Quiz 23

1. $$AA^* = id$$

$$\Rightarrow \det(A)\det(\bar{A}^t) = 1$$

$$\Rightarrow \det(A)\overline{\det(A)} = 1$$

$$\Rightarrow |\det(A)|^2 = 1$$

$$\Rightarrow |\det(A)| = 1$$

2. $$\langle v_i, v_j \rangle = \langle Tv_i, Tv_j \rangle = \delta_{ij}$$

We wish to show that $\forall v, u \in V,$

$$\langle (T^*T - id)v, u \rangle = 0$$

Then,

$$\langle T^* T v, u \rangle - \langle v, u \rangle$$

$$= \langle Tv, Tu \rangle - \langle v, u \rangle$$

$$= \langle T \sum a_i v_i , T \sum b_j v_j \rangle$$

$$- \langle \sum a_i v_i , \sum b_j v_j \rangle$$

$$= \sum_{i,j} a_i \overline{b_j} \langle T v_i , T v_j \rangle$$

$$- \sum_{i,j} a_i \overline{b_j} \langle v_i , v_j \rangle$$

$$= \sum_i a_i \overline{b_i} - \sum_i a_i \overline{b_i}$$

$$= 0$$

Digression

$\mathbb{K} = \mathbb{R}$

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- $\{ v_1, v_2, \dots, v_n \}$ orthogonal basis

  when $\langle v_i, v_j \rangle = 0 \quad \forall\, i \neq j$

- $\{ v_1, v_2, \dots, v_n \}$ orthonormal basis

  when $\langle v_i, v_j \rangle = \delta_{ij} \quad \forall\, i \neq j$

- $A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \in O(n)$

orthogonal matrix when $AA^t = I$

$$\{A^t A\}_{i,j} = \langle v_i, v_j \rangle$$

$K = \mathbb{C}$

- $$A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \in U(n)$$

unitary matrix when $AA^* = I$

1. A

$\chi_A(x)$, eigenvalues, eigenvectors ?

$\{ v_1, v_2, v_3, v_4 \}$

Gram Schmidt

$\{ w_1, w_2, w_3, w_4 \}$

$$S = \begin{bmatrix} | & | & & | \\ w_1 & w_2 & \cdots & w_4 \\ | & | & & | \end{bmatrix}$$

$$A = S^{-1} G S = S^t G S$$

2 $S = S^*$ and $T = T^*$

show $(ST)^* = ST \iff ST = TS$

$$(ST)^* = ST$$

$$\Rightarrow T^* S^* = ST$$

$$\Rightarrow TS = ST$$

3 T is normal $W \subseteq V$

$P_W(v) :=$ projection of v onto W.

a T has an orthogonal basis of eigenvectors

$\lambda_1, \lambda_2, \ldots, \lambda_k$ eigenvalues of T

$v_1, v_2, \ldots, v_{l_1}$ are eigenvectors corresponding to λ_1

$v_{l_1+1}, v_{l_1+2}, \ldots, v_{l_2}$ correspond to λ_2

and so on ...

We just group all the eigenvectors by the eigenvalue they correspond to.

$W_i = \text{span}\{v_{l_{i-1}+1}, v_{l_{i-1}+2}, \ldots, v_{l_i}\} = E_{\lambda_i}$

$=$ eigenspace corresponding to λ_i

$$v = \sum_i a_i v_i \in V$$

$$\Rightarrow Tv = \sum a_i \, T v_i$$

how do you group this?

remember

$$T v_i = \lambda_1 v_i \quad \forall \; 1 \leq i \leq l_1$$

$$T v_i = \lambda_2 v_i \quad \forall \; l_1 + 1 \leq i \leq l_2$$

and so on...

b need to show $P_U = P_U^*$

this is equivalent to showing that

$$\langle v, P_U(w) \rangle = \langle P_U(v), w \rangle$$

Let basis of U be $\{u_1, u_2, \dots, u_K\}$

Then, $P_U(w) = \sum_i \langle w, u_i \rangle u_i$

now show above

4 T is normal

show T is self-adjoint

$\Leftrightarrow$ all eigenvalues are real

Let $\lambda \in \mathbb{C}$ be eigenvalue and v an eigenvector corresponding to it

$$\langle Tv, v \rangle = \langle v, Tv \rangle$$

$$\Rightarrow \langle \lambda v, v \rangle = \langle v, \lambda v \rangle$$

$$\Rightarrow \lambda \langle v, v \rangle = \bar{\lambda} \langle v, v \rangle$$

conversely, all eigenvalues are real

$B = \{ e_1, e_2, \ldots, e_n \}$ be an orthonormal basis of eigenvectors with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

we wish to show $T = T^*$

it is equivalent to showing

$$\langle (T - T^*)v, v \rangle = 0 \qquad \forall v$$

$$\langle (T - T^*)v, v \rangle$$

$$= \langle Tv, v \rangle - \langle T^*v, v \rangle$$

$$= \langle Tv, v \rangle - \overline{\langle Tv, v \rangle}$$

show that $\langle Tv, v \rangle \in \mathbb{R} \quad \forall v$

how does that help?

5 U has a basis of eigenvectors
$\Leftrightarrow \exists$ an IP that makes U
self - adjoint

reverse direction follows from Real Spectral theorem

Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of eigenvectors that have been normalised

Define,

$$\left\langle \sum_i a_i e_i, \sum_i b_i e_i \right\rangle := \sum a_i b_i$$

now just check if

$$\langle v, Tw \rangle = \langle Tv, w \rangle \quad \forall v, w$$

6 $A, B \in F \Rightarrow AB = BA$ and $AA^* = A^*A$

show $\exists$ an orthonormal basis $\{v_1, v_2, \dots, v_n\}$ such v_j is an eigenvector for all $A \in F$

a given $AU \subseteq U$

show that A has an eigenvector in U

let $\{e_1, e_2, \dots, e_d\}$ be a basis for U

$$A e_j = \sum a_{i,j} e_i$$

why is this special, you might think at first glance.

Ae_j is representable using a basis for U and you do not need a basis for V

Let $w = \sum_{j=1}^{d} x_j e_j \in U$

$Aw = ?$

is there some condition that you see being satisfied that will help you claim that w could be an eigenvector? that solves the problem.

b $\quad AU \subseteq U \quad \forall A \in G \subset \mathrm{Hom}(V)$

where G is a family of commuting operators

show $\exists$ a non-zero vector $v \in V$ that is an eigenvector for every $A \in G$

Really understand eigenvectors and representations of transformations which have eigenvectors in them

we show the below statement and that is enough

$\exists$ a linear subspace W of U which has dimension ≥ 1 such that for all $A \in G$, $A|_W = cI$

i.e. A restricted to W is a multiple of the identity matrix

why is showing this enough?

We prove by induction on dimension of U

$\dim(U) = 1$: $W = U$

let the statement hold for all dimensions up to d. We check for $\dim(U) = d$

if A restricted to U is identity for all $A \in G$, we can take $W = U$ and be done. If not, pick an A such that A restricted to U is not identity

By (a), A has an eigenvector $v \in U$ and corresponding eigenvalue λ

Let $U' = U \cap \{ v \in V, \ Av = \lambda v \}$
$\Rightarrow \dim(U') < \dim(U) = d$

All you need to do apply the induction hypothesis on U' is to check that $BU' \subseteq U'$ $\forall B \in G$

By the nature of our choice of U', do you see how the induction hypothesis proves that $A|_U$ is a multiple of identity?

MCQ

1. $(iA)^* = -iA \neq iA$

3. $A^i (A^*)^j = (A^*)^j A^i$

$$\overbrace{A\,A\cdots A}^{i}\,\overbrace{A^* A^* \cdots A^*}^{j}$$

$$= \overbrace{A\,A\cdots A}^{i-1}\,\overbrace{A^* A^* \cdots A^*}^{j}\,A$$

⋮