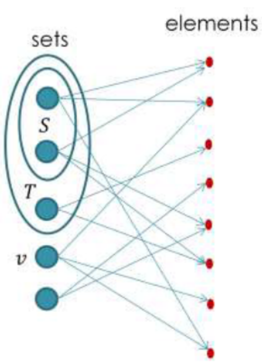

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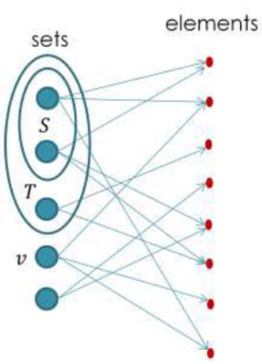

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- Find k subsets that maximizes their total coverage;

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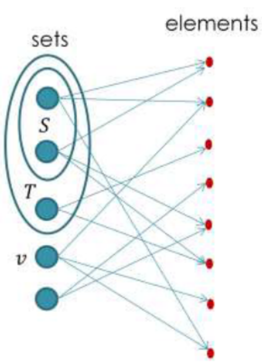

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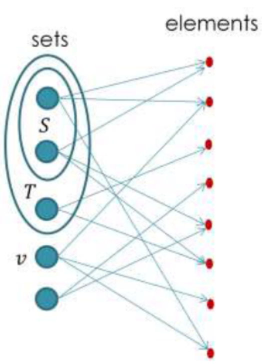

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Example of 2$-$max cover

Ground set $\{a, b, c, d, e, f, g, h, i, j, k, l\}$

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Subsets $A_1 = \{a, b, c, d\}, A_2 = \{e, f, g, h\}, A_3 = \{i, j, k, l\}$
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- Collection $C = \{A_5, A_6\}$ is a 2−max cover since it covers nine elements.

Text summarization via extracting text

Definition

Given a keyword set denoted as $V = \{w_1, w_2, \cdots, w_n\}$, and sentence set $S = \{S_1, S_2, \cdots, S_m\}$, where $S_j = \{w_k | w_k \in V\}$, then text summarization is to find k sentences from C such that maximizes the coverage.

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- Let $C \subset D$, and $S_k \in D$, we have

$$f(C \cup \{S_k\}) - f(C) = \Big| S_k - \bigcup_{S_i \in C} S_i \Big|$$

$$\geq \Big| S_k - \bigcup_{S_i \in D} S_i \Big| = f(D \cup \{S_k\}) - f(D).$$

In addition, since $\bigcup_{S_i \in C} S_i \subset \bigcup_{S_i \in D} S_i$, we therefore have

$$f(C) \leq f(D).$$

Outline

Hill-climbing algorithm

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$$f(S) \geq (1 - \frac{1}{e}) \max_{S' \subseteq V, |S'| = k} f(S'),$$

where f is monotonicity if $f(S) \leq f(T)$ for all $S \subseteq T \subseteq V$.

Example of Hill-climbing algorithm: first iteration

keyword set	$W = \{w_1, w_2, \cdots, w_8\}$
sentences	$s_1 = \{w_1, w_2, w_8\}, s_2 = \{w_1, w_3, w_7\}, s_3 = \{w_1, w_6\}$
	$s_4 = \{w_1, w_3, w_7, w_8\}, s_5 = \{w_1, w_5, w_6\}, s_6 = \{w_1, w_5, w_8\}$
	$s_7 = \{w_5\}, s_8 = \{w_1, w_4, w_6\}, s_9 = \{w_2, w_8\}$

Example of Hill-climbing algorithm: first iteration

keyword set $W = \{w_1, w_2, \cdots, w_8\}$

sentences $s_1 = \{w_1, w_2, w_8\}, s_2 = \{w_1, w_3, w_7\}, s_3 = \{w_1, w_6\}$
$s_4 = \{w_1, w_3, w_7, w_8\}, s_5 = \{w_1, w_5, w_6\}, s_6 = \{w_1, w_5, w_8\}$
$s_7 = \{w_5\}, s_8 = \{w_1, w_4, w_6\}, s_9 = \{w_2, w_8\}$

Sentence	$f(C)$	$f(C \cup \{S_i\})$	$\Delta(S_i)$
s_1	0	3	3
s_2	0	3	3
s_3	0	2	2
s_4	0	4	4
s_5	0	3	3
s_6	0	3	3
s_7	0	1	1
s_8	0	3	3
s_9	0	2	2

Table: First iteration

Example of Hill-climbing algorithm: first iteration

keyword set $W = \{w_1, w_2, \cdots, w_8\}$

sentences $s_1 = \{w_1, w_2, w_8\}, s_2 = \{w_1, w_3, w_7\}, s_3 = \{w_1, w_6\}$
$s_4 = \{w_1, w_3, w_7, w_8\}, s_5 = \{w_1, w_5, w_6\}, s_6 = \{w_1, w_5, w_8\}$
$s_7 = \{w_5\}, s_8 = \{w_1, w_4, w_6\}, s_9 = \{w_2, w_8\}$

Sentence	$f(C)$	$f(C \cup \{S_i\})$	$\Delta(S_i)$
s_1	0	3	3
s_2	0	3	3
s_3	0	2	2
s_4	0	4	4
s_5	0	3	3
s_6	0	3	3
s_7	0	1	1
s_8	0	3	3
s_9	0	2	2

Table: First iteration

Sentence S_4 is selected in the first iteration since it has maximal coverage gain.

Example of Hill-climbing algorithm: second iteration

keyword set	$W = \{w_1, w_2, \cdots, w_8\}$
sentences	$s_1 = \{w_1, w_2, w_8\}, s_2 = \{w_1, w_3, w_7\}, s_3 = \{w_1, w_6\}$
	$s_5 = \{w_1, w_5, w_6\}, s_6 = \{w_1, w_5, w_8\}, s_7 = \{w_5\}$
	$s_8 = \{w_1, w_4, w_6\}, s_9 = \{w_2, w_8\}$

Example of Hill-climbing algorithm: second iteration

keyword set $W = \{w_1, w_2, \cdots, w_8\}$

sentences $s_1 = \{w_1, w_2, w_8\}, s_2 = \{w_1, w_3, w_7\}, s_3 = \{w_1, w_6\}$
$s_5 = \{w_1, w_5, w_6\}, s_6 = \{w_1, w_5, w_8\}, s_7 = \{w_5\}$
$s_8 = \{w_1, w_4, w_6\}, s_9 = \{w_2, w_8\}$

Sentence	$f(C)$	$f(C \cup \{S_i\})$	$\Delta(S_i)$
s_1	4	5	1
s_2	4	4	0
s_3	4	5	1
s_5	4	6	2
s_6	4	5	1
s_7	4	5	1
s_8	4	6	2
s_9	4	5	1

Table: The second iteration

Example of Hill-climbing algorithm: second iteration

keyword set $W = \{w_1, w_2, \cdots, w_8\}$

sentences $s_1 = \{w_1, w_2, w_8\}, s_2 = \{w_1, w_3, w_7\}, s_3 = \{w_1, w_6\}$
$s_5 = \{w_1, w_5, w_6\}, s_6 = \{w_1, w_5, w_8\}, s_7 = \{w_5\}$
$s_8 = \{w_1, w_4, w_6\}, s_9 = \{w_2, w_8\}$

Sentence	$f(C)$	$f(C \cup \{S_i\})$	$\Delta(S_i)$
s_1	4	5	1
s_2	4	4	0
s_3	4	5	1
s_5	4	6	2
s_6	4	5	1
s_7	4	5	1
s_8	4	6	2
s_9	4	5	1

Table: The second iteration

Sentence S_5 is selected in the second iteration since it has maximal coverage gain.

Example of Hill-climbing algorithm: third iteration

keyword set	$W = \{w_1, w_2, \cdots, w_8\}$
sentences	$s_1 = \{w_1, w_2, w_8\}, s_2 = \{w_1, w_3, w_7\}, s_3 = \{w_1, w_6\}$
	$s_6 = \{w_1, w_5, w_8\}, s_7 = \{w_5\}$
	$s_8 = \{w_1, w_4, w_6\}, s_9 = \{w_2, w_8\}$

Example of Hill-climbing algorithm: third iteration

keyword set	$W = \{w_1, w_2, \cdots, w_8\}$
sentences	$s_1 = \{w_1, w_2, w_8\}, s_2 = \{w_1, w_3, w_7\}, s_3 = \{w_1, w_6\}$
	$s_6 = \{w_1, w_5, w_8\}, s_7 = \{w_5\}$
	$s_8 = \{w_1, w_4, w_6\}, s_9 = \{w_2, w_8\}$

Sentence	$f(C)$	$f(C \cup \{S_i\})$	$\Delta(S_i)$
s_1	6	7	1
s_2	6	6	0
s_3	6	6	0
s_6	6	6	0
s_7	6	6	0
s_8	6	7	1
s_9	6	7	1

Table: The third iteration

Example of Hill-climbing algorithm: third iteration

keyword set $W = \{w_1, w_2, \cdots, w_8\}$

sentences $s_1 = \{w_1, w_2, w_8\}, s_2 = \{w_1, w_3, w_7\}, s_3 = \{w_1, w_6\}$

$s_6 = \{w_1, w_5, w_8\}, s_7 = \{w_5\}$

$s_8 = \{w_1, w_4, w_6\}, s_9 = \{w_2, w_8\}$

Sentence	$f(C)$	$f(C \cup \{S_i\})$	$\Delta(S_i)$
s_1	6	7	1
s_2	6	6	0
s_3	6	6	0
s_6	6	6	0
s_7	6	6	0
s_8	6	7	1
s_9	6	7	1

Table: The third iteration

Sentence S_1 is selected in the third iteration since it has maximal coverage gain.

Example of Hill-climbing algorithm: third iteration

keyword set $W = \{w_1, w_2, \cdots, w_8\}$

sentences $s_1 = \{w_1, w_2, w_8\}, s_2 = \{w_1, w_3, w_7\}, s_3 = \{w_1, w_6\}$

$s_6 = \{w_1, w_5, w_8\}, s_7 = \{w_5\}$

$s_8 = \{w_1, w_4, w_6\}, s_9 = \{w_2, w_8\}$

Sentence	$f(C)$	$f(C \cup \{S_i\})$	$\Delta(S_i)$
s_1	6	7	1
s_2	6	6	0
s_3	6	6	0
s_6	6	6	0
s_7	6	6	0
s_8	6	7	1
s_9	6	7	1

Table: The third iteration

Sentence S_1 is selected in the third iteration since it has maximal coverage gain.

Finally, it outputs text summarization $C = \{S_4, S_5, S_1\}$.

Project assignment: Only for undergraduate students

Task

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- Crawl a corpus, which contains some plain text in the Web under a topic;

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- Crawl a corpus, which contains some plain text in the Web under a topic;
- Extract keywords from them;

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Task

- Crawl a corpus, which contains some plain text in the Web under a topic;
- Extract keywords from them;
- Implement an algorithm to generate 100 summarizations from the corpus;

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Submissions

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- Crawled corpus;

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- Crawl a corpus, which contains some plain text in the Web under a topic;
- Extract keywords from them;
- Implement an algorithm to generate 100 summarizations from the corpus;

Submissions

- Crawled corpus;
- Source code and readme file;

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Task

- Crawl a corpus, which contains some plain text in the Web under a topic;
- Extract keywords from them;
- Implement an algorithm to generate 100 summarizations from the corpus;

Submissions

- Crawled corpus;
- Source code and readme file;
- A technique report;

Project assignment: Only for undergraduate students

Task

- Crawl a corpus, which contains some plain text in the Web under a topic;
- Extract keywords from them;
- Implement an algorithm to generate 100 summarizations from the corpus;

Submissions

- Crawled corpus;
- Source code and readme file;
- A technique report;
- All data and documents submit to Baoli Gao, Email: 1760001992@qq.com;

Take-home messages

- Motivation of Submodular
- Submodular
- Set Covering Problem
  - Problem Formulation
  - Hill-climbing Algorithm