ESE 253 – Assignment 1

SRIRAM RAVINDRAN
A53208651
sriram@ucsd.edu

1) We have $\int_{-\infty}^{\infty} \exp\left\{-\frac{\lambda}{2} x^2\right\} dx = \left(\frac{2\pi}{\lambda}\right)^{1/2}$

We need to show

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \quad \text{given}$$

$$\prod_{i=1}^{d} \int_{-\infty}^{\infty} e^{-x_i^2} dx_i = S_d \int_0^{\infty} e^{-r^2} r^{d-1} dr$$

$$\Rightarrow \prod_{i=1}^{d} \left(\frac{2\pi}{2}\right)^{1/2} = S_d \int_0^{\infty} e^{-r^2} r^{d-1} dr$$

Put $u = r^2$

$\Rightarrow du = 2r\, dr \quad \Rightarrow \quad \frac{u^{-1/2}}{2} du = dr$

$$\therefore (\pi)^{d/2} = S_d \times \frac{1}{2} \int e^{-u} u^{d/2 - 1} du$$

$$\Rightarrow \frac{2\pi^{d/2}}{\Gamma(d/2)} = S_d$$

We know $\Gamma(1) = 1$ and $\Gamma(3/2) = \sqrt{\pi}/2$

Put $d = 2$, we have

$$S_2 = \frac{2\pi}{1}$$

Put $d = 3$, we have

$$S_3 = \frac{2\pi^{3/2}}{\sqrt{\pi}} \times 2 = 4\pi$$

which match with expected S_2 and S_3
(circle) (sphere

2. $\frac{dV_d}{dr} = S_d r^{d-1}$

Volume - surface area relationship for any hypersphere.

$$dV = \int_0^a S_d r^{d-1} dr$$

$$= \left(\frac{S_d r^d}{d} \right)_0^a$$

$$\Rightarrow \boxed{V_d = \frac{S_d a^d}{d}}$$

$$\therefore \frac{\text{Volume of sphere}}{\text{Volume of cube}} = \frac{S_d \cancel{a^d}}{d \times 2^d \cancel{a^d}} = \frac{2 \pi^{d/2}}{2^d d \Gamma(d/2)}$$

$$= \frac{\pi^{d/2}}{d 2^{d-1} \Gamma(d/2)}$$

Using Stirlings' approximation,

$$\Gamma(x+1) \simeq (2\pi)^{1/2} e^{-x} x^{x+1/2}$$

$$\Gamma(d/2) = \Gamma\left(d/2 - 1 + 1 \right)$$

$$\simeq (2\pi)^{1/2} e^{-[d/2 - 1]} \left(\frac{d}{2} - 1 \right)^{d/2 - 1/2}$$

$$\Rightarrow \text{Ratio} = \frac{\pi^{d/2}}{d 2^{d-1} (2\pi)^{1/2} e^{-[d/2 - 1]} [d/2 - 1]^{(d/2 - 1/2)}}$$

$$\propto \frac{\pi^{d/2} \, e^{d/2}}{d \, 2^{d} \left[\frac{d}{2}-1\right]^{d/2 - 1/2}}$$

Removing constants.

$$= \frac{\left(\frac{\pi e}{4}\right)^{d/2} \left(\frac{d}{2}-1\right)^{1/2}}{d \left(\frac{d}{2}-1\right)^{d/2}}$$

Ratio $\propto \dfrac{(d/2 - 1)^{1/2}}{d} \times \left[\dfrac{\pi e}{4(d/2 - 1)}\right]^{d/2}$

$\downarrow$ < 1 (always)

$\downarrow$ Let it be γ

As $d \to \infty$ $\quad \gamma \to 0$

$\Rightarrow \gamma^{d/2} \to 0$

$\therefore$ As $d \to \infty$,

Ratio, $\dfrac{\text{Volume of sphere}}{\text{Volume of cube}} \longrightarrow 0$ $\quad$ [Since $\gamma^{d/2} \to 0$ as $d \to \infty$]

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1.3 Volume of sphere $\propto a^d$

$\Rightarrow \dfrac{\text{Volume of sphere w. radius } a - \text{Volume of sphere w. radius } a-\epsilon}{\text{Volume of sphere w. radius } a}$

$$= f = \frac{a^d - (a-\epsilon)^d}{a^d}$$

$$= 1 - \underbrace{\left(1 - \frac{\epsilon}{a}\right)^d}_{< 1}$$

Since $0 < \left(1 - \frac{\epsilon}{a}\right)^d < 1 \Rightarrow \left(1 - \frac{\epsilon}{a}\right)^d \rightarrow 0$ as $d \rightarrow \infty$

$\forall\ 0 < \epsilon < a$

$\Rightarrow f \rightarrow 1$ as $d \rightarrow \infty$

Assume $\frac{\epsilon}{a} = 0.01 \Rightarrow f = 1 - (0.99)^d$

When $\frac{\epsilon}{a} = 0.5, f = 1 - (0.5)^d$

ϵ/a / d	2	10	1000
0.01	0.0199	0.0956	0.9999
0.5	0.75	0.999	~1

1.4 We have $p(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{\frac{-\|x\|^2}{2\sigma^2}}$

Since p is only radially dependent, we can easily write it in polar form as $\hat{p}(r)$

$$= \frac{1}{(2\pi\sigma^2)^{1/2}} e^{\frac{-|r|^2}{2\sigma^2}}$$

(~~Volume~~ Probability Density at distance r)

∴ Probability mass inside thin shell of width dr at r

$$= \hat{p}(r) \times S_d \, r^{d-1} \, dr$$

$$= \frac{S_d \, r^{d-1}}{(2\pi\sigma^2)^{1/2}} e^{-r^2/2\sigma^2} dr$$

$$\Rightarrow P(r) = \frac{S_d \, r^{d-1}}{(2\pi\sigma^2)^{1/2}} e^{\left(-r^2/2\sigma^2\right)}$$

$$\frac{d P(r)}{dr} = 0 \Rightarrow (d-1)\,\hat{r}^{d-2} e^{-\hat{r}^2/2\sigma^2} + e^{-\hat{r}^2/2\sigma^2} \times \frac{-2\hat{r}}{2\sigma^2} \hat{r}^{d-1} = 0$$

$$\Rightarrow \hat{r} = \sqrt{d-1}\,\sigma \qquad \text{i.e} \quad d-1 = \frac{r^2}{\sigma^2}$$

$$\Rightarrow \boxed{\hat{r} \simeq \sqrt{d}\,\sigma}$$

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Now, consider

$$\frac{p(\hat{r}+\epsilon)}{p(\hat{r})},$$

$$= \frac{(\hat{r}+\epsilon)^{d-1}}{(\hat{r})^{d-1}} e^{-\left(\frac{(\epsilon+\hat{r})^2 + \hat{r}^2}{2\sigma^2}\right)}$$

$$= \left(1+\frac{\epsilon}{\hat{r}}\right)^{\hat{r}^2/\sigma^2} e^{-\left(\frac{\epsilon^2+2r\epsilon}{2\sigma^2}\right)}$$

Since $d-1 = \frac{\hat{r}^2}{\sigma^2}$

$$= e^{\frac{\hat{r}^2}{\sigma^2}\ln\left(1+\frac{\epsilon}{\hat{r}}\right) - \left(\frac{\epsilon^2+2r\epsilon}{2\sigma^2}\right)}$$

$$= e^{\frac{\hat{r}^2}{\sigma^2}\left[\frac{\epsilon}{r} - \frac{\epsilon^2}{2r^2} - \dots\right] - \frac{\epsilon^2+2r\epsilon}{2\sigma^2}}$$

Expand $\ln(1+x)$

$$= x - \frac{x^2}{2} + \dots$$

$$= e^{\left[\cancel{\frac{\epsilon\hat{r}}{\sigma^2}} - \frac{\epsilon^2}{2\sigma^2} - \frac{\epsilon^2}{2\sigma^2} - \cancel{\frac{r\epsilon}{\sigma^2}}\right]}$$

$$= e^{-\epsilon^2/\sigma^2}$$

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OJAS GUPTA
A53201624

Q 1.1) Given $\int_{-\infty}^{\infty} \exp\left\{\frac{-\lambda}{2}x^2\right\} dx = \left(\frac{2\pi}{\lambda}\right)^{1/2}$ —— ①

$$\prod_{i=1}^{d} \int_{-\infty}^{\infty} e^{-x_i^2} dx_i = S_d \int_0^{\infty} e^{-r^2} r^{d-1} dr. \quad \text{—— ②}$$

<u>To Prove</u> $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ here $\Gamma(x) = \int_0^{\infty} u^{x-1} e^{-u} du$

<u>Proof:</u>

from eq ①

$$\int_{-\infty}^{\infty} e^{-x_i^2} dx_i = (\pi)^{1/2}$$

$$\Rightarrow \prod_{i=1}^{d} (\pi)^{1/2} = S_d \int_0^{\infty} e^{-r^2} r^{d-1} dr.$$

$$\Rightarrow (\pi)^{d/2} = S_d \int_0^{\infty} e^{-r^2} r^{d-1} dr$$

using $u = r^2$

$$\Rightarrow (\pi)^{d/2} = \frac{1}{2} S_d \int_0^{\infty} e^{-u} u^{d/2-1} du$$

$$\Rightarrow S_d = \frac{2(\pi)^{d/2}}{\int_0^{\infty} e^{-u} u^{d/2-1} du} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$ where $\Gamma(x) = \int_0^{\infty} u^{x-1} e^{-u} du$

Here Given $\Gamma(1) = 1$

$\Gamma(3/2) = \sqrt{\pi}/2$

* at $d=2$ $\quad S_2 = \dfrac{2(\pi)^{2/2}}{\Gamma(2/2)} = 2\pi.$ unit Circle

* at $d=3$ $\quad S_3 = \dfrac{2\pi^{3/2}}{\Gamma(3/2)} = \dfrac{2\pi^{3/2}}{\sqrt{\pi}} \times 2 = 4\pi$

unit Sphere

Hence Proved.

Q.1.2

Given: $S_d = \frac{2\pi^{d/2}}{\int_0^\infty u^{d/2-1} e^{-u}\, du}$

To Prove: $V_d = \frac{S_d a^d}{d}$ $\quad a = \text{radius}$

Proof:

Since we know

$$\frac{dV_d}{dr} = S$$

V= Volume of a sphere
S = Surface area of sphere.

$$dV_d = \int_0^R S\, dr$$

$$S = S_d r^{d-1}$$

Surface area relationship with unit surface area

$$dV_d = \int_0^R S_d\, r^{d-1}\, dr$$

$$\Rightarrow V_d = \frac{S_d r^d}{d}\Big|_0^R$$

$$\Rightarrow \boxed{V_d = \frac{S_d R^d}{d}}$$

Hence Proved

Hypersphere radius a
hypercube ~~so~~ side 2a

$$\frac{\text{Volume of hypersphere}}{\text{Volume of hypercube}} = \frac{S_d a^d}{d(2a)^d} = \frac{S_d \cancel{a^d}}{d\, 2^d \cancel{a^d}}$$

$$\Rightarrow \text{Ratio} = \frac{S_d}{d\, 2^d} = \frac{2\pi^{d/2}}{d\, 2^d\, \Gamma(d/2)}$$

$$= \frac{\pi^{d/2}}{d\, 2^{d-1}\, \Gamma(d/2)}$$

Hence Proved.

Given Sterling's approximation

$$\Gamma(x+1) \simeq (2\pi)^{1/2} e^{-x} x^{x+1/2} \qquad \Gamma(d/2) = \Gamma(d/2 - 1 + 1$$

$$\text{Ratio} = \frac{\pi^{d/2}}{d\, 2^{d-1} (2\pi)^{1/2} e^{-(d/2-1)} \left(\frac{d}{2}-1\right)^{d/2-1+1/2}}$$

$$\cancel{\propto} \quad \frac{\pi^{d/2} e^{d/2}}{d\, 2^d \left[d/2 - 1\right)^{d/2 - 1/2}}$$

$$\simeq \left(\frac{\pi e}{4}\right)^{d/2} \frac{(d/2-1]^{1/2}}{(d/2-1)^{d/2}} \simeq \Bigg(\underbrace{\frac{\pi e}{4(d/2-1)}}_{\text{Deno. } \infty}\Bigg)^{d/2} \times \underbrace{\frac{(d/2-1)^{1/2}}{d}}_{<1}$$

as d tends to ∞
Ratio tends to $\underline{0}$ Hence Proved

Q.1.3

Given: $V_d = \dfrac{S_d a^d}{d}$

To Prove $f = 1 - \left(1 - \varepsilon/a\right)^d$

$$f = \frac{\text{Volume}(a) - \text{Volume}(a-\varepsilon)}{\text{Volume}(a)}$$

$$f = \frac{\dfrac{S_d a^d}{\cancel{d}} - \dfrac{S_d(a-\varepsilon)^d}{\cancel{d}}}{\dfrac{\cancel{S_d} a^d}{\cancel{d}}} = \frac{a^d - (a-\varepsilon)^d}{a^d}$$

$$\Rightarrow f = 1 - \left(1 - \varepsilon/a\right)^d$$ Hence Proved

	$d = 2$	$d = 10$	$d = 1000$
$\varepsilon/a = 1/2$	0.75	0.9990	1
$\varepsilon/a = 0.01$	0.0199	0.0956	0.9999

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Q.1.4 Given Probability Density function

$$P(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(\frac{-\|x\|^2}{2\sigma^2}\right) \quad \text{--- (1)}$$

To Prove

Probab. mass inside a thin shell of radius r and thickness ϵ

$P(r)\epsilon$

$$P(r) = \frac{S_d r^{d-1}}{(2\pi d^2)^{1/2}} \exp\left(\frac{-r^2}{2\sigma^2}\right)$$

Proof writing eq (1) in polar form

$$\hat{P}(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(\frac{-r^2}{2\sigma^2}\right) \quad \{ \|x\| = r$$

Probability mass inside shell of dr at radius $= r$.

$$\hat{P}(x) \times S_d r^{d-1} dr.$$

$$= \frac{S_d r^{d-1}}{(2\pi\sigma^2)^{1/2}} \exp\left(\frac{-r^2}{2\sigma^2}\right) dr.$$

$dr = \epsilon$

$$P(r) = \frac{S_d r^{d-1}}{(2\pi\sigma^2)^{1/2}} \exp\left(\frac{-r^2}{2\sigma^2}\right).$$ Hence Proved.

To find Max of $f(r)$ diff it with respect to r

$$\frac{d\,f(r)}{dr} = \frac{S_d}{(2\pi\sigma^2)^{1/2}}\left[d-1\, r^{d-2}\, e^{-r^2/2\sigma^2} + \frac{r^{d-1}}{2\sigma^2}\, e^{-\frac{r^2}{2\sigma^2}} \times 2r\right] = 0$$

$$r^{d-1}\, e^{-r^2/2\sigma^2}\left[d-1 - \frac{r^2}{\sigma^2}\right] = 0$$

$$\frac{r^2}{\sigma^2} = d-1$$

$$\boxed{r \simeq \sqrt{d}\,\sigma}$$ Hence Proved.

<u>To prove</u>: for large d $\quad d >> 0$.

$$f(\hat{r}+e) = f(\hat{r})\,.\,\exp\left(-\frac{3\varepsilon^2}{2\sigma^2}\right).$$

let us find

$$\frac{f(\hat{r}+\varepsilon)}{f(\hat{r})} = \frac{\dfrac{S_d\,(\hat{r}+\varepsilon)^{d-1}}{(2\pi\sigma^2)^{1/2}}\, e^{-\frac{(\hat{r}+\varepsilon)^2}{2\sigma^2}}}{\dfrac{S_d\,(\hat{r})^{d-1}}{(2\pi\sigma^2)^{1/2}}\, e^{-\frac{\hat{r}^2}{2\sigma^2}}}$$

$$= \left(1+\varepsilon/\hat{r}\right)^{d-1} e^{\frac{-(\hat{r}+\varepsilon)^2 + r^2}{2\sigma^2}}$$

$$\Rightarrow \left(1+\varepsilon/\hat{r}\right)^{r^2/\sigma^2}\, e^{-\left(\frac{\varepsilon^2+2r\varepsilon}{2\sigma^2}\right)}.$$

$$\Rightarrow e^{\frac{\hat{\gamma}^2}{\sigma^2}\ln\left(1+\varepsilon/\hat{\gamma}\right)-\left(\frac{\varepsilon^2+2\gamma\varepsilon}{2\sigma^2}\right)}$$

$$\Rightarrow e^{\frac{\hat{\gamma}^2}{\sigma^2}\left[\varepsilon/\hat{\gamma}-\frac{\varepsilon^2}{2\hat{\gamma}^2}+\right]-\left(\frac{\varepsilon^2+2\gamma\varepsilon}{2\sigma^2}\right)}$$

Expansion of $\ln(1+x)$
$= x - x^2/2 +$

$$\Rightarrow e^{\frac{1}{\sigma^2}\left(\cancel{\hat{\gamma}\varepsilon}-\frac{\varepsilon^2}{2}-\frac{\varepsilon^2}{2}-\cancel{\gamma\varepsilon}\right)}$$

$$\Rightarrow e^{-\frac{\varepsilon^2}{\sigma^2}}$$

<u>Here Proved</u>

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