Notes for *Foundations of Modern Analysis* by Avner Friedman

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Chapter 1 – Measure Theory

Section 1.1 – Rings and algebras

Problems

1.1.1

$$\left(\underline{\lim}_n E_n\right)^c = \overline{\lim}_n E_n^c, \quad \left(\overline{\lim}_n E_n\right)^c = \underline{\lim}_n E_n^c.$$

Solution. Note that

$$\begin{aligned} x \in \underline{\lim}_n E_n &\iff x \in E_n \text{ for all but finitely many } n \\ &\iff x \in E_n^c \text{ for only finitely many } n. \end{aligned}$$

Hence

$$\begin{aligned} x \in \left(\underline{\lim}_n E_n\right)^c &\iff x \in E_n^c \text{ for infinitely many } n \\ &\iff x \in \overline{\lim}_n E_n^c, \end{aligned}$$

proving the first identity.

Next, let $F_n = E_n^c$ for every n. Then

$$\overline{\lim}_n E_n = \overline{\lim}_n F_n^c = \left(\underline{\lim}_n F_n\right)^c = \left(\underline{\lim}_n E_n^c\right)^c$$

by the first identity, and the second identity follows.

1.1.2

$$\overline{\lim}_n E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n, \quad \underline{\lim}_n E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n.$$

Solution. Suppose $x \in \overline{\lim}_n E_n$. Then $x \in E_n$ for infinitely many n. It follows that $x \in \bigcup_{n=k}^{\infty} E_n$ for all $k \in \mathbb{N}$, and hence that $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$.

Conversely, assume that $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$. Then $x \in \bigcup_{n=k}^{\infty} E_n$ for all $k \in \mathbb{N}$. It follows that $x \in E_n$ for infinitely many n, and thus that $x \in \overline{\lim}_n E_n$. This proves the first identity.

Next, suppose that $x \in \underline{\lim}_n E_n$. Then $x \in E_n$ for all but finitely many n, so there is some $k' \in \mathbb{N}$ such that $x \in E_n$ for all $n \geq k'$. It follows that $x \in \bigcap_{n=k'}^{\infty} E_n$, and hence that $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$.

Conversely, assume that $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$. Then $x \in \bigcap_{n=k'}^{\infty} E_n$ for some $k' \in \mathbb{N}$, which means that $x \in E_n$ for all $n \geq k'$. It follows that $x \in E_n$ for all but finitely many n; that is, $x \in \underline{\lim}_n E_n$.

1.1.3

If $\mathscr{R}$ is a σ-ring and $E_n \in \mathscr{R}$, then

$$\bigcap_{n=1}^{\infty} E_n \in \mathscr{R}, \quad \overline{\lim_n} E_n \in \mathscr{R}, \quad \underline{\lim_n} E_n \in \mathscr{R}.$$

Solution. Let $Y = \bigcup_{n=1}^{\infty} E_n \in \mathscr{R}$. Then $E_n \subset Y$ for all Y, and it follows that

$$\bigcap_{n=1}^{\infty} E_n = Y \cap \left(\bigcap_{n=1}^{\infty} E_n \right) = Y - \left(Y - \bigcap_{n=1}^{\infty} E_n \right).$$

Notice that

$$Y - \bigcap_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (Y - E_n) \in \mathscr{R},$$

by properties (b) and (e). (The equality is analogous to the identity (1.1.2), but with Y in place of X.) It follows (by (b) again) that $\bigcap_{n=1}^{\infty} E_n \in \mathscr{R}$. For later reference, let us call this result (x).

Given $k \in \mathbb{N}$, let $A_n = \varnothing$ for $n < k$, and let $A_n = E_n$ for $n \geq k$. Then $A_n \in \mathscr{R}$ for all n by (a), hence

$$\bigcup_{n=k}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$$

by (e). It then follows by (x) that

$$\overline{\lim_n} E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \in \mathscr{R}.$$

By a similar argument we find that (x) implies

$$\bigcap_{n=k}^{\infty} E_n \in \mathscr{R}$$

for all $k \in \mathbb{N}$. Thus

$$\varliminf_n E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n \in \mathscr{R}$$

by (e).

1.1.4

The intersection of any collection of rings (algebras, σ-rings, or σ-algebras) is also a ring (an algebra, σ-ring, or σ-algebra).

Solution. Let $\mathscr{C}$ be a collection of classes. Let $\bigcap \mathscr{C}$ denote the intersection of all classes in $\mathscr{C}$. We will show that if one of the properties (a)-(e) is satisfied by all classes in $\mathscr{C}$, then $\bigcap \mathscr{C}$ satisfies that property as well. The result requested in the problem then follows as an immediate corollary.

It is clear that if every $\mathscr{R} \in \mathscr{C}$ satisfies (a), then so does $\bigcap \mathscr{C}$. Suppose every $\mathscr{R} \in \mathscr{C}$ satisfies (b). If $A, B \in \bigcap \mathscr{C}$ then $A, B \in \mathscr{R}$ for every $\mathscr{R} \in \mathscr{C}$. Hence $A - B \in \mathscr{R}$ for all $\mathscr{R} \in \mathscr{C}$, and it follows that $A - B \in \bigcap \mathscr{C}$. The argument for (c) is similar (with $A \cup B$ in place of $A - B$), and (d) is obvious.

Finally, suppose that every $\mathscr{R} \in \mathscr{C}$ satisfies (e). If $A_1, A_2, \ldots \in \bigcap \mathscr{C}$ then $A_1, A_2, \ldots \in \mathscr{R}$ for every $\mathscr{R} \in \mathscr{C}$. Hence $\bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$ for all $\mathscr{R} \in \mathscr{C}$, and it follows that $\bigcup_{n=1}^{\infty} A_n \in \bigcap \mathscr{C}$.

1.1.5

If $\mathscr{D}$ is any class of sets, then there exists a unique ring $\mathscr{R}_0$ such that (i) $\mathscr{R}_0 \supset \mathscr{D}$, and (ii) any ring $\mathscr{R}$ containing $\mathscr{D}$ contains also $\mathscr{R}_0$. $\mathscr{R}_0$ is called the *ring generated* by $\mathscr{D}$, and is denoted by $\mathscr{R}(\mathscr{D})$.

Solution. Let $\mathscr{R}_0$ be the intersection of all rings containing $\mathscr{D}$. This is a ring by the previous exercise, and it satisfies the properties (i) and (ii). To see that it is unique, let $\mathscr{R}_0'$ also by a ring satisfying (i) and (ii). Then $\mathscr{R}_0 \subset \mathscr{R}_0'$ and $\mathscr{R}_0' \subset \mathscr{R}_0$ by property (ii), so $\mathscr{R}_0 = \mathscr{R}_0'$.

1.1.6

If $\mathscr{D}$ is any class of sets, then there exists a unique σ-ring $\mathscr{S}_0$ such that (i) $\mathscr{S}_0 \supset \mathscr{D}$, and (ii) any σ-ring containing $\mathscr{D}$ contains also $\mathscr{S}_0$. We call $\mathscr{S}_0$ the *σ-ring generated* by $\mathscr{D}$, and denote it by $\mathscr{S}(\mathscr{D})$. A similar result holds for σ-algebras, and we speak of the *σ-algebra generated* by $\mathscr{D}$.

Solution. By the same argument as in the previous exercise, $\mathscr{S}_0$ is the intersection of all σ-rings containing $\mathscr{D}$. Similarly the σ-algebra generated by $\mathscr{D}$ is the intersection of all σ-algebras containing $\mathscr{D}$.

1.1.7

If $\mathscr{D}$ is any class of sets, then every set in $\mathscr{R}(\mathscr{D})$ can be covered by (that is, is contained in) a finite union of sets of $\mathscr{D}$. [Hint: The class $\mathscr{K}$ of sets that can be covered by finite unions of sets of $\mathscr{D}$ forms a ring.]

Solution. Let $\mathscr{K}$ be the class of all sets that can be covered by a finite union of sets in $\mathscr{D}$. Certainly $\varnothing \in \mathscr{K}$, since $\varnothing$ is a subset of the empty union. If $A, B \in \mathscr{K}$, then

$$A \subset \bigcup_{i=1}^{m} E_i, \quad B \subset \bigcup_{i=1}^{n} F_i,$$

for some sets $E_1, \ldots, E_m, F_1, \ldots, F_n \in \mathscr{D}$. (Note that m or n can be zero, in which case the corresponding union is empty.) Thus

$$A - B \subset A \subset \bigcup_{i=1}^{m} E_i$$

and

$$A \cup B \subset \left(\bigcup_{i=1}^{m} E_i \right) \cup \left(\bigcup_{j=1}^{n} F_j \right),$$

so both $A - B$ and $A \cup B$ are elements of $\mathscr{K}$.

The above shows that $\mathscr{K}$ is a ring, and certainly $\mathscr{D} \subset \mathscr{K}$. Hence $\mathscr{R}(\mathscr{D}) \subset \mathscr{K}$ by Problem 1.1.5, and it follows that every set in $\mathscr{R}(\mathscr{D})$ can be covered by a finite union of sets in $\mathscr{D}$.

Section 1.1 – Definition of measure

Problems

1.2.1

If μ satisfies the properties (i)-(iii) in Definition 1.2.1, and if $\mu(E) < \infty$ for at least one set E, then (iv) is also satisfied.

Solution. We have

$$\mu(E) = \mu(E \cup \varnothing) = \mu(E) + \mu(\varnothing),$$

hence $\mu(\varnothing) = 0$.

1.2.2

Let X be an infinite space. Let $\mathcal{A}$ be the class of all subsets of X. Define $\mu(E) = 0$ if E is finite and $\mu(E) = \infty$ if E is infinite. Then μ is finitely additive but not completely additive.

Solution. Suppose $A, B \in \mathcal{A}$. Note that $A \cup B$ is finite if both A and B are finite, but infinite otherwise. Hence

$$\mu(A \cup B) = 0 = \mu(A) + \mu(B)$$

in the former case, and

$$\mu(A \cup B) = \infty = \mu(A) + \mu(B)$$

in the latter. This proves that μ is additive; *finite* additivity follows by a simple induction argument.

Let (x_n) be a sequence of distinct points in X. Then $\bigcup_{n=1}^{\infty}\{x_n\}$ is an infinite set, so

$$\mu\left(\bigcup_{n=1}^{\infty}\{x_n\}\right) = \infty,$$

but

$$\sum_{n=1}^{\infty}\mu(\{x_n\}) = 0.$$

Thus μ is not completely additive.

1.2.3

If μ is a measure on a σ-algebra $\mathcal{A}$, and if E, F are sets of $\mathcal{A}$, then

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

Solution. If $\mu(F) = \infty$, then $\mu(E \cup F) = \infty$ by Theorem 1.2.1(i), and the given equality holds. If $\mu(F) < \infty$, then

$$\begin{aligned}\mu(E \cup F) &= \mu[E \cup (F - (E \cap F))] \\ &= \mu(E) + \mu[F - (E \cap F)] \\ &= \mu(E) + \mu(F) - \mu(E \cap F),\end{aligned}$$

with the last equality following from Theorem 1.2.1(ii). Note that $E \cap F \subset F$ so that $\mu(E \cap F) \leq \mu(F) < \infty$. Hence we can rearrange the above to yield

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

1.2.6

Give an example of a measure μ and a monotone-decreasing sequence $\{E_n\}$ of $\mathcal{A}$ such that $\mu(E_n) = \infty$ for all n, and $\mu(\lim_n E_n) = 0$.

Solution. Let $X = \mathbb{R}$ and let $\mathcal{A} = \mathcal{P}(\mathbb{R})$ (the power set of $\mathbb{R}$; this is easily seen to be a σ-algebra). Define $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(E)$ is the number of points in E (with $\mu(E) = \infty$ if E is infinite). This is easily seen to be a measure.

For each $n \in \mathbb{N}$, let $E_n = (0, 1/n)$. Then (E_n) is a monotone decreasing sequence of sets in $\mathcal{A}$, $\mu(E_n) = \infty$ for all n, and

$$\mu\left(\lim_n E_n\right) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \mu(\varnothing) = 0.$$

Section 1.3 – Outer measure

Problems

1.3.1

Define $\mu^*(E)$ as the number of points in E if E is finite and $\mu^*(E) = \infty$ if E is infinite. Show that μ^* is an outer measure. Determine the measurable sets.

Solution. Of the properties listed in Definition 1.3.1, only countable subadditivity is non-obvious for μ^*. But let us start with proving finite subadditivity.

Let A and B be sets. If either is infinite, then so is $A \cup B$, hence

$$\mu^*(A \cup B) = \infty = \mu(A) + \mu(B).$$

If both A and B are finite sets, then

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B - A) \leq \mu(A) + \mu(B)$$

by basic set-theoretic considerations. Thus μ^* is subadditive, and finite subadditivity follows by induction on the number of sets in the union.

Now, let (E_n) be a sequence of sets. If infinitely many of the sets E_n are nonempty, then $\sum_n \mu^*(E_n) = \infty$, and

$$\mu^*\left(\bigcup_n E_n\right) \leq \sum_n \mu^*(E_n)$$

follows. If only finitely many of the sets E_n are nonempty, let $E_{n_1}, E_{n_2}, \ldots, E_{n_k}$ be those sets. Then

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu^*\left(\bigcup_{i=1}^{k} E_{n_i}\right) \leq \sum_{i=1}^{k} \mu^*(E_{n_i}) = \sum_{n=1}^{\infty} \mu^*(E_n),$$

by finite subadditivity. This proves that μ^* is countably subadditive, and hence that μ^* is an outer measure.

Note that μ^* is *additive* on disjoint sets; if $A \cup B = \varnothing$, then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$. In particular,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A - E)$$

for all sets A, E. That is, all sets (really subsets of X) are measurable.

1.3.2

Define $\mu^*(\varnothing) = 0$, $\mu^*(E) = 1$ if $E \neq \varnothing$. Show that μ^* is an outer measure, and determine the measurable sets.

Solution. As in the previous excercise, the only slightly non-obvious property is countable subadditivity. Hence, let (E_n) be a sequence of sets. If all the sets E_n are empty, then certainly

$$\mu^*\left(\bigcup_n E_n\right) = 0 = \sum_n \mu^*(E_n).$$

If not, then there is some m such that $E_m \neq \varnothing$, and it follows that

$$\mu^*\left(\bigcup_n E_n\right) = 1 = \mu^*(E_m) \leq \sum_n \mu^*(E_n).$$

Thus μ^* is indeed countably subadditive, and therefore also an outer measure.

The empty set is measurable:

$$\mu^*(A \cap \varnothing) + \mu^*(A - \varnothing) = \mu^*(\varnothing) + \mu^*(A) = \mu^*(A)$$

for all sets A. It follows by Theorem 1.3.1 that $X = \varnothing^c$ is measurable as well (the measurable sets make up a σ-algebra, and algebras are closed under conjugation). Indeed $\varnothing$ and X are the only measurable sets. To see this, let E be any set other than those two. Then both E and E^c are nonempty, so

$$\mu^*(X \cap E) + \mu^*(X - E) = \mu^*(E) + \mu^*(E^c) = 2 > 1 = \mu^*(X).$$