Notes for *Foundations of Modern Analysis* by Avner Friedman

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Chapter 1 – Measure theory

Section 1.1 – Rings and algebras

Problems

1.1.1

$$\left(\underline{\lim}_n E_n\right)^c = \overline{\lim}_n E_n^c, \quad \left(\overline{\lim}_n E_n\right)^c = \underline{\lim}_n E_n^c.$$

Solution. Note that

$$\begin{aligned} x \in \underline{\lim}_n E_n &\iff x \in E_n \text{ for all but finitely many } n \\ &\iff x \in E_n^c \text{ for only finitely many } n. \end{aligned}$$

Hence

$$\begin{aligned} x \in \left(\underline{\lim}_n E_n\right)^c &\iff x \in E_n^c \text{ for infinitely many } n \\ &\iff x \in \overline{\lim}_n E_n^c, \end{aligned}$$

proving the first identity.

Next, let $F_n = E_n^c$ for every n. Then

$$\overline{\lim}_n E_n = \overline{\lim}_n F_n^c = \left(\underline{\lim}_n F_n\right)^c = \left(\underline{\lim}_n E_n^c\right)^c$$

by the first identity, and the second identity follows.

1.1.2

$$\overline{\lim}_n E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n, \quad \underline{\lim}_n E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n.$$

Solution. Suppose $x \in \overline{\lim}_n E_n$. Then $x \in E_n$ for infinitely many n. It follows that $x \in \bigcup_{n=k}^{\infty} E_n$ for all $k \in \mathbb{N}$, and hence that $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$.

Conversely, assume that $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$. Then $x \in \bigcup_{n=k}^{\infty} E_n$ for all $k \in \mathbb{N}$. It follows that $x \in E_n$ for infinitely many n, and thus that $x \in \overline{\lim}_n E_n$. This proves the first identity.

Next, suppose that $x \in \underline{\lim}_n E_n$. Then $x \in E_n$ for all but finitely many n, so there is some $k' \in \mathbb{N}$ such that $x \in E_n$ for all $n \geq k'$. It follows that $x \in \bigcap_{n=k'}^{\infty} E_n$, and hence that $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$.

Conversely, assume that $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$. Then $x \in \bigcap_{n=k'}^{\infty} E_n$ for some $k' \in \mathbb{N}$, which means that $x \in E_n$ for all $n \geq k'$. It follows that $x \in E_n$ for all but finitely many n; that is, $x \in \underline{\lim}_n E_n$.

1.1.3

If $\mathscr{R}$ is a σ-ring and $E_n \in \mathscr{R}$, then

$$\bigcap_{n=1}^{\infty} E_n \in \mathscr{R}, \quad \overline{\lim_{n}}\, E_n \in \mathscr{R}, \quad \underline{\lim_{n}}\, E_n \in \mathscr{R}.$$

Solution. Let $Y = \bigcup_{n=1}^{\infty} E_n \in \mathscr{R}$. Then $E_n \subset Y$ for all Y, and it follows that

$$\bigcap_{n=1}^{\infty} E_n = Y \cap \left(\bigcap_{n=1}^{\infty} E_n \right) = Y - \left(Y - \bigcap_{n=1}^{\infty} E_n \right).$$

Notice that

$$Y - \bigcap_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (Y - E_n) \in \mathscr{R},$$

by properties (b) and (e). (The equality is analogous to the identity (1.1.2), but with Y in place of X.) It follows (by (b) again) that $\bigcap_{n=1}^{\infty} E_n \in \mathscr{R}$. For later reference, let us call this result (x).

Given $k \in \mathbb{N}$, let $A_n = \varnothing$ for $n < k$, and let $A_n = E_n$ for $n \geq k$. Then $A_n \in \mathscr{R}$ for all n by (a), hence

$$\bigcup_{n=k}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$$

by (e). It then follows by (x) that

$$\overline{\lim_{n}}\, E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \in \mathscr{R}.$$

By a similar argument we find that (x) implies

$$\bigcap_{n=k}^{\infty} E_n \in \mathscr{R}$$

for all $k \in \mathbb{N}$. Thus

$$\underline{\lim_{n}}\, E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n \in \mathscr{R}$$

by (e).

1.1.4

The intersection of any collection of rings (algebras, σ-rings, or σ-algebras) is also a ring (an algebra, σ-ring, or σ-algebra).

Solution. Let $\mathscr{C}$ be a collection of classes. Let $\bigcap \mathscr{C}$ denote the intersection of all classes in $\mathscr{C}$. We will show that if one of the properties (a)-(e) is satisfied by all classes in $\mathscr{C}$, then $\bigcap \mathscr{C}$ satisfies that property as well. The result requested in the problem then follows as an immediate corollary.

It is clear that if every $\mathscr{R} \in \mathscr{C}$ satisfies (a), then so does $\bigcap \mathscr{C}$. Suppose every $\mathscr{R} \in \mathscr{C}$ satisfies (b). If $A, B \in \bigcap \mathscr{C}$ then $A, B \in \mathscr{R}$ for every $\mathscr{R} \in \mathscr{C}$. Hence $A - B \in \mathscr{R}$ for all $\mathscr{R} \in \mathscr{C}$, and it follows that $A - B \in \bigcap \mathscr{C}$. The argument for (c) is similar (with $A \cup B$ in place of $A - B$), and (d) is obvious.

Finally, suppose that every $\mathscr{R} \in \mathscr{C}$ satisfies (e). If $A_1, A_2, \ldots \in \bigcap \mathscr{C}$ then $A_1, A_2, \ldots \in \mathscr{R}$ for every $\mathscr{R} \in \mathscr{C}$. Hence $\bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$ for all $\mathscr{R} \in \mathscr{C}$, and it follows that $\bigcup_{n=1}^{\infty} A_n \in \bigcap \mathscr{C}$.

1.1.5

If $\mathscr{D}$ is any class of sets, then there exists a unique ring $\mathscr{R}_0$ such that (i) $\mathscr{R}_0 \supset \mathscr{D}$, and (ii) any ring $\mathscr{R}$ containing $\mathscr{D}$ contains also $\mathscr{R}_0$. $\mathscr{R}_0$ is called the *ring generated* by $\mathscr{D}$, and is denoted by $\mathscr{R}(\mathscr{D})$.

Solution. Let $\mathscr{R}_0$ be the intersection of all rings containing $\mathscr{D}$. This is a ring by the previous exercise, and it satisfies the properties (i) and (ii). To see that it is unique, let $\mathscr{R}_0'$ also by a ring satisfying (i) and (ii). Then $\mathscr{R}_0 \subset \mathscr{R}_0'$ and $\mathscr{R}_0' \subset \mathscr{R}_0$ by property (ii), so $\mathscr{R}_0 = \mathscr{R}_0'$.

1.1.6

If $\mathscr{D}$ is any class of sets, then there exists a unique σ-ring $\mathscr{S}_0$ such that (i) $\mathscr{S}_0 \supset \mathscr{D}$, and (ii) any σ-ring containing $\mathscr{D}$ contains also $\mathscr{S}_0$. We call $\mathscr{S}_0$ the *σ-ring generated* by $\mathscr{D}$, and denote it by $\mathscr{S}(\mathscr{D})$. A similar result holds for σ-algebras, and we speak of the *σ-algebra generated* by $\mathscr{D}$.

Solution. By the same argument as in the previous exercise, $\mathscr{S}_0$ is the intersection of all σ-rings containing $\mathscr{D}$. Similarly the σ-algebra generated by $\mathscr{D}$ is the intersection of all σ-algebras containing $\mathscr{D}$.

1.1.7

If $\mathscr{D}$ is any class of sets, then every set in $\mathscr{R}(\mathscr{D})$ can be covered by (that is, is contained in) a finite union of sets of $\mathscr{D}$. [*Hint:* The class $\mathscr{K}$ of sets that can be covered by finite unions of sets of $\mathscr{D}$ forms a ring.]

Solution. Let $\mathscr{K}$ be the class of all sets that can be covered by a finite union of sets in $\mathscr{D}$. Certainly $\varnothing \in \mathscr{K}$, since $\varnothing$ is a subset of the empty union. If $A, B \in \mathscr{K}$, then

$$A \subset \bigcup_{i=1}^{m} E_i, \quad B \subset \bigcup_{i=1}^{n} F_i,$$

for some sets $E_1, \ldots, E_m, F_1, \ldots, F_n \in \mathscr{D}$. (Note that m or n can be zero, in which case the corresponding union is empty.) Thus

$$A - B \subset A \subset \bigcup_{i=1}^{m} E_i$$

and

$$A \cup B \subset \left(\bigcup_{i=1}^{m} E_i \right) \cup \left(\bigcup_{j=1}^{n} F_j \right),$$

so both $A - B$ and $A \cup B$ are elements of $\mathscr{K}$.

The above shows that $\mathscr{K}$ is a ring, and certainly $\mathscr{D} \subset \mathscr{K}$. Hence $\mathscr{R}(\mathscr{D}) \subset \mathscr{K}$ by Problem 1.1.5, and it follows that every set in $\mathscr{R}(\mathscr{D})$ can be covered by a finite union of sets in $\mathscr{D}$.

Section 1.1 – Definition of measure

Problems

1.2.1

If μ satisfies the properties (i)-(iii) in Definition 1.2.1, and if $\mu(E) < \infty$ for at least one set E, then (iv) is also satisfied.

Solution. We have

$$\mu(E) = \mu(E \cup \varnothing) = \mu(E) + \mu(\varnothing),$$

hence $\mu(\varnothing) = 0$.

1.2.2

Let X be an infinite space. Let $\mathcal{A}$ be the class of all subsets of X. Define $\mu(E) = 0$ if E is finite and $\mu(E) = \infty$ if E is infinite. Then μ is finitely additive but not completely additive.

Solution. Suppose $A, B \in \mathcal{A}$. Note that $A \cup B$ is finite if both A and B are finite, but infinite otherwise. Hence

$$\mu(A \cup B) = 0 = \mu(A) + \mu(B)$$

in the former case, and

$$\mu(A \cup B) = \infty = \mu(A) + \mu(B)$$

in the latter. This proves that μ is additive; *finite* additivity follows by a simple induction argument.

Let (x_n) be a sequence of distinct points in X. Then $\bigcup_{n=1}^{\infty}\{x_n\}$ is an infinite set, so

$$\mu\left(\bigcup_{n=1}^{\infty}\{x_n\}\right) = \infty,$$

but

$$\sum_{n=1}^{\infty}\mu(\{x_n\}) = 0.$$

Thus μ is not completely additive.

1.2.3

If μ is a measure on a σ-algebra $\mathcal{A}$, and if E, F are sets of $\mathcal{A}$, then

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

Solution. If $\mu(F) = \infty$, then $\mu(E \cup F) = \infty$ by Theorem 1.2.1(i), and the given equality holds. If $\mu(F) < \infty$, then

$$\begin{aligned}\mu(E \cup F) &= \mu[E \cup (F - (E \cap F))] \\ &= \mu(E) + \mu[F - (E \cap F)] \\ &= \mu(E) + \mu(F) - \mu(E \cap F),\end{aligned}$$

with the last equality following from Theorem 1.2.1(ii). Note that $E \cap F \subset F$ so that $\mu(E \cap F) \leq \mu(F) < \infty$. Hence we can rearrange the above to yield

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

1.2.6

Give an example of a measure μ and a monotone-decreasing sequence $\{E_n\}$ of $\mathcal{A}$ such that $\mu(E_n) = \infty$ for all n, and $\mu(\lim_n E_n) = 0$.

Solution. Let $X = \mathbb{R}$ and let $\mathcal{A} = \mathcal{P}(\mathbb{R})$ (the power set of $\mathbb{R}$; this is easily seen to be a σ-algebra). Define $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(E)$ is the number of points in E (with $\mu(E) = \infty$ if E is infinite). This is easily seen to be a measure.

For each $n \in \mathbb{N}$, let $E_n = (0, 1/n)$. Then (E_n) is a monotone decreasing sequence of sets in $\mathcal{A}$, $\mu(E_n) = \infty$ for all n, and

$$\mu\left(\lim_n E_n\right) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \mu(\varnothing) = 0.$$

Section 1.3 – Outer measure

Problems

1.3.1

Define $\mu^*(E)$ as the number of points in E if E is finite and $\mu^*(E) = \infty$ if E is infinite. Show that μ^* is an outer measure. Determine the measurable sets.

Solution. Of the properties listed in Definition 1.3.1, only countable subadditivity is non-obvious for μ^*. But let us start with proving finite subadditivity.

Let A and B be sets. If either is infinite, then so is $A \cup B$, hence

$$\mu^*(A \cup B) = \infty = \mu^*(A) + \mu^*(B).$$

If both A and B are finite sets, then

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B - A) \leq \mu^*(A) + \mu^*(B)$$

by basic set-theoretic considerations. Thus μ^* is subadditive, and finite subadditivity follows by induction on the number of sets in the union.

Now, let (E_n) be a sequence of sets. If infinitely many of the sets E_n are nonempty, then $\sum_n \mu^*(E_n) = \infty$, and

$$\mu^*\left(\bigcup_n E_n\right) \leq \sum_n \mu^*(E_n)$$

follows. If only finitely many of the sets E_n are nonempty, let $E_{n_1}, E_{n_2}, \ldots, E_{n_k}$ be those sets. Then

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu^*\left(\bigcup_{i=1}^{k} E_{n_i}\right) \leq \sum_{i=1}^{k} \mu^*(E_{n_i}) = \sum_{n=1}^{\infty} \mu^*(E_n),$$

by finite subadditivity. This proves that μ^* is countably subadditive, and hence that μ^* is an outer measure.

Note that μ^* is *additive* on disjoint sets; if $A \cap B = \varnothing$, then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$. In particular,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A - E)$$

for all sets A, E. That is, all sets are measurable.

1.3.2

Define $\mu^*(\varnothing) = 0$, $\mu^*(E) = 1$ if $E \neq \varnothing$. Show that μ^* is an outer measure, and determine the measurable sets.

Solution. As in the previous excercise, the only slightly non-obvious property is countable subadditivity. Hence, let (E_n) be a sequence of sets. If all the sets E_n are empty, then certainly

$$\mu^*\left(\bigcup_n E_n\right) = 0 = \sum_n \mu^*(E_n).$$

If not, then there is some m such that $E_m \neq \varnothing$, and it follows that

$$\mu^*\left(\bigcup_n E_n\right) = 1 = \mu^*(E_m) \leq \sum_n \mu^*(E_n).$$

Thus μ^* is indeed countably subadditive, and therefore also an outer measure.

The empty set is measurable:

$$\mu^*(A \cap \varnothing) + \mu^*(A - \varnothing) = \mu^*(\varnothing) + \mu^*(A) = \mu^*(A)$$

for all sets A. It follows by Theorem 1.3.1 that X is measurable as well (the measurable sets make up a σ-algebra). Indeed $\varnothing$ and X are the only measurable sets. To see this, let E be any set other than those two (this requires that X contains at least two elements). Then both E and E^c are nonempty, so

$$\mu^*(X \cap E) + \mu^*(X - E) = \mu^*(E) + \mu^*(E^c) = 2 > 1 = \mu^*(X).$$

Section 1.4 – Construction of outer measures

Problems

1.4.4

If $\mathscr{K}$ is a σ-algebra and λ is a measure on $\mathscr{K}$, then $\mu^*(A) = \lambda(A)$ for any $A \in \mathscr{K}$. [*Hint:* $\mu^*(A) = \inf\{\lambda(E); E \in \mathscr{K}, E \supset A\}$.]

Solution. Note that the description of μ^* can be simplified when $\mathscr{K}$ is a σ-algebra and λ is a measure. For suppose that $A \subset X$, $E_n \in \mathscr{K}$ $(n = 1, 2, \dots)$, and $A \subset \bigcup_n E_n$. Then $E := \bigcup_n E_n \in \mathscr{K}$, and $\lambda(E) \leq \sum_n \lambda(E_n)$ by Theorem 1.2.2. Hence

$$\mu^*(A) = \inf\{\lambda(E); E \in \mathscr{K}, E \supset A\}.$$

Now, suppose that $A \in \mathscr{K}$. Certainly $\lambda(A)$ is an element of $\{\lambda(E); E \in \mathscr{K}, E \supset A\}$. And if $E \in \mathscr{K}$ and $E \supset A$, then $\lambda(E) \geq \lambda(A)$ by Theorem 1.2.1(i). Thus

$$\lambda(A) = \inf\{\lambda(E); E \in \mathscr{K}, E \supset A\} = \mu^*(A).$$

1.4.5

If $\mathscr{K}$ is a σ-algebra and λ is a measure on $\mathscr{K}$, then every set in $\mathscr{K}$ is μ^*-measurable.

Solution. Recall the simplified description of μ^* from the previous problem. Let $E \in \mathscr{K}$ and $A \subset X$. For every $\epsilon > 0$ there exists $F \in \mathscr{K}$ such that $F \supset A$ and

$$\mu^*(A) + \epsilon > \lambda(F);$$

else $\mu^*(A)$ would not be the greatest lower bound of $\{\lambda(E); E \in \mathscr{K}, E \supset A\}$. Moreover,

$$\lambda(F) = \lambda(F \cap E) + \lambda(F - E)$$

since λ is a measure on $\mathscr{K}$,

$$\lambda(F \cap E) + \lambda(F - E) = \mu^*(F \cap E) + \mu^*(F - E)$$

by what we found in the previous exercise, and finally

$$\mu^*(F \cap E) + \mu^*(F - E) \geq \mu^*(A \cap E) + \mu^*(A - E)$$

by monotonicity of the outer measure μ^*. Putting all of this together, we have

$$\mu^*(A) + \epsilon > \mu^*(A \cap E) + \mu^*(A - E)$$

for all $\epsilon > 0$, and thus

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A - E).$$

It follows that every set $E \in \mathscr{K}$ is μ^*-measurable.

Section 1.6 – The Lebesgue and the Lebesgue-Stieltjes measures

Problems

1.6.3

The outer Lebesgue measure of a closed bounded interval $[a, b]$ on the real line is equal to $b - a$. [*Hint:* Use the Heine-Borel theorem to replace a countable covering by a finite covering.]

Solution. Suppose (E_n) is a sequence of elements of $\mathscr{K}$ (i.e. a sequence of open intervals) such that $[a, b] \subset \bigcup_{n=1}^{\infty} E_n$. The collection $\{E_n\}$ constitutes an *open cover* of $[a, b]$. By the Heine-Borel theorem $[a, b]$ is compact, hence there exists a *finite subcover* $\{E_{n_1}, \ldots, E_{n_k}\}$, such that $[a, b] \subset \bigcup_{i=1}^{k} E_{n_i}$.

Assume without loss of generality that $E_{n_i} \cap [a, b] \neq \varnothing$ for all i; otherwise we can simply remove those E_{n_i} that are disjoint with $[a, b]$ and still have a finite subcover. Write $E_{n_i} = (a_i, b_i)$ for each i, and define

$$\alpha = \min\{a_1, \ldots, a_k\}, \quad \beta = \max\{b_1, \ldots, b_k\}.$$

It is clear that α and β are the infimum and supremum, respectively, of $\bigcup_{i=1}^{k} E_{n_i}$. Note that $\alpha = a_j$ for some j, and $a_j < a < b_j$ since E_{n_j} and $[a, b]$ have nonempty intersection. Thus $(\alpha, a] \subset \bigcup_{i=1}^{k} E_{n_i}$, and similarly $[b, \beta) \subset \bigcup_{i=1}^{k} E_{n_i}$. It follows that

$$\bigcup_{i=1}^{k} E_{n_i} = (\alpha, \beta) \in \mathscr{K}.$$

Finally note that λ is finitely subadditive. (This is easily proven with induction.) (TODO: This is not convincing; use better proof from Rosenthal notes.) Thus,

$$\sum_{n=1}^{\infty} \lambda(E_n) \geq \sum_{i=1}^{k} \lambda(E_{n_i}) \geq \lambda\left[(\alpha, \beta)\right] = \beta - \alpha > b - a.$$

It follows that $b-a$ is a lower bound of the set

$$\Lambda([a,b]) := \left\{ \sum_{n=1}^{\infty} \lambda(E_n);\ E_n \in \mathscr{K},\ \bigcup_{n=1}^{\infty} E_n \supset [a,b] \right\}.$$

Moreover, for every $\epsilon > 0$ we have

$$[a,b] \subset \left(a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2}\right) \in \mathscr{K}$$

and

$$\lambda\left[\left(a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2}\right)\right] = b - a + \epsilon.$$

Hence $b-a$ is the *greatest* lower bound of $\Lambda([a,b])$, and $\mu^*([a,b]) = b-a$.

1.6.4

The outer Lebesgue measure of each of the intervals $(a,b), [a,b), (a,b]$ is equal to $b-a$.

Solution. Recall that μ^* is monotone, on account of being an outer measure. Hence $\mu^*[(a,b)] \leq \mu^*([a,b]) = b-a$, the latter equality being the result of the previous problem. Moreover, for all $\epsilon \in (0, b-a)$ we have

$$\left(a + \frac{\epsilon}{2}, b - \frac{\epsilon}{2}\right) \subset (a,b),$$

so that

$$\mu^*[(a,b)] \geq \mu^*\left[\left(a + \frac{\epsilon}{2}, b - \frac{\epsilon}{2}\right)\right] = b - a + \epsilon.$$

Thus $\mu^*[(a,b)] \geq b-a$, and it follows that $\mu^*[(a,b)] = b-a$.

The outer measures of $[a,b)$ and $(a,b]$ follow immediately by monotonicity:

$$\mu^*[(a,b)] \leq \mu^*([a,b)) \leq \mu^*([a,b]),$$

so that $\mu^*([a,b)) = b-a$. Similarly for $(a,b]$.

1.6.5

Consider the transformation $Tx = \alpha x + \beta$ from the real line onto itself, where α, β are real numbers and $\alpha \neq 0$. It maps sets E onto sets $T(E)$. Denote by μ (μ^*) the Lebesgue measure (outer measure) on the real line. Prove

(a) For any set E, $\mu^*(T(E)) = |\alpha|\mu^*(E)$.

(b) E is Lebesgue-measurable if and only if $T(E)$ is Lebesgue-measureable.

(c) If E is Lebesgue-measurable, then $\mu(T(E)) = |\alpha|\mu(E)$.

Solution. Let us start with a couple of simple observations:

- T is bijective, with inverse given by

$$T^{-1}(x) = \frac{x-\beta}{\alpha}.$$

- Suppose $I = (a, b)$. Then
$$T(I) = (\alpha a + \beta, \alpha b + \beta)$$
if $\alpha > 0$, and
$$T(I) = (\beta b + \beta, \alpha a + \beta)$$
if $\alpha < 0$. Either way,
$$\mu^*[T(I)] = |\alpha|(b - a) = |\alpha|\mu^*(I),$$
where we have used one of the results of the previous exercise. Similarly, $T^{-1}(I)$ is an open interval and
$$\mu^*[T^{-1}(I)] = |\alpha|^{-1}\mu^*(I).$$
Of course, the latter two identities still hold if $I = \varnothing$. Hence they hold for all $I \in \mathscr{K}$.

Also, let us use the notation
$$\Lambda(E) = \left\{ \sum_{n=1}^{\infty} \lambda(I_n);\ I_n \in \mathscr{K},\ \bigcup_{n=1}^{\infty} I_n \supset E \right\}$$
for all $E \subset \mathbb{R}$.

(a) Suppose (I_n) is a sequence in $\mathscr{K}$ (i.e. a sequence of open intervals) and $E \subset \bigcup_n I_n$. Then $T(I_n) \in \mathscr{K}$ for every n,
$$T(E) \subset T\left(\bigcup_n I_n\right) = \bigcup_n T(I_n),$$
and
$$\sum_n \lambda[T(I_n)] = |\alpha| \sum_n \lambda(I_n).$$
Thus, if $s \in \Lambda(E)$, then $|\alpha| s \in \Lambda[T(E)]$. It follows that
$$\mu^*[T(E)] = \inf \Lambda[T(E)] \leq |\alpha| \inf \Lambda(E) = |\alpha|\mu^*(E).$$

Conversely, suppose (J_n) is a sequence in $\mathscr{K}$ and $T(E) \subset \bigcup_n J_n$. Then $T^{-1}(J_n) \in \mathscr{K}$ for all n,
$$E = T^{-1}[T(E)] \subset T^{-1}\left(\bigcup_n J_n\right) = \bigcup_n T^{-1}(J_n),$$
and
$$\sum_n \lambda[T^{-1}(J_n)] = |\alpha|^{-1} \sum_n \lambda(J_n).$$
Hence, by the same logic as above, we find that $\mu^*(E) \leq |\alpha|^{-1}\mu^*[T(E)]$, and it follows that
$$\mu^*[T(E)] = |\alpha|\mu^*(E).$$

(b) Note that if $f : X \to Y$ is a bijective function (between arbitrary sets X, Y), then

$$
\begin{aligned}
f^{-1}[f(A)] &= A, \\
f(A \cup B) &= f(A) \cup f(B), \\
f(A - B) &= f(A) - f(B), \\
f[f^{-1}(C)] &= C,
\end{aligned}
$$

for all $A, B \subset X$ and $C \subset Y$.

Suppose that E is measureable:

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A - E)$$

for all $A \subset \mathbb{R}$. Then, for all $B \subset \mathbb{R}$, we have

$$
\begin{aligned}
&\mu^*[B \cap T(E)] + \mu^*[B - T(E)] \\
&\qquad = \mu^*[T(T^{-1}(B) \cap E)] + \mu^*[T(T^{-1}(B) - E)] \\
&\qquad = |\alpha|\mu^*[T^{-1}(B) \cap E] + |\alpha|\mu^*[T^{-1}(B) - E] \\
&\qquad = |\alpha|\mu^*[T^{-1}(B)] \\
&\qquad = \mu^*(B),
\end{aligned}
$$

so that $T(E)$ is measurable.

Conversely, suppose that $T(E)$ is measurable. Then, for all $A \subset \mathbb{R}$,

$$
\begin{aligned}
&\mu^*(A \cap E) + \mu^*(A - E) \\
&\qquad = \mu^*[T^{-1}(T(A) \cap T(E))] + \mu^*[T^{-1}(T(A) - T(E))] \\
&\qquad = |\alpha|^{-1}\mu^*[T(A) \cap T(E)] + |\alpha|^{-1}\mu^*[T(A) - T(E)] \\
&\qquad = |\alpha|^{-1}\mu^*[T(A)] \\
&\qquad = \mu^*(A),
\end{aligned}
$$

so that E is measurable.

(c) This is immediate given (a), (b), and the definition of the Lebesgue-measure. First, $T(E)$ is Lebesgue-measurable by (b). Next, $\mu(E) = \mu^*(E)$ and $\mu[T(E)] = \mu^*[T(E)]$ since μ is simply the restriction of μ^* to the measurable sets. Finally, $\mu^*[T(E)] = |\alpha|\mu^*(E)$ by (a).

Chapter 2 – Integration

Section 2.1 – Definition of Measurable Functions

Problems

2.1.6

The *characteristic function* of a set E is the function χ_E defined by

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

Prove that the set E is measurable if and only if the function χ_E is measurable.

Solution. Suppose $E \in \mathcal{A}$. For all $c \in \mathbb{R}$,

$$\chi_E^{-1}\{(-\infty, c)\} = \{x \in X; \chi_E(x) < c\} = \begin{cases} \varnothing & (c \leq 0), \\ E^c & (0 < c \leq 1), \\ X & (c > 1), \end{cases}$$

so that $\chi_E^{-1}\{(-\infty, c)\} \in \mathcal{A}$. By Theorem 2.1.1, χ_E is measurable.

Conversely, suppose χ_E is measurable. Then E is measurable, since

$$E = X - E^c = \chi^{-1}\{(-\infty, 2)\} - \chi^{-1}\{(-\infty, 1)\}.$$

2.1.9

If f is measurable, then $|f|$ and $|f|^2$ are measurable.

Solution. If $c \leq 0$, then

$$(|f|)^{-1}\{(-\infty, c)\} = (|f|^2)^{-1}\{(-\infty, c)\} = \emptyset \in \mathcal{A},$$

since $|f|$ and $|f|^2$ are nonnegative functions.

Let $c > 0$. Then

$$(|f|)^{-1}\{(-\infty, c)\} = \{x \in X; -c < f(x) < c\} = f^{-1}\{(-c, c)\}.$$

The set $(-c, c)$ is open, hence $f^{-1}\{(-c, c)\} \in \mathcal{A}$ by the measurability of f. Similarly,

$$(|f|^2)^{-1}\{(-\infty, c)\} = f^{-1}\{(-\sqrt{c}, \sqrt{c})\} \in \mathcal{A}.$$

Finally,

$$(|f|)^{-1}\{+\infty\} = (|f|^2)^{-1}\{+\infty\} = f^{-1}\{+\infty\} \cup f^{-1}\{-\infty\} \in \mathcal{A}$$

by the measurability of f, and

$$(|f|)^{-1}\{-\infty\} = (|f|^2)^{-1}\{-\infty\} = \emptyset \in \mathcal{A}$$

since $|f|$ and $|f|^2$ are nonnegative. Thus, both $|f|$ and $|f|^2$ are measurable by Theorem 2.1.1.

2.1.10

A monotone function defined on the real line is Lebesgue-measurable.

Solution. Let f be a monotone increasing extended real-valued function on $\mathbb{R}$;

$$(\forall x, y \in \mathbb{R}) : \quad x < y \implies f(x) \le f(y).$$

Given any $c \in \mathbb{R}$, let

$$\xi_c = \inf\{x \in X; f(x) \ge c\}.$$

We need to consider two cases: $f(\xi_c) < c$ and $f(\xi_c) \ge c$. In the former case, $f(x) < c$ for all $x \le \xi_c$ and $f(x) \ge c$ for all $x > \xi_c$ (by monotonicity). Hence

$$f^{-1}\{(-\infty, c)\} = (-\infty, \xi_c].$$

This is a Borel set, hence also a Lebesgue set (see Problem 1.9.3). In the latter case, $f(x) < c$ for all $x < \xi_c$ and $f(x) \ge c$ for all $x \ge \xi_c$, so that

$$f^{-1}\{(-\infty, c)\} = (-\infty, \xi_c),$$

which is Lebesgue-measurable. Since c was arbitrary, we conclude that f is measurable, by Theorem 2.1.1.

The proof for f monotone decreasing is similar.

Section 2.2 – Operations on Measurable Functions

Problems

2.2.2

Let $g(u_1, \ldots, u_k)$ be a continuous function in $\mathbb{R}^k$, and let $\varphi_1, \ldots, \varphi_k$ be measurable functions. Prove that the composite function $h(x) = g[\varphi_1(x), \ldots, \varphi_k(x)]$ is a measurable function. Note that as a special case we may conclude that

$$\max(\varphi, \ldots, \varphi_n) \quad \text{and} \quad \min(\varphi, \ldots, \varphi_n)$$

are measurable functions.

Solution. We will use the following fact, which may be proven in a course in topology:

> $\mathbb{R}^k$ has a countable basis of product open subsets. Hence, if U is an open subset of $\mathbb{R}^k$, then there are open subsets $U_{ni} \subset \mathbb{R}$ for $n = 1, 2, \ldots$ and $i = 1, \ldots, k$ such that
>
> $$U = \bigcup_{n=1}^{\infty} (U_{n1} \times \cdots \times U_{nk}).$$

We are assuming that g is real-valued, likewise for the functions φ_i. Let $c \in \mathbb{R}$. Note that $g^{-1}\{(-\infty, c)\}$ is open by continuity of g. Thus

$$g^{-1}\{(-\infty, c)\} = \bigcup_{n=1}^{\infty} (U_{n1} \times \cdots \times U_{nk})$$

for some open subsets $U_{ni} \subset \mathbb{R}$. Hence

$$\begin{aligned}
h^{-1}\{(-\infty, c)\} &= \{x \in X; g(\varphi_1(x), \ldots, \varphi_k(x)) \leq c\} \\
&= \{x \in X; (\varphi_1(x), \ldots, \varphi_k(x)) \in g^{-1}\{(-\infty, c)\}\} \\
&= \bigcup_{n=1}^{\infty} \{x \in X; (\varphi_1(x), \ldots, \varphi_k(x)) \in U_{n1} \times \cdots \times U_{nk}\} \\
&= \bigcup_{n=1}^{\infty} \bigcap_{i=1}^{k} \{x \in X; \varphi_i(x) \in U_{ni}\} \\
&= \bigcup_{n=1}^{\infty} \bigcap_{i=1}^{k} \varphi_i^{-1}(U_{ni}).
\end{aligned}$$

The sets $\varphi_i(U_{ni})$ are measurable since the functions φ_i are measurable. It follows that $h^{-1}\{(-\infty, c)\}$ is measurable, and thus that h is measurable, by Theorem 2.1.1.

To apply the above to the max and min functions $\mathbb{R}^k \to \mathbb{R}$ we must show that they are continuous. Let $a < b$ and note that

$$\begin{aligned}
\max{}^{-1}\{(a, b)\} = \{&(x_1, \ldots, x_k) \in \mathbb{R}^k; x_i > a \text{ for some } i\} \\
&\cap \{(x_1, \ldots, x_k) \in \mathbb{R}^k; x_i < b \text{ for all } i\}.
\end{aligned}$$

Both sets in the above binary intersection are easily seen to be open by considering ϵ-neighborhoods about their points. It follows that $\max^{-1}(U)$ is open for all open subsets $U \in \mathbb{R}^k$, since every such U can be written as a countable union of open intervals. Thus max is continuous, and one similarly shows that min is continuous.

2.2.3

Let $f(x)$ be a measurable function and define

$$g(x) = \begin{cases} \frac{1}{f(x)}, & \text{if } f(x) \neq 0, \\ 0, & \text{if } f(x) = 0. \end{cases}$$

Prove that g is measurable.

Solution. For $c < 0$,

$$g^{-1}\{(-\infty, c)\} = \{x; 1/f(x) < c\} = \{x; 1/c < f < 0\} = f^{-1}\{(1/c, 0)\},$$

which is measurable by the measurability of f. Next,

$$g^{-1}\{(-\infty, 0)\} = \{x; 1/f(x) < 0\} = \{x; f(x) < 0\} = f^{-1}\{(-\infty, 0)\},$$

also measurable. Note that if we take the natural convention (unfortunately not addressed in the text) that $x/(\pm\infty) = 0$ for all $x \in \mathbb{R}$, then

$$g^{-1}(\{0\}) = \{x; f(x) = 0\} \cup \{x; f(x) = \pm\infty\} = f^{-1}(\{0\}) \cup f^{-1}(\{\pm\infty\}).$$

Hence, for $c > 0$,

$$\begin{aligned} g^{-1}\{(-\infty, c)\} &= g^{-1}\{(-\infty, 0)\} \cup g^{-1}(\{0\}) \cup g^{-1}\{(0, c)\} \\ &= f^{-1}\{(-\infty, 0)\} \cup f^{-1}(\{0\}) \cup f^{-1}(\{\pm\infty\}) \cup f^{-1}\{(1/c, \infty)\} \\ &= f^{-1}\{(-\infty, 0]\} \cup f^{-1}(\{\pm\infty\}) \cup f^{-1}\{(1/c, \infty)\}, \end{aligned}$$

which is measurable by the measurability of f (see Problem 2.1.4). Finally, $g^{-1}(\{\pm\infty\}) = \emptyset$, and it follows by Theorem 2.1.1 that g is measurable.

Section 2.3 – Egoroff's Theorem

Problems

2.3.2

Let $\{f_n\}$ be a sequence of measurable functions in a finite measure space X. Suppose that for almost every x, $\{f_n(x)\}$ is a bounded set. Then for any $\epsilon > 0$ there exist a positive number c and a measurable set E with $\mu(X - E) < \epsilon$, such that $|f_n(x)| \leq c$ for all $x \in E$, $n = 1, 2, \ldots$.

Solution. The definition we have for 'bounded set' applies to metric spaces, and it does not make much sense here since the functions f_n may be extended real-valued. Hence we will assume that '$\{f_n(x)\}$ is a bounded set' means that $\sup_n |f_n(x)| < \infty$.

Let $g = \sup_n |f_n|$, and note that g is measurable by Problem 2.1.9 and Theorem 2.2.3. Let $F = \{x;\, g(x) < \infty\}$. Notice that $g(x) < \infty$ if and only if $\{f_n(x)\}$ is bounded. Hence $\mu(X - F) = 0$.

For $k = 1, 2, \dots,$ define $F_k = \{x;\, g(x) \le k\}$. Then $F_1 \subset F_2 \subset \cdots$ and $\lim_k F_k = \bigcup_{k=1}^{\infty} F_k = F$. By Theorem 1.2.1(iv),

$$\lim_k \mu(X - F_k) = \mu(X - F) = 0.$$

Given any $\epsilon > 0$, there exists a positive integer K such that $\mu(X - F_k) < \epsilon$ for all $k \ge K$. In particular $\mu(X - F_K) < \epsilon$, and $g(x) \le K$ for all $x \in F_K$, which means that $|f_n(x)| \le K$ for all $x \in F_K$.

Section 2.4 – Convergence in Measure

Problems

2.4.3

Prove the following result (which immediately yields another proof of Corollary 2.4.2): Let f_n $(n = 1, 2, \dots)$ and f be a.e. real-valued measurable functions in a finite measure space. For any $\epsilon > 0$, $n \ge 1$, let

$$E_n(\epsilon) = \{x;\, |f_n(x) - f(x)| \ge \epsilon\}.$$

Then $\{f_n\}$ converges a.e. to f if and only if

$$\lim_{n\to\infty} \mu\left[\bigcup_{m=n}^{\infty} E_m(\epsilon)\right] = 0 \quad \text{for any } \epsilon > 0. \tag{2.4.2}$$

[*Hint:* Let $F = \{x;\, \{f_n(x)\}$ is not convergent to $f(x)\}$. Then $F = \bigcup_{k=1}^{\infty} \overline{\lim}_n E_n(1/k)$. Show that $\mu(F) = 0$ if and only if (2.4.2) holds.]

Solution. Define

$$F = \bigcup_{k=1}^{\infty} \overline{\lim_{n}}\, E_n\left(\frac{1}{k}\right) = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m\left(\frac{1}{k}\right).$$

Note that

$$\begin{aligned} x \in F &\iff \exists k, \forall n, \exists m \ge n,\, |f_m(x) - f(x)| \ge \frac{1}{k} \\ &\iff \neg\left(\forall k, \exists n, \forall m \ge n,\, |f_m(x) - f(x)| < \frac{1}{k}\right) \\ &\iff f_n(x) \not\to f(x), \end{aligned}$$

so that

$$F = \{x;\, f_n(x) \not\to f(x)\}.$$

Suppose (2.4.2) holds. Fix $\delta > 0$. For every positive integer k, there exists a positive integer n_k such that $n \ge n_k$ implies

$$\mu\left[\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{k}\right)\right] < \frac{\delta}{2^k}.$$

By subadditivity and monotonicity,

$$\mu(F) = \mu\left[\bigcup_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{k}\right)\right] \le \sum_{k=1}^{\infty}\mu\left[\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{k}\right)\right]$$
$$\le \sum_{k=1}^{\infty}\mu\left[\bigcup_{m=n_k}^{\infty} E_m\left(\frac{1}{k}\right)\right] < \sum_{k=1}^{\infty}\frac{\delta}{2^k} = \delta.$$

Since δ was arbitrary, $\mu(F) = 0$, and it follows that $f_n \to f$ a.e.

Conversely, suppose $f_n \to f$ a.e., so that $\mu(F) = 0$. By monotonicity and Theorem 1.2.2,

$$0 = \mu(F) = \mu\left[\bigcup_{k=1}^{\infty}\overline{\lim_n} E_n\left(\frac{1}{k}\right)\right] \ge \mu\left[\overline{\lim_n} E_n\left(\frac{1}{l}\right)\right] \ge \overline{\lim_n}\,\mu\left[E_n\left(\frac{1}{l}\right)\right]$$

for all positive integers l. But of course $\overline{\lim}_n\,\mu\left[E_n\left(1/l\right)\right] \ge \underline{\lim}_n\,\mu\left[E_n\left(1/l\right)\right] \ge 0$ since μ is nonnegative, so $\lim_n \mu\left[E_n\left(1/l\right)\right]$ exists and is equal to zero. Note that the sets $\bigcup_{m=n}^{\infty} E_m(1/l)$ are decreasing, so their limit as $n \to \infty$ exists. Hence we can apply Corollary 1.2.3 and monotonicity to find that

$$\lim_n \mu\left[\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{l}\right)\right] = \mu\left[\lim_n \bigcup_{m=n}^{\infty} E_m\left(\frac{1}{l}\right)\right] = \mu\left[\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{l}\right)\right]$$
$$\le \mu\left[\bigcup_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{k}\right)\right] = \mu(F) = 0.$$

Finally, given $\epsilon > 0$, note that

$$E_n(\epsilon) \subset E_n\left(\frac{1}{\lceil 1/\epsilon\rceil}\right).$$

Hence

$$\lim_n \mu\left[\bigcup_{m=n}^{\infty} E_m(\epsilon)\right] \le \lim_n \mu\left[\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{\lceil 1/\epsilon\rceil}\right)\right] \le 0$$

by monotonicity, and (2.4.2) follows.