1.a $\quad \mathrm{Im}(T^*) = \ker(T)^\perp$

$T : V \longrightarrow W \qquad T^* : W \longrightarrow V$

- $\langle Tv, w \rangle_W = \langle v, T^* w \rangle_V$

$v \in \mathrm{Im}(T^*)$

$\Leftrightarrow \exists\, w \in W$ s.t. $T^* w = v$

$\Leftrightarrow \forall\, u \in \ker(T), \quad \langle u, v \rangle$

$= \langle u, T^* w \rangle$

$= \cdots$

2 use 1

3 $T \in \mathrm{End}(V)$ $U \subseteq V$

$T(U) \subseteq U$ iff $T^*(U^\perp) \subseteq U^\perp$

$\forall u \in U, \ \forall w \in U^\perp$

$\langle Tu, w \rangle = 0$

$\Leftrightarrow \langle u, T^* w \rangle = 0$

4. a exercise

b $\mathcal{B} = \{ e_1, e_1 + e_2 \}$

c $\mathcal{B} = \{ e_1, e_2, e_3 \}$

$T(e_i) = \lambda_i e_i$

pick λ_i appropriately

d similar to b

e eigenvalues ?

5. a orthonormal basis $\{w_1, w_2\}$

$$Tw_1 = \sqrt{2}\, w_1$$

$$Tw_2 = w_2$$

check for $\{e_2, e_1 + e_2\}$

b what is the matrix of T under the orthonormal basis ?

6 $\quad u, x \in V \qquad Tv = \langle v, u \rangle x$

$\forall w_1, w_2 \in V,$

$$\langle Tw_1, w_2 \rangle = \langle w_1, T^* w_2 \rangle$$

$$= \langle \langle w_1, u \rangle x, w_2 \rangle$$

$$= \langle w_1, u \rangle \langle x, w_2 \rangle$$

$$= \langle w_1, \overline{\langle x, w_2 \rangle} u \rangle$$

$$= \langle w_1, \langle w_2, x \rangle u \rangle$$

$$\therefore \ T^* w_2 = \langle w_2, x \rangle u$$

$\forall v, \quad Tv = T^* v$

$\Rightarrow \langle v, u \rangle x = \langle v, x \rangle u$

b $\quad \forall v, \quad TT^* v = T^* T v$

$\cdots$

7 $T(a_0 + a_1 x + a_2 x^2) = a_1 x$

a contradiction, assume $T = T^*$

$$p = a_0 + a_1 x + a_2 x^2$$
$$q = b_0 + b_1 x + b_2 x^2$$

$$\langle Tp, q \rangle = \langle p, T^* q \rangle$$

$$\Rightarrow \langle a_1 x, q \rangle = \langle p, Tq \rangle$$

$$= \langle p, b_1 x \rangle$$

b what can you say about the basis

$\{1, x, x^2\}$

8 $G = (V, E)$

$V =$ set of vertices.
$\{1, 2, \dots, 5\}$ in given example

$\vec{E} \subseteq V \times V$

ordering matters!

in the example, $(1, 2) \in \vec{E}$ but
$(2, 1) \notin \vec{E}$

$\mathbb{R}^V = \{ f : V \to \mathbb{R} \}$

set of functions which assign some value to the vertices

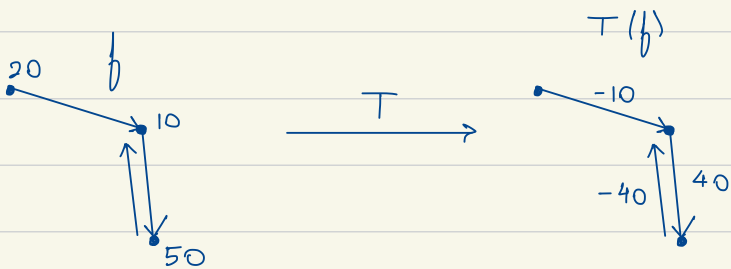
$$\mathbb{R}^E = \{ \phi : \vec{E} \longrightarrow \mathbb{R} \}$$

set of functions which assign some value to the edges

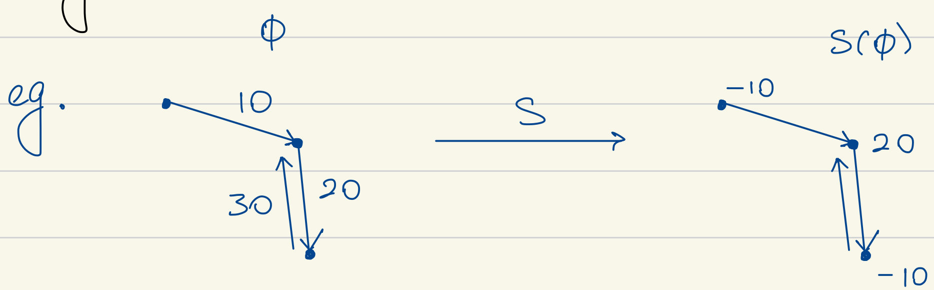
Inner products are defined on these vector space as described

T takes functions from $\mathbb{R}^V$ to functions in $\mathbb{R}^E$ by assigning to each edge the difference of the values of the function at the two vertices

eg.



S takes functions from $\mathbb{R}^E$ to functions in $\mathbb{R}^V$ by assigning to each vertex the net "flux" through the vertex



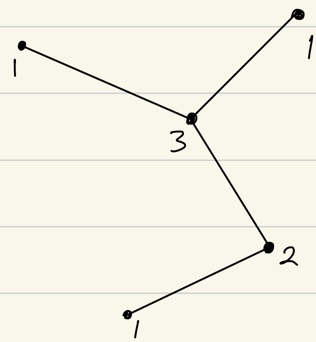
if you think of the edges as the amount of water being given from one country to another, S gives you the function describing the amount of water in each country.

a to show $T^* = S$, you have to show

$$\langle f, S\phi \rangle_r = \langle Tf, \phi \rangle_E$$

A question on Graph Theory ?

$n(v) = \#$ neighbours of V



$$\therefore \sum_{v \in V} n(v) = 8$$

how is related to the number of edges ? why ?