Notes for *Foundations of Modern Analysis* by Avner Friedman

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January 10, 2025

Chapter 1 – Measure Theory

Section 1.1 – Rings and Algebras

Problems

1.1.1

$$\left(\underline{\lim}_n E_n\right)^c = \overline{\lim}_n E_n^c, \quad \left(\overline{\lim}_n E_n\right)^c = \underline{\lim}_n E_n^c.$$

Solution. Note that

$$\begin{aligned} x \in \underline{\lim}_n E_n &\iff x \in E_n \text{ for all but finitely many } n \\ &\iff x \in E_n^c \text{ for only finitely many } n. \end{aligned}$$

Hence

$$\begin{aligned} x \in \left(\underline{\lim}_n E_n\right)^c &\iff x \in E_n^c \text{ for infinitely many } n \\ &\iff x \in \overline{\lim}_n E_n^c, \end{aligned}$$

proving the first identity.

Next, let $F_n = E_n^c$ for every n. Then

$$\overline{\lim}_n E_n = \overline{\lim}_n F_n^c = \left(\underline{\lim}_n F_n\right)^c = \left(\underline{\lim}_n E_n^c\right)^c$$

by the first identity, and the second identity follows.

1.1.2

$$\overline{\lim}_n E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n, \quad \underline{\lim}_n E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n.$$

Solution. Suppose $x \in \overline{\lim}_n E_n$. Then $x \in E_n$ for infinitely many n. It follows that $x \in \bigcup_{n=k}^{\infty} E_n$ for all $k \in \mathbb{N}$, and hence that $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$.

Conversely, assume that $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$. Then $x \in \bigcup_{n=k}^{\infty} E_n$ for all $k \in \mathbb{N}$. It follows that $x \in E_n$ for infinitely many n, and thus that $x \in \overline{\lim}_n E_n$. This proves the first identity.

Next, suppose that $x \in \underline{\lim}_n E_n$. Then $x \in E_n$ for all but finitely many n, so there is some $k' \in \mathbb{N}$ such that $x \in E_n$ for all $n \geq k'$. It follows that $x \in \bigcap_{n=k'}^{\infty} E_n$, and hence that $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$.

Conversely, assume that $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$. Then $x \in \bigcap_{n=k'}^{\infty} E_n$ for some $k' \in \mathbb{N}$, which means that $x \in E_n$ for all $n \geq k'$. It follows that $x \in E_n$ for all but finitely many n; that is, $x \in \underline{\lim}_n E_n$.

1.1.3

If $\mathscr{R}$ is a σ-ring and $E_n \in \mathscr{R}$, then

$$\bigcap_{n=1}^{\infty} E_n \in \mathscr{R}, \quad \overline{\lim_n} E_n \in \mathscr{R}, \quad \underline{\lim_n} E_n \in \mathscr{R}.$$

Solution. Note that

$$\bigcap_{n=1}^{\infty} E_n = E_1 \cap \left(\bigcap_{n=1}^{\infty} E_n \right) = E_1 - \left(E_1 - \bigcap_{n=1}^{\infty} E_n \right),$$

and that

$$E_1 - \bigcap_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (E_1 - E_n) \in \mathscr{R},$$

using one of De Morgan's laws along with properties (b) and (e). It follows (by (b) again) that $\bigcap_{n=1}^{\infty} E_n \in \mathscr{R}$. That is, $\mathscr{R}$ is closed under countable intersections.

Given $k \in \mathbb{N}$, let $A_n = \varnothing$ for $n < k$, and let $A_n = E_n$ for $n \geq k$. Then $A_n \in \mathscr{R}$ for all n by (a), hence

$$\bigcup_{n=k}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$$

by (e). Since $\mathscr{R}$ is closed under countable intersections, we then have

$$\overline{\lim_n} E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \in \mathscr{R}.$$

By a similar argument we find that

$$\bigcap_{n=k}^{\infty} E_n \in \mathscr{R}$$

for all $k \in \mathbb{N}$. Thus

$$\varliminf_n E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n \in \mathscr{R}$$

by (e).

1.1.4

The intersection of any collection of rings (algebras, σ-rings, or σ-algebras) is also a ring (an algebra, σ-ring, or σ-algebra).

Solution. Let $\mathscr{C}$ be a collection of classes. Let $\bigcap \mathscr{C}$ denote the intersection of all classes in $\mathscr{C}$. We will show that if one of the properties (a)-(e) is satisfied by all classes in $\mathscr{C}$, then $\bigcap \mathscr{C}$ satisfies that property as well. The result requested in the problem then follows as an immediate corollary.

It is clear that if every $\mathscr{R} \in \mathscr{C}$ satisfies (a), then so does $\bigcap \mathscr{C}$. Suppose every $\mathscr{R} \in \mathscr{C}$ satisfies (b). If $A, B \in \bigcap \mathscr{C}$ then $A, B \in \mathscr{R}$ for every $\mathscr{R} \in \mathscr{C}$. Hence $A - B \in \mathscr{R}$ for all $\mathscr{R} \in \mathscr{C}$, and it follows that $A - B \in \bigcap \mathscr{C}$. The argument for (c) is similar (with $A \cup B$ in place of $A - B$), and (d) is obvious.

Finally, suppose that every $\mathscr{R} \in \mathscr{C}$ satisfies (e). If $A_1, A_2, \ldots \in \bigcap \mathscr{C}$ then $A_1, A_2, \ldots \in \mathscr{R}$ for every $\mathscr{R} \in \mathscr{C}$. Hence $\bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$ for all $\mathscr{R} \in \mathscr{C}$, and it follows that $\bigcup_{n=1}^{\infty} A_n \in \bigcap \mathscr{C}$.

1.1.5

If $\mathscr{D}$ is any class of sets, then there exists a unique ring $\mathscr{R}_0$ such that (i) $\mathscr{R}_0 \supset \mathscr{D}$, and (ii) any ring $\mathscr{R}$ containing $\mathscr{D}$ contains also $\mathscr{R}_0$. $\mathscr{R}_0$ is called the *ring generated* by $\mathscr{D}$, and is denoted by $\mathscr{R}(\mathscr{D})$.

Solution. Let $\mathscr{R}_0$ be the intersection of all rings containing $\mathscr{D}$. This is a ring by the previous exercise, and it satisfies the properties (i) and (ii). To see that it is unique, let $\mathscr{R}_0'$ also by a ring satisfying (i) and (ii). Then $\mathscr{R}_0 \subset \mathscr{R}_0'$ and $\mathscr{R}_0' \subset \mathscr{R}_0$ by property (ii), so $\mathscr{R}_0 = \mathscr{R}_0'$.

1.1.6

If $\mathscr{D}$ is any class of sets, then there exists a unique σ-ring $\mathscr{S}_0$ such that (i) $\mathscr{S}_0 \supset \mathscr{D}$, and (ii) any σ-ring containing $\mathscr{D}$ contains also $\mathscr{S}_0$. We call $\mathscr{S}_0$ the *σ-ring generated* by $\mathscr{D}$, and denote it by $\mathscr{S}(\mathscr{D})$. A similar result holds for σ-algebras, and we speak of the *σ-algebra generated* by $\mathscr{D}$.

Solution. By the same argument as in the previous exercise, $\mathscr{S}_0$ is the intersection of all σ-rings containing $\mathscr{D}$. Similarly the σ-algebra generated by $\mathscr{D}$ is the intersection of all σ-algebras containing $\mathscr{D}$.

1.1.7

If $\mathscr{D}$ is any class of sets, then every set in $\mathscr{R}(\mathscr{D})$ can be covered by (that is, is contained in) a finite union of sets of $\mathscr{D}$. [*Hint:* The class $\mathscr{K}$ of sets that can be covered by finite unions of sets of $\mathscr{D}$ forms a ring.]

Solution. Let $\mathscr{K}$ be the class of all sets that can be covered by a finite union of sets in $\mathscr{D}$. Certainly $\varnothing \in \mathscr{K}$, since $\varnothing$ is a subset of the empty union. If $A, B \in \mathscr{K}$, then

$$A \subset \bigcup_{i=1}^{m} E_i, \quad B \subset \bigcup_{i=1}^{n} F_i,$$

for some sets $E_1, \ldots, E_m, F_1, \ldots, F_n \in \mathscr{D}$. (Note that m or n can be zero, in which case the corresponding union is empty.) Thus

$$A - B \subset A \subset \bigcup_{i=1}^{m} E_i$$

and

$$A \cup B \subset \left(\bigcup_{i=1}^{m} E_i \right) \cup \left(\bigcup_{j=1}^{n} F_j \right),$$

so both $A - B$ and $A \cup B$ are elements of $\mathscr{K}$.

The above shows that $\mathscr{K}$ is a ring, and certainly $\mathscr{D} \subset \mathscr{K}$. Hence $\mathscr{R}(\mathscr{D}) \subset \mathscr{K}$ by Problem 1.1.5, and it follows that every set in $\mathscr{R}(\mathscr{D})$ can be covered by a finite union of sets in $\mathscr{D}$.

Section 1.2 – Definition of Measure

Problems

1.2.1

If μ satisfies the properties (i)-(iii) in Definition 1.2.1, and if $\mu(E) < \infty$ for at least one set E, then (iv) is also satisfied.

Solution. We have

$$\mu(E) = \mu(E \cup \varnothing) = \mu(E) + \mu(\varnothing),$$

hence $\mu(\varnothing) = 0$.

1.2.2

Let X be an infinite space. Let $\mathcal{A}$ be the class of all subsets of X. Define $\mu(E) = 0$ if E is finite and $\mu(E) = \infty$ if E is infinite. Then μ is finitely additive but not completely additive.

Solution. Suppose $A, B \in \mathcal{A}$. Note that $A \cup B$ is finite if both A and B are finite, but infinite otherwise. Hence

$$\mu(A \cup B) = 0 = \mu(A) + \mu(B)$$

in the former case, and

$$\mu(A \cup B) = \infty = \mu(A) + \mu(B)$$

in the latter. This proves that μ is additive; *finite* additivity follows by a simple induction argument.

Let (x_n) be a sequence of distinct points in X. Then $\bigcup_{n=1}^{\infty}\{x_n\}$ is an infinite set, so

$$\mu\left(\bigcup_{n=1}^{\infty}\{x_n\}\right) = \infty,$$

but

$$\sum_{n=1}^{\infty}\mu(\{x_n\}) = 0.$$

Thus μ is not completely additive.

1.2.3

If μ is a measure on a σ-algebra $\mathcal{A}$, and if E, F are sets of $\mathcal{A}$, then

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

Solution. If $\mu(F) = \infty$, then $\mu(E \cup F) = \infty$ by Theorem 1.2.1(i), and the given equality holds. If $\mu(F) < \infty$, then

$$\begin{aligned}\mu(E \cup F) &= \mu[E \cup (F - (E \cap F))] \\ &= \mu(E) + \mu[F - (E \cap F)] \\ &= \mu(E) + \mu(F) - \mu(E \cap F),\end{aligned}$$

with the last equality following from Theorem 1.2.1(ii). Note that $E \cap F \subset F$ so that $\mu(E \cap F) \leq \mu(F) < \infty$. Hence we can rearrange the above to yield

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

1.2.6

Give an example of a measure μ and a monotone-decreasing sequence $\{E_n\}$ of $\mathcal{A}$ such that $\mu(E_n) = \infty$ for all n, and $\mu(\lim_n E_n) = 0$.

Solution. Let $X = \mathbb{R}$ and let $\mathcal{A} = \mathcal{P}(\mathbb{R})$ (the power set of $\mathbb{R}$; this is easily seen to be a σ-algebra). Define $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(E)$ is the number of points in E (with $\mu(E) = \infty$ if E is infinite). This is easily seen to be a measure.

For each $n \in \mathbb{N}$, let $E_n = (0, 1/n)$. Then (E_n) is a monotone decreasing sequence of sets in $\mathcal{A}$, $\mu(E_n) = \infty$ for all n, and

$$\mu\left(\lim_n E_n\right) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \mu(\varnothing) = 0.$$

Section 1.3 – Outer Measure

Problems

1.3.1

Define $\mu^*(E)$ as the number of points in E if E is finite and $\mu^*(E) = \infty$ if E is infinite. Show that μ^* is an outer measure. Determine the measurable sets.

Solution. Of the properties listed in Definition 1.3.1, only countable subadditivity is non-obvious for μ^*. But let us start with proving finite subadditivity.

Let A and B be sets. If either is infinite, then so is $A \cup B$, hence

$$\mu^*(A \cup B) = \infty = \mu^*(A) + \mu^*(B).$$

If both A and B are finite sets, then

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B - A) \leq \mu^*(A) + \mu^*(B)$$

by basic set-theoretic considerations. Thus μ^* is subadditive, and finite subadditivity follows by induction on the number of sets in the union.

Now, let (E_n) be a sequence of sets. If infinitely many of the sets E_n are nonempty, then $\sum_n \mu^*(E_n) = \infty$, and

$$\mu^*\left(\bigcup_n E_n\right) \leq \sum_n \mu^*(E_n)$$

follows. If only finitely many of the sets E_n are nonempty, let $E_{n_1}, E_{n_2}, \ldots, E_{n_k}$ be those sets. Then

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu^*\left(\bigcup_{i=1}^{k} E_{n_i}\right) \leq \sum_{i=1}^{k} \mu^*(E_{n_i}) = \sum_{n=1}^{\infty} \mu^*(E_n),$$

by finite subadditivity. This proves that μ^* is countably subadditive, and hence that μ^* is an outer measure.

Note that μ^* is *additive* on disjoint sets; if $A \cap B = \varnothing$, then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$. In particular,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A - E)$$

for all sets A, E. That is, all sets are measurable.

1.3.2

Define $\mu^*(\varnothing) = 0$, $\mu^*(E) = 1$ if $E \neq \varnothing$. Show that μ^* is an outer measure, and determine the measurable sets.

Solution. As in the previous excercise, the only slightly non-obvious property is countable subadditivity. Hence, let (E_n) be a sequence of sets. If all the sets E_n are empty, then certainly

$$\mu^*\left(\bigcup_n E_n\right) = 0 = \sum_n \mu^*(E_n).$$

If not, then there is some m such that $E_m \neq \varnothing$, and it follows that

$$\mu^*\left(\bigcup_n E_n\right) = 1 = \mu^*(E_m) \leq \sum_n \mu^*(E_n).$$

Thus μ^* is indeed countably subadditive, and therefore also an outer measure.

The empty set is measurable:

$$\mu^*(A \cap \varnothing) + \mu^*(A - \varnothing) = \mu^*(\varnothing) + \mu^*(A) = \mu^*(A)$$

for all sets A. It follows by Theorem 1.3.1 that X is measurable as well (the measurable sets make up a σ-algebra). Indeed $\varnothing$ and X are the only measurable sets. To see this, let E be any set other than those two (this requires that X contains at least two elements). Then both E and E^c are nonempty, so

$$\mu^*(X \cap E) + \mu^*(X - E) = \mu^*(E) + \mu^*(E^c) = 2 > 1 = \mu^*(X).$$

Section 1.4 – Construction of Outer Measures

Problems

1.4.4

If $\mathscr{K}$ is a σ-algebra and λ is a measure on $\mathscr{K}$, then $\mu^*(A) = \lambda(A)$ for any $A \in \mathscr{K}$. [*Hint:* $\mu^*(A) = \inf\{\lambda(E); E \in \mathscr{K}, E \supset A\}$.]

Solution. Note that the description of μ^* can be simplified when $\mathscr{K}$ is a σ-algebra and λ is a measure. For suppose that $A \subset X$, $E_n \in \mathscr{K}$ $(n = 1, 2, \ldots)$, and $A \subset \bigcup_n E_n$. Then $E := \bigcup_n E_n \in \mathscr{K}$, and $\lambda(E) \leq \sum_n \lambda(E_n)$ by Theorem 1.2.2. Hence

$$\mu^*(A) = \inf\{\lambda(E); E \in \mathscr{K}, E \supset A\}.$$

Now, suppose that $A \in \mathscr{K}$. Certainly $\lambda(A)$ is an element of $\{\lambda(E); E \in \mathscr{K}, E \supset A\}$. And if $E \in \mathscr{K}$ and $E \supset A$, then $\lambda(E) \geq \lambda(A)$ by Theorem 1.2.1(i). Thus

$$\lambda(A) = \inf\{\lambda(E); E \in \mathscr{K}, E \supset A\} = \mu^*(A).$$

1.4.5

If $\mathscr{K}$ is a σ-algebra and λ is a measure on $\mathscr{K}$, then every set in $\mathscr{K}$ is μ^*-measurable.

Solution. Recall the simplified description of μ^* from the previous problem. Let $E \in \mathscr{K}$ and $A \subset X$. For every $\epsilon > 0$ there exists $F \in \mathscr{K}$ such that $F \supset A$ and

$$\mu^*(A) + \epsilon > \lambda(F);$$

else $\mu^*(A)$ would not be the greatest lower bound of $\{\lambda(E); E \in \mathscr{K}, E \supset A\}$. Moreover,

$$\lambda(F) = \lambda(F \cap E) + \lambda(F - E)$$

since λ is a measure on $\mathscr{K}$,

$$\lambda(F \cap E) + \lambda(F - E) = \mu^*(F \cap E) + \mu^*(F - E)$$

by what we found in the previous exercise, and finally

$$\mu^*(F \cap E) + \mu^*(F - E) \geq \mu^*(A \cap E) + \mu^*(A - E)$$

by monotonicity of the outer measure μ^*. Putting all of this together, we have

$$\mu^*(A) + \epsilon > \mu^*(A \cap E) + \mu^*(A - E)$$

for all $\epsilon > 0$, and thus

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A - E).$$

It follows that every set $E \in \mathscr{K}$ is μ^*-measurable.

Section 1.6 – The Lebesgue and the Lebesgue-Stieltjes Measures

Problems

1.6.3

The outer Lebesgue measure of a closed bounded interval $[a, b]$ on the real line is equal to $b - a$. [*Hint:* Use the Heine-Borel theorem to replace a countable covering by a finite covering.]

Solution. Suppose (E_n) is a sequence of elements of $\mathscr{K}$ (i.e. a sequence of open intervals) such that $[a, b] \subset \bigcup_{n=1}^{\infty} E_n$. The collection $\{E_n\}$ constitutes an *open cover* of $[a, b]$. By the Heine-Borel theorem $[a, b]$ is compact, hence there exists a *finite subcover* $\{E_{n_1}, \ldots, E_{n_k}\}$, such that $[a, b] \subset \bigcup_{i=1}^{k} E_{n_i}$.

Assume without loss of generality that $E_{n_i} \cap [a, b] \neq \varnothing$ for all i; otherwise we can simply remove those E_{n_i} that are disjoint with $[a, b]$ and still have a finite subcover. Write $E_{n_i} = (a_i, b_i)$ for each i, and define

$$\alpha = \min\{a_1, \ldots, a_k\}, \quad \beta = \max\{b_1, \ldots, b_k\}.$$

It is clear that α and β are the infimum and supremum, respectively, of $\bigcup_{i=1}^k E_{n_i}$. Note that $\alpha = a_j$ for some j, and $a_j < a < b_j$ since E_{n_j} and $[a,b]$ have nonempty intersection. Thus $(\alpha, a] \subset \bigcup_{i=1}^k E_{n_i}$, and similarly $[b, \beta) \subset \bigcup_{i=1}^k E_{n_i}$. It follows that

$$\bigcup_{i=1}^k E_{n_i} = (\alpha, \beta) \in \mathscr{K}.$$

Finally note that λ is finitely subadditive. (This is easily proven with induction.) (TODO: This is not convincing; use better proof from Rosenthal notes.) Thus,

$$\sum_{n=1}^\infty \lambda(E_n) \geq \sum_{i=1}^k \lambda(E_{n_i}) \geq \lambda\left[(\alpha, \beta)\right] = \beta - \alpha > b - a.$$

It follows that $b - a$ is a lower bound of the set

$$\Lambda([a,b]) := \left\{ \sum_{n=1}^\infty \lambda(E_n);\ E_n \in \mathscr{K},\ \bigcup_{n=1}^\infty E_n \supset [a,b] \right\}.$$

Moreover, for every $\epsilon > 0$ we have

$$[a,b] \subset \left(a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2}\right) \in \mathscr{K}$$

and

$$\lambda\left[\left(a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2}\right)\right] = b - a + \epsilon.$$

Hence $b - a$ is the *greatest* lower bound of $\Lambda([a,b])$, and $\mu^*([a,b]) = b - a$.

1.6.4

The outer Lebesgue measure of each of the intervals $(a,b), [a,b), (a,b]$ is equal to $b - a$.

Solution. Recall that μ^* is monotone, on account of being an outer measure. Hence $\mu^*[(a,b)] \leq \mu^*([a,b]) = b - a$, the latter equality being the result of the previous problem. Moreover, for all $\epsilon \in (0, b-a)$ we have

$$\left[a + \frac{\epsilon}{2}, b - \frac{\epsilon}{2}\right] \subset (a,b),$$

so that

$$\mu^*[(a,b)] \geq \mu^*\left(\left[a + \frac{\epsilon}{2}, b - \frac{\epsilon}{2}\right]\right) = b - a - \epsilon.$$

Thus $\mu^*[(a,b)] \geq b - a$, and it follows that $\mu^*[(a,b)] = b - a$.

The outer measures of $[a,b)$ and $(a,b]$ follow immediately by monotonicity:

$$\mu^*[(a,b)] \leq \mu^*([a,b)) \leq \mu^*([a,b]),$$

so that $\mu^*([a,b)) = b - a$. Similarly for $(a,b]$.

1.6.5

Consider the transformation $Tx = \alpha x + \beta$ from the real line onto itself, where α, β are real numbers and $\alpha \neq 0$. It maps sets E onto sets $T(E)$. Denote by μ (μ^*) the Lebesgue measure (outer measure) on the real line. Prove

(a) For any set E, $\mu^*(T(E)) = |\alpha|\mu^*(E)$.

(b) E is Lebesgue-measurable if and only if $T(E)$ is Lebesgue-measureable.

(c) If E is Lebesgue-measurable, then $\mu(T(E)) = |\alpha|\mu(E)$.

Solution. Let us start with a couple of simple observations:

- T is bijective, with inverse given by

$$T^{-1}(x) = \frac{x - \beta}{\alpha}.$$

- Suppose $I = (a, b)$. Then

$$T(I) = (\alpha a + \beta, \alpha b + \beta)$$

  if $\alpha > 0$, and

$$T(I) = (\beta b + \beta, \alpha a + \beta)$$

  if $\alpha < 0$. Either way,

$$\mu^*[T(I)] = |\alpha|(b - a) = |\alpha|\mu^*(I),$$

  where we have used one of the results of the previous exercise. Similarly, $T^{-1}(I)$ is an open interval and

$$\mu^*[T^{-1}(I)] = |\alpha|^{-1}\mu^*(I).$$

  Of course, the latter two identities still hold if $I = \varnothing$. Hence they hold for all $I \in \mathscr{K}$.

Also, let us use the notation

$$\Lambda(E) = \left\{ \sum_{n=1}^{\infty} \lambda(I_n);\ I_n \in \mathscr{K},\ \bigcup_{n=1}^{\infty} I_n \supset E \right\}$$

for all $E \subset \mathbb{R}$.

(a) Suppose (I_n) is a sequence in $\mathscr{K}$ (i.e. a sequence of open intervals) and $E \subset \bigcup_n I_n$. Then $T(I_n) \in \mathscr{K}$ for every n,

$$T(E) \subset T\left(\bigcup_n I_n\right) = \bigcup_n T(I_n),$$

and

$$\sum_n \lambda[T(I_n)] = |\alpha| \sum_n \lambda(I_n).$$

Thus, if $s \in \Lambda(E)$, then $|\alpha|s \in \Lambda[T(E)]$. It follows that

$$\mu^*[T(E)] = \inf \Lambda[T(E)] \leq |\alpha| \inf \Lambda(E) = |\alpha|\mu^*(E).$$

Conversely, suppose (J_n) is a sequence in $\mathscr{K}$ and $T(E) \subset \bigcup_n J_n$. Then $T^{-1}(J_n) \in \mathscr{K}$ for all n,

$$E = T^{-1}[T(E)] \subset T^{-1}\left(\bigcup_n J_n\right) = \bigcup_n T^{-1}(J_n),$$

and

$$\sum_n \lambda[T^{-1}(J_n)] = |\alpha|^{-1} \sum_n \lambda(J_n).$$

Hence, by the same logic as above, we find that $\mu^*(E) \leq |\alpha|^{-1}\mu^*[T(E)]$, and it follows that

$$\mu^*[T(E)] = |\alpha|\mu^*(E).$$

(b) Note that if $f : X \to Y$ is a bijective function (between arbitrary sets X, Y), then

$$\begin{aligned} f^{-1}[f(A)] &= A, \\ f(A \cup B) &= f(A) \cup f(B), \\ f(A - B) &= f(A) - f(B), \\ f[f^{-1}(C)] &= C, \end{aligned}$$

for all $A, B \subset X$ and $C \subset Y$.

Suppose that E is measureable:

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A - E)$$

for all $A \subset \mathbb{R}$. Then, for all $B \subset \mathbb{R}$, we have

$$\begin{aligned} &\mu^*[B \cap T(E)] + \mu^*[B - T(E)] \\ &\qquad = \mu^*[T(T^{-1}(B) \cap E)] + \mu^*[T(T^{-1}(B) - E)] \\ &\qquad = |\alpha|\mu^*[T^{-1}(B) \cap E] + |\alpha|\mu^*[T^{-1}(B) - E] \\ &\qquad = |\alpha|\mu^*[T^{-1}(B)] \\ &\qquad = \mu^*(B), \end{aligned}$$

so that $T(E)$ is measurable.

Conversely, suppose that $T(E)$ is measurable. Then, for all $A \subset \mathbb{R}$,

$$\begin{aligned} &\mu^*(A \cap E) + \mu^*(A - E) \\ &\qquad = \mu^*[T^{-1}(T(A) \cap T(E))] + \mu^*[T^{-1}(T(A) - T(E))] \\ &\qquad = |\alpha|^{-1}\mu^*[T(A) \cap T(E)] + |\alpha|^{-1}\mu^*[T(A) - T(E)] \\ &\qquad = |\alpha|^{-1}\mu^*[T(A)] \\ &\qquad = \mu^*(A), \end{aligned}$$

so that E is measurable.

(c) This is immediate given (a), (b), and the definition of the Lebesgue-measure. First, $T(E)$ is Lebesgue-measurable by (b). Next, $\mu(E) = \mu^*(E)$ and $\mu[T(E)] = \mu^*[T(E)]$ since μ is simply the restriction of μ^* to the measurable sets. Finally, $\mu^*[T(E)] = |\alpha|\mu^*(E)$ by (a).

Chapter 2 – Integration

Section 2.1 – Definition of Measurable Functions

Problems

2.1.6

The *characteristic function* of a set E is the function χ_E defined by

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

Prove that the set E is measurable if and only if the function χ_E is measurable.

Solution. Suppose $E \in \mathcal{A}$. For all $c \in \mathbb{R}$,

$$\chi_E^{-1}\{(-\infty, c)\} = \{x \in X; \chi_E(x) < c\} = \begin{cases} \varnothing & (c \leq 0), \\ E^c & (0 < c \leq 1), \\ X & (c > 1), \end{cases}$$

so that $\chi_E^{-1}\{(-\infty, c)\} \in \mathcal{A}$. By Theorem 2.1.1, χ_E is measurable.

Conversely, suppose χ_E is measurable. Then E is measurable, since

$$E = X - E^c = \chi^{-1}\{(-\infty, 2)\} - \chi^{-1}\{(-\infty, 1)\}.$$

2.1.9

If f is measurable, then $|f|$ and $|f|^2$ are measurable.

Solution. If $c \leq 0$, then

$$(|f|)^{-1}\{(-\infty, c)\} = (|f|^2)^{-1}\{(-\infty, c)\} = \emptyset \in \mathcal{A},$$

since $|f|$ and $|f|^2$ are nonnegative functions.

Let $c > 0$. Then

$$(|f|)^{-1}\{(-\infty, c)\} = \{x \in X; -c < f(x) < c\} = f^{-1}\{(-c, c)\}.$$

The set $(-c, c)$ is open, hence $f^{-1}\{(-c, c)\} \in \mathcal{A}$ by the measurability of f. Similarly,

$$(|f|^2)^{-1}\{(-\infty, c)\} = f^{-1}\{(-\sqrt{c}, \sqrt{c})\} \in \mathcal{A}.$$

Finally,

$$(|f|)^{-1}\{+\infty\} = (|f|^2)^{-1}\{+\infty\} = f^{-1}\{+\infty\} \cup f^{-1}\{-\infty\} \in \mathcal{A}$$

by the measurability of f, and

$$(|f|)^{-1}\{-\infty\} = (|f|^2)^{-1}\{-\infty\} = \emptyset \in \mathcal{A}$$

since $|f|$ and $|f|^2$ are nonnegative. Thus, both $|f|$ and $|f|^2$ are measurable by Theorem 2.1.1.

2.1.10

A monotone function defined on the real line is Lebesgue-measurable.

Solution. Let f be a monotone increasing extended real-valued function on $\mathbb{R}$;

$$(\forall x, y \in \mathbb{R}): \quad x < y \implies f(x) \leq f(y).$$

Given any $c \in \mathbb{R}$, let

$$\xi_c = \inf\{x \in X; f(x) \geq c\}.$$

We need to consider two cases: $f(\xi_c) < c$ and $f(\xi_c) \geq c$. In the former case, $f(x) < c$ for all $x \leq \xi_c$ and $f(x) \geq c$ for all $x > \xi_c$ (by monotonicity). Hence

$$f^{-1}\{(-\infty, c)\} = (-\infty, \xi_c].$$

This is a Borel set, hence also a Lebesgue set (see Problem 1.9.3). In the latter case, $f(x) < c$ for all $x < \xi_c$ and $f(x) \geq c$ for all $x \geq \xi_c$, so that

$$f^{-1}\{(-\infty, c)\} = (-\infty, \xi_c),$$

which is Lebesgue-measurable. Since c was arbitrary, we conclude that f is measurable, by Theorem 2.1.1.

The proof for f monotone decreasing is similar.

Section 2.2 – Operations on Measurable Functions

Problems

2.2.2

Let $g(u_1, \ldots, u_k)$ be a continuous function in $\mathbb{R}^k$, and let $\varphi_1, \ldots, \varphi_k$ be measurable functions. Prove that the composite function $h(x) = g[\varphi_1(x), \ldots, \varphi_k(x)]$ is a measurable function. Note that as a special case we may conclude that

$$\max(\varphi, \ldots, \varphi_n) \quad \text{and} \quad \min(\varphi, \ldots, \varphi_n)$$

are measurable functions.

Solution. We will use the following fact, which may be proven in a course in topology:

> $\mathbb{R}^k$ has a countable basis of product open subsets. Hence, if U is an open subset of $\mathbb{R}^k$, then there are open subsets $U_{ni} \subset \mathbb{R}$ for $n = 1, 2, \dots$ and $i = 1, \dots, k$ such that
>
> $$U = \bigcup_{n=1}^{\infty} (U_{n1} \times \cdots \times U_{nk}).$$

We are assuming that g is real-valued, likewise for the functions φ_i. Let $c \in \mathbb{R}$. Note that $g^{-1}\{(-\infty, c)\}$ is open by continuity of g. Thus

$$g^{-1}\{(-\infty, c)\} = \bigcup_{n=1}^{\infty} (U_{n1} \times \cdots \times U_{nk})$$

for some open subsets $U_{ni} \subset \mathbb{R}$. Hence

$$\begin{aligned}
h^{-1}\{(-\infty, c)\} &= \{x \in X; g(\varphi_1(x), \dots, \varphi_k(x)) \le c\} \\
&= \{x \in X; (\varphi_1(x), \dots, \varphi_k(x)) \in g^{-1}\{(-\infty, c)\}\} \\
&= \bigcup_{n=1}^{\infty} \{x \in X; (\varphi_1(x), \dots, \varphi_k(x)) \in U_{n1} \times \cdots \times U_{nk}\} \\
&= \bigcup_{n=1}^{\infty} \bigcap_{i=1}^{k} \{x \in X; \varphi_i(x) \in U_{ni}\} \\
&= \bigcup_{n=1}^{\infty} \bigcap_{i=1}^{k} \varphi_i^{-1}(U_{ni}).
\end{aligned}$$

The sets $\varphi_i(U_{ni})$ are measurable since the functions φ_i are measurable. It follows that $h^{-1}\{(-\infty, c)\}$ is measurable, and thus that h is measurable, by Theorem 2.1.1.

To apply the above to the max and min functions $\mathbb{R}^k \to \mathbb{R}$ we must show that they are continuous. Let $a < b$ and note that

$$\begin{aligned}
\max{}^{-1}\{(a, b)\} = \{&(x_1, \dots, x_k) \in \mathbb{R}^k; x_i > a \text{ for some } i\} \\
&\cap \{(x_1, \dots, x_k) \in \mathbb{R}^k; x_i < b \text{ for all } i\}.
\end{aligned}$$

Both sets in the above binary intersection are easily seen to be open by considering ϵ-neighborhoods about their points. It follows that $\max^{-1}(U)$ is open for all open subsets $U \in \mathbb{R}^k$, since every such U can be written as a countable union of open intervals. Thus max is continuous, and one similarly shows that min is continuous.

2.2.3

Let $f(x)$ be a measurable function and define

$$g(x) = \begin{cases} \frac{1}{f(x)}, & \text{if } f(x) \neq 0, \\ 0, & \text{if } f(x) = 0. \end{cases}$$

Prove that g is measurable.

Solution. For $c < 0$,

$$g^{-1}\{(-\infty, c)\} = \{x; 1/f(x) < c\} = \{x; 1/c < f < 0\} = f^{-1}\{(1/c, 0)\},$$

which is measurable by the measurability of f. Next,

$$g^{-1}\{(-\infty, 0)\} = \{x; 1/f(x) < 0\} = \{x; f(x) < 0\} = f^{-1}\{(-\infty, 0)\},$$

also measurable. Note that if we take the natural convention (unfortunately not addressed in the text) that $x/(\pm\infty) = 0$ for all $x \in \mathbb{R}$, then

$$g^{-1}(\{0\}) = \{x; f(x) = 0\} \cup \{x; f(x) = \pm\infty\} = f^{-1}(\{0\}) \cup f^{-1}(\{\pm\infty\}).$$

Hence, for $c > 0$,

$$\begin{aligned} g^{-1}\{(-\infty, c)\} &= g^{-1}\{(-\infty, 0)\} \cup g^{-1}(\{0\}) \cup g^{-1}\{(0, c)\} \\ &= f^{-1}\{(-\infty, 0)\} \cup f^{-1}(\{0\}) \cup f^{-1}(\{\pm\infty\}) \cup f^{-1}\{(1/c, \infty)\} \\ &= f^{-1}\{(-\infty, 0]\} \cup f^{-1}(\{\pm\infty\}) \cup f^{-1}\{(1/c, \infty)\}, \end{aligned}$$

which is measurable by the measurability of f (see Problem 2.1.4). Finally, $g^{-1}(\{\pm\infty\}) = \emptyset$, and it follows by Theorem 2.1.1 that g is measurable.

Section 2.3 – Egoroff's Theorem

Problems

2.3.2

Let $\{f_n\}$ be a sequence of measurable functions in a finite measure space X. Suppose that for almost every x, $\{f_n(x)\}$ is a bounded set. Then for any $\epsilon > 0$ there exist a positive number c and a measurable set E with $\mu(X - E) < \epsilon$, such that $|f_n(x)| \leq c$ for all $x \in E$, $n = 1, 2, \ldots$.

Solution. The definition we have for 'bounded set' applies to metric spaces, and it does not make much sense here since the functions f_n may be extended real-valued. Hence we will assume that '$\{f_n(x)\}$ is a bounded set' means that $\sup_n |f_n(x)| < \infty$.

Let $g = \sup_n |f_n|$, and note that g is measurable by Problem 2.1.9 and Theorem 2.2.3. Let $F = \{x;\, g(x) < \infty\}$. Notice that $g(x) < \infty$ if and only if $\{f_n(x)\}$ is bounded. Hence $\mu(X - F) = 0$.

For $k = 1, 2, \dots,$ define $F_k = \{x;\, g(x) \le k\}$. Then $F_1 \subset F_2 \subset \cdots$ and $\lim_k F_k = \bigcup_{k=1}^{\infty} F_k = F$. By Theorem 1.2.1(iv),

$$\lim_k \mu(X - F_k) = \mu(X - F) = 0.$$

Given any $\epsilon > 0$, there exists a positive integer K such that $\mu(X - F_k) < \epsilon$ for all $k \ge K$. In particular $\mu(X - F_K) < \epsilon$, and $g(x) \le K$ for all $x \in F_K$, which means that $|f_n(x)| \le K$ for all $x \in F_K$.

Section 2.4 – Convergence in Measure

Problems

2.4.3

Prove the following result (which immediately yields another proof of Corollary 2.4.2): Let f_n $(n = 1, 2, \dots)$ and f be a.e. real-valued measurable functions in a finite measure space. For any $\epsilon > 0$, $n \ge 1$, let

$$E_n(\epsilon) = \{x;\, |f_n(x) - f(x)| \ge \epsilon\}.$$

Then $\{f_n\}$ converges a.e. to f if and only if

$$\lim_{n\to\infty} \mu\left[\bigcup_{m=n}^{\infty} E_m(\epsilon)\right] = 0 \quad \text{for any } \epsilon > 0. \tag{2.4.2}$$

[*Hint:* Let $F = \{x;\, \{f_n(x)\}$ is not convergent to $f(x)\}$. Then $F = \bigcup_{k=1}^{\infty} \overline{\lim}_n E_n(1/k)$. Show that $\mu(F) = 0$ if and only if (2.4.2) holds.]

Solution. Define

$$F = \bigcup_{k=1}^{\infty} \overline{\lim_n} E_n\left(\frac{1}{k}\right) = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m\left(\frac{1}{k}\right).$$

Note that

$$\begin{aligned} x \in F &\iff \exists k, \forall n, \exists m \ge n,\ |f_m(x) - f(x)| \ge \frac{1}{k} \\ &\iff \neg\left(\forall k, \exists n, \forall m \ge n,\ |f_m(x) - f(x)| < \frac{1}{k}\right) \\ &\iff f_n(x) \not\to f(x), \end{aligned}$$

so that

$$F = \{x;\, f_n(x) \not\to f(x)\}.$$

Suppose (2.4.2) holds. Fix $\delta > 0$. For every positive integer k, there exists a positive integer n_k such that $n \ge n_k$ implies

$$\mu\left[\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{k}\right)\right] < \frac{\delta}{2^k}.$$

By subadditivity and monotonicity,

$$\begin{aligned}\mu(F) = \mu\left[\bigcup_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{k}\right)\right] &\leq \sum_{k=1}^{\infty}\mu\left[\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{k}\right)\right] \\ &\leq \sum_{k=1}^{\infty}\mu\left[\bigcup_{m=n_k}^{\infty} E_m\left(\frac{1}{k}\right)\right] < \sum_{k=1}^{\infty}\frac{\delta}{2^k} = \delta.\end{aligned}$$

Since δ was arbitrary, $\mu(F) = 0$, and it follows that $f_n \to f$ a.e.

Conversely, suppose $f_n \to f$ a.e., so that $\mu(F) = 0$. By monotonicity and Theorem 1.2.2,

$$0 = \mu(F) = \mu\left[\bigcup_{k=1}^{\infty}\overline{\lim_n} E_n\left(\frac{1}{k}\right)\right] \geq \mu\left[\overline{\lim_n} E_n\left(\frac{1}{l}\right)\right] \geq \overline{\lim_n}\,\mu\left[E_n\left(\frac{1}{l}\right)\right]$$

for all positive integers l. But of course $\overline{\lim}_n\,\mu\left[E_n\left(1/l\right)\right] \geq \underline{\lim}_n\,\mu\left[E_n\left(1/l\right)\right] \geq 0$ since μ is nonnegative, so $\lim_n \mu\left[E_n\left(1/l\right)\right]$ exists and is equal to zero. Note that the sets $\bigcup_{m=n}^{\infty} E_m(1/l)$ are decreasing, so their limit as $n \to \infty$ exists. Hence we can apply Corollary 1.2.3 and monotonicity to find that

$$\begin{aligned}\lim_n \mu\left[\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{l}\right)\right] = \mu\left[\lim_n \bigcup_{m=n}^{\infty} E_m\left(\frac{1}{l}\right)\right] &= \mu\left[\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{l}\right)\right] \\ &\leq \mu\left[\bigcup_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{k}\right)\right] = \mu(F) = 0.\end{aligned}$$

Finally, given $\epsilon > 0$, note that

$$E_n(\epsilon) \subset E_n\left(\frac{1}{\lceil 1/\epsilon\rceil}\right).$$

Hence

$$\lim_n \mu\left[\bigcup_{m=n}^{\infty} E_m(\epsilon)\right] \leq \lim_n \mu\left[\bigcup_{m=n}^{\infty} E_m\left(\frac{1}{\lceil 1/\epsilon\rceil}\right)\right] \leq 0$$

by monotonicity, and (2.4.2) follows.

2.4.4

Let X be the set of all positive integers, $\mathcal{A}$ the class of all subsets of X, and $\mu(E)$ (for any $E \in \mathcal{A}$) the number of points in E. Prove that in this measure space, convergence in measure is equivalent to uniform convergence.

Solution. Uniform convergence always implies convergence in measure. Conversely, suppose (f_n) converges in measure to f. Given any $\epsilon > 0$, there exists a positive integer N such that $n \geq N$ implies

$$\mu\left[\{x;\, |f_n(x) - f(x)| \geq \epsilon\}\right] < 1.$$

That is, for $n \geq N$ the set $\{x;\, |f_n(x) - f(x)| \geq \epsilon\}$ is empty, which in particular means that $\sup_x |f_n(x) - f(x)| \leq \epsilon$. It follows that $f_n \to f$ uniformly.

Section 2.5 – Integrals of Simple Functions

Problems

2.5.2

An integrable simple function f is equal a.e. to zero if and only if $\int_E f d\mu = 0$ for any measurable set E.

Solution. Let f be an integrable simple function. Then f can be written in the form

$$f = \sum_{i=1}^{n} \alpha_i \chi_{E_i},$$

for mutually disjoint sets $E_1, \ldots E_n$, with all $\alpha_i \neq 0$, and all $\mu(E_i) < \infty$.

Suppose $f = 0$ a.e., and let E be any measurable set. By Theorem 2.5.1(b) and (g),

$$0 \leq \int_E f d\mu \leq \int f d\mu = \sum_{i=1}^{n} \alpha_i \mu(E_i).$$

But $\mu(E_i) = 0$ since $f = 0$ a.e., so $\int_E f d\mu = 0$.

Conversely, suppose $\int_E f d\mu = 0$ for all measurable sets E. Then

$$\alpha_i \mu(E_i) = \int_{E_i} f d\mu = 0,$$

so that $\mu(E_i) = 0$, for all $i \in \{1, \ldots, n\}$. It follows that $f = 0$ a.e.

Section 2.6 – Definition of the Integral

Problems

2.6.3

Let f be a measurable function. Prove that f is integrable if and only if f^+ and f^- are integrable, or if and only if $|f|$ is integrable.

Solution. Let f be measurable. We must prove the equivalence of the following statements:

(i) f is integrable.

(ii) f^+ and f^- are integrable.

(iii) $|f|$ is integrable.

We will first show that (iii) $\Longrightarrow$ (ii), then that (ii) $\Longrightarrow$ (i), and finally that (i) $\Longrightarrow$ (iii).

Suppose that $|f|$ is integrable. Let $E = \{x;\ f(x) \geq 0\} = f^{-1}[0, \infty)$, and note that E is measurable since f is. There exists a Cauchy in the mean sequence (g_n) of integrable simple functions converging to $|f|$ a.e., and the sequence $(\chi_E g_n)$

is easily seen to satisfy the corresponding properties with respect to f^+. Since f^+ is measurable by Problem 2.1.8, this implies that it is integrable. The proof that f^- is integrable is similar.

Next, suppose that f^+ and f^- are integrable. Then there exist Cauchy in the mean sequences (g_n) and (h_n) of integrable simple functions converging a.e. to f^+ and f^-, respectively. Define a new sequence (f_n) of integrable simple functions by $f_n = g_n - h_n$. Then (f_n) is Cachy in the mean, since

$$|f_n - f_m| = |g_n - h_n - g_m + h_m| \leq |g_n - g_m| + |h_n - h_m|.$$

It also converges to f a.e. since

$$|f_n - f| = |g_n - h_n - f^+ + f^-| \leq |g_n - f^+| + |h_n - f^-|.$$

It follows that f is integrable.

Finally, assume that f is integrable. There is a Cauchy in the mean sequence (f_n) of integrable simple functions converging to f a.e. The sequence $(|f_n|)$ consists of integrable simple functions. It is Cauchy in the mean since

$$||f_n| - |f_m|| \leq |f_n - f_m|,$$

and it converges to $|f|$ a.e. since

$$||f_n| - |f|| \leq |f_n - f|.$$

Since $|f|$ is measurable by Problem 2.1.9, it follows that $|f|$ is integrable.

2.6.4

Let X be the measure space described in Problem 2.4.4. Then f is integrable if and only if the series $\sum_{n=1}^{\infty} |f(n)|$ is convergent. If f is integrable, then

$$\int f \, d\mu = \sum_{n=1}^{\infty} f(n).$$

Solution. Suppose f is integrable. Then there is a Cauchy in the mean sequence (f_n) of integrable simple functions converging to f a.e. We saw in the previous problem that this implies that $|f|$ is integrable, and that $(|f_n|)$ is a Cauchy in the mean sequence of integrable simple functions converging to $|f|$ a.e. Note that in this particular space convergence a.e. is the same as convergence everywhere (since the only subset with measure zero is $\emptyset$).

By Theorem 2.5.1(h),

$$\int |f_n| \, d\mu = \sum_{i=1}^{\infty} \int_{\{i\}} |f_n| \, d\mu = \sum_{i=1}^{\infty} |f_n(i)|.$$

Hence, in particular,

$$\int |f| \, d\mu = \lim_{n\to\infty} \sum_{i=1}^{\infty} |f_n(i)|.$$

Given any positive integer m, there exists n' such that

$$|f(i) - f_{n'}(i)| < 1/m \quad (i = 1, 2, \ldots, m)$$

(since $f_n \to f$) and

$$\left| \sum_{i=1}^{\infty} |f_{n'}(i)| - \int |f| \, d\mu \right| < 1$$

(since $\sum_i |f_n(i)| \to \int |f| \, d\mu$). Thus

$$\sum_{i=1}^{m} |f(i)| \leq \sum_{i=1}^{m} |f(i) - f_{n'}(i)| + \sum_{i=1}^{m} |f_{n'}(i)| < 1 + \sum_{i=1}^{\infty} |f_{n'}(i)| < 2 + \int |f| \, d\mu,$$

and it follows that the series $\sum_{i=1}^{\infty} |f(i)|$ converges (to a finite number).

Conversely, assume that the series $\sum_{i=1}^{\infty} |f(i)|$ converges. Define a sequence of integrable simple functions (g_n) by

$$g_n = \sum_{i=1}^{n} f(i) \chi_{\{i\}}.$$

It is clear that $g_n \to f$ everywhere. Moreover, if $m > n$, then

$$\int |g_m - g_n| \, d\mu = \int \left| \sum_{i=n+1}^{m} f(i) \chi_{\{i\}} \right| d\mu = \sum_{n+1}^{m} |f(i)| \leq \sum_{n+1}^{\infty} |f(i)|.$$

The right-hand side goes to zero as $n \to \infty$ since $\sum_{i=1}^{\infty} |f(i)|$ is convergent, which means that $\int |g_m - g_n| \, d\mu \to 0$ as $n, m \to \infty$; i.e., (g_n) is Cauchy in the mean. It follows that f is integrable, with

$$\int f \, d\mu = \lim_{n \to \infty} \int g_n \, d\mu = \lim_{n \to \infty} \sum_{i=1}^{n} f(i) = \sum_{i=1}^{\infty} f(i).$$

Section 2.7 – Elementary Properties of Integrals

Problems

2.7.3

Let f be an integrable function. Prove: (a) if $\int_E f d\mu \geq 0$ for all measurable sets E, then $f \geq 0$ a.e.; (b) if $\mu(X) < \infty$ and if $\int_E f d\mu \leq \mu(E)$ for all measurable sets E, then $f \leq 1$ a.e.

Solution.

(a) Let $F = \{x; \, f(x) < 0\}$. Then $-f$ is positive on F, so

$$-\int_F f d\mu = \int_F (-f) d\mu \geq 0$$

by Theorem 2.7.1(b). But

$$\int_F f d\mu \geq 0$$

by our hypothesis on f, so

$$\int_F f d\mu = 0,$$

and it follows by Theorem 2.7.5 that $\mu(F) = 0$. That is, $f \geq 0$ a.e.

(b) Let $G = \{x : f(x) > 1\}$. Then $f - 1$ is positive on G, so

$$\int_G (f-1) d\mu \geq 0$$

by Theorem 2.7.1(b). Note that χ_G is integrable since $\mu(G) \leq \mu(X) < \infty$. Hence we also have

$$\int_G (f-1) d\mu = \int_G f d\mu - \int_G d\mu = \int_G f d\mu - \mu(G) \leq 0,$$

with the inequality following from our hypothesis on f. Thus

$$\int_G (f-1) d\mu = 0,$$

and Theorem 2.7.5 yields $\mu(G) = 0$. That is, $f \leq 1$ a.e.

Section 2.8 – Sequences of Integrable Functions

Problems

2.8.1

A measurable function f is acalled a *null function* if $f = 0$ a.e. We shall say that f is *equivalent* to g (and write $f \sim g$) if $f - g$ is a null function. Denote by $\bar{f}$ the class of all measurable functions that are equivalent to f. We denote by $L^1(X, \mathcal{A}, \mu)$, or, more briefly, by $L^1(X, \mu)$, the set of all classes $\bar{f}$ for which f is integrable, and define on it the function

$$\rho(\bar{f}, \bar{g}) = \rho(f, g) = \int |f - g| d\mu.$$

[Note that if $f_0 \in \bar{f}$, $g_0 \in \bar{g}$, then $\rho(f_0, g_0) = \rho(f, g)$.] Prove that $L^1(X, \mu)$ is a complete metric space with the metric ρ.

Solution. Let $(\bar{f}_n)$ be a Cauchy sequence in $L^1(X, \mu)$. Then

$$\int |f_m - f_n| d\mu = \rho(\bar{f}_m, \bar{f}_n) \to 0$$

as $m, n \to \infty$, so the sequence (f_n) (of representative functions) is Cauchy in the mean. By Theorem 2.8.3 there is an integrable function f such that $f_n \to f$ in the mean. Hence

$$\rho(\bar{f}_n, \bar{f}) = \int |f_n - f| d\mu \to 0$$

as $n \to \infty$. That is, $\bar{f}_n \to \bar{f}$ in $L^1(X, \mu)$.

2.8.2

TODO.

Section 2.9 – Lebesgue's Bounded Convergence Theorem

Problems

2.9.1

Let $f_n(x) = n$ if $0 \leq x < 1/n$, $f_n(x) = 0$ if $1 \geq x \geq 1/n$. Then $\lim_n f_n(x) = 0$ on $[0, 1]$, but $\lim_n \int f_n d\mu = 1$ (μ is the Lebesgue integral). This example shows that the boundedness condition $|f_n| \leq g$ in Theorem 2.9.1 is essential.

Solution. Indeed $f_n(x) \to 0$ for all $x > 0$, and

$$\mu([0, 1] - (0, 1]) = \mu(\{0\}) = 0,$$

so $f_n(x) \to 0$ a.e. Note that $f_n = n\chi_{[0,1/n)}$ for each n, so that

$$\int f_n d\mu = n\mu([0, 1/n)) = n\frac{1}{n} = 1.$$

Hence $\lim_n \int f_n d\mu = 1$.

2.9.4

If $\mu(X) < \infty$ and if $\{f_n\}$ is a sequence of measurable functions that converges uniformly to a function f, then f is integrable and $\lim_n \int f_n d\mu = \int f d\mu$.

Solution. The problem statement, as written, is wrong. For a counterexample, let $X = \{1, 2, \ldots\}$ and define the measure by $\mu(\{x\}) = 2^{-x}$ (extended additively to all subsets of X). Then $\mu(X) = 1$. Define f by $f(x) = 2^x$ and let $f_n = f$ for all n. These definitions match the assumptions of the problem, but

$$f \geq g_n := \sum_{i=1}^{n} f(i)\chi_{\{i\}}$$

for all n, and

$$\int g_n d\mu = n \to \infty,$$

so f cannot possibly be integrable.

We will therefore assume that the functions f_n are integrable, not just measurable. Given any $\epsilon > 0$, there is an integer N such that $n \geq N$ implies $|f_n - f| < \epsilon$. Hence

$$|f_m - f_n| \leq |f_m - f| + |f_n - f| < 2\epsilon$$

whenever $m, n \geq N$, and

$$\int |f_m - f_n| d\mu \leq 2\epsilon\mu(X).$$

It follows (since $\mu(X) < \infty$) that the sequence (f_n) is Cauchy in the mean, hence that f is integrable and $\int f d\mu = \lim_n \int f_n d\mu$ by Theorem 2.8.2

Remark: Assuming that the functions f_n are integrable might not be the right way to fix to this problem, in particular since we did not even need to use the bounded convergence theorem to solve this version of it.

Section 2.10 – Applications of Lebesgue's Bounded Convergence Theorem

Problems

2.10.2

Derive the Lebesgue monotone convergence theorem from Fatou's Lemma.

Solution. Let (f_n) be a monotone increasing sequence of non-negative integrable functions. Note that

$$\varliminf_n f_n(x) = \lim_n \inf_{j \geq n} f_j(x) = \lim_n f_n(x)$$

for all x, since $f_1(x) \leq f_2(x) \leq \cdots$. Similarly,

$$\varliminf_n \int f_n d\mu = \lim_n \int f_n d\mu.$$

By Fatou's lemma (Theorem 2.10.5),

$$\int \lim_n f_n d\mu = \int \varliminf_n f_n d\mu \leq \varliminf_n \int f_n d\mu = \lim_n \int f_n d\mu.$$

(Note that either the right-hand side or both sides of this inequality may be infinite.) On the other hand we also have

$$\lim_n \int f_n d\mu \leq \int \lim_n f_n d\mu$$

(again with the possibility of infinite values), since $f_j \leq \lim_n f_n$ for all j. Combining these results, we obtain

$$\lim_n \int f_n d\mu = \int \lim_n f_n d\mu,$$

with both sides either finite or infinite.

2.10.7

Let $\{f_n\}$ be a sequence of integrable functions. Prove that if $\sum_{n=1}^{\infty} \int |f_n| d\mu < \infty$, then the series $\sum_{n=1}^{\infty} f_n(x)$ is convergent to an integrable function $f(x)$, and

$$\int f \, d\mu = \sum_{n=1}^{\infty} \int f_n d\mu.$$

Solution. Assume that

$$\sum_{n=1}^{\infty} \int |f_n| \, d\mu < \infty.$$

For each $n \geq 1$, let

$$g_n = \sum_{i=1}^{n} f_i^+.$$

Then (g_n) is a monotone-increasing sequence of nonnegative integrable functions. Also define $g = \lim_n g_n$. By the monotone convergence theorem (Theorem 2.10.4),

$$\int g \, d\mu = \lim_n \int g_n d\mu = \lim_n \int \sum_{i=1}^{n} f_i^+ d\mu = \lim_n \sum_{i=1}^{n} \int f_i^+ d\mu = \sum_{n=1}^{\infty} \int f_n^+ d\mu,$$

which is finite, since

$$\sum_{n=1}^{\infty} \int f_n^+ d\mu \leq \sum_{n=1}^{\infty} \int |f_n| \, d\mu < \infty.$$

Similarly defining $h_n = \sum_{i=1}^{n} f_i^-$ and $h = \lim_n h_n$, we find that

$$\int h \, d\mu = \sum_{n=1}^{\infty} \int f_i^- d\mu < \infty.$$

Next, define $f = g - h$. Note that f is integrable by the above, and that

$$f = \sum_{n=1}^{\infty} (f_n^+ - f_n^-) = \sum_{n=1}^{\infty} f_n.$$

Also by what we found above,

$$\int f \, d\mu = \int g \, d\mu - \int h \, d\mu = \sum_{n=1}^{\infty} \int f_n^+ d\mu - \sum_{n=1}^{\infty} \int f_n^- d\mu,$$

from which we easily get

$$\int f \, d\mu = \sum_{n=1}^{\infty} \int f_n d\mu.$$

2.10.8

Let $\{f_n\}$ be a sequence of nonnegative integrable functions. Prove that if the series $f(x) = \sum f_n(x)$ is an integrable function, then $\sum_{n=1}^{\infty} \int f_n d\mu < \infty$.

Solution. Assume that f is integrable. Then

$$\sum_{n=1}^{m} \int f_n d\mu = \int \sum_{n=1}^{m} f_n d\mu \leq \int f\, d\mu < \infty$$

for every m. Also,

$$\sum_{n=1}^{m} \int f_n d\mu \leq \sum_{n=1}^{m+1} \int f_n d\mu$$

for every m, since the functions f_n are nonnegative. Hence the partial sums make up a monotone-increasing sequence of real numbers that is bounded above, and it follows that they converge to a finite number.

2.10.9

Let f and f_n $(n = 1, 2, \dots)$ be integrable functions such that $0 \leq f_n(x) \leq f(x)$ a.e. Then

$$\int \left(\overline{\lim_n} f_n \right) d\mu \geq \overline{\lim_n} \int f_n d\mu \geq \underline{\lim_n} \int f_n d\mu \geq \int \left(\underline{\lim_n} f_n \right) d\mu.$$

Solution. For each $n \geq 1$, define

$$\tilde{f}_n(x) = \begin{cases} f_n(x) & \text{if } 0 \leq f_n(x) \leq f(x), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } 0 \leq f_n(x) \leq f(x), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\tilde{f}_n(x) = f_n(x)$ a.e., $\tilde{f}(x) = f(x)$ a.e., and $0 \leq \tilde{f}_n(x) \leq f(x)$ *everywhere.* By Problem 2.6.1, the functions $\tilde{f}_n$ and $\tilde{f}$ are integrable with

$$\int \tilde{f}_n d\mu = \int f_n d\mu, \quad \int \tilde{f}\, d\mu = \int f\, d\mu.$$

Moreover, since

$$\underline{\lim_n} \tilde{f}_n(x) \neq \underline{\lim_n} f_n(x) \quad \Longrightarrow \quad \exists n : \tilde{f}_n(x) \neq f_n(x)$$

we have

$$\{x;\, \underline{\lim_n} \tilde{f}_n(x) \neq \underline{\lim_n} f_n(x)\} \subset \bigcup_{n=1}^{\infty} \{x;\, \tilde{f}_n(x) \neq f_n(x)\}.$$

Each of the sets in the union on the right-hand side has measure zero, so $\underline{\lim}_n \tilde{f}_n(x) = \underline{\lim}_n f_n(x)$ a.e. Similarly, $\overline{\lim}_n \tilde{f}_n(x) = \overline{\lim}_n f_n(x)$ a.e. These functions are all integrable by Theorem 2.10.1 (since they are bounded a.e. by f), and Problem 2.6.1 again gives

$$\int \underline{\lim}_n \tilde{f}_n d\mu = \int \underline{\lim}_n f_n d\mu, \quad \int \overline{\lim}_n \tilde{f}_n d\mu = \int \overline{\lim}_n f_n d\mu.$$

The upshot of the preceding discussion is that we can safely assume that $0 \leq f_n(x) \leq f(x)$ holds *everywhere*; otherwise we simply work with the functions $\tilde{f}_n$ and $\tilde{f}$ instead. With such an assumption the inequality

$$\int \underline{\lim}_n f_n d\mu \leq \underline{\lim}_n \int f_n d\mu$$

follows immediately from Fatou's lemma. The next inequality,

$$\underline{\lim}_n \int f_n d\mu \leq \overline{\lim}_n \int f_n d\mu,$$

is trivial. For the final inequality, note that

$$f(x) - \overline{\lim}_n f_n(x) = \underline{\lim}_n (f(x) - f_n(x))$$

for all x. Hence, again by Fatou's lemma,

$$\begin{aligned}
\int f\, d\mu - \int \overline{\lim}_n f_n d\mu &= \int \underline{\lim}_n (f - f_n) d\mu \\
&\leq \underline{\lim}_n \int (f - f_n) d\mu \\
&= \underline{\lim}_n \left(\int f\, d\mu - \int f_n d\mu \right) \\
&= \int f\, d\mu - \overline{\lim}_n \int f_n d\mu,
\end{aligned}$$

whereby

$$\overline{\lim}_n \int f_n d\mu \leq \int \overline{\lim}_n f_n d\mu.$$

2.10.11

Let $X = \bigcup_{n=1}^{\infty} E_n$, $E_n \subset E_{n+1}$ for all n. Let f be a nonnegative measurable function. Prove that

$$\int f\, d\mu = \lim_{n \to \infty} \int_{E_n} f\, d\mu.$$

Solution. Note that $(\chi_{E_n} f)$ is a monotone-increasing sequence of nonnegative measurable functions, and that

$$\int_{E_n} f\, d\mu = \int \chi_{E_n} f\, d\mu.$$

Consider first the case that there is some n such that $\chi_{E_n} f$ is not integrable. Then f cannot be integrable, and $\chi_{E_m} f$ cannot be integrable for $m \geq n$, by Theorem 2.10.1 (since these functions bound $\chi_{E_n} f$ from above). Hence

$$\int f \, d\mu = \lim_n \int \chi_{E_n} f \, d\mu = \infty$$

in this case. If on the other hand all of the functions $\chi_{E_n} f$ are integrable, then the monotone convergence theorem tells us that

$$\lim_n \int \chi_{E_n} f \, d\mu = \int f \, d\mu,$$

since $\lim_n \chi_{E_n} f = f$.

2.10.12

Let f be a nonnegative measurable function and let

$$f_n(x) = \begin{cases} f(x), & \text{if } f(x) \leq n, \\ n, & \text{if } f(x) > n. \end{cases}$$

Prove that $\lim_n \int f_n d\mu = \int f d\mu$.

Solution. Note that (f_n) is a monotone-increasing sequence of nonnegative measurable functions, with $\lim_n f_n = f$. If f_n is not integrable for some n, then

$$\int f \, d\mu = \lim_n \int f_n d\mu = \infty$$

by Theorem 2.10.1. Otherwise the result follows by the monotone convergence theorem.

2.10.14

Give an example where Fatou's lemma holds with strict inequality.

Solution. Consider $[0,1] \subset \mathbb{R}$ with Lebesgue measure. Let $f_n = \chi_{[0,1/2]}$ for odd n, and $f_n = \chi_{(1/2,1]}$ for even n. Then

$$\int f_n d\mu = \frac{1}{2}$$

for all n, so that

$$\underline{\lim}_n \int f_n d\mu = \frac{1}{2},$$

but $\underline{\lim}_n f_n = 0$ so that

$$\int \underline{\lim}_n f_n d\mu = 0.$$

Chapter 3 – Metric Spaces

Section 3.1 – Topological and Metric Spaces

Problems

3.1.1

Prove that if (X, p) is a metric space, and if

$$\hat{\rho}(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)},$$

then also $(X, \hat{\rho})$ is a metric space. [*Hint:* Cf. the proof of (3.1.3).]

Solution. The only nonobvious property is the triangle inequality. Let x, y, z be arbitrary points of X. Since $t \mapsto t/(1+t)$ is monotone increasing on $[0, \infty)$, and since $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$, we have

$$\frac{\rho(x, z)}{1 + \rho(x, z)} \leq \frac{\rho(x, y) + \rho(y, z)}{1 + \rho(x, y) + \rho(y, z)}.$$

Moreover, by equation (3.1.3),

$$\frac{\rho(x, y) + \rho(y, z)}{1 + \rho(x, y) + \rho(y, z)} \leq \frac{\rho(x, y)}{1 + \rho(x, y)} + \frac{\rho(y, z)}{1 + \rho(y, z)}.$$

It follows that $\hat{\rho}(x, z) \leq \hat{\rho}(x, y) + \rho(y, z)$.

3.1.2

Let $X, \rho, \hat{\rho}$ be as in Problem 3.1.1. Prove that $\rho(x_n, x) \to 0$ if and only if $\hat{\rho}(x_n, x) \to 0$. Give an example showing that ρ and $\hat{\rho}$ are not equivalent in general.

Solution. It is clear that $\rho(x_n, x) \to 0$ implies $\hat{\rho}(x_n, x) \to 0$, since $\hat{\rho} \leq \rho$.

Conversely, suppose $\hat{\rho}(x_n, x) \to 0$. Given any $\epsilon > 0$, there is a positive integer N such that

$$\hat{\rho}(x_n, x) < \frac{\epsilon}{1 + \epsilon} \quad (n \geq N).$$

Substituting the definition of $\hat{\rho}$ and rearranging yields $\rho(x_n, x) < \epsilon$.

If ρ and $\hat{\rho}$ are equivalent then, in particular, there exists a positive constant β such that

$$\frac{\rho(x,y)}{\hat{\rho}(x,y)} \leq \beta$$

whenever $x \neq y$. But

$$\frac{\rho(x,y)}{\hat{\rho}(x,y)} = 1 + \rho(x,y),$$

so this is impossible if X is unbounded (w.r.t. ρ), say if $X = \mathbb{R}^n$ and ρ is the Euclidean metric.

3.1.6

Prove that the spaces l^1, s, c, c_0 are separable metric spaces.

Solution. For each of the spaces X we will take an arbitrary element $x \in X$ and demonstrate that every ϵ-ball around x contains a point y of a certain countable subset $Y \subset X$. It will then follow that Y is dense in X, and hence that X is separable.

Let $x = (x_i) \in l^1$. Fix $\epsilon > 0$. Since $\sum_i |x_i| < \infty$, there exists n such that

$$\sum_{i=n+1}^{\infty} |x_i| < \frac{\epsilon}{2}.$$

For $i = 1, \dots, n$, choose $y_i \in \mathbb{Q}$ (or y_i with rational real and imaginary parts in the complex case) such that $|x_i - y_i| < \epsilon/2n$, and let $y = (y_1, \dots, y_n, 0, 0, \dots)$. Then

$$\rho(x,y) = \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=n+1}^{\infty} |x_i| < \epsilon.$$

Moreover, y is an element of the subset of l^1 consisting of sequences with rational components, with only finitely many being nonzero. This subset is easily seen to be countable, and it follows that l^1 is separable.

Next, let $x = (x_i) \in s$, and fix $\epsilon > 0$. Choose n such that

$$\sum_{i=n+1}^{\infty} \frac{1}{2^i} \frac{|x_i|}{1 + |x_i|} < \frac{\epsilon}{2}.$$

For $i = 1, \dots, n$, choose $y_i \in \mathbb{Q}$ such that $|x_i - y_i| < \epsilon/2$, and let $y = (y_1, \dots, y_n, 0, 0, \dots)$. Then

$$\rho(x,y) = \sum_{i=1}^{n} \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|} + \sum_{i=n+1}^{\infty} \frac{1}{2^i} \frac{|x_i|}{1 + |x_i|} < \epsilon.$$

Similarly to the above, it follows that s is dense.

Finally, let $x = (x_i) \in c$. (This argument will also cover c_0.) Let $\xi = \lim_i x_i$ and fix $\epsilon > 0$. Choose n such that $|x_i - \xi| < \epsilon/2$ for all $i \geq n$. For $i = 1, \dots, n-1$, choose $y_i \in \mathbb{Q}$ such that $|x_i - y_i| < \epsilon$. Also choose $\eta \in \mathbb{Q}$ such that $|\xi - \eta| < \epsilon/2$ (take $\eta = \xi = 0$ in the c_0 case), so that

$$|x_i - \eta| \leq |x_i - \xi| + |\xi - \eta| < \epsilon$$

for all $i \geq n$. Let $y = (y_1, \dots, y_{n-1}, \eta, \eta, \dots)$. Then

$$\rho(x, y) = \sup_i |x_i - y_i| \leq \epsilon.$$

It follows that c is separable (and c_0 as well).

3.1.10

If $\rho(x_n, x) \to 0$, $\rho(y_n, y) \to 0$, then $\rho(x_n, y_n) \to \rho(x, y)$.

Solution. The triangle inequality yields

$$\rho(x_n, y_n) \leq \rho(x_n, x) + \rho(x, y) + \rho(y, y_n)$$

and

$$\rho(x, y) \leq \rho(x, x_n) + \rho(x_n, y_n) + \rho(y_n, y).$$

Hence

$$|\rho(x_n, y_n) - \rho(x, y)| \leq \rho(x_n, x) + \rho(y_n, y),$$

and the result follows.

Section 3.2 – L^p Spaces

Problems

3.2.4

Prove that l^p is separable if $1 \leq p < \infty$.

Solution. The argument is of the same sort as in Problem 3.1.6. Given $x = (x_i) \in l^p$ $(1 \leq p < \infty)$ and $\epsilon > 0$, choose n such that

$$\sum_{i=n+1}^{\infty} |x_i|^p < \frac{\epsilon^p}{2}.$$

(This is possible since $\sum_i |x_i|^p < \infty$ for $x \in l^p$.) For $i = 1, \dots, n$, choose $y_i \in \mathbb{Q}$ such that $|x_i - y_i|^p < \epsilon^p/2n$ and let $y = (y_1, \dots, y_n, 0, 0, \dots)$. Then $\|x - y\|_p < \epsilon$, and the conclusion follows as in Problem 3.1.6.

3.2.5

Prove that l^p is not a metric space if $0 < p < 1$.
Solution. Let $x = (1, 0, 0, \dots)$, $y = (0, 0, \dots)$, $z = (0, 1, 0, 0, \dots)$. Then

$$\|x - z\|_p = 2^{1/p}$$

while

$$\|x - y\|_p + \|y - z\|_p = 2.$$

But $2^{1/p} > 2$ if $0 < p < 1$, so the triangle inequality does not hold.

3.2.6

Prove that the space $C[a, b]$ with the metric

$$\rho(f, g) = \int_a^b |f(t) - g(t)|\, dt$$

is not a complete metric space.
Solution. Note that the integral defining ρ is guaranteed to exist by Theorem 2.11.1 and the extreme value theorem from elementary analysis, and it can be taken in the Riemann sense. It is easily seen that ρ is indeed a metric.

For simplicity, let us assume $a = 0$ and $b = 1$. (The general argument is identical modulo a coordinate transformation.) Define a sequence of functions in $C[0, 1]$ by $f_n(x) = x^n$. Clearly (f_n) converges pointwise to the discontinuous function

$$f(x) = \begin{cases} 0 & (0 \le x < 1), \\ 1 & (x = 1). \end{cases}$$

For $n \le m$ we have $f_n \ge g_n$, hence

$$\rho(f_n, f_m) = \int_0^1 f_n(t)\, dt - \int_0^1 f_m(t)\, dt = \frac{1}{n+1} - \frac{1}{m+1},$$

which goes to zero as $n, m \to \infty$. Thus (f_n) is a Cauchy sequence in $C[0, 1]$ which converges to a function f which is not in $C[0, 1]$, so $(C[0, 1], \rho)$ is not a complete metric space.

Section 3.4 – Complete Metric Spaces

Problems

3.4.5

A set Y in a metric space X is said to be *of the first category in* X if it is contained in a countable union of nowhere dense sets of X. If Y is not of the first category in X, then it is said to be *of the second category in* X. The real line with the Euclidean metric is a space of the second category. Prove, however, that, as a subset of the Euclidean plane, the real line is a set of the first category.

Solution. We identify the real line with the subset $L = \{(x, y) \in \mathbb{R}^2 : y = 0\}$, with the induced metric topology. Indeed L is homeomorphic to $\mathbb{R}$ via the map $(x, 0) \mapsto x$, as can easily be shown.

The sets of the form $B((0,0), r) \cap L = \{(x, 0) : |x| < r\}$ are easily seen to be nowhere dense in $\mathbb{R}^2$, and we have

$$L = \bigcup_{n=1}^{\infty} B((0,0), n) \cap L.$$

Hence L is of the first category in $\mathbb{R}^2$.

3.4.7

Let $f(x)$ be a real-valued function on the real line. Prove that there is a nonempty interval (a, b) and a positive number c such that for any $x \in (a, b)$ there is a sequence $\{x_n\}$ such that $x_n \to x$ and $|f(x_n)| \leq c$.

Solution. Note that

$$\mathbb{R} = |f|^{-1}(\mathbb{R}) = |f|^{-1}\left(\bigcup_{n=1}^{\infty}(-\infty, n]\right) = \bigcup_{n=1}^{\infty} |f|^{-1}\left((-\infty, n]\right).$$

Since $\mathbb{R}$ is of the second category (by Theorem 3.4.2), the sets $|f|^{-1}((-\infty, n])$ cannot all be nowhere dense. Hence there is a positive integer c such that the closure of $|f|^{-1}((-\infty, c])$ has nonempty interior. On account of being a nonempty open set, this interior contains an open interval (a, b).

Let $x \in (a, b)$. Then, since x is contained in the closure of $|f|^{-1}((-\infty, c])$, every neighborhood of x contains a point of $|f|^{-1}((-\infty, c])$. It follows that we can construct a sequence (x_n) in $|f|^{-1}((-\infty, c])$ converging to x, and this sequence will satisfy $|f(x_n)| \leq c$ for all n.

Section 3.5 – Compact Metric Spaces

Problems

3.5.4

A subset F of a compact metric space is compact if and only if it is closed.

Solution. Let X be a compact metric space, and F a subset of X. If F is compact then it is also closed by Corollary 3.5.5.

Conversely, suppose that F is closed. Let $\mathscr{C}$ be any open cover of F. The collection $\mathscr{C} \cup \{X - F\}$ is an open cover of X, hence it has a finite subcover, consisting of some sets $E_1, \ldots, E_n \in \mathscr{C}$ and perhaps $\{X - F\}$. The sets $E_1, \ldots, E_n$ cover F, hence F is compact.

3.5.5

A subset Y of a metric space is totally bounded if and only if its closure $\overline{Y}$ is totally bounded.

Solution. Clearly Y is totally bounded whenever $\overline{Y}$ is. Conversely, suppose Y is totally bounded, and let $\epsilon > 0$ be given. Then Y admits a finite $\epsilon/2$-covering,

$$\{B(x_1, \epsilon/2), \dots, B(x_n, \epsilon/2)\}.$$

The union of the closed balls,

$$\bigcup_{i=1}^{n} \overline{B}(x_i, \epsilon/2),$$

is a closed set containing Y, hence it contains $\overline{Y}$. It follows that

$$\overline{Y} \subset \bigcup_{i=1}^{n} \overline{B}(x_i, \epsilon/2) \subset \bigcup_{i=1}^{n} B(x_i, \epsilon),$$

so $\overline{Y}$ admits a finite ϵ-covering.

3.5.6

The intersection of any number of compact subsets of a metric space is a compact space.

Solution. Let $\mathscr{C}$ be a collection of compact subsets of a metric space X. By Corollary 3.5.5 each $C \in \mathscr{C}$ is closed, hence the intersection $K = \bigcap \mathscr{C}$ is closed as well. Let C' be some particular member of $\mathscr{C}$. Then K is a closed subset of the compact space C', thus itself compact by Problem 3.5.4.

Remark: Really the above shows that K is compact *in* C'. However, one easily proves the general result that if F is compact in G, and G is compact in H, then F is compact in H. Hence K is indeed compact in X.

3.5.8

Show that a metric space is compact if and only if it has the following property: for every collection of closed subsets $\{F_\alpha\}$, if any finite subcollection has nonempty intersection, then the whole collection has a nonempty intersection.

Solution. The wording of the problem is potentially misleading. Replace "any" with "every" to make it nonambiguous. Also, let us introduce some useful terminology. A collection of subsets of a topological space is said to have the *finite intersection property* iff every finite subcollection has nonempty intersection. Our task is therefore to show that a metric space X is compact iff every collection $\mathscr{F}$ of closed subsets of X having the finite intersection property has nonempty intersection. However, the proof we will use is valid for general topological spaces, not just metric spaces.

Let X be a topological space, and suppose first that X is compact. Let $\mathscr{F}$ be a collection of closed subsets of X with the finite intersection property. Assume towards a contradiction that $\bigcap \mathscr{F} = \emptyset$. Then

$$\bigcup_{F \in \mathscr{F}} (X - F) = X - \bigcap_{F \in \mathscr{F}} F = X,$$

so $\{X - F;\, F \in \mathscr{F}\}$ is an open cover of X. Since X is compact, there are sets $F_1, \dots, F_n \in \mathscr{F}$ such that

$$X = \bigcup_{i=1}^{n} (X - F_i) = X - \bigcap_{i=1}^{n} F_i.$$

But then $\bigcap_{i=1}^{n} F_i = \emptyset$, contradicting the hypothesis that $\mathscr{F}$ has the finite intersection property. We conclude that $\bigcap \mathscr{F} \neq \emptyset$.

Conversely, suppose that X has the property described in the problem statement: every collection of closed subsets of X with the finite intersection property has nonempty intersection. Let $\mathscr{U}$ be any open cover of X, and assume towards a contradiction that $\mathscr{U}$ has no finite subcover. Then

$$\bigcap_{i=1}^{n} (X - U_i) = X - \bigcup_{i=1}^{n} U_i \neq \emptyset$$

for every finite subcollection $\{U_1, \dots, U_n\} \subset \mathscr{U}$. Hence $\mathscr{F} = \{X - U;\, U \in \mathscr{U}\}$ is a collection of closed subsets with the finite intersection property. By hypothesis $\mathscr{F}$ has nonempty intersection, so

$$\emptyset = X - \bigcup_{U \in \mathscr{U}} U = \bigcap_{U \in \mathscr{U}} (X - U) = \bigcap_{F \in \mathscr{F}} F \neq \emptyset,$$

a contradiction. We conclude that $\mathscr{U}$ does indeed have a finite subcover, and that X is compact.

Chapter 4 – Elements of Functional Analysis in Banach Spaces

Section 4.1 – Linear Normed Spaces

Problems

4.1.4

If $\{x_n\}$ is a convergent sequence in a normed linear space, with limit x, then also the sequence with elements $(x_1 + \cdots + x_n)/n$ is convergent to x.

Solution. For $n = 1, 2, \ldots$, define

$$\sigma_n = \frac{x_1 + \cdots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Let $\epsilon > 0$ be given. Since (x_n) is convergent there exists a positive integer m such that $\|x_n - x\| < \epsilon/2$ for all $n \geq m + 1$. For such n we have

$$\begin{aligned}
\|\sigma_n - x\| &= \left\| \frac{1}{n}\sum_{i=1}^{n}(x_i - x) \right\| \\
&\leq \frac{1}{n}\left\| \sum_{i=1}^{m}(x_i - x) \right\| + \frac{1}{n}\left\| \sum_{i=m+1}^{n}(x_i - x) \right\| \\
&\leq \frac{1}{n}\left\| \sum_{i=1}^{m}(x_i - x) \right\| + \frac{1}{n}\sum_{i=m+1}^{n}\|x_i - x\| \\
&\leq \frac{1}{n}\left\| \sum_{i=1}^{m}(x_i - x) \right\| + \frac{n - m - 1}{n} \cdot \frac{\epsilon}{2}.
\end{aligned}$$

The first term on the right-hand side goes to zero as $n \to \infty$, and the other goes to $\epsilon/2$. Hence $\|\sigma_n - x\| < \epsilon$ for sufficiently large n, and it follows that $\sigma_n \to x$.

4.1.6

A normed linear space is a Banach space if the following property is satisfied: every absolutely convergent series is convergent.

Solution. Let X be a normed linear space with the property that every absolutely convergent series is convergent. Let (x_n) be a Cauchy sequence in X, and choose integers $N_1 < N_2 < \cdots$ such that $m, n \geq N_k$ implies $\|x_m - x_n\| < 2^{-k}$. Define a new sequence (y_k) by $y_1 = x_{N_1}$ and $y_k = x_{N_k} - x_{N_{k-1}}$ for $k > 1$. Then

$$\sum_{k=1}^{\infty} \|y_k\| = \|x_{N_1}\| + \sum_{k=2}^{\infty} \|x_{N_k} - x_{N_{k-1}}\| \leq \|x_{N_1}\| + \sum_{k=1}^{\infty} 2^{-k} = \|x_{N_1}\| + 1.$$

It follows by our hypothesis on X that

$$x_{N_k} = \sum_{i=1}^{k} y_i$$

converges to some $x \in X$. Given $\epsilon > 0$, choose an integer $j \geq 1$ such that $2^{-j} < \epsilon$, and $k > j$ such that

$$\|x_{N_k} - x\| < \epsilon - 2^{-j}.$$

Then, for all $n \geq N_k$,

$$\|x_n - x\| \leq \|x_n - x_{N_k}\| + \|x_{N_k} - x\| < 2^{-k} + \epsilon - 2^{-j} < \epsilon.$$

We conclude that every Cauchy sequence in X is convergent, and hence that X is a Banach space.

Section 4.2 – Subspaces and Bases

Problems

4.2.6

If a linear vector space is infinite-dimensional, then there exist on it norms that are not equivalent. [*Hint:* Let $\{y_\alpha\}$ be a Hamel basis and define norms by $\|x\|^2 = \sum_\alpha c_\alpha |\lambda_\alpha|^2$, where x has the form (4.2.2) and c_α are positive numbers.]

Solution. Let $\{y_\alpha\}_{\alpha \in I}$ be a Hamel basis for an infinite-dimensional linear space X. For every $x = \sum_{\alpha \in I} \lambda_\alpha y_\alpha$, define

$$\|x\|_1 = \left(\sum_{\alpha \in I} |\lambda_\alpha|^2 \right)^{1/2}.$$

Then $\| \ \|_1$ is easily seen to be a norm on X.

Let $\{\alpha_i\}_{i=1}^{\infty}$ be a countable subset of the index set I. Define a set $\{c_\alpha\}_{\alpha \in I}$ of positive constants by $c_{\alpha_i} = 2^{-i}$ for $i = 1, 2, \dots,$ and $c_\alpha = 1$ for $\alpha \notin \{\alpha_i\}$. Define another norm $\| \ \|_2$ by

$$\|x\|_2 = \left(\sum_{\alpha \in I} c_\alpha |\lambda_\alpha|^2 \right)^{1/2},$$

for $x = \sum_{\alpha \in I} \lambda_\alpha y_\alpha$.

To see that these two norms are not equivalent, note that

$$\|y_{\alpha_i}\|_2^2 = c_{\alpha_i} = 2^{-i} = 2^{-i} \|y_{\alpha_i}\|_1^2$$

for all i. Hence there exists no $\beta > 0$ such that $\|x\|_1 \le \beta \|x\|_2$ for all $x \in X$.

Section 4.3 – Finite-Dimensional Normed Linear Spaces

Problems

4.3.1

Let X be a finite-dimensional linear space. Then any two norms on X are equivalent. (According to Problem 4.2.6, the assertion is false if X is infinite-dimensional.)

Solution. Let $e_1, \dots, e_n$ be a basis of X. For every $x = \sum_{i=1}^{n} \lambda_i e_i \in X$, let

$$\|x\|_1 = \sum_{i=1}^{n} |\lambda_i|.$$

It is easily verified that $\| \ \|_1$ is a norm on X. We will show that every norm on X is equivalent to $\| \ \|_1$, and hence (by transitivity) that any two norms on X are equivalent.

Given an arbitrary norm $\| \ \|$, we must prove the existence of positive constants α and β such that

$$\alpha \|x\|_1 \le \|x\| \le \beta \|x\|_1$$

for all $x \in X$. These inequalities hold trivially for $x = 0$, so it suffices to consider nonzero x. In fact it is sufficient to consider x in the "$\| \ \|_1$-sphere" $S_1 = \{x \in X : \|x\|_1 = 1\}$, where the inequalities reduce to

$$\alpha \le \|x\| \le \beta.$$

The inequality for general, nonzero x then follows upon division by $\|x\|_1$.

We start by showing that that the map $x \mapsto \|x\|$ is continuous with respect to the metric $\rho_1(x,y) = \|x-y\|_1$. Let $\epsilon > 0$ be given, and write $M = \max(\|e_1\|, \ldots, \|e_n\|)$. Given

$$x = \sum_{i=1}^{n} \lambda_i e_i, \quad y = \sum_{i=1}^{n} \mu_i e_i$$

satisfying $\rho_1(x,y) < \epsilon/M$, we have

$$\|x-y\| \le \sum_{i=1}^{n} |\lambda_i - \mu_i| \, \|e_i\| \le \sum_{i=1}^{n} |\lambda_i - \mu_i| \, M = \rho_1(x,y)M < \epsilon,$$

and continuity follows.

The sphere S_1 is obviously closed and bounded under $\| \ \|_1$, hence compact by Theorem 4.3.3. By Theorem 3.6.2 and the continuity established above, the map $x \mapsto \|x\|$ attains a maximum and a minimum on S_1. Let

$$\alpha = \inf_{x \in S_1} \|x\|, \quad \beta = \sup_{x \in S_1} \|x\|,$$

and note that $\alpha > 0$ since $\|x\| = \alpha$ is attained for some nonzero x. It follows that α and β are positive constants such that $\alpha \le \|x\| \le \beta$ for all $x \in S_1$, so we are done.

4.3.2

Let Y be a finite-dimensional linear subspace of a normed linear space X, and let $x_0 \in X$, $x_0 \notin Y$. Then there exists a point $y_0 \in Y$ such that

$$\inf_{y \in Y} \|x_0 - y\| = \|x_0 - y_0\|.$$

Solution. Note that Y is closed by Theorem 4.3.2. Let $L = \inf_{y \in Y} \|x_0 - y\|$. For $n = 1, 2, \ldots$, choose $y_n \in Y$ such that

$$\|x_0 - y_n\| < L + \frac{1}{n}.$$

Note that $y_n \in B(x_0, L+1) \cap Y$ for all n. This is a bounded subset of Y, so by Theorems 4.3.3 and 3.5.4, the sequence (y_n) has a subsequence (y_{n_k}) converging to a point

$$y_0 \in \overline{B(x_0, L+1) \cap Y} \subset \overline{Y} = Y.$$

For all k we have

$$\|x_0 - y_0\| \le \|x_0 - y_{n_k}\| + \|y_{n_k} - y_0\|.$$

The left-hand side of this inequality is bounded below by L, and the right-hand side converges to L as $k \to \infty$. It follows that $\|x_0 - y_0\| = L$.

4.3.3

A norm $\| \ \|$ is said to be *strictly convex* if $\|x\| = 1$, $\|y\| = 1$, $\|x + y\| = 2$ imply that $x = y$. Prove that if the norm of X is strictly convex, then the point y_0 occurring in the assertion of Problem 4.3.2 is unique.

Solution. Let $\| \ \|$ be a strictly convex norm on a linear space X, and let Y be a finite-dimensional linear subspace of X. Let $x_0 \in X - Y$, $L = \inf_{y \in Y} \|x_0 - y\|$, and suppose that there are elements $y_0, y_0' \in Y$ such that

$$\|x_0 - y_0\| = \|x_0 - y_0'\| = L.$$

It is clear that $\|(x_0 - y_0)/L\| = \|(x_0 - y_0')/L\| = 1$. Moreover,

$$\|(x_0 - y_0)/L + (x_0 - y_0')/L\| = \frac{2}{L}\|x_0 - (y_0 + y_0')/2\| \geq \frac{2}{L}L = 2,$$

and

$$\|(x_0 - y_0)/L + (x_0 - y_0')/L\| \leq \frac{1}{L}\left(\|x_0 - y_0\| + \|x_0 - y_0'\|\right) = \frac{1}{L}2L = 2,$$

so $\|(x_0 - y_0)/L + (x_0 - y_0')/L\| = 2$. By strict convexity,

$$\frac{1}{L}(x_0 - y_0) = \frac{1}{L}(x_0 - y_0'),$$

and it follows that $y_0 = y_0'$.

4.3.4

Prove that the norm of $L^p(X, \mu)$ is strictly convex if $1 < p < \infty$, and is not strictly convex if $p = 1$ or if $p = \infty$.

Solution. Let $f, g \in \mathscr{L}^p(X, \mu)$, $1 < p < \infty$, and suppose $\|f\|_p = \|g\|_p = 1$ and $\|f + g\|_p = 2$. We then have equality in Minkowski's inequality:

$$\|f + g\|_p = \|f\|_p + \|g\|_p.$$

By Problem 3.2.7 this implies that $f = 0$ a.e., or $g = 0$ a.e., or $f = \lambda g$ a.e. for some positive constant λ. The first two possibilities are ruled out since $\|f\|_p = \|g\|_p = 1$, so the third alternative must hold. But then

$$1 = \|f\|_p = |\lambda| \, \|g\|_p = |\lambda|,$$

so $\lambda = 1$. Thus $f = g$ a.e., so that $\tilde{f} = \tilde{g}$ in $L^p(X, \mu)$. It follows that $\| \ \|_p$ is strictly convex if $1 < p < \infty$.

For $p = 1, \infty$, surely the problem is supposed to say 'not *necessarily* strictly convex', because we can come up with examples where $\| \ \|_p$ *is* strictly convex, such as an empty measure space $X = \emptyset$, or any space with identically zero measure $\mu = 0$. (Perhaps less trivial examples exist.) Hence we will only demonstrate that there are examples where $\| \ \|_p$ ($p \in \{1, \infty\}$) is not strictly convex.

For $p = 1$, consider the L^1-space l^1. The sequences $x = (1, 0, 0, \dots)$ and $y = (0, 1, 0, 0, \dots)$ satisfy $\|x\|_1 = \|y\|_1 = 1$ and $\|x + y\|_1 = 2$, yet $x \neq y$. For $p = \infty$, consider the L^∞-space l^∞. The sequences $x = (1, 0, 0, \dots)$ and $y = (1, 1, 1, \dots)$ satisfy $\|x\|_\infty = \|y\|_\infty = 1$ and $\|x + y\|_\infty = 2$, but $x \neq y$.

4.3.5

Prove that in $C[a, b]$ the uniform norm is not equivalent to the L^p norm (for $1 \leq p < \infty$).

Solution. For simplicity, let $a = 0$ and $b = 1$. For $n = 1, 2, \dots,$ define $f_n : [0, 1] \to \mathbb{R}$ (or $\mathbb{C}$) by $f_n(x) = x^n$. Let $\| \ \|_u$ denote the uniform norm. For all n we have

$$\|f_n\|_u = \max_{0 \leq x \leq 1} |f(x)| = 1$$

and

$$\|f_n\|_p = \left(\int_0^1 x^n dx \right)^{1/p} = (n + 1)^{-1/p}.$$

(We are assuming that the L^p norm is defined with the standard Lebesgue measure.) In particular $\|f_n\|_p \to 0$ as $n \to \infty$, whatever the value of p ($\neq \infty$). It follows that there exists no $\beta > 0$ such that $\|f\|_u \leq \beta \|f\|_p$ for all $f \in C[0, 1]$, and hence that the two norms are not equivalent.

4.3.7

Let n be a positive integer, $1 \leq p < \infty$, and let $f(x)$ be a continuous function on $0 \leq x \leq 1$. Then there exists a unique polynomial Q_n of degree n such that for any other polynomial P_n of degree n

$$\int_0^1 |f(x) - P_n(x)|^p dx > \int_0^1 |f(x) - Q_n(x)|^p dx.$$

Solution. There is an error in the problem statement; it should be polynomials of degree $\leq n$, not *exactly* n. To see that the written claim is false, consider the case $f(x) = 0$ and $n = 1$. Then f can be approximated arbitrarily closely by degree 1 polynomials $ax + b$, $a \neq 0$, but no such polynomial will make $\|f - P\|_p$ vanish completely.

Let $\mathscr{P}_n$ denote the set of polynomial functions on $[0, 1]$ with degree $\leq n$. Then $\mathscr{P}_n$ is a linear subspace of the normed linear space $(C[0, 1], \| \ \|_p)$, and is finite-dimensional since it is spanned by the polynomials $1, x, x^2, \dots, x^n$.

If $f \in \mathscr{P}_n$, then $Q = f$ satisfies the claim, since $\|f - P\|_p > 0$ for all $P \neq f$ (else $\| \ \|_p$ would not be a norm). If $f \notin \mathscr{P}_n$, then the conclusion of Problem 4.3.2 tells us that there exists $Q \in \mathscr{P}_n$ such that

$$\inf_{P \in \mathscr{P}_n} \|f - P\| = \|f - Q\|.$$

This Q is unique by Problem 4.3.3 if the norm $\| \ \|_p$ is strictly convex, whereupon the claim follows. By Problem 4.3.4 this is the case for $1 < p < \infty$. In fact the norm is strictly convex even for $p = 1$ over $C[0,1]$, as is easily verified directly from the definition.

Section 4.4 – Linear Transformations

Problems

4.4.2

Let T be an additive operator [that is, $T(x_1 + x_2) = Tx_1 + Tx_2$] from a real normed linear space X into a normed linear space Y. If T is continuous, then T is homogeneous [that is, $T(\lambda x) = \lambda Tx$]. [*Hint:* Prove that $T[(m/n)x] = (m/n)Tx$, where m, n are integers.]

Solution. Let x be an arbitrary element of X. By induction, additivity implies $T(mx) = m\,Tx$ for positive integers m. Moreover,

$$T0 = T(0 + 0) = T0 + T0$$

implies that $T0 = 0$, so that

$$0 = T0 = T(mx - mx) = T(mx) + T(-mx),$$

which shows that $T(-mx) = -T(mx) = -m\,Tx$, thus extending the earlier result to nonpositive integers. Finally, if m and n are integers, $n \neq 0$, then

$$n\,T\left(\frac{m}{n}x\right) = T\left(n\frac{m}{n}x\right) = T(mx) = m\,Tx,$$

so that $T[(m/n)x] = (m/n)Tx$, further extending the result to rationals.

Now, let $\lambda \in \mathbb{R}$. There exists a sequence (λ_n) in $\mathbb{Q}$ such that $\lambda_n \to \lambda$. Clearly $\lambda_n x \to \lambda x$ in X, so $T(\lambda_n x) \to T(\lambda x)$ in Y by continuity of T. But, by what we found above,

$$T(\lambda_n x) = \lambda_n\,Tx \to \lambda\,Tx,$$

so $T(\lambda x) = \lambda\,Tx$.

4.4.6

Find the norm of the operator $A \in \mathscr{B}(X)$ given by $(Af)(t) = tf(t)$ $(0 \le t \le 1)$, where (a) $X = C[0,1]$, (b) $X = L^p(0,1)$ and $(1 \le p \le \infty)$.

Solution.

(a) For $0 \le t \le 1$ we have $|tf(t)| = |t||f(t)| \le |f(t)|$, hence $\|Af\| \le \|f\|$, and

$$\|A\| = \sup_{f \neq 0} \frac{\|Af\|}{\|f\|} \le 1.$$

The upper bound $\|Af\|/\|f\| = 1$ is attained with f constant, so $\|A\| = 1$.

(b) Let us first verify that $A \in \mathscr{B}(X)$, i.e. that it is a bounded linear operator $L^p(0,1) \to L^p(0,1)$. Linearity is immediate. If $f \in \mathscr{L}^p(0,1)$, then $|f|^p$ is integrable, so $|tf(t)|^p = |t|^p|f(t)|^p$ is integrable by Corollary 2.10.2, and we see that A does indeed map into $L^p(0,1)$. Finally, since $|tf(t)|^p = |t|^p|f(t)|^p \le |f(t)|^p$ for $0 < t < 1$, we have $\|Af\|_p \le \|f\|_p$, so that $\|A\| \le 1$.

We will now show that $\|A\| \ge 1$, so that $\|A\| = 1$. The case $p = \infty$ is similar to (a), so we will assume $1 \le p < \infty$. For $n = 1, 2, \ldots$, define simple functions $f_n : (0,1) \to \mathbb{R}$ (or $\mathbb{C}$) by

$$f_n(t) = \begin{cases} 0 & \text{if } 0 < t < 1 - 1/n, \\ n^{1/p} & \text{if } 1 - 1/n \le t < 1. \end{cases}$$

Then one easily finds that $\|f_n\|_p = 1$ and $\|Af_n\|_p \ge 1 - 1/n$, so that

$$\|A\| \ge \frac{\|Af_n\|_p}{\|f_n\|_p} \ge 1 - \frac{1}{n}$$

for all n. Indeed it follows that $\|A\| \ge 1$.

4.4.7

A linear operator from a normed linear space X into a normed linear space Y is bounded if and only if it maps bounded sets onto bounded sets.

Solution. Let $T : X \to Y$ be a linear operator between normed linear spaces.

Suppose first that T is bounded, and let A be a bounded subset of X. Then $\|T\| < \infty$, and there is some L such that $\|x\| \le L < \infty$ for all $x \in A$. Hence

$$\|Tx\| \le \|T\|\,\|x\| \le \|T\|L$$

for all $x \in A$, so $T(A)$ is bounded.

Conversely, suppose that T maps bounded sets onto bounded sets. The set $\{x \in X : \|x\| = 1\}$ is bounded, so there exists M such that $\|Tx\| \le M < \infty$ whenever $\|x\| = 1$. It follows that

$$\|T\| = \sup_{\|x\|=1} \|Tx\| \le M,$$

and hence that T is bounded.

4.4.8

A linear operator from a normed linear space X into a normed linear space Y is continuous if and only if it maps sequences converging to 0 into bounded sequences.

Solution. Let $T : X \to Y$ be a linear operator between normed linear spaces. The claim is immediate if X is the trivial space (containing only the 0 vector), so let us assume that X is nontrivial.

Suppose first that T is continuous, and therefore bounded. Any sequence in X converging to 0 is easily seen to be bounded, hence is mapped to a bounded sequence by the conclusion of the previous problem.

Conversely, suppose T has the property that it maps sequences converging to 0 into bounded sequences. Assume towards a contradiction that T is unbounded. Then it is possible to construct a sequence (x_n) in X such that $\|x_n\| = 1$ and $\|Tx_n\| > n$ for every n. The sequence $(x_n/\sqrt{n})$ converges to 0, so $\{T(x_n/\sqrt{n})\}$ is bounded by hypothesis. But

$$\left\| T\left(\frac{x_n}{\sqrt{n}}\right)\right\| = \frac{1}{\sqrt{n}}\|Tx_n\| > \frac{1}{\sqrt{n}} \cdot n = \sqrt{n} \to \infty$$

as $n \to \infty$, yielding a contradiction. We conclude that T must be bounded, and thus continuous.

Section 4.6 – The Open-Mapping Theorem and the Closed-Graph Theorem

Problems

4.6.1

If T, S, T^{-1}, S^{-1} belong to $\mathscr{B}(X)$, then $(TS)^{-1} \in \mathscr{B}(X)$ and $(TS)^{-1} = S^{-1}T^{-1}$.

Solution. Note in particular that T, S, T^{-1}, S^{-1} belonging to $\mathscr{B}(X)$ implies that all of these transformations are bijective, in addition to bounded. (If T is not surjective, then $D_{T^{-1}} = T(X) \neq X$, hence $T^{-1} \notin \mathscr{B}(X)$.)

It is clear that $\|TS\| \le \|T\|\,\|S\| < \infty$, hence $TS \in \mathscr{B}(X)$, and it is bijective on account of being a composition of bijections. Thus $(TS)^{-1}$ exists and is equal to $S^{-1}T^{-1}$. It follows that $\|(TS)^{-1}\| \le \|S^{-1}\|\,\|T^{-1}\| < \infty$, hence $(TS)^{-1} \in \mathscr{B}(X)$.

4.6.2

Let X be a Banach space and let $A \in \mathscr{B}(X)$, $\|A\| < 1$. Prove that $(I + A)^{-1}$ exists and is given by

$$(I + A)^{-1} = \sum_{n=0}^{\infty} (-1)^n A^n,$$

where the series is absolutely convergent [in $\mathscr{B}(X)$]. Show also that

$$\|(I + A)^{-1}\| \le 1/(1 - \|A\|).$$

Solution. We note first that

$$\sum_{n=0}^{\infty} \|(-1)^n A^n\| = \sum_{n=0}^{\infty} \|A^n\| \le \sum_{n=0}^{\infty} \|A\|^n = \frac{1}{1 - \|A\|},$$

since $\|A\| < 1$. Hence the series $\sum_{n=0}^{\infty}(-1)^n A^n$ is strongly convergent to an operator $B \in \mathscr{B}(X)$, by Theorem 4.5.2. (Recall that $\mathscr{B}(X)$ is a Banach space, hence absolute convergence implies convergence. Convergence in $\mathscr{B}(X)$ is the uniform convergence of operators, and uniform convergence implies strong convergence.)

For all $x \in X$, we have

$$ABx = A\left(\sum_{n=0}^{\infty}(-1)^n A^n x\right) = \sum_{n=0}^{\infty}(-1)^n A^{n+1}x = x - \sum_{n=0}^{\infty}(-1)^n A^n x = (I-B)x,$$

where we have used the continuity of A to interchange limiting processes. Also,

$$BAx = \sum_{n=0}^{\infty}(-1)^n A^{n+1}x = ABx.$$

It follows that

$$B(I+A) = (I+A)B = B + AB = B + I - B = I,$$

so that $B = (I+A)^{-1}$.

Finally,

$$\|(I+A)^{-1}\| \le \sum_{n=0}^{\infty} \|(-1)^n A^n\| \le \frac{1}{1-\|A\|}$$

by what we found earlier.

4.6.3

Let X be a Banach space and let T and T^{-1} belong to $\mathscr{B}(X)$. Prove that if $S \in \mathscr{B}(X)$ and $\|S - T\| < 1/\|T^{-1}\|$, then S^{-1} exists and is a bounded operator, and

$$\|S^{-1} - T^{-1}\| < \frac{\|T^{-1}\|}{1 - \|S-T\|\,\|T^{-1}\|}.$$

[*Hint:* $S = [(S-T)T^{-1} + I]T$.]

Solution. Note that $ST^{-1} = I + (S-T)T^{-1}$, and that

$$\|(S-T)T^{-1}\| \le \|S-T\|\|T^{-1}\| < 1.$$

By the previous problem $(ST^{-1})^{-1}$ exists, and

$$\|(ST^{-1})^{-1}\| \le \frac{1}{1 - \|(S-T)T^{-1}\|} \le \frac{1}{1 - \|(S-T)\|\,\|T^{-1}\|}.$$

Moreover, since $ST^{-1}(ST^{-1})^{-1} = I$ and

$$T^{-1}(ST^{-1})^{-1}S = T^{-1}(ST^{-1})^{-1}ST^{-1}T = T^{-1}T = I,$$

we have $S^{-1} = T^{-1}(ST^{-1})^{-1}$. Finally,

$$\|S^{-1}\| = \|T^{-1}(ST^{-1})^{-1}\| \le \|T^{-1}\| \, \|(ST^{-1})^{-1}\| \le \frac{\|T^{-1}\|}{1 - \|(S-T)\| \, \|T^{-1}\|},$$

and

$$\|S^{-1} - T^{-1}\| = \|S^{-1}(S-T)T^{-1}\| \le \|S^{-1}\| \, \|S-T\| \, \|T^{-1}\| < \|S^{-1}\|,$$

so that

$$\|S^{-1} - T^{-1}\| < \frac{\|T^{-1}\|}{1 - \|(S-T)\| \, \|T^{-1}\|}.$$

4.6.4

Let X and Y be two linear vector spaces. Find necessary and sufficient conditions for a subset G of $X \times Y$ to be the graph of a linear operator from X into Y.

Solution. We claim that G is the graph of a linear operator from X into Y if and only if

(i) G is a linear subspace of $X \times Y$.

(ii) The set $G \cap (\{0\} \times Y)$ is a singleton.

It is clear that these conditions are necessary, so we need only prove that they are sufficient.

Since G is nonempty by (ii), it contains an element (x, y). By (i) it also contains $(0,0) = 0 \cdot (x, y)$, and it follows that $G \cap (\{0\} \times Y) = \{(0,0)\}$.

If $(u, v), (u, v') \in G$, then

$$(0, v - v') = (u, v) - (u, v') \in G \cap (\{0\} \times Y)$$

by (i). It follows that $(0, v - v') = (0, 0)$, hence that $v = v'$.

By the above, G is a functional relation on $X \times Y$, so it defines a partial function $T : D \subset X \to Y$, where

$$D = \{x \in X; \, \exists y \in Y \text{ such that } (x, y) \in G\},$$

and $T(x) = y$ if $(x, y) \in G$.

Suppose $x_1, x_2 \in D$, and λ_1, λ_2 are scalars. Then

$$(\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 T(x_1) + \lambda_2 T(x_2)) = \lambda_1 (x_1, T(x_1)) + \lambda_2 (x_2, T(x_2)) \in G$$

by (i). Hence $\lambda_1 x_1 + \lambda_2 x_2 \in D$, which shows that D is a linear subspace of $X \times Y$, and

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2),$$

so that T is a linear operator.

Finally, note that G is precisely the graph of T. This completes the proof.

4.6.5

Let X and Y be Banach spaces and let T be a bounded linear map from X into Y. If $T(X)$ is of the second category (in Y), then $T(X) = Y$.

Solution. We will use the following lemma:

> If W is a linear subspace of a normed space V, and if W contains a nonempty open subset of V, then $W = V$.

To see that this is true, note that W contains an open ball $B(x_0, r)$. Given any $y \in V$, let

$$x = \frac{r}{2\|y\|} y + x_0.$$

Then $x \in B(x_0, r) \subset W$, so that

$$y = \frac{2\|y\|}{r}(x - x_0) \in W$$

by closure under linear combinations.

Now, assume that $T(X)$ is of the second category, and follow the steps of parts (a) and (b) of the proof of Theorem 4.6.1, but with $T(X)$ in place of Y. We find that $T(X)$ contains an open ball (see equation (4.6.2)) and hence that $T(X) = Y$ by our lemma.

4.6.6

Let X and Y be Banach spaces and let T be a linear map from a linear subspace D_T of X into Y. If D_T (in X) and the graph of T (in $X \times Y$) are closed, then T is bounded—that is, $\|Tx\| \leq K\|x\|$ for all $x \in D_T$ (K constant).

Solution. Note first that D_T is complete; this is true for any closed subset of any complete space. Indeed, if (x_n) is a Cauchy sequence in D_T, then it is also a Cauchy sequence in X, so it converges to a point $x \in X$, and since D_T is closed we have $x \in D_T$. It follows that D_T is a Banach space. Hence we can regard T as a map $D_T \to Y$ and apply Theorem 4.6.4 to conclude that T is continuous, thus bounded by Theorem 4.4.2.

4.6.7

Let X be a normed linear space with any one of two norms $\| \ \|_1$, $\| \ \|_2$. If $\|x_n\|_2 \to 0$ implies $\|x_n\|_1 \to 0$, then (4.6.5) holds.

Solution. Assume towards a contradiction that (4.6.5) does not hold. Then, for every $n \in \{1, 2, \dots\}$, there exists $x_n \in X$ such that $\|x_n\|_1 > n\|x_n\|_2$. The sequence $y_n = x_n/(n\|x_n\|_2)$ satisfies

$$\|y_n\|_1 = \frac{\|x_n\|_1}{n\|x_n\|_2} > \frac{n\|x_n\|_2}{n\|x_n\|_2} = 1$$

for all n. But $\|y_n\|_2 = 1/n \to 0$, which by hypothesis implies $\|y_n\|_1 \to 0$, yielding a contradiction.

Section 4.8 – The Hahn-Banach Theorem

Problems

4.8.1

Let X be a normed linear space and let $\{x_n\} \subset X$. A point y_0 is the limit of linear combinations $\sum_{j=1}^{n} c_j x_j$ if and only if $x^*(y_0) = 0$ for all x^* for which $x^*(x_j) = 0$ for $1 \leq j < \infty$.

Solution. Note that "y_0 is the limit of linear combinations $\sum_{j=1}^{n} c_j x_j$" is not meant to imply $y_0 = \lim_{n\to\infty} \sum_{j=1}^{n} c_j x_j$ for some sequence of coefficients (c_j). Rather, it just says that y_0 is the limit of a sequence of finite linear combinations of elements in $\{x_n\}$; i.e., that y_0 is a limit point of $S := \operatorname{span}\{x_n\}$. Also, the statement "$x^*(x_j) = 0$ for $1 \leq j < \infty$" can be simplified to "x^* vanishes on S."

Suppose that y_0 is a limit point of S. Then there is a sequence (s_n) in S such that $s_n \to y_0$. If $x^* \in X^*$ vanishes on S, then

$$x^*(y_0) = x^*\left(\lim_{n\to\infty} s_n\right) = \lim_{n\to\infty} x^*(s_n) = 0$$

by continuity of x^*.

Conversely, suppose that $x^*(y_0) = 0$ for all $x^* \in X^*$ that vanish on S. Let

$$d = \inf_{s\in S} \|s - y_0\|.$$

If $d > 0$, then Theorem 4.8.3 tells us that there exists $x^* \in X^*$ such that $x^*(y_0) = 1$, but which vanishes on S. This contradicts our assumptions, so we conclude that $d = 0$, which in turn shows that y_0 is a limit point of S.

4.8.5

Let X be an infinite-dimensional Banach space. Prove that there exists an infinite, strictly decreasing sequence $\{Y_n\}$ of infinite-dimensional closed linear subspaces of X. [*Hint:* Take Y_1 to be the null space of some $x_1^* \neq 0$ in X^*. Take Y_2 to be the null space of some $x_2^* \neq 0$ in Y_1^*, and so on.]

Solution. We will construct such a sequence by induction. For the base case, let $Y_0 = X$. Certainly Y_0 is an infinite-dimensional closed linear subspace of X.

For the inductive step, assume that we have infinite-dimensional closed linear subspaces $Y_0 \supset Y_1 \supset \cdots \supset Y_n$, with the inclusions being strict. Fix an element $x_0 \in Y_n$ such that $\|x_0\| = 1$. (Such an element is guaranteed to exist since Y_n is infinite-dimensional.) Corollary 4.8.4 provides a continuous linear functional $x^* \in Y_n$ such that $x^*(x_0) = 1$. Let Y_{n+1} be the null space of x^*; clearly a proper linear subspace of Y_n, hence also of X. As discussed after Corollary 4.8.7, each element $x \in Y_n$ can be written as $x = z + \lambda x_0$, where $\lambda = x^*(x)$ and $z = x - \lambda x_0 \in Y_{n+1}$. It follows that Y_{n+1} is infinite-dimensional. Finally, if (y_i) is a sequence in Y_{n+1} and $y_i \to y \in X$, then $x^*(y) = \lim_{i\to\infty} x^*(y_i) = 0$ by continuity, so $y \in Y_{n+1}$, making Y_{n+1} a closed subset of X. This concludes the inductive step.

4.8.9

Let $u(t)$ be a function defined on $a < t < b$ with values in a Banach space X. We say that $u(t)$ is *strongly differentiable at* t [*on* (a, b)] if $\lim_{h\to 0}\{[u(t+h) - u(t)]/h\}$ exists [for all $t \in (a, b)$]. The limit is denoted by $du(t)/dt$ and is called the derivative of $u(t)$. For functions $A(t)$ with values in $\mathscr{B}(X)$, if $\lim_{h\to 0}\{[A(t+h)x - A(t)x]/h\}$ exists for any $x \in X$, then we say that $A(t)$ has a *strong derivative.* If $\lim_{h\to 0}\{[A(t+h) - A(t)]/h\}$ exists (in the uniform topology), then we say that $A(t)$ is *uniformly differentiable.* Prove that e^{tA} $[A \in \mathscr{B}(X)]$ is uniformly differentiable and $de^{tA}/dt = Ae^{tA}$.

Solution. First note that, given any $t \in \mathbb{R}$,

$$\sum_{n=0}^{\infty} \left\| \frac{(tA)^n}{n!} \right\| \leq \sum_{n=0}^{\infty} \frac{(|t|^n \|A\|^n)}{n!} = e^{|t|\,\|A\|} < \infty,$$

so that the series $\sum_{n=0}^{\infty}(tA)^n/(n!)$ is strongly convergent (to an operator) in $\mathscr{B}(X)$ (c.f. Theorems 4.1.2, 4.5.2). Hence we can define a map $E_A : \mathbb{R} \to \mathscr{B}(X)$ by $E_A(t) = \sum_{n=0}^{\infty}(tA)^n/(n!)$. Instead of $E_A(t)$ we typically write e^{tA}.

With a little work we find that

$$\frac{e^{(t+h)A} - e^{tA}}{h} - Ae^{tA} = \sum_{n=2}^{\infty} \frac{1}{n!} \frac{(t+h)^n - t^n - nht^{n-1}}{h} A^n.$$

Further simplification with the binomial formula yields

$$\frac{(t+h)^n - t^n - nht^{n-1}}{h} = \sum_{k=2}^{n} \binom{n}{k} t^{n-k} h^{k-1}.$$

Hence

$$\left\| \frac{e^{(t+h)A} - e^{tA}}{h} - Ae^{tA} \right\| \leq \sum_{n=2}^{\infty} \frac{\|A\|^n}{n!} \sum_{k=2}^{n} \binom{n}{k} |t|^{n-k} |h|^{k-1}.$$

If $t = 0$, then the right-hand side becomes

$$|h| \cdot \|A\|^2 \sum_{n=0}^{\infty} \frac{(|h|\,\|A\|)^n}{(n+2)!} \leq |h|\,\|A\|^2\, e^{|h|\,\|A\|},$$

which goes to zero as $h \to 0$. If $t \neq 0$, then

$$\sum_{k=2}^{n} \binom{n}{k} |t|^{n-k} |h|^{k-1} \leq |h|\,|t|^{n-2} \sum_{k=2}^{n} \binom{n}{k} = |h|\,|t|^{n-2}\, 2^n$$

whenever $|h| \leq |t|$, yielding

$$\left\| \frac{e^{(t+h)A} - e^{tA}}{h} - Ae^{tA} \right\| \leq \frac{|h|}{|t|^2} \sum_{n=2}^{\infty} \frac{(2\,|t|\,\|A\|)^n}{n!} \leq \frac{|h|}{|t|^2} e^{2\,|t|\,\|A\|}$$

for sufficiently small h, which also goes to zero as $h \to 0$. It follows that e^{tA} is uniformly differentiable on $\mathbb{R}$, with derivative Ae^{tA}.

4.8.10

Let X be a real normed linear space, and let $u(t)$ be continuous and strongly differentiable in (a, b). Then for any $a < \alpha < \beta < b$,

$$\|u(\beta) - u(\alpha)\| \le (\beta - \alpha) \sup_{\alpha \le t \le \beta} \left\| \frac{du(t)}{dt} \right\|.$$

[*Hint:* Apply x^* to $u(\beta) - u(\alpha)$.]

Solution. If $u(\alpha) = u(\beta)$ then there is nothing to prove, so assume $u(\beta) \neq u(\alpha)$. By Corollary 4.8.4 there is a bounded linear operator $x^* \in X^*$ such that $\|x^*\| = 1$ and $x^*(u(\beta) - u(\alpha)) = \|u(\beta) - u(\alpha)\|$. Define $f = x^* \circ u : (a, b) \to \mathbb{R}$. Since x^* is linear and continuous,

$$\lim_{h \to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h \to 0} x^* \left[\frac{u(t+h) - u(h)}{h} \right] = x^* \left[\lim_{h \to 0} \frac{u(t+h) - u(h)}{h} \right],$$

so that f is differentiable with derivative

$$\frac{df(t)}{dt} = x^* \left[\frac{du(t)}{dt} \right].$$

By the mean value theorem from elementary real analysis, we have

$$f(\beta) - f(\alpha) = (\beta - \alpha) \frac{df(\gamma)}{dt}$$

for some $\gamma \in (\alpha, \beta)$. Thus,

$$\begin{aligned}
\|u(\beta) - u(\alpha)\| &= x^*(u(\beta) - u(\alpha)) \\
&= f(\beta) - f(\alpha) \\
&= (\beta - \alpha) \frac{df(\gamma)}{dt} \\
&= (\beta - \alpha) x^* \left[\frac{du(\gamma)}{dt} \right] \\
&\le (\beta - \alpha) \|x^*\| \cdot \left\| \frac{du(\gamma)}{dt} \right\| \\
&\le (\beta - \alpha) \sup_{\alpha \le t \le \beta} \left\| \frac{du(t)}{dt} \right\|.
\end{aligned}$$

4.8.12

For every normed linear space X there is a set A such that X is isomorphic to a subspace of the Banach space of functions f on A with norm $\|f\| = \sup_{t \in A} |f(t)|$. If X is separable, A is countable. [*Hint:* Let $\{x_\alpha : \alpha \in A\}$ be dense in X. Let $f(x, \alpha)$ be the bounded linear functional (in $x \in X$) satisfying $\|f(\,.\,, \alpha)\| = 1$, $f(x_\alpha, \alpha) = \|x_\alpha\|$. Define the isomorphism $x \to g_x(\alpha)$, where $g_x(\alpha) = f(x, \alpha)$. Prove: $\big| |f(x, \alpha)| - \|x\| \big| \le 2\|x_\alpha - x\|$.]

Solution. The conclusion is immediate if X is a trivial space, so assume $X \neq \{0\}$. Let $\{x_\alpha\}_{\alpha\in A}$ be any dense subset of X (possibly X itself), indexed by a set A. (Recall that any space can be indexed by itself.) For each $\alpha \in A$, Corollary 4.8.4 fashions a bounded linear functional $f_\alpha \in X^*$ such that $\|f_\alpha\| = 1$ and $f_\alpha(x_\alpha) = \|x_\alpha\|$.

For each $x \in X$, define $g_x : A \to \mathbb{F}$ (where $\mathbb{F}$ is the field associated with X) by $g_x(\alpha) = f_\alpha(x)$. We will prove the following properties of the functions g_x:

(i) If $x, y \in X$ and $\lambda, \mu \in \mathbb{F}$, then $\lambda g_x + \mu g_y = g_{\lambda x+\mu y}$.

(ii) Each function g_x is bounded, with $\sup_{\alpha\in A} |g_x(\alpha)| = \|x\|$.

To prove (i), simply let $\alpha \in A$ and compute

$$(\lambda g_x+\mu g_y)(\alpha) = \lambda g_x(\alpha)+\mu g_y(\alpha) = \lambda f_\alpha(x)+\mu f_\alpha(y) = f_\alpha(\lambda x+\mu y) = g_{\lambda x+\mu y}(\alpha),$$

using the linearity of f_α.

Proving (ii) takes more work. Note first that

$$\sup_{\alpha\in A} |g_x(\alpha)| = \sup_{\alpha\in A} |f_\alpha(x)| \leq \sup_{\alpha\in A} \|f_\alpha\|\,\|x\| = \|x\|.$$

Next, given any $\epsilon > 0$, fix $\alpha' \in A$ such that $\|x_{\alpha'} - x\| < \epsilon/2$. (This is possible because $\{x_\alpha\}_{\alpha\in A}$ is dense in X.) By the triangle inequality,

$$\|x\| \leq \big|\|x\| - \|x_{\alpha'}\|\big| + \big|\|x_{\alpha'}\| - |f_{\alpha'}(x)|\big| + |f_{\alpha'}(x)|.$$

Now, the "reverse triangle inequality" yields

$$\big|\|x\| - \|x_{\alpha'}\|\big| \leq \|x_{\alpha'} - x\| < \frac{\epsilon}{2}$$

and

$$\begin{aligned}\big|\|x_{\alpha'}\| - |f_{\alpha'}(x)|\big| &\leq \big|\|x_{\alpha'}\| - f_{\alpha'}(x)\big| \\ &= |f_{\alpha'}(x_{\alpha'}) - f_{\alpha'}(x)| \\ &= |f_{\alpha'}(x_{\alpha'} - x)| \\ &\leq \|f_{\alpha'}\|\,\|x_{\alpha'} - x\| \\ &= \|x_{\alpha'} - x\| \\ &< \frac{\epsilon}{2}.\end{aligned}$$

Also,

$$|f_{\alpha'}(x)| \leq \sup_{\alpha\in A} |f_\alpha(x)| = \sup_{\alpha\in A} |g_x(\alpha)|.$$

Putting it all together, we have

$$\|x\| \leq \sup_{\alpha\in A} |g_x(\alpha)| + \epsilon,$$

and since this holds for every ϵ we finally arrive at (ii).

Let F_A be the Banach space of all bounded functions $A \to \mathbb{F}$, with the supremum norm $\|g\| = \sup_{\alpha \in A} |g(\alpha)|$. Since the functions g_x are bounded by (ii), we can define a map $\sigma : X \to F_A$ by $\sigma(x) = g_x$. The image of σ is a linear subspace of F_A by (i). Moreover, σ is an imbedding, since

$$\|\sigma(x) - \sigma(y)\| = \|g_x - g_y\| = \|g_{x-y}\| = \sup_{\alpha \in A} |g_{x-y}(\alpha)| = \|x - y\|$$

for all $x, y \in X$, by (i) and (ii). Every imbedding injective, so σ is an isomorphism (in the sense of Section 3.3) onto its image.

Finally, if X is separable, then we can take A to be countable.

Chapter 6 – Hilbert Spaces and Spectral Theory

Section 6.2 – The Projection Theorem

Problems

6.2.1

If M and N are closed linear spaces and $M \perp N$, then $M \oplus N$ is a closed linear space.

Solution. We start by noting that an analogue of the Pythagorean theorem holds in Hilbert spaces, and in fact more generally in inner product spaces. Namely, if $u \perp v$ then

$$\|u+v\|^2 = (u+v, u+v) = (u,u) + (u,v) + (v,u) + (v,v) = \|u\|^2 + \|v\|^2.$$

We will assume that $M, N \subset H$, with H a Hilbert space. It is clear that $M \oplus N$ is a linear subspace, so it remains only to show that it is closed. Let (x_n) be a sequence in $M \oplus N$ with limit $x \in H$. For each n we have $x_n = y_n + z_n$ for some $y_n \in M$ and $z_n \in N$, and $x = y + z$ for some $y \in M$ and $z \in M^\perp$ by the projection theorem (Theorem 6.2.2). Note that $y_n - y \in M$ and $z_n - z \in M^\perp$, so that $(y_n - y) \perp (z_n - z)$. Thus

$$\|z_n - z\| \leq \|y_n - y + z_n - z\| = \|x_n - x\|$$

by the analogue of the Pythagorean theorem with $u = y_n - y$ and $v = z_n - z$. It follows that $z_n \to z$, so that $z \in N$ since N is closed. This in turn means that $x \in M \oplus N$, hence that $M \oplus N$ is closed.

6.2.2

Let M be any subset of a Hilbert space H. Then $(M^\perp)^\perp$ is the closed linear space spanned by M.

Solution. We begin with a small lemma:

If C is a closed linear subspace of a Hilbert space, then $(C^\perp)^\perp = C$.

Indeed, it is clear that $C \subset (C^{\perp})^{\perp}$. Conversely, suppose $x \in (C^{\perp})^{\perp}$. By the projection theorem $x = y + z$ for some $y \in C$ and $z \in C^{\perp}$. But

$$\|z\|^2 = (z, z) = (x - y, z) = (x, z) - (y, z) = 0$$

by orthogonality, so $z = 0$ and $x = y \in C$. Hence $(C^{\perp})^{\perp} \subset C$, and the lemma follows.

Note that "the closed linear space spanned by M" is the intersection of all linear subspaces of H that contain M (by definition; see Section 4.2). Let us denote this subspace by C. We immediately see that $C \subset (M^{\perp})^{\perp}$, since the latter is a closed linear subspace containing M. Moreover, since $M \subset C$, we have $C^{\perp} \subset M^{\perp}$. By taking orthogonal complements once more and applying the lemma, we obtain

$$(M^{\perp})^{\perp} \subset (C^{\perp})^{\perp} = C,$$

thus proving that $(M^{\perp})^{\perp} = C$.