Notes for *Foundations of Modern Analysis* by Avner Friedman

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Chapter 1 – Measure Theory

Section 1.1 – Rings and algebras

Problems

1.1.1

$$\left(\underline{\lim}_n E_n\right)^c = \overline{\lim}_n E_n^c, \quad \left(\overline{\lim}_n E_n\right)^c = \underline{\lim}_n E_n^c.$$

Solution. For the first identity, note that

$$\begin{aligned} x \in \left(\underline{\lim}_n E_n\right)^c &\iff x \notin \underline{\lim}_n E_n \\ &\iff x \notin E_n \text{ for infinitely many } n \\ &\iff x \in E_n^c \text{ for infinitely many } n \\ &\iff x \in \overline{\lim}_n E_n^c. \end{aligned}$$

For the second,

$$\begin{aligned} x \in \left(\overline{\lim}_n E_n\right)^c &\iff x \notin \overline{\lim}_n E_n \\ &\iff x \in E_n \text{ for finitely many } n \\ &\iff x \in E_n^c \text{ for all but finitely many } n \\ &\iff x \in \underline{\lim}_n E_n^c. \end{aligned}$$

1.1.2

$$\overline{\lim}_n E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n, \quad \underline{\lim}_n E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n.$$

Solution. Suppose $x \in \overline{\lim}_n E_n$. Then $x \in E_n$ for infinitely many n. It follows that $x \in \bigcup_{n=k}^{\infty} E_n$ for all $k \in \mathbb{N}$, and hence that $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$.

Conversely, assume that $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$. Then $x \in \bigcup_{n=k}^{\infty} E_n$ for all $k \in \mathbb{N}$. It follows that $x \in E_n$ for infinitely many n, and thus that $x \in \overline{\lim}_n E_n$. This proves the first identity.

Next, suppose that $x \in \underline{\lim}_n E_n$. Then $x \in E_n$ for all but finitely many n, so there is some $k' \in \mathbb{N}$ such that $x \in E_n$ for all $n \geq k'$. It follows that $x \in \bigcap_{n=k'}^{\infty} E_n$, and hence that $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$.

Conversely, assume that $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$. Then $x \in \bigcap_{n=k'}^{\infty} E_n$ for some $k' \in \mathbb{N}$, which means that $x \in E_n$ for all $n \geq k'$. It follows that $x \in E_n$ for all but finitely many n; that is, $x \in \underline{\lim}_n E_n$.

1.1.3

If $\mathscr{R}$ is a σ-ring and $E_n \in \mathscr{R}$, then

$$\bigcap_{n=1}^{\infty} E_n \in \mathscr{R}, \quad \overline{\lim_n} E_n \in \mathscr{R}, \quad \underline{\lim_n} E_n \in \mathscr{R}.$$

Solution. Let $Y = \bigcup_{n=1}^{\infty} E_n$. Then $E_n \subset Y$ for all Y, and it follows that

$$\bigcap_{n=1}^{\infty} E_n = Y \cap \left(\bigcap_{n=1}^{\infty} E_n \right) = Y - \left(Y - \bigcap_{n=1}^{\infty} E_n \right).$$

Notice that

$$Y - \bigcap_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (Y - E_n) \in \mathscr{R},$$

by properties (b) and (e). (The equality is analogous to the identity (1.1.2), but with Y in place of X.) Hence, applying (b) again, we find that

$$\bigcap_{n=1}^{\infty} E_n = Y - \left(Y - \bigcap_{n=1}^{\infty} E_n \right) \in \mathscr{R}.$$

For later reference, let us call this result (x).

Given $k \in \mathbb{N}$, let $A_n = \varnothing$ for $n < k$, and let $A_n = E_n$ for $n \geq k$. Then $A_n \in \mathscr{R}$ for all n by (a), hence

$$\bigcup_{n=k}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$$

by (e). It then follows by (x) that

$$\overline{\lim_n} E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \in \mathscr{R}.$$

By a similar argument we find that (x) implies

$$\bigcap_{n=k}^{\infty} E_n \in \mathscr{R}$$

for all $k \in \mathbb{N}$. Hence

$$\varliminf_n E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n \in \mathscr{R}$$

by (e).

1.1.4

The intersection of any collection of rings (algebras, σ-rings, or σ-algebras) is also a ring (an algebra, σ-ring, or σ-algebra).

Solution. Let $\mathscr{C}$ be a collection of classes. Let $\bigcap \mathscr{C}$ denote the intersection of all classes in $\mathscr{C}$. We will show that if one of the properties (a)-(e) is satisfied by all classes in $\mathscr{C}$, then $\bigcap \mathscr{C}$ satisfies that property as well. The result requested in the problem then follows as an immediate corollary.

It is clear that if every $\mathscr{R} \in \mathscr{C}$ satisfies (a), then so does $\bigcap \mathscr{C}$. Suppose every $\mathscr{R} \in \mathscr{C}$ satisfies (b). If $A, B \in \bigcap \mathscr{C}$ then $A, B \in \mathscr{R}$ for every $\mathscr{R} \in \mathscr{C}$. Hence $A - B \in \mathscr{R}$ for all $\mathscr{R} \in \mathscr{C}$, and it follows that $A - B \in \bigcap \mathscr{C}$. The argument for (c) is similar (with $A \cup B$ in place of $A - B$), and (d) is obvious.

Finally, suppose that every $\mathscr{R} \in \mathscr{C}$ satisfies (e). If $A_1, A_2, \ldots \in \bigcap \mathscr{C}$ then $A_1, A_2, \ldots \in \mathscr{R}$ for every $\mathscr{R} \in \mathscr{C}$. Hence $\bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$ for all $\mathscr{R} \in \mathscr{C}$, and it follows that $\bigcup_{n=1}^{\infty} A_n \in \bigcap \mathscr{C}$.

1.1.5

If $\mathscr{D}$ is any class of sets, then there exists a unique ring $\mathscr{R}_0$ such that (i) $\mathscr{R}_0 \supset \mathscr{D}$, and (ii) any ring $\mathscr{R}$ containing $\mathscr{D}$ contains also $\mathscr{R}_0$. $\mathscr{R}_0$ is called the *ring generated* by $\mathscr{D}$, and is denoted by $\mathscr{R}(\mathscr{D})$.

Solution. Let $\mathscr{R}_0$ be the intersection of all rings containing $\mathscr{D}$. This is a ring by the previous exercise, and it satisfies the properties (i) and (ii). To see that it is unique, let $\mathscr{R}_0'$ also by a ring satisfying (i) and (ii). Then $\mathscr{R}_0 \subset \mathscr{R}_0'$ and $\mathscr{R}_0' \subset \mathscr{R}_0$ by property (ii), so $\mathscr{R}_0 = \mathscr{R}_0'$.

1.1.6

If $\mathscr{D}$ is any class of sets, then there exists a unique σ-ring $\mathscr{S}_0$ such that (i) $\mathscr{S}_0 \supset \mathscr{D}$, and (ii) any σ-ring containing $\mathscr{D}$ contains also $\mathscr{S}_0$. We call $\mathscr{S}_0$ the *σ-ring generated* by $\mathscr{D}$, and denote it by $\mathscr{S}(\mathscr{D})$. A similar result holds for σ-algebras, and we speak of the *σ-algebra generated* by $\mathscr{D}$.

Solution. By the same argument as in the previous exercise, $\mathscr{S}_0$ is the intersection of all σ-rings containing $\mathscr{D}$. Similarly the σ-algebra generated by $\mathscr{D}$ is the intersection of all σ-algebras containing $\mathscr{D}$.