2 $\langle A, B \rangle = \text{trace}\,(A^t B)$

Gram Schmidt

3. Let $B_S = \{v_1, v_2, \dots, v_m\}$ be an orthonormal basis for S

It can be extended to a basis $B_V = \{v_1, v_2, \dots, v_m, v_{m+1}, \dots, v_n\}$ for V

orthonormalise !

4. a orthonormalise $\{v_1, v_2\}$ for an orthonormal basis, $\{w_1, w_2\}$ of U

Solve $\langle w_1, v \rangle = \langle w_2, v \rangle = 0$

How does that help?

b $\pi : \mathbb{R}^3 \longrightarrow U$

$v \rightsquigarrow \langle v, w_1 \rangle w_1 + \langle v, w_2 \rangle w_2$

$\pi(e_1), \pi(e_2), \pi(e_3) = (?) w_1 + (?) w_2$

what would the matrix look like?

Digression :

$$L_A : \underset{\substack{\uparrow \text{basis} \\ \{u_1, u_2, \ldots, u_n\}}}{U} \longrightarrow \underset{\substack{\uparrow \text{basis} \\ \{v_1, v_2, \ldots, v_m\}}}{V}$$

$$L_A(u_i) = \sum_{j=1}^{m} a_{ji} \, v_j \qquad \forall \; 1 \leq i \leq n$$

$$\Rightarrow A = (a_{ij})$$

5 Prove :

given a basis, $B = (v_1, v_2, \dots, v_n)$
T is upper Triangular $\Leftrightarrow$
$\{v_1, v_2, \dots, v_j\}$ is a T - invariant subspace for all j

use this along with another property from Gram Schmidt.

6. a) $V = \{ (a_0, a_1, \dots) \mid \sum_{n=0}^{\infty} |a_n|^2 < \infty \}$

$$\langle (a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty} \rangle = \sum_{n=0}^{\infty} a_n b_n$$

$U_1 = \{ (a_n) \mid \exists N \geq 0$ such that $a_m = 0 \quad \forall m \geq N \}$

eg. $(0, 1, 3, 0, 0, 5, 0, 0, 0, \dots) \in U_1$

Let $v \in U_1^{\perp}$

$\{ u_1, u_2, \dots \} \subseteq U_1$

where $u_1 = (1, 0, 0, \dots)$,

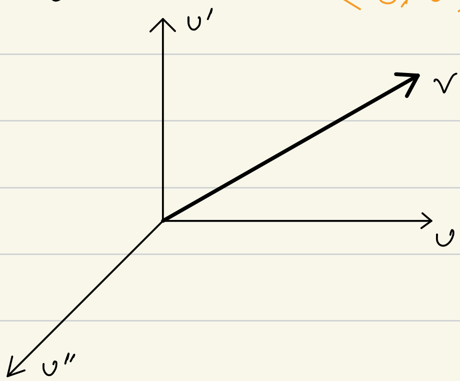
$u_2 = (0, 1, 0, 0, \dots)$,

$u_3 = (0, 0, 1, 0, 0, \dots)$ and so on...

Then, $\langle v, u_k \rangle = 0 \quad \forall k$

$\Rightarrow \quad v_k = 0 \quad \forall k$

$$\pi_u(v) = \frac{\langle u, v \rangle}{\langle u, u \rangle} u$$



$$v = au + bu' + cu''$$

$$\pi_u(v) = a$$

Gram Schmidt

$\{v_1, v_2, \ldots, v_n\}$ be a basis for V

$$w_1 = v_1$$

$$w_2 = v_2 - \pi_{w_2}(v_2)$$

$$w_3 = v_3 - \pi_{w_2}(v_3) - \pi_{w_1}(v_3)$$

$$\vdots$$

$\{ w_1, w_2, \ldots, w_n \}$ is an orthogonal basis for V

$\left\{ \frac{w_1}{\| w_1 \|}, \frac{w_2}{\| w_2 \|}, \ldots, \frac{w_n}{\| w_n \|} \right\}$ is an orthonormal basis for V.

Alternately,

$$w_1 = \frac{v_1}{\| v_1 \|}$$

$$w_2 = \frac{v_2 - \langle v_2, w_1 \rangle w_1}{\| v_2 - \langle v_2, w_1 \rangle w_1 \|}$$

$\vdots$

to get an orthonormal basis directly.

From Last Week

2. V : polynomials of degree $\leqslant n$

$$\langle p, q \rangle = \int_0^\infty p(t)\, q(t)\, e^{-t}\, dt$$

$$p = \sum_{i=0}^{n} p_i x^i$$

$$p_c = [\, p_0 \;\; p_1 \; \cdots \; p_n \,]$$ coordinate vector

$$\langle p, q \rangle = p_c^t \, A \, q_c$$

where $A = \{ a_{ij} \}$ and

$$a_{ij} = \langle x^{i-1}, x^{j-1} \rangle$$

More generally, for a vector space V with basis $\{ v_1, v_2, \ldots, v_n \}$,

$$a_{ij} = \langle v_i, v_j \rangle$$

Here, $v_1 = 1,\ v_2 = x,\ \ldots,\ v_{n+1} = x^n$

$$a_{ij} = \langle x^{i-1}, x^{j-1} \rangle$$

$$= \int_0^\infty t^{i+j-2} e^{-t} \, dt$$

$$= (i+j-2)!$$

3 $T : \mathrm{Hom}(V) \quad \forall v, \quad \|Tv\| \leqslant \|v\|$

show that $T - \sqrt{2} I$ is invertible

assume not invertible,

$\exists \ v \neq 0,$

$$(T - \sqrt{2} I) v = 0$$

$$\Rightarrow Tv = \sqrt{2} v$$

$$\Rightarrow \|Tv\| = \sqrt{2} \|v\|$$

which is a contradiction

$$\langle \ , \ \rangle : V \times V \longrightarrow \mathbb{R}$$

- $\langle v_1 + v_2 , w \rangle = \langle v_1 , w \rangle + \langle v_2 , w \rangle$
- $\langle v , w_1 + w_2 \rangle = \langle v , w_1 \rangle + \langle v , w_2 \rangle$
- $\langle \alpha v , w \rangle = \alpha \langle v , w \rangle$

  $\langle v , \alpha w \rangle = \alpha \langle v , w \rangle$
- $\langle v , w \rangle = \langle w , v \rangle$
- $\langle v , v \rangle > 0$

$$\langle \ , \ \rangle : V \times V \longrightarrow \mathbb{C}$$

- $\langle v_1 + v_2 , w \rangle = \langle v_1 , w \rangle + \langle v_2 , w \rangle$
- $\langle v , w_1 + w_2 \rangle = \langle v , w_1 \rangle + \langle v , w_2 \rangle$
- $\langle \alpha v , w \rangle = \alpha \langle v , w \rangle$

  $\langle v , \alpha w \rangle = \overline{\alpha} \langle v , w \rangle$
- $\langle v , w \rangle = \overline{\langle w , v \rangle}$
- $\langle v , v \rangle > 0$

$$\langle v, w_1 + w_2 \rangle$$

$$= \overline{\langle w_1 + w_2 , v \rangle}$$

$$= \overline{\langle w_1 , v \rangle} + \overline{\langle w_2 , v \rangle}$$

$$= \langle v , w_1 \rangle + \langle v , w_2 \rangle$$

linearity in the second variable follows from the first and conjugation property

$$\langle v, \alpha w \rangle = \overline{\alpha} \langle v , w \rangle$$

also follows