

Notes for *Foundations of Modern Analysis* by
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Chapter 1 – Measure Theory

Section 1.1 – Rings and algebras

Problems

1.1.1

$$\left(\varliminf_n E_n\right)^c = \overline{\varliminf_n E_n^c}, \quad \left(\overline{\varliminf_n E_n}\right)^c = \varliminf_n E_n^c.$$

Solution. Note that

$$\begin{aligned} x \in \varliminf_n E_n &\iff x \in E_n \text{ for all but finitely many } n \\ &\iff x \in E_n^c \text{ for only finitely many } n. \end{aligned}$$

Hence

$$\begin{aligned} x \in \left(\varliminf_n E_n\right)^c &\iff x \in E_n^c \text{ for infinitely many } n \\ &\iff x \in \overline{\varliminf_n E_n^c}, \end{aligned}$$

proving the first identity.

Next, let $F_n = E_n^c$ for every n . Then

$$\overline{\varliminf_n E_n} = \overline{\varliminf_n F_n^c} = \left(\varliminf_n F_n\right)^c = \left(\varliminf_n E_n^c\right)^c$$

by the first identity, and the second identity follows.

1.1.2

$$\overline{\varliminf_n E_n} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n, \quad \varliminf_n E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n.$$

Solution. Suppose $x \in \overline{\lim}_n E_n$. Then $x \in E_n$ for infinitely many n . It follows that $x \in \bigcup_{n=k}^{\infty} E_n$ for all $k \in \mathbb{N}$, and hence that $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$.

Conversely, assume that $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$. Then $x \in \bigcup_{n=k}^{\infty} E_n$ for all $k \in \mathbb{N}$. It follows that $x \in E_n$ for infinitely many n , and thus that $x \in \overline{\lim}_n E_n$. This proves the first identity.

Next, suppose that $x \in \underline{\lim}_n E_n$. Then $x \in E_n$ for all but finitely many n , so there is some $k' \in \mathbb{N}$ such that $x \in E_n$ for all $n \geq k'$. It follows that $x \in \bigcap_{n=k'}^{\infty} E_n$, and hence that $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$.

Conversely, assume that $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$. Then $x \in \bigcap_{n=k'}^{\infty} E_n$ for some $k' \in \mathbb{N}$, which means that $x \in E_n$ for all $n \geq k'$. It follows that $x \in E_n$ for all but finitely many n ; that is, $x \in \underline{\lim}_n E_n$.

1.1.3

If \mathcal{R} is a σ -ring and $E_n \in \mathcal{R}$, then

$$\bigcap_{n=1}^{\infty} E_n \in \mathcal{R}, \quad \overline{\lim}_n E_n \in \mathcal{R}, \quad \underline{\lim}_n E_n \in \mathcal{R}.$$

Solution. Let $Y = \bigcup_{n=1}^{\infty} E_n \in \mathcal{R}$. Then $E_n \subset Y$ for all n , and it follows that

$$\bigcap_{n=1}^{\infty} E_n = Y \cap \left(\bigcap_{n=1}^{\infty} E_n \right) = Y - \left(Y - \bigcap_{n=1}^{\infty} E_n \right).$$

Notice that

$$Y - \bigcap_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (Y - E_n) \in \mathcal{R},$$

by properties (b) and (e). (The equality is analogous to the identity (1.1.2), but with Y in place of X .) It follows (by (b) again) that $\bigcap_{n=1}^{\infty} E_n \in \mathcal{R}$. For later reference, let us call this result (x).

Given $k \in \mathbb{N}$, let $A_n = \emptyset$ for $n < k$, and let $A_n = E_n$ for $n \geq k$. Then $A_n \in \mathcal{R}$ for all n by (a), hence

$$\bigcup_{n=k}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$$

by (e). It then follows by (x) that

$$\overline{\lim}_n E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \in \mathcal{R}.$$

By a similar argument we find that (x) implies

$$\bigcap_{n=k}^{\infty} E_n \in \mathcal{R}$$

for all $k \in \mathbb{N}$. Thus

$$\varliminf_n E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n \in \mathcal{R}$$

by (e).

1.1.4

The intersection of any collection of rings (algebras, σ -rings, or σ -algebras) is also a ring (an algebra, σ -ring, or σ -algebra).

Solution. Let \mathcal{C} be a collection of classes. Let $\bigcap \mathcal{C}$ denote the intersection of all classes in \mathcal{C} . We will show that if one of the properties (a)-(e) is satisfied by all classes in \mathcal{C} , then $\bigcap \mathcal{C}$ satisfies that property as well. The result requested in the problem then follows as an immediate corollary.

It is clear that if every $\mathcal{R} \in \mathcal{C}$ satisfies (a), then so does $\bigcap \mathcal{C}$. Suppose every $\mathcal{R} \in \mathcal{C}$ satisfies (b). If $A, B \in \bigcap \mathcal{C}$ then $A, B \in \mathcal{R}$ for every $\mathcal{R} \in \mathcal{C}$. Hence $A - B \in \mathcal{R}$ for all $\mathcal{R} \in \mathcal{C}$, and it follows that $A - B \in \bigcap \mathcal{C}$. The argument for (c) is similar (with $A \cup B$ in place of $A - B$), and (d) is obvious.

Finally, suppose that every $\mathcal{R} \in \mathcal{C}$ satisfies (e). If $A_1, A_2, \dots \in \bigcap \mathcal{C}$ then $A_1, A_2, \dots \in \mathcal{R}$ for every $\mathcal{R} \in \mathcal{C}$. Hence $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$ for all $\mathcal{R} \in \mathcal{C}$, and it follows that $\bigcup_{n=1}^{\infty} A_n \in \bigcap \mathcal{C}$.

1.1.5

If \mathcal{D} is any class of sets, then there exists a unique ring \mathcal{R}_0 such that (i) $\mathcal{R}_0 \supset \mathcal{D}$, and (ii) any ring \mathcal{R} containing \mathcal{D} contains also \mathcal{R}_0 . \mathcal{R}_0 is called the *ring generated by \mathcal{D}* , and is denoted by $\mathcal{R}(\mathcal{D})$.

Solution. Let \mathcal{R}_0 be the intersection of all rings containing \mathcal{D} . This is a ring by the previous exercise, and it satisfies the properties (i) and (ii). To see that it is unique, let \mathcal{R}'_0 also be a ring satisfying (i) and (ii). Then $\mathcal{R}_0 \subset \mathcal{R}'_0$ and $\mathcal{R}'_0 \subset \mathcal{R}_0$ by property (ii), so $\mathcal{R}_0 = \mathcal{R}'_0$.

1.1.6

If \mathcal{D} is any class of sets, then there exists a unique σ -ring \mathcal{S}_0 such that (i) $\mathcal{S}_0 \supset \mathcal{D}$, and (ii) any σ -ring containing \mathcal{D} contains also \mathcal{S}_0 . We call \mathcal{S}_0 the *σ -ring generated by \mathcal{D}* , and denote it by $\mathcal{S}(\mathcal{D})$. A similar result holds for σ -algebras, and we speak of the *σ -algebra generated by \mathcal{D}* .

Solution. By the same argument as in the previous exercise, \mathcal{S}_0 is the intersection of all σ -rings containing \mathcal{D} . Similarly the σ -algebra generated by \mathcal{D} is the intersection of all σ -algebras containing \mathcal{D} .

1.1.7

If \mathcal{D} is any class of sets, then every set in $\mathcal{R}(\mathcal{D})$ can be covered by (that is, is contained in) a finite union of sets of \mathcal{D} . [Hint: The class \mathcal{K} of sets that can be covered by finite unions of sets of \mathcal{D} forms a ring.]

Solution. Let \mathcal{K} be the class of all sets that can be covered by a finite union of sets in \mathcal{D} . Certainly $\emptyset \in \mathcal{K}$, since \emptyset is a subset of the empty union. If $A, B \in \mathcal{K}$, then

$$A \subset \bigcup_{i=1}^m E_i, \quad B \subset \bigcup_{i=1}^n F_i,$$

for some sets $E_1, \dots, E_m, F_1, \dots, F_n \in \mathcal{D}$. (Note that m or n can be zero, in which case the corresponding union is empty.) Thus

$$A - B \subset A \subset \bigcup_{i=1}^m E_i$$

and

$$A \cup B \subset \left(\bigcup_{i=1}^m E_i \right) \cup \left(\bigcup_{j=1}^n F_j \right),$$

so both $A - B$ and $A \cup B$ are elements of \mathcal{K} .

The above shows that \mathcal{K} is a ring, and certainly $\mathcal{D} \subset \mathcal{K}$. Hence $\mathcal{R}(\mathcal{D}) \subset \mathcal{K}$ by Problem 1.1.5, and it follows that every set in $\mathcal{R}(\mathcal{D})$ can be covered by a finite union of sets in \mathcal{D} .

Section 1.1 – Definition of measure

Problems

1.2.1

If μ satisfies the properties (i)-(iii) in Definition 1.2.1, and if $\mu(E) < \infty$ for at least one set E , then (iv) is also satisfied.

Solution. We have

$$\mu(E) = \mu(E \cup \emptyset) = \mu(E) + \mu(\emptyset),$$

hence $\mu(\emptyset) = 0$.

1.2.2

Let X be an infinite space. Let \mathcal{A} be the class of all subsets of X . Define $\mu(E) = 0$ if E is finite and $\mu(E) = \infty$ if E is infinite. Then μ is finitely additive but not completely additive.

Solution. Suppose $A, B \in \mathcal{A}$. Note that $A \cup B$ is finite if both A and B are finite, but infinite otherwise. Hence

$$\mu(A \cup B) = 0 = \mu(A) + \mu(B)$$

in the former case, and

$$\mu(A \cup B) = \infty = \mu(A) + \mu(B)$$

in the latter. This proves that μ is additive; *finite* additivity follows by a simple induction argument.

Let (x_n) be a sequence of distinct points in X . Then $\bigcup_{n=1}^{\infty} \{x_n\}$ is an infinite set, so

$$\mu\left(\bigcup_{n=1}^{\infty} \{x_n\}\right) = \infty,$$

but

$$\sum_{n=1}^{\infty} \mu(\{x_n\}) = 0.$$

Thus μ is not completely additive.

1.2.3

If μ is a measure on a σ -algebra \mathcal{A} , and if E, F are sets of \mathcal{A} , then

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

Solution. If $\mu(F) = \infty$, then $\mu(E \cup F) = \infty$ by Theorem 1.2.1(i), and the given equality holds. If $\mu(F) < \infty$, then

$$\begin{aligned} \mu(E \cup F) &= \mu[E \cup (F - (E \cap F))] \\ &= \mu(E) + \mu[F - (E \cap F)] \\ &= \mu(E) + \mu(F) - \mu(E \cap F), \end{aligned}$$

with the last equality following from Theorem 1.2.1(ii). Note that $E \cap F \subset F$ so that $\mu(E \cap F) \leq \mu(F) < \infty$. Hence we can rearrange the above to yield

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

1.2.6

Give an example of a measure μ and a monotone-decreasing sequence $\{E_n\}$ of \mathcal{A} such that $\mu(E_n) = \infty$ for all n , and $\mu(\lim_n E_n) = 0$.

Solution. Let $X = \mathbb{R}$ and let $\mathcal{A} = \mathcal{P}(\mathbb{R})$ (the power set of \mathbb{R} ; this is easily seen to be a σ -algebra). Define $\mu : \mathcal{A} \rightarrow [0, \infty]$ such that $\mu(E)$ is the number of points in E (with $\mu(E) = \infty$ if E is infinite). This is easily seen to be a measure.

For each $n \in \mathbb{N}$, let $E_n = (0, 1/n)$. Then (E_n) is a monotone decreasing sequence of sets in \mathcal{A} , $\mu(E_n) = \infty$ for all n , and

$$\mu\left(\lim_n E_n\right) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \mu(\emptyset) = 0.$$