$$v_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$w_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$w_3 = \text{normalised}\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{3}{\sqrt{5}} \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$= \text{normalised}\left(\begin{bmatrix} -\frac{1}{5} \\ \frac{2}{5} \\ 0 \end{bmatrix} \right) = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$\therefore \quad Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}$$

$$A = QR$$

$$\Rightarrow R = Q^{-1} A = Q^{t} A$$

$$\therefore R = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}^t \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{5} & 0 & \frac{3}{\sqrt{5}} \\ 0 & 1 & 2 \\ 0 & 0 & \frac{1}{\sqrt{5}} \end{bmatrix}$$

1. $$Ax = b = \begin{bmatrix} 0 & 3 & -3 \end{bmatrix}^t$$

$$\Rightarrow QRx = b$$

$$\Rightarrow Rx = Q^{-1}b = Q^t b$$

2 skipping

3 $A|_U = id_U$, $A(U^\perp) \subset U^\perp$.

$A|_{U^\perp}$ has 0 has the only fixed point

think of eigenvalues and eigenspace
is there an eigenspace that would solve this ?

Let $w \in U^\perp$
show $A(w) \in U^\perp$

if $A(w) = w$, then $w = 0$

suppose another subspace V exists
$\Rightarrow V \subseteq U$

$V \subseteq U$

4. $$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$$

consider the functional, $\varphi : V \longrightarrow \mathbb{R}$

$$f \rightsquigarrow f(0)$$

show that $\exists\, g$ such that $\forall f$,
$\varphi(f) = \langle f, g \rangle$

proof by contradiction. Assume g exists.

choose different functions f that give more ideas.

$$f \equiv 1$$

$$\Rightarrow \int_{-1}^{1} g(x)\, dx = \langle f, g \rangle = f(0) = 1$$

$\exists$ some interval U_0 where $g(U_0) > 0$

you need another function h

shrink U_0 to some $U_1 \subseteq U_0$ such that $0 \in U_1$

does this give you a hint of what you can define as h.
drawing helps !

5. a $\quad \delta : V \longrightarrow V^*$

$$v \rightsquigarrow \langle v, \cdot \rangle$$

δ is an isometry for the inner product $\langle , \rangle^* \iff$

$\forall v, w \in V, \quad \langle \delta(v), \delta(w) \rangle^* = \langle v, w \rangle$

$$\langle \lambda, \mu \rangle^* = \langle \delta^{-1}(\lambda), \delta^{-1}(\mu) \rangle$$

why is this valid and unique

b) Digression : From last class, remember example

$$\langle f, g \rangle = \int_0^\infty f(x)\, g(x)\, e^{-x}\, dx$$

$$= c_f^t \, A \, c_g$$

Let $\langle v_1, v_2, \ldots, v_n \rangle$ be a basis for V.

$\langle \delta(v_1), \delta(v_2), \ldots, \delta(v_n) \rangle$

– is this a basis for V^*

$a_{ij} = \langle v_i, v_j \rangle$ where $A = \{a_{ij}\}$

$a^*_{ij} = \langle \delta(v_i), \delta(v_j) \rangle^*$ where $A^* = \{a^*_{ij}\}$

Expand A^* in terms of A by expanding δ and using the definition of $\langle , \rangle^*$

Rough Notes

$A \in GL_n(\mathbb{R})$

$A = QR, \quad Q \in O(n), \quad R \in M_{n \times n}(\mathbb{R})$
upper triangular

$\langle v_1, v_2, \ldots, v_n \rangle$ is a basis for A

$$v_1 = pr_{w_1}(v_1) = \langle w_1, v_1 \rangle w_1$$

$$v_2 = pr_{w_1}(v_2) + pr_{w_2}(v_2)$$

$$= \langle w_1, v_2 \rangle w_1 + \langle w_1, v_2 \rangle w_2$$

$$v_n = \sum_j \langle w_j, v_n \rangle w_j$$

$$v_2 = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} \langle w_1, v_2 \rangle \\ \langle w_2, v_2 \rangle \end{bmatrix}$$

$$Q = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$$

$$R = \begin{bmatrix} \langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \langle u_1, v_3 \rangle & \cdots \\ & \langle u_2, v_2 \rangle & \langle u_2, v_3 \rangle & \cdots \\ & & \langle u_3, v_3 \rangle & \cdots \\ & 0 & & \ddots \end{bmatrix}$$

$$A = QR$$

$$Q \in O(n)$$
$$Q^{-1} \in O(n)$$

$$V^* = \mathrm{Hom}(V, K)$$

$$\phi_v(\cdot) = \langle \cdot, v \rangle$$

Let $\phi \in V^*$

Then $\phi = \phi_v$ for some $v \in V$

$$\phi_u(\cdot) + \phi_v(\cdot) = \phi_{u+v}(\cdot)$$

$$\alpha \phi_u(\cdot) = \phi_{\alpha u}(\cdot)$$

$\langle e_1, e_2, \ldots, e_n \rangle$ basis for V

$$v = \sum_j \langle v, e_j \rangle e_j$$

$$\Phi(v) = \Phi\left(\sum_j \langle v, e_j \rangle e_j\right)$$

$$= \sum_j \Phi(e_j) \langle v, e_j \rangle$$

$$= \left\langle v, \sum_j \Phi(e_j) e_j \right\rangle$$

$$= \Phi_u(v)$$

where $u = \sum_j \Phi(e_j) e_j$