Exercises

(1) Compute, by induction, the determinant of the matrix,

$$\begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & & & \\ & & & & \ddots & & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix}_{n\times n}$$

To make it more explicit, $a_{i,i} = 2 \ \forall i$, $a_{i,i+1} = -1 \ \forall i \in \{1, 2, \ldots, n-1\}$ and $a_{i,i-1} = -1 \ \forall i \in \{2, \ldots, n\}$

(2) Let A be a 3×3 matrix over the field of complex numbers. We form the matrix $xI_3 - A$ with polynomial entries, the i, j entry of this matrix being the polynomial $\delta_{ij}x - a_{ij}$. If $f = \det(xI_3 - A)$, show that f is a monic polynomial of degree 3. If we write

$$f = (x - c_1)(x - c_2)(x - c_3)$$

with c_1, c_2, c_3 being complex numbers, prove that

$$c_1 + c_2 + c_3 = \operatorname{trace}(A) \text{ and } c_1c_2c_3 = \det(A)$$

Hint: You can show this by explicit calculation. It is a bit of a nasty calculation but it is a useful result (that also works more generally on $n \times n$ matrices) that you'll find very useful later on during our discussion on eigenvalues.