THE GEOMETRY OF WEYL REDUCIBILITY

PETER XU

We state below the standard "algebraic" proof of Weyl's famous reducibility theorem, and give a geometric interpretation. We also attempt to tie it into the classical proof of Maschke's theorem.

In this entire section, every Lie algebra is over characteristic zero.

Theorem (Weyl's semisimplicity theorem)**.** *Every finite-dimensional representation of a semisimple Lie algebra* $\mathfrak{g}$ *is itself semisimple.*

Proof. The idea of the proof is as follows: for a reducible representation, any time we have a subrepresentation, we want to be able to split it as a direct sum decomposition - that is, we want the corresponding short exact sequence to be trivial, as measured by Ext^1. By some homological nonsense and a devissage argument, we need only consider extensions of the trivial representation $\mathbb{C}$ by irreducible representations. For such extensions, we find an element in $U\mathfrak{g}$ which is a "projection" onto the nontrivial rep; i.e. acts by zero on the trivial rep and by a nonzero scalar on the nontrivial one.

The element which accomplishes precisely this is the **Casimir element**, which can be defined coordinate-free as the image C f the identity in the composite $\hom(\mathfrak{g},\mathfrak{g}) \to \mathfrak{g}\otimes\mathfrak{g}^* \to \mathfrak{g}\otimes\mathfrak{g} \to U(\mathfrak{g})$ where the first map is the canonical isomorphism, the second map is given by the dual identification arising from the Killing form B, and the third map is the multiplication. If we fix a basis e_i of $\mathfrak{g}$, tracing the arrows through shows that the resulting element is $C = \sum e_i f_i \in U(\mathfrak{g})$, where f_i is the dual basis under B.

Some formal properties of the Casimir are easily verified: since the Killing form is $\mathfrak{g}$-invariant, the Casimir lies in the center of the algebra. Further, it is obviously in the (left) ideal generated by $\mathfrak{g}$ from writing it out above. Hence it always acts by a scalar on an irreducible representation, and by zero on the trivial representation.

Moreover, the scalar for any non-trivial irrep is nonzero, as can be verified via highest-weight theory. We will not go into the details, as one can find an adequate exposition of this proof anywere.[1]

Now let V be any reducible representation, W a subrepresentation, so that we have an exact sequence $0 \to W \to V \to U \to 0$. It suffices to show that $\mathrm{Ext}^1(W,U) = 0$.[2] From deriving the tensor-hom adjunction in the first argument, this is equivalent to $\mathrm{Ext}^1(\hom(U,W),\mathbb{C}) = 0$; treating the former entry as an arbitrary representation, we can interpret this as saying it suffices to show that extensions of any representation V by the trivial $\mathbb{C}$ are direct sums; equivalently, that any extension of $\mathbb{C}$ by V is trivial.

Indeed, choose a composition series for V (which obviously has finite length since it has finite dimension); since Ext is right-exact in the first argument since hom is, by devissage we immediately see that it suffices to prove the result for each quotient and submodule; by induction, it suffices to prove it for simple V.

[1] To avoid the technology of highest weights, we can directly take the Casimir element with respect to the trace pairing on the representation, rather than with respect to the Killing form. A bit of structure theory on simple modules shows that it still has all the properties we need. We avoid this easier viewpoint because it adapts less readily to the geometric viewpoint we will need later.

[2] In the abelian category of $U\mathfrak{g}$-modules enriched over itself; equivalently, of internal hom in the category of $\mathfrak{g}$-representations.

Since C acts trivially on $\mathbb{C}$, left multiplication by C maps any extension into the submodule V. V is a simple module, so the action of C either takes V to itself or kills it. If it takes it to itself, the exact sequence splits and we have a direct sum. Otherwise, by above, V must be the trivial rep.

Finally, if $V = \mathbb{C}$, suppose we have a nontrivial extension E of $\mathbb{C}$ by $\mathbb{C}$. Then $U\mathfrak{g}$ sends the "inclusion" copy of $\mathbb{C}$ sitting inside to zero. The whole thing can't be sent to zero otherwise it would be trivial, so the whole extension is sent only to the "inclusion" copy of $\mathbb{C}$ - this is a one-dimensional family of endomorphisms. Hence the kernel of the associated map $\mathfrak{g} \to \mathrm{End}E$ has codimension ≤ 1, but a semisimple Lie algebra has no ideals of codimension 1, since as a sum of simple Lie algebras, it would then have a dimension-1 (hence abelian) factor. Contradiction. □

To get at the geometric content hidden inside the homological nonsense of this proof, the key is clearly to interpret the action of the Casimir in a geometric way. Of course, we now restrict to real Lie algebras, so that we can use the Lie correspondence.

The correct starting point is the insight that the universal enveloping algebra is actually a very geometric object: notice that the Lie algebra $\mathfrak{g}$ of a Lie group G can be put in correspondence with left-invariant vector fields on G; differentiating functions along these vector fields yields all the (left-invariant) first-order differential operators. Correspondingly, the universal enveloping algebra is all the left-invariant differential operators on G.

The Casimir's equivariance is precisely its left-invariance as a differential operator. Moreover, as its symmetry suggests, it is actually a very special differential operator. To see this, we need the first of a few facts we'll use.

Proposition. *The Killing form on a compact Lie group is negative definite.*[3]

Proof. We will merely sketch the proof: choose an arbitrary metric on $\mathfrak{g}$, and average it so that it is ad-invariant (possible by compactness). Under this metric, it is then clear that the adjoint representation of G is orthogonal and hence $\mathrm{ad}(\mathfrak{g})$ consists of skew-symmetric matrices. As a result, for any nonzero $a \in \mathfrak{g}$, $K(a, a) = \mathrm{tr}(\mathrm{ad}(a)\mathrm{ad}(a)) = -\mathrm{tr}(\mathrm{ad}(a)^T \mathrm{ad}(a)) < 0$. □

Corollary. *The negative Killing form $-K$ on a compact semisimple Lie group induces a bi-invariant Riemannian metric.*[4]

Proof. Nondegeneracy from simplicity, symmetry is obvious, positive definiteness from above. Bi-invariance is obvious. Hence G is endowed a "canonical" Riemannian manifold structure.[5] □

This Riemannian structure is really nice: its Levi-Civita connection is satisfies $\nabla_X Y = \frac{1}{2}[X, Y]$, the exponential geodesics of the connection are the exponential geodesics of the Lie group, parallel transport is given in the obvious way along geodesics, and we can define normal coordinates using the geodesics/one-parameter subgroups so that the metric tensor locally is the Kronecker delta in those coordinates. None of this is too difficult to verify, but is besides the point for now.

What is even more impressive is the following payoff:

[3] It is not too hard to see that this is actually if and only if.

[4] In fact it's not hard to show any such metric is a multiple of the Killing form.

[5] In fact with only the semisimplicity condition, we obtain a pseudo-Riemannian manifold which also satisfies many nice properties, but we will not pursue this viewpoint here.

Proposition. *The negative Casimir element $-C$ is the Laplace-Beltrami operator with respect to this metric.*

Proof. Proof sketch: under normal coordinates at the origin, the ordinary total derivative d corresponds to the sum of the partials under a Killing-orthogonal basis, and δ corresponds to the sum of the partials under the dual basis; since $\nabla = \delta d$ we have our result globally by translation by the group action. □

Given a representation $\rho : G \to \mathrm{GL}(V)$ which differentiates to give $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, functions on $\mathrm{GL}(V)$ pull back, so the Laplacian operator acts on them. The last twist then need is a cancellation of double homs - said differently, a vector $v \in V$ is actually a function from $\mathrm{GL}(V)$ to $V \cong \mathbb{C}^{\dim V}$, given by $v(g) := g(v)$. What is more, the differential operator action of an element of $\mathfrak{gl}(V)$ is precisely the action of matrix left multiplication!

This picture doesn't really add anything to the fact that the Casimir must act by a scalar on any irrep; the notion of irreducible representation is not very geometric.

However, the scalar by which it acts no longer needs to be shown to be nonzero by clever algebraic devilry or highest-weight theory: it is simply the natural consequence of the fact that there are no harmonic functions on a compact manifold!

To pass from the compact case to the general one, all we need is the following lemma:

Lemma. *Any simple Lie group has a nontrivial compact subgroup.*

Proof. This is a fairly technical result due to Cartan, following (for example) from his Cartan decomposition. Interestingly, we don't actually need the nontriviality of the compact factor - the two factors in the Cartan decomposition are respectively negative/positive definite with respect to the Killing form, so the argument can be replicated in either case by cobbling together negative/positive Killing form summands for the full semisimple group. □

Corollary. *Any semisimple Lie group has a nontrivial compact subgroup whose normal closure is the entire group.*

Proof. Write the group as a product of simple groups; in each simple group there is a nontrivial compact subgroup; the product of these subgroups is a compact subgroup of the whole Lie group whose normal closure is clearly the entire group. □

Thus, since again the pullback of each vector is the zero function, the map ρ must be trivial on a subgroup whose normal closure is the whole group - a sort of "density" result for compact subgroups in semisimple ones. Hence again the scalar is zero iff we are dealing with the trivial representation.

Finally, we note how this geometric interpretation provides a connection to the proof of Maschke's theorem. The Laplacian, in normal coordinates, can be written

$$(\nabla f)(g) = \lim_{r \to 0^+} \frac{2n}{r^2} \frac{1}{\mu(S^r)} \int_{S^r} f(g+h) - f(g) d\mu(h)$$

In Maschke's theorem, one of the equivalent ways to see the "averaging" approach is to average a projection onto one simple factor over the group to make it equivariant, thus splitting any extension. In the case of projecting an extension by the trivial representation onto another simple representation, up to scalars this yields $1 - \frac{1}{\mu(G)} \int_G g d\mu$. Considering the action of this operator L as a differential operator by left multiplication on a function f, we find that

$$(Lf)(g) = \frac{1}{\mu(G)} \int_G f(g+h) - f(g) d\mu(h)$$

and the local-global "averaging" analogy is clear.