

1.

$$1.(1) \quad A^T A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$$

$$|\lambda E - A^T A| = \begin{vmatrix} \lambda - 5 & -5 \\ -5 & \lambda - 5 \end{vmatrix} = (\lambda - 5)^2 - 25 = \lambda(\lambda - 10) = 0.$$

$$\lambda_1 = 10, \lambda_2 = 0.$$

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{10}, \sigma_2 = \sqrt{\lambda_2} = 0$$

$$1.(2) \quad \text{由 (1) 知 } \Sigma = \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$(i) \lambda = 10 \quad \lambda E - A^T A = \begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$y_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(ii) \lambda = 0, \lambda E - A^T A = \begin{pmatrix} -5 & -5 \\ -5 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$y_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$v_1 = \frac{1}{\|y_1\|_2} y_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$v_2 = \frac{1}{\|y_2\|_2} y_2 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

$$\text{当 } \sigma \neq 0 \text{ 时即 } \sigma_1 = \sqrt{10}, \quad u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix}$$

u_2, u_3 为 $\text{Nul}(A)$ 的一组标准正交基.

$$\begin{aligned}
 A^T y = 0 \quad & \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0. \quad A^T \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
 y_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, y_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad & u_2 = \frac{1}{\|y_2\|_2} y_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}; u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
 U = & \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 U = UV^T = & \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^T
 \end{aligned}$$

2.

$$\begin{aligned}
 Cov(x, x) = Var(x) &= \frac{1}{2} \left[\left(\frac{1}{3}\right)^2 + \left(\frac{7}{3}\right)^2 + \left(\frac{8}{3}\right)^2 \right] = \frac{57}{9} \\
 Cov(y, y) = Var(y) &= \frac{1}{2} [0^2 + 1^2 + 1^2] = 1 \\
 Cov(z, z) = Var(z) &= \frac{1}{2} [0^2 + 2^2 + 2^2] = 4 \\
 Cov(x, y) = Cov(y, x) &= \frac{1}{2} \left[\frac{-1}{3} \times 0 + \frac{7}{3} \times 1 + \frac{8}{3} \times 1 \right] = \frac{5}{2} \\
 (1) \quad Cov(x, z) = Cov(z, x) &= \frac{1}{2} \left[\frac{-1}{3} \times 0 + \frac{-7}{3} \times 2 - \frac{8}{3} \times 2 \right] = -5 \\
 Cov(y, z) = Cov(z, y) &= \frac{1}{2} [0 - 1 \times 2 - 1 \times 2] = -2 \\
 \Sigma &= \begin{pmatrix} \frac{57}{9} & \frac{5}{2} & -5 \\ \frac{5}{2} & 1 & -2 \\ -5 & -2 & 4 \end{pmatrix}
 \end{aligned}$$

$$(2) P^{-1} \Sigma P = P^T \Sigma P = \Lambda$$

$$\begin{aligned}
 P^T &= \begin{bmatrix} 0 & -0.664 & -0.748 \\ 0.894 & 0.334 & -0.297 \\ 0.447 & -0.669 & 0.594 \end{bmatrix} \\
 \Lambda &= \begin{bmatrix} -0 & 0 & 0 \\ 0 & 0.037 & 0 \\ 0 & 0 & 11.296 \end{bmatrix} \quad \text{取 } k=1, \text{ 则第一主成分对应的特征向量为 } [- \\
 & 0.748, -0.297, 0.594]
 \end{aligned}$$

7.

A: $\lambda_1 = 0 \neq \lambda_2 = -2$, 因此 A 可对角化

B: $\lambda_1 = \lambda_2 = 2, \lambda_3 = 1$

$\dim(\text{Nul}(2I - B)) = 2, \dim(\text{Nul}(I - B)) = 1$
 $\dim(\text{Nul}(2I - B)) + \dim(\text{Nul}(I - B)) = 3$ 因此 B 可对角化。