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Farey Sequences

Duality and Maps Between Subsequences

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that is,

$$\begin{aligned}
\min\Big\{\frac{m+h_{j+2}}{h_{j+1}}, \frac{n+k_{j+2}}{k_{j+1}}, \frac{(n-m)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}}\Big\}
&= \begin{cases}
\frac{m+h_{j+2}}{h_{j+1}} & \text{if } h_{j+1}n - k_{j+1}m \geq 1, \\
\frac{m+h_{j+2}}{h_{j+1}} = \frac{n+k_{j+2}}{k_{j+1}} = \frac{(n-m)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}} & \text{if } h_{j+1}n - k_{j+1}m = 1, \\
\frac{(n-m)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}} & \text{if } h_{j+1}n - k_{j+1}m \leq 1
\end{cases} \\
&= \min\Big\{\frac{m+h_{j+2}}{h_{j+1}}, \frac{(n-m)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}}\Big\}.
\end{aligned}$$

On the one hand, if $\min\{\frac{m+h_{j+2}}{h_{j+1}}, \frac{n+k_{j+2}}{k_{j+1}}\} = \frac{m+h_{j+2}}{h_{j+1}}$, that is, $h_{j+1}n - k_{j+1}m \geq 1$, then we have

$$\Big\lfloor \frac{m+h_{j+2}}{h_{j+1}} \Big\rfloor = \Big\lfloor \frac{h_j+h_{j+2}+(m-h_j)}{h_{j+1}} \Big\rfloor = \frac{h_j+h_{j+2}}{h_{j+1}} + \Big\lfloor \frac{m-h_j}{h_{j+1}} \Big\rfloor.$$

But the summand $\lfloor \frac{m-h_j}{h_{j+1}} \rfloor$ vanishes, since $h_j + h_{j+1} > m$, in view of Remark 1.22 (i). As a consequence, we have

$$\frac{h_j+h_{j+2}}{h_{j+1}} = \Big\lfloor \frac{m+h_{j+2}}{h_{j+1}} \Big\rfloor.$$

On the other hand, if

$$\min\Big\{\frac{n+k_{j+2}}{k_{j+1}}, \frac{(n-m)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}}\Big\} = \frac{(n-m)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}},$$

that is, $h_{j+1}n - k_{j+1}m \leq 1$, then

$$\begin{aligned}
\Big\lfloor \frac{(n-m)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}} \Big\rfloor &= \Big\lfloor \frac{(k_j-h_j)+(k_{j+2}-h_{j+2})+((n-m)-(k_j-h_j))}{k_{j+1}-h_{j+1}} \Big\rfloor \\
&= \frac{(k_j-h_j)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}} + \Big\lfloor \frac{(n-m)-(k_j-h_j)}{k_{j+1}-h_{j+1}} \Big\rfloor.
\end{aligned}$$

But $\lfloor \frac{(n-m)-(k_j-h_j)}{k_{j+1}-h_{j+1}} \rfloor = 0$, since $(k_j-h_j)+(k_{j+1}-h_{j+1}) > n-m$, in view of Remark 1.22 (ii). As a consequence, we have

$$\frac{(k_j-h_j)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}} = \Big\lfloor \frac{(n-m)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}} \Big\rfloor.$$

Thus, we have

$$\begin{aligned}
\frac{h_j+h_{j+2}}{h_{j+1}} = \frac{k_j+k_{j+2}}{k_{j+1}} &= \frac{(k_j-h_j)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}} \\
&=: \gcd(h_j+h_{j+2}, k_j+k_{j+2}) \\
&= \Big\lfloor \min\Big\{\frac{m+h_{j+2}}{h_{j+1}}, \frac{(n-m)+(k_{j+2}-h_{j+2})}{k_{j+1}-h_{j+1}}\Big\}\Big\rfloor. \qquad (1.115)
\end{aligned}$$

Hence, (1.111) and (1.112) follow.

Table 2.7. The neighbors of the fraction $\frac{1}{3}$ in Farey subsequences; see also Table 2.6.

Sequence	The predecessor of $\frac{1}{3}$		See
$\mathcal{F}(\mathbb{B}(2m), m)$, $m > 1$	$\frac{\lfloor (m-1)/2 \rfloor}{3\lfloor (m-1)/2 \rfloor + 1}$		Remark 2.24
$\mathcal{F}(\mathbb{B}(n), m)$, $n < 2m$, $(n-m) > 1$	$\frac{\lfloor (n-m-1)/2 \rfloor}{3\lfloor (n-m-1)/2 \rfloor + 1}$		Remark 2.42 (i)
$\mathcal{F}(\mathbb{B}(n), m)$, $n > 2m$, $m > 1$	$\frac{m}{3m+1}$	if $n - 3m \geq 1$	Remark 2.42 (ii)
	$\frac{\lfloor (n-m-1)/2 \rfloor}{3\lfloor (n-m-1)/2 \rfloor + 1}$	if $n - 3m \leq 1$	

Sequence	The successor of $\frac{1}{3}$		See
$\mathcal{F}(\mathbb{B}(2m), m)$, $m > 1$	$\frac{\lceil m/2 \rceil}{3\lceil m/2 \rceil - 1}$		Remark 2.24
$\mathcal{F}(\mathbb{B}(n), m)$, $n < 2m$, $(n-m) > 1$	$\frac{\lceil (n-m)/2 \rceil}{3\lceil (n-m)/2 \rceil - 1}$		Remark 2.42 (i)
$\mathcal{F}(\mathbb{B}(n), m)$, $n > 2m$, $m > 1$	$\frac{\lceil (n-m)/2 \rceil}{3\lceil (n-m)/2 \rceil - 1}$	if $3m - n \geq 1$	Remark 2.42 (ii)
	$\frac{m}{\cancel{2}m-1}$	if $3m - n \leq 1$	

precedes $\frac{1}{3}$ in $\mathcal{F}(\mathbb{B}(2m), m)$, and the fraction

$$\frac{\lceil m/2 \rceil}{3\lceil m/2 \rceil - 1} = \frac{\lfloor (m-1)/2 \rfloor + 1}{3\lfloor (m-1)/2 \rfloor + 2}$$

succeeds $\frac{1}{3}$ in $\mathcal{F}(\mathbb{B}(2m), m)$.

Thus, if m is even, then the fractions

$$\frac{(m-2)/2}{(3m-4)/2} < \frac{1}{3} < \frac{m/2}{(3m-2)/2}$$

are consecutive in $\mathcal{F}(\mathbb{B}(2m), m)$. If m is odd, then the fractions

$$\frac{(m-1)/2}{(3m-1)/2} < \frac{1}{3} < \frac{(m+1)/2}{(3m+1)/2}$$

are consecutive in $\mathcal{F}(\mathbb{B}(2m), m)$.

2.2.7.4 The neighbors of $\frac{2}{3}$

Now that we have found in the previous subsection the neighbors of the fraction $\frac{1}{3}$ in the Farey subsequences $\mathcal{F}(\mathbb{B}(2m), m)$, it is natural to ask which fractions are the neighbors of the fraction $\frac{2}{3}$. Let us apply the order-reversing bijection $\mathcal{F}(\mathbb{B}(2m), m) \to \mathcal{F}(\mathbb{B}(2m), m)$ defined in (1.8) to the observations made in Remark 2.24.

Remark 2.25 (see also Table 2.8). If $m > 1$, then the fraction

$$\frac{2\lceil m/2 \rceil - 1}{3\lceil m/2 \rceil - 1} = \frac{2\lfloor (m-1)/2 \rfloor + 1}{3\lfloor (m-1)/2 \rfloor + 2}$$

precedes $\frac{2}{3}$ in $\mathcal{F}(\mathbb{B}(2m), m)$, and the fraction

$$\frac{2\lfloor (m-1)/2 \rfloor + 1}{3\lfloor (m-1)/2 \rfloor + 1} = \frac{2\lceil m/2 \rceil - 1}{3\lceil m/2 \rceil - 2}$$

succeeds $\frac{2}{3}$ in $\mathcal{F}(\mathbb{B}(2m), m)$.

Table 2.8. The neighbors of the fraction $\frac{2}{3}$ in Farey subsequences; see also Table 2.6.

Sequence	**The predecessor of $\frac{2}{3}$**		**See**
$\mathcal{F}(\mathbb{B}(2m), m)$, $m > 1$	$\frac{2\lceil m/2\rceil-1}{3\lceil m/2\rceil-1}$		Remark 2.25
$\mathcal{F}(\mathbb{B}(n), m)$, $n < 2m$, $(n-m) > 1$	$\frac{2\lceil m/2\rceil-1}{3\lceil m/2\rceil-1}$	if $2n - 3m \geq 1$	Remark 2.43 (i)
	$\frac{\cancel{n-m}-1}{\cancel{2}(n-m)-1}$	if $2n - 3m \leq 1$	
$\mathcal{F}(\mathbb{B}(n), m)$, $n > 2m$, $m > 1$	$\frac{2\lceil m/2\rceil-1}{3\lceil m/2\rceil-1}$		Remark 2.43 (ii)

Sequence	**The successor of $\frac{2}{3}$**		**See**
$\mathcal{F}(\mathbb{B}(2m), m)$, $m > 1$	$\frac{2\lfloor(m-1)/2\rfloor+1}{3\lfloor(m-1)/2\rfloor+1}$		Remark 2.25
$\mathcal{F}(\mathbb{B}(n), m)$, $n < 2m$, $(n-m) > 1$	$\frac{2(n-m)+1}{3(n-m)+1}$	if $3m - 2n \geq 1$	Remark 2.43 (i)
	$\frac{2\lfloor(m-1)/2\rfloor+1}{3\lfloor(m-1)/2\rfloor+1}$	if $3m - 2n \leq 1$	
$\mathcal{F}(\mathbb{B}(n), m)$, $n > 2m$, $m > 1$	$\frac{2\lfloor(m-1)/2\rfloor+1}{3\lfloor(m-1)/2\rfloor+1}$		Remark 2.43 (ii)

Thus, if m is even, then the fractions

$$\frac{m-1}{(3m-2)/2} < \frac{2}{3} < \frac{m-1}{(3m-4)/2}$$

are consecutive in $\mathcal{F}(\mathbb{B}(2m), m)$. If m is odd, then the fractions

$$\frac{m}{(3m+1)/2} < \frac{2}{3} < \frac{m}{(3m-1)/2}$$

are consecutive in $\mathcal{F}(\mathbb{B}(2m), m)$.

2.2.8 More on neighboring fractions in $\mathcal{F}_m^\ell$ and $\mathcal{G}_m^\ell$

In this subsection we will describe explicitly the neighbors of fractions of the form $\frac{1}{j}$, $\frac{j-1}{j}$, $\frac{2}{j}$ and $\frac{j-2}{j}$ in the Farey subsequences $\mathcal{F}_m^\ell$ and $\mathcal{G}_m^\ell$, with the help of the formulas provided by Lemmas 2.13 and 2.15. Recall that we have already found in Section 2.2.6 the neighbors of these fractions in the Farey sequences $\mathcal{F}_m$. See Table 2.6 on the selected pairs of neighboring fractions that are considered in this book.

2.2.8.1 The neighbors of $\frac{1}{j}$

Let us find the neighbors of a fraction $\frac{1}{j}$ in the Farey subsequences $\mathcal{F}_m^\ell$ and $\mathcal{G}_m^\ell$.

Corollary 2.26. (i) *Consider a fraction $\frac{1}{j} \in \mathcal{F}_m^\ell$, where $0 < \ell < m$.*

(a) *If $m - j\ell \geq 1$, then the fraction*

$$\frac{\ell}{j\ell+1} \tag{2.51}$$

precedes the fraction $\frac{1}{j}$ in $\mathcal{F}_m^\ell$.

Thus, if $3m - n \geq 1$ and $(n - m)$ is even, then

$$\frac{(n-m)/2}{(3(n-m)-2)/2}$$

succeeds $\frac{1}{3}$ in $\mathcal{F}^{\leq\frac{1}{2}}(\mathbb{B}(n), m)$. If $3m - n \geq 1$ and $(n - m)$ is odd, then

$$\frac{(n-m+1)/2}{(3(n-m)-1)/2}$$

succeeds $\frac{1}{3}$ in $\mathcal{F}^{\leq\frac{1}{2}}(\mathbb{B}(n), m)$.

If $3m - n \leq 1$, then $\frac{m}{2m-1}$ succeeds $\frac{1}{3}$ in $\mathcal{F}^{\leq\frac{1}{2}}(\mathbb{B}(n), m)$.

2.2.9.10 The neighbors of $\frac{2}{3}$

In order to find the neighbors of the fraction $\frac{2}{3}$ in the Farey subsequences $\mathcal{F}(\mathbb{B}(n), m)$, with $n \neq 2m$, let us apply the order-reversing bijections $\mathcal{F}(\mathbb{B}(n), m) \leftrightarrow \mathcal{F}(\mathbb{B}(n), n - m)$ defined in (1.8) to the observations, made in Remark 2.42, on the neighbors of the fraction $\frac{1}{3}$.

Remark 2.43 (see also Table 2.8). (i) Consider a Farey subsequence $\mathcal{F}(\mathbb{B}(n), m)$ such that $n < 2m$ and $n - m > 1$.

If $2n - 3m \geq 1$, then

$$\frac{2\lceil m/2\rceil - 1}{3\lceil m/2\rceil - 1} = \frac{2\lfloor (m-1)/2\rfloor + 1}{3\lfloor (m-1)/2\rfloor + 2}$$

precedes $\frac{2}{3}$ in $\mathcal{F}^{\geq\frac{1}{2}}(\mathbb{B}(n), m)$.

Thus, if $2n - 3m \geq 1$ and m is even, then

$$\frac{m-1}{(3m-2)/2}$$

precedes $\frac{2}{3}$ in $\mathcal{F}^{\geq\frac{1}{2}}(\mathbb{B}(n), m)$. If $2n - 3m \geq 1$ and m is odd, then

$$\frac{m \cancel{- 1}}{(3m-1)/2}$$

precedes $\frac{2}{3}$ in $\mathcal{F}^{\geq\frac{1}{2}}(\mathbb{B}(n), m)$.

If $2n - 3m \leq 1$, then

$$\frac{\cancel{n-m} - 1}{\cancel{2}(n-m) - 1}$$

precedes $\frac{2}{3}$ in $\mathcal{F}^{\geq\frac{1}{2}}(\mathbb{B}(n), m)$.

If $3m - 2n \geq 1$, then

$$\frac{2(n-m)+1}{3(n-m)+1}$$

succeeds $\frac{2}{3}$ in $\mathcal{F}^{\geq\frac{1}{2}}(\mathbb{B}(n), m)$. If $3m - 2n \leq 1$, then

$$\frac{2\lfloor (m-1)/2\rfloor + 1}{3\lfloor (m-1)/2\rfloor + 1} = \frac{2\lceil m/2\rceil - 1}{3\lceil m/2\rceil - 2}$$

succeeds $\frac{2}{3}$ in $\mathcal{F}^{\geq\frac{1}{2}}(\mathbb{B}(n), m)$.

(ii) *The maps*

$$\mathcal{F}^{\leq\frac{1}{2}}(\mathbb{B}(2m), m) \to \mathcal{F}^{\geq\frac{1}{2}}(\mathbb{B}(2m), m), \quad \tfrac{h}{k} \mapsto \tfrac{k-h}{2k-3h}, \quad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -1 & 1 \\ -3 & 2 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right], \tag{3.3}$$

$$\mathcal{F}^{\geq\frac{1}{2}}(\mathbb{B}(2m), m) \to \mathcal{F}^{\leq\frac{1}{2}}(\mathbb{B}(2m), m), \quad \tfrac{h}{k} \mapsto \tfrac{2h-k}{3h-k}, \quad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 2 & -1 \\ 3 & -1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right], \tag{3.4}$$

are order-preserving and bijective.

Proof. (i) The maps in (3.1) and (3.2) can be regarded as the composite maps

$$\mathcal{F}^{\leq\frac{1}{2}}(\mathbb{B}(2m), m) \xrightarrow{(2.12)} \mathcal{F}_m \xrightarrow{(2.17)} \mathcal{F}^{\leq\frac{1}{2}}(\mathbb{B}(2m), m), \quad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -1 & 1 \\ -1 & 2 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right],$$

and

$$\mathcal{F}^{\geq\frac{1}{2}}(\mathbb{B}(2m), m) \xrightarrow{(2.14)} \mathcal{F}_m \xrightarrow{(2.19)} \mathcal{F}^{\geq\frac{1}{2}}(\mathbb{B}(2m), m), \quad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right],$$

respectively.

(ii) The maps in (3.3) and (3.4) can be regarded as the composite maps

$$\mathcal{F}^{\leq\frac{1}{2}}(\mathbb{B}(2m), m) \xrightarrow{(2.12)} \mathcal{F}_m \xrightarrow{(2.15)} \mathcal{F}^{\leq\frac{1}{2}}(\mathbb{B}(2m), m), \quad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \boldsymbol{Z}\left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right],$$

and

$$\mathcal{F}^{\geq\frac{1}{2}}(\mathbb{B}(2m), m) \xrightarrow{(2.14)} \mathcal{F}_m \xrightarrow{(2.13)} \mathcal{F}^{\geq\frac{1}{2}}(\mathbb{B}(2m), m), \quad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \boldsymbol{W}\left[\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right],$$

respectively. □

Note that the observations made in Theorem 3.1 on the properties of the sequences $\mathcal{F}^{\leq\frac{1}{2}}(\mathbb{B}(2m), m)$ and $\mathcal{F}^{\geq\frac{1}{2}}(\mathbb{B}(2m), m)$ are no longer valid for the entire half sequences of the Farey sequence $\mathcal{F}_{2m}$.

Example 3.2. Suppose $m := 3$. We have

$$\begin{aligned} \mathcal{F}_{2m} &= \left(\tfrac{0}{1} < \tfrac{1}{6} < \tfrac{1}{5} < \tfrac{1}{4} < \tfrac{1}{3} < \tfrac{2}{5} < \tfrac{1}{2} < \tfrac{3}{5} < \tfrac{2}{3} < \tfrac{3}{4} < \tfrac{4}{5} < \tfrac{5}{6} < \tfrac{1}{1}\right), \\ \mathcal{F}(\mathbb{B}(2m), m) &= \left(\tfrac{0}{1} < \quad < \quad < \tfrac{1}{4} < \tfrac{1}{3} < \tfrac{2}{5} < \tfrac{1}{2} < \tfrac{3}{5} < \tfrac{2}{3} < \tfrac{3}{4} < \quad < \quad < \tfrac{1}{1}\right). \end{aligned}$$

Also,

$$\mathcal{F}^{\geq\frac{1}{2}}(\mathbb{B}(2m), m) \ni \tfrac{3}{4} \overset{(3.2)}{\longmapsto} \tfrac{3}{3\cdot 3-4} = \tfrac{3}{5} \in \mathcal{F}^{\geq\frac{1}{2}}(\mathbb{B}(2m), m),$$

but

$$([\![\tfrac{1}{2}, 1]\!] \cap \mathcal{F}_{2m}) - \mathcal{F}^{\geq\frac{1}{2}}(\mathbb{B}(2m), m) \ni \tfrac{4}{5} \overset{(3.2)}{\longmapsto} \tfrac{4}{3\cdot 4-5} = \tfrac{4}{7} \notin \mathcal{F}_{2m}.$$

Further,

$$\mathcal{F}^{\leq\frac{1}{2}}(\mathbb{B}(2m), m) \ni \tfrac{1}{4} \overset{(3.3)}{\longmapsto} \tfrac{4-1}{2\cdot 4-3\cdot 1} = \tfrac{3}{5} \in \mathcal{F}^{\geq\frac{1}{2}}(\mathbb{B}(2m), m),$$

but

$$([\![0, \tfrac{1}{2}]\!] \cap \mathcal{F}_{2m}) - \mathcal{F}^{\leq\frac{1}{2}}(\mathbb{B}(2m), m) \ni \tfrac{1}{6} \overset{(3.3)}{\longmapsto} \tfrac{6-1}{2\cdot 6-3\cdot 1} = \tfrac{5}{9} \notin \mathcal{F}_{2m}.$$