$$A = \begin{bmatrix} 4 & 0 & -1 \\ -4 & 2 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\det(xI - A) = (x-2)^2(x-3)$$

$$x = 2 \text{ and } 3$$

$$\ker(2I - A) = \left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\rangle$$

$$\ker(3I - A) = \left\langle \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\rangle$$

diagonalisable!

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & & \\ & 2 & \\ & & 3 \end{bmatrix}$$

1 $A = \begin{bmatrix} 7 & 1 & & & \\ & 7 & 2 & & \\ & & 7 & 3 & \\ & & & 7 & 4 \\ & & & & 7 \end{bmatrix}$

$\lambda = 7$ only eigenvalue ; $g_7 = 1$

should be able to conclude what the JNF is at this point.

Let v be an eigenvector

$$(7I - A)v_1 = v$$
$$(7I - A)v_2 = v_1$$
$$(7I - A)v_3 = v_2$$
$$\vdots$$

$(v_4 , v_3 , v_2 , v_1 , v)$

2. $A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ & 3 & 0 & 2 \\ & & 3 & 2 \\ & & & 3 \end{bmatrix}$; $\chi_A(x) = (x-2)(x-3)^3$

JNF has $J_1(2)$

$E_3 = \langle v_1, v_2 \rangle$

JNF would be $\begin{bmatrix} 2 & & & \\ & 3 & & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}$

try to compute 4^{th} vector by

$(3I - A)v' = v_1 \longrightarrow$ fail

$(3I - A)v' = v_2 \longrightarrow$ fail

huh ...

$\ker(3I - A)^3 = ?$

can you find a vector there that works?

vector p such that

$(p, (3I - A)p)$ is our chain

and $(3I - A)^2 p = 0$

3. a Find V and T such that

$$V = \ker(T) \oplus \operatorname{Im}(T)$$

does not hold!

$$V = K[x]_{\leq 3}$$

polynomials of degree at most 3

$$T : V \longrightarrow V$$

$$f \rightsquigarrow f'$$

b $T^2 = T$

5. a $\quad A = I + \frac{1}{2}N - \frac{1}{8}N^2$

Show, $\quad A^2 = I + N$

b $\quad \lambda \neq 0$, square root of $\lambda I_3 + N$?

$\lambda I_3 + N$

$= \lambda (I_3 + \lambda^{-1} N)$

$A^2 = I_3 + \lambda^{-1} N$

what is A?

$\lambda^{\frac{1}{2}} A$?

c $B = P^{-1} J P$ where J is the JNF of B

does J have a square root ?
how does that help ?

6 a $(x-1)^2(x+2)^2$

b $(x-1)^3(x+2)$

a possible Jordan Normal Forms ?

$x=1$: $J_2(1)$, $J_1(1) \oplus J_1(1)$

$x=-2$: $J_2(-2)$, $J_1(-2) \oplus J_1(-2)$

4 possibilities

b exercise

7 $\quad \chi_A(x) = (x-3)^2 (x+5)^3$

$$m_A(x) = (x-3)(x+5)^2$$

Theorem 14.2.3

8 an endomorphism r is unipotent if

$$r = id_V + u$$

where u is nilpotent

a n is nilpotent

$$\exp(n) := \sum_{m \geq 0} \frac{n^m}{m!}$$

show well-defined, unipotent

$$n^k = 0 \quad \forall k \geq p$$

$$\exp(n) = \sum_{m=0}^{p-1} \frac{n^m}{m!}$$

$$\exp(n) - id_V = \sum_{m=1}^{p-1} \frac{n^m}{m!}$$

why is the above nilpotent ?

b $\quad u = \mathrm{id}_V + n$

where n is nilpotent

$$\log(\mathrm{id}_V + n) = \sum_{k=1}^{p-1} (-1)^{k-1} \frac{n^k}{k}$$

d $\quad u = \mathrm{id}_V + n$, where n is nilpotent

$\exists\, p$ st

$n^p = 0$ and $(\log(\mathrm{id}_V + n))^p = 0$

Define $p(x) = \sum_{m=0}^{p-1} \frac{1}{m!} \left(\sum_{k=1}^{p-1} (-1)^{k-1} \frac{x^k}{k} \right)^m$

compare with Taylor expansion of $\exp(\log x)$ to conclude that

$$p(x) = 1 + x + x^p q(x)$$

how to use this?