Notes for *Foundations of Modern Analysis* by Avner Friedman

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Chapter 1 – Measure Theory

Section 1.1 – Rings and algebras

Problems

1.1.1

$$\left(\varliminf_n E_n\right)^c = \varlimsup_n E_n^c, \quad \left(\varlimsup_n E_n\right)^c = \varliminf_n E_n^c.$$

Solution. For the first identity, note that

$$\begin{aligned} x \in \left(\varliminf_n E_n\right)^c &\iff x \notin \varliminf_n E_n \\ &\iff x \notin E_n \text{ for infinitely many } n \\ &\iff x \in \varlimsup_n E_n^c. \end{aligned}$$

For the second,

$$\begin{aligned} x \in \left(\varlimsup_n E_n\right)^c &\iff x \notin \varlimsup_n E_n \\ &\iff x \in E_n \text{ for finitely many } n \\ &\iff x \in E_n^c \text{ for all but finitely many } n \\ &\iff x \in \varliminf_n E_n^c. \end{aligned}$$

1.1.2

$$\varlimsup_n E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n, \quad \varliminf_n E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n.$$

Solution. Suppose $x \in \varlimsup_n E_n$. Then $x \in E_n$ for infinitely many n. It follows that $x \in \bigcup_{n=k}^{\infty} E_n$ for all $k \in \mathbb{N}$, and hence that $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$.

Conversely, assume that $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$. Then $x \in \bigcup_{n=k}^{\infty} E_n$ for all $k \in \mathbb{N}$. It follows that $x \in E_n$ for infinitely many n, and thus that $x \in \varlimsup_n E_n$. This proves the first identity.

Next, suppose that $x \in \underline{\lim}_n E_n$. Then $x \in E_n$ for all but finitely many n, so there is some $k' \in \mathbb{N}$ such that $x \in E_n$ for all $n \geq k'$. It follows that $x \in \bigcap_{n=k'}^{\infty} E_n$, and hence that $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$.

Conversely, assume that $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$. Then $x \in \bigcap_{n=k'}^{\infty} E_n$ for some $k' \in \mathbb{N}$, which means that $x \in E_n$ for all $n \geq k'$. It follows that $x \in E_n$ for all but finitely many n; that is, $x \in \underline{\lim}_n E_n$.