$$A = \begin{bmatrix} 2 & 0 & 2 \\ 4 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\det(x I - A)$$

$$= \begin{vmatrix} x-2 & 0 & -2 \\ -4 & x-1 & -2 \\ 0 & 0 & x \end{vmatrix}$$

$$= x(x-1)(x-2)$$

diagonalizable because splits into linear factors

$$\begin{bmatrix} 2 & 0 & 2 \\ 4 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{aligned} 2x + 2z &= x \\ 4x + y + 2z &= y \\ z &= 0 \end{aligned} \Rightarrow \begin{aligned} x = z &= 0 \\ y &= y \end{aligned}$$

$$E_1 = \left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle$$

1. $v = u + w$

$\Rightarrow T_A(v) = T_A(u) + T_A(w)$

$= T_B(u) + T_C(w)$ (how?)

how does T_B and T_C look like in K^n?

use above to show both directions

2 a) Recall that $p_A(x) = \det(xI - A)$

$p_A(0) = ?$

b) use (a) and transform the equation

c) compute $p_A(x)$

check condition (a)

use the same idea as in (b)

3

$$\dim(\operatorname{Im}(L_A)) = r$$
$$\Rightarrow \dim(\ker(L_A)) = n - r$$

Let $\{v_1, v_2, \ldots, v_{n-r}\}$ be a basis of $\ker L_A$.

Extend to basis
$B = \{v_1, v_2, \ldots, v_{n-r}, u_{n-r+1}, \ldots, u_n\}$ for K^n

What does $[L_A]^B_B$ look like?

$$\begin{bmatrix} \underbrace{\begin{matrix} 0 & 0 & & 0 \\ 0 & 0 & & 0 \\ 0 & \vdots & & \vdots \\ \vdots & \vdots & & \end{matrix}}_{n-r} & \underbrace{C}_{r} \end{bmatrix}$$

$L_A v_i = 0$

$\forall i \leq n-r$

$$P_A(x) = x^{n-r}\left(c_0 + c_1 x + \ldots + c_r x^r\right)$$

propose g_A where $\partial g_A = r+1$
and $g_A(A) = 0$

then $m_A \mid g_A$

do show that $g_A(A) = 0$

do this by showing $g_A(A)v = 0$
$\forall$ basis elements v.

4. Start with the subspace

$$W = \langle I_n, A, A^2, \dots, A^{n-1} \rangle$$

W has dimension $\leqslant n$

Show that $\forall k \geqslant 0$, $A^k \in W$

characteristic polynomial of A ?

how would it generally look like?

Cayley Hamilton

5 a) $E_\lambda = \langle v \rangle$

$Bv = \lambda v$

$ABv = \lambda Av$

$\Rightarrow B(Av) = \lambda(Av)$

b) $P_\sigma A = A P_\sigma$

use (a)

characteristic polynomial of P_σ ?

what are the geometric multiplicity of the roots ?

6. three cases :

i) two distinct real roots

ii) 1 real root with algebraic multiplicity 2

iii) no real roots

Let A be a 2×2 real matrix with complex eigenvalue, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and v is a corresponding eigenvector

Then $A = CBC^{-1}$ for

$$C = \begin{bmatrix} | & | \\ \text{Real}(v) & \text{Im}(v) \\ | & | \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \text{Real}(\lambda) & \text{Im}(\lambda) \\ -\text{Im}(\lambda) & \text{Real}(\lambda) \end{bmatrix}$$

where $\text{Real}\left(\begin{bmatrix} x+iy \\ z+iw \end{bmatrix}\right) = \begin{bmatrix} x \\ z \end{bmatrix}$ and

$$\text{Im}\left(\begin{bmatrix} x+iy \\ z+iw \end{bmatrix}\right) = \begin{bmatrix} y \\ w \end{bmatrix}$$

proove!

Minimal Polynomial

- minimal polynomial of T is the monic polynomial of least degree such that $m_T(T) = 0_V$
- unique
- $m_T \mid g \quad \forall g \in K[x]$ such that $g(T) = 0_V$
- (Cayley Hamilton) $\chi_T(T) = 0$
- implies $m_T \mid \chi_T$

Jordan Normal Form

- The building blocks, $J_n(\lambda) =$

$$\begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$$

  - only eigenvalue is λ
  - $g_\lambda = 1$ and $a_\lambda = n$
  - $E_\lambda = \langle e_1 \rangle$
  - $m_{J_n(\lambda)}(x) = (x - \lambda)^n$

- (Jordan Normal Form)

$$\begin{bmatrix} J_{n_1}(\alpha_1) & & & \\ & J_{n_2}(\alpha_2) & & \\ & & \ddots & \\ & & & J_{n_k}(\alpha_k) \end{bmatrix}$$

$$
B = \begin{pmatrix}
\lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & \lambda_1 & 1 & 0 & 0 & 0 & 0 \\
 & & & & \lambda_1 & 0 & 0 & 0 & 0 \\
 & & & & & \lambda_2 & 1 & 0 & 0 \\
 & & & & & & \lambda_2 & 0 & 0 \\
 & & & & & & & \lambda_2 & 0 \\
 & & & & & & & & \lambda_2
\end{pmatrix}
$$

e_1 e_4 e_6 e_8 e_9

$E_{\lambda_1} = \langle e_1, e_4 \rangle$

ist zusammengesetzt aus Jordanbloecken $J_3(\lambda_1)$, $J_2(\lambda$

$$B = J_3(\lambda_1) \oplus J_2(\lambda_1) \oplus J_2(\lambda_2) \oplus J_1(\lambda_2) \oplus J_1(\lambda_2)$$

$$(x-\lambda_1)^3 \, (x-\lambda_1)^2 \, (x-\lambda_2)^2 \, (x-\lambda_2)^2$$

$$\chi_T = (x-\lambda_1)^5 (x-\lambda_2)^4$$

$$m_T = (x-\lambda_1)^3 (x-\lambda_2)^2$$