Algorithm Foundations of Data Science and Engineering

Lecture 10: Integer Programming

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Outline

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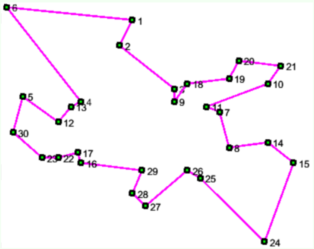

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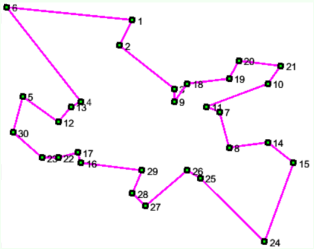

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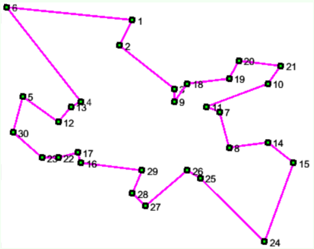

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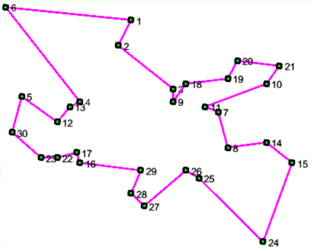

- **Input:** a set of n points;
- **Feasible solutions:** a tour that passes through each point exactly once, the possible feasible solutions is given as $\frac{(n-1)!}{2}$ for symmetric TSP.
- **Objective function:** minimize the length of the tour.

It can be applied into scheduling problem, vehicle routing, aircraft routing, etc.

TSP formulation

The TSP can be defined on an undirected graph $G = (V, E)$ if it is symmetric (directed VS. asymmetric), $V = \{1, \cdots, n\}$ is the vertex set, $E \subset V \times V$ is an edge set, and a cost matrix C_{ij} is defined on E.

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Set covering problem: SCP

Input: Universe set $U = \{u_1, u_2, \cdots, u_n\}$
Subsets $S = \{s_i | s_i \subset U, 1 \leq i \leq m\}$
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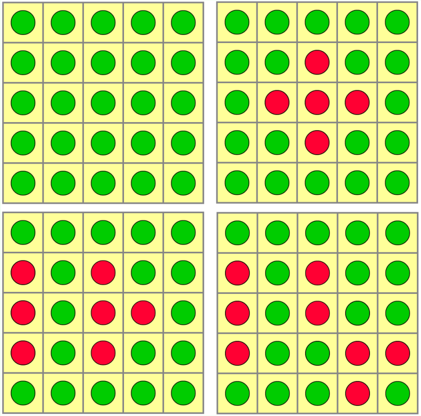
- It can be applied in the edge covering, vertex covering, text summarization, etc.

A game of fiver



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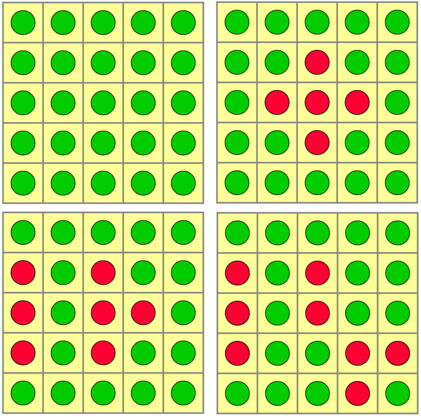

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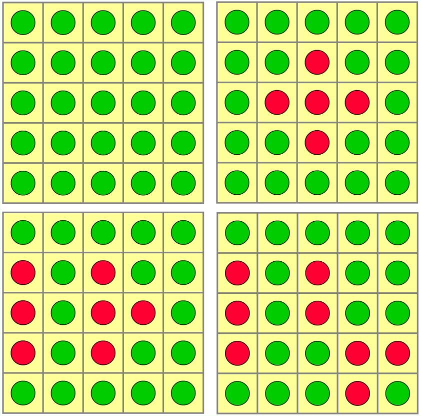

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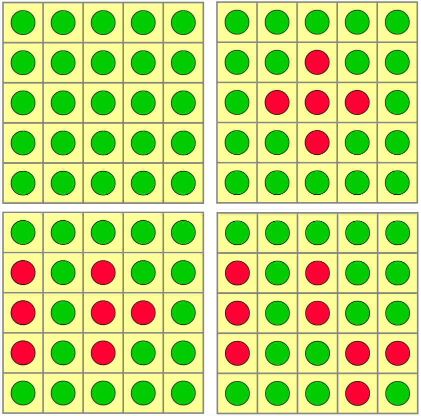

- Click on a circle, and flip its color and that of adjacent colors;
- Can you make all of the circles red?
- Click on $(3,3),(3,1)$ and $(4,4)$, sequentially.

Next: an optimization problem whose solution solves the problem in the fewest moves.

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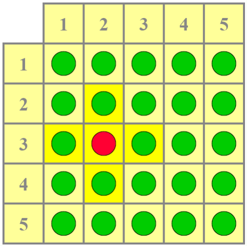

Let

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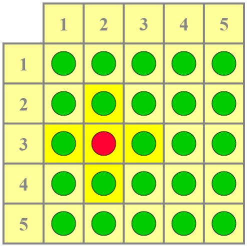

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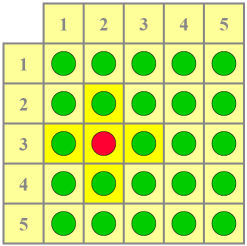

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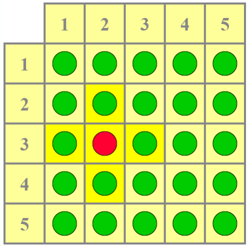

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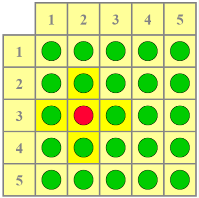

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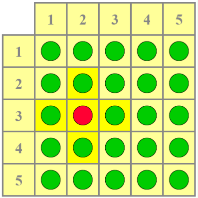

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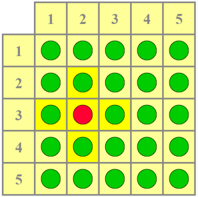

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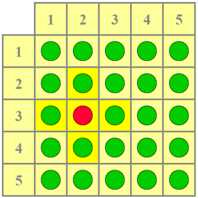

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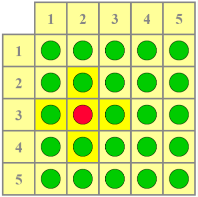

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$0 \leq y_{ij} \leq 2$, and $y_{ij} \in Z^+$ for all $1 \leq i, j \leq 5$

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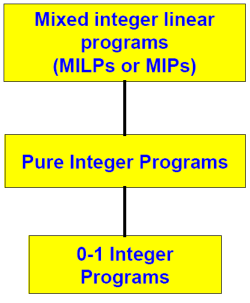
& 20x_1 + 10x_2 \geq 110 \\
& 0 \leq x_1, x_2 \in Z
\end{aligned}
$$

20 TVs + 20 laundries
Cost: 360

40 TVs + 10 laundries
Cost: 400

Task: ship 180 TVs and 110 laundries.

Type of Integer programming



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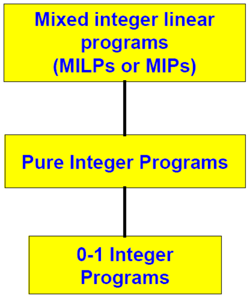

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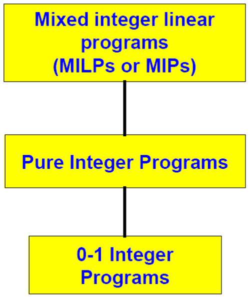

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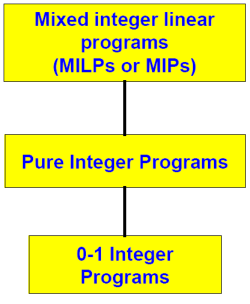

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Note, pure integer programming instances that are unbounded can have an infinite number of solutions. But they have a finite number of solutions if the variables are bounded.

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  - You must select x_1 or x_2 or both, then $x_1 + x_2 \geq 1$;
- Modeling logical constraints that involve non-binary variables is much harder. But we will try to make it as simple as possible.

Logical constraint

If constraint $x \leq 2$ or $x \geq 6$, choose a binary variable w s.t.,

$$w = \begin{cases} 1, & x \leq 2; \\ 0, & x \geq 6. \end{cases}$$

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$$\begin{aligned} x &\leq 2 + M(1 - w) \\ x &\geq 6 - Mw \\ w &\in \{0, 1\} \end{aligned}$$

If $x \leq 2$, then let $w = 1$.	$\Rightarrow$	**$x \leq 2$ and** **$x \geq 6 - M$**
If $x \geq 6$, **then let $w = 0$.**	$\Rightarrow$	**$x \leq 2 + M$ and** **$x \geq 6$**

In both cases, the IP constraints are satisfied.

Modeling "or" constraint

$$x_1 + 2x_2 \geq 12 \text{ or}$$
$$4x_2 - 10x_3 \leq 1$$

Logical constraints

$\boxed{\text{If } w = 1\text{, then } x_1 + 2x_2 \geq 12}$

$\boxed{\text{If } w = 0\text{, then } 4x_2 - 10x_3 \leq 1}$

$$x_1 + 2x_2 \geq 12 - M(1 - w) \text{ or}$$
$$4x_2 - 10x_3 \leq 1 + Mw$$
$$w \in \{0, 1\}$$

IP constraints

Suppose that (x, w) is feasible, for the IP.

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Suppose that (x, w) is feasible, for the IP. Therefore, the logical constraints are satisfied.

If $x_1 + 2x_2 \geq 12$, then let w = 1	⇒	**$x_1 + 2x_2 \geq 12$ AND $4x_2 - 10x_3 \leq 1 + M$.**
Else $4x_2 - 10x_3 \leq 1$ then let w = 0	⇒	**$x_1 + 2x_2 \geq 12 - M$ AND $4x_2 - 10x_3 \leq 1$.**

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$x_1 + 2x_2 \geq 12 - M(1-w)$ or
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If w = 1, then $x_1 + 2x_2 \geq 12$

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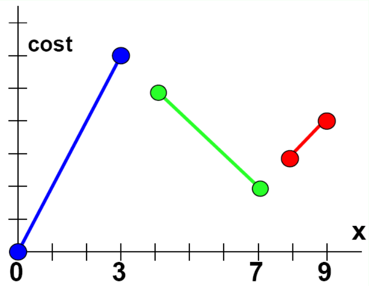
Suppose that (x, w) is feasible, for the IP. Therefore, the logical constraints are satisfied.

If $x_1 + 2x_2 \geq 12$, then let w = 1	⇒	$x_1 + 2x_2 \geq 12$ AND $4x_2 - 10x_3 \leq 1 + M$.
Else $4x_2 - 10x_3 \leq 1$ then let w = 0	⇒	$x_1 + 2x_2 \geq 12 - M$ AND $4x_2 - 10x_3 \leq 1$.

The logical constraints are equivalent to the IP constraints.

Outline

Modeling piecewise linear functions



$$y = \begin{cases} 2x, & \text{if } 0 \le x \le 3 \\ 9 - x, & \text{if } 4 \le x \le 7 \\ -5 + x, & \text{if } 8 \le x \le 9 \end{cases}$$

Assume that x is integer valued. We will create an IP formulation so that the variable y is correctly modeled.

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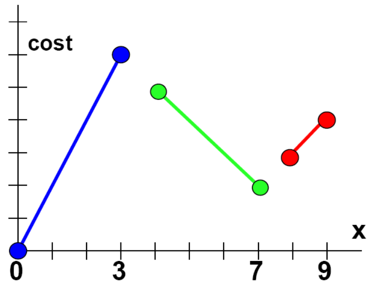

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Assume that x is integer valued. We will create an IP formulation so that the variable y is correctly modeled.

Create new binary and integer variables

$w_1 = \begin{cases} 1 & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$	$x_1 = \begin{cases} x & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$
$w_2 = \begin{cases} 1 & 4 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$	$x_2 = \begin{cases} x & 4 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$
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$w_1 + w_2 + w_3 = 1$
$x = x_1 + x_2 + x_3$ x_i integer $\forall\ i$

IP constraints

Modeling piecewise linear functions Cont'd

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$$y = 2x_1 + (9w_2 - x_2) + (-5w_3 + x_3).$$

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A 2-variable integer program

Maximize: $\quad z = 3x + 4y$

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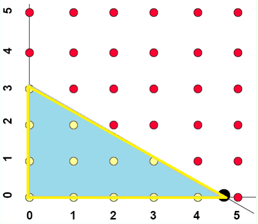

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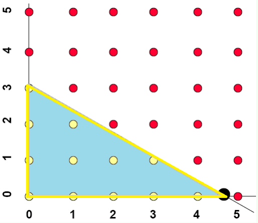

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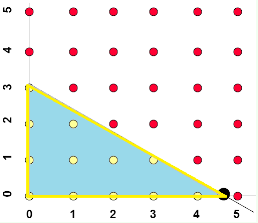

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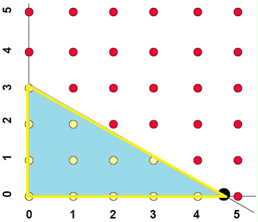

Q1: What is the optimal integer solution?

Q2: Can one use linear programming to solve IP problem?

Solve LP (ignore integrality) get $x = \frac{24}{5}, y = 0$ and $z = 14.4$.

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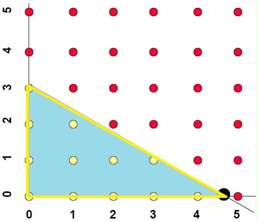

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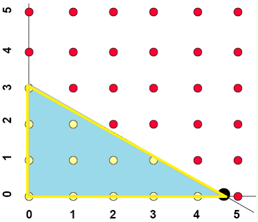

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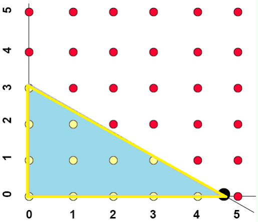

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Truncate, get $x = 4, y = 0$, and $z = 12$. Same solution value at $x = 0, y = 3$.
Optimal is $x = 3, y = 1$, and $z = 13$.

Feasible region for two constraints

Maximize: $z = 3x + 4y$

Feasible region for two constraints

Maximize: $z = 3x + 4y$

Subject to: $x + y \leq 4$

Feasible region for two constraints

Maximize: $z = 3x + 4y$

Subject to: $x + y \leq 4$
$2x + 3y \leq 9$

Feasible region for two constraints

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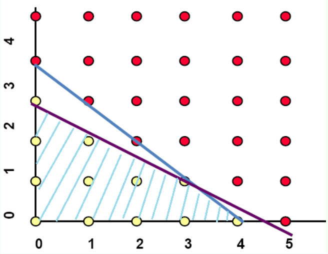

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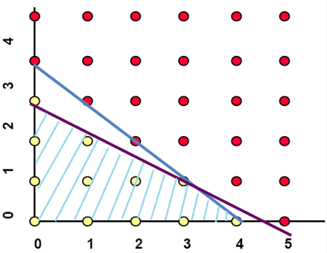

- More constraints will result in a smaller feasible region;

Feasible region for two constraints

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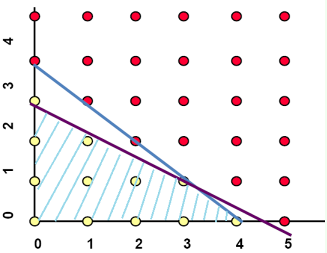

- More constraints will result in a smaller feasible region;
- That is, the search space will be reduced;

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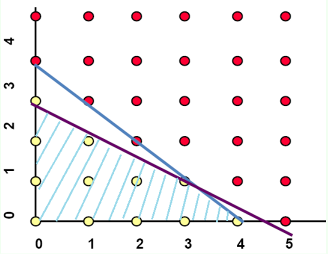

- More constraints will result in a smaller feasible region;
- That is, the search space will be reduced;
- Much, much harder than solving linear programs.

Outline

Complete enumeration

Prize	iPad	server	Brass Rat	Au Bon Pain	6.041 tutoring	15.053 dinner
Points	**5**	**7**	**4**	**3**	**4**	**6**
Utility	**16**	**22**	**12**	**8**	**11**	**19**

Maximize: $16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$

Complete enumeration

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$x_i \in \{0, 1\}$ for $1 \leq i \leq 6$

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- Systematically considers all possible values of the decision variables, i.e., $n \to 2^n$;

Complete enumeration

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$x_i \in \{0, 1\}$ for $1 \leq i \leq 6$

- Systematically considers all possible values of the decision variables, i.e., $n \to 2^n$;
- Usual idea: iteratively break the problem in two. At the first iteration, we consider separately the case that $x_1 \in \{0, 1\}$;

Complete enumeration

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Utility	**16**	**22**	**12**	**8**	**11**	**19**

Maximize: $16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$

Subject to: $5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14$

$x_i \in \{0, 1\}$ for $1 \leq i \leq 6$

- Systematically considers all possible values of the decision variables, i.e., $n \to 2^n$;
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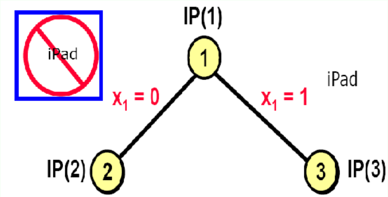
- Each node of the tree represents the original problem plus additional constraints.

An enumeration tree



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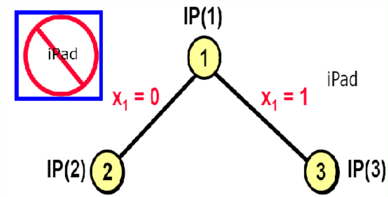

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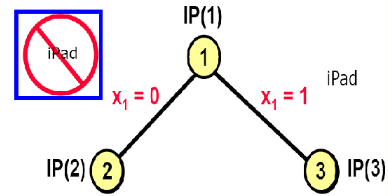

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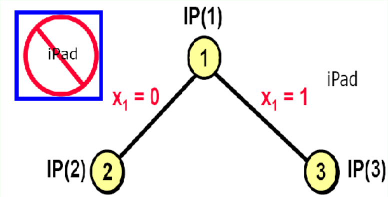

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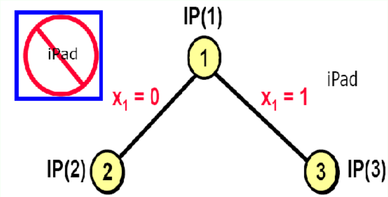

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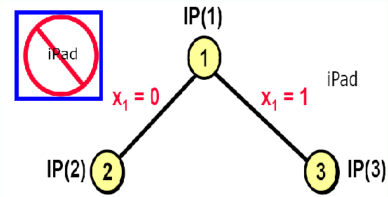

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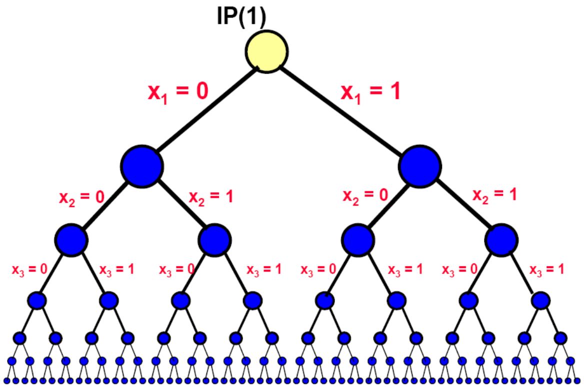

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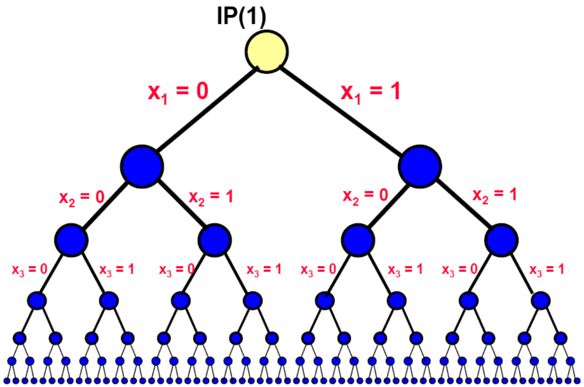

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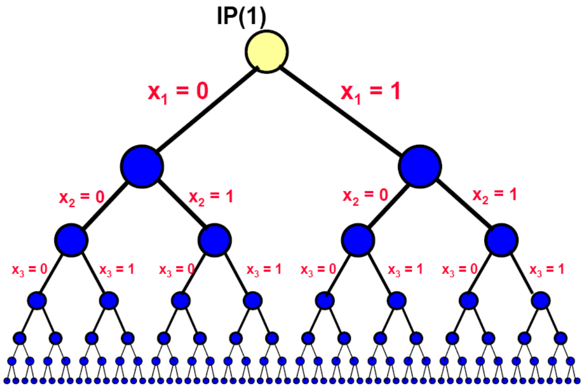

- Number of leaves of the tree: 64;
- If there are n variables, the number of leaves is 2^n.

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- Let n = number of binary variables;

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- Suppose that we could evaluate 1 trillion solutions per second, and instantaneously eliminate 99.9999999% of all solutions as not worth considering

n=70	1 sec.	n=100	31 years
n=80	17 minutes	n=110	31,000 years
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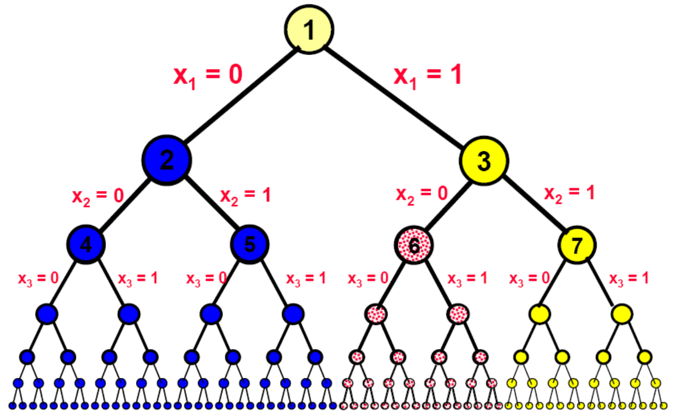

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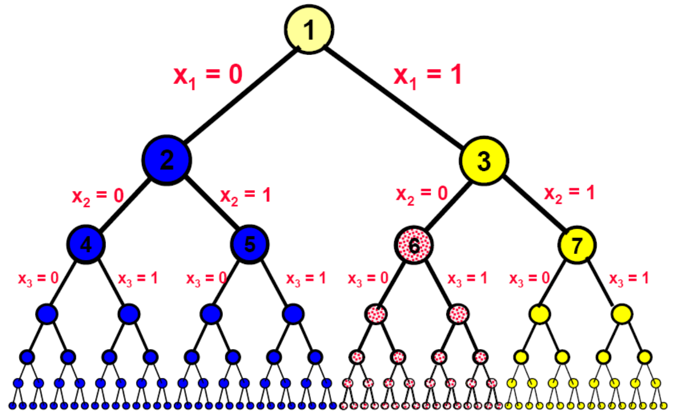

If we can eliminate an entire subtree in one step, we can eliminate a fraction of all complete solutions at in a single step.

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$$\text{Maximize:} \qquad 24x_1 + 2x_2 + 20x_3 + 4x_4$$

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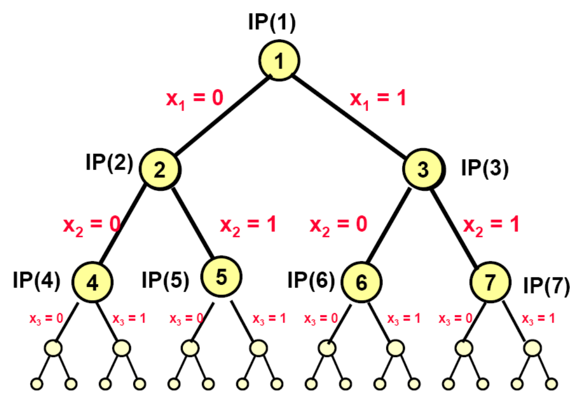

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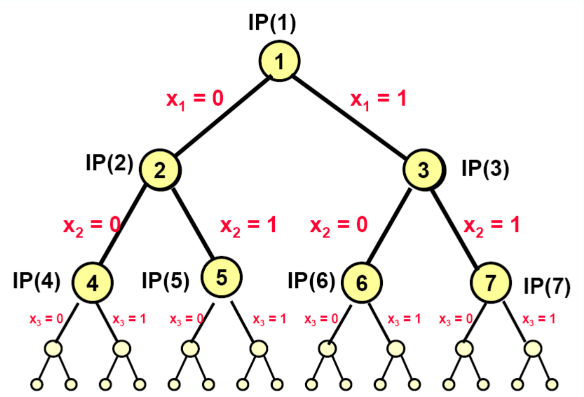

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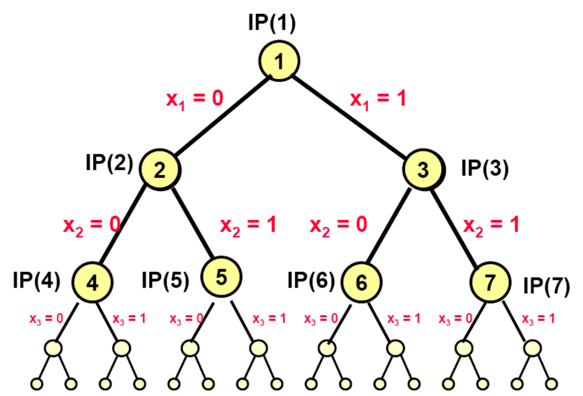

- In a branch and bound tree, the nodes represent IPs;
- What is the optimal objective value for IP(4)?

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We eliminate a subtree if

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- For example, after we solved IP(4), you don't need to look at its children.

Outline

The LP relaxation of the IP

If we drop the requirements that variables be integer, we call it the **LP relaxation of the IP**.

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If the optimal solution for LP(k) is feasible for IP(k), then it is also optimal for IP(k).

The LP relaxation solves the IP Cont'd

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- If LP(k) is infeasible, then IP(k) is infeasible.
- In this example, the LHS of the constraint is at least 13. There is no way that the constraint can be satisfied by fractional values or integer values of x_3 and x_4.

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Recall that we don't solve IP(k) directly. Instead, we solve its LP relaxation. We can use this to obtain bounds.

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Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$
$x_1 = 0$
$0 \leq x_i \leq 1$ for i $= 2$ to 4

- The optimal solution for LP(2) is $z_{LP}(2) = 25$;
- Suppose that the incumbent is $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0, z_I = 26$;

Pruning branches

Based on the observation

- We can prune the active node k IP(k) if $z_{LP(k)} \leq z_I$, where z_I is the objective value of the incumbent.
- A node is active if it has not been pruned and if LP(k) has not been solved yet.

LP(2)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

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- The optimal solution for LP(2) is $z_{LP}(2) = 25$;
- Suppose that the incumbent is $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0, z_I = 26$;

Recall that we don't solve IP(k) directly. Instead, we solve its LP relaxation. We can use this to obtain bounds.

The branch and bound algorithm

while there is some **active** nodes **do**

select an **active** node *j*

mark j as **inactive**

Solve LP(j): denote solution as x(j);

Case 1 -- **if** $z_{LP}(j) \le z_I$ **then** ***prune*** node j;

Case 2 -- **if** $z_{LP}(j) > z_I$ and
if x(j) is feasible for IP(j)

then Incumbent := x(j), and $z_I := z_{LP}(j)$;

then ***prune*** node j;

Case 3 -- If **if** $z_{LP}(j) > z_I$ and
if x(j) is not feasible for IP(j) **then**

mark the children of node *j* as **active**

endwhile

The branch and bound algorithm

while there is some **active** nodes **do**

select an **active** node *j*

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Solve LP(j): denote solution as x(j);

Case 1 -- **if** $z_{LP}(j) \leq z_I$ **then** ***prune*** node j;

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if x(j) is feasible for IP(j)
then Incumbent := x(j), and $z_I := z_{LP}(j)$;
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Case 3 -- If **if** $z_{LP}(j) > z_I$ and
if x(j) is not feasible for IP(j) **then**
mark the children of node *j* as **active**

endwhile

Under which condition can we not prune active node *j* from the B&B Tree for a maximization problem?

Example of B&B algorithm

LP(1)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$0 \leq x_i \leq 1$ for i = 1 to 4

No incumbent $z_I = -\infty$ and $z_{LP(1)} = 32$.

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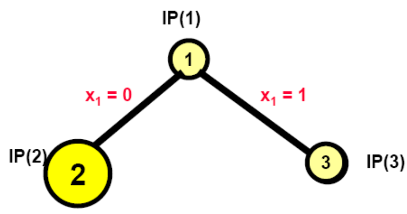

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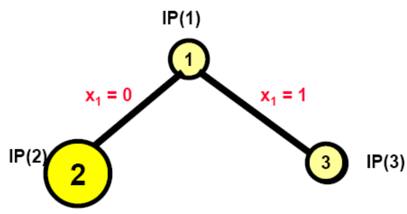

Optimal solution for LP(2) is: $x_1 = 0, x_2 = 1, x_3 = 1, x_4 = \frac{3}{4}, z_{LP}(2) = 25$;

Example: Node 3

LP(3)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Example: Node 3

LP(3)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1$

$0 \leq x_i \leq 1$ for i $= 2$ to 4

No incumbent $z_I = -\infty$ and $z_{LP(1)} = 32$.

Example: Node 3

LP(3)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

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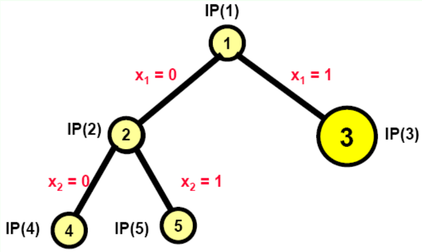

Example: Node 3

LP(3)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1$

$0 \leq x_i \leq 1$ for i = 2 to 4

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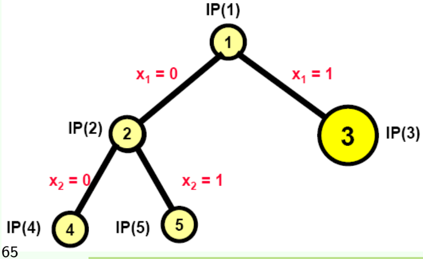


Optimal solution for LP(3) is:

$x_1 = 1, x_2 = 0, x_3 = \frac{1}{4}, x_4 = 0, z_{LP}(3) = 28;$

Example: Node 4

LP(4)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 0,\ x_2 = 0$

$0 \leq x_i \leq 1$ for i $= 3$ to 4

No incumbent $z_I = -\infty$ and $z_{LP(1)} = 32$.

Example: Node 4

LP(4)

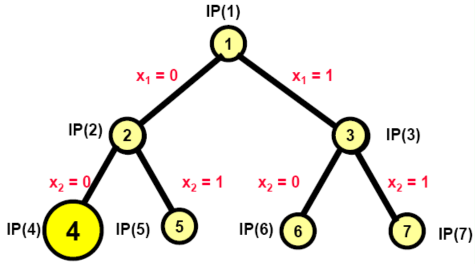
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 0,\ x_2 = 0$

$0 \leq x_i \leq 1$ for i $= 3$ to 4

No incumbent $z_I = -\infty$ and $z_{LP(1)} = 32$.



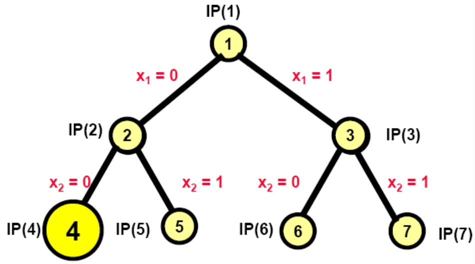
Example: Node 4

LP(4)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$
$x_1 = 0,\ x_2 = 0$
$0 \leq x_i \leq 1$ for i $= 3$ to 4

No incumbent $z_I = -\infty$ and $z_{LP(1)} = 32$.



Optimal solution for LP(4) is:
$x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1, z_{LP}(4) = 24$;

Example: Node 4

LP(4)

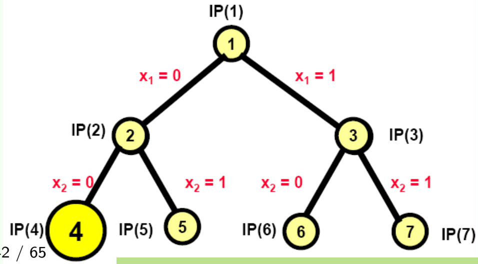
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 0,\ x_2 = 0$

$0 \leq x_i \leq 1$ for i $= 3$ to 4

No incumbent $z_I = -\infty$ and $z_{LP(1)} = 32$.



Optimal solution for LP(4) is: $x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1, z_{LP}(4) = 24$; Pruned.

Example: Node 5

LP(5)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 0,\ x_2 = 1$

$0 \leq x_i \leq 1$ for $i = 3$ to 4

Incumbent solution $z_I = 24$.

Example: Node 5

LP(5)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

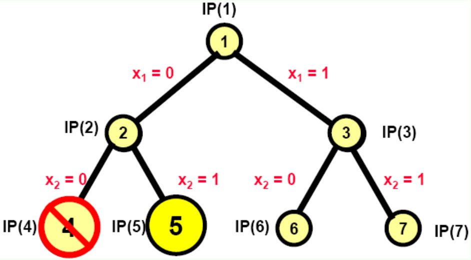
Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 0,\ x_2 = 1$

$0 \leq x_i \leq 1$ for $i = 3$ to 4

Incumbent solution

$z_I = 24$.



Example: Node 5

LP(5)

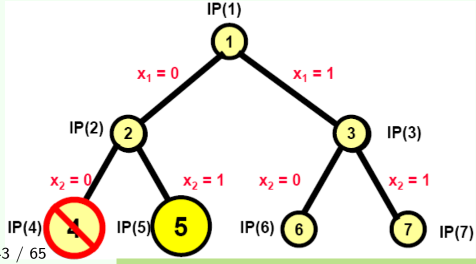
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 0,\ x_2 = 1$

$0 \leq x_i \leq 1$ for i = 3 to 4

Incumbent solution $z_I = 24$.



Optimal solution for LP(5) is:
$x_1 = 0, x_2 = 1, x_3 = 1, x_4 = \frac{3}{4}, z_{LP}(5) = 25;$

Example: Node 6

LP(6)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1,\ x_2 = 0$

$0 \leq x_i \leq 1$ for i = 3 to 4

Incumbent solution

$z_I = 24$.

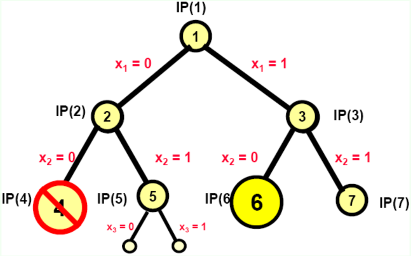
Example: Node 6

LP(6)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$
$x_1 = 1,\ x_2 = 0$
$0 \leq x_i \leq 1$ for i = 3 to 4

Incumbent solution
$z_I = 24.$



Example: Node 6

LP(6)

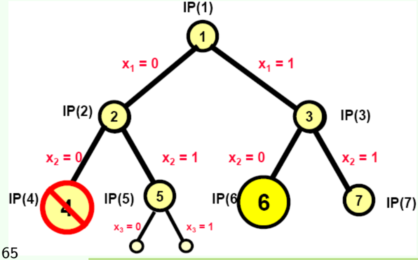
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1,\ x_2 = 0$

$0 \leq x_i \leq 1$ for i = 3 to 4

Incumbent solution $z_I = 24$.



Optimal solution for LP(6) is: $x_1 = 1, x_2 = 0, x_3 = \frac{1}{5}, x_4 = 0, z_{LP}(6) = 28;$

Example: Node 7

LP(7)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1,\ x_2 = 1$

$0 \leq x_i \leq 1$ for $i = 3$ to 4

Incumbent solution $z_I = 24$.

Example: Node 7

LP(7)

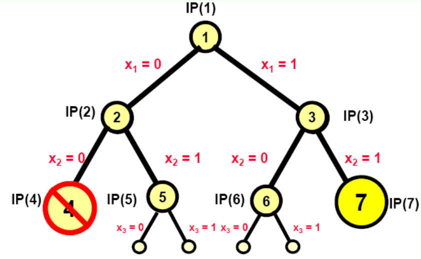
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1,\ x_2 = 1$

$0 \leq x_i \leq 1$ for i = 3 to 4

Incumbent solution $z_I = 24$.



Example: Node 7

LP(7)

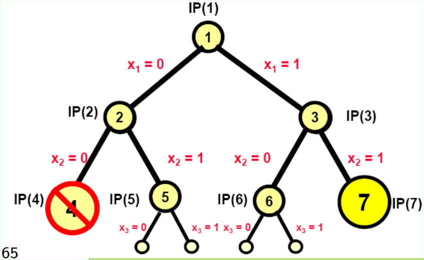
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1,\ x_2 = 1$

$0 \leq x_i \leq 1$ for i = 3 to 4

Incumbent solution $z_I = 24$.



Optimal solution for LP(7) is: $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0, z_{LP}(7) = 26;$

Example: Node 8

LP(8)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 0,\ x_2 = 1,\ x_3 = 0$

$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.

Example: Node 8

LP(8)

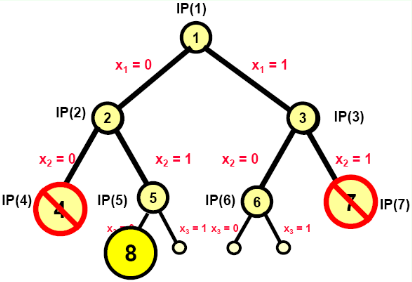
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 0,\ x_2 = 1,\ x_3 = 0$

$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.



Example: Node 8

LP(8)

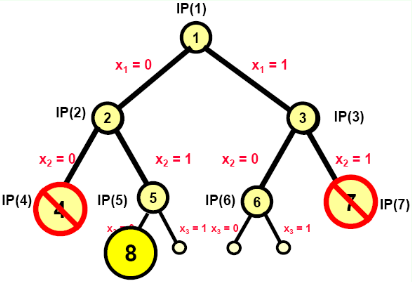
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 0, \; x_2 = 1, \; x_3 = 0$

$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.



Optimal solution for LP(8) is: $x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 1, z_{LP}(8) = 6$;

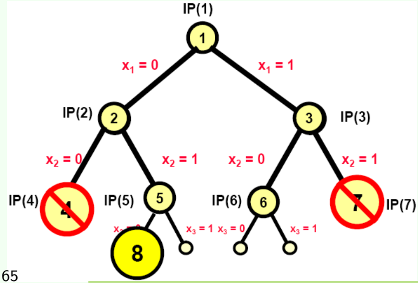
Example: Node 8

LP(8)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$
$x_1 = 0,\ x_2 = 1,\ x_3 = 0$
$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.



Optimal solution for LP(8) is:
$x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 1, z_{LP}(8) = 6$;
Pruned.

Example: Node 9

LP(9)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 0,\ x_2 = 1,\ x_3 = 1$

$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.

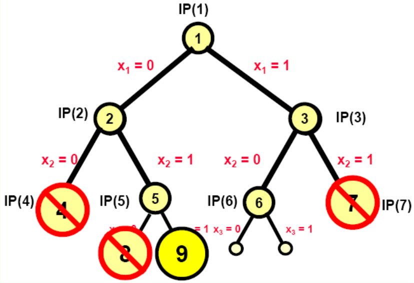
Example: Node 9

LP(9)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$
$x_1 = 0,\ x_2 = 1,\ x_3 = 1$
$0 \leq x_4 \leq 1$

Incumbent solution
$z_I = 26$.



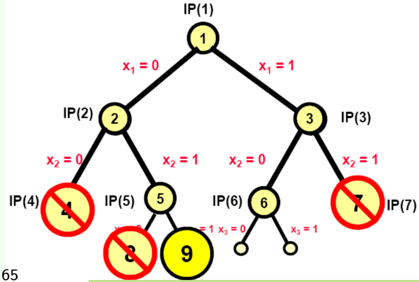
Example: Node 9

LP(9)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$
$x_1 = 0,\ x_2 = 1,\ x_3 = 1$
$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.



Optimal solution for LP(9) is: $x_1 = 0, x_2 = 1, x_3 = 1, x_4 = \frac{3}{4}, z_{LP}(9) = 25$;

Example: Node 10

LP(10)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1,\ x_2 = 0,\ x_3 = 0$

$0 \leq x_4 \leq 1$

Incumbent solution

$z_I = 26$.

Example: Node 10

LP(10)

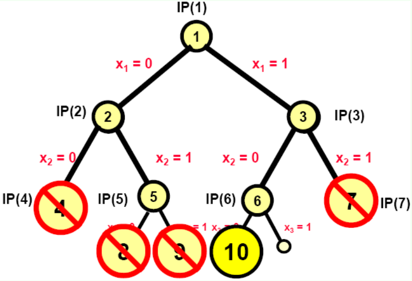
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1, \; x_2 = 0, \; x_3 = 0$

$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.



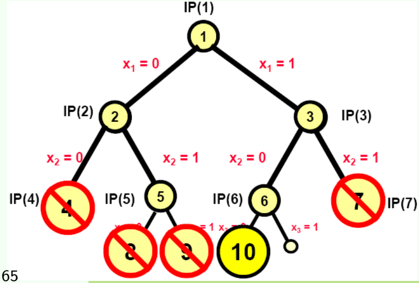
Example: Node 10

LP(10)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$
$x_1 = 1,\ x_2 = 0,\ x_3 = 0$
$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.



Optimal solution for LP(10) is:
$x_1 = 1, x_2 = 0, x_3 = 0, x_4 = \frac{1}{4}, z_{LP}(10) = 25;$

Example: Node 11

LP(11)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1,\ x_2 = 0,\ x_3 = 1$

$0 \leq x_4 \leq 1$

Incumbent solution

$z_I = 26.$

Example: Node 11

LP(11)

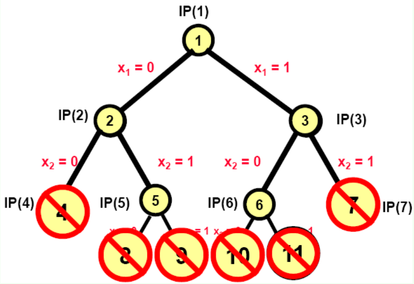
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1,\ x_2 = 0,\ x_3 = 1$

$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.



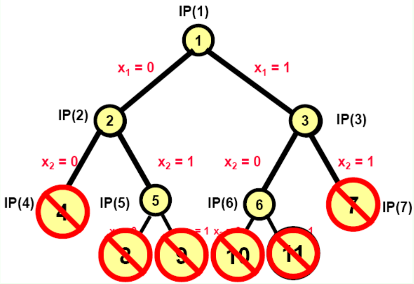
Example: Node 11

LP(11)

Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$
$x_1 = 1,\ x_2 = 0,\ x_3 = 1$
$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.



Optimal solution for LP(11): there is no feasible solution for LP(11).

Example: Node 11

LP(11)

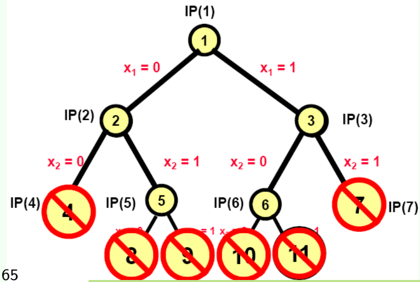
Maximize: $24x_1 + 2x_2 + 20x_3 + 4x_4$

Subject to: $8x_1 + 1x_2 + 5x_3 + 4x_4 \leq 9$

$x_1 = 1,\ x_2 = 0,\ x_3 = 1$

$0 \leq x_4 \leq 1$

Incumbent solution $z_I = 26$.



Optimal solution for LP(11): there is no feasible solution for LP(11). Pruned.

Lessons learned

- Branch and Bound can speed up the search. Only 11 nodes (LPs) out of 31 were evaluated.

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- In practice, there are lots of ways to make Branch and Bound even faster.

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- Branch and Bound can speed up the search. Only 11 nodes (LPs) out of 31 were evaluated.
- Branch and Bound relies on eliminating subtrees, either because the IP at the node was solved, or else because the IP solution cannot possibly be optimum;
- Complete enumerations not possible (because of the running time) if there are more than 100 variables. (Even 50 variables would take too long.)
- In practice, there are lots of ways to make Branch and Bound even faster.
  - There are several ways. One way is for the B&B algorithm to have heuristics that "intelligently" choose the best variable to branch on;

Lessons learned

- Branch and Bound can speed up the search. Only 11 nodes (LPs) out of 31 were evaluated.
- Branch and Bound relies on eliminating subtrees, either because the IP at the node was solved, or else because the IP solution cannot possibly be optimum;
- Complete enumerations not possible (because of the running time) if there are more than 100 variables. (Even 50 variables would take too long.)
- In practice, there are lots of ways to make Branch and Bound even faster.
  - There are several ways. One way is for the B&B algorithm to have heuristics that "intelligently" choose the best variable to branch on;
  - Another technique is to use "rounding", e.g., $x_1 + x_2 \leq 1.5 \rightarrow x_1 + x_2 \leq 1$, or $z_{IP} \leq Z_{LP} = 5.5 \rightarrow z_{IP} \leq 5$.

Outline

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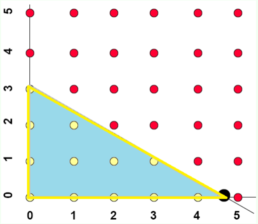

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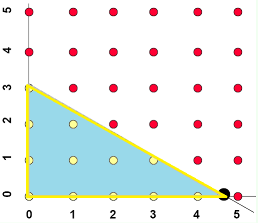

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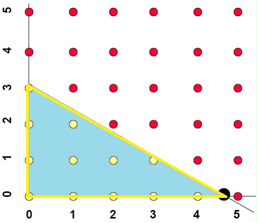

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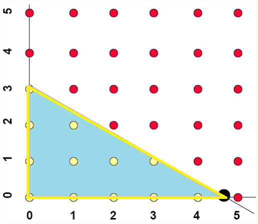

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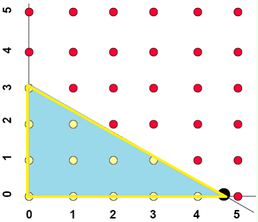

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Note that LHS is integral, so RHS can be truncated, while it does not necessarily dominate original constraint.

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Valid inequality (subtract (2) from (1)):

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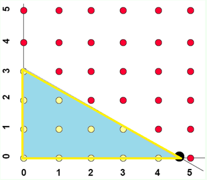

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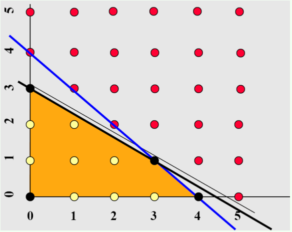

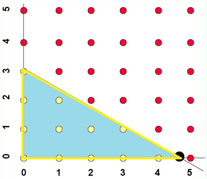

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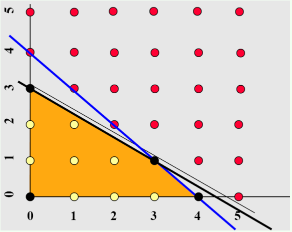

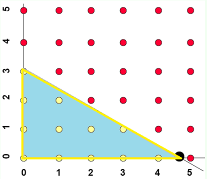

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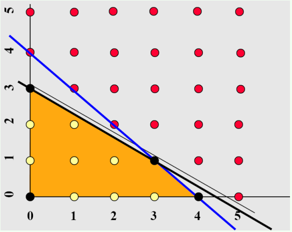

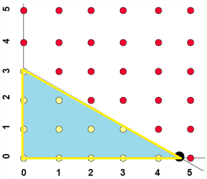

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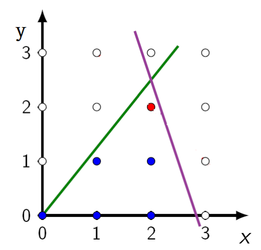

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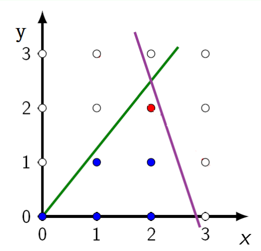

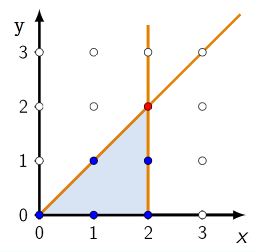

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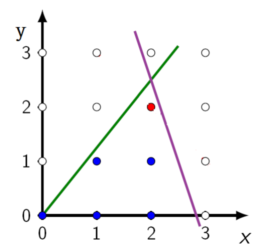

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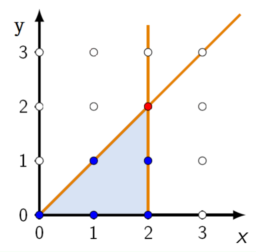

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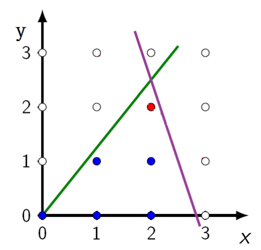

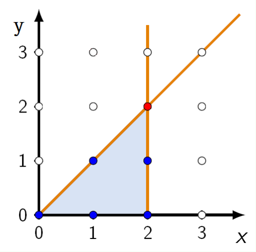

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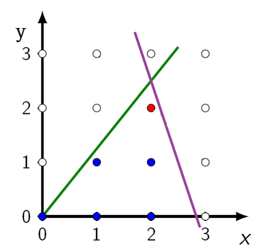

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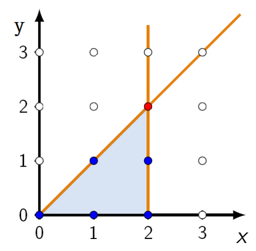

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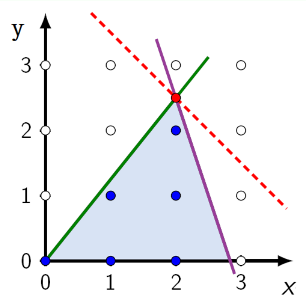

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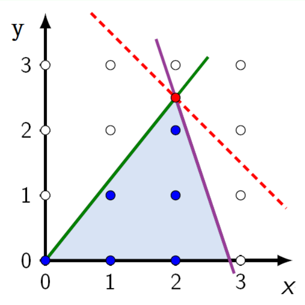

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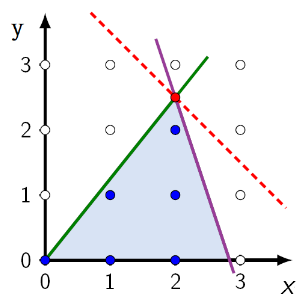

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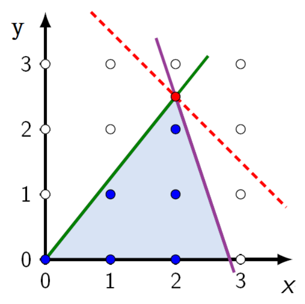

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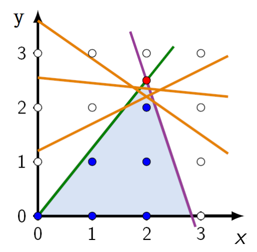

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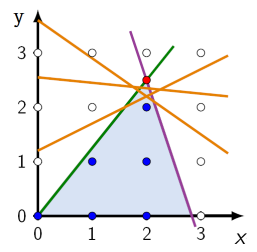
- Step 3: If LP solution is not integral, find a linear constraint that excludes the LP solution but does not exclude any integer points (always possible);
- Step 4: Add the cut constraint to the problem.
  Return to step 1.



Maximize: $z = x + y$

Subject to:
$$-5x + 4y \leq 0$$
$$6x + 2y \leq 17$$

Cutting plane method

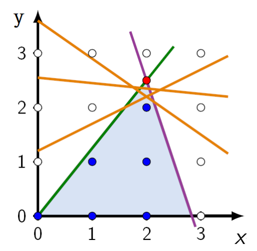
Cutting plane algorithm:

Step 1: Solve LP relaxation

Step 2: If LP solution is integral, it is optimal for the original problem. We are done!

Step 3: If LP solution is not integral, find a linear constraint that excludes the LP solution but does not exclude any integer points (always possible);

Step 4: Add the cut constraint to the problem. Return to step 1.



Maximize: $z = x + y$

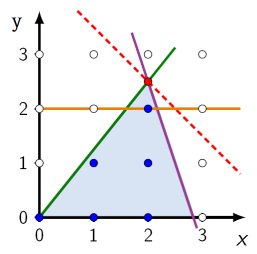
Subject to: $-5x + 4y \leq 0$

$6x + 2y \leq 17$

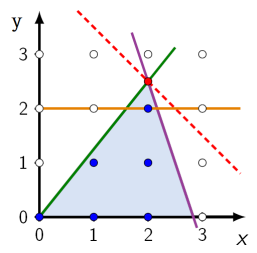
valid inequality

$0 \leq x, y \in Z$

Example of cutting plane



Example of cutting plane

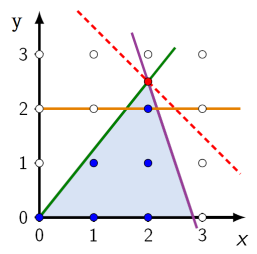


Maximize: $z = x + y$

Subject to: $-5x + 4y \leq 0$

$6x + 2y \leq 17$

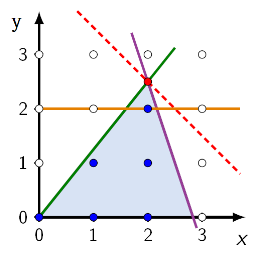
Example of cutting plane



Maximize: $z = x + y$

Subject to:
$$-5x + 4y \leq 0$$
$$6x + 2y \leq 17$$
$$y \leq 2$$
$$0 \leq x, y \in Z$$

Example of cutting plane



Maximize: $z = x + y$

Subject to: $-5x + 4y \leq 0$

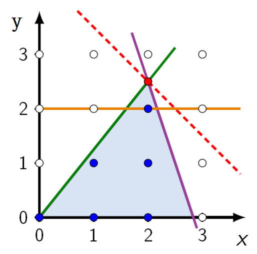
$6x + 2y \leq 17$

$y \leq 2$

$0 \leq x, y \in Z$

- The constraint $y \leq 2$ is a valid cut because it excludes the optimal LP solution but does not exclude any integer points;

Example of cutting plane



Maximize: $z = x + y$

Subject to: $-5x + 4y \leq 0$

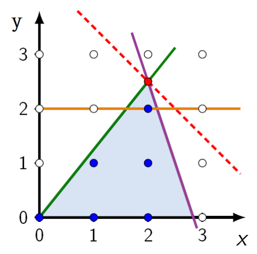
$6x + 2y \leq 17$

$y \leq 2$

$0 \leq x, y \in Z$

- The constraint $y \leq 2$ is a valid cut because it excludes the optimal LP solution but does not exclude any integer points;
- Now solve the LP relaxation for this new problem.

Example of cutting plane



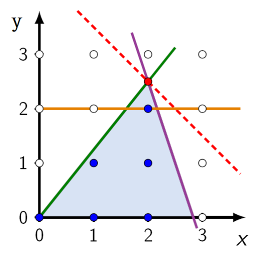
Maximize: $z = x + y$

Subject to:
$-5x + 4y \leq 0$
$6x + 2y \leq 17$
$y \leq 2$
$0 \leq x, y \in Z$

- The constraint $y \leq 2$ is a valid cut because it excludes the optimal LP solution but does not exclude any integer points;
- Now solve the LP relaxation for this new problem.

A cut must simultaneously exclude the LP solution while keeping all the feasible integer points.

Example of cutting plane



Maximize: $z = x + y$

Subject to: $-5x + 4y \leq 0$

$6x + 2y \leq 17$

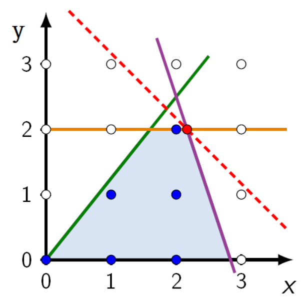
$y \leq 2$

$0 \leq x, y \in Z$

- The constraint $y \leq 2$ is a valid cut because it excludes the optimal LP solution but does not exclude any integer points;
- Now solve the LP relaxation for this new problem.

A cut must simultaneously exclude the LP solution while keeping all the feasible integer points. There always exists at least one valid cut.

Example of cutting plane Cont'd



Maximize: $z = x + y$

Subject to: $-5x + 4y \leq 0$

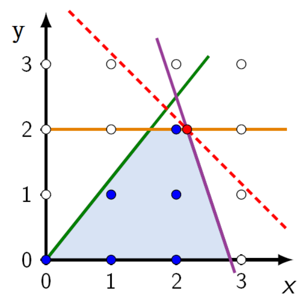
$6x + 2y \leq 17$

$y \leq 2$

$0 \leq x, y \in Z$

Optimal solution $= 4.1667$.

Example of cutting plane Cont'd



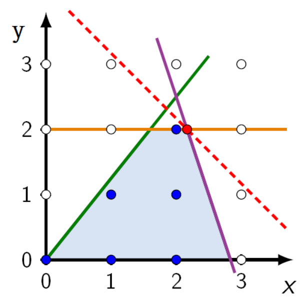
Maximize: $z = x + y$

Subject to:
$$-5x + 4y \leq 0$$
$$6x + 2y \leq 17$$
$$y \leq 2$$
$$0 \leq x, y \in Z$$

Optimal solution $= 4.1667$.

- Adding a cut reduces our upper bound because we are shrinking the feasible set (we added another constraint).

Example of cutting plane Cont'd



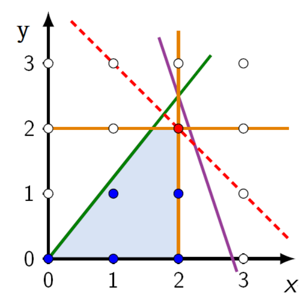
Maximize: $z = x + y$

Subject to: $-5x + 4y \leq 0$
$6x + 2y \leq 17$
$y \leq 2$
$0 \leq x, y \in Z$

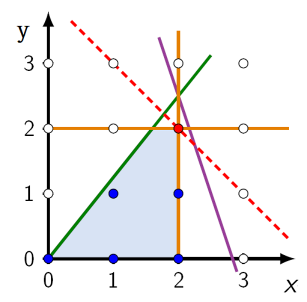
Optimal solution $= 4.1667$.

- Adding a cut reduces our upper bound because we are shrinking the feasible set (we added another constraint).
- Solution is still not an integer. Add another cut!

Example of cutting plane Cont'd



Example of cutting plane Cont'd



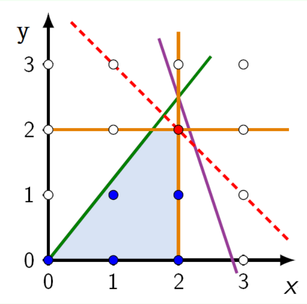
Maximize: $z = x + y$

Subject to: $-5x + 4y \leq 0$

$6x + 2y \leq 17$

$y \leq 2$

Example of cutting plane Cont'd



Maximize: $z = x + y$

Subject to: $-5x + 4y \leq 0$

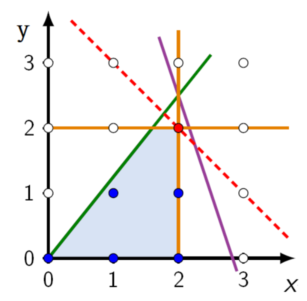
$6x + 2y \leq 17$

$y \leq 2$

$x \leq 2$

$0 \leq x, y \in Z$

Example of cutting plane Cont'd

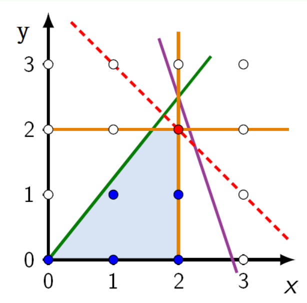


Maximize: $z = x + y$

Subject to:
$-5x + 4y \leq 0$
$6x + 2y \leq 17$
$y \leq 2$
$x \leq 2$
$0 \leq x, y \in Z$

- Optimal solution $x = 2, y = 2, z = 4$;

Example of cutting plane Cont'd



Maximize: $z = x + y$

Subject to:
$$-5x + 4y \leq 0$$
$$6x + 2y \leq 17$$
$$y \leq 2$$
$$x \leq 2$$
$$0 \leq x, y \in Z$$

- Optimal solution $x = 2, y = 2, z = 4$;
- LP solution is integral, so it must also be optimal for the original integer problem.

Take-home messages

- Combinatorial Optimization
  - Motivated Examples
  - Constraint
  - Piecewise Objective Function
  - Feasible Region
- Branch and Bound
  - Enumeration Tree
  - LP Relation
  - Branch and Bound
- Cutting Planes
  - Valid Inequalities
  - Cutting Planes