Notes for *Foundations of Modern Analysis* by Avner Friedman

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Chapter 1 – Measure Theory

Section 1.1 – Rings and Algebras

Problems

1.1.1

$$\left(\varliminf_n E_n\right)^c = \varlimsup_n E_n^c, \quad \left(\varlimsup_n E_n\right)^c = \varliminf_n E_n^c.$$

Solution. Note that

$$\begin{aligned} x \in \varliminf_n E_n &\iff x \in E_n \text{ for all but finitely many } n \\ &\iff x \in E_n^c \text{ for only finitely many } n. \end{aligned}$$

Hence

$$\begin{aligned} x \in \left(\varliminf_n E_n\right)^c &\iff x \in E_n^c \text{ for infinitely many } n \\ &\iff x \in \varlimsup_n E_n^c, \end{aligned}$$

proving the first identity.

Next, let $F_n = E_n^c$ for every n. Then

$$\varlimsup_n E_n = \varlimsup_n F_n^c = \left(\varliminf_n F_n\right)^c = \left(\varliminf_n E_n^c\right)^c$$

by the first identity, and the second identity follows.

1.1.2

$$\varlimsup_n E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n, \quad \varliminf_n E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n.$$

Solution. Suppose $x \in \overline{\lim}_n E_n$. Then $x \in E_n$ for infinitely many n. It follows that $x \in \bigcup_{n=k}^{\infty} E_n$ for all $k \in \mathbb{N}$, and hence that $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$.

Conversely, assume that $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$. Then $x \in \bigcup_{n=k}^{\infty} E_n$ for all $k \in \mathbb{N}$. It follows that $x \in E_n$ for infinitely many n, and thus that $x \in \overline{\lim}_n E_n$. This proves the first identity.

Next, suppose that $x \in \underline{\lim}_n E_n$. Then $x \in E_n$ for all but finitely many n, so there is some $k' \in \mathbb{N}$ such that $x \in E_n$ for all $n \geq k'$. It follows that $x \in \bigcap_{n=k'}^{\infty} E_n$, and hence that $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$.

Conversely, assume that $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$. Then $x \in \bigcap_{n=k'}^{\infty} E_n$ for some $k' \in \mathbb{N}$, which means that $x \in E_n$ for all $n \geq k'$. It follows that $x \in E_n$ for all but finitely many n; that is, $x \in \underline{\lim}_n E_n$.

1.1.3

If $\mathscr{R}$ is a σ-ring and $E_n \in \mathscr{R}$, then

$$\bigcap_{n=1}^{\infty} E_n \in \mathscr{R}, \quad \overline{\lim_n} E_n \in \mathscr{R}, \quad \underline{\lim_n} E_n \in \mathscr{R}.$$

Solution. Let $Y = \bigcup_{n=1}^{\infty} E_n \in \mathscr{R}$. Then $E_n \subset Y$ for all Y, and it follows that

$$\bigcap_{n=1}^{\infty} E_n = Y \cap \left(\bigcap_{n=1}^{\infty} E_n \right) = Y - \left(Y - \bigcap_{n=1}^{\infty} E_n \right).$$

Notice that

$$Y - \bigcap_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (Y - E_n) \in \mathscr{R},$$

by properties (b) and (e). (The equality is analogous to the identity (1.1.2), but with Y in place of X.) It follows (by (b) again) that $\bigcap_{n=1}^{\infty} E_n \in \mathscr{R}$. For later reference, let us call this result (x).

Given $k \in \mathbb{N}$, let $A_n = \varnothing$ for $n < k$, and let $A_n = E_n$ for $n \geq k$. Then $A_n \in \mathscr{R}$ for all n by (a), hence

$$\bigcup_{n=k}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$$

by (e). It then follows by (x) that

$$\overline{\lim_n} E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \in \mathscr{R}.$$

By a similar argument we find that (x) implies

$$\bigcap_{n=k}^{\infty} E_n \in \mathscr{R}$$

for all $k \in \mathbb{N}$. Thus

$$\varliminf_n E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n \in \mathscr{R}$$

by (e).

1.1.4

The intersection of any collection of rings (algebras, σ-rings, or σ-algebras) is also a ring (an algebra, σ-ring, or σ-algebra).

Solution. Let $\mathscr{C}$ be a collection of classes. Let $\bigcap \mathscr{C}$ denote the intersection of all classes in $\mathscr{C}$. We will show that if one of the properties (a)-(e) is satisfied by all classes in $\mathscr{C}$, then $\bigcap \mathscr{C}$ satisfies that property as well. The result requested in the problem then follows as an immediate corollary.

It is clear that if every $\mathscr{R} \in \mathscr{C}$ satisfies (a), then so does $\bigcap \mathscr{C}$. Suppose every $\mathscr{R} \in \mathscr{C}$ satisfies (b). If $A, B \in \bigcap \mathscr{C}$ then $A, B \in \mathscr{R}$ for every $\mathscr{R} \in \mathscr{C}$. Hence $A - B \in \mathscr{R}$ for all $\mathscr{R} \in \mathscr{C}$, and it follows that $A - B \in \bigcap \mathscr{C}$. The argument for (c) is similar (with $A \cup B$ in place of $A - B$), and (d) is obvious.

Finally, suppose that every $\mathscr{R} \in \mathscr{C}$ satisfies (e). If $A_1, A_2, \ldots \in \bigcap \mathscr{C}$ then $A_1, A_2, \ldots \in \mathscr{R}$ for every $\mathscr{R} \in \mathscr{C}$. Hence $\bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$ for all $\mathscr{R} \in \mathscr{C}$, and it follows that $\bigcup_{n=1}^{\infty} A_n \in \bigcap \mathscr{C}$.

1.1.5

If $\mathscr{D}$ is any class of sets, then there exists a unique ring $\mathscr{R}_0$ such that (i) $\mathscr{R}_0 \supset \mathscr{D}$, and (ii) any ring $\mathscr{R}$ containing $\mathscr{D}$ contains also $\mathscr{R}_0$. $\mathscr{R}_0$ is called the *ring generated* by $\mathscr{D}$, and is denoted by $\mathscr{R}(\mathscr{D})$.

Solution. Let $\mathscr{R}_0$ be the intersection of all rings containing $\mathscr{D}$. This is a ring by the previous exercise, and it satisfies the properties (i) and (ii). To see that it is unique, let $\mathscr{R}_0'$ also by a ring satisfying (i) and (ii). Then $\mathscr{R}_0 \subset \mathscr{R}_0'$ and $\mathscr{R}_0' \subset \mathscr{R}_0$ by property (ii), so $\mathscr{R}_0 = \mathscr{R}_0'$.

1.1.6

If $\mathscr{D}$ is any class of sets, then there exists a unique σ-ring $\mathscr{S}_0$ such that (i) $\mathscr{S}_0 \supset \mathscr{D}$, and (ii) any σ-ring containing $\mathscr{D}$ contains also $\mathscr{S}_0$. We call $\mathscr{S}_0$ the *σ-ring generated* by $\mathscr{D}$, and denote it by $\mathscr{S}(\mathscr{D})$. A similar result holds for σ-algebras, and we speak of the *σ-algebra generated* by $\mathscr{D}$.

Solution. By the same argument as in the previous exercise, $\mathscr{S}_0$ is the intersection of all σ-rings containing $\mathscr{D}$. Similarly the σ-algebra generated by $\mathscr{D}$ is the intersection of all σ-algebras containing $\mathscr{D}$.

1.1.7

If $\mathscr{D}$ is any class of sets, then every set in $\mathscr{R}(\mathscr{D})$ can be covered by (that is, is contained in) a finite union of sets of $\mathscr{D}$. [*Hint:* The class $\mathscr{K}$ of sets that can be covered by finite unions of sets of $\mathscr{D}$ forms a ring.]

Solution. Let $\mathscr{K}$ be the class of all sets that can be covered by a finite union of sets in $\mathscr{D}$. Certainly $\varnothing \in \mathscr{K}$, since $\varnothing$ is a subset of the empty union. If $A, B \in \mathscr{K}$, then

$$A \subset \bigcup_{i=1}^{m} E_i, \quad B \subset \bigcup_{i=1}^{n} F_i,$$

for some sets $E_1, \ldots, E_m, F_1, \ldots, F_n \in \mathscr{D}$. (Note that m or n can be zero, in which case the corresponding union is empty.) Thus

$$A - B \subset A \subset \bigcup_{i=1}^{m} E_i$$

and

$$A \cup B \subset \left(\bigcup_{i=1}^{m} E_i \right) \cup \left(\bigcup_{j=1}^{n} F_j \right),$$

so both $A - B$ and $A \cup B$ are elements of $\mathscr{K}$.

The above shows that $\mathscr{K}$ is a ring, and certainly $\mathscr{D} \subset \mathscr{K}$. Hence $\mathscr{R}(\mathscr{D}) \subset \mathscr{K}$ by Problem 1.1.5, and it follows that every set in $\mathscr{R}(\mathscr{D})$ can be covered by a finite union of sets in $\mathscr{D}$.

Section 1.1 – Definition of Measure

Problems

1.2.1

If μ satisfies the properties (i)-(iii) in Definition 1.2.1, and if $\mu(E) < \infty$ for at least one set E, then (iv) is also satisfied.

Solution. We have

$$\mu(E) = \mu(E \cup \varnothing) = \mu(E) + \mu(\varnothing),$$

hence $\mu(\varnothing) = 0$.

1.2.2

Let X be an infinite space. Let $\mathcal{A}$ be the class of all subsets of X. Define $\mu(E) = 0$ if E is finite and $\mu(E) = \infty$ if E is infinite. Then μ is finitely additive but not completely additive.

Solution. Suppose $A, B \in \mathcal{A}$. Note that $A \cup B$ is finite if both A and B are finite, but infinite otherwise. Hence

$$\mu(A \cup B) = 0 = \mu(A) + \mu(B)$$

in the former case, and

$$\mu(A \cup B) = \infty = \mu(A) + \mu(B)$$

in the latter. This proves that μ is additive; *finite* additivity follows by a simple induction argument.

Let (x_n) be a sequence of distinct points in X. Then $\bigcup_{n=1}^{\infty}\{x_n\}$ is an infinite set, so

$$\mu\left(\bigcup_{n=1}^{\infty}\{x_n\}\right) = \infty,$$

but

$$\sum_{n=1}^{\infty}\mu(\{x_n\}) = 0.$$

Thus μ is not completely additive.

1.2.3

If μ is a measure on a σ-algebra $\mathcal{A}$, and if E, F are sets of $\mathcal{A}$, then

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

Solution. If $\mu(F) = \infty$, then $\mu(E \cup F) = \infty$ by Theorem 1.2.1(i), and the given equality holds. If $\mu(F) < \infty$, then

$$\begin{aligned}\mu(E \cup F) &= \mu[E \cup (F - (E \cap F))] \\ &= \mu(E) + \mu[F - (E \cap F)] \\ &= \mu(E) + \mu(F) - \mu(E \cap F),\end{aligned}$$

with the last equality following from Theorem 1.2.1(ii). Note that $E \cap F \subset F$ so that $\mu(E \cap F) \leq \mu(F) < \infty$. Hence we can rearrange the above to yield

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

1.2.6

Give an example of a measure μ and a monotone-decreasing sequence $\{E_n\}$ of $\mathcal{A}$ such that $\mu(E_n) = \infty$ for all n, and $\mu(\lim_n E_n) = 0$.

Solution. Let $X = \mathbb{R}$ and let $\mathcal{A} = \mathcal{P}(\mathbb{R})$ (the power set of $\mathbb{R}$; this is easily seen to be a σ-algebra). Define $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(E)$ is the number of points in E (with $\mu(E) = \infty$ if E is infinite). This is easily seen to be a measure.

For each $n \in \mathbb{N}$, let $E_n = (0, 1/n)$. Then (E_n) is a monotone decreasing sequence of sets in $\mathcal{A}$, $\mu(E_n) = \infty$ for all n, and

$$\mu\left(\lim_n E_n\right) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \mu(\varnothing) = 0.$$

Section 1.3 – Outer Measure

Problems

1.3.1

Define $\mu^*(E)$ as the number of points in E if E is finite and $\mu^*(E) = \infty$ if E is infinite. Show that μ^* is an outer measure. Determine the measurable sets.

Solution. Of the properties listed in Definition 1.3.1, only countable subadditivity is non-obvious for μ^*. But let us start with proving finite subadditivity.

Let A and B be sets. If either is infinite, then so is $A \cup B$, hence

$$\mu^*(A \cup B) = \infty = \mu^*(A) + \mu^*(B).$$

If both A and B are finite sets, then

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B - A) \leq \mu^*(A) + \mu^*(B)$$

by basic set-theoretic considerations. Thus μ^* is subadditive, and finite subadditivity follows by induction on the number of sets in the union.

Now, let (E_n) be a sequence of sets. If infinitely many of the sets E_n are nonempty, then $\sum_n \mu^*(E_n) = \infty$, and

$$\mu^*\left(\bigcup_n E_n\right) \leq \sum_n \mu^*(E_n)$$

follows. If only finitely many of the sets E_n are nonempty, let $E_{n_1}, E_{n_2}, \ldots, E_{n_k}$ be those sets. Then

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu^*\left(\bigcup_{i=1}^{k} E_{n_i}\right) \leq \sum_{i=1}^{k} \mu^*(E_{n_i}) = \sum_{n=1}^{\infty} \mu^*(E_n),$$

by finite subadditivity. This proves that μ^* is countably subadditive, and hence that μ^* is an outer measure.

Note that μ^* is *additive* on disjoint sets; if $A \cap B = \varnothing$, then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$. In particular,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A - E)$$

for all sets A, E. That is, all sets are measurable.

1.3.2

Define $\mu^*(\varnothing) = 0$, $\mu^*(E) = 1$ if $E \neq \varnothing$. Show that μ^* is an outer measure, and determine the measurable sets.

Solution. As in the previous excercise, the only slightly non-obvious property is countable subadditivity. Hence, let (E_n) be a sequence of sets. If all the sets E_n are empty, then certainly

$$\mu^*\left(\bigcup_n E_n\right) = 0 = \sum_n \mu^*(E_n).$$

If not, then there is some m such that $E_m \neq \varnothing$, and it follows that

$$\mu^*\left(\bigcup_n E_n\right) = 1 = \mu^*(E_m) \leq \sum_n \mu^*(E_n).$$

Thus μ^* is indeed countably subadditive, and therefore also an outer measure.

The empty set is measurable:

$$\mu^*(A \cap \varnothing) + \mu^*(A - \varnothing) = \mu^*(\varnothing) + \mu^*(A) = \mu^*(A)$$

for all sets A. It follows by Theorem 1.3.1 that X is measurable as well (the measurable sets make up a σ-algebra). Indeed $\varnothing$ and X are the only measurable sets. To see this, let E be any set other than those two (this requires that X contains at least two elements). Then both E and E^c are nonempty, so

$$\mu^*(X \cap E) + \mu^*(X - E) = \mu^*(E) + \mu^*(E^c) = 2 > 1 = \mu^*(X).$$

Section 1.4 – Construction of Outer Measures

Problems

1.4.4

If $\mathscr{K}$ is a σ-algebra and λ is a measure on $\mathscr{K}$, then $\mu^*(A) = \lambda(A)$ for any $A \in \mathscr{K}$. [*Hint:* $\mu^*(A) = \inf\{\lambda(E); E \in \mathscr{K}, E \supset A\}$.]

Solution. Note that the description of μ^* can be simplified when $\mathscr{K}$ is a σ-algebra and λ is a measure. For suppose that $A \subset X$, $E_n \in \mathscr{K}$ $(n = 1, 2, \dots)$, and $A \subset \bigcup_n E_n$. Then $E := \bigcup_n E_n \in \mathscr{K}$, and $\lambda(E) \leq \sum_n \lambda(E_n)$ by Theorem 1.2.2. Hence

$$\mu^*(A) = \inf\{\lambda(E); E \in \mathscr{K}, E \supset A\}.$$

Now, suppose that $A \in \mathscr{K}$. Certainly $\lambda(A)$ is an element of $\{\lambda(E); E \in \mathscr{K}, E \supset A\}$. And if $E \in \mathscr{K}$ and $E \supset A$, then $\lambda(E) \geq \lambda(A)$ by Theorem 1.2.1(i). Thus

$$\lambda(A) = \inf\{\lambda(E); E \in \mathscr{K}, E \supset A\} = \mu^*(A).$$

1.4.5

If $\mathscr{K}$ is a σ-algebra and λ is a measure on $\mathscr{K}$, then every set in $\mathscr{K}$ is μ^*-measurable.

Solution. Recall the simplified description of μ^* from the previous problem. Let $E \in \mathscr{K}$ and $A \subset X$. For every $\epsilon > 0$ there exists $F \in \mathscr{K}$ such that $F \supset A$ and

$$\mu^*(A) + \epsilon > \lambda(F);$$

else $\mu^*(A)$ would not be the greatest lower bound of $\{\lambda(E); E \in \mathscr{K}, E \supset A\}$. Moreover,

$$\lambda(F) = \lambda(F \cap E) + \lambda(F - E)$$

since λ is a measure on $\mathscr{K}$,

$$\lambda(F \cap E) + \lambda(F - E) = \mu^*(F \cap E) + \mu^*(F - E)$$

by what we found in the previous exercise, and finally

$$\mu^*(F \cap E) + \mu^*(F - E) \geq \mu^*(A \cap E) + \mu^*(A - E)$$

by monotonicity of the outer measure μ^*. Putting all of this together, we have

$$\mu^*(A) + \epsilon > \mu^*(A \cap E) + \mu^*(A - E)$$

for all $\epsilon > 0$, and thus

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A - E).$$

It follows that every set $E \in \mathscr{K}$ is μ^*-measurable.

Section 1.6 – The Lebesgue and the Lebesgue-Stieltjes Measures

Problems

1.6.3

The outer Lebesgue measure of a closed bounded interval $[a, b]$ on the real line is equal to $b - a$. [*Hint:* Use the Heine-Borel theorem to replace a countable covering by a finite covering.]

Solution. Suppose (E_n) is a sequence of elements of $\mathscr{K}$ (i.e. a sequence of open intervals) such that $[a, b] \subset \bigcup_{n=1}^{\infty} E_n$. The collection $\{E_n\}$ constitutes an *open cover* of $[a, b]$. By the Heine-Borel theorem $[a, b]$ is compact, hence there exists a *finite subcover* $\{E_{n_1}, \ldots, E_{n_k}\}$, such that $[a, b] \subset \bigcup_{i=1}^{k} E_{n_i}$.

Assume without loss of generality that $E_{n_i} \cap [a, b] \neq \varnothing$ for all i; otherwise we can simply remove those E_{n_i} that are disjoint with $[a, b]$ and still have a finite subcover. Write $E_{n_i} = (a_i, b_i)$ for each i, and define

$$\alpha = \min\{a_1, \ldots, a_k\}, \quad \beta = \max\{b_1, \ldots, b_k\}.$$

It is clear that α and β are the infimum and supremum, respectively, of $\bigcup_{i=1}^k E_{n_i}$. Note that $\alpha = a_j$ for some j, and $a_j < a < b_j$ since E_{n_j} and $[a, b]$ have nonempty intersection. Thus $(\alpha, a] \subset \bigcup_{i=1}^k E_{n_i}$, and similarly $[b, \beta) \subset \bigcup_{i=1}^k E_{n_i}$. It follows that

$$\bigcup_{i=1}^{k} E_{n_i} = (\alpha, \beta) \in \mathscr{K}.$$

Finally note that λ is finitely subadditive. (This is easily proven with induction.) (TODO: This is not convincing; use better proof from Rosenthal notes.) Thus,

$$\sum_{n=1}^{\infty} \lambda(E_n) \geq \sum_{i=1}^{k} \lambda(E_{n_i}) \geq \lambda\left[(\alpha, \beta)\right] = \beta - \alpha > b - a.$$

It follows that $b - a$ is a lower bound of the set

$$\Lambda([a, b]) := \left\{ \sum_{n=1}^{\infty} \lambda(E_n);\ E_n \in \mathscr{K},\ \bigcup_{n=1}^{\infty} E_n \supset [a, b] \right\}.$$

Moreover, for every $\epsilon > 0$ we have

$$[a, b] \subset \left(a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2}\right) \in \mathscr{K}$$

and

$$\lambda\left[\left(a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2}\right)\right] = b - a + \epsilon.$$

Hence $b - a$ is the *greatest* lower bound of $\Lambda([a, b])$, and $\mu^*([a, b]) = b - a$.

1.6.4

The outer Lebesgue measure of each of the intervals $(a, b), [a, b), (a, b]$ is equal to $b - a$.

Solution. Recall that μ^* is monotone, on account of being an outer measure. Hence $\mu^*[(a, b)] \leq \mu^*([a, b]) = b - a$, the latter equality being the result of the previous problem. Moreover, for all $\epsilon \in (0, b - a)$ we have

$$\left(a + \frac{\epsilon}{2}, b - \frac{\epsilon}{2}\right) \subset (a, b),$$

so that

$$\mu^*[(a, b)] \geq \mu^*\left[\left(a + \frac{\epsilon}{2}, b - \frac{\epsilon}{2}\right)\right] = b - a + \epsilon.$$

Thus $\mu^*[(a, b)] \geq b - a$, and it follows that $\mu^*[(a, b)] = b - a$.

The outer measures of $[a, b)$ and $(a, b]$ follow immediately by monotonicity:

$$\mu^*[(a, b)] \leq \mu^*([a, b)) \leq \mu^*([a, b]),$$

so that $\mu^*([a, b)) = b - a$. Similarly for $(a, b]$.

1.6.5

Consider the transformation $Tx = \alpha x + \beta$ from the real line onto itself, where α, β are real numbers and $\alpha \neq 0$. It maps sets E onto sets $T(E)$. Denote by μ (μ^*) the Lebesgue measure (outer measure) on the real line. Prove

(a) For any set E, $\mu^*(T(E)) = |\alpha|\mu^*(E)$.

(b) E is Lebesgue-measurable if and only if $T(E)$ is Lebesgue-measureable.

(c) If E is Lebesgue-measurable, then $\mu(T(E)) = |\alpha|\mu(E)$.

Solution. Let us start with a couple of simple observations:

- T is bijective, with inverse given by

$$T^{-1}(x) = \frac{x - \beta}{\alpha}.$$

- Suppose $I = (a, b)$. Then

$$T(I) = (\alpha a + \beta, \alpha b + \beta)$$

  if $\alpha > 0$, and

$$T(I) = (\beta b + \beta, \alpha a + \beta)$$

  if $\alpha < 0$. Either way,

$$\mu^*[T(I)] = |\alpha|(b - a) = |\alpha|\mu^*(I),$$

  where we have used one of the results of the previous exercise. Similarly, $T^{-1}(I)$ is an open interval and

$$\mu^*[T^{-1}(I)] = |\alpha|^{-1}\mu^*(I).$$

  Of course, the latter two identities still hold if $I = \varnothing$. Hence they hold for all $I \in \mathscr{K}$.

Also, let us use the notation

$$\Lambda(E) = \left\{ \sum_{n=1}^{\infty} \lambda(I_n);\ I_n \in \mathscr{K},\ \bigcup_{n=1}^{\infty} I_n \supset E \right\}$$

for all $E \subset \mathbb{R}$.

(a) Suppose (I_n) is a sequence in $\mathscr{K}$ (i.e. a sequence of open intervals) and $E \subset \bigcup_n I_n$. Then $T(I_n) \in \mathscr{K}$ for every n,

$$T(E) \subset T\left(\bigcup_n I_n\right) = \bigcup_n T(I_n),$$

and
$$\sum_n \lambda[T(I_n)] = |\alpha| \sum_n \lambda(I_n).$$
Thus, if $s \in \Lambda(E)$, then $|\alpha| s \in \Lambda[T(E)]$. It follows that
$$\mu^*[T(E)] = \inf \Lambda[T(E)] \leq |\alpha| \inf \Lambda(E) = |\alpha| \mu^*(E).$$

Conversely, suppose (J_n) is a sequence in $\mathscr{K}$ and $T(E) \subset \bigcup_n J_n$. Then $T^{-1}(J_n) \in \mathscr{K}$ for all n,
$$E = T^{-1}[T(E)] \subset T^{-1}\left(\bigcup_n J_n\right) = \bigcup_n T^{-1}(J_n),$$
and
$$\sum_n \lambda[T^{-1}(J_n)] = |\alpha|^{-1} \sum_n \lambda(J_n).$$
Hence, by the same logic as above, we find that $\mu^*(E) \leq |\alpha|^{-1}\mu^*[T(E)]$, and it follows that
$$\mu^*[T(E)] = |\alpha|\mu^*(E).$$

(b) Note that if $f : X \to Y$ is a bijective function (between arbitrary sets X, Y), then
$$\begin{aligned} f^{-1}[f(A)] &= A, \\ f(A \cup B) &= f(A) \cup f(B), \\ f(A - B) &= f(A) - f(B), \\ f[f^{-1}(C)] &= C, \end{aligned}$$
for all $A, B \subset X$ and $C \subset Y$.

Suppose that E is measureable:
$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A - E)$$
for all $A \subset \mathbb{R}$. Then, for all $B \subset \mathbb{R}$, we have
$$\begin{aligned} &\mu^*[B \cap T(E)] + \mu^*[B - T(E)] \\ &\qquad = \mu^*[T(T^{-1}(B) \cap E)] + \mu^*[T(T^{-1}(B) - E)] \\ &\qquad = |\alpha|\mu^*[T^{-1}(B) \cap E] + |\alpha|\mu^*[T^{-1}(B) - E] \\ &\qquad = |\alpha|\mu^*[T^{-1}(B)] \\ &\qquad = \mu^*(B), \end{aligned}$$
so that $T(E)$ is measurable.

Conversely, suppose that $T(E)$ is measurable. Then, for all $A \subset \mathbb{R}$,
$$\begin{aligned} &\mu^*(A \cap E) + \mu^*(A - E) \\ &\qquad = \mu^*[T^{-1}(T(A) \cap T(E))] + \mu^*[T^{-1}(T(A) - T(E))] \\ &\qquad = |\alpha|^{-1}\mu^*[T(A) \cap T(E)] + |\alpha|^{-1}\mu^*[T(A) - T(E)] \\ &\qquad = |\alpha|^{-1}\mu^*[T(A)] \\ &\qquad = \mu^*(A), \end{aligned}$$

so that E is measurable.

(c) This is immediate given (a), (b), and the definition of the Lebesgue-measure. First, $T(E)$ is Lebesgue-measurable by (b). Next, $\mu(E) = \mu^*(E)$ and $\mu[T(E)] = \mu^*[T(E)]$ since μ is simply the restriction of μ^* to the measurable sets. Finally, $\mu^*[T(E)] = |\alpha|\mu^*(E)$ by (a).

Chapter 2 – Integration

Section 2.1 – Definition of Measurable Functions

Problems

2.1.6

The *characteristic function* of a set E is the function χ_E defined by

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

Prove that the set E is measurable if and only if the function χ_E is measurable.

Solution. Suppose $E \in \mathcal{A}$. For all $c \in \mathbb{R}$,

$$\chi_E^{-1}\{(-\infty, c)\} = \{x \in X; \chi_E(x) < c\} = \begin{cases} \varnothing & (c \le 0), \\ E^c & (0 < c \le 1), \\ X & (c > 1), \end{cases}$$

so that $\chi_E^{-1}\{(-\infty, c)\} \in \mathcal{A}$. By Theorem 2.1.1, χ_E is measurable.

Conversely, suppose χ_E is measurable. Then E is measurable, since

$$E = X - E^c = \chi^{-1}\{(-\infty, 2)\} - \chi^{-1}\{(-\infty, 1)\}.$$

2.1.9

If f is measurable, then $|f|$ and $|f|^2$ are measurable.

Solution. If $c \le 0$, then

$$(|f|)^{-1}\{(-\infty, c)\} = (|f|^2)^{-1}\{(-\infty, c)\} = \emptyset \in \mathcal{A},$$

since $|f|$ and $|f|^2$ are nonnegative functions.

Let $c > 0$. Then

$$(|f|)^{-1}\{(-\infty, c)\} = \{x \in X; -c < f(x) < c\} = f^{-1}\{(-c, c)\}.$$

The set $(-c, c)$ is open, hence $f^{-1}\{(-c,c)\} \in \mathcal{A}$ by the measurability of f. Similarly,

$$(|f|^2)^{-1}\{(-\infty, c)\} = f^{-1}\{(-\sqrt{c}, \sqrt{c})\} \in \mathcal{A}.$$

Finally,

$$(|f|)^{-1}\{+\infty\} = (|f|^2)^{-1}\{+\infty\} = f^{-1}\{+\infty\} \cup f^{-1}\{-\infty\} \in \mathcal{A}$$

by the measurability of f, and

$$(|f|)^{-1}\{-\infty\} = (|f|^2)^{-1}\{-\infty\} = \emptyset \in \mathcal{A}$$

since $|f|$ and $|f|^2$ are nonnegative. Thus, both $|f|$ and $|f|^2$ are measurable by Theorem 2.1.1.

2.1.10

A monotone function defined on the real line is Lebesgue-measurable.

Solution. Let f be a monotone increasing extended real-valued function on $\mathbb{R}$;

$$(\forall x, y \in \mathbb{R}): \quad x < y \implies f(x) \leq f(y).$$

Given any $c \in \mathbb{R}$, let

$$\xi_c = \inf\{x \in X; f(x) \geq c\}.$$

We need to consider two cases: $f(\xi_c) < c$ and $f(\xi_c) \geq c$. In the former case, $f(x) < c$ for all $x \leq \xi_c$ and $f(x) \geq c$ for all $x > \xi_c$ (by monotonicity). Hence

$$f^{-1}\{(-\infty, c)\} = (-\infty, \xi_c].$$

This is a Borel set, hence also a Lebesgue set (see Problem 1.9.3). In the latter case, $f(x) < c$ for all $x < \xi_c$ and $f(x) \geq c$ for all $x \geq \xi_c$, so that

$$f^{-1}\{(-\infty, c)\} = (-\infty, \xi_c),$$

which is Lebesgue-measurable. Since c was arbitrary, we conclude that f is measurable, by Theorem 2.1.1.

The proof for f monotone decreasing is similar.

Section 2.2 – Operations on Measurable Functions

Problems

2.2.2

Let $g(u_1, \ldots, u_k)$ be a continuous function in $\mathbb{R}^k$, and let $\varphi_1, \ldots, \varphi_k$ be measurable functions. Prove that the composite function $h(x) = g[\varphi_1(x), \ldots, \varphi_k(x)]$ is a measurable function. Note that as a special case we may conclude that

$$\max(\varphi, \ldots, \varphi_n) \quad \text{and} \quad \min(\varphi, \ldots, \varphi_n)$$

are measurable functions.

Solution. We will use the following fact, which may be proven in a course in topology:

> $\mathbb{R}^k$ has a countable basis of product open subsets. Hence, if U is an open subset of $\mathbb{R}^k$, then there are open subsets $U_{ni} \subset \mathbb{R}$ for $n = 1, 2, \ldots$ and $i = 1, \ldots, k$ such that
>
> $$U = \bigcup_{n=1}^{\infty} (U_{n1} \times \cdots \times U_{nk}).$$

We are assuming that g is real-valued, likewise for the functions φ_i. Let $c \in \mathbb{R}$. Note that $g^{-1}\{(-\infty, c)\}$ is open by continuity of g. Thus

$$g^{-1}\{(-\infty, c)\} = \bigcup_{n=1}^{\infty} (U_{n1} \times \cdots \times U_{nk})$$

for some open subsets $U_{ni} \subset \mathbb{R}$. Hence

$$\begin{aligned} h^{-1}\{(-\infty, c)\} &= \{x \in X; g(\varphi_1(x), \ldots, \varphi_k(x)) \leq c\} \\ &= \{x \in X; (\varphi_1(x), \ldots, \varphi_k(x)) \in g^{-1}\{(-\infty, c)\}\} \\ &= \bigcup_{n=1}^{\infty} \{x \in X; (\varphi_1(x), \ldots, \varphi_k(x)) \in U_{n1} \times \cdots \times U_{nk}\} \\ &= \bigcup_{n=1}^{\infty} \bigcap_{i=1}^{k} \{x \in X; \varphi_i(x) \in U_{ni}\} \\ &= \bigcup_{n=1}^{\infty} \bigcap_{i=1}^{k} \varphi_i^{-1}(U_{ni}). \end{aligned}$$

The sets $\varphi_i(U_{ni})$ are measurable since the functions φ_i are measurable. It follows that $h^{-1}\{(-\infty, c)\}$ is measurable, and thus that h is measurable, by Theorem 2.1.1.

To apply the above to the max and min functions $\mathbb{R}^k \to \mathbb{R}$ we must show that they are continuous. Let $a < b$ and note that

$$\begin{aligned} \max{}^{-1}\{(a, b)\} = \{(x_1, \ldots, x_k) \in \mathbb{R}^k; x_i > a \text{ for some } i\} \\ \cap \{(x_1, \ldots, x_k) \in \mathbb{R}^k; x_i < b \text{ for all } i\}. \end{aligned}$$

Both sets in the above binary intersection are easily seen to be open by considering ϵ-neighborhoods about their points. It follows that $\max^{-1}(U)$ is open for all open subsets $U \in \mathbb{R}^k$, since every such U can be written as a countable union of open intervals. Thus max is continuous, and one similarly shows that min is continuous.

2.2.3

Let $f(x)$ be a measurable function and define

$$g(x) = \begin{cases} \frac{1}{f(x)}, & \text{if } f(x) \neq 0, \\ 0, & \text{if } f(x) = 0. \end{cases}$$

Prove that g is measurable.

Solution. For $c < 0$,

$$g^{-1}\{(-\infty, c)\} = \{x; 1/f(x) < c\} = \{x; 1/c < f < 0\} = f^{-1}\{(1/c, 0)\},$$

which is measurable by the measurability of f. Next,

$$g^{-1}\{(-\infty, 0)\} = \{x; 1/f(x) < 0\} = \{x; f(x) < 0\} = f^{-1}\{(-\infty, 0)\},$$

also measurable. Note that if we take the natural convention (unfortunately not addressed in the text) that $x/(\pm\infty) = 0$ for all $x \in \mathbb{R}$, then

$$g^{-1}(\{0\}) = \{x; f(x) = 0\} \cup \{x; f(x) = \pm\infty\} = f^{-1}(\{0\}) \cup f^{-1}(\{\pm\infty\}).$$

Hence, for $c > 0$,

$$\begin{aligned} g^{-1}\{(-\infty, c)\} &= g^{-1}\{(-\infty, 0)\} \cup g^{-1}(\{0\}) \cup g^{-1}\{(0, c)\} \\ &= f^{-1}\{(-\infty, 0)\} \cup f^{-1}(\{0\}) \cup f^{-1}(\{\pm\infty\}) \cup f^{-1}\{(1/c, \infty)\} \\ &= f^{-1}\{(-\infty, 0]\} \cup f^{-1}(\{\pm\infty\}) \cup f^{-1}\{(1/c, \infty)\}, \end{aligned}$$

which is measurable by the measurability of f (see Problem 2.1.4). Finally, $g^{-1}(\{\pm\infty\}) = \emptyset$, and it follows by Theorem 2.1.1 that g is measurable.

Section 2.3 – Egoroff's Theorem

Problems

2.3.2

Let $\{f_n\}$ be a sequence of measurable functions in a finite measure space X. Suppose that for almost every x, $\{f_n(x)\}$ is a bounded set. Then for any $\epsilon > 0$ there exist a positive number c and a measurable set E with $\mu(X - E) < \epsilon$, such that $|f_n(x)| \leq c$ for all $x \in E$, $n = 1, 2, \ldots$.

Solution. The definition we have for 'bounded set' applies to metric spaces, and it does not make much sense here since the functions f_n may be extended real-valued. Hence we will assume that '$\{f_n(x)\}$ is a bounded set' means that $\sup_n |f_n(x)| < \infty$.

Let $g = \sup_n |f_n|$, and note that g is measurable by Problem 2.1.9 and Theorem 2.2.3. Let $F = \{x;\ g(x) < \infty\}$. Notice that $g(x) < \infty$ if and only if $\{f_n(x)\}$ is bounded. Hence $\mu(X - F) = 0$.

For $k = 1, 2, \ldots$, define $F_k = \{x;\, g(x) \le k\}$. Then $F_1 \subset F_2 \subset \cdots$ and $\lim_k F_k = \bigcup_{k=1}^{\infty} F_k = F$. By Theorem 1.2.1(iv),

$$\lim_k \mu(X - F_k) = \mu(X - F) = 0.$$

Given any $\epsilon > 0$, there exists a positive integer K such that $\mu(X - F_k) < \epsilon$ for all $k \ge K$. In particular $\mu(X - F_K) < \epsilon$, and $g(x) \le K$ for all $x \in F_K$, which means that $|f_n(x)| \le K$ for all $x \in F_K$.

Section 2.4 – Convergence in Measure

Problems

2.4.3

Prove the following result (which immediately yields another proof of Corollary 2.4.2): Let f_n $(n = 1, 2, \ldots)$ and f be a.e. real-valued measurable functions in a finite measure space. For any $\epsilon > 0$, $n \ge 1$, let

$$E_n(\epsilon) = \{x;\, |f_n(x) - f(x)| \ge \epsilon\}.$$

Then $\{f_n\}$ converges a.e. to f if and only if

$$\lim_{n\to\infty} \mu\left[\bigcup_{m=n}^{\infty} E_m(\epsilon)\right] = 0 \quad \text{for any } \epsilon > 0. \tag{2.4.2}$$

[*Hint:* Let $F = \{x;\, \{f_n(x)\}$ is not convergent to $f(x)\}$. Then
$F = \bigcup_{k=1}^{\infty} \overline{\lim}_n E_n(1/k)$. Show that $\mu(F) = 0$ if and only if (2.4.2) holds.]

Solution. Define

$$F = \bigcup_{k=1}^{\infty} \overline{\lim_n} E_n\left(\frac{1}{k}\right) = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m\left(\frac{1}{k}\right).$$

Note that

$$\begin{aligned} x \in F &\iff \exists k, \forall n, \exists m \ge n,\, |f_m(x) - f(x)| \ge \frac{1}{k} \\ &\iff \neg\left(\forall k, \exists n, \forall m \ge n,\, |f_m(x) - f(x)| < \frac{1}{k}\right) \\ &\iff f_n(x) \not\to f(x), \end{aligned}$$

so that

$$F = \{x;\, f_n(x) \not\to f(x)\}.$$

Suppose (2.4.2) holds. Fix $\delta > 0$. For every positive integer k, there exists a positive integer n_k such that $n \ge n_k$ implies

$$\mu\left[\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{k}\right)\right] < \frac{\delta}{2^k}.$$

By subadditivity and monotonicity,

$$\begin{aligned}\mu(F) = \mu\left[\bigcup_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{k}\right)\right] &\leq \sum_{k=1}^{\infty}\mu\left[\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{k}\right)\right] \\ &\leq \sum_{k=1}^{\infty}\mu\left[\bigcup_{m=n_k}^{\infty} E_m\left(\frac{1}{k}\right)\right] < \sum_{k=1}^{\infty}\frac{\delta}{2^k} = \delta.\end{aligned}$$

Since δ was arbitrary, $\mu(F) = 0$, and it follows that $f_n \to f$ a.e.

Conversely, suppose $f_n \to f$ a.e., so that $\mu(F) = 0$. By monotonicity and Theorem 1.2.2,

$$0 = \mu(F) = \mu\left[\bigcup_{k=1}^{\infty}\overline{\lim_n} E_n\left(\frac{1}{k}\right)\right] \geq \mu\left[\overline{\lim_n} E_n\left(\frac{1}{l}\right)\right] \geq \overline{\lim_n}\,\mu\left[E_n\left(\frac{1}{l}\right)\right]$$

for all positive integers l. But of course $\overline{\lim}_n \mu\left[E_n\left(1/l\right)\right] \geq \underline{\lim}_n \mu\left[E_n\left(1/l\right)\right] \geq 0$ since μ is nonnegative, so $\lim_n \mu\left[E_n\left(1/l\right)\right]$ exists and is equal to zero. Note that the sets $\bigcup_{m=n}^{\infty} E_m(1/l)$ are decreasing, so their limit as $n \to \infty$ exists. Hence we can apply Corollary 1.2.3 and monotonicity to find that

$$\begin{aligned}\lim_n \mu\left[\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{l}\right)\right] = \mu\left[\lim_n \bigcup_{m=n}^{\infty} E_m\left(\frac{1}{l}\right)\right] &= \mu\left[\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{l}\right)\right] \\ &\leq \mu\left[\bigcup_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{k}\right)\right] = \mu(F) = 0.\end{aligned}$$

Finally, given $\epsilon > 0$, note that

$$E_n(\epsilon) \subset E_n\left(\frac{1}{\lceil 1/\epsilon \rceil}\right).$$

Hence

$$\lim_n \mu\left[\bigcup_{m=n}^{\infty} E_m(\epsilon)\right] \leq \lim_n \mu\left[\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{\lceil 1/\epsilon \rceil}\right)\right] \leq 0$$

by monotonicity, and (2.4.2) follows.

2.4.4

Let X be the set of all positive integers, $\mathcal{A}$ the class of all subsets of X, and $\mu(E)$ (for any $E \in \mathcal{A}$) the number of points in E. Prove that in this measure space, convergence in measure is equivalent to uniform convergence.

Solution. Uniform convergence always implies convergence in measure. Conversely, suppose (f_n) converges in measure to f. Given any $\epsilon > 0$, there exists a positive integer N such that $n \geq N$ implies

$$\mu\left[\{x;\, |f_n(x) - f(x)| \geq \epsilon\}\right] < 1.$$

That is, for $n \geq N$ the set $\{x;\, |f_n(x) - f(x)| \geq \epsilon\}$ is empty, which in particular means that $\sup_x |f_n(x) - f(x)| \leq \epsilon$. It follows that $f_n \to f$ uniformly.

Section 2.5 – Integrals of Simple Functions

Problems

2.5.2

An integrable simple function f is equal a.e. to zero if and only if $\int_E f d\mu = 0$ for any measurable set E.

Solution. Let f be an integrable simple function. Then f can be written in the form

$$f = \sum_{i=1}^{n} \alpha_i \chi_{E_i},$$

for mutually disjoint sets $E_1, \ldots E_n$, with all $\alpha_i \neq 0$, and all $\mu(E_i) < \infty$.

Suppose $f = 0$ a.e., and let E be any measurable set. By Theorem 2.5.1(b) and (g),

$$0 \leq \int_E f d\mu \leq \int f d\mu = \sum_{i=1}^{n} \alpha_i \mu(E_i).$$

But $\mu(E_i) = 0$ since $f = 0$ a.e., so $\int_E f d\mu = 0$.

Conversely, suppose $\int_E f d\mu = 0$ for all measurable sets E. Then

$$\alpha_i \mu(E_i) = \int_{E_i} f d\mu = 0,$$

so that $\mu(E_i) = 0$, for all $i \in \{1, \ldots, n\}$. It follows that $f = 0$ a.e.

Section 2.6 – Definition of the Integral

Problems

2.6.3

Let f be a measurable function. Prove that f is integrable if and only if f^+ and f^- are integrable, or if and only if $|f|$ is integrable.

Solution. Let f be measurable. We must prove the equivalence of the following statements:

(i) f is integrable.

(ii) f^+ and f^- are integrable.

(iii) $|f|$ is integrable.

We will first show that (iii) $\implies$ (ii), then that (ii) $\implies$ (i), and finally that (i) $\implies$ (iii).

Suppose that $|f|$ is integrable. Let $E = \{x;\, f(x) \geq 0\} = f^{-1}[0, \infty)$, and note that E is measurable since f is. There exists a Cauchy in the mean sequence (g_n) of integrable simple functions converging to $|f|$ a.e., and the sequence $(\chi_E g_n)$

is easily seen to satisfy the corresponding properties with respect to f^+. Since f^+ is measurable by Problem 2.1.8, this implies that it is integrable. The proof that f^- is integrable is similar.

Next, suppose that f^+ and f^- are integrable. Then there exist Cauchy in the mean sequences (g_n) and (h_n) of integrable simple functions converging a.e. to f^+ and f^-, respectively. Define a new sequence (f_n) of integrable simple functions by $f_n = g_n - h_n$. Then (f_n) is Cachy in the mean, since

$$|f_n - f_m| = |g_n - h_n - g_m + h_m| \leq |g_n - g_m| + |h_n - h_m|.$$

It also converges to f a.e. since

$$|f_n - f| = |g_n - h_n - f^+ + f^-| \leq |g_n - f^+| + |h_n - f^-|.$$

It follows that f is integrable.

Finally, assume that f is integrable. There is a Cauchy in the mean sequence (f_n) of integrable simple functions converging to f a.e. The sequence $(|f_n|)$ consists of integrable simple functions. It is Cauchy in the mean since

$$||f_n| - |f_m|| \leq |f_n - f_m|,$$

and it converges to $|f|$ a.e. since

$$||f_n| - |f|| \leq |f_n - f|.$$

Since $|f|$ is measurable by Problem 2.1.9, it follows that $|f|$ is integrable.

2.6.4

Let X be the measure space described in Problem 2.4.4. Then f is integrable if and only if the series $\sum_{n=1}^{\infty} |f(n)|$ is convergent. If f is integrable, then

$$\int f \, d\mu = \sum_{n=1}^{\infty} f(n).$$

Solution. Suppose f is integrable. Then there is a Cauchy in the mean sequence (f_n) of integrable simple functions converging to f a.e. We saw in the previous problem that this implies that $|f|$ is integrable, and that $(|f_n|)$ is a Cauchy in the mean sequence of integrable simple functions converging to $|f|$ a.e. Note that in this particular space convergence a.e. is the same as convergence everywhere (since the only subset with measure zero is $\emptyset$).

By Theorem 2.5.1(h),

$$\int |f_n| \, d\mu = \sum_{i=1}^{\infty} \int_{\{i\}} |f_n| \, d\mu = \sum_{i=1}^{\infty} |f_n(i)|.$$

Hence, in particular,

$$\int |f| \, d\mu = \lim_{n \to \infty} \sum_{i=1}^{\infty} |f_n(i)|.$$

Given any positive integer m, there exists n' such that

$$|f(i) - f_{n'}(i)| < 1/m \quad (i = 1, 2, \ldots, m)$$

(since $f_n \to f$) and

$$\left| \sum_{i=1}^{\infty} |f_{n'}(i)| - \int |f| \, d\mu \right| < 1$$

(since $\sum_i |f_n(i)| \to \int |f| \, d\mu$). Thus

$$\sum_{i=1}^{m} |f(i)| \le \sum_{i=1}^{m} |f(i) - f_{n'}(i)| + \sum_{i=1}^{m} |f_{n'}(i)| < 1 + \sum_{i=1}^{\infty} |f_{n'}(i)| < 2 + \int |f| \, d\mu,$$

and it follows that the series $\sum_{i=1}^{\infty} |f(i)|$ converges (to a finite number).

Conversely, assume that the series $\sum_{i=1}^{\infty} |f(i)|$ converges. Define a sequence of integrable simple functions (g_n) by

$$g_n = \sum_{i=1}^{n} f(i) \chi_{\{i\}}.$$

It is clear that $g_n \to f$ everywhere. Moreover, if $m > n$, then

$$\int |g_m - g_n| \, d\mu = \int \left| \sum_{i=n+1}^{m} f(i) \chi_{\{i\}} \right| d\mu = \sum_{n+1}^{m} |f(i)| \le \sum_{n+1}^{\infty} |f(i)|.$$

The right-hand side goes to zero as $n \to \infty$ since $\sum_{i=1}^{\infty} |f(i)|$ is convergent, which means that $\int |g_m - g_n| \, d\mu \to 0$ as $n, m \to \infty$; i.e., (g_n) is Cauchy in the mean. It follows that f is integrable, with

$$\int f \, d\mu = \lim_{n \to \infty} \int g_n \, d\mu = \lim_{n \to \infty} \sum_{i=1}^{n} f(i) = \sum_{i=1}^{\infty} f(i).$$

Chapter 3 – Metric Spaces

Section 3.1 – Topological and Metric Spaces

Problems

3.1.1

Prove that if (X, p) is a metric space, and if

$$\hat{\rho}(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)},$$

then also $(X, \hat{\rho})$ is a metric space. [*Hint:* Cf. the proof of (3.1.3).]

Solution. The only nonobvious property is the triangle inequality. Let x, y, z be arbitrary points of X. Since $t \mapsto t/(1+t)$ is monotone increasing on $[0, \infty)$, and since $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$, we have

$$\frac{\rho(x, z)}{1 + \rho(x, z)} \leq \frac{\rho(x, y) + \rho(y, z)}{1 + \rho(x, y) + \rho(y, z)}.$$

Moreover, by equation (3.1.3),

$$\frac{\rho(x, y) + \rho(y, z)}{1 + \rho(x, y) + \rho(y, z)} \leq \frac{\rho(x, y)}{1 + \rho(x, y)} + \frac{\rho(y, z)}{1 + \rho(y, z)}.$$

It follows that $\hat{\rho}(x, z) \leq \hat{\rho}(x, y) + \rho(y, z)$.

3.1.2

Let $X, \rho, \hat{\rho}$ be as in Problem 3.1.1. Prove that $\rho(x_n, x) \to 0$ if and only if $\hat{\rho}(x_n, x) \to 0$. Give an example showing that ρ and $\hat{\rho}$ are not equivalent in general.

Solution. It is clear that $\rho(x_n, x) \to 0$ implies $\hat{\rho}(x_n, x) \to 0$, since $\hat{\rho} \leq \rho$.

Conversely, suppose $\hat{\rho}(x_n, x) \to 0$. Given any $\epsilon > 0$, there is a positive integer N such that

$$\hat{\rho}(x_n, x) < \frac{\epsilon}{1 + \epsilon} \quad (n \geq N).$$

Substituting the definition of $\hat{\rho}$ and rearranging yields $\rho(x_n, x) < \epsilon$.

If ρ and $\hat{\rho}$ are equivalent then, in particular, there exists a positive constant β such that

$$\frac{\rho(x,y)}{\hat{\rho}(x,y)} \leq \beta$$

whenever $x \neq y$. But

$$\frac{\rho(x,y)}{\hat{\rho}(x,y)} = 1 + \rho(x,y),$$

so this is impossible if X is unbounded (w.r.t. ρ), say if $X = \mathbb{R}^n$ and ρ is the Euclidean metric.

3.1.6

Prove that the spaces l^1, s, c, c_0 are separable metric spaces.

Solution. For each of the spaces X we will take an arbitrary element $x \in X$ and demonstrate that every ϵ-ball around x contains a point y of a certain countable subset $Y \subset X$. It will then follow that Y is dense in X, and hence that X is separable.

Let $x = (x_i) \in l^1$. Fix $\epsilon > 0$. Since $\sum_i |x_i| < \infty$, there exists n such that

$$\sum_{i=n+1}^{\infty} |x_i| < \frac{\epsilon}{2}.$$

For $i = 1, \ldots, n$, choose $y_i \in \mathbb{Q}$ (or y_i with rational real and imaginary parts in the complex case) such that $|x_i - y_i| < \epsilon/2n$, and let $y = (y_1, \ldots, y_n, 0, 0, \ldots)$. Then

$$\rho(x,y) = \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=n+1}^{\infty} |x_i| < \epsilon.$$

Moreover, y is an element of the subset of l^1 consisting of sequences with rational components, with only finitely many being nonzero. This subset is easily seen to be countable, and it follows that l^1 is separable.

Next, let $x = (x_i) \in s$, and fix $\epsilon > 0$. Choose n such that

$$\sum_{i=n+1}^{\infty} \frac{1}{2^i} \frac{|x_i|}{1 + |x_i|} < \frac{\epsilon}{2}.$$

For $i = 1, \ldots, n$, choose $y_i \in \mathbb{Q}$ such that $|x_i - y_i| < \epsilon/2$, and let $y = (y_1, \ldots, y_n, 0, 0, \ldots)$. Then

$$\rho(x,y) = \sum_{i=1}^{n} \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|} + \sum_{i=n+1}^{\infty} \frac{1}{2^i} \frac{|x_i|}{1 + |x_i|} < \epsilon.$$

Similarly to the above, it follows that s is dense.

Finally, let $x = (x_i) \in c$. (This argument will also cover c_0.) Let $\xi = \lim_i x_i$ and fix $\epsilon > 0$. Choose n such that $|x_i - \xi| < \epsilon/2$ for all $i \geq n$. For $i = 1, \ldots, n-1$, choose $y_i \in \mathbb{Q}$ such that $|x_i - y_i| < \epsilon$. Also choose $\eta \in \mathbb{Q}$ such that $|\xi - \eta| < \epsilon/2$ (take $\eta = \xi = 0$ in the c_0 case), so that

$$|x_i - \eta| \leq |x_i - \xi| + |\xi - \eta| < \epsilon$$

for all $i \geq n$. Let $y = (y_1, \dots, y_{n-1}, \eta, \eta, \dots)$. Then

$$\rho(x, y) = \sup_i |x_i - y_i| \leq \epsilon.$$

It follows that c is separable (and c_0 as well).