$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\chi_A(\lambda) = (\lambda - 1)^3$$

$$I - A = \begin{bmatrix} 0 & -2 & 0 \\ -1 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$(I - A)^2 = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 0 & 0 \\ -1 & 0 & -2 \end{bmatrix}$$

$$(I - A)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$v \in \tilde{E}_\lambda$ such that $(\lambda I - A)^n v = 0$

$\{ v, (\lambda I - A) v, \ldots, (\lambda I - A)^{n-1} v \}$
is a Jordan chain of length n

\# of Jordan blocks $J_n(\lambda) = g_\lambda$

Find v_2 such that

$$(I - A) v_2 = v = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ works

Similarly, find v_3 such that

$$(I - A) v_3 = v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$ works

Then, $\{ v_3, v_2, v \}$ is a Jordan chain of length 3 (why?)

1. skipping as the exercise is clear

2. $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$\mathbb{R}$ $\quad \chi_A(x) = (x-1)^2 (x-4)$

$$(x-1)^2 \downarrow \quad J_2(1)\ ? \qquad (x-4) \downarrow \quad J_1(4)$$

$\mathbb{F}_3$ $\quad \chi_A(x) = (x-1)^3$

$g_1 = ?$

3. A : $\chi_A(x) = x^4$

$$g_0 = 3$$

D : $\chi_D{}^2 = (x-2)^2(x-1)^2$

$$g_2 = 1$$

$$g_1 = 1$$

$$\tilde{E}_2 = \ker(2I-A)^2$$

$$\tilde{E}_1 = \ker(I-A)^2$$

JNF :
$$\begin{bmatrix} 2 & 1 & & \\ & 2 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$$

4. a) $D_1 : K[x]_n \to K[x]_n$

$p(x) \to p'(x)$

$B = \{1, x, x^2, \ldots, x^n\}$

basis for $K[x]_n$

$$[D_1]_B^B = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 2 & & \vdots \\ & \vdots & 0 & & 0 \\ \vdots & \vdots & \vdots & & n \\ 0 & 0 & 0 & & 0 \end{bmatrix}_{(n+1)\times(n+1)}$$

$\ker(\ \ -[D_1]_B^B) = ?$

$g_0 = 1$ $\qquad J_{n+1}(0)$

b) $K[x]_n \longrightarrow K[x]_n$

$$p(x) \longrightarrow p''(x)$$

$$[D_{B_2}] \begin{bmatrix} 0 & 0 & 2\times1 & 0 & 0 \\ \vdots & \vdots & \vdots & 3\times2 & 0 \\ \vdots & \vdots & \vdots & \vdots & n(n-1) \\ \vdots & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\lambda = 0$$

$$g_0 = 2$$

$$D_{B_2} = D_{B_1}^2$$

$$J_{\underline{\lfloor \frac{n}{2} \rfloor}}(0), \quad J_{\underline{\lceil \frac{n}{2} \rceil}}(0)$$

5. $S^k = 0$

$n_1, n_2, \ldots, n_k \geq 1$

$\Sigma n_k = n$

$S^{n_1-1} v_1, S^{n_1-2} v_1, \ldots, v_1$

$S^{n_2-1} v_2, S^{n_2-2} v_2, \ldots, v_2$

...

$S^{n_i} v_i = 0$

14.3.2

$n \;,\quad v = \tilde{E}_n$

$$\{ v, Sv, S^2v, \ldots, S^{n-1}v \}$$

$$a_0 v + a_1 Sv + \ldots + a_{n-1} S^{n-1} v = 0$$

S^{n-1}

$$a_0 S^{n-1} v + a_1 \cancel{S^n} v + \ldots a_{n-1} \cancel{S^{2n-2}} v = 0$$

$\Rightarrow \quad a_0 S^{n-1} v = 0$

$\Rightarrow \quad a_0 = 0 \qquad\qquad S^{n-1} v \neq 0$

$a_1 Sv + a_2 S^2 v + \ldots$

S^{n-2}

$a_1 S^{n-1} v + a_2 S^n v +$

$S^{n_1-1} v_1, S^{n_1-2} v_1, \dots, v_1$

$S^{n_2-1} v_2, S^{n_2-2} v_2, \dots, v_2$

$\dots$

$$a_0^1 v + \underset{\substack{\uparrow \\ 0}}{a_1^1 S v} + \dots + \underset{\uparrow}{a_{n_1-1} S^{n_1-1} v_1}$$

$+ \dots$

S^{n_1-1}

$$a_0^1 S^{n_1-1} v_1 + a_0^2 S^{n_2-1} v_2$$

$n_1 \geq n_2 \geq \dots \geq n_k$ WLOG

how does this argument help you?

6.

$$A^4 = 0$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$W \qquad T : W \to W$

$$\ker(T^k) \subsetneq \ker(T^{k+1})$$

and $$\operatorname{Im}(T^k) \supsetneq \operatorname{Im}(T^{k+1})$$

$$W : \quad K^\infty \times K^\infty$$

$$T(u, v) = (\uparrow, \uparrow)$$

$u, v \in K^\infty$

left shift   right shift

$$(x_1, x_2, \ldots) \longrightarrow (x_2, x_3, \ldots$$
$$(x_1, x_2, \ldots) \longrightarrow (0, x_1, x_2, \ldots)$$

$T^k_{(1)}$

$$(x_1, x_2, \ldots) \longrightarrow (x_{k+1}, x_{k+2}, \ldots)$$

$\ker(T^k)_{(1)}$:

$$x_1, x_2, \ldots, x_k, 0, 0, \ldots$$

$\ker(T^{k+1})_{(1)}$:

$$x_1, x_2, \ldots, x_{k+1}, 0, 0, \ldots$$

$$\ker(S^1) \subsetneq \ker(S^2) \subsetneq$$

$$\ker(S^n) = \ker(S^{n+1}) = \ker(S^{n+2})$$

$$= \dots$$

$$S = \lambda I - A$$

- $$\ker(S^k) \subseteq \ker(S^{k+1})$$
- $$U \subset V$$

$$\dim U < \dim V$$

- $$\ker(S^k) = \ker(S^{k+1})$$