

# Properties of the Bayesian Knowledge Tracing Model

BRETT VAN DE SANDE  
Arizona State University  
bvds@asu.edu

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Bayesian Knowledge Tracing is used very widely to model student learning. It comes in two different forms: The first form is the Bayesian Knowledge Tracing *hidden Markov model* which predicts the probability of correct application of a skill as a function of the number of previous opportunities to apply that skill and the model parameters. We present an analytical solution to this model and show that it is, in fact, a function of three parameters and has the functional form of an exponential. The second form is the Knowledge Tracing *algorithm* which uses student performance at each opportunity to apply a skill to update the conditional probability that the student has learned that skill. We use a fixed point analysis to study solutions of this model and find a range of parameters where it has the desired behavior.

Additional Key Words and Phrases: Bayesian Knowledge Tracing, student modelling, hidden Markov model

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## 1. INTRODUCTION

Since its introduction by Corbett and Anderson [1995], Bayesian Knowledge Tracing (BKT) has been widely applied to studies of student learning and to various tutor systems. In addition, BKT often serves as the starting point for more complicated models of learning, for example [Baker and Aleven 2008; Lee and Brunskill 2012]. Researchers have applied BKT in two distinct ways: The *Hidden Markov Model* (HMM) form and the *Knowledge Tracing Algorithm* form. The purpose of this paper is to investigate solutions of both forms of the BKT model.

The HMM form [Beck and Chang 2007] of BKT predicts the probability that a student will correctly apply a skill when they have an opportunity to apply it. Typically, this model is fit to student performance data using either a residual sum of squares (RSS) test (sometimes described as “curve fitting”) [Corbett and Anderson 1995] or a Maximum Likelihood test [Pardos and Heffernan 2010], with several studies comparing the two approaches [Baker et al. 2011; Chang et al. 2006; Pardos et al. 2011; 2012]. For the RSS test, many authors in these studies use a grid search over the parameter space, or “BKT-BF” [Baker et al. 2011], to perform the fit. Likewise, for the Maximum Likelihood test, the most widely used fitting technique is the Expectation Maximization (EM) algorithm [Dempster et al. 1977; Chang et al. 2006].

A typical HMM must be solved numerically to find its functional form. However, the BKT model is simple enough that it can be solved analytically, which we will do

here. We will see that the model, in functional form, is an exponential. Moreover, we will see that it is, in fact, a three parameter model. As we shall see, this explains the “Identifiability Problem” first noted by Beck and Chang [2007].

Knowledge Tracing Algorithm form [Corbett and Anderson 1995] of BKT is used to determine in real time whether a student has learned a skill. The probability that the student has learned a skill is updated by student performance at each opportunity to apply that skill. The model has four parameters which must be supplied externally. Although, this form of BKT is too complicated to solve analytically, we will use a fixed point analysis to study the behavior of its solutions. Demanding that the model behaves in a reasonable manner leads to significant constraints on the model parameters.

## 2. HIDDEN MARKOV MODEL FORM

The Bayesian Knowledge Trading model [Corbett and Anderson 1995] has four parameters:

- $P(L_0)$  is the initial probability that the student knows a particular skill.
- $P(G)$  is probability of guessing correctly, if the student doesn’t know the skill.
- $P(S)$  is probability of making a slip, if student does know the skill.
- $P(T)$  is probability of learning the skill if the student does not know the skill.

Note that  $P(T)$  is assumed to be constant over time.

We define step  $j$  to be the  $j^{\text{th}}$  opportunity for a student to apply a given skill. Let  $P(L_j)$  be the probability that the student knows the skill at step  $j$ . According to the model,  $P(L_j)$  can be determined from the previous opportunity:

$$P(L_j) = P(L_{j-1}) + P(T) (1 - P(L_{j-1})) . \quad (1)$$

In this model, the probability that the student actually gets opportunity  $j$  correct is:

$$P(C_j) = P(G) (1 - P(L_j)) + (1 - P(S)) P(L_j) . \quad (2)$$

Eqns. (1) and (2) define a hidden Markov model, where  $P(L_j)$  is the “hidden” variable.

This model can be solved exactly. First, we rewrite (1) in a more suggestive form:

$$1 - P(L_j) = (1 - P(T)) (1 - P(L_{j-1})) . \quad (3)$$

One can show that this recursion relation has solutions of the form:

$$1 - P(L_j) = (1 - P(T))^j (1 - P(L_0)) . \quad (4)$$

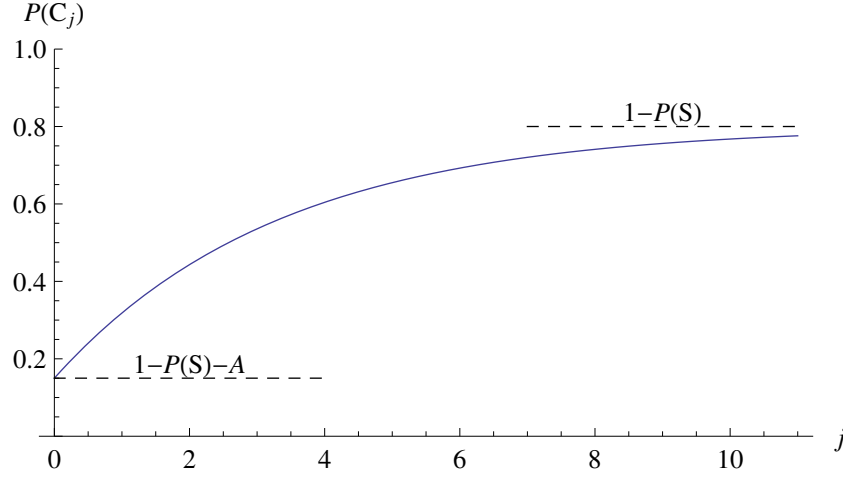


Fig. 1. Solution of the hidden Markov model form of BKT, Eqn. (7).  $P(C_j)$  is the probability of the student getting step  $j$  correct. Note that the solution has the functional form of an exponential.

Substituting (4) into (2), we get:

$$P(C_j) = 1 - P(S) - (1 - P(S) - P(G)) (1 - P(L_0)) (1 - P(T))^j. \quad (5)$$

Note that the form of  $P(C_j)$ , as a function of  $j$ , depends on only *three* parameters:  $P(S)$ ,  $P(T)$ , and  $(1 - P(S) - P(G)) (1 - P(L_0))$ . If we define

$$A = (1 - P(S) - P(G)) (1 - P(L_0)) \quad (6)$$

and  $\beta = -\log(1 - P(T))$ , then we can rewrite (5) in a clearer form:

$$P(C_j) = 1 - P(S) - Ae^{-\beta j}. \quad (7)$$

Thus,  $P(C_j)$  is an exponential; see Fig. 1.

### 3. IDENTIFIABILITY AND PARAMETER CONSTRAINTS

The fact that the Markov model is a function of three parameters was first noticed by Beck and Chang [2007] where they call it the “Identifiability Problem.” In their paper, they noted that multiple combinations of  $P(G)$  and  $P(L_0)$  give exactly the same error rate  $P(C_j)$ , but do not explain the origin of these degenerate solutions. From Eqn. (6), we see that different combinations of  $P(G)$  and  $P(L_0)$  that give the same value for  $A$  will result in models that have the exact same functional form. An example showing the relation between  $P(G)$  and  $P(L_0)$  for a given model is shown in Fig. 2. Thus, we can see that Eqn. (6) provides an explanation of the Identifiability problem.

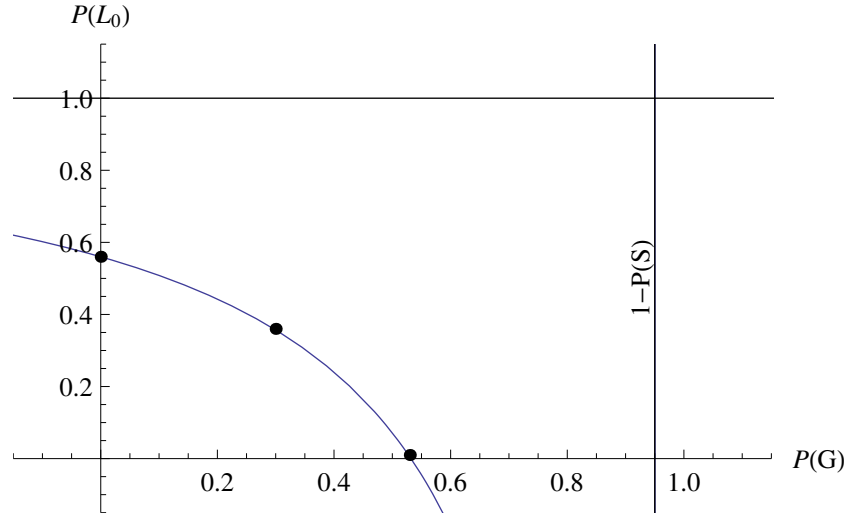


Fig. 2. The relation between the guess rate  $P(G)$  and the initial probability of knowing a skill  $P(L_0)$  for the hidden Markov model form of BKT. All points along the curve correspond to identical models. The curve is a plot of the solutions of Eqn. (6), with  $P(S) = 0.05$  and  $A = 0.416$ . The three points are the three parameter sets listed in Table 1 of [Beck and Chang 2007].

In general, for a model to make sense, all probabilities must lie between zero and one. For the HMM form of BKT, this means that  $0 \leq P(L_0) \leq 1$  and  $0 \leq P(G) \leq 1$ . If we further demand that learning is positive (which may or may not be empirically justified), then we have the constraint  $A > 0$  or  $P(G) + P(S) < 1$ . These constraints on  $P(L_0)$  and  $P(G)$  correspond to the rectangular region shown in Fig. 2. In terms of  $A$ , valid values of  $P(G)$  and  $P(L_0)$  occur when  $0 < A < 1 - P(S)$ ; for negative learning, meaningful values occur when  $-P(S) < A < 0$ . This sets the range of physically meaningful values of  $A$  when fitting to student data.

Note that it is reasonable for the HMM form of the model to find negative learning when fitting to student data. If the student does well on the first opportunities to apply a skill and then poorly on latter opportunities, then the best fit to the student data will be a decreasing function, which corresponds to  $A < 0$ . However, we will see that such model parameters are problematic when used as input for the Knowledge Tracing Algorithm.

#### 4. BAYESIAN KNOWLEDGE TRACING ALGORITHM

In order to predict  $P(L_j)$  for an individual student in real time, the Knowledge Tracing Algorithm [Corbett and Anderson 1995] may be employed. In this case, we find  $P(L_j|O_j)$ , the probability that the student has learned a skill just after

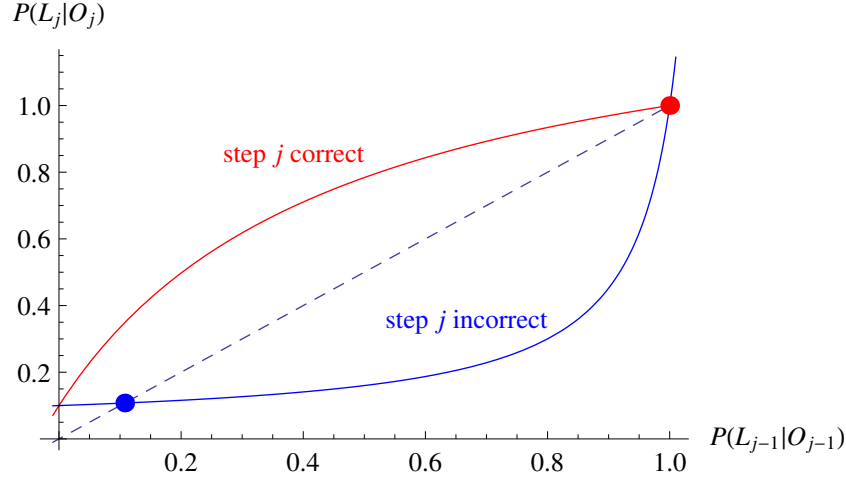


Fig. 3. Plot of the recursion relations for the Knowledge Tracing Algorithm, Eqns. (11) and (12). The stable fixed point for each equation is marked by a dot. The dashed line represents the boundary between increasing and decreasing solutions. Since the “step  $j$  correct” curve is above the dashed line, Eqn. (11) causes the sequence  $\{P(L_j|O_j)\}$  to converge to the fixed point at 1. Likewise, since the “step  $j$  incorrect” curve is below the line, Eqn. (12) causes the sequence  $\{P(L_j|O_j)\}$  to converge to the lower fixed point. The model parameters are  $P(S) = 0.05$ ,  $P(G) = 0.3$ ,  $P(T) = 0.1$ , and  $P(L_0) = 0.36$ .

completing step  $j$  given student performance  $O_j$  on previous steps, where  $O_j = \{o_1, \dots, o_j\}$  is the student performance on the first  $j$  opportunities and  $o_i$  can be either correct or incorrect. This conditional probability obeys the recursion [Baker et al. 2008]:

$$P(L_{j-1}|O_j) = \frac{P(L_{j-1}|O_{j-1})(1 - P(S))}{P(L_{j-1}|O_{j-1})(1 - P(S)) + [1 - P(L_{j-1}|O_{j-1})]P(G)}, \quad o_j = \text{correct} \quad (8)$$

$$P(L_{j-1}|O_j) = \frac{P(L_{j-1}|O_{j-1})P(S)}{P(L_{j-1}|O_{j-1})P(S) + [1 - P(L_{j-1}|O_{j-1})](1 - P(G))}, \quad o_j = \text{incorrect} \quad (9)$$

$$P(L_j|O_j) = P(L_{j-1}|O_j) + [1 - P(L_{j-1}|O_j)]P(T). \quad (10)$$

These equations can be combined to give:

$$P(L_j|O_j) = 1 - \frac{(1 - P(T))[1 - P(L_{j-1}|O_{j-1})]P(G)}{P(G) + (1 - P(S) - P(G))P(L_{j-1}|O_{j-1})}, \quad o_j = \text{correct} \quad (11)$$

$$P(L_j|O_j) = 1 - \frac{(1 - P(T))[1 - P(L_{j-1}|O_{j-1})](1 - P(G))}{1 - P(G) - (1 - P(S) - P(G))P(L_{j-1}|O_{j-1})}, \quad o_j = \text{incorrect} . \quad (12)$$

Their functional forms are plotted in Fig. 3.

While this recursion relation cannot be solved analytically, we can learn much about its solutions by conducting a fixed point analysis. This is a technique that is covered in many differential equations textbooks; for example, see [Blanchard et al. 2006]. The goal of a fixed point analysis is to determine the qualitative behavior of the sequence  $\{P(L_j|O_j)\}_{j=0}^n$  as a function of  $j$ . If  $P(L_j|O_j) > P(L_{j-1}|O_{j-1})$ , then we know that  $P(L_j|O_j)$  increases with  $j$ . Likewise, if  $P(L_j|O_j) < P(L_{j-1}|O_{j-1})$ , then we know that  $P(L_j|O_j)$  decreases with  $j$ . Thus, the boundary between increasing and decreasing solutions is  $P(L_j|O_j) = P(L_{j-1}|O_{j-1})$ , shown as the dashed line in Fig. 3. A *fixed point* is a value of  $P(L_j|O_j)$  such that the recursion relation obeys  $P(L_j|O_j) = P(L_{j-1}|O_{j-1})$ . In the example at hand, there are two kinds of fixed points:

*Stable fixed point.* If  $P(L_j|O_j)$  is near the fixed point, then  $P(L_j|O_j)$  converges to the fixed point as  $j$  increases;

*Unstable fixed point.* If  $P(L_j|O_j)$  is near the fixed point, then  $P(L_j|O_j)$  moves away from the fixed point with increasing  $j$ .

Let us apply these ideas to Eqns. (11) and (12). For Eqn. (11), we find a stable fixed point at 1 and an unstable fixed point at

$$-\frac{P(G)P(T)}{1 - P(G) - P(S)} . \quad (13)$$

Similarly, Eqn. (12) has an unstable fixed point at 1 and a stable fixed point at

$$\frac{(1 - P(G)) P(T)}{1 - P(G) - P(S)} . \quad (14)$$

The two stable fixed points are plotted as dots in Fig. 3.

In order for  $P(L_j|O_j)$  to remain in the interval  $[0, 1]$  for any starting value  $P(L_0) \in [0, 1]$  and any sequence of correct/incorrect steps  $O_j$ , we need the stable fixed point (14) to lie in the interval  $[0, 1]$  and the unstable fixed point (13) to remain negative. This gives us the following constraints on allowed values for the model parameters:

$$P(G) + P(S) < 1 , \quad (15)$$

$$0 < P(T) < 1 - \frac{P(S)}{1 - P(G)} . \quad (16)$$

If we instead choose parameters consistent with negative learning,  $P(S) + P(G) > 1$ , we find that the behavior of  $P(L_j|O_j)$  becomes inverted:  $P(L_j|O_j)$  decreases for correct steps and increases for incorrect steps.

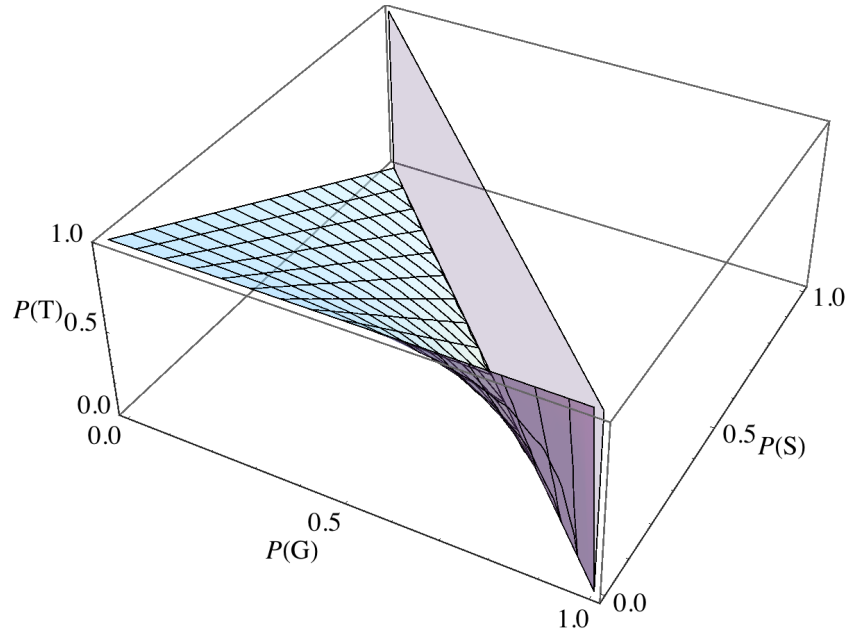


Fig. 4. Plot of the constraint on the learning rate  $P(T)$  as a function of the guess rate  $P(G)$  and the slip rate  $P(S)$  given by Eqn. (16). Valid parameters for the knowledge tracing algorithm lie in the region that is below this surface and above the  $P(T) = 0$  plane. The vertical surface represents the other constraint  $P(G) + P(S) < 1$ , Eqn. (15); valid model parameters lie to the left of this surface.

Baker, Corbett, and Aleven [2008] discuss the need to choose model parameters such that the BKT algorithm behaves in a sensible fashion. In particular, they point out that correct actions by the student should cause the estimate of learning  $P(L_j|O_j)$  to increase. They define models that do not have this property to be “empirically degenerate.” Our constraint (15) identifies precisely which models will be “empirically degenerate.” Note that [Baker et al. 2008] impose the somewhat stronger constraints  $P(S) < 1/2$  and  $P(G) < 1/2$  on grounds that slip and guess rates that are very large are simply nonsensical.

In terms of the parameter space, Eqn. (16) provides a significant constraint on allowed values of model parameters; see Fig. 4. In fact, it completely supersedes the other constraint  $P(G) + P(S) < 1$ .

Finally, the “Identifiability Problem” for the Knowledge Tracing Algorithm does not exist, so long as there are both correct and incorrect steps. This fact can be seen by examining Eqns. (11) and (12) where we see that  $P(G)$  has a different functional form in the numerator of each equation. Thus, any redefinition of  $1 - P(L_j|O_j)$  cannot be compensated for by a redefinition of other parameters in both (11) and

(12). That is, all four model parameters are needed to define the behavior of the Knowledge Tracing Algorithm.

## 5. CONCLUSION

In conclusion, the hidden Markov model form of BKT, when expressed in functional form, is an exponential function with three parameters. Expressing the model in terms of three parameters should improve the speed and stability of any RSS or maximum likelihood fit to student data. In that case, if  $P(G)$  or  $P(L_0)$  is needed as output, then we suggest finding  $A$  by the fitting process, fixing  $P(G)$  or  $P(L_0)$  by some external constraint, and using Eqn. (6) to find the remaining variable. This would be an alternative to the Dirichlet priors [Beck and Chang 2007] approach used by some authors, but is much easier to implement and does not modify the model itself. Typically, Dirichlet priors are applied to all four model parameters, which adds further complexity to the model.

Our other result is that the functional form of the Markov model corresponds to an exponential, rather than a power law. Heathcote, Brown, and Mewhort [2000] argue that learning for individuals is better described by an exponential while (as shown in earlier studies) learning averaged over individuals is better described by a power law function. However, this analysis was performed using mainly reaction time (latency) for tasks that were already learned, while we are interested in the initial acquisition of a skill. A recent study that focused on students learning physics skills found that student learning was better explained by a power law function, even for individual students [Chi et al. 2011]. Together, these results call into question the popular practice of fitting the Markov model form of BKT to student data. Perhaps a model that has power law behavior would work better?

We see that the Knowledge Tracing Algorithm does not suffer from the Identifiability Problem: all four model parameters affect model behavior separately. We also see that, for the model to behave properly, there are significant constraints on the allowed values of  $P(S)$ ,  $P(G)$ , and  $P(T)$  and that parameters consistent with negative learning are not allowed. Violations of these constraints should correspond to the “empirically degenerate” models found in [Baker et al. 2008]. Also, it would be interesting to apply to see how often these constraints are violated in situations where the Knowledge Tracing algorithm is employed, such as in the Cognitive Tutors [Ritter et al. 2007]. A violation of these constraints could lead to incorrect estimates of student learning.

Finally, a fixed point analysis like the one that we conducted here could be applied to more complicated models of learning, such as [Baker and Alevan 2008; Lee and



Brunskill 2012]. This may give us greater insight into the qualitative behaviors of these models as well as constrain model parameters.

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