Exercises

(1) Let T be a linear operator on a finite-dimensional vector space for which every non-zero vector is an eigenvector. Prove that T is multiplication by a scalar.

(2) Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0_V\}$. Prove that for each vector α in V there are *unique* vectors α_1 in W_1 and α_2 in W_2 such that $\alpha = \alpha_1 + \alpha_2$

(3) *Generalisation of above*

Let $W_1, W_2, \ldots, W_k$ be subspaces of a vector space V such that $V = \sum W_i$. Assume that $W_1 \cap W_2 = (W_1 + W_2) \cap W_3 = \cdots = (W_1 + W_2 + \ldots + W_{k-1}) \cap W_k = \{0_V\}$. Prove that

$$V = \bigoplus W_i$$

i.e. V is the direct sum of the subspaces $W_1, W_2, \ldots, W_k$