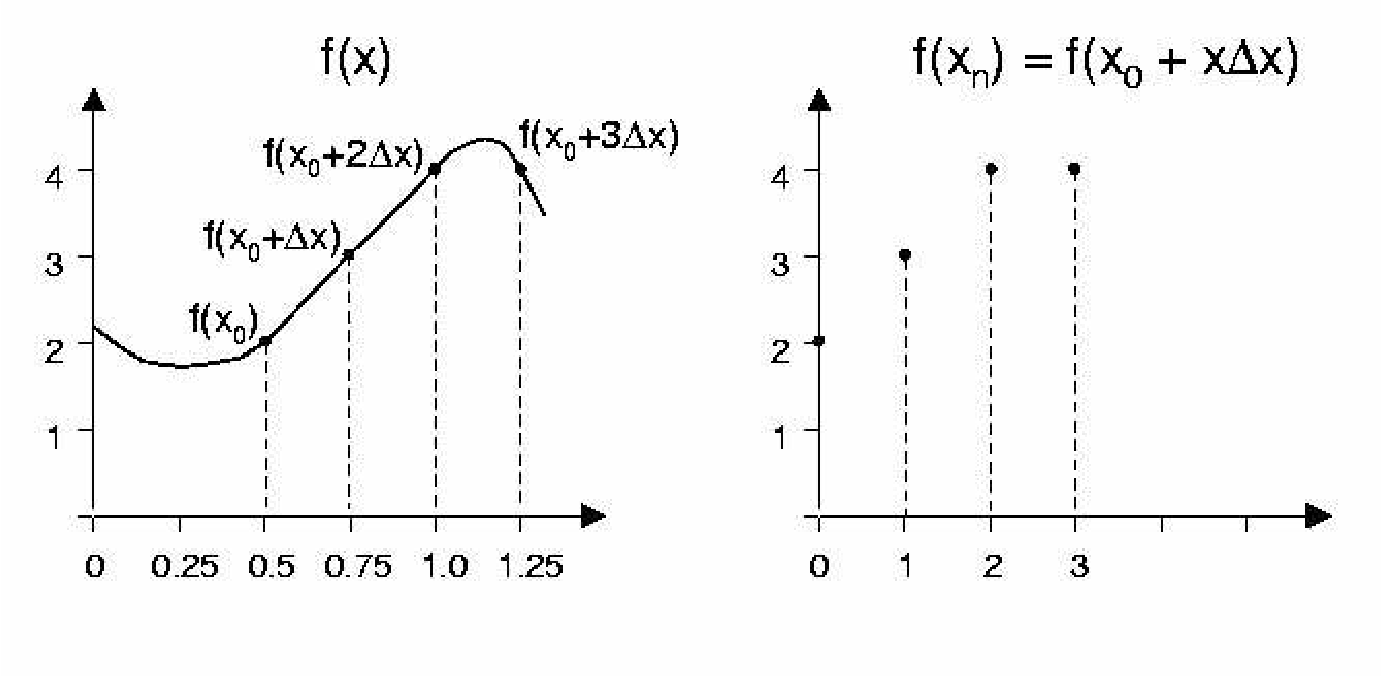
TF DISCRÈTE

TF DISCRÈTE 1D (1)

CAS CONTINU (RAPPEL)

$$\boxed{\mathcal{F}[f(x)] = F(u) = \int_{-\infty}^{+\infty} f(x) \exp\left(-2\pi j u x\right) dx \qquad u \in I\!R.}$$

CAS DISCRET



$$f(x) = f(x_0 + x\Delta x) \qquad x = 0, 1, 2, \ldots N-1$$

TFD et TFD inverse

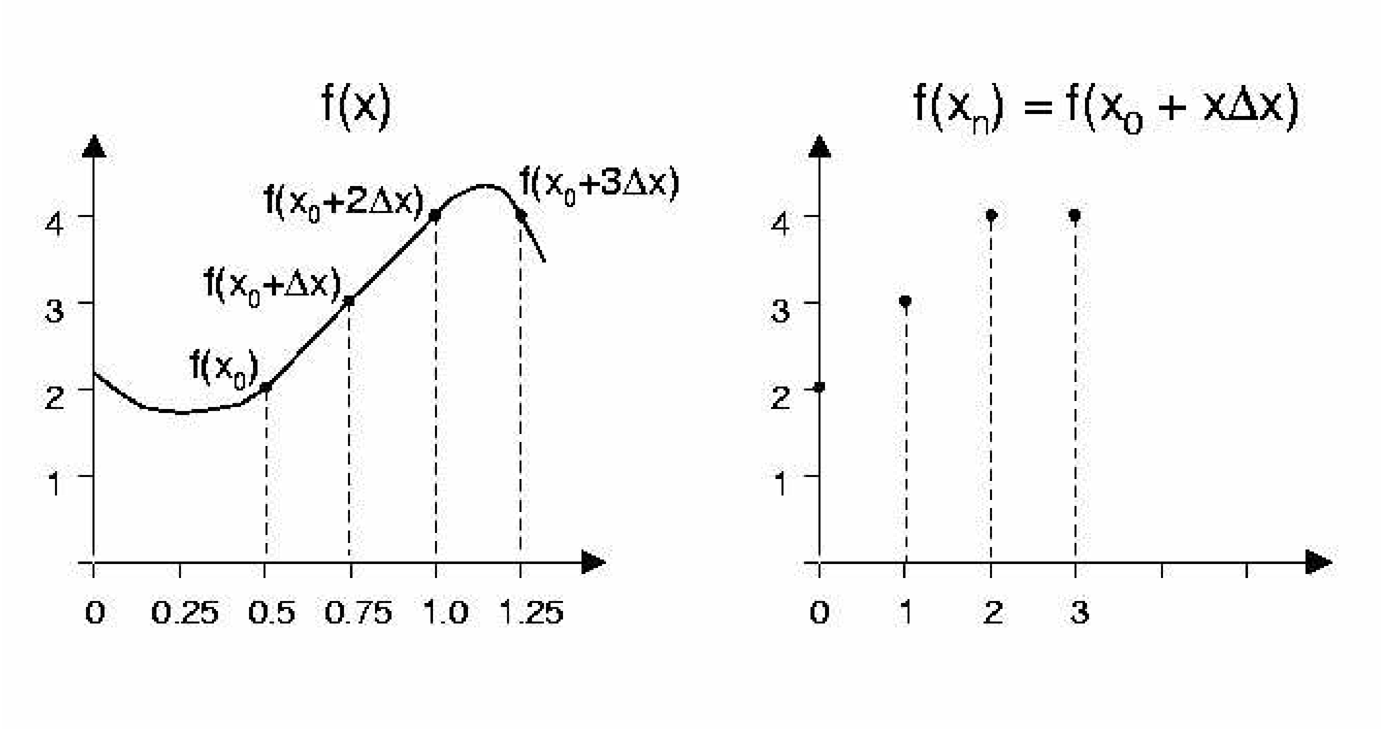
$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) \exp\left(\frac{-2\pi j u x}{N}\right) \qquad u = 0, 1, 2, \ldots, N-1$$

$$f(x) = \sum_{u=0}^{N-1} F(u) \exp\left(\frac{2\pi j u x}{N}\right) \qquad x = 0, 1, 2, \ldots, N-1$$

$$\Delta u = \frac{1}{N\Delta x}$$

TF DISCRÈTE

TF DISCRÈTE 1D -EXEMPLE- (2)



$$
\begin{aligned}
F(0) &= 1/4 \sum_{x=0}^{3} f(x) \exp\left(\frac{-2\pi j 0 x}{4}\right) = 1/4 \sum_{x=0}^{3} f(x)\, 1 \\
&= 1/4\big(f(0) + f(1) + f(2) + f(3)\big) = 1/4(2+3+4+4) = 3.25 \\
F(1) &= 1/4 \sum_{x=0}^{3} f(x) \exp\left(\frac{-2\pi j x}{4}\right) \\
&= 1/4\big(2\exp(0) + 3\exp(-j\pi/2) + 4\exp(-j\pi) + 4\exp(-3j\pi/2)\big) \\
&= \frac{1}{4}(-2+j) \\
F(2) &= 1/4 \sum_{x=0}^{3} f(x) \exp\left(\frac{-4\pi j x}{4}\right) = \ldots = 1/4 \\
F(3) &= 1/4 \sum_{x=0}^{3} f(x) \exp\left(\frac{-6\pi j 0 x}{4}\right) = \ldots = 1/4(-2-j)
\end{aligned}
$$

Spectre d'amplitude

$|F(0)| = 3.25 \quad |F(1)| = \sqrt{5}/4 \quad |F(2)| = 1/4 \quad |F(3)| = \sqrt{5}/4$

TF DISCRÈTE
TF DISCRÈTE 2D (1)

TFD 2D et TFD 2D inverse

$$F(u,\nu) = \frac{1}{MN}\sum_{x=0}^{N-1}\sum_{y=0}^{M-1} f(x,y)\exp\left(-2\pi j(\frac{ux}{N}+\frac{\nu y}{M})\right)$$

$$u = 0,1,2,\ldots,N-1$$

$$\nu = 0,1,2,\ldots,M-1$$

$$f(x,y) = \sum_{u=0}^{N-1}\sum_{\nu=0}^{M-1} F(u,\nu)\exp\left(2\pi j(\frac{ux}{N}+\frac{\nu y}{M})\right)$$

$$x = 0,1,2,\ldots,N-1$$

$$y = 0,1,2,\ldots,M-1$$

Le plus souvent, l'image à traiter est carrée ◀▶ $M = N$

PROPRIÉTÉS

- Les mêmes que ceux énoncé pour la TF continue
- Cyclique (périodique)

$$F(u,\nu) = F(u+N,\nu) = F(u,\nu+N) = F(u+N,\nu+N)$$

$$f(x,y) = f(x+N,y+N)$$

car

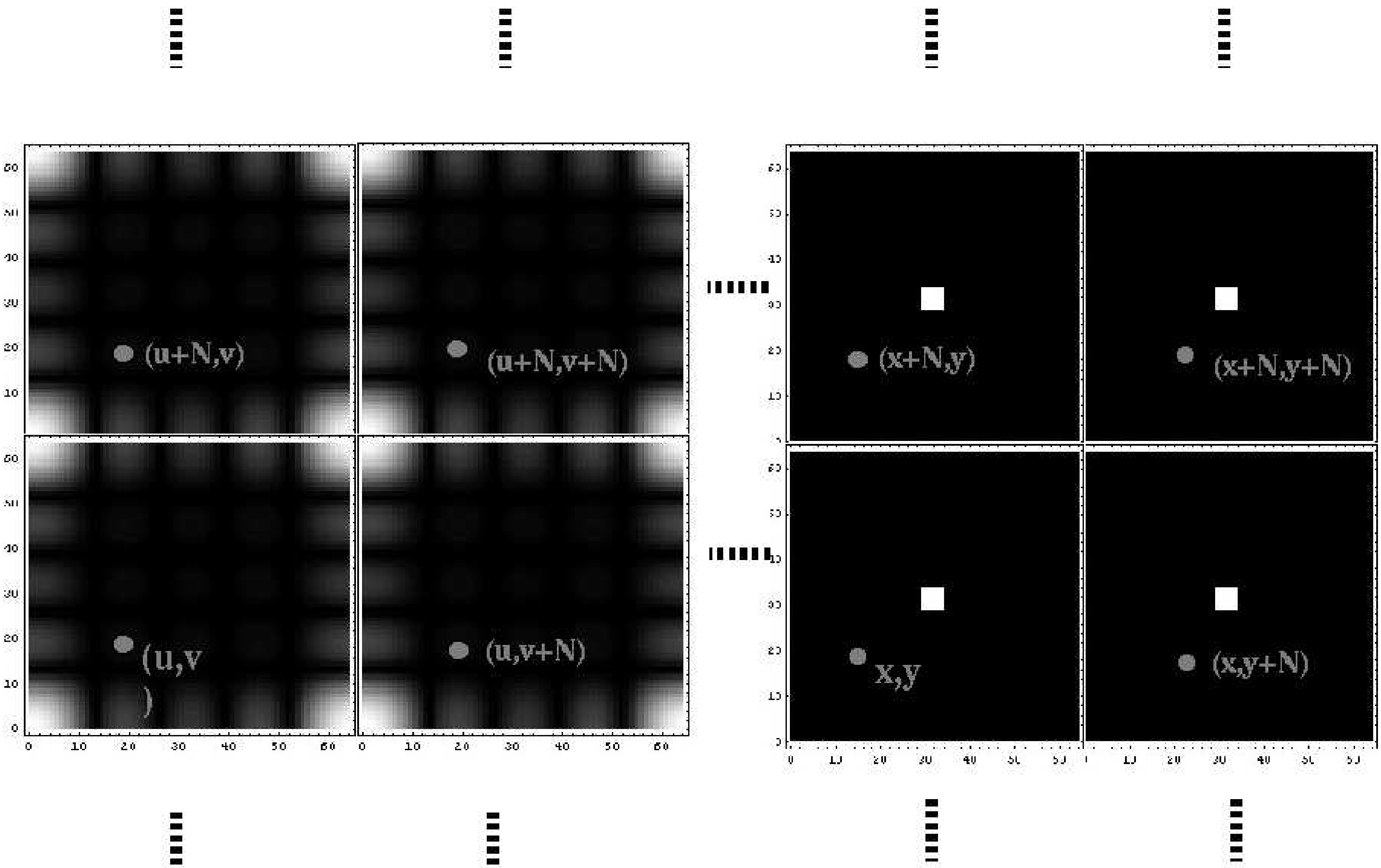
Signal périodique $-\mathcal{F}$ -> Spectre de raies

Signal échantillonné $-\mathcal{F}$ -> Spectre périodique

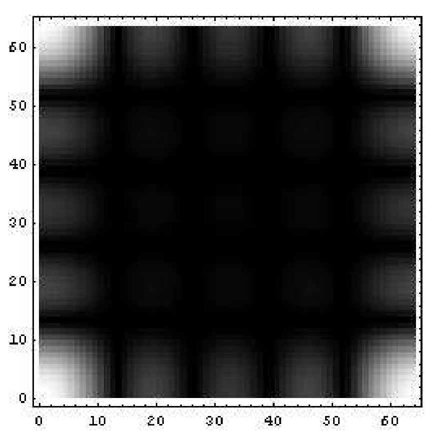
TF DISCRÈTE

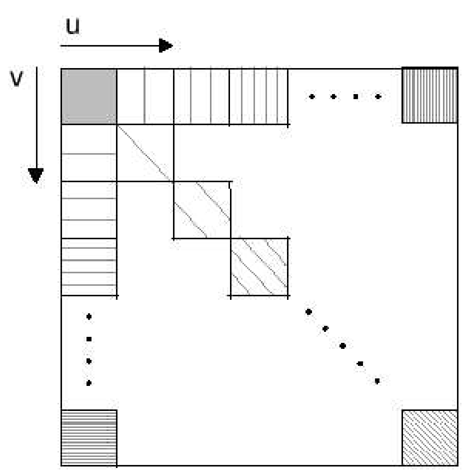
TF DISCRÈTE 2D (4)

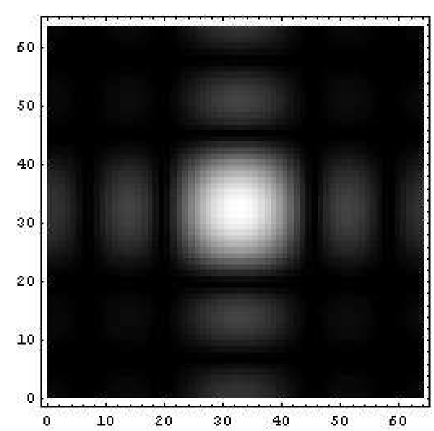
- Périodicité



- **Nota**







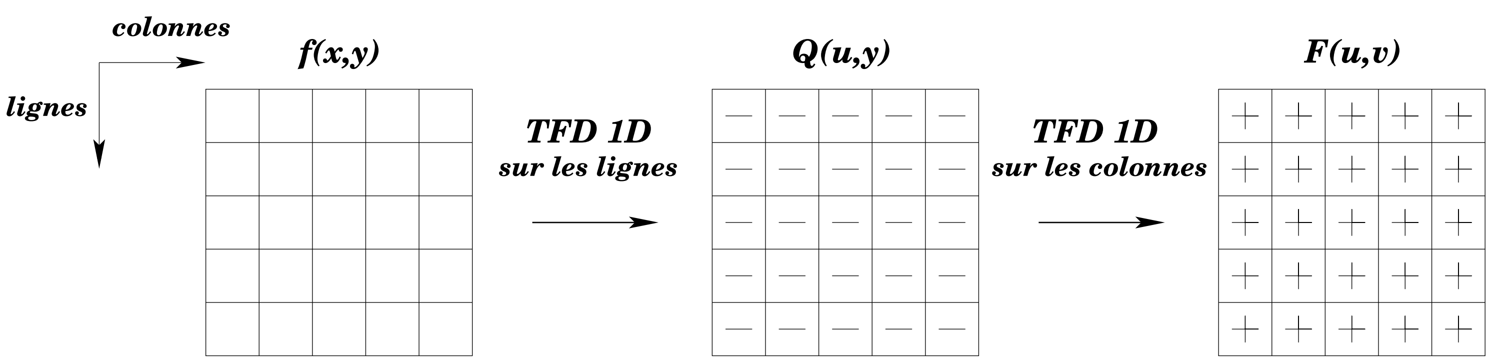
TF DISCRÈTE

TF DISCRÈTE 2D (5)

- Séparabilité

Pour une image carrée ◀▶ $M = N$

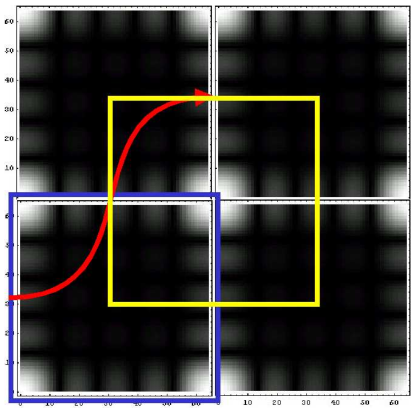
$$
\begin{aligned}
F(u,\nu) &= \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\,\mathrm{exp}\left(-2\pi j\left(\frac{ux+\nu y}{N}\right)\right) \\
&= \frac{1}{N}\sum_{x=0}^{N-1}\left(\frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\,\mathrm{exp}\left(-2\pi j\frac{\nu y}{N}\right)\right)\mathrm{exp}\left(-2\pi j\frac{ux}{N}\right) \\
&= \underbrace{\frac{1}{N}\sum_{x=0}^{N-1} \text{[TF 1D de la x ième colonne]}\,\mathrm{exp}\left(-2\pi j\frac{ux}{N}\right)}_{\text{TF 1D de la y ième ligne}}
\end{aligned}
$$



TF DISCRÈTE

TF DISCRÈTE 2D (6)

Recalage Cyclique

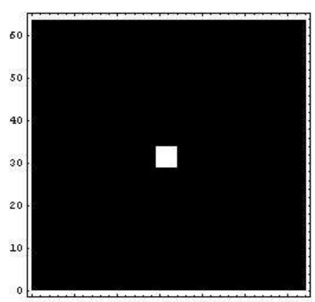


Translation -Rappel-

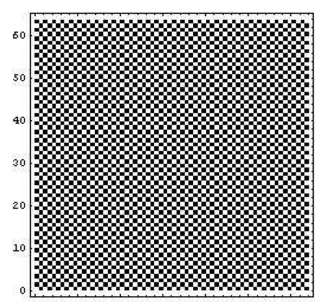
$$f(x-x_0, y-y_0) \quad \blacktriangleleft \mathcal{F} \blacktriangleright \quad F(u,\nu)\,\text{exp}\left(\frac{-2\pi j(ux_0+\nu y_0)}{N}\right)$$

$$f(x,y)\,\text{exp}\left(\frac{2\pi j(u_0 x+\nu_0 y)}{N}\right) \quad \blacktriangleleft \mathcal{F} \blacktriangleright \quad F(u-u_0, \nu-\nu_0)$$

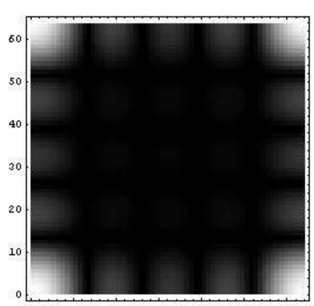
$$u_0 = \nu_0 = \tfrac{N}{2} \blacktriangleright \text{exp}\left(\tfrac{2\pi j(u_0 x+\nu_0 y)}{N}\right) = \text{exp}\left(\pi j(x+y)\right) = (-1)^{(x+y)}$$

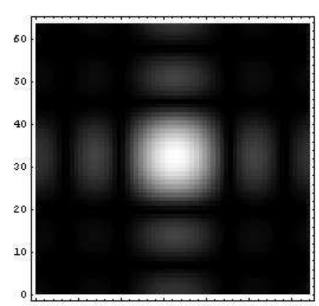


$\times(-1)^{(x+y)}$



$\mathcal{F}$

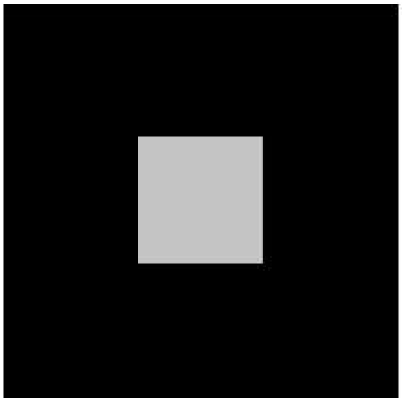




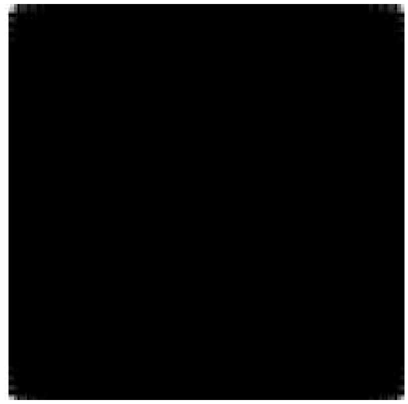
TF DISCRÈTE

TF DISCRÈTE 2D -VISUALISATION- (7)

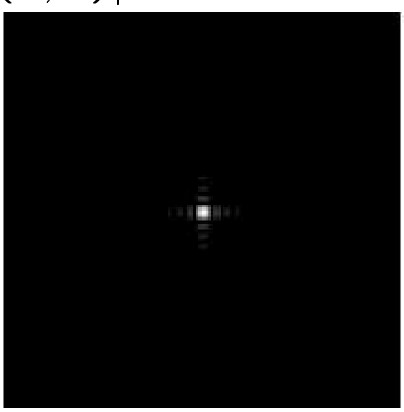
Image originale



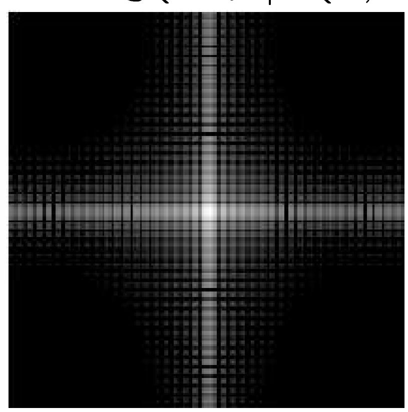
$|F(u,\nu)|$



$|F(0,0)|$ au centre



$k\log(1+|F(u,\nu)|)$



Nota

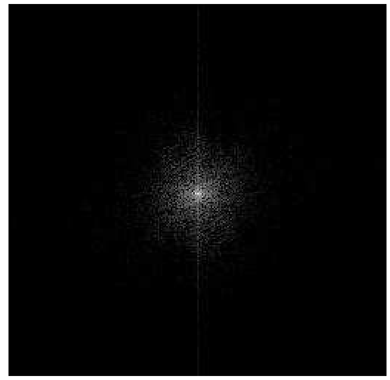
- On affiche généralement $|F(u,\nu)|$ ($F(u,\nu)$ complexe)

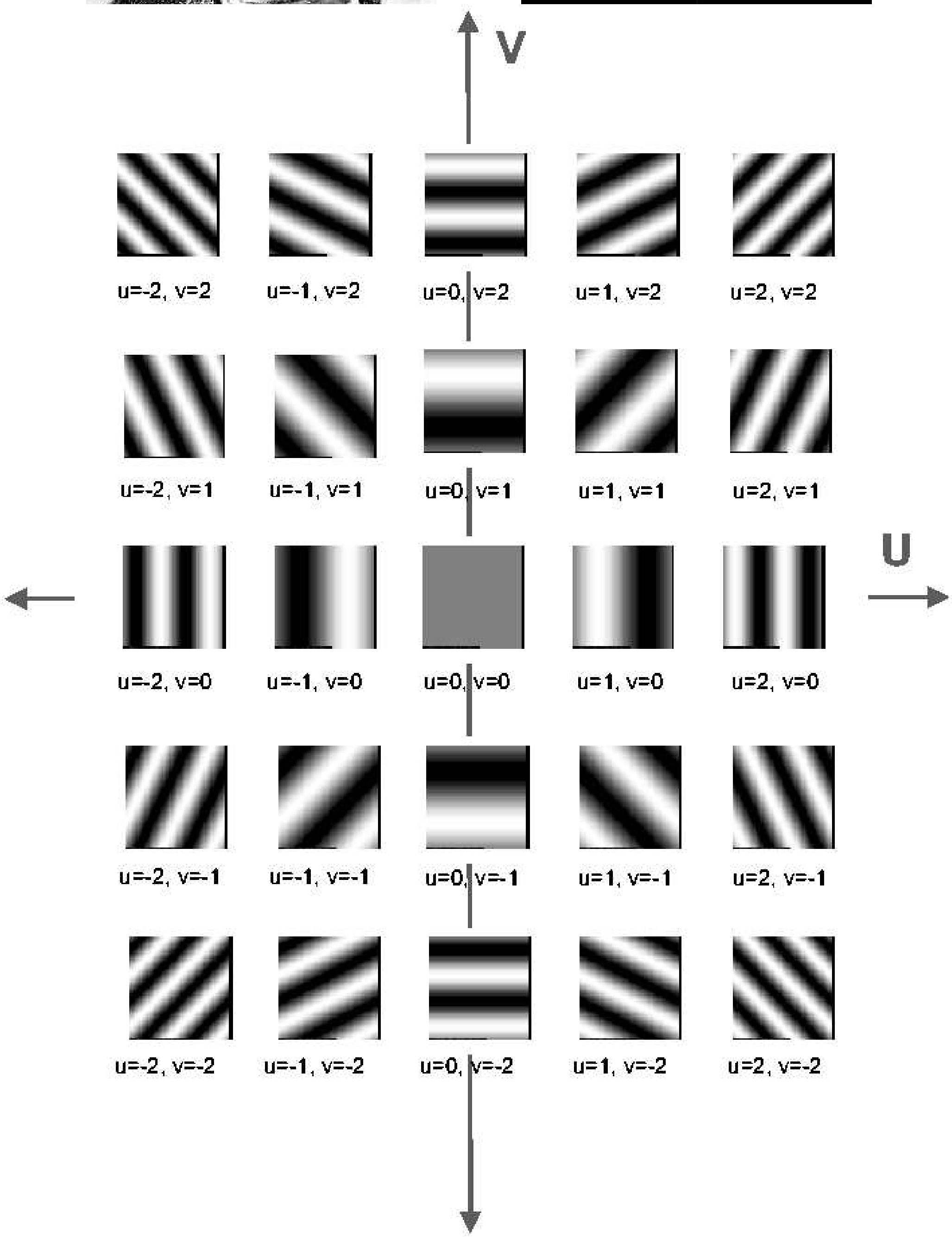
- Les fréquences élevées sont bcp plus faible que les fréquences plus basses. On affiche donc plutot $k\log(1+|F(u,\nu)|)$ (k est une constante de normalisation pour recaler les niveau de gris dans $[0,255]$)

- On met l'origine au centre de l'image en effectuant un décalage cyclique

TF DISCRÈTE

TF DISCRÈTE 2D -INTERPRÉTATION- (8)

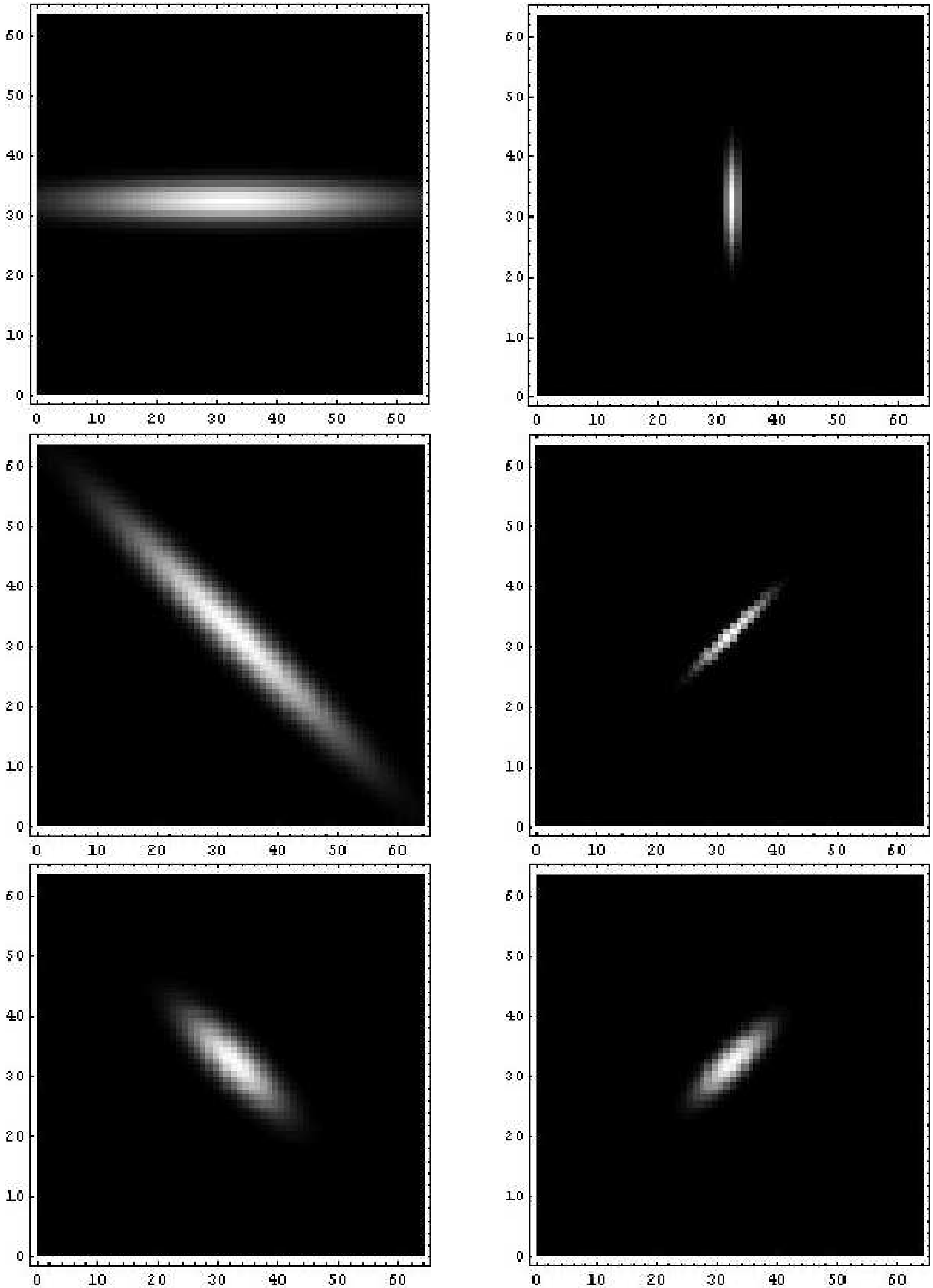




TF DISCRÈTE

TF DISCRÈTE 2D -EXEMPLES- (9)

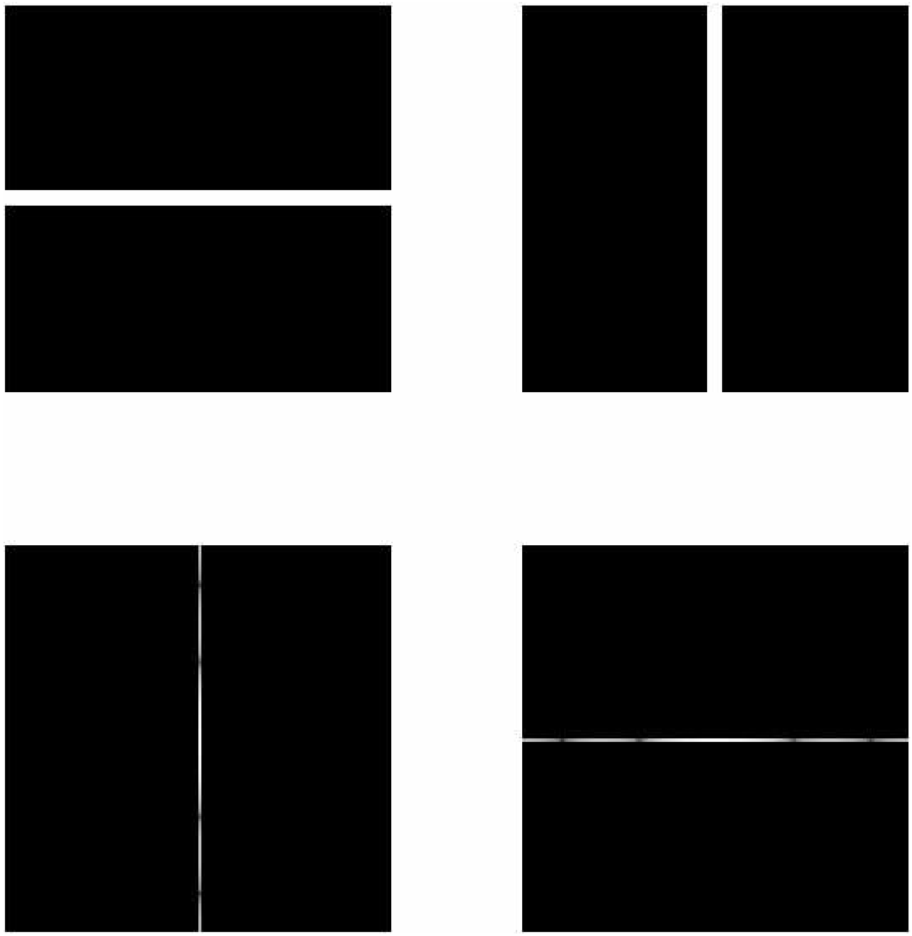
Image $\qquad\qquad$ $\mathcal{F}$(image)



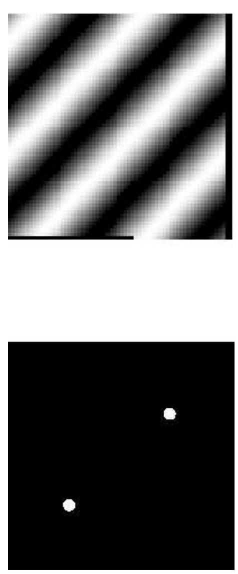
TF DISCRÈTE 2D -EXEMPLES- (10)

- Rotation

Image $\mathcal{F}$(image)

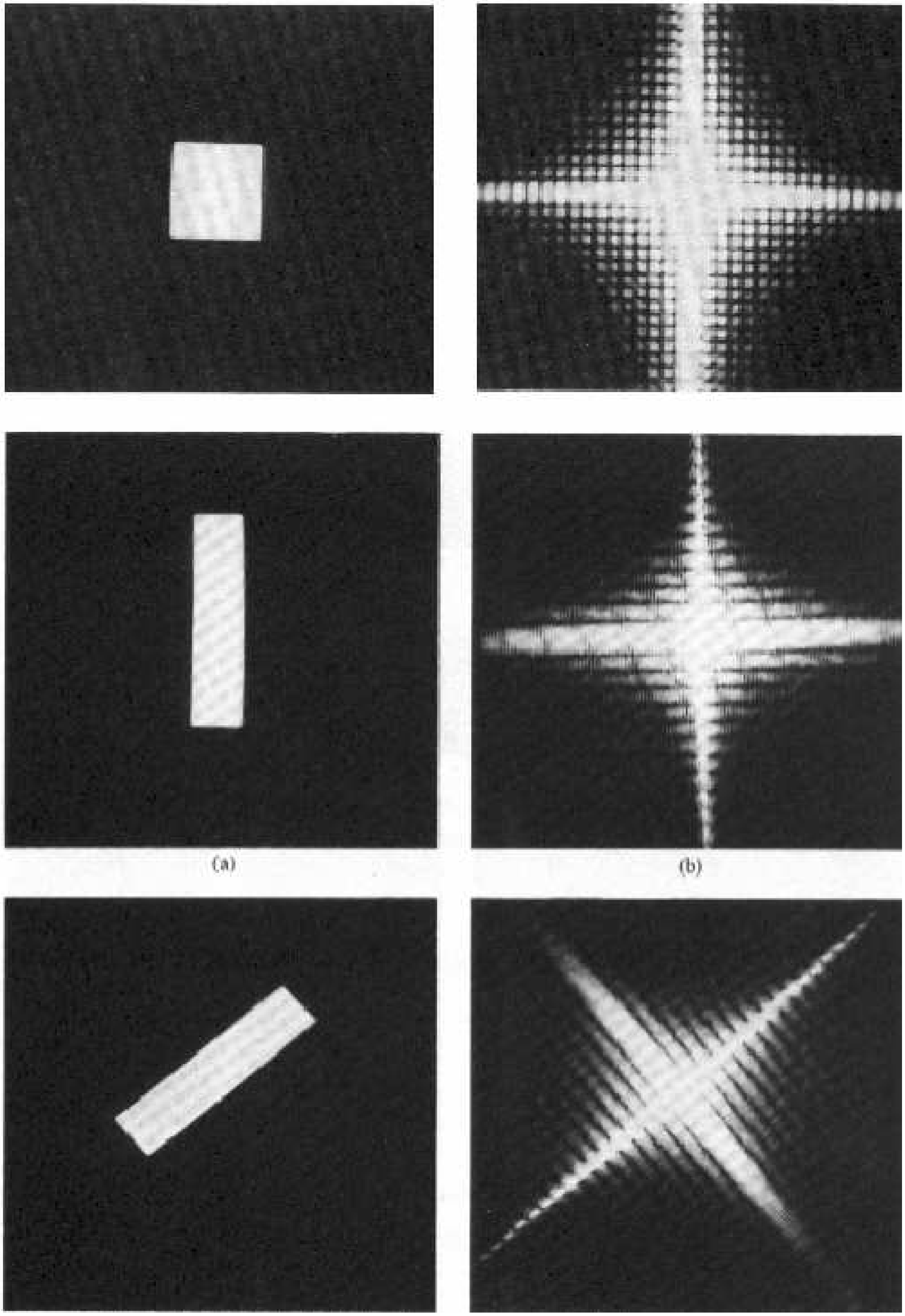


- Périodicité



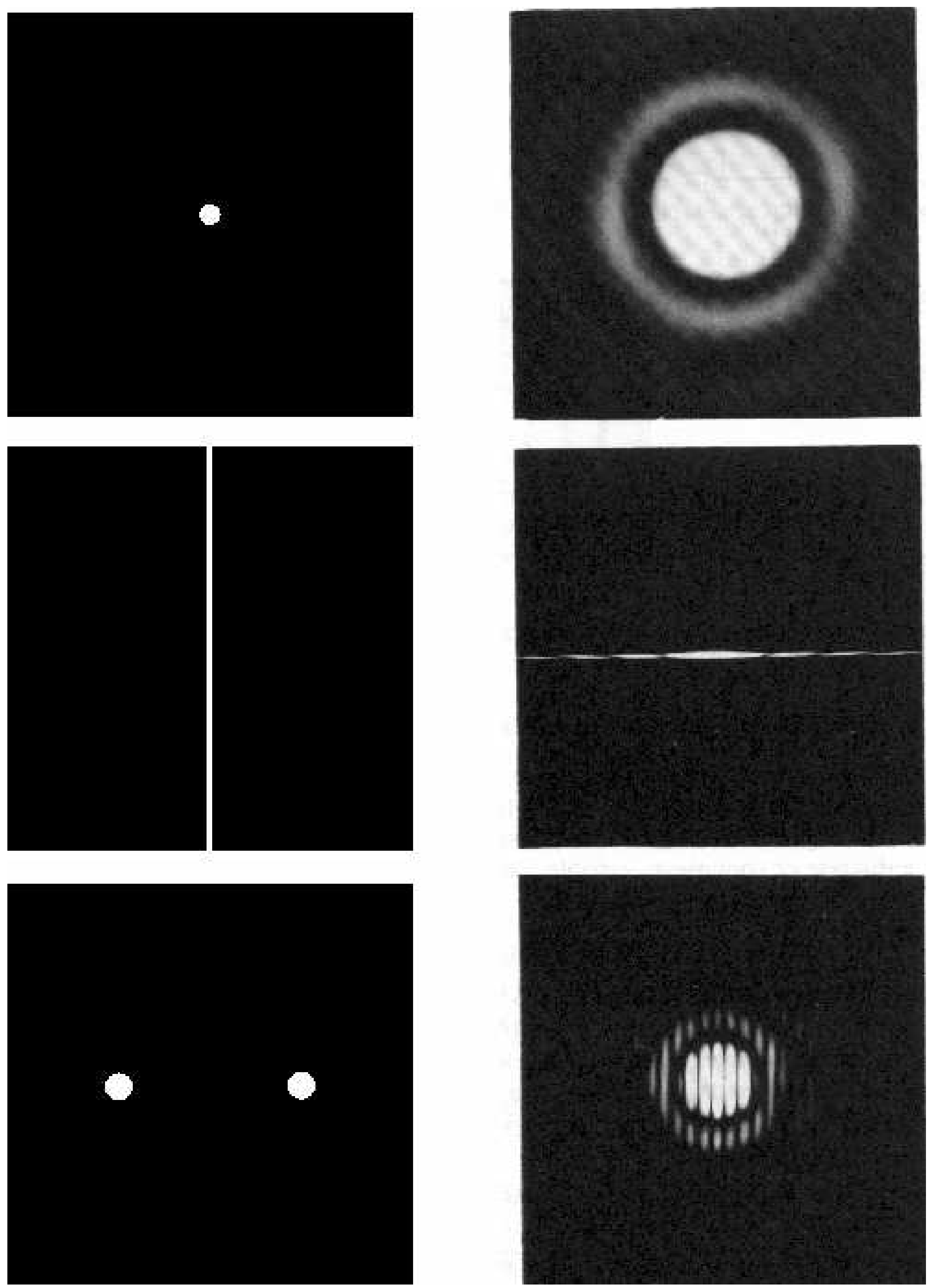
TF DISCRÈTE
TF DISCRÈTE 2D -EXEMPLES- (11)

Image $\mathcal{F}$(image)



TF DISCRÈTE
TF DISCRÈTE 2D -EXEMPLES- (12)

Image $\mathcal{F}$(image)



TF DISCRÈTE

CONVOLUTION DISCRÈTE 1D & 2D (1)

Théorème de convolution 2D

$$\begin{aligned} f(x,y) * g(x,y) \quad & -\mathcal{F}- > \quad F(u,\nu)\cdot G(u,\nu) \\ f(x,y)\cdot g(x,y) \quad & -\mathcal{F}- > \quad F(u,\nu) * G(u,\nu) \end{aligned}$$

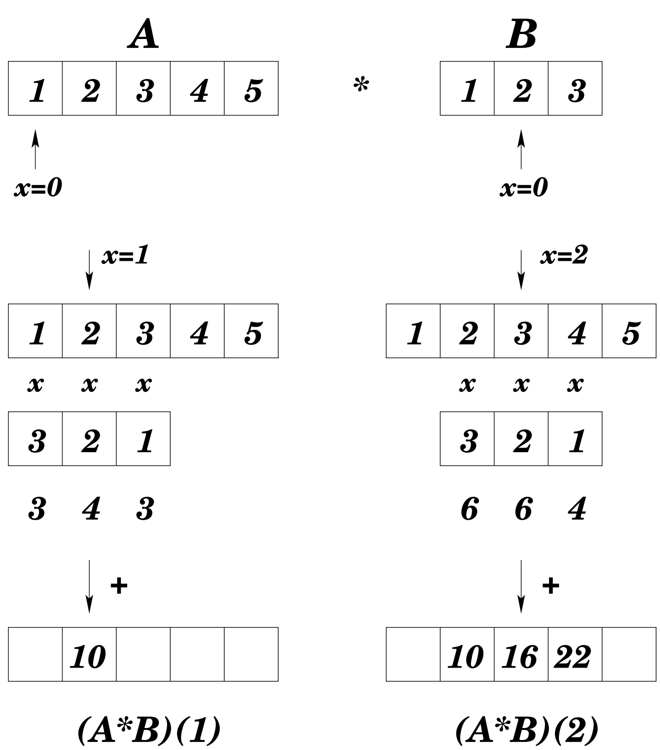
Convolution 2D -Version continue-

$$\boxed{(f*g)(x,y) = \int_{x=0}^{\infty}\int_{y=0}^{\infty} f(\alpha,\beta)g(x-\alpha,y-\beta)\,d\alpha\,d\beta}$$

Convolution 1D -Version discrète-

$$\boxed{(A*B)(x) = \sum_i A(i)B(x-i)}$$

-Exemple-



TF DISCRÈTE

CONVOLUTION DISCRÈTE 1D & 2D (2)

Convolution des bords

A) Ajout de zéro

0	*0*	*1*	*2*	*3*	*4*	*5*	*0*	*0*

4	*10*	*16*	*22*	*22*

B) Enroulement

4	*5*	*1*	*2*	*3*	*4*	*5*	*1*	*2*

19	*10*	*16*	*22*	*23*

C) Réflexion

3	*2*	*1*	*2*	*3*	*4*	*5*	*4*	*3*

7	*10*	*16*	*22*	*27*

D) Défaut

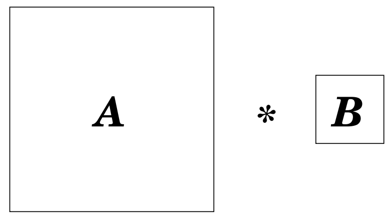
1	*2*	*3*	*4*	*5*

	10	*16*	*22*	

TF DISCRÈTE

CONVOLUTION DISCRÈTE 1D & 2D (3)

Convolution 2D -Version discrète-

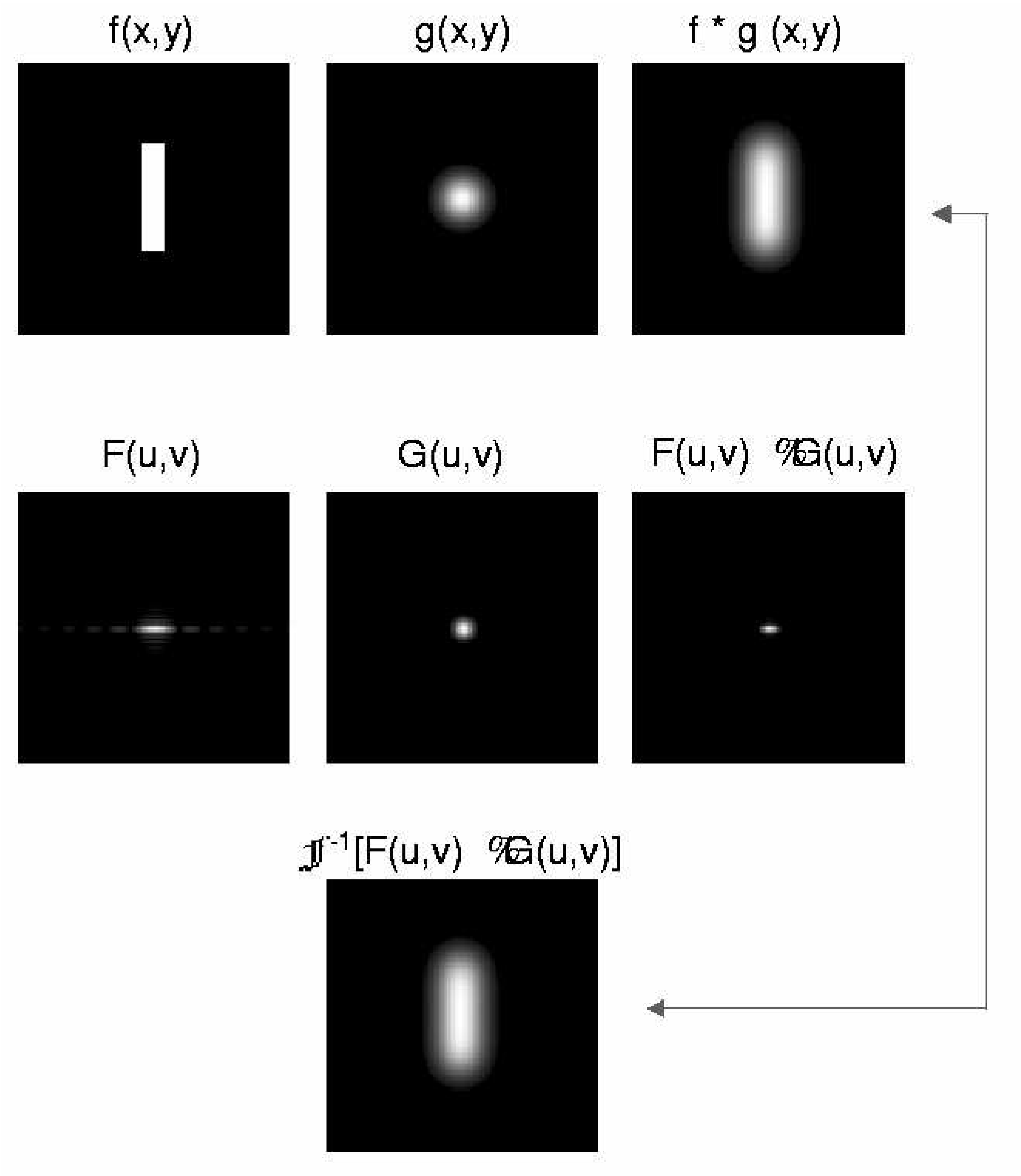


$$(A * B)(x, y) = \sum_i \sum_j A(i, j) B(x - i, y - j)$$

PROPRIÉTÉS

– Commutatif ▶ $f_1 * f_2 * f = f_2 * f_1 * f$
– Associatif ▶ $(f_1 * f_2) * f = f_1 * (f_2 * f)$
– Distributif ▶ $(f_1 + f_2) * f = f_1 * f + f_2 * f$
▶ $f * (f_1 + f_2) = f * f_1 + f * f_2$

TF DISCRÈTE
CONVOLUTION DISCRÈTE 1D & 2D (4)



TF DISCRÈTE

FFT (1)

$$\boxed{F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x) \exp\left(\frac{-2\pi jux}{N}\right) \qquad u = 0, 1, 2, \ldots, N-1}$$

$O(N^2)$ opérations

$$F(u) = \underbrace{\frac{1}{N}\sum_{x=0}^{N/2-1} f(2x) \exp\left(\frac{-2\pi ju2x}{N}\right)}_{\text{x pair}} + \underbrace{\frac{1}{N}\sum_{x=0}^{N/2-1} f(2x+1) \exp\left(\frac{-2\pi ju(2x+1)}{N}\right)}_{\text{x impair}}$$

$$= \frac{1}{2}\Bigg[\underbrace{\frac{1}{N/2}\sum_{x=0}^{N/2-1} f(2x) \exp\left(\frac{-2\pi jux}{N/2}\right)}_{\text{TF de la partie paire}} + \exp\left(\frac{-2\pi ju}{N}\right) \underbrace{\frac{1}{N/2}\sum_{x=0}^{N/2-1} f(2x+1) \exp\left(\frac{-2\pi jux}{N/2}\right)}_{\text{TF de la partie impaire}}\Bigg]$$

- TF [N éléments] = 2 TF [$N/2$ éléments] + $N/2$ multiplications et $N/2$ additions complexes (en fait $N/4$ pour les multiplications)
- $\boxed{Op(N) = 2Op(N/2) + N}$

$$F_N(u) = \frac{1}{2}\left[F_{N/2}^{\text{paire}}(u) + \exp\left(-\frac{2\pi ju}{N}\right) F_{N/2}^{\text{impaire}}(u)\right]$$

Pour $\quad u' = u + \frac{N}{2}$

$$\exp\left(-\frac{2\pi ju'}{N}\right) = \exp\left(-\frac{2\pi j(u+\frac{N}{2})}{N}\right) = \exp\left(-\frac{2\pi ju}{N}\right)\exp(-\pi j) = -\exp\left(-\frac{2\pi ju}{N}\right)$$

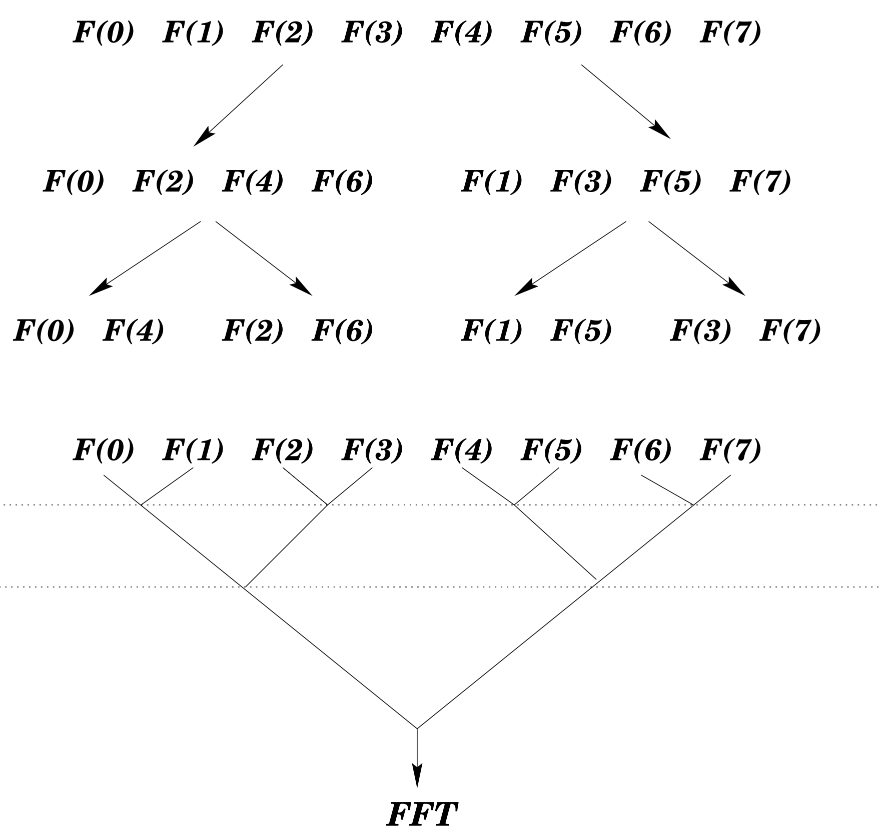
TF DISCRÈTE
FFT (2)

Ainsi,

$$
\begin{aligned}
F_N(u) &= \frac{1}{2}\Big[F_{N/2}^{\text{paire}}(u) + \exp\big(-\frac{2\pi j u}{N}\big) F_{N/2}^{\text{impaire}}(u)\Big] \\
F_N(u + \frac{N}{2}) &= \frac{1}{2}\Big[F_{N/2}^{\text{paire}}(u) - \exp\big(-\frac{2\pi j u}{N}\big) F_{N/2}^{\text{impaire}}(u)\Big]
\end{aligned}
$$

Pour $\quad u = 0, 1, 2, \ldots, N/2 - 1$

1 multiplications est nécessaire pour deux termes



FFT 1D ▶ $\qquad O(N \log N)$

FFT 2D ▶ $\qquad O(N^2 \log N)$

TF DISCRÈTE

FFT (3)

$\boxed{\text{Vérifions que } O(N \log N)}$

(Par induction, sachant que $\mathrm{Op}(N) = 2\,\mathrm{Op}(N/2) + N$)

- Vrai pour $N = 1$? ▶ $\mathrm{Op}(1) = 1 \log 1 = 0$ ▶ oui
- Vrai pour $N = 2$? ▶ $\mathrm{Op}(2) = 2 \log 2 = 2$ ▶ oui
- Vrai pour $N \log N$?

  – $\mathrm{Op}(N) = N \log N$?
  – $\mathrm{Op}(N) = 2N/2 \log(2N/2)$
  – $\mathrm{Op}(N) = 2N/2(\log(N/2) + \log 2)$
  – $\mathrm{Op}(N) = 2N/2 \log N/2 + 2N/2 \log 2$
  – $\mathrm{Op}(N) = 2\,\mathrm{Op}(N/2) + N$

C.Q.F.D.