Algorithm Foundations of Data Science and Engineering

Lecture 5: Random Walk

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Outline

Joint Probability

The r.v.s $X_1, \cdots, X_n$ are a sequence of discrete r.v.s, then joint pmf $P(X_1 = x_1, \cdots, X_n = x_n)$ can be computed as

- If $X_1, \cdots, X_n$ are mutually independent r.v.s, $\prod_{i=1}^{n} P(X_i = x_i)$;
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  - The first-order correlation in language model can be defined as:

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    - For financial applications, the stock price can be modeled by $t-$order correlation $P(X_i = x_i | X_{i-1} = x_{i-1}, \cdots, X_1 = x_1) = P(X_i = x_i | X_{i-1} = x_{i-1}, \cdots, X_{i-t} = x_{i-t})$.

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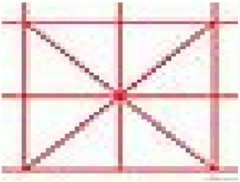
- This guess is not improved by the added knowledge that you started with \$10, then went up to \$11, down to \$10, up to \$11, and then to \$12.

In this example, your money satisfies the first-order correlation, i.e.,

$$P(X_n = x_n | X_{n-1} = x_{n-1}, \cdots, X_1 = x_1) = P(X_n = x_n | X_{n-1} = x_{n-1}).$$

Chess



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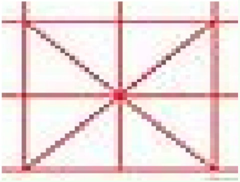

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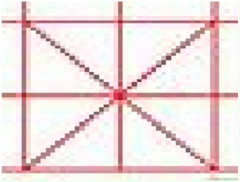

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- A walk of king in the graph is a sequence of vertices $v_1, v_2, \cdots, v_n, \cdots$, not necessarily distinct, such that (v_i, v_{i+1}) is an edge in the graph.

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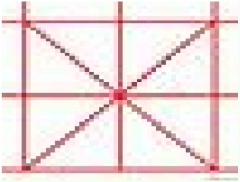

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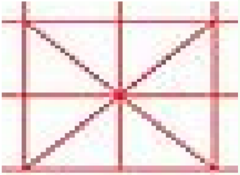

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In this example, the location of king also satisfies the first-order correlation, i.e.,

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- Currently, you locate in the page w_i. Next step, you will visit the page w_{i+1}. A walk of you in the Web graph is a sequence of pages $w_1, w_2, \cdots, w_n, \cdots$, such that (w_i, w_{i+1}) is an edge in the graph.

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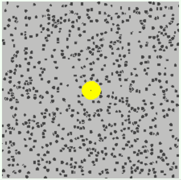

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  for all n and all $x_i \in \Omega$.
- The term Markov property refers to the memoryless property of a stochastic process.

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- If moreover $P(X_{n+1} = j | X_n = i) = P_{ij}$ is independent of n, then X is said to be time homogeneous Markov chain.

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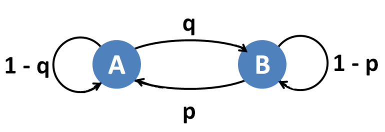

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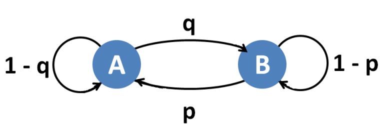

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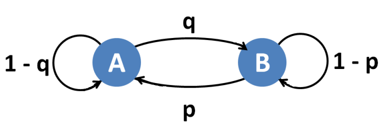
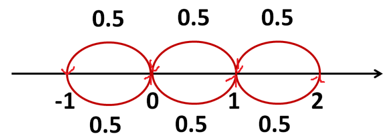


$\Omega = \mathbb{Z}$;

$$\mu(a, a-1) = \frac{1}{2}, \mu(a, a+1) = \frac{1}{2},$$
$$\mu(a, b) = 0, \text{ if } b \neq a \pm 1, \forall a \in \mathbb{Z}$$

and $\pi(10) = 1$, $\pi(a) = 0$ if $a \neq 10$, so at time 0 we always start at 10.



Example of random walk

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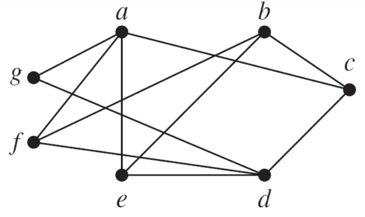

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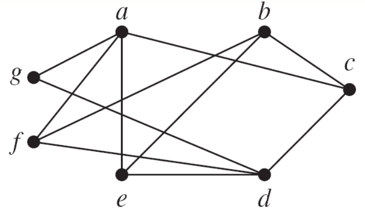
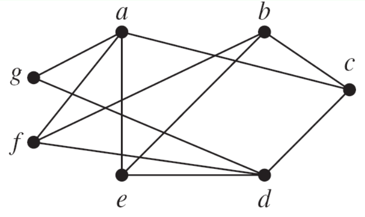


$deg(a) = 4, deg(b) = 3$
$deg(c) = 3, deg(d) = 4$
$deg(e) = 3, deg(f) = 3$
$deg(g) = 2$

Example of random walk

A graph $G = (V, E)$ consists of a vertex set V and an edge set E, where the elements of E are unordered pairs of vertices:
$E \subset \{(x, y) : x, y \in V, x \neq y\}$.
The degree $deg(x)$ of a vertex x is the number of neighbours of x.
In this case, $\Omega = V$ and is finite, $\mu(x, y) = \frac{1}{deg(x)}$ if y is a neighbour of x and $\mu(x, y) = 0$ otherwise.
$\pi(a) = 1$, and $\pi(x) = 0$ for every $x \in V \backslash \{a\}$, so at time 0 we start from vertex a.



$deg(a) = 4, deg(b) = 3$

$deg(c) = 3, deg(d) = 4$

$deg(e) = 3, deg(f) = 3$

$deg(g) = 2$

Thus, a graph can be considered as a random walk, i.e., a Markov chain.

Probability transition matrix

Definition

Let a Markov chain have $P_{x,y}^{(t+1)} = P[X_{t+1} = y | X_t = x]$, and the finite state space be $\Omega = [n]$. This gives us a **probability transition matrix** $P^{(t+1)}$ at time t.

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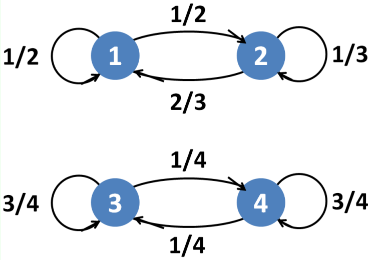

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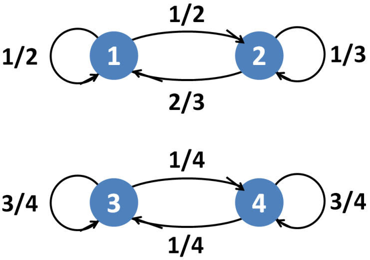

- For the time homogeneous Markov chain,

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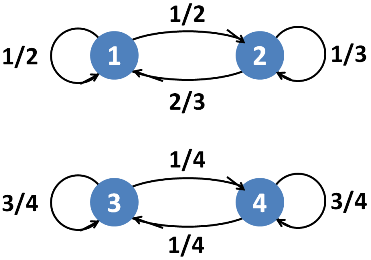

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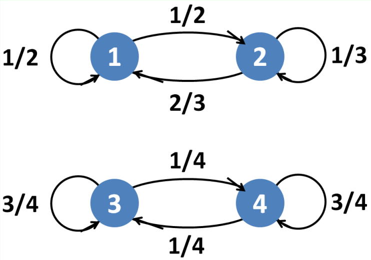

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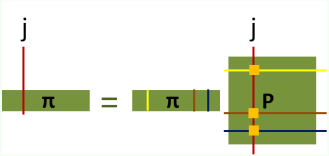

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State y is accessible from state x if it is possible for the chain to visit state y if the chain starts in state x, in other words, $P^n(x, y) > 0, \forall n$. State x **communicates** with state y if y is accessible from x and x is accessible from y.

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Examples of irreducibility

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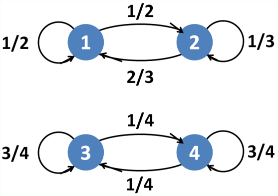

Whether is the random walk irreducibility or not?

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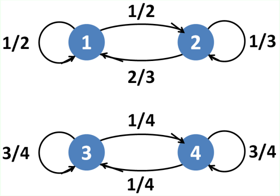

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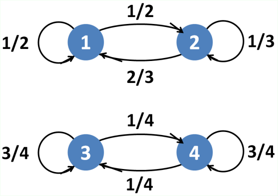

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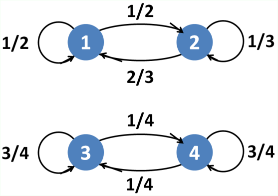

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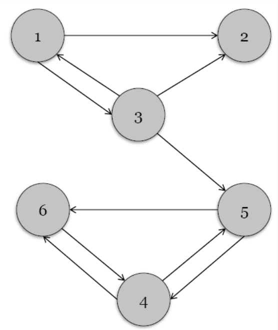

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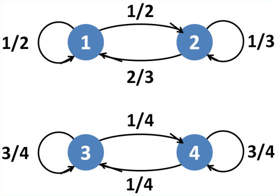

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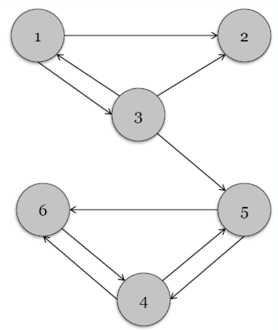

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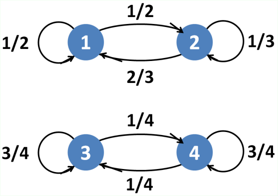

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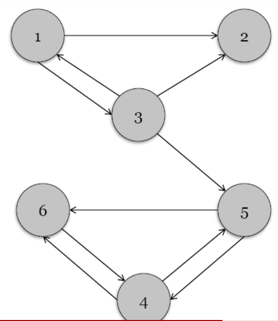

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For v_2, it only has the in-edges, i.e., there is not path from v_2 to another vertices, e.g., v_1.

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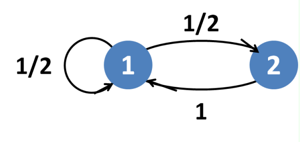
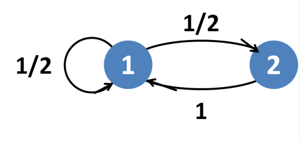
A state is **aperiodic** if its period is 1. A Markov chain is **aperiodic** if all its states are aperiodic.

Example period



What is the period of state 1?

Example period



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Period of a state x will be 1, if it has a self-loop.

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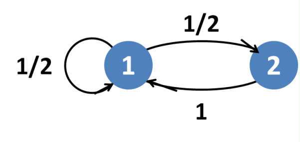

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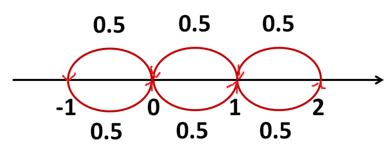
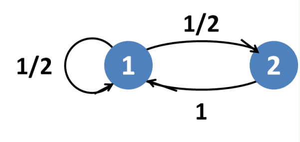


What is the period of state 0?

Example period



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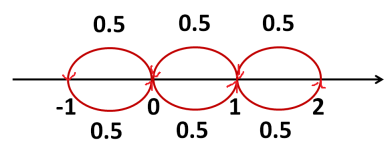

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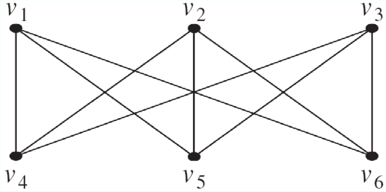

What is the period of vertex v_1?

Properties of period

Theorem

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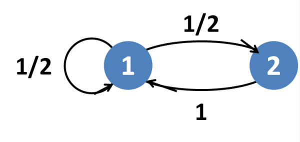

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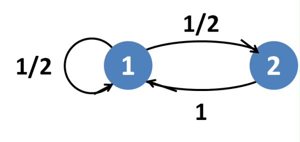

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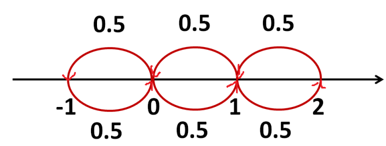

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Existence of the stationary distribution

Theorem

Let $X_0, X_1, \cdots$, be an **irreducible** and **aperiodic** Markov chain with transition matrix P. Then, $\lim_{n\to\infty}(P^n)_{ij}$ exists and independent of i, denoted as $\lim_{n\to\infty}(P^n)_{ij} = \pi(j)$. We also have

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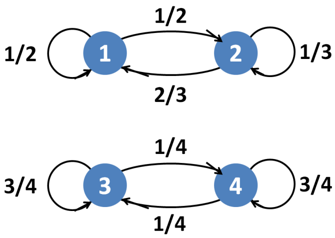
Counter examples



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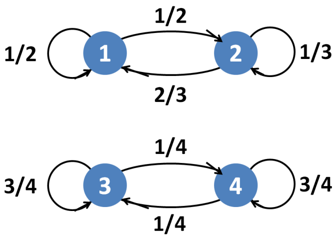

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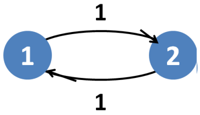

The transition matrix is $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For $n \in N$, we have

$$P^{2n} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } P^{2n+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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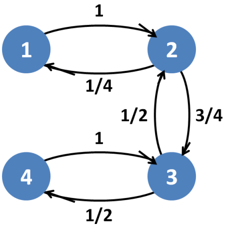

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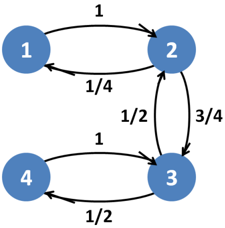

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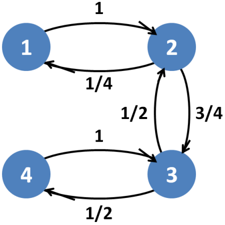

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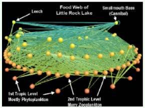
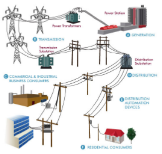
b. Period of every vertex is 2.

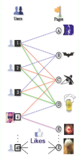
Outline

Graphs - why should we care?

Motivation







Graphs in real world

- “YahooWeb graph”: 1B vertices(Web sites), 6B edges (links)

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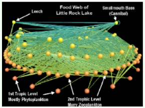

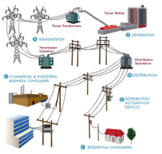

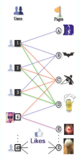

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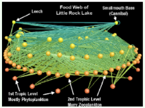

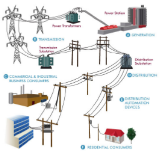

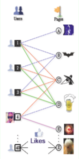

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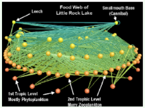

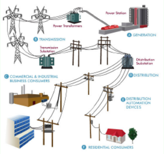

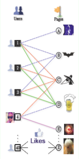

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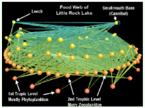

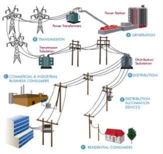


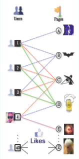

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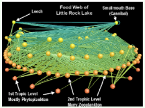

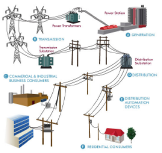

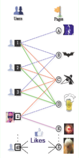

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- Adoption: users purchase products, adopt services, etc.

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Graphs model the relations between entities, e.g., person VS. person, Web page VS. Web pages, airport VS. airport, paper VS. paper, person VS. item, and entity VS. entity, etc.

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  - What is the "best" answer to query "DaSE"? (Trick: Ranking the related documents)

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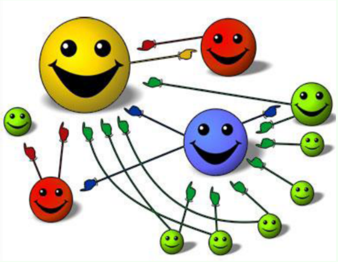

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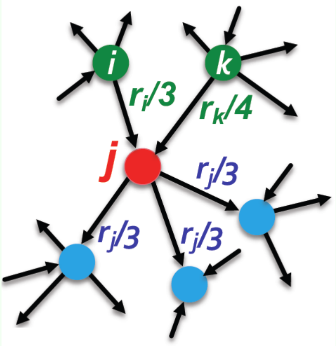
- The problem is formulated to compute the stationary distribution of the Web graph.

Recursive formulation



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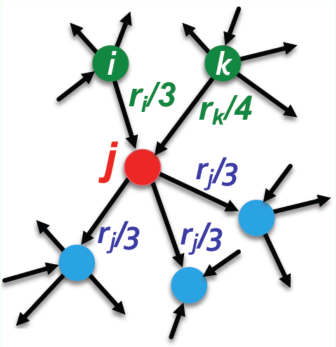

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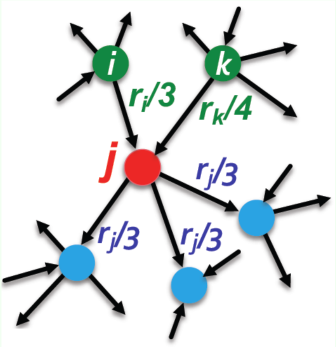

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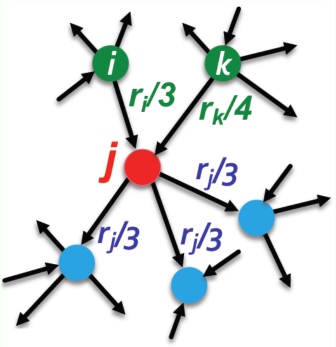

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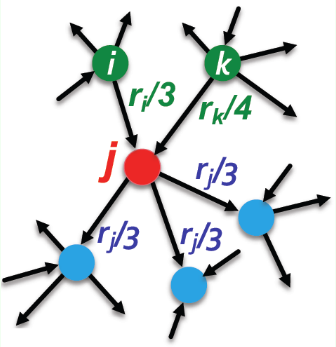

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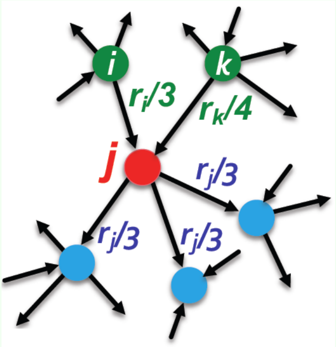

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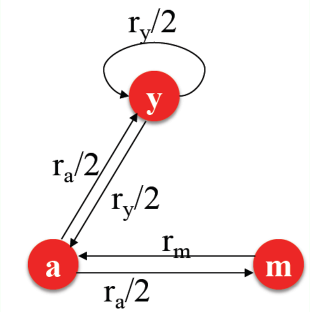
"Flow" equations



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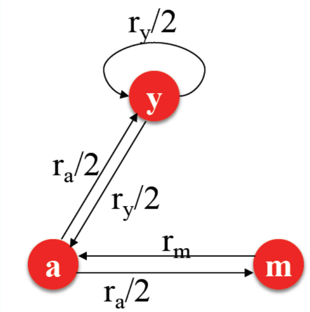

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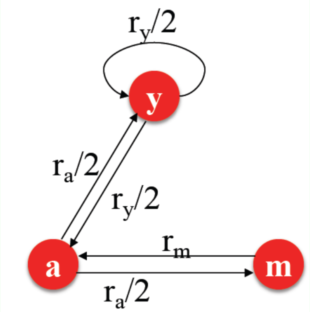

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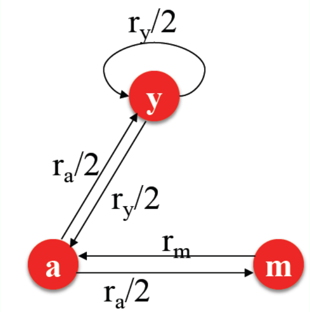

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Gaussian elimination method works for small examples, but we need a better method for large web-size graphs.

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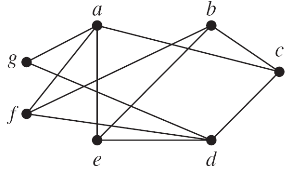

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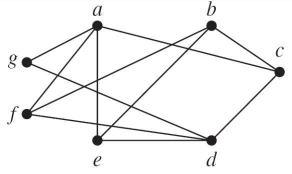

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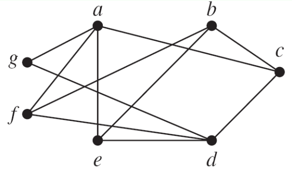

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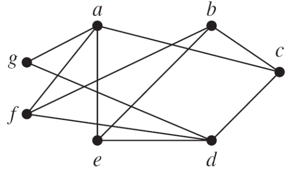

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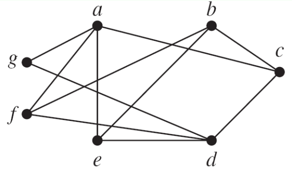

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That is, r is the stationary distribution of the random walk.

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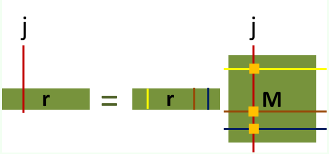

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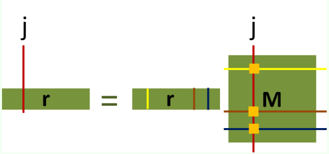

The improved algorithm has less time and space consumption.

Outline

1. Motivation

2. Markov Chain and Random Walk

3. Page Rank
   - Problem Formulation
   - PageRank Algorithm
   - Improvements of PageRank Algorithm

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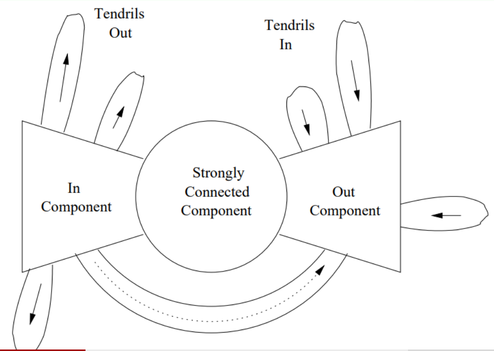

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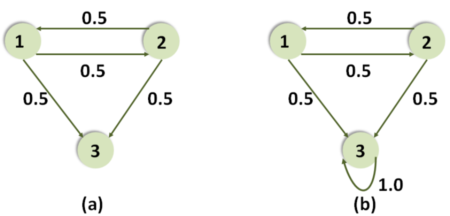

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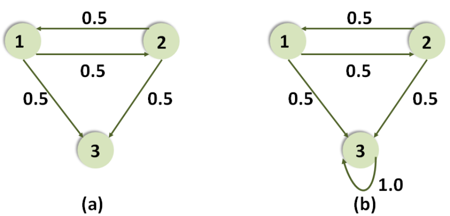

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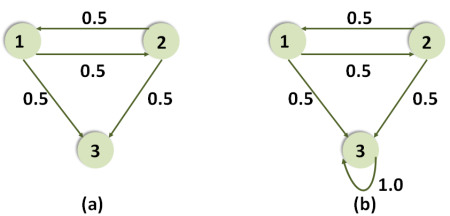

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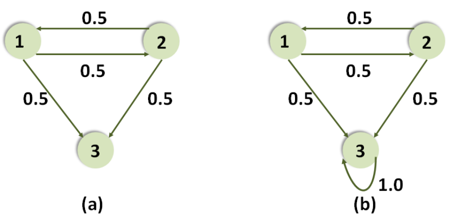

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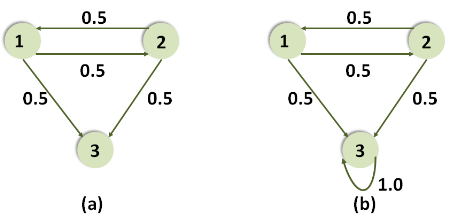

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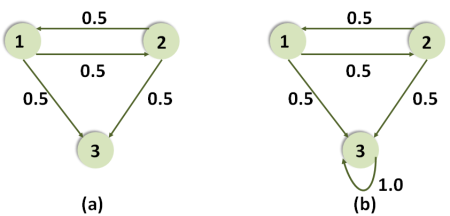

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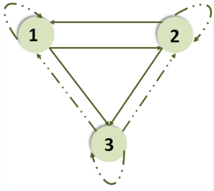

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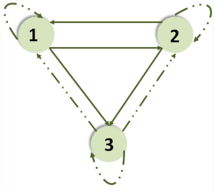

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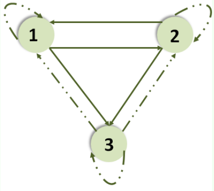

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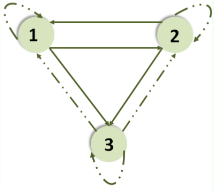

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Does this random walk converge? **May not converge due to leakage!**

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We rearranging the equation $\widetilde{M} = \beta M + (1-\beta)\left[\frac{1}{n}\right]_{n\times n}$ as

$$\begin{aligned} r_j &= \sum_{i=1}^{n} \widetilde{M}_{ij} r_i = \sum_{i=1}^{n} (\beta M_{ij} + \frac{1-\beta}{n}) r_i \\ &= \sum_{i=1}^{n} \beta M_{ij} r_i + \sum_{i=1}^{n} \frac{1-\beta}{n} r_i = \sum_{i=1}^{n} \beta M_{ij} r_i + \frac{1-\beta}{n} \end{aligned}$$

Thus, we get a new matrix formulation

$$r^T = \beta r^T \cdot M + [\frac{1-\beta}{n}]_n$$

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- Output: PageRank vector $\mathbf{r}^{new}$.

1: set $\mathbf{r}^{old} = (\frac{1}{n}, \cdots, \frac{1}{n})$;
2: repeat until convergence: $\sum_j |r_j^{new} - r_j^{old}| < \epsilon$;
3: $\forall j : r_j'^{new} = \sum_{i \to j} \frac{r_i^{old}}{n_i}$;
$r_j'^{new} = 0$ if in-degree of j is 0;
Now revise the random walk:
4: $\forall j : r_j^{new} = \beta r_j'^{new} + \frac{1-\beta s}{n}$, where $s = \sum_j r_j'^{new}$;
5: $\mathbf{r}^{old} = \mathbf{r}^{new}$;

where

$$\beta\frac{1-s}{n} + \frac{1-\beta}{n} = \frac{1-\beta s}{n}.$$

Take-home messages

- Motivation
- Markov Chain and Random Walk
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  - PageRank Algorithm
  - Improvements of PageRank Algorithm