Notes for *Foundations of Modern Analysis* by Avner Friedman

Anton Ottosson – antonott@kth.se

August 27, 2024

Chapter 1 − Measure Theory

Section 1.1 − Rings and algebras

Problems

1.1.1

$$\left(\varliminf_{n} E_n\right)^c = \varlimsup_{n} E_n^c, \quad \left(\varlimsup_{n} E_n\right)^c = \varliminf_{n} E_n^c.$$

Solution. For the first identity, note that

$$
\begin{aligned}
x \in \left(\varliminf_{n} E_n\right)^c &\iff x \notin \varliminf_{n} E_n \\
&\iff x \notin E_n \text{ for infinitely many } n \\
&\iff x \in E_n^c \text{ for infinitely many } n \\
&\iff x \in \varlimsup_{n} E_n^c.
\end{aligned}
$$

For the second,

$$
\begin{aligned}
x \in \left(\varlimsup_{n} E_n\right)^c &\iff x \notin \varlimsup_{n} E_n \\
&\iff x \in E_n \text{ for finitely many } n \\
&\iff x \in E_n^c \text{ for all but finitely many } n \\
&\iff x \in \varliminf_{n} E_n^c.
\end{aligned}
$$

1.1.2

$$\varlimsup_{n} E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n, \quad \varliminf_{n} E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n.$$

Solution. Suppose $x \in \varlimsup_{n} E_n$. Then $x \in E_n$ for infinitely many n. It follows that $x \in \bigcup_{n=k}^{\infty} E_n$ for all $k \in \mathbb{N}$, and hence that $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$.

Conversely, assume that $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$. Then $x \in \bigcup_{n=k}^{\infty} E_n$ for all $k \in \mathbb{N}$. It follows that $x \in E_n$ for infinitely many n, and thus that $x \in \overline{\lim}_n E_n$. This proves the first identity.

Next, suppose that $x \in \underline{\lim}_n E_n$. Then $x \in E_n$ for all but finitely many n, so there is some $k' \in \mathbb{N}$ such that $x \in E_n$ for all $n \geq k'$. It follows that $x \in \bigcap_{n=k'}^{\infty} E_n$, and hence that $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$.

Conversely, assume that $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$. Then $x \in \bigcap_{n=k'}^{\infty} E_n$ for some $k' \in \mathbb{N}$, which means that $x \in E_n$ for all $n \geq k'$. It follows that $x \in E_n$ for all but finitely many n; that is, $x \in \underline{\lim}_n E_n$.

1.1.3

If $\mathscr{R}$ is a σ-ring and $E_n \in \mathscr{R}$, then

$$\bigcap_{n=1}^{\infty} E_n \in \mathscr{R}, \quad \overline{\lim_n} E_n \in \mathscr{R}, \quad \underline{\lim_n} E_n \in \mathscr{R}.$$

Solution. Let $Y = \bigcup_{n=1}^{\infty} E_n$. Then $E_n \subset Y$ for all Y, and it follows that

$$\bigcap_{n=1}^{\infty} E_n = Y \cap \left(\bigcap_{n=1}^{\infty} E_n \right) = Y - \left(Y - \bigcap_{n=1}^{\infty} E_n \right).$$

Notice that

$$Y - \bigcap_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (Y - E_n) \in \mathscr{R},$$

by properties (b) and (e). (The equality is analogous to the identity (1.1.2), but with Y in place of X.) Hence, applying (b) again, we find that

$$\bigcap_{n=1}^{\infty} E_n = Y - \left(Y - \bigcap_{n=1}^{\infty} E_n \right) \in \mathscr{R}.$$

For later reference, let us call this result (x).

Given $k \in \mathbb{N}$, let $A_n = \varnothing$ for $n < k$, and let $A_n = E_n$ for $n \geq k$. Then $A_n \in \mathscr{R}$ for all n by (a), hence

$$\bigcup_{n=k}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$$

by (e). It then follows by (x) that

$$\overline{\lim_n} E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \in \mathscr{R}.$$

By a similar argument we find that (x) implies

$$\bigcap_{n=k}^{\infty} E_n \in \mathscr{R}$$

for all $k \in \mathbb{N}$. Hence

$$\varliminf_n E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n \in \mathscr{R}$$

by (e).

1.1.4

The intersection of any collection of rings (algebras, σ-rings, or σ-algebras) is also a ring (an algebra, σ-ring, or σ-algebra).

Solution. Let $\mathscr{C}$ be a collection of classes. Let $\bigcap \mathscr{C}$ denote the intersection of all classes in $\mathscr{C}$. We will show that if one of the properties (a)-(e) is satisfied by all classes in $\mathscr{C}$, then $\bigcap \mathscr{C}$ satisfies that property as well. The result requested in the problem then follows as an immediate corollary.

It is clear that if every $\mathscr{R} \in \mathscr{C}$ satisfies (a), then so does $\bigcap \mathscr{C}$. Suppose every $\mathscr{R} \in \mathscr{C}$ satisfies (b). If $A, B \in \bigcap \mathscr{C}$ then $A, B \in \mathscr{R}$ for every $\mathscr{R} \in \mathscr{C}$. Hence $A - B \in \mathscr{R}$ for all $\mathscr{R} \in \mathscr{C}$, and it follows that $A - B \in \bigcap \mathscr{C}$. The argument for (c) is similar (with $A \cup B$ in place of $A - B$), and (d) is obvious.

Finally, suppose that every $\mathscr{R} \in \mathscr{C}$ satisfies (e). If $A_1, A_2, \ldots \in \bigcap \mathscr{C}$ then $A_1, A_2, \ldots \in \mathscr{R}$ for every $\mathscr{R} \in \mathscr{C}$. Hence $\bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$ for all $\mathscr{R} \in \mathscr{C}$, and it follows that $\bigcup_{n=1}^{\infty} A_n \in \bigcap \mathscr{C}$.

1.1.5

If $\mathscr{D}$ is any class of sets, then there exists a unique ring $\mathscr{R}_0$ such that (i) $\mathscr{R}_0 \supset \mathscr{D}$, and (ii) any ring $\mathscr{R}$ containing $\mathscr{D}$ contains also $\mathscr{R}_0$. $\mathscr{R}_0$ is called the *ring generated* by $\mathscr{D}$, and is denoted by $\mathscr{R}(\mathscr{D})$.

Solution. Let $\mathscr{R}_0$ be the intersection of all rings containing $\mathscr{D}$. This is a ring by the previous exercise, and it satisfies the properties (i) and (ii). To see that it is unique, let $\mathscr{R}_0'$ also by a ring satisfying (i) and (ii). Then $\mathscr{R}_0 \subset \mathscr{R}_0'$ and $\mathscr{R}_0' \subset \mathscr{R}_0$ by property (ii), so $\mathscr{R}_0 = \mathscr{R}_0'$.

1.1.6

If $\mathscr{D}$ is any class of sets, then there exists a unique σ-ring $\mathscr{S}_0$ such that (i) $\mathscr{S}_0 \supset \mathscr{D}$, and (ii) any σ-ring containing $\mathscr{D}$ contains also $\mathscr{S}_0$. We call $\mathscr{S}_0$ the σ-*ring generated* by $\mathscr{D}$, and denote it by $\mathscr{S}(\mathscr{D})$. A similar result holds for σ-algebras, and we speak of the σ-*algebra generated* by $\mathscr{D}$.

Solution. By the same argument as in the previous exercise, $\mathscr{S}_0$ is the intersection of all σ-rings containing $\mathscr{D}$. Similarly the σ-algebra generated by $\mathscr{D}$ is the intersection of all σ-algebras containing $\mathscr{D}$.

1.1.7

If $\mathscr{D}$ is any class of sets, then every set in $\mathscr{R}(\mathscr{D})$ can be covered by (that is, is contained in) a finite union of sets of $\mathscr{D}$. [Hint: The class $\mathscr{K}$ of sets that can be covered by finite unions of sets of $\mathscr{D}$ forms a ring.]

Solution. Let $\mathscr{K}$ be the class of all sets that can be covered by a finite union of sets in $\mathscr{D}$. Certainly $\varnothing \in \mathscr{K}$, since $\varnothing$ is a subset of the empty union. If $A, B \in \mathscr{K}$, then

$$A \subset \bigcup_{i=1}^{m} E_i, \quad B \subset \bigcup_{i=1}^{n} F_i,$$

for some sets $E_1, \ldots, E_m, F_1, \ldots, F_n \in \mathscr{D}$. (Note that m or n can be zero, in which case the corresponding union is empty.) Thus

$$A - B \subset A \subset \bigcup_{i=1}^{m} E_i$$

and

$$A \cup B \subset \left(\bigcup_{i=1}^{m} E_i \right) \cup \left(\bigcup_{j=1}^{n} F_j \right),$$

so both $A - B$ and $A \cup B$ are elements of $\mathscr{K}$.

The above shows that $\mathscr{K}$ is a ring, and certainly $\mathscr{D} \subset \mathscr{K}$. Hence $\mathscr{R}(\mathscr{D}) \subset \mathscr{K}$ by Problem 1.1.5, and it follows that every set in $\mathscr{R}(\mathscr{D})$ can be covered by a finite union of sets in $\mathscr{D}$.

Section 1.1 – Definition of measure

Problems

1.2.1

If μ satisfies the properties (i)-(iii) in Definition 1.2.1, and if $\mu(E) < \infty$ for at least one set E, then (iv) is also satisfied.

Solution. We have

$$\mu(E) = \mu(E \cup \varnothing) = \mu(E) + \mu(\varnothing),$$

hence $\mu(\varnothing) = 0$.

1.2.2

Let X be an infinite space. Let $\mathcal{A}$ be the class of all subsets of X. Define $\mu(E) = 0$ if E is finite and $\mu(E) = \infty$ if E is infinite. Then μ is finitely additive but not completely additive.

Solution. Suppose $A, B \in \mathcal{A}$. Note that $A \cup B$ is finite if both A and B are finite, but infinite otherwise. Hence

$$\mu(A \cup B) = 0 = \mu(A) + \mu(B)$$

in the former case, and

$$\mu(A \cup B) = \infty = \mu(A) + \mu(B)$$

in the latter. This proves that μ is additive; *finite* additivity follows by a simple induction argument.

Let (x_n) be a sequence of distinct points in X. Then $\bigcup_{n=1}^{\infty}\{x_n\}$ is an infinite set, so

$$\mu\left(\bigcup_{n=1}^{\infty}\{x_n\}\right)=\infty,$$

but

$$\sum_{n=1}^{\infty}\mu(\{x_n\})=0.$$

Thus μ is not completely additive.

1.2.3

If μ is a measure on a σ-algebra $\mathcal{A}$, and if E, F are sets of $\mathcal{A}$, then

$$\mu(E)+\mu(F)=\mu(E\cup F)+\mu(E\cap F).$$

Solution. If $\mu(F)=\infty$, then $\mu(E\cup F)=\infty$ by Theorem 1.2.1(i), and the given equality holds. If $\mu(F)<\infty$, then

$$\begin{aligned}\mu(E\cup F)&=\mu[E\cup(F-(E\cap F))]\\&=\mu(E)+\mu[F-(E\cap F)]\\&=\mu(E)+\mu(F)-\mu(E\cap F),\end{aligned}$$

with the last equality following from Theorem 1.2.1(ii). Note that $E\cap F\subset F$ so that $\mu(E\cap F)\leq\mu(F)<\infty$. Hence we can rearrange the above to yield

$$\mu(E)+\mu(F)=\mu(E\cup F)+\mu(E\cap F).$$

1.2.6

Give an example of a measure μ and a monotone-decreasing sequence $\{E_n\}$ of $\mathcal{A}$ such that $\mu(E_n)=\infty$ for all n, and $\mu(\lim_n E_n)=0$.

Solution. Let $X=\mathbb{R}$ and let $\mathcal{A}=\mathcal{P}(\mathbb{R})$ (the power set of $\mathbb{R}$; this is easily seen to be a σ-algebra). Define $\mu:\mathcal{A}\to[0,\infty]$ such that $\mu(E)$ is the number of points in E (with $\mu(E)=\infty$ if E is infinite). This is easily seen to be a measure.

For each $n\in\mathbb{N}$, let $E_n=(0,1/n)$. Then (E_n) is a monotone decreasing sequence of sets in $\mathcal{A}$, $\mu(E_n)=\infty$ for all n, and

$$\mu\left(\lim_n E_n\right)=\mu\left(\bigcap_{n=1}^{\infty}E_n\right)=\mu(\varnothing)=0.$$