Lecture Notes for

Algebraic Topology I

Lecturer
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This document will (hopefully) contain lecture notes for the course *Algebraic Topology I* given by Prof. Stefan Schwede at Bonn University during the winter semester 2021/22.

Thanks to Álvaro for the illustrations!

Everything in these notes should be taken with a grain of salt (at least for now), I'm new to the material, to real time TEXing and I tend to be late to class more often than not.

Eventually I plan to make these notes into a nice reference, maybe adding some (well written) solutions to some important exercises or useful comments, so any feedback on how to improve the notes is appreciated! I have a GitHub repository for the notes, you are welcome to use the Issues tab to report any errors or typos, or make any correction/remark/suggestion (or you can just tell me).

When the "Álvaro pls" Signal /\ appears in the margin, it indicates that I would like some picture to be added in that place eventually.

Also a lot of thanks to Paul, Yikai, Zhu, for lending me their notes/photos of the blackboard when I was missing stuff.

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Introduction

Lecture 1
11^{th} Oct, 2021

In the Topology I class given by Prof. Schwede last year, two important homotopy invariant functors were defined:

- The singular homology groups $H_n(X;\mathbb{Z})$. The definition of these groups is quite involved, but they are relatively easy to compute (e.g. by cellular homology). In the case of the spheres we have:

$$\tilde{H}_n(S^k;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$

- The homotopy groups $\pi_n(S^k, *)$. These groups are instead easy to define, but really difficult to compute. In the case of the spheres their calculation becomes complicated already for $n > k$:

$$\pi_n(S^k) = \begin{cases} 0 & \text{if } n < k \\ \mathbb{Z} & \text{if } n = k \\ ??? & \text{if } n > k \end{cases}$$

  As of today (and most likely as of tomorrow too) still a lot is unknown about the higher homotopy groups of the spheres, and those we do know display an apparently erratic behaviour.

Homotopy groups are so hard to compute in general that, as a matter of fact, *there is no non-contractible, simply connected finite CW-complex for which all homotopy groups are known.*

A First Look to Hurewicz Theorem

An important result about homotopy groups is a theorem due to Hurewicz relating the first non-trivial homotopy and homology groups under certain hypotheses:

I.1. Theorem (Hurewicz)**.** — *Let $n \geqslant 2$ and let X be an $(n-1)$-connected based space. Then $H_i(X;A) = 0$ for all $0 < i < n$ and any abelian group A and the Hurewicz map*

$$h : \pi_n(X, x_0) \to H_n(X;\mathbb{Z})$$

is an isomorphism.

Where the **Hurewicz map** is defined in the following way. Let $n \geqslant 1$ and let $c \in H_n(S^n;\mathbb{Z})$ be a generator. For a based space (X, x_0) define

$$h : \pi_n(X, x_0) \to H_n(X;\mathbb{Z}),\ [f : S^n \to X] \mapsto f_*(c)$$

where $f_* : H_n(S^n;\mathbb{Z}) \to H_n(X,\mathbb{Z})$ is the map induced by f on homology groups. I.e. the Hurewicz map h is the evaluation at the fundamental class of S^n.

Proving this theorem will keep us busy for the next few lectures.

Remark. — Choosing the other generator of $H^n(S^n;\mathbb{Z})$, the Hurewicz map changes into its negative which is still an isomorphism, i.e. the map itself slightly depends on the choice of the generator, but the fact that it is an isomorphism does not.

Remark. — Recall: for path connected X, $h : \pi_1(X, x_0) \to H_1(X,\mathbb{Z})$ is surjective with kernel the commutator subgroup, so it factors to an isomorphism $\pi_1(X, x_0)^{\text{ab}} \to H_1(X;\mathbb{Z})$. Hurewicz theorem is a generalization of this fact, whose first proof is due to Poincaré.

We know prove two properties of the Hurewicz map, namely its naturality and the fact that it is actually a group homomorphism.

Naturality of the Hurewicz map. — Let $f : X \to Y$ be a based map between based spaces. Then the following square commutes

$$\begin{array}{ccc} \pi_n(X, x_0) & \xrightarrow{h^X} & H_n(X;\mathbb{Z}) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \pi_n(Y, f(x_0)) & \xrightarrow{h^Y} & \pi_n(Y;\mathbb{Z}) \end{array}$$

Proof. Let $\alpha : S^n \to X$ represent a class in $\pi_n(X, x_0)$. Then

$$f_*(h^X[\alpha]) = f_*(\alpha_*(c)) = (f\alpha)_*(c) = h^Y[f \circ \alpha] = h^Y(f_*[\alpha])$$

□

The Hurewicz map is a group homomorphism. — Let $p : S^n \to S^n \vee S^n$ be a pinch map, i.e. a continuous based map such that both compositions with the projections $S^n \vee S^n \rightrightarrows S^n$ are based-homotopic to the identity. The group structure on $\pi_n(X, x_0)$ (for $n \geqslant 2$) is as follows (thinking of spheres):

The pinch map is the subject of one of the exercises in the first exercise sheet ("The" pinch map, because as we will see it is unique up to homotopy).

$$[f] + [f'] := [(f \vee f') \circ p].$$

It will be an exercise this week to show that if $i_1, i_2 : S^n \to S^n \vee S^n$ are the two summand inclusions the following relations holds:

$$p_*(c) = (i_1)_*(c) + (i_2)_*(c) \text{ in } H_n(S^n \vee S^n;\mathbb{Z})$$

with $c \in H_n(S^n;\mathbb{Z})$ generator. Now we can show that the Hurewicz map is in fact a group homomorphism.

Proof. If $[f], [f'] \in \pi_n(X, x_0)$ we have:

$$h([f]+[f']) = h[(f\vee f')\circ p] = ((f\vee f')\circ p)_*(c) = (f\vee f')_*(p_*(c)) = (f\vee f')_*((i_1)_*(c)+(i_2)_*(c)_*)$$

but since $(f \vee f') \circ i_1 = f$ and $(f \vee f') \circ i_2 = f'$,

$$(f \vee f')_*((i_1)_*(c)) + (f \vee f')_*((i_2)_*(c)_*) = f_*(c) + f'_*(c) = h[f] + h[f']$$

□

We will actually prove a stronger version of the Hurewicz theorem, the relative Hurewicz theorem.

Recall the definition of the relative homotopy groups. We identify I^{n-1} with the subspace of I^n with $x_1 = 0$. Define $J^{n-1} = \partial(I^n) \smallsetminus \mathring{I}n-1$. Then $I^{n-1} \cap Jn-1 = \partial(I^{n-1})$. The **relative homotopy groups** of the triple (X, A, x_0) are defined as triple homotopy classes of triple maps:

$$\pi_n(X, A, x_0) = [(I^n, I^{n-1}, J^{n-1}), (X, A, x_0)].$$

Addition on $\pi_n(X, A, x_0)$ for $n \geqslant 2$ is defined by "juxtaposition and reparametrization in the first coordinate" as follows:

$$[f] + [g] = [f + g],\ (f + g)(t_1, \ldots, t_n) = \begin{cases} f(2t_1, \ldots, t_n) & t_1 \in [0, 1/2] \\ g(2t_1 - 1, \ldots, t_n) & t_1 \in [1/2, 1] \end{cases}$$

this is easily seen to be well defined on homotopy classes.

The **relative Hurewicz map** is defined similarly to the absolute one: with $c \in H_n(I^n, \partial I^n; \mathbb{Z})$ a generator, define

$$h : \pi_n(X, A, x_0) \to H_n(X, A; \mathbb{Z}),\ [f] \mapsto f_*(c)$$

Recall that $\pi_1(A, x_0) = [(I', \partial I'), (A, x_0)]$ acts on $\pi_n(X, A, x_0)$ in a non-trivial fashion. This poses a problem, because for all $[f] \in \pi_n(X, A, x_0)$ and $\omega \in \pi_1(S, x_0)$ the maps representing $[\omega] * [f]$ and $[f]$ are pair-homotopic as maps $(I^n, \partial I^n) \to (X, A)$, hence the relative Hurewicz map takes them to the same class in $H_n(X, A; \mathbb{Z})$.

This leads to the definition of a **modified relative Hurewicz map**. For $n \geqslant 2$ define $\pi_n(X, A, x_0)^\dagger$ to be the quotient of $\pi_n(X, A, x_0)$ by the normal subgroup generated by elements of the form $([\omega] * [f])[f]^{-1}$ for all $[\omega] \in \pi_1(A, x_0)$, $[f] \in \pi_n(X, A, x_0)$. By design the relative Hurewicz map factors through this quotient:

$$\begin{tikzcd} \pi_n(X, A, x_0) \arrow[r, "h"] \arrow[d] & H_n(X, A; \mathbb{Z}) \\ \pi_n(X, A, x_0)^\dagger \arrow[ur, dashed, "h^\dagger"'] & \end{tikzcd}$$

Now we can state the relative Hurewicz theorem.

I.2. Theorem (Hurewicz)**.** — *Let (X, A) be a pair of path connected spaces such that for all $x_0 \in A$, the map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ is an isomorphism. Let $n \geqslant 2$ and suppose that $\pi_i(X, A, x_0) = 0$ for $1 \leqslant i \leqslant n - 1$. Then the modified relative Hurewicz map $h^\dagger$ is an isomorphism.*

Remark. — For $A = \{x_0\}$, the relative version recovers the absolute version.

Remark. — The hypothesis of the relative Hurewicz theorem refers to $\pi_i(X, A, x_0)$ but the conclusion refers to $\pi_n(X, A, x_0)^\dagger$. This makes the relative version not as manageable as the absolute one.

Some Consequences of Hurewicz Theorem

Lecture 2
13^{th} Oct, 2021

Before resuming with the proof of the relative Hurewicz theorem we prove an application of it, a version of Whitehead's theorem which uses homology groups in place of homotopy groups.

I.3. Theorem. — *Let $f : X \to Y$ be a map between simply connected CW-complexes such that $f_* : H_i(X;\mathbb{Z}) \to H_i(Y;\mathbb{Z})$ is an isomorphism for all $i \geqslant 0$. Then f is an homotopy equivalence.*

Proof. By cellular approximation we can assume f cellular. Let $Z(f) = X \times [0,1] \cup_{X\times 1, f} Y$ be the mapping cylinder of f. This inherits a CW-structure such that $X \cong X \times 0$ and Y are subcomplexes. The projection $Z(f) \to Y$ is a homotopy equivalence (fact check this), hence by replacing Y by $Z(f)$ we can assume wlog that $f : X \to Y$ is the inclusion of a subcomplex. Since X and Y are simply-connected the relative Hurewicz theorem applies for all $n \geqslant 2$, but all relative homology groups vanish because f_* is an isomorphism (by the long exact sequence), hence all relative homotopy groups vanish and by Whitehead's theorem we can conclude that f is an homotopy equivalence. □

An elaboration of the previous results leads to the following proposition.

I.4. Proposition. — *Let $f : X \to Y$ be a map of path-connected CW-complexes. The following are equivalent:*

- *(i) f is a homotopy equivalence,*
- *(ii) f induces an isomorphism on fundamental groups and the induced map $\tilde{f} : \tilde{X} \to \tilde{Y}$ on universal covers induces an isomorphism on all integral homology groups.*

Proof. $(i) \implies (ii)$ Since f is an homotopy equivalence, f_* is an isomorphism on all homotopy groups. Then $\tilde{f}$ induces an isomorphism on all homotopy groups, hence it is an homotopy equivalence, thus it induces an isomorphism on all homology groups.

$(ii) \implies (i)$ Since $\tilde{f}$ induces an isomorphism on integral homology groups it is a homotopy equivalence by the version of Whitehead's theorem we just proved, hence it induces an isomorphism on all homotopy groups. This in turn means that f induces an isomorphism on all homotopy groups, i.e. it is a homotopy equivalence. □

Getting Rid of the Basepoint

We now return to the proof of the relative Hurewicz theorem

Recall: the **degree** of a map $f : (D^n, \partial D^n) \to (D^n, \partial D^n)$ is the integer $\deg(f)$ such that $f_*(x) = \deg(f)x$ for all $x \in H_n(D^n, \partial D^n; \mathbb{Z})$.

I.5. Lemma. — *Let $n \geqslant 1$. For $n > 1$ assume known that $\pi_{n-1}(S^{n-1}, z)$ is free abelian of rank 1. Let f be a continuous self map of $(D^n, \partial D^n)$ of degree ± 1. Then f is pair-homotopic to the identity if $\deg(f) = 1$ and to any reflection if $\deg(f) = -1$.*

Proof. We first see the case when $\deg(f) = 1$.

($n = 1$) Since $\partial D^1 = \{\pm 1\}$ and $\deg(f) = 1$ we have $f|_{\partial D^1} = \mathrm{id}_{\partial D^1}$. Then the linear homotopy $H(x, t) = tf(x) + (1 - t)x$ is a relative homotopy between f and the identity id_{D^1}.

($n \geqslant 2$) Consider the commutative square:

$$\begin{array}{ccc} H_n(D^n, \partial D^n; \mathbb{Z}) & \xrightarrow[\cong]{\partial} & H_{n-1}(S^{n-1}; \mathbb{Z}) \\ \Big\downarrow{\scriptstyle f_* = \mathrm{id}} & & \Big\downarrow{\scriptstyle (f|_{S^{n-1}})_* = \mathrm{id}} \\ H_n(D^n, \partial D^n; \mathbb{Z}) & \xrightarrow[\cong]{\partial} & H_{n-1}(S^{n-1}; \mathbb{Z}) \end{array}$$

Since $\pi_{n-1}(S^{n-1}, z)$ is free of rank 1, the Hurewicz map $h : \pi_{n-1}(S^{n-1}, z) \to H_{n-1}(S^{n-1}; \mathbb{Z})$ is an isomorphism. Then $(f|_{\partial D^n})_* : \pi_{n-1}(S^{n-1}, z) \to \pi_{n-1}(S^{n-1}, z)$ is the identity, therefore $f|_{\partial D^n}$ is homotopic to the identity of S^{n-1}. Now let $H : S^{n-1} \times [0, 1] \to S^{n-1}$ be a homotopy. This gives a map $D^n \times 0 \cup S^{n-1} \times [0, 1] \cup D^n \times 1 \xrightarrow{f \cup H \cup \mathrm{id}} D^n$. Since $D^n \times [0, 1]$ can be obtained from $D^n \times 0 \cup S^{n-1} \times [0, 1] \cup D^n \times 1$ by attaching an $(n + 1)$-cell and D^n is contractible, there is a continuous extension $\bar{H} : D^n \times [0, 1] \to D^n$. This is the desired pair homotopy from f to id_{D^n}.

Why do we need D^n contractible? Think of $S^1 \hookrightarrow D^2$

If $\deg(f) = -1$, we let $r : D^n \to D^n$ be the reflection in the first coordinate. Then $\deg(r \circ f) = 1$, hence $r \circ f$ is pair homotopic to id_{D^n} and so $f = r \circ r \circ f$ is pair homotopic to r. □

Now let (X, A) be a based space. Define the group $\pi_n(X, A)^\#$ as the quotient of the free abelian group generated by pair homotopic maps $(I^n, \partial I^n) \to (X, A)$ by the relation $[f] + [f'] = [f + f']$ when the right hand side is defined.

The "forgetful" map $\pi_n(X, A, x_0) \to \pi_n(X, A)^\#$ is a group homomorphism and it factors through a homomorphism $\pi_n(X, A, x_0)^\dagger \to \pi_n(X, A)^\#$ (because $\omega * f$ and f are always pair homotopic).

I.6. Proposition. — *Let (X, A) be a pair of path-connected spaces. Let $n \geqslant 2$ or $n = 1$ and A a point. Then the "forgetful" homomorphism $\pi_n(X, A, x_0)^\dagger \to \pi_n(X, A)^\#$ is an isomorphism.*

Proof. We will define a homomorphism in the opposite direction. Let $f : (I^n, \partial I^n) \to (X, A)$ be a pair map. f need not send J^{n-1} to x_0, but J^{n-1} is contractible and A path-connected. So $f|_{J^{n-1}}$ is homotopic in A to the constant map at the basepoint. Let $H : J^{n-1} \times [0, 1] \to A$ be such a homotopy from $f|_{J^{n-1}}$ to the constant map x_0. The HEP for $(\partial I^n, J^{n-1})$ lets us extend H to a homotopy $H' : \partial I^n \times [0, 1] \to A$ from $f|_{\partial I^n}$ to some map $H'(1, -)$ that sends J^{n-1} to x_0. The HEP for $(I^n, \partial I^n)$ with target space X lets us extend H' to

$H'' : I^n \times [0,1] \to X$ from f to a map that sends J^{n-1} to x_0. Moreover H'' is a pair homotopy of maps $(I^n, \partial I^n) \to (X, A)$.

We now define a map $\Psi : [(I^n, \partial I^n) \to (X, A)] \to \pi_n(X, A, x_0)^\dagger$ by sending $[f]$ to $[H''(-,1)]$. We claim that this is well defined.

Claim. Let $f, f' : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ be triple maps that are pair homotopic as maps $(I^n, \partial I^n) \to (X, A)$. Then they represent the same element in $\pi_n(X, A, x_0)^\dagger$.

Proof of the claim. Let $H : I^n \times [0,1] \to X$ be a pair homotopy from f to f'. We choose a point $z \in J^{n-1}$ and a triple homotopy $K : (I^n, \partial I^n, J^{n-1}) \times [0,1] \to (I^n, \partial I^n, J^{n-1})$ from the identity to a map with $K(J^{n-1} \times 1) = \{z\}$. Formally what we are applying two times the HEP (for $(\partial I^n, J^{n-1})$ and for $(I^n, \partial I^n)$, as before). Then f is triple homotopic to $f \circ K(-,1)$, f' is triple homotopic to $f' \circ K(-,1)$, so that $f \circ K(-,1)$ is pair homotopic to $f' \circ K(-,1)$. In particular $\tilde{H} = H \circ K(-,1)$ satisfies $\tilde{H}(J^{n-1}, t) = H(z, t)$ for all $t \in [0,1]$. Now for all $x \in J^{n-1}$, the loop at x_0 in A, $\tilde{H}(x, -)$, is independent of the point $x \in J^{n-1}$ and it always agrees with $\omega = \tilde{H}(z, -)$. By identifying I^{n+1} as $I^n \times [0,1]$ we can view $\tilde{H}$ as a triple homotopy between $\omega * (f \circ K(-,1))$ and $f' \circ K(-,1)$. In the end we have $[f] = [\omega * (f \circ K(-,1))] = [f' \circ K(-,1)] = [f']$ in $\pi_n(X, A, x_0)^\dagger$ (the second equality holds in $\pi_n(X, A, x_0)^\dagger$ by construction, the third one is because of the homotopy we found).

I'm not *too* sure I really get this...

Corollary of the claim. The map $\Psi : [(I^n, \partial I^n), (X, A)] \to \pi_n(X, A, x_0)^\dagger$ we defined before the claim is well defined, so it has a unique extension on the free abelian group which factors to a homomorphism $\pi_n(X, A)^\# \to \pi_n(X, A, x_0)^\dagger$ which is then an isomorphism by design. $\square$

Punchline: In the situation of the relative Hurewicz theorem it suffices to show that the map $\pi_n(X, A)^\# \to H_n(X, A; \mathbb{Z})$ is an isomorphism (i.e. we don't have to deal with basepoints!).

The Homotopy Addition Theorem

Lecture 3
18th Oct, 2021

This lecture was given by Tobias Lenz, a PhD student of Schwede. I was late to the class, so most of the notes for this lecture are copied from Qi Zhu's notes, thank you Qi Zhu!

Remark. — There's a standing assumption for all of today's lecture: $\pi_k(S^k) \cong \mathbb{Z}$ for all $1 \leqslant k < n$.

The main goal of today's lesson is to prove the following theorem.

I.7. Theorem (Homotopy Addition Theorem)**.** — *Assume we have $f_1, \ldots, f_k : I^n \to I^n$ such that $f_i|_{\mathring{I}^n}$ is an open embedding and the sets $f_i(\mathring{I}^n)$ are pairwise disjoint. Furthermore, let $g : (I^n, \partial I^n) \to (X, A)$ such that $g(I^n \smallsetminus \bigcup_{i=1}^k f_i(\mathring{I}^n)) \subset A$. Then*

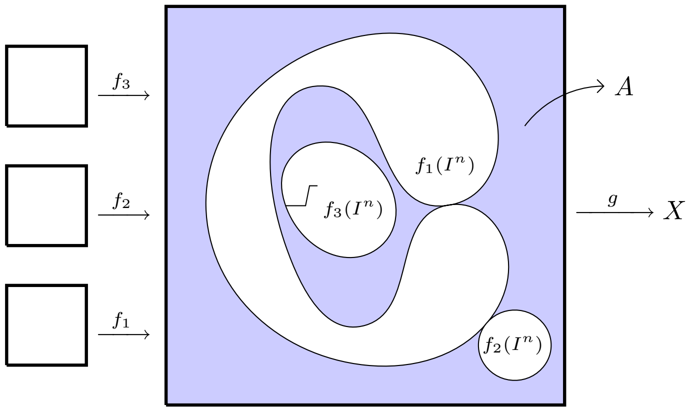
$$[g] = \sum_{i=1}^k (\deg f_i)[g \circ f_i]$$

in $\pi_n(I^n, \partial I^n)^\#$.

Remark. — Note that we have $f_i(\partial I^n) \cap f_j(\mathring{I}^n) = \varnothing$ for all i, j. This will play a (small) role later in the lecture.

Remark. — Two remarks about the theorem:

- When homotopy theory was new, people thought this was obvious and didn't feel the need for a proof until Eilenberg suggested so.
- Tobias: *"I'm not sure if I can finish the proof today but I was promised an award if I do!"*



The strategy to prove this theorem is inductive: we want to reduce the problem to the case $k = 1$ which is easy.

Definition. — Let $f : \mathring{I}^n \to \mathring{I}^n$ be an open embedding, $p \in \mathring{I}^n$. We have that f induces a commutative diagram:

$$\begin{tikzcd}
H_n(\mathring{I}^n, \mathring{I}^n \setminus \{p\}) \arrow[r, "i_*"] \arrow[d, "f_*"'] \arrow[dr, "f_*"] & H_n(I^n, I^n \setminus \{p\}) & H_n(I^n, \partial I^n) \arrow[l, "i_*"'] \arrow[d, dashed, "d-"] \\
H_n(f(\mathring{I}^n), f(\mathring{I}^n) \setminus \{f(p)\}) \arrow[r, "i_*"'] & H_n(f(I^n), f(I^n) \setminus \{f(p)\}) & H_n(I^n, \partial I^n) \arrow[l, "i_*"]
\end{tikzcd}$$

where the maps are all isomorphisms by homotopies and excision, hence they induce the dashed arrow. This is an automorphism of $H_n(I^n, \partial I^n) \cong \mathbb{Z}$ and thus $d = \pm 1$. One can show this is independent of p, hence we call it the local degree of f, and write $\deg(f) = d$.

I.8. Lemma. — *For $1 \leqslant i \leqslant k$, let $\mathcal{U}_i \subset f_i(\mathring{I}^n)$ be any non-empty open set. Then g is homotopic relative to $I^n \setminus f(\mathring{I}^n)$ to a map that sends $f(I^n) \setminus \mathcal{U}_i$ to A.*

Remark. — If $[g] = [g']$ in $\pi_n(X, A)^{\#}$ then $[g \circ f_j] = [g' \circ f_j]$ for all $1 \leqslant j \leqslant k$.

Proof. There is a homotopy relative ∂I^n from the identity to a map that sends everything outside of $\mathring{Q}$ to ∂I^n, where Q is a cube inside $f_i^{-1}(\mathcal{U}_i)$ (basically we take a cube Q inside $f_i^{-1}(\mathcal{U}_i)$ and we blow it to the big cube ∂I^n containing $f_i^{-1}(\mathcal{U}_i)$). Composing with gf_i yields a homotopy H of maps of pairs $(I^n, \partial I^n) \to (X, A)$ from gf_i to a map that sends everything outside $\mathring{Q}$ to A. We have a map of sets:

$$H' : I^n \times I \to X, \quad H'(x, t) = \begin{cases} H(f^{-1}(x), t) & x \in f_i(I^n) \\ g(x) & x \notin f_i(\mathring{I}^n) \end{cases}$$

which we can show is well defined. Let $x \in f_i(\partial I^n)$, with preimage y, then

It suffices to check this because $f_i(\partial I^n) \cap f_j(\mathring{I}^n)$ is empty for all i, j.

$$g(x) = H(y,t) = H(y,0) = gf_i(y)$$

so that it is well defined as a map of sets. Then $H'(x,0) = g(x)$ and $H'(x,1) \in A$ for $x \notin f(\mathring{Q})$, in particular $H'(x,1) \in A$ for $x \notin \mathcal{U}_i$. It remains to prove that H' is continuous.

Claim. We have a pushout

$$\begin{tikzcd}
\partial I^n \times I \arrow[r, "f_i|_{\partial I^n} \times \mathrm{id}"] \arrow[d, "i"] & (I^n \smallsetminus f_i(\mathring{I}^n)) \times I \arrow[d] \\
I^n \times I \arrow[r] & (I^n \times I) \amalg_{\partial I^n \times I^n} ((I^n \smallsetminus f_i(\mathring{I}^n)) \times I)
\end{tikzcd}$$

Then,

$$(I^n \times I) \amalg_{\partial I^n \times I^n} ((I^n \smallsetminus f_i(\mathring{I}^n)) \times I) \xrightarrow{(f_i \times \mathrm{id}, i)} I^n \times I$$

is a homeomorphism.

Proof of the claim. Well-definedness and continuity of the function follow from the universal property of the pushout. One can check that it is is bijective by a direct computation. Hence we have a continuous bijection from a quasi-compact space to an Hausdorff space, which is then a homeomorphism. ■

Thus to show that H' is continuous, it suffices to show that the maps $H' \circ (f_i \times I)$ and $H'|_{(I^n \smallsetminus f(\mathring{I}^n)) \times I}$ are continuous, which follows from the construction. □

Not sure how this works, to be honest

Proof of the homotopy addition theorem. Induction on k.

($k = 1$) Take a cube $Z_1 \subset f_i(\mathring{I}^n)$. By the lemma we may assume that $g(I^n \smallsetminus \mathring{Z}_1) \subset A$. Our goal is to construct some $f_1' : (I^n, \partial I^n) \to (I^n, \partial I^n)$ with $[gf_1] = [gf_1']$. There exists a homotopy Q from id_{I^n} to a map p such that:

The homotopy Q is the same kind of "pushing out" homotopy that we already considered in the proof of the lemma.

- $p|_{Z_1} = \mathrm{id}_{Z_1}$
- $p(I^n \smallsetminus \mathring{Z}_1) \subset \partial I^n$
- $Q((I^n \smallsetminus \mathring{Z}_1) \times I) \subset I^n \smallsetminus \mathring{Z}_1$

Now, Qf_1 is a homotopy of maps of pairs $(I^n, \partial I^n) \to (I^n, I^n \smallsetminus \mathring{Z}_1)$. Then gQf_1 will be a homotopy of maps of pairs $(I^n, \partial I^n) \to (X, A)$, hence $[gpf_1] = [gf_1]$ in $\pi_n(X,A)^{\#}$. Then $f_1' := pf_1$ is a map of pairs $(I^n, \partial I^n) \to (I^n, \partial I^n)$ and $f_1 \simeq f_1'$ as maps of pairs $(I^n, \partial I^n) \to (I^n, I^n \smallsetminus \mathring{Z}_1)$.

We are dealing with two different notions of degree, the usual one for f_1' and the local degree for f_1.

Claim. $\deg f_1$ equals the degree of f_1' as a map $(I^n, \partial I^n) \to (I^n, \partial I^n)$.

Proof of the claim. Consider the diagram:

$$\begin{tikzcd}
H_n(\mathring{I}^n, \mathring{I}^n \smallsetminus \{x\}) \arrow[r, "i_*"] \arrow[d, "(f_1)_*"'] & H_n(I^n, I^n \smallsetminus \{x\}) \arrow[d, "(f_1)_*"] & H_n(I^n, \partial I^n) \arrow[l, "i_*"'] \arrow[d, shift right, "(f_1)_*"'] \arrow[d, shift left, "(f_1')_*"] \arrow[dr, "(f_1')_*"] & \\
H_n(\mathring{I}^n, \mathring{I}^n \smallsetminus \{f_1(x)\}) \arrow[r, "i_*"] & H_n(I^n, I^n \smallsetminus \{f_1(x)\}) & H_n(I^n, I^n \smallsetminus \mathring{Z}_1) \arrow[l, "i_*"'] & H_n(I^n, \partial I^n) \arrow[l, "i_*"]
\end{tikzcd}$$

we have that the parallel arrows agree, which proves the claim. ∎

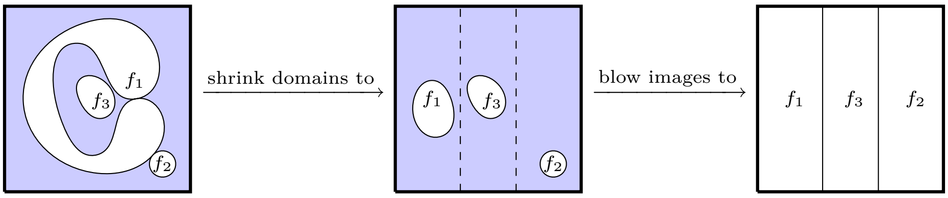
Now, if $\deg f_1 = 1$, then $f_1' \simeq \mathrm{id}$ as maps of pairs, hence $(\deg f_1)[gf_1] = [gf_1'] = [g]$.

If instead $\deg f_1 = -1$, then $f_1' \sim r$ where r is reflection in the first coordinate by lemma I.5, hence $[g] = -[gr] = -[gf_1'] = -[gf_1] = (\deg f_1)[gf_1]$.

To be continued...

Lecture 4
25th Oct, 2021

$(k \geqslant 2)$ Set $u_i = f_i(\text{center of } I^n) \in I^n$. Assume without loss of generality that the first coordinates of $u_1, \dots, u_k$ are not all equal (if some of them are, we can "wiggle" f_1).



$\rightsquigarrow [g] = \sum [g \circ f_i]$.

Choose $t \in (0,1)$ such that

- t is different from the first coordinates of $u_1, \dots, u_k$,
- for some $1 \leqslant i \leqslant k$, the first coordinate of u_i is smaller than t,
- for some $1 \leqslant i \leqslant k$, the first coordinate of u_i is larger than t.

Choose neighborhoods $\mathcal{U}_i$ of u_i inside $f_i(\mathring{I}^n)$ that do not intersect $\{t\} \times I^{n-1}$, that is such that $\mathcal{U}_i$ lies "on the same side respect to t" as u_i.

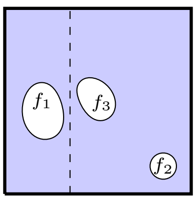
Choose subcubes Q_i inside $\mathring{I}^n$ that contain the center and lie in in $f^{-1}(\mathcal{U}_i)$.

Last time we proved that g is pair-homotopic to some $g' : (I^n, \partial I^n) \to (X, A)$ such that

$$g'(I^n \smallsetminus \bigcup_{i=1,\dots,k} f_i(Q_i)) \subset A \quad \text{and} \quad [g \circ f_i] = [g' \circ f_i] \text{ in } \pi_n(X, A)^{\#}.$$

We precompose each $f_i : I^n \to I^n$ with the linear shrinking homotopy relative Q_i. Set $f_i' = f_i \circ$ end of shrinking. Then $g' \circ f_i$ is pair homotopic to $g' \circ f_i'$ and $f_i'(I^n) \subset \mathcal{U}_i$.

By replacing g by g' and f_i by f_i' we can therefore assume without loss of generality that $f_i(\mathring{I}^n)$ lies on one side of $\{t\} \times I^{n-1}$.



Write $g = g_1 +_t g_2$ by "cutting along $\{x_1 = t\}$".

Formally, $g_1(x_1, \dots, x_n) = g(tx_1, \dots, x_n)$ and $g_2 = g((1-t)x_1 + t, \dots, x_n)$.

Set $I_1 = \{i \in I \mid u_i \text{ lies left of } \{x_1 = t\}\}$ and $I_2 = \{i \in I \mid u_i \text{ lies right of } \{x_1 = t\}\}$, where $I = \{1, \ldots, k\}$.

Then by the inductive hypothesis:

$$[g] = [g_1] + [g_2] = \sum_{i \in I_1} \deg(f_i)[g_1 \circ f_i] + \sum_{i \in I_2} \deg(f_i)[g_2 \circ f_i] = \sum_{i \in I} \deg(f_i)[g \circ f_i].$$

□

My First Non-Trivial Homotopy Group

I.9. Theorem. — *Let $n \geqslant 2$ and assume the HAT in dimension n. Then $\pi_n(S^n, *)$ is infinite cyclic.*

Proof. Choose some point $z \in S^n$. Set $U = S^n \smallsetminus \{-z\}$. Then

$$\pi_n(S^n, z) = \pi_n(S^n, \{z\}, z) \underset{U \cong *}{\cong} \pi_n(S^n, U, z) \underset{\pi_1(U,z) = \{1\}}{\cong} \pi_n(S^n, U, z)^\dagger \cong \pi_n(S^n, U)^\#$$

So we may show that $\pi_n(S^n, U)^\# \cong \mathbb{Z}$.

We show that $\pi_n(S^n, U)^\#$ is generated by the class of any pair map $\psi : (I^n, \partial I^n) \to (S^n, U)$ such that $\psi(\partial I^n) = \{z\}$ and ψ factors out a homeomorphism $I^n / \partial I^n \cong S^n$.

Let $f : (I^n, \partial I^n) \to (S^n, U)$ be any pair map. Set $V = S^n \smallsetminus \{z\}$ so that $S^n = U \cup V$ is an open cover.

The Lebesgue Number lemma provides an $m \geqslant 1$ so that each subcube of I^n of side length $1/m$ is mapped by f into U or into V. Decompose I^n into m^n subcubes of side length $1/m$.

We define subspaces of I^n in this way:

- $A_{-1} = \partial I^n$
- $A_0 = A_{-1} \cup$ vertices
- $A_1 = A_0 \cup$ edges
- $A_2 = \cdots$
- $A_n = I^n$

We want to "improve" f successively by pair homotopies to maps $f = f_{-1}, f_0, f_1, \ldots, f_{n-1}$ such that:

- each f_j is homotopic to f relative ∂I^n,
- each f_j is admissible, i.e. it sends every subcube of side length $1/m$ to U or to V,
- $f_j(A_j) \subset U$ for $j = -1, 0, 1, \ldots, n-1$.

We proceed by induction on j. There is nothing to show for $j = -1$. Let now $j \geqslant 0$. We first modify f_{j-1} and the faces of the j-cube. If such a face Q is "good", i.e. sent by f_{j-1} into U, we do not do anything to Q. Otherwise the $(j-1)$-subcube is mapped to V and the restriction of f_{j-1} to it is a pair map $(Q, \partial Q) \to (V, V \cap U)$.

Claim. For $j < n$, any pair map $(I^j, \partial I^j) \to (V, U \cap V)$ is homotopic relative ∂I^j to a map with image in $U \cap V$.

Proof of the claim. By stereographic projection $(U, U \cap V)$ is pair homotopic to $(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{0\})$, i.e. we can costruct a pair map $g : (I^j, \partial I^j) \to (\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{0\})$. Because ∂I^j is compact and $0 \notin f(\partial I^j)$, there is an $\varepsilon > 0$ such that $g(\partial I^j) \cap (\varepsilon\text{-ball around } 0) = \varnothing$.

So $g : (I^j, \partial I^j) \to (\mathbb{R}^n, \mathbb{R}^n \smallsetminus \mathring{B}(\varepsilon, 0))$. Now, $\mathbb{R}^n$ can be obtained from $\mathbb{R}^n \smallsetminus \mathring{B}(\varepsilon, 0)$ by attaching an n-cell. The cellular approximation theorem and the fact that I^j is a j-dimensional CW-complex gives us a relative homotopy from g to a cellular map. Since $j < n$, the cellular map has image in $\mathbb{R}^n \smallsetminus \mathring{B}(\varepsilon, 0) \subset \mathbb{R}^n \smallsetminus \{0\}$. ■

We can now change $f_{j-1}|A_j$ into $f_j|A_j$ by a homotopy relative A_{j-1} into a map that sends all j-cells to U.

We use the HEP for (I^n, A_j) to extend f_j to all of I^n; we use the HEP with target U or with target V to ensure that the map f_j is again admissible.

After this inductive construction we can replace f by f_{n-1} and we have arranged without loss of generality that $f(A_{n-1}) \subset U$.

We can now assume that $g : (I^n, \partial I^n) \to (S^n, U)$ satisfies $g(A_{n-1}) \subset U$ and each top-dimensional subcube is mapped to U or to V.

We apply the HAT to this map g with $f_1, \dots, f_k$ the reparametrization of those subcubes that are *not* mapped into U (and hence into V).

Then by the HAT we have:

$$[g] = \sum_i \pm[g \circ f_i] \text{ in } \pi_n(S^n, U)^\# \cong \pi_n(S^n, \{z\}, z)^\dagger \cong \pi_n(S, z)$$

We have gained that each summand on the right hand side is in the image of the homomorphism $\pi_n(V, V \cap U)^\# \to \pi_n(S^n, U)^\#$, which is then surjective. By the long exact homotopy group sequence of the pair $(V, V \cap U)$, we obtain:

$$\pi_n(V, V \cap U, z) = \pi_n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{0\}, z) \cong \pi_{n-1}(\mathbb{R}^{n-1} \smallsetminus \{0\}, z) \cong \pi_{n-1}(S^{n-1}, z) \cong \mathbb{Z}$$

so $\pi_n(S^n, U)^\#$ is infinite cyclic.

□

Reminder on Simplicial Sets

We denote by Δ the category with objects the sets $[n] = \{0, 1, \dots, n\}$, $n \geqslant 0$ and morphisms the weakly monotone maps.

A **simplicial set** is a contravariant functor from Δ to sets, $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$. The set of the n-simplices is denoted $X_n = X([n])$, for a morphism $\alpha : [n] \to [m]$ write $\alpha^* = X(\alpha) : X_m \to X_n$.

The **singular complex** (singular simplicial set) of a space Y is the simplicial set

$$\mathcal{S}(Y) = \{\text{all continuous maps } f : \nabla^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geqslant 0, \ \sum x_i = 1\} \to Y\}.$$

For $\alpha : [n] \to [m]$, the map $\alpha^* : \mathcal{S}(Y)_m \to \mathcal{S}(Y)_n$ is precomposition with the affine linear map $\alpha_* : \nabla^n \to \nabla^m$, $e_i \mapsto e_{\alpha(i)}$.

For a continuous map $\psi : Y \to Z$, a morphism of simplicial sets $\psi_* = \mathcal{S}(f) : \mathcal{S}(Y) \to \mathcal{S}(Z)$ is given by $\mathcal{S}(\psi)_n(f) = \psi \circ f$. This yelds a functor $\mathcal{S} : \mathrm{Top} \to \mathrm{sSet}$.

The **geometric realization** is a functor $|-| : \mathrm{sSet} \to \mathrm{Top}$ defined as follows. For a simplicial set X, $|X| = (\coprod X_n \times \nabla^n)/\sim$, where X_n is endowed with the discrete topology and the equivalence relation is the one generated by:

The equivalence relation is only *generated* by the condition stated, which is not (what?). The actual equivalence relation is not easy to understand, apparently.

$$X_m \times \nabla^m \ni (x, \alpha_*(t)) \sim (\alpha^*(x), t) \in X_n \times \nabla^n \quad \text{for all } \alpha : [n] \to [m],\ x \in X_m,\ t \in \nabla^m.$$

Lecture 5
27th Oct, 2021

Given two simplicial sets X and Y, their product $X \times Y$ is the functor

$$\Delta^{\mathrm{op}} \xrightarrow{(X,Y)} \mathrm{Set} \times \mathrm{Set} \xrightarrow{\times} \mathrm{Set}$$

i.e. $(X \times Y)_n = X_n \times Y_n$ and $\alpha^*_{X\times Y} = \alpha^*_X \times \alpha^*_Y$.

The **simplicial n-simplex** is the represented simplicial set $\Delta[n] := \Delta(-, [n])$. By the Yoneda lemma, for every simplicial set X, the map $\mathrm{sSet}(\Delta[n], X) \to X_n$, $(f : \Delta[n] \to X) \mapsto f_n(\mathrm{id}_{[n]})$, is bijective.

A **homotopy between morphisms** $f, g : X \to Y$ in sSet is a morphism $H : X \times \Delta[1] \to Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$, where $i_0, i_1 : X \to X \times \Delta[1]$ (note: the morphism i_j has components $(i_j)_n : X_n \to X_n \times \Delta([n],[1])$, $x \mapsto (x, \mathrm{cost}_j)$).

The topological simplex $\nabla^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geqslant 0,\ \sum x_i = 1\}$ has a preferred CW-structure with $\mathrm{sk}_k(\nabla^n) = \{(x_0, \dots, x_n) \in \nabla^n \mid \text{at most } k+1 \text{ cordinates are non-zero}\}$, i.e. $\mathrm{sk}_k(\nabla^n)$ is the union of all k-dimensional faces of ∇^n:

- $\mathrm{sk}_0(\nabla^n) = \{e_0, \dots, e_n\}$,
- $\mathrm{sk}_j = \dots$
- $\mathrm{sk}_{n-1}(\nabla^n) = \partial\nabla^n$,
- $\mathrm{sk}_n(\nabla^n) = \nabla^n$.

Let (X, A) be a space pair and $k \geqslant -1$. We define a **simplicial subset** $\mathcal{S}(X, A, k)$ of $\mathcal{S}(X)$ by setting

$$S(X, A, k)_n = \{f : \nabla^n \to X \mid f(\mathrm{sk}_k(\nabla^n)) \subset A\}.$$

This is indeed a simplicial subset because the affine linear maps $\alpha_* : \nabla^n \to \nabla^m$ are cellular.

Note that we have:

$$\mathcal{S}(X) = \mathcal{S}(X, A, -1) \supset \mathcal{S}(X, A, 0) \supset \mathcal{S}(X, A, 1) \supset \cdots \supset \bigcap_{k \geqslant 1} \mathcal{S}(X, A, k) = \mathcal{S}(A).$$

A space pair (X, A) is **k-connected**, $h \geqslant 0$, if the following equivalent conditions hold:

(a) The inclusion $A \hookrightarrow X$ is a bijection on π_i for all $i < k$ and all basepoints in A and a surjection on π_k,

(b) Every pair map $(D^n, \partial D^n) \to (X, A)$ for $0 \leqslant n \leqslant k$ is homotopic relative ∂D^n to a map with image in A,

(c) The map $\pi_0(A) \to \pi_0(X)$ is surjective and $\pi_i(X, A, x) = \{*\}$ for all $1 \leqslant i \leqslant k$ and all $x \in A$.

We want to prove the following theorem.

Theorem. — *Let (X, A) be a k-connected pair of spaces. The inclusion $\mathcal{S}(X, A, k) \hookrightarrow \mathcal{S}(X)$ is then a deformation retraction of simplicial sets.*

This means there is a simplicial homotopy

$$H : S(X) \times \Delta[1] \to S(X)$$

from the identity to a morphism with image in $\mathcal{S}(X, A, k)$ that is relative to $\mathcal{S}(X, A, k)$, i.e. the restriction of H to $S(X, A, k)$ is the composite

$$\mathcal{S}(X, A, k) \times \Delta[1] \xrightarrow{\text{proj}} \mathcal{S}(X, A, k) \xhookrightarrow{\text{incl}} \mathcal{S}(X).$$

We first prove a proposition.

Let X be a simplicial set and $x \in X_n$, for $n \geqslant 0$. Then the n-simplex x is **degenerate** if there is a surjective morphism $\sigma : [n] \to [k]$ with $k < n$, and $y \in X_k$ such that $X = \sigma^*(y)$ (i.e. $x \in \operatorname{im}(\sigma^* : X_k \to X_n)$).

I.10. Proposition. — *Let X be a simplicial set and $x \in X_n$. Then there is a unique pair (σ, y) consisting of:*

- *a surjective morphism $\sigma : [n] \to [k]$ and*
- *a non-degenerate simplex $y \in X_k$*

such that $X = \sigma^*(y)$.

Proof.

Existence. By induction on n. If $n = 0$ then X is non-degenerate and $(\mathrm{id}_{[0]}, x)$ does the job.

For $n \geqslant 1$: if x is non-degenerate, then $(\mathrm{id}_{[n]}, x)$ does the job. Otherwise $x = \sigma^*(x')$ for some $\sigma : [n] \to [k]$, $k < n$, $x' \in X_k$. Then $x' = (\sigma')^*(y)$ for some surjective morphism $\sigma' : [k] \to [l]$ and $y \in X_l$ non-degenerate, by induction. Then

$$x = \sigma^*(x') = \sigma^*((\sigma')^*(y)) = (\sigma' \circ \sigma)^*(y)$$

which is the desired expression.

Uniqueness. Let $x = \sigma^*(y) = \bar{\sigma}^*(\bar{y})$ for surjective morphisms $\sigma : [n] \twoheadrightarrow [k]$, $\bar{\sigma} : [n] \twoheadrightarrow [l]$ and $y \in X_k$, $\bar{y} \in X_l$ non-degenerate.

Let $\delta : [k] \to [n]$ be a morphism in Δ such that $\sigma \circ \delta = \mathrm{id}_{[k]}$. Then

$$y = (\sigma\delta)^*(y) = \delta^*(\sigma^*(y)) = \delta^*(\bar{\sigma}^*(\bar{y})) = (\bar{\sigma}\delta)^*(\bar{y})$$

We write

$$\bar{\sigma}\delta = \delta'\sigma' : [k] \to [l]$$

where $\sigma' : [k] \twoheadrightarrow [a]$ is surjective and $\delta' : [a] \hookrightarrow [l]$ is injective. Then

$$y = (\bar{\sigma}\delta)(\bar{y}) = (\delta'\sigma')^*(\bar{y}) = (\sigma')^*((\delta')^*(\bar{y})) \tag{$*$}$$

Since y is non-degenerate, we must have $a = k$ and $\sigma' = \mathrm{id}_{[k]}$. Hence $k = a \leqslant l$. By interchanging the roles of (σ, y) and $(\bar{\sigma}, \bar{y})$ we obtain $l \leqslant k$, hence $l = k$.

Then by $(*)$ we have $y = (\delta')^*(\bar{y})$ so $\delta' = \mathrm{id}$ and hence $y = \bar{y}$ and $\sigma = \bar{\sigma}$. □

I.11. Theorem. — *Let (X, A) be k-connected. Then $S(X, A, k) \hookrightarrow S(X)$ is a simplicial deformation retraction.*

Proof. We will construct the following data. Continuous maps $\psi_f : \nabla^n \times \nabla^1 \to X$ for all $f : \nabla^n \to X$, $n \geqslant 0$, such that:

(a) $\psi_f(-, e_0) = f$, $\psi_f(\mathrm{sk}_k(\nabla^n), e_1) \subset A$.

(b) If $f(\mathrm{sk}_k(\nabla^n)) \subset A$, then $\psi_f = f \circ \mathrm{pr}_1$.

(c) The maps ψ_f are compatible in the simplicial direction, i.e. for all $\alpha : [n] \to [m]$, $g : \nabla^m \to X$, the following commutes:

$$\begin{tikzcd} \nabla^n \times \nabla^1 \arrow[r, "\psi_{\alpha^*(g)}"] \arrow[d, "\alpha_* \times \mathrm{id}"'] & X \\ \nabla^m \times \nabla^1 \arrow[ur, "\psi_g"'] & \end{tikzcd}$$

Construction of the ψ_f's. By induction on $n \geqslant 0$.

$(n = 0)$ The map $f : \nabla^0 \to X$ is determined by its image $f(e_0) \in X$. Since $\pi_0(A) \to \pi_0(X)$ is surjective, we can choose a path from $f(e_0)$ to some point in A. We view the path as a continuous map $\psi_f : \nabla^0 \times \nabla^1 \to X$ with $\psi_f(e_0, e_0) = f(e_0)$, $\psi_f(e_0, e_1) \in A$. If $f(e_0) \in A$, we take the constant path at $f(e_0)$.

$(n \geqslant 1)$ We distinguish three cases:

Case 1. $f : \nabla^n \to X_n$ is degenerate as a simplex of the simplicial set $\mathcal{S}(X)$. Then $f = \sigma^*(g)$ for a unique pair (σ, g) with $\sigma : [n] \twoheadrightarrow [k]$, $k < n$ and $g : \nabla^k \to X$ continuous. By (c) we have to define ψ_f as the composite

$$\nabla^n \times \nabla^1 \xrightarrow{\sigma_* \times \mathrm{id}} \nabla^k \times \nabla^1 \xrightarrow{\psi_g} X$$

Case 2. $f : \nabla^n \to X$ is non-degenerate and $k < n$. We note that by property (c), the map $\psi_f : \nabla^n \times \nabla^1 \to X$ is already fixed on $(\partial\nabla^n) \times \nabla^1$; by (a) it is also determined on $\nabla^n \times e_0$.

We extend the data to $\nabla^n \times \nabla^1$ by a choice of continuous retraction:

$$r : \nabla^n \times \nabla^1 \to (\nabla^n \times e_0) \cup (\partial\nabla^n \times \nabla^1)$$

hence set ψ_f to be the composite of r and $\tilde{f} = f \cup \bigcup_{i=0,\ldots,n} \psi_{d_i^*(f)}$:

$d_i : [n-1] \to [n]$ is the unique monotone injection with $i \notin \mathrm{im}(d_i)$).

$$\nabla^n \times \nabla^1 \xrightarrow{r} (\nabla^n \times e_0) \cup (\partial\nabla^n \times \nabla^1) \xrightarrow{\tilde{f}} X$$

The definition satisfies $\psi_f(-, e_0) = f$ by design and $\psi_f(\mathrm{sk}_k(\nabla^n), e_1) \subset \psi_f(\partial\nabla^n, e_1) \subset A$ by induction because $\psi_{d_i^*(f)}$ have property (a).

Case 3. $f : \nabla^n \to X$ is non-degenerate and $n \leqslant k$. First, note that we can show the pair $((\nabla^n \times e_0) \cup (\partial\nabla^n \times \nabla^1), \partial\nabla^n \times e_1)$ to be pair homeomorphic to $(D^n, \partial D^n)$.

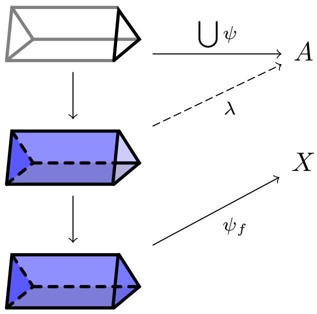
We assumed that (X, A) is k-connected, so there are a continuous map λ and a homotopy from λ to the map $\tilde{f} = f \cup \bigcup_{i=0,\ldots,n} \psi_{d_i^*(f)}$

$$\lambda : (\nabla^n \times e_0) \cup (\partial\nabla^n \times \nabla^1) \to A$$

$$H : (\nabla^n \times e_0) \cup (\partial\nabla^n \times \nabla^1) \times [0,1] \to X$$

which combine in the diagram (where the lower triangle is commutative up to homotopy):

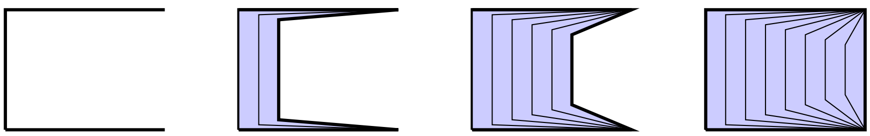
$$\begin{tikzcd}
(\partial\nabla^n \times e_1) \arrow[r, "\bigcup_{i=0,\dots,n} \psi_{d_i^*(f)}"] \arrow[d, hook] & A \arrow[d, hook] \\
(\nabla^n \times e_0) \cup (\partial\nabla^n \times \nabla^1) \arrow[r, "\tilde{f}"'] \arrow[ur, dashed, "\lambda"'] & X
\end{tikzcd}$$



We reparametrize the relative homotopy into the desired map ψ_f as follows:

$$\begin{tikzcd}
(\nabla^n \times e_0) \cup (\partial\nabla^n \times \nabla^1) \times [0,1] \arrow[r, "H"] \arrow[d] & X \\
\nabla^n \times \nabla^1 \arrow[ur, dashed, "\psi_f"']
\end{tikzcd}$$

considering the continuous quotient map.
The following picture illustrates the reparametrization for the $n = 1$ case.



Now we "adjoin" the continuous maps ψ_f into the simplicial deformation retraction

$$H : S(X) \times \Delta[1] \to S(X),$$

that is, in the simplicial dimension n, we need to specify a map

$$S(X)_n \times \Delta([n],[1]) \to S(X)_n.$$

We do this via an adjunction bijection:

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{sSet}}(Z\times\Delta[1],S(X))\cong\\&\qquad\cong\{\psi_f:\nabla^n\times\nabla^1\to X \text{ for all } n\geqslant 0, z\in Z_n, \text{ such that } \psi_{\alpha^*(z)}=\psi_Z(\alpha_*\circ\nabla^1)\}\end{aligned}$$

Sketch (full argument as an exercise).

Step 1. Given $H:Z\times\Delta[1]\to Y$ and $z\in Z_n$, we define $\psi_z:\Delta[n]\times\Delta[1]\to Y$ by $(\psi_z)_m(\alpha,k)=H_m(\alpha^*(z),k)$ (this is bijective by the Yoneda lemma).

Step 2. $\mathcal{S}:\mathrm{Top}\to\mathrm{sSet}$ is right adjoint to $|-|:\mathrm{sSet}\to\mathrm{Top}$ and there is a preferred homeomorphism $|\Delta[n]|\cong\nabla^n$, $(\alpha:[m]\to[n],t)\mapsto\alpha_*(t)$, with inverse $s\mapsto[\mathrm{id}_{[n]},s]$.

Step 3. The map $|\Delta[n]\times\Delta[1]|\xrightarrow{(|\mathrm{pr}_1|,|\mathrm{pr}_2|)}|\Delta[n]|\times|\Delta[1]|\cong\nabla^n\times\nabla^1$ is a homeomorphism.

Step 4. Combine steps 1-3.

$$\begin{aligned}\mathrm{Hom}_{\mathrm{sSet}}(Z\times\Delta[1],\mathcal{S}(X))&\underset{\text{Step 1}}{\cong}\{\psi_z:\Delta[n]\times\Delta[1]\to\mathcal{S}(X)\mid(\psi_z)_m(\alpha,k)=H_m(\alpha^*(z),k)\}\\&\underset{\text{Step 2}}{\cong}\{\psi_z^\#:|\Delta[n]\times\Delta[1]|\to X\}\\&\underset{\text{Step 3}}{\cong}\{\hat{\psi}_z:\nabla^n\times\nabla^1\to X\}\end{aligned}$$

I might have written some dumb things

$\square$

Proof of Hurewicz Theorem

Lecture 6
3rd Nov, 2021

Proof of Hurewicz theorem. We modify the definition of $\pi_n(X,A)^\#$ by replacing $(I^n,\partial I^n)$ by the homeomorphic pair $(\nabla^n,\partial\nabla^n)$.

Then the fundamental class $i\in H_n(\nabla^n,\partial\nabla^n;\mathbb{Z})$ is represented by the map $\mathrm{id}^n_\nabla\in\mathcal{S}(\nabla^n)_n$. The inclusion of simplicial sets (the $\cong$ comes from theorem I.11 of last lecture)

$$\mathcal{S}(A)\hookrightarrow\mathcal{S}(X,A,n-1)\xhookrightarrow{\cong}\mathcal{S}(X)$$

induces morphisms of chain complexes

$$C(\mathcal{S}(A))\to C(\mathcal{S}(X,A,n-1))\xrightarrow{\sim}C(\mathcal{S}(X))$$

where " $\sim$ " is a chain homotopy equivalence.

We compare the long exact homology sequences:

$$\begin{tikzcd}\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, equal] & H_n(C(\mathcal{S}(X,A,n-1))) \arrow[r] \arrow[d, "\cong"] & H_n(\frac{C(\mathcal{S}(X,A,n-1))}{C(\mathcal{S}(A))}) \arrow[r, "\partial"] \arrow[d, "\cong\text{ (by 5-lemma!)}"] & \cdots \\ & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(\frac{C(\mathcal{S}(X))}{C(\mathcal{S}(A))})=H_n(X,A) \arrow[r, "\partial"] & \cdots\end{tikzcd}$$

Conclusion: the inclusion $\mathcal{S}(X,A,n-1)\hookrightarrow\mathcal{S}(X)$ induces an isomorphism

$$H_n\left(\frac{C(\mathcal{S}(X,A,n-1))}{C(\mathcal{S}(A))}\right)\xrightarrow{\cong}H_n(X,A)$$

so that we reduced the problem to finding an isomorphism

$$\pi_n(X,A)^{\#} \xrightarrow{"\cong"} H_n\left(\frac{C(\mathcal{S}(X,A,n-1))}{C(\mathcal{S}(A))}\right)$$

We note that $\mathcal{S}(X,A,n-1)_{n-1} = \mathcal{S}(A)_{n-1}$ so

$$\left(\frac{C(\mathcal{S}(X,A,n-1))}{C(\mathcal{S}(A))}\right)_{n-1} = 0$$

which implies

$$\begin{aligned} H_n(X,A) &\cong H_n\left(\frac{C(\mathcal{S}(X,A,n-1))}{C(\mathcal{S}(A))}\right) \\ &= \operatorname{coker}\left(\frac{\mathbb{Z}[\mathcal{S}(X,A,n-1)_{n+1}]}{\mathbb{Z}[\mathcal{S}(A)_{n+1}]} \xrightarrow{\quad d \quad} \frac{\mathbb{Z}[\mathcal{S}(X,A,n-1)_n]}{\mathbb{Z}[\mathcal{S}(A)_n]}\right) \\ &= \mathbb{Z}[f:(\nabla^n,\partial\nabla^n)\to(X,A)]/E'' \end{aligned}$$

where E'' is the subgroup generated by:

- the classes of all $f:(\nabla^n,\partial\nabla^n)\to(X,A)$ with $f(\nabla^n)\subset A$,
- elements of the form $\sum_0^{n+1}(-1)^i d_i^*(g)$ for all $g:\nabla^{n+1}\to X$ with $g(\mathrm{sk}_{n-1}(\nabla^{n+1}))\subset A$.

On the other hand, $\pi_n(X,A)^{\#} = \mathbb{Z}[f:(\nabla^n,\partial\nabla^n)\to(X,A)]/E'$ where E' is generated by:

- $f-f'$ for all pair homotopic $f\sim f'$,
- $f_1+f_2-(f_1\oplus f_2)$ whenever f_1 and f_2 are "addible".

To add maps on simplices of the same dimension, we divide ∇^n into two sub-simplices by a procedure defined inductively, using the hyperplane in ∇^n through e_0 and $d_0(T_{n-1})$ to divide T_n (this is best explained with pictures).

/\

Claim. The canonical homomorphism

$$\mathbb{Z}[f(\nabla^n,\partial\nabla^n)\to(X,A)]\to\pi_n(X,A)^{\#},\quad [f]\mapsto[f]$$

factors through a homomorphism

$$\Phi: H_n\left(\frac{C(\mathcal{S}(X,A,n-1))}{C(\mathcal{S}(A))}\right)\to\pi_n(X,A)^{\#}$$

(which is equivalent to saying that $E''\subset E$).

Proof of the claim. We need to show that the two kinds of relations that generate E'' are sent to 0.

- If $f:(\nabla^n,\partial\nabla^n)\to(X,A)$ has image in A, we contract ∇^n onto e_0 and postcompose this contraction homotopy with f. The result is a pair homotopy from f to a constant map with value $f(e_0)$. So $[f]=[\mathrm{const}_{f(e_0)}]$, which is the zero element in $\pi_n(X,A)^{\#}$.
- Now we consider all maps $g:\nabla^{n+1}\to X$ with $g(\mathrm{sk}_{n-1}(\nabla^{n+1}))\subset A$. We want to show that $\sum_0^{n+1}(-1)^i[g\circ(d_i)_*]=0$ in $\pi_n(X,A)^{\#}$.

We consider the space $B = \nabla^n \cup_{\nabla^{n-1}} \cdots \cup_{\nabla^{n-1}} \nabla^n$. It is a quotient space of a disjoint union of $n+2$ copies of ∇^n. If we number these copies from 0 to $n+1$, we glue the i-th copy to the $(i+1)$-st copy by the maps:

$$\begin{tikzcd} \nabla^{n-1} \arrow[r, "(d_i)_*"] \arrow[d, "(d_i)_*"] & \nabla^n_{((i+1)\text{-st})} \\ \nabla^n_{(i\text{-th})} & \end{tikzcd}$$

Informally, B is $\partial\nabla^{n+1}$ "cut open".

We define $p : B \to \partial\nabla^{n+1}$ by defining the restriction to the i-th copy of ∇^n as $(d_i)_*$. The map p is compatible with the equivalence relation (and hence well defined on B) thanks to the simplicial relations:

$$d_i \circ d_i = d_{i+1} \circ d_i$$

the upshot is that p is a quotient map onto $\partial\nabla^{n+1}$.

Because g is defined on all of ∇^{n+1} its restriction to $\partial\nabla^{n+1}$ represents the 0 element in $\pi_n(X, A)^{\#}$.

$$\nabla^n \cong B \xrightarrow{p} \partial\nabla^{n+1} \hookrightarrow \nabla^{n+1} \xrightarrow{g} X$$

We apply the homotopy addition theorem (in simplex version) for the maps $f_0, \ldots, f_{n+1} : \nabla^n \to B = \nabla^n \cup_{\nabla^{n-1}} \cdots \cup_{\nabla^{n-1}} \nabla^n$, where f_i is the inclusion of the i-th copy. By the HAT:

$$0 = [g|_{\partial\nabla^{m+1}} \circ p] = \sum_{i=0}^{m+1} (-1)^i [g \circ (d_i)_*]$$

■

Let's finish once and for all the proof:

$$H_n\left(\frac{C(\mathcal{S}(X, A, n-1))}{C(\mathcal{S}(A))}\right) \xrightarrow{\Phi} \pi_n(X, A)^{\#} \xrightarrow{h^{\#}} H_n(X, A; \mathbb{Z})$$

the composite $h^{\#} \circ \Phi$ is the homomorphism induced by the inclusion $\mathcal{S}(X, A, n-1) \hookrightarrow \mathcal{S}(X)$, which is an isomorphism. So Φ is injective. But Φ is also surjective since it hits all generators. So Φ is an isomorphism, hence $h^{\#}$ is an isomorphism. □

Chapter II.

Fibre Bundles and Fibrations

II

Generalities on Fibre Bundles

A **fibre bundle** over a space B is a continuous map $\pi : E \to B$ that is locally trivial in the following sense: for every point $b \in B$ there is a space F, a neighbourhood $U \subset B$ of b and a homeomorphism such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

B is called the **base**, E the **total space**, F is the **fibre**, π is the **projection**, the maps $\pi^{-1}(U) \xrightarrow{\cong} U \times F$ the **local trivialisations**.

If we fix F, the set of points $b \in B$ such that $F_b = \pi^{-1}(b)$ is homeomorphic to F is open. So in particular, if B is connected, then all fibres are homeomorphic.

Examples. — Trivial fiber bundles: $\pi = \mathrm{pr}_1 : E = B \times F \to B$.

Covering spaces: locally trivial fibre bundle with discrete fiber.

Vector bundles: particular fibre bundles with fibre $\mathbb{R}^n$.

Hopf fibration: $\eta : S^3 \to S^2$.

Remark. — Suppose $\pi : E \to B$ is a locally trivial fibre bundle with fibre $\mathbb{R}^n$. For it to be a **vector bundle** there must be:

- additional structure, as each fibre $F_b = \pi^{-1}(b)$ is given the structure of an $\mathbb{R}$ vector space,
- additional conditions, i.e. the local trivialisation $\pi^{-1}(U)$ are fiberwise linear isomorphisms.

An equivalent perspective is the following. Suppose we chose a cover of B by open subsets $\{U_i\}_{i\in I}$ and local trivializations for each U_i, $u_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{R}^n$. For each pair of indices i, j the "change of charts"

$$(U_i \cap U_j) \times \mathbb{R}^n \xrightarrow{u_i^{-1}} \pi^{-1}(U_i \cap U_j) \xrightarrow{u_j} (U_i \cap U_j) \times \mathbb{R}^n$$

is a homeomorphism on the projection to the first factors. So $u_j \circ u_i^{-1}$ is of the form

$$(u_j \circ u_i^{-1})(x, v) = (x, \Psi(x))$$

for some map $\Psi : (U_i \cap U_j) \times \mathbb{R}^n \to \mathbb{R}^n$. The map Φ is adjoint to a function

$$U_i \cap U_j \to \mathrm{Homeo}(\mathbb{R}^n, \mathbb{R}^n), x \mapsto \Psi(x, -)$$

In a vector bundle, the map factors through $\mathrm{GL}_n(\mathbb{R}) =$ linear isos $(\mathbb{R}^n, \mathbb{R}^n)$.

Several related concepts/refinements of fibre bundles can also be conveniently formulated this way, by specifying a **structure group**, for example there is a hierarchy:

- locally trivial fibre bundles with structure group $\mathrm{Homeo}(\mathbb{R}^n, \mathbb{R}^n)$,
- smooth bundles with structure group $\mathrm{Diffeo}(\mathbb{R}^n, \mathbb{R}^n)$,
- vector bundles with structure group $\mathrm{GL}_n(\mathbb{R}^n)$,
  - vector bundles can be equipped with an inner product, in which case the structure group is required to be $\mathrm{O}(n)$,
- oriented bundles with structure group $\mathrm{GL}_n^+(\mathbb{R}^n)$,
  - oriented vector bundles can be equipped with an inner product, in which case the structure group is required to be $\mathrm{SO}(n)$.

Hopf Fibration

Lecture 7
8th Nov, 2021

The goals of the following two lectures are:

1. to construct the **Hopf fibration**, a fibre bundle $\eta : S^3 \to S^2$ with fibre S^1,
2. to associate to every fibre bundle $p : E \to B$ a long exact sequence of the form

$$\cdots \to \pi_n(p^{-1}(b), e) \xrightarrow{i_*} \pi_n(E, e) \xrightarrow{p_*} \pi_n(B, b) \to \pi_{n-1}(p^{-1}(b), e) \to \cdots$$

When we are done, we will have as a corollary the computation of our first *really* non-trivial homotopy group.

II.1. Corollary. — *For every $n \geqslant 3$ there is an isomorphism $\pi_n(S^3, *) \cong \pi_n(s^2, *)$. In particular, $\pi_3(S^2, *) \cong \mathbb{Z}$, generated by the class of the Hopf fibration.*

Proof. For $n \geqslant 3$ we have

$$\cdots \to \pi_n(S^1, *) = 0 \to \pi_n(S^3, *) \xrightarrow{\eta_*} \pi_n(S^3, *) \to \pi_{n-1}(S^1, *) = 0 \to \cdots$$

which yields the claim.

In particular, for $n = 3$ we get that $\eta_* : \pi_3(S^3, *) \cong \mathbb{Z} \to \pi_3(S^2, *)$ is an isomorphism which sends $[\mathrm{id}_{S^3}]$, generator of $\pi_3(S^3, *)$, to $[\eta \circ \mathrm{id}_{S^3}] = [\eta]$. □

The Hopf fibration is part of a family of fibre bundles.

Let $K = \mathbb{R}$ or $\mathbb{C}$ and recall the projective spaces:

$$K\mathbf{P}^n = (K^{n+1} \smallsetminus \{0\})/K^\times$$

where $x \sim \lambda x$ for all $x, \lambda \in K^\times$. In particular, we have that:

- $\mathbb{R}\mathbf{P}^n$ is an n-dimensional manifold,
- $\mathbb{C}\mathbf{P}^n$ is a $2n$-dimensional manifold.

Moreover, recall that $\mathbb{R}\mathbf{P}^1 \cong S^1$ and $\mathbb{C}\mathbf{P}^1 \cong S^2$.

We consider now the projections $p : K^{n+1} \setminus \{0\} \to K\mathbf{P}^n$.

Let G be a topological group. A principal G-bundle is a G-space E such that:

(1) for every $e \in E$ the map $G \to Ge = \{ge \mid g \in G\}$, $g \to ge$, is a homeomorphism,

(2) the quotient map $p : E \to E/G = E/\sim$, where $e \sim ge$ for all $e \in E$, $g \in G$, is a fibre bundle.

Example. — The group action of the additive group of the real numbers with the discrete topology on itself (with the standard topology):

$$\mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$$

is an example of free action which does not satisfy property (2).

II.2. Proposition. — *For $K = \mathbb{R}$ or $\mathbb{C}$, the $K^\times$ action on $K^{n+1} \setminus \{0\}$ is a $K^\times$-principal bundle.*

Proof. Let $e \in K^{n+1} \setminus \{0\}$. Then the map

$$K^\times \to K^\times e, \quad \lambda \mapsto \lambda e$$

is continuous and satisfies $\|\lambda_1 e - \lambda_2 e\| = |\lambda_1 - \lambda_2| \|e\|$. It follows that the inverse is also continuous.

It remains to show that $K^{n+1} \setminus \{0\} \to K\mathbf{P}^n$ is a locally trivial fibre bundle. For $1 \leqslant i \leqslant n+1$ let $X_i \subset K^{n+1} \setminus \{0\}$ be the subspace of tuples $(x_1, \ldots, x_{n+1})$ such that $x_i \neq 0$, i.e. $x_i \in K^\times$. Then $K^{n+1} \setminus \{0\} = \cup_{i=1}^{n+1} X_i$. Let $Y_i = p(X_i) \subset K\mathbf{P}^n$. This is open since $p^{-1}(Y_i) = X_i$. We define $u : p^{-1}(Y_i) = X_i \to Y_i \times K^\times$ by $u(x) = (p(x), x_i)$, with inverse

$$u^{-1}([x], \lambda) = (x_1/x_i, \ldots, x_{i-1}/x_i, \lambda, x_{i+1}/x_i, \ldots, x_{n+1}/x_i).$$

□

For $K = \mathbb{C}$ and $n = 1$ this gives a fibre bundle

$$p : \mathbb{C}^2 \setminus \{0\} \cong S^3 \to CP1 \cong S^2$$

with fibre $\mathbb{C}^\times \cong S^1$. This is already the Hopf fibration "up to homotopy".

Let $S(K^n) \subset K^n$ be the unit sphere ($(n-1)$-dimensional if $K = \mathbb{R}$, $(2n-1)$-dimensional if $K = \mathbb{C}$). Further, let $G(K) \subset K^\times$ be the subgroup of elements of norm 1 ($G(\mathbb{R}) = \{\pm 1\}$, $G(\mathbb{C}) = S^1$). Then the $K^\times$-action on $K^{n+1} \setminus \{0\}$ restricts to a $G(K)$-action on $S(K^{n+1})$ and the induced map

$$\begin{tikzcd} S(K^{n+1}) \arrow[r] \arrow[dr] & K^{n+1} \setminus \{0\} \arrow[r] & K\mathbf{P}^n \\ & S(K^{n+1})/G(K) \arrow[ur, "\cong"'] & \end{tikzcd}$$

is a homeomorphism.

II.3. Proposition. — *The $G(K)$-action on $S(K^{n+1})$ defines a $G(K)$-principal bundle with base space $K\mathbf{P}^n$.*

Proof. Let $X_i \subset K^{n+1} \setminus \{0\}$, $Y_i \subset K\mathbf{P}^n$ as before. We obtain a homeomorphism

$$v : q^{-1}(Y_i) = S(K^{n+1}) \cap X_i \to Y_i \times G(K), \quad x \mapsto (q(x), x_i/|x_i|)$$

$\square$

For $K = \mathbb{R}$ we obtain the covering space $S^n \to \mathbb{R}\mathbf{P}^n$ we already knew.

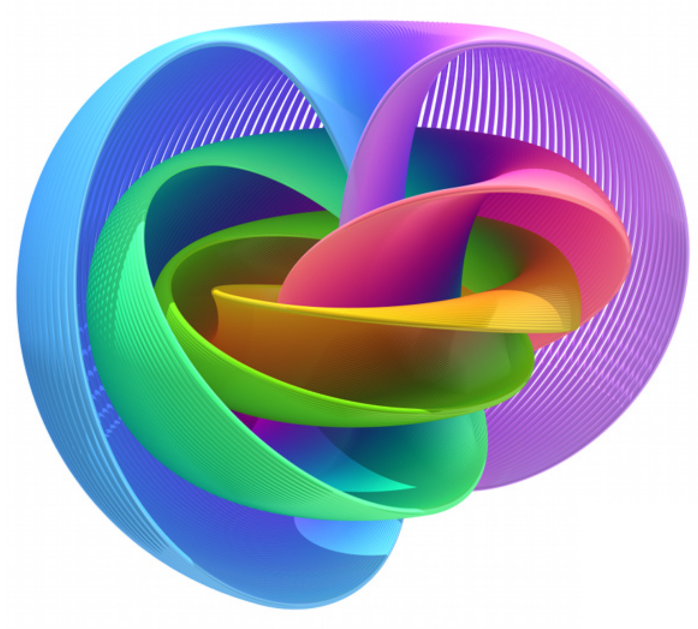
For $K = \mathbb{C}$ we get a fibre bundle $S^{2n+1} \to \mathbb{C}\mathbf{P}^n$ with fibre S^1. For $n = 1$ we get the Hopf fibration $\eta : S^3 \to S^2$.

Remark. — The Hopf fibration decomposes S^3 as a disjoint union of circles, continuously indexed over S^2. It can be shown that any two of them are linked!

Considering $\eta : S^3 \to S^2$ and $x_1 \neq x_2 \in S^2$

$$s : S^3 \supset \eta^{-1}(x_1) \times \eta^{-1}(x_2) \to S^2, \; (x, y) \to \frac{x - y}{||x - y||}$$

is continuous. Choosing orientations on S^2 and $\eta^{-1}(x_1) \times \eta^{-1}(x_2) \cong S^1 \times S^1$, we can consider the mapping degree of s. This is an example of an invariant for links called the **linking number** and it will be ± 1 in this case.



Someday I'll really wrap my head around this picture... (but maybe it's just not *that* illuminating?)

The Long Exact Sequence Associated to a Serre Fibration

We turn now to our second goal. If $p : E \to B$ is a fibre sequence, $b \in B$, $e \in p^{-1}(b)$, then we want to show that there is a long exact sequence of the form:

$$\cdots \to \pi_n(p^{-1}(b), e) \to \pi_n(E, e) \to \pi_n(B, b) \to \cdots$$

Example. — Let $p : E \to B$ be a covering space. We get a long exact sequence (assuming E is connected):

$$\cdots \to \pi_n(p^{-1}(b), e) = 0 \to \pi_n(E, e) \to \pi_n(B, b) \to \pi_{n-1}(p^{-1}(b), e) = 0 \to \cdots$$

$$\cdots \to 0 \to \pi_1(E, e) \to \pi_1(B, b) \to \pi_0(p^{-1}(b), e) \to 0$$

This amounts to the known statements that $p_* : \pi_1(E, e) \to \pi_1(B, b)$ is injective (and that the fiber can be identified, via path lifting, with the set of cosets of $p_*\pi_n(E, e)$ in $\pi_1(B, b)$) and that $p_* : \pi_n(E, e) \to \pi_n(B, b)$ is an isomorphism for $n \geqslant 2$. Both facts have been already proven using the lifting properties of covering spaces.

He adds a reminder on map lifting here.

Let $p : E \to B$ be a continuous map. A **test situation** for the homotopy lifting property (HLP) consists of a space X and a commutative square:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "H"] & B \end{tikzcd}$$

A solution to the test situation is a map $\tilde{H} : X \times [0, 1] \to E$ such that $p \circ \tilde{H} = H$ and $\tilde{H} \circ i_0 = f$.

there is also a relative version: a pair of spaces (X, A) and a commutative diagram:

$$\begin{tikzcd} X \times \{0\} \cup (A \times [0,1]) \arrow[r, "f"] \arrow[d, hook, "i_0"] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "H"] & B \end{tikzcd}$$

A solution is again a map $\tilde{H} : X \times [0, 1] \to E$ such that $p \circ \tilde{H} = H$ and $\tilde{H} \circ i_0 = f$.

A map $p : E \to B$ is called a **Hurewicz fibration** if it has the HLP with respect to all X and all absolute test-situations for X.

A map $p : E \to B$ is called a **Serre fibration** if it has the HLP with respect to every CW-complex X and all absolute test-situations for X.

For the proof of the existence of the long exact sequence associated to a fibre bundle we need two lemmas.

II.4. Lemma. — *Let $p : E \to B$ be a Serre fibration, $Y \subset B$ a subspace and $x \in p^{-1}(Y)$. Then the projection induces an isomorphism (for $n \geqslant 1$):*

$$p_* : \pi_n(E, p^{-1}(Y), *) \xrightarrow{\cong} \pi_n(B, Y, p(x))$$

II.5. Corollary. — *Let $Y = \{b\}$. We get a long exact sequence:*

$$\cdots \to \pi_n(p^{-1}(b), x) \to \pi_n(E, x) \to \pi_n(E, p^{-1}(b), x) \cong \pi_n(B, b) \to \cdots$$

II.6. Lemma. — *Every fiber bundle is a Serre fibration.*

Lecture 8
10^{th} Nov, 2021

Before we get to the two promised lemmas, we prove an auxiliary one.

⚠ Didn't sleep much the night before this one, I hope I didn't type anything too stupid!

II.7. Lemma. — *Let $p : E \to B$ a map. The following are equivalent:*

(1) *p is a Serre fibration,*

(2) *p has the absolute HLP for D^n for all n,*

(3) *p has the relative HLP for $(D^n, \partial D^n)$ for all n,*

(4) *p has the relative HLP for all relative CW-complexes.*

Proof. (1) $\implies$ (2) This is true because D^n is a CW-complex.

(2) $\implies$ (3) The space pairs $(D^n \times [0,1], D^n \times \{0\})$ and $(D^n \times [0,1], D^n \times \{0\} \cup \partial D^n \times [0,1])$ are homeomorphic, hence any test situation for one HLP can be translated into a test situation for the other, and similarly for the solutions.

(3) $\implies$ (4) Let (X, X') be a relative CW-complex. Consider the test situation:

$$\begin{tikzcd}
X \times \{0\} \cup X' \times [0,1] \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\
X \times [0,1] \arrow[r] & B
\end{tikzcd}$$

We first assume that $X = X' \cup_{\partial D^n} D^n$ is obtained from X' by attaching a single cell, with characteristic map $\alpha : D^n \to X$.

We obtain:

$$\begin{tikzcd}
D^n \times \{0\} \cup \partial D^n \times [0,1] \arrow[r] \arrow[d] & X \times \{0\} \cup X' \times [0,1] \arrow[r, "f"] \arrow[d] & E \arrow[d, "P"] \\
D^n \times [0,1] \arrow[r, "\alpha \times \mathrm{id}_{[0,1]}"'] \arrow[urr, dashed, "H'"] & X \times [0,1] \arrow[r, "H"'] & B
\end{tikzcd}$$

By assumption, there exists a lift H' as in the diagram.

Then the desired solution is

$$X \times [0,1] = (X' \times [0,1]) \cup_{\partial D^n \times [0,1]} D^n \times [0,1] \xrightarrow{f \cup H'} E$$

The case where (X, X') has finitely many relative cells follows by induction, the infinite case by passing to the colimit.

(4) $\implies$ (1) This is the special case $(X, \varnothing)$.

$\square$

Remark. — Note: CW-complexes are colimits of their skeleta.

$$\begin{tikzcd}
\operatorname{sk}_n X \times [0,1] \arrow[r] \arrow[d, hook] & E \\
\operatorname{sk}_{n+1} X \times [0,1] \arrow[ur] &
\end{tikzcd}$$

We have $X \cong \operatorname{colim}_n \operatorname{sk}_n X$ and $X \times [0,1] \cong \operatorname{colim}_n(\operatorname{sk}_n X \times [0,1])$.

II.8. Proposition. — *Let $p : E \to B$ be a Serre fibration, $Y \subset B$ and $x \in p^{-1}(Y)$. Then p induces an isomorphism*

$$p_* \pi_n(E, p^{-1}(Y), x) \xrightarrow{\cong} \pi_n(B, Y, p(x))$$

for all $n \geqslant 1$.

Proof.

Surjectivity. Let $[\beta] \in \pi_n(B, Y, p(x))$ be represented by $\beta(I^n, \partial I^n, s_0) \to (B, Y, p(x))$ with $s_0 = (0, \dots, 0)$.

There's a useful drawing here.

This already looks like a homotopy but we need to lift $\beta|_{I^n \times \{0\}}$. There's different ways to go about this, for example repeated applications of the HEP would work, but since we're working with a contractible space, there's an easier and faster way.

Applying the HEP first to the relative CW-complex $(\partial I^n, I^{n-1} \times \{0\})$ (for maps into Y), and second for the relative CW-complex $(I^n, \partial I^n)$ mapping into B, we can replace β by an homotopic map β' which sends all of $I^{n-1} \times \{0\}$ to $p(x)$.

The constant map $c_{p(x)} : I^{n-1} \times \{0\} \to Y$ has a canonical lift to E via the constant map $c_x : I^{n-1} \times \{0\} \to p^{-1}(Y)$. Hence we obtain:

$$\begin{tikzcd}
I^{n-1} \times \{0\} \arrow[r, "c_x"] \arrow[d] & E \arrow[d, "p"] \\
I^n \arrow[r, "\beta'"'] \arrow[ur, dashed, "\exists \tilde{\beta}"] & B
\end{tikzcd}$$

Since p is a Serre fibration, there exists $\tilde{\beta} : I^n \to E$ such that $\tilde{\beta}|_{I^{n-1} \times \{0\}} = c_x$ (in particular $\tilde{\beta}(s_0) = x$) and $p \circ \tilde{\beta} = \beta'$ (in particular $\tilde{\beta}(\partial I^n) \subset p^{-1}(Y)$). Hence $\tilde{\beta}$ represents an element $[\tilde{\beta}]$ of $\pi_n(E, p^{-1}(Y), x)$, which by construction maps to $[\beta'] = [\beta]$ under p_*.

Injectivity. Let $\alpha_1, \alpha_2 : (I^n, \partial I^n, s_0) \to (E, p^{-1}(Y), x)$ represent elements of $\pi_n(E, p^{-1}(Y), x)$ which are sent to the same element under p_*. Then there exists a homotopy of triple maps $H : I^n \times I \to B$ from $p \circ \alpha_1$ to $p \circ \alpha_2$.

There's a useful drawing here.

Again we can assume that $\alpha_1(I^{n-1} \times \{0\}) = \alpha_2(I^{n-1} \times \{0\}) = \{x\}$. In addition we can assume that H sends $(I^{n-1} \times \{0\}) \times I$ constantly to $p(x)$. We again lift $c_{p(x)}$ to c_x and view α_1 and α_2 as lifts of H on the subspace $I^{n-1} \times \{0\} \times \{0\} \cup I^{n-1} \times \{0\} \times \{1\}$. Since $(I^{n-1} \times \{0\} \times I, I^{n-1} \times \{0\} \times \{0\} \cup I^{n-1} \times \{0\} \times \{1\})$ is a relative CW-complex, we can apply the relative HLP to lift H to a map $\tilde{H} : I^n \times I \to E$, giving a relative homotopy from α_1 to α_2. □

II.9. Theorem. — *Every fibre bundle is a Serre fibration.*

Proof. Let $p : E \to B$ be a fibre bundle and a lifting problem

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"] & B \end{tikzcd}$$

Easy case. Let p be globally trivial, i.e. of the form $\mathrm{pr}_B : B \times F \to B$ for some space F. Then we have

$$\begin{tikzcd} X \times \{0\} \arrow[r, "{(f_1,f_2)}"] \arrow[d] & B \times F \arrow[d, "\mathrm{pr}_B"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B \end{tikzcd}$$

We can define a lift $\tilde{H}$ explicitly via $\tilde{H}(x,t) = (H(x,t), f_2(x))$ (this works for any space X).

General case. We have to glue local lifts together systematically.

By lemma II.7, it suffices to check the HLP for disks D^n, or equivalently for cubes I^n. Hence we are given:

$$\begin{tikzcd} I^n \times \{0\} \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "H"] & B \end{tikzcd}$$

Let $\{U_i\}_{i \in I}$ be an open covering of B, such that $p^{-1}(U_i) \xrightarrow{p} U_i$ is a trivial fibre bundle for all i. Pulling back along H, we get an open cover of $I^n \times I$. By Lebesgue's lemma, we can divide $I^n \times I$ into smaller cubes of side length $1/k$, such that each cube is contained in some $p^{-1}(U_i)$.

There's a useful drawing here, an explanation of how "row by row" is fine and otherwise not.

We can then extend H iteratively over the smaller cubes "row by row". In every situation this amounts to choosing a solution to the relative lifting problem for a globally trivial fibre bundle. □

Remark. — Not every fibre bundle is a Hurewicz fibration (but actual counter-examples are complicated). A sufficient condition is that the base space be paracompact.

Remark. — An interesting question: are lifting of homotopies unique? It turns out that they are unique up to homotopy!

$$
\begin{tikzcd}
X \times \{0\} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
X \times I \arrow[r] \arrow[ur, dashed, "\tilde{H}_1", "\tilde{H}_2"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times I \times \{0\} \cup X \times I \times \{1\} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
X \times I \times I \arrow[r] & B
\end{tikzcd}
$$

I wasn't paying attention, it might be interesting to reconstruct the argument!

Chapter III. Appendix

III

Interesting exercises

add text

III.1. Exercise (Homology and homotopy groups of telescope)**.** — Write it!

add solution

Proof. Do it! □

Things to see

References

[Owe00] U.N. Owen. *Whoknows*. Whocares, 3000.