Homework 4

Graded

Student

Boyuan Deng

Total Points

5 / 5 pts

Question 1

1 **1** / 1 pt

✔ **– 0 pts** Correct

Question 2

2 **1** / 1 pt

✔ **– 0 pts** Correct

Question 3

3 **1** / 1 pt

✔ **– 0 pts** Correct

Question 4

4 **1** / 1 pt

✔ **– 0 pts** Correct

Question 5

5 **1** / 1 pt

✔ **– 0 pts** Correct

No questions assigned to the following page.

PDE
Homework 4
Due Date: March 01

Saman H. Esfahani

- 1. Find the general solution of the following equation

$$x^2y^2 = \boxed{x^3y\frac{dy}{dx} + y^2(\frac{dy}{dx})^2 + xy(\frac{dy}{dx})} + 9x^2.$$

$$\frac{dy}{dx} = \left[\frac{\ }{dx}(x^3y\frac{dy}{dx}) + \frac{\ }{dx}(y\frac{dy}{dx})^2 + \frac{\ }{dx}9x^2 - \frac{\ }{dx}x^2y^2\right]$$

$$= \left[\left(x^3y\frac{d^2y}{dx^2} + x^3(\frac{dy}{dx})^2 + 3x^2y\frac{dy}{dx}\right) + \left(2y(\frac{dy}{dx})^3 + 2y^2\frac{dy}{dx}\frac{d^2y}{dx^2}\right) + 18x\right.$$

$$- \left(2xy^2 + 2x^2y\frac{dy}{dx}\right)$$

- 2. Find the general solution of the following equation

y order

$$\cos^2(y)(\frac{dy}{dx})^2 + \sin(x)\cos(x)\cos(y)(\frac{dy}{dx}) - \sin(y)\cos^2(x) = 0.$$

- 3- Solve the following equation using the method of characteristics:

$$x_1u_{x_1} + x_2u_{x_2} = 2u, \quad u(x_1, 1) = g(x_1),$$

for a given function g. <u>F - linear</u>

- 4- Solve the following equation using the method of characteristics:

$$uu_{x_1} + u_{x_2} = 1, \quad u(x_1, x_1) = \frac{1}{2}x_1.$$

5- Let $H : \mathbb{R}^n \to \mathbb{R}$ be a convex function. We say q belongs to the subdifferential of H at p, and write $q \in \partial H(p)$, if

$$H(r) \geq H(p) + q \cdot (r - p),$$

for all $r \in \mathbb{R}^n$. Prove $q \in \partial H(p)$ if and only if

$$p \cdot q = H(p) + L(q),$$

where $L = H^*$ is the Legendre transform of H.

1. Simple form $y(x) = x\cdot\frac{dy}{dx} + f(\frac{dy}{dx})$

$y = x\,g(p) + f(p)$

$x^3 y\frac{dy}{dx} + y^2(\frac{dy}{dx})^2 + xy(\frac{dy}{dx}) + 9x^2 - x^2y^2 = 0$

p

$y^2(\frac{dy}{dx})^2 + xy(x^2+1)(\frac{dy}{dx}) + 9x^2 - x^2y^2 = 0$

$y^2(p)^2 + xy(x^2+1)p + 9x^2 - x^2y^2 = 0$

$x^2 = u \Rightarrow x = \sqrt{u}$

$2x\,dx = du$

$dx = \frac{du}{2x}$

$dx = \frac{du}{2\sqrt{u}}$ ...①

$y^2 = v \Rightarrow y = \sqrt{v}$

$2y\,dy = dv$

$dy = \frac{dv}{2y}$

$dy = \frac{dv}{2\sqrt{v}}$ ...②

$\frac{②}{①} = \frac{dy}{dx} = \frac{dv/2\sqrt{v}}{du/2\sqrt{u}} = p = \frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}$ P $\Rightarrow$ $\boxed{p = \frac{\sqrt{u}}{\sqrt{v}}P}$

$\cancel{v}\frac{u}{\cancel{v}}P^2 + \sqrt{u}\cancel{\sqrt{v}}(u+1)\frac{\sqrt{u}}{\cancel{\sqrt{v}}}P + 9u - uv = 0$

$\cancel{u}P^2 + \cancel{u}(u+1)P + 9\cancel{u} - \cancel{u}v = 0$

$P^2 + (u+1)P + 9 - v = 0$

$v = uP + P^2 + P + 9$

So that the general solution

$y^2 = x^2c + c^2 + c + 9$ for $c \in \mathbb{R}^n$

2. $\cos^2(y)(\frac{dy}{dx})^2 + \sin(x)\cos(x)\cos(y)(\frac{dy}{dx}) - \sin(y)\cos^2(x) = 0$ p

$(\cos(y)\frac{dy}{dx})^2 + \sin(x)\cos(x)(\cos(y)(\frac{dy}{dx})) - \sin(y)\cos^2(x) = 0$

$(\boxed{\sin(y)})' \xrightarrow{' = \frac{\partial}{\partial x}} \cos(y)\frac{dy}{dx}$

$: z$

$\Rightarrow (\frac{dz}{dx})^2 + \sin(x)\cos(x)(\frac{dz}{dx}) - z\cos^2(x) = 0$

p p

$(p)^2 + \sin(x)\cdot\cos(x)\,p = z\cos^2(x)$

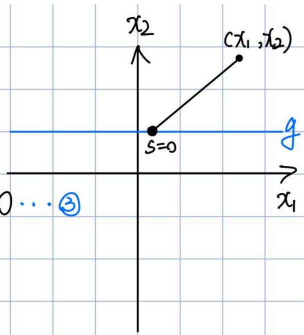
So that the general solution is

$z = c\cdot\tan(x) + \frac{1}{\cos^2(x)}c^2$

$\sin(y)$

Questions assigned to the following page: 4 and 3

3. $$\begin{cases} x_1 u_{x_1} + x_2 u_{x_2} = 2u \cdots ① \\ u(x_1,1) = g(x_1) \cdots ② \end{cases}$$



$F(p,z,x) = x_1 p_1 + x_2 p_2 - 2z = 0 \cdots ③$

$$\begin{cases} \nabla_p F = (x_1, x_2) \\ \nabla_z F = -2 \\ \nabla_x F = (p_1, p_2) \end{cases}$$

$\Downarrow$

$$\begin{cases} p'(s) = -\nabla_x F - \nabla_z F p \\ z'(s) = p \cdot \nabla_p F \\ x'(s) = \nabla_p F \end{cases}$$

$\Downarrow$

$$\begin{cases} (p_1', p_2') = (-p_1, -p_2) + 2(p_1, p_2) \\ z'(s) = x_1 p_1 + x_2 p_2 = 2z \Rightarrow z(s) = z_0 e^{2s} \to \text{when } s=0 \\ x'(s) = (x_1, x_2) \quad \begin{cases} \dot{x}^1 = x_1 \\ \dot{x}^2 = x_2 \end{cases} \Rightarrow \begin{cases} x_1(s) = x_0^1 e^s \\ x_2(s) = x_0^2 e^s \end{cases} \\ \dot{x}(s) \quad \Downarrow \end{cases}$$

Applying bc:

We have $u(x_1(s), 1) = g(x_1(s))$

① $x_2(s) = 1 \Rightarrow x_0^2 e^s = 1$, at the boundary, s is set to be 0, $x_0^2 = 1$

Also, $x_1(0) = x_0^1 \overset{1}{e^0} = x_0^1 \cdots ④$

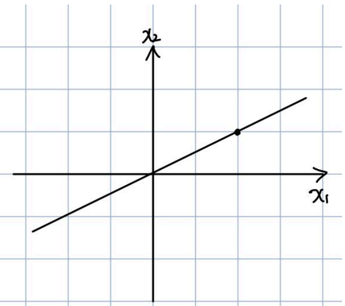
$u(x_1(0), 1) = \overset{z_0}{u_0} = g(x_0^1) = g(x_1)$ (from ④)

$x_2(s) = \overset{=1}{x_0^2} e^s \Rightarrow s = \ln(x_2)$

$u = u_0 e^{2s} = g(x_1) e^{2\ln(x_2)}$

$= x_2^2 \, g(x_1)$

4. $$\begin{cases} u u_{x_1} + u_{x_2} = 1 \cdots ① \\ u(x_1, x_1) = \frac{1}{2} x_1 \cdots ② \end{cases}$$



$F(p,z,x) =$

$z p_1 + p_2 - 1 = 0 \cdots ③$

$$\begin{cases} \Delta_p F = (z, 1) \\ \Delta_z F = p_1 \\ \Delta_x F = 0 \end{cases}$$

Then the characteristic functions

$$\begin{cases} p'(s) = -\nabla_x F - \nabla_z F p \\ z'(s) = p \cdot \nabla_p F \\ x'(s) = \nabla_p F \end{cases}$$

$\Downarrow$

$$\begin{cases} (p_1', p_2') = -p_1 \\ z'(s) = z p_1 + p_2 \overset{③}{=} 1 \Rightarrow z(s) = s + C_1 \\ x'(s) = (z, 1) \Longrightarrow \begin{cases} \dot{x}_1 = z \Rightarrow x_1(s) = \frac{1}{2}s^2 + C_1 s + C_3 \\ \dot{x}_2 = 1 \Rightarrow x_2(s) = s + C_2 \end{cases} \end{cases}$$

Applying BC we have $u(x_1, x_1) = \frac{1}{2} x_1$

$x_2(s) = x_1(s) \xrightarrow{s=0} C_2 = C_3 := a$

$u(x_1(0), x_1(0)) = C_1 = \underbrace{\frac{1}{2} x_1(0)}_{\frac{1}{2} C_3} \Rightarrow C_1 = \frac{1}{2} a$

$$\begin{cases} z(s) = \frac{a + 2s}{2} \cdots ④ \\ x_1(s) = \frac{s^2 + as + 2a}{2} \cdots ⑤ \\ x_2(s) = s + a \longrightarrow s = x_2 - a \cdots ⑥ \end{cases}$$

Given the fact that a is derived from the boundary

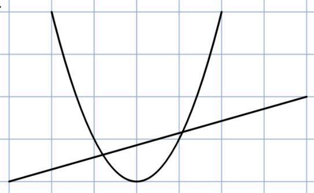
$\frac{1}{2} a = \frac{1}{2} x_1 \Rightarrow a = x_1$

Then $s = x_2 - x_1$

$u(x_1, x_2) = \frac{x_1 + 2x_2 - 2x_1}{2} = \frac{2x_2 - x_1}{2}$

Question assigned to the following page: 5

5.



H is a convex

$\underline{q \in \partial H(p)}$ if & only if $p \cdot q = H(p) + L(q)$ $\quad H = L^*$

$H(r) \geq H(p) + q(r-p)$

$r \in \mathbb{R}^n$

for any $r \in \mathbb{R}^n$ we have

$$H(r) \geq H(p) + rq - pq \quad \cdots ①$$

$$rq - H(r) \leq q \cdot p - H(p)$$

Thus

$$L(q) = \sup_{r \in \mathbb{R}^n} \{rq - H(r)\} \leq q \cdot p - H(p)$$

Since $p \cdot q = H(p) + L(q) \quad \cdots ②$

$$\Downarrow$$

$$\begin{aligned} L(q) &= p \cdot q - H(p) \\ &= \sup_{s \in \mathbb{R}^n} \{s q - H(s)\} \\ &\geq rq - H(r) \end{aligned}$$

so that $\Rightarrow H(r) \geq H(p) + rq - pq = ①$