Homework 7

● Graded

Student

Boyuan Deng

Total Points

8 / 10 pts

Question 1

(no title) **1** / 1 pt

✔ **– 0 pts** Correct

Question 2

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Question 3

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Question 4

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✔ **– 0 pts** Correct

Question 5

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Question 6

(no title) **1** / 1 pt

✔ **– 0 pts** Correct

– 0.5 pts not complete

Question 7

(no title) **0.5** / 1 pt

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Question 8

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Question 9

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Question 10

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No questions assigned to the following page.

PDE
Homework 7
Due Date: April 12

Saman H. Esfahani

- 1. Show that when $b_i = c = 0$, the non-divergence form of a linear second-order PDE can be written as

$$\mathrm{tr}(A(x)D^2u(x)) = f(x),$$

for some matrix-valued function $A(x)$, where 'tr' stands for the trace of a matrix, and the divergence form can be written as

$$\mathrm{div}(A(x)\nabla u(x)) = f(x),$$

for some matrix-valued function $A(x)$.

- 2. We learned two definitions for uniform ellipticity. Show that they are equivalent. Moreover, show that the smallest eigenvalue of $A(x) = (a_{i,j}(x))$ is greater than or equal to θ, the constant which appears in the first definition of the uniform ellipticity.

- 3- Let $1 \le p < \infty$ and $u \in W^{1,p}(U)$ for bounded open set $U \subset \mathbb{R}^n$. We define the i$^{\text{th}}$-difference quotient of u with size h by

direction

$$D_i^h u(x) := \frac{u(x + he_i) - u(x)}{h}.$$

This resembles the directional derivative; however, h is fixed here rather than $h \to 0$. Moreover, let

$$D^h u(x) = (D_1^h u(x), \ldots, D_n^h u(x)).$$

(i) Prove for each $V \subset\subset U$, there is a constant C, such that

$$\|D^h u\|_{L^p(V)} \le C\|Du\|_{L^p(V)},$$

for all h where $0 < |h| < \frac{1}{2}\mathrm{distance}(V, \partial U)$. The notation $V \subset\subset U$ means V is an open set where $\bar{V} \subset U$.

(ii) Suppose $1 < p < \infty$ and $u \in L^p(V)$. Show that there is a constant $C > 0$ such that if

$$\|D^h u\|_{L^p(V)} \le C,$$

for all h where $0 < |h| < \frac{1}{2}\mathrm{distance}(V, \partial U)$, then, $u \in W^{1,p}(V)$ with $\|Du\|_{L^p(V)} \le C$.

Definition 0.1. *Let X and Y be Banach spaces. A linear map*

$$T : X \to Y,$$

is called a ***bounded linear map*** *if the operator norm of T, as defined below, is finite;*

$$\|T\|_{op} := \sup\{\|T(x)\|_Y \mid \|x\|_X \le 1\} < \infty.$$

A special case is when $Y = \mathbb{R}$. A bounded linear map $T : X \to \mathbb{R}$ is called a bounded linear functional.

We define the ***dual space*** *of a Banach space X, denoted by X^*, by*

$$X^* = \{T : X \to \mathbb{R} \mid T \text{ is a bounded linear functional}\}$$

Let H be a ***Hilbert space*** *with an inner product $(\cdot,\cdot)_H$.*

Theorem 1 (Riesz representation theorem)**.** *Let H be a Hilbert space. For any $T \in H^*$, there exists a unique element $u \in H$ such that*

$$T(v) = (u, v)_H,$$

for all $v \in H$.

H can't have many A

Riesz representation theorem shows that the dual space H^* can be identified with H, since for any element $T \in H^*$, there is a unique element $u \in H$. The mapping

$$H^* \to H, \quad T \to u,$$

H can't have many H* Every u has some T

is injective and surjective, and therefore, we have the identification $H^* \cong H$.

No questions assigned to the following page.

4- Prove the Riesz representation theorem when H is a finite-dimensional space, $H = \mathbb{R}^n$.

Consider the case where

$$X = H_0^1(U) = W_0^{1,2}(U).$$

We denote the dual space of $H_0^1(U)$ by $H^{-1}(U)$; hence,

$$H^{-1}(U) := (H_0^1(U))^*$$
$$= \{f : H_0^1(U) \to \mathbb{R} \mid f \text{ is bounded linear } \},$$

equipped with the operator norm.

- 4- Let $U \subset \mathbb{R}^n$. Use the Riesz representation theorem to show that for any $f \in H^{-1}(U)$, there are functions $f_0, \dots, f_n \in L^2(U)$ such that

$$f(v) = \int_U f_0 v \, dx_1 \dots dx_n + \sum_{i=1}^n \int_U f_i \partial_{x_i} v \, dx_1 \dots dx_n, \qquad (1)$$

for all $v \in H_0^1(U)$.

Moreover, show that the operator norm equals

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$$\|f\|_{H^{-1}(U)} \underset{\leq}{=} \inf\{(\int_U \sum_{i=0}^n |f_i|^2 dx_1 \dots dx_n)^{\frac{1}{2}} \mid \text{for any set of } L^2 \text{ functions } f_0, \dots, f_n \text{ which satisfy (1)}\}.$$

- 5- Let L be a linear second-order elliptic differential operator defined on $W_0^{1,2}(U) = H_0^1(U)$. What is the appropriate target space Y for $L : H_0^1(U) \to Y$? Justify your answer.

- 6- Prove the Lax-Milgrim theorem using the Riesz representation theorem.

- 7- Prove the second existence theorem for weak solutions of linear second-order elliptic equations.

- 8- Prove the third existence theorem for weak solutions of linear second-order elliptic equations.

- 9- The regularity theorem for elliptic operators can be understood as the generalization of the regularity/ smoothness of solution to Laplace and Poisson equations, which we learned in Chapter 3 of lecture notes (Chapter 2 of the book by Evans).

  Here, we prove the regularity theorem we saw in the class for linear second-order elliptic operators in a special case by reducing it to the Laplace equation.

  Let's consider the case where $f = 0$ and $b_i = c = 0$, the coefficient functions $a_{i,j}(x) = a_{i,j}$ are constant, and $U = B_1(0)$. Suppose we have the uniformly elliptic equation

$$\sum_{i,j=1}^n a_{i,j} u_{x_j, x_i} = 0, \quad \text{in} \quad B_1(0). \qquad (2)$$

  Therefore, the matrix $A = (a_{i,j})$ is positive definite. Show there exists a ~~unique~~ positive definite square root matrix B such that

$$B^2 = A.$$

  We denote $B = A^{\frac{1}{2}}$. Introduce change of variable

$$z = A^{\frac{1}{2}} x.$$

  Show the equation (2) reduces to a Laplace equation and conclude the smoothness of the function u.

- 10- Let $U \subset \mathbb{R}^n$ be a bounded open set, and let $u \in C^0(\bar{U}) \cap C^2(U)$ be a function such that it satisfies the following uniformly elliptic PDE,

$$Lu = \sum_{i,j=1}^n a_{i,j}(x) u_{x_i, x_j}(x) = f, \quad \text{in} \quad U,$$

  and $u = 0$ on ∂U. Show that there exists a constant C depending only on U and $a_{i,j}$ (and not u or f) such that

$$\|u\|_{L^\infty(U)} \leq C \|f\|_{L^\infty(U)}.$$

  To prove this, you can use the following maximum principle.

No questions assigned to the following page.

Theorem 2 (Maximum Principle for elliptic operators)**.** *Consider a bounded open set $U \subset \mathbb{R}^n$. Let $u \in C^0(\bar{U}) \cap C^2(U)$. Suppose u satisfies the following,*

$$Lu = \sum_{i,j=1}^{n} a_{i,j}(x) u_{x_i,x_j}(x) = 0, \quad \textit{in} \quad U,$$

where the matrix $A(x)$ satisfies the uniform ellipticity condition. Then,

$$\sup_U u = \sup_{\partial U} u.$$

Questions assigned to the following page: 2 and 1

1. $b_i = c = 0$, the non-divergence linear 2nd PDE

$\Rightarrow \operatorname{tr}(A(x) D^2 u(x)) = f(x)$

matrix-valued

divergence: $\operatorname{div}(A(x) \nabla u(x)) = f(x)$

$a^{ij}(x)$

$Lu = f$

Non-dirergence form:

$$Lu = -\sum_{i,j}^{n} a^{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} \cancel{b^i}(x) u_{x_i} + \cancel{c}(x) u$$

Divergence form

$$Lu = -\sum_{i,j}^{n} (a^{ij}(x) u_{x_i})_{x_j}$$

Define $A(x) : (a^{ij}(x))$

Divergence form $\Rightarrow$ $\operatorname{div}(A(x) \nabla u(x)) = f(x)$

Non-divergence $\Rightarrow$ Assume that $a^{ij} \underset{\text{delta}}{\overset{\text{kronecker}}{=}} \delta_{ij}$

where $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & j \neq j \end{cases}$

Then $A(x)$ is a diagonal matrix

$$Lu = -\sum_{i,j}^{n} a^{ij}(x) u_{x_i x_j}$$

$$\overset{i=j}{=} -\sum_{i}^{n} a^{ii} u_{x_i x_i}$$

$A(x) = (a^{ij}(x))$

$\Rightarrow \operatorname{tr}(A(x) D^2 u(x)) = f(x)$

2. Assume that $a_{ij} \in C^1(U)$

We have the non-divergence form

$$Lu = -\sum_{i,j}^{n} a^{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} b^i(x) u_{x_i} + c(x) u$$

$$= -\sum_{i,j}^{n} (a^{ij} u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} + c u$$

$$= -\sum_{i,j}^{n} (a^{ij} u_{x_i})_{x_j} + \sum_{i,j}^{n} a^{ij}_{x_j} u_{x_i} + \sum_{i=1}^{n} b^i u_{x_i} + c u$$

$$= -\sum_{i,j}^{n} (a^{ij} u_{x_i})_{x_j} + \underbrace{\sum_{i=1}^{n} (b^i + \sum_{j=1}^{n} a^{ij}_{x_j})}_{\hat{b}^i} u_{x_i} + c u$$

$$= -\sum_{i,j}^{n} (a^{ij} u_{x_i})_{x_j} + \hat{b}^i u_{x_i} + c u$$

Divergence form

We say L is uniformly elliptic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2, \ x \in U \text{ and all } \xi \in \mathbb{R}^n$$

By defination $\Downarrow$

$\xi = (\xi_1, \xi_2 \cdots \xi_n)$

Assume that $a^{ij} \underset{\text{delta}}{\overset{\text{kronecker}}{=}} \delta_{ij}$

diagonal matrix

$A(x) : (a^{ij}(x))_{1 \leq i,j \leq n}$, A is a positive, definite matrix

$$\Rightarrow \sum_{i=1}^{n} a^{ii}(x) \xi_i \xi_i \geq \theta |\xi|^2$$

$$A(x) \cancel{|\xi|^2} \geq \xi^T A \xi \geq \theta \cancel{|\xi|^2}$$

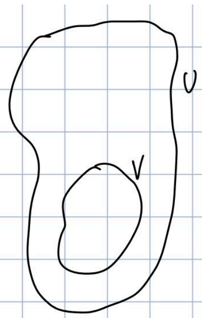
$A(x) \geq \theta$

This implies that every eigenvalue $\lambda_{ii} \geq \theta$

$\Rightarrow$ Smallest eigenvalue is greater than or equal to θ

Question assigned to the following page: 3

3. (i) Assume $1 \le p < \infty$ and u is smooth



$0 < |h| < \frac{1}{2}$ distance $(V, \partial U)$

Let's fix x, assume that

$v(t) = u(x + he_i t)$

$v(1) = u(x + he_i)$, $v(0) = u(x)$

$v(1) - v(0) = \int_0^1 v'(t)\,dt$

Then we have

$u(x+he_i) - u(x) = \int_0^1 h\, u_{x_i}(x + the_i)\,dt$

$u(x+he_i) - u(x) = h\int_0^1 u_{x_i}(x + the_i)\,dt$

$\underbrace{|u(x+he_i) - u(x)|}_{|h|} \le \cancel{|h|} \left|\int_0^1 |u_{x_i}(x+the_i)|dt\right|$

$\Longrightarrow$

$$\int_V |D^h u|^p\,dx \le C\sum_{i=1}^n \int_V \int_0^1 |Du(x+the_i)|^p\,dt\,dx$$

$$= C\sum_{i=1}^n \int_0^1 \int_V |Du(x+the_i)|^p\,dx\,dt$$

$$\int_V |D^h u|^p\,dx \le C\int_V |Du|^p\,dx$$

since $u \in W^{1,p}(U)$

$$\|D^h u\|_{L^p(V)} \le C\,\|Du\|_{L^p(V)}$$

(ii) Since $0 < |h| < \frac{1}{2}\mathrm{dist}(V, \partial U)$

Choose $i = 1, \cdots, n$ $\phi \in C_c^\infty(V)$

$$\int_V u(x)\left[\frac{\phi(x+he_i) - \phi(x)}{h}\right]dx = -\int_V \left[\frac{u(x) - u(x-he_i)}{h}\right]\phi(x)\,dx$$

$$\int_V u(D_i^h \phi)\,dx = -\int_V (D_i^{-h} u)\phi\,dx$$

Then $\|D^h u\|_{L^p(V)} \le C$

$$\sup_h \|D_i^{-h} u\| < \infty$$

Since $1 < p < \infty$, there exists a function $v_i \in L^p(V)$ $h_k \to 0$ such that

$D_i^{-h_k} u \to v_i$ weakly in $L^p(U)$

$$\int_V u\,\phi_{x_i}\,dx = \int_U u\,\phi_{x_i}\,dx = \lim_{h_k \to 0}\int_U u\,D_i^{h_k}\phi$$

$$= -\lim_{h_k \to 0}\int_V D_i^{-h_k} u\,\phi\,dx$$

$$= -\int_V v_i \phi\,dx$$

$v_i = u_{x_i}$ in the weak sense

so $Du \in L^p(V)$

as $u \in L^p(V) \Rightarrow u \in W^{1,p}(V)$

Question assigned to the following page: 4

4、 $T \in H^*$, there exists a unique element $u \in H$ such that $T(v) = (u, v)_H$ for all $v \in H$

$X = H_0^1(U) = W_0^{1,2}(U)$

dual space $H^{-1}(U) : (H_0^1(U))^* = \{f : X \to \mathbb{R} \mid f \text{ is bounded linear}\}$

f is a bounded linear functional on a Hilbert space $H^{-1}(U)$

H is a finite space & $H \in \mathbb{R}^n$

$\Rightarrow$ Define $\phi_i, i \geq 1$, as orthonormal basis for H

and $a_i = f(\phi_i)$ coefficients associated with f

Choose $v \in H$, let $c_i = (v, \phi_i)$ decomposition of v

$v_n = \sum_{i=1}^{n} c_i \phi_i$

Since ϕ_i is the basis, $\|v - v_n\| \to 0$ as $n \to +\infty$

$f(v_n) = f(\sum_{i=1}^{n} c_i \phi_i)$

Since f is bounded $\|f\| < \infty$, and since it's linear

$|f(v) - f(v_n)| \leq \|f\| \; \|v - v_n\|$

$f(v) = \lim_{n \to \infty} f(v_n) = \sum_{i=1}^{\infty} a_i c_i$

$|\sum_{i=1}^{\infty} a_i c_i| \leq \|f\| \left(\sum_{i=1}^{\infty} c_i^2\right)^{1/2}$

Fix a positive integer N & define a sequence $c_j = a_j$ for $j \leq N$, $c_j = 0$ for $j > N$

$\Downarrow$

$|\sum_{j=1}^{N} a_j^2| \leq \|f\| \left(\sum_{j=1}^{N} a_j^2\right)^{1/2}$

$\left(\sum_{j=1}^{N} a_j^2\right)^{1/2} \leq \|f\|$

Then $u = \sum_j a_j \phi_j \in X$ is well-defined

$f(v) = \sum_j a_j c_j = \langle v, u \rangle$

$\|u\| \leq \|f\|$

By Cauchy-Schwarz

$|f(v)| = |\langle u, v \rangle| \leq \|u\| \; \|v\| \Rightarrow \frac{|f|}{\|v\|} \leq \|u\|$

$\Rightarrow \|f\| \leq \|u\|$ so that $\|f\| = \|u\|$

4-

For u and $v \in H_0^1$

$u = \left(\Sigma_{|\alpha| \leq k} \int_U |D^\alpha u| \, dx\right)_{L^2}$

$= \left(\int_U u \, dx + \sum_{i=1}^{n} \int_U \partial x_i u \, dx\right)_{L^2}$

$v = \left(\Sigma_{|\alpha| \leq k} \int_U |D^\alpha v| \, dx\right)_{L^2}$

$= \left(\int_U v \, dx + \sum_{i=1}^{n} \int_U \partial x_i v \, dx\right)_{L^2}$

$\langle u, v \rangle = \left(\int_U uv \, dx + \sum_{i=1}^{n} \int_U \partial x_i v \, \partial x_i u \, dx\right)_{L^2} \cdots ①$

Since $f \in H^{-1}(U)$, by the Riesz Representation Thm

$f(v) = \langle u, v \rangle$

$= \left(\int_U uv \, dx + \sum_{i=1}^{n} \int_U \partial x_i v \, \partial x_i u \, dx\right)_{L^2}$

Since $f_0, f_1 \cdots f_n \in L^2$

Let's define $f_0 := u$, $f_{i, i \geq 1} := \partial x_i u$

Then we have $f(v) = \int_U f_0 v \, dx_1 \cdots dx_n$

$+ \sum_{i=1}^{n} \int_U f_i \partial x_i v \, dx_1 \cdots dx_n \cdots ②$

The dual norm of f could be expressed as

$\|f(v)\| = \sup_{v \in H_0^1(U) \,\&\, \|v\|_{H_0^1} \neq 0} \left\{\frac{|f(v)|}{\|v\|}\right\}$

We have proved that ②

$\frac{|f(v)|}{\|v\|} = \int_U |f_0 v \, dx| + \sum_{i=1}^{n} \int_U |f_i \partial x_i v \, dx|$

$\leq \left(\int_U |f_0|^2 dx\right)^{1/2} \left(\int_U |v|^2 dx\right)^{1/2} + \sum_{i=1}^{n} \left(\int_U |f_i|^2 dx\right)^{1/2} \left(\int_U |\partial x_i v|^2 dx\right)^{1/2}$

Devided by $\|v\|$ both side

$\|f(v)\| \leq \int_U \sum_{i=0}^{n} |f_i|^2 dx$ for $f_i \in L^2(U)$

So we have

$\|f(v)\| = \inf \left\{ \left(\int_U \sum_{i=0}^{n} |f_i|^2 dx_1 \cdots dx_n\right)^{\frac{1}{2}} \;\middle|\; \text{for any set of } L^2(U) \text{ \& satisfies } ② \right\}$

Questions assigned to the following page: 6 and 5

5. $Lu=f$

$L \in W_0^{1,2}(U) = H_0^1(U)$ to be a linear 2nd order elliptic differential operator

$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)\, u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x)\, u_{x_i} + c(x)u$

highest order term

However $u \in H_0^1(U)$, the 2nd derivative would not be guaranteed in the $L^2(U)$ space

Then we multiply a test function $v \in C_0^\infty(U)$ so that only one derivative of u appears in the divergence form (By Lax-Milgram thm we know the existence of weak solution)

So that f should always be $L^2(U)$, that's why we assume $f \in L^2(U)$ when proving Existence thms

$Lu=f$

6.

$B[u,v] = \int_U \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + c u v \, dx$

$B: H \times H \to \mathbb{R}$

For each fixed element $u \in H$, the mapping $v \to B[u,v]$ is a bounded linear functional on H

Riesz Representation Thm asserts the existence of a unique element $w \in H$ satisfying

$B[u,v] = (w, v) \quad (v \in H)$ ⋯ ①

Let's write $Au = w$ that ① still holds

$\Rightarrow$

$B[u,v] = (Au, v)$ ⋯ ②

since $B: H \times H \to \mathbb{R}$

We then claim that $A: H \to H$ is a bounded linear operator.

$\lambda_1, \lambda_2 \in \mathbb{R}$ & $u_1, u_2 \in H$, we see for each $v \in H$ that

$(A(\lambda_1 u_1 + \lambda_2 u_2), v) = B[\lambda_1 u_1 + \lambda_2 u_2, v]$ by ②

$= \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v]$

$= \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v)$ by ②

$= (\lambda_1 Au_1 + \lambda_2 Au_2, v)$

Since A is linear, $v \in H$ $B[u,v] = (Au, v)$

$\|Au\|^2 = (Au, Au) = B[u, Au] \le \|u\| \|Au\|$

$\Rightarrow \|Au\| \le \alpha \|u\|$ for all $u \in H$, and so A is bounded

Next assert that

$\begin{cases} A \text{ is one-to-one and} \\ R(A), \text{ the range of } A, \text{ is closed} \end{cases}$ ⋯ ③

To prove this, let us compute

$\beta \|u\|^2 \le B[u,u] = (Au, u) \le \|Au\| \|u\|$

$\Rightarrow \beta \|u\| \le \|Au\|$, then ③ holds

We demonstrate $R(A) = H$, and since $R(A)$ is closed there would exist a non-zero element $w \in H$ with $w \in R(A)^\perp$. But this fact in turn implies the contradiction $\beta \|w\|^2 \le B[w,w] = (Aw, w) = 0$

More from the Riesz Representation Thm that

$\langle f, v \rangle = \langle w, v \rangle$ for all $v \in H$

for some elemet $w \in H$. Combine (3) & (4) $u \in H$ satisfying $Au = w$. Then

$B[u,v] = (Au, v) = (w, v) = \langle f, v \rangle \quad (v \in H)$

For $B[u,v] = \langle f, v \rangle$ and $B[\hat{u}, v] = \langle f, v \rangle$

then $B[u - \hat{u}, v] = 0 \quad (v \in H)$ set $v = u - \hat{u}$

we have $\beta \|u - \hat{u}\|^2 \le B[u - \hat{u}, u - \hat{u}] = 0$

Questions assigned to the following page: 7 and 8

7 $B_r[u,v] := B[u,v] + r(u,v)$

$\Rightarrow \underbrace{L_r[u,v] := Lu + ru}$, then for each $g \in L^2(U)$ there exists a unique function $u \in H_0^1(U)$

$$B_r[u,v] = (g,v) \text{ for } v \in H_0^1(U) \cdots ①$$

$$u = L_r^{-1} g \cdots ②$$

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

Observe that $u \in H_0^1(U)$ is a weak solution of above iff

$$B_r[u,v] = (ru+f, v) \text{ for all } v \in H_0^1(U)$$

$$\Downarrow$$

$$u = L_r^{-1}(ru+f)$$

We rewrite this equality to read

$$u - \underset{:rL_r^{-1}u}{Ku} = \underset{L_r^{-1}f}{h}$$

$K: L^2(U) \to L^2(U)$ is a bounded, linear, compact operator

$$\beta \|u\|^2_{H_0^1(U)} \underbrace{\le B_r[u,u] = (g,u)}_{\text{from } ①} \le \|g\|_{L^2(U)} \|u\|_{L^2(U)}$$

$$\le \|g\|_{L^2(U)} \cancel{\|u\|_{H_0^1(U)}}$$

$$\beta \|u\|_{H_0^1(U)} \le \|g\|_{L^2(U)}$$

$u = L_r^{-1} g \qquad Ku = rL_r^{-1}u$

$$\|Kg\|_{H_0^1(U)} \le C \|g\|_{L^2(U)} \quad (g \in L^2(U))$$

By Rellich-Kondrachov compactness thm, K is a compact operator

(α) $\begin{cases} \text{for each } h \in L^2(U) \text{ the equation} \\ u - Ku = h \\ \text{has a unique solution } u \in L^2(U) \end{cases}$

or else

(β) $\begin{cases} \text{for the homogeneous equation} \\ u - Ku = 0 \\ \text{has non-zero solution in } L^2(U) \end{cases}$

Recall that the dimension of space N of the solution of β is finite & equals the dimension of the space N^* of solutions of $\underline{v - K^* v = 0}$

Only holds u & v is a weak solution

(α) has a solution iff $(h, v) = 0$

$$(h,v) = \frac{1}{r}(Kf, v) = \frac{1}{r}(f, K^* v) = \frac{1}{r}(f, v)$$

So the boundary-value problem only has a solution iff $(f,v) = 0$ for all weak solutions v.

8. Let r be the constant

$\lambda > -r$

Assume also with no loss of generality that $r > 0$

According to the Fredholm alternative, the boundary value problem has a unique weak solution for each $f \in L^2(U)$ iff $u \equiv 0$ is the only weak solution of the homogeneous problem

$$\begin{cases} Lu = \lambda u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \cdots ①$$

$$\begin{cases} \underline{Lu + \gamma u = (r + \gamma) u} & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \cdots ②$$

$$u = L^{-1}(\gamma + \lambda) u = \frac{\gamma + \lambda}{\gamma} Ku \cdots ③$$

Now if $u \equiv 0$ is the only solution

$\dfrac{\gamma}{\gamma + \lambda}$ is not an eigenvalue of $K \cdots ④$

Consequently, ① has a unique weak solution for each $f \in L^2(U)$ iff ④ holds

9.

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} = 0 \quad \text{in} \quad B_1(0)$$

$A = (a_{ij})$ is positive and definite

$\Rightarrow$ all eigenvalues are positive

$A = PDP^{-1}$

Diagonalize

Since $B^2 = A \Rightarrow B = A^{1/2}$

$B = P D^{1/2} P^{-1}$

$$B^2 = PD^{1/2}\underbrace{P^{-1} P}_{1} D^{1/2} P^{-1} = PDP^{-1} = A$$

$A = (a_{ij})$ positive & definite $\Rightarrow$ $D^{1/2}$ positive & definite

So matrix B is unique positive definite square root matrix

Since $z = A^{\frac{1}{2}} x$

$x = A^{-\frac{1}{2}} z$

$$\frac{\partial}{\partial x_i} = \sum_k (A^{-\frac{1}{2}})_{ik} \frac{\partial}{\partial z_k}$$

$$\frac{\partial^2}{\partial x_i \partial x_j} = \sum_{k,l} (A^{-\frac{1}{2}})_{ik} (A^{-\frac{1}{2}})_{jl} \frac{\partial^2}{\partial z_k \partial z_l}$$

Then $\sum_{i,j} a_{ij} u_{x_j, x_i} = \sum_{i,j,k,l} \underbrace{a_{ij}}_{A} (A^{-\frac{1}{2}})_{ik} (A^{-\frac{1}{2}})_{jl} \frac{\partial^2 u}{\partial z_k \partial z_l}$

$$= \sum_{k,l} \delta_{kl} \frac{\partial u}{\partial z_k \partial z_l}$$

$$= \Delta_z u = 0$$

harmonic function

So we know that $u \in C^\infty(U)$

10.

$U \subset \mathbb{R}^n$ and $u \in C^0(\bar{U}) \cap C^2(U)$

$$\begin{cases} Lu = \sum_{i,j}^{n} a_{ij}(x) u_{x_i x_j}(x) = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

Uniformly elliptic implies that there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2$$

Since $u \in C^0(\bar{U}) \cap C^2(U)$

$\Rightarrow$ u will attain its max & min on ∂U

So both min & max of $u = 0$

Assuming that f in $L^2(U)$

$$\int_U u\, Lu = \int_U uf$$

$$\Rightarrow \int_U \sum_{ij} a_{ij} u_{x_i} u_{x_j}\, dx = \int_U fu\, dx$$

$$\int_U \underbrace{\sum_{ij} a_{ij}}_{A} u_{x_i} u_{x_j}\, dx \geq \lambda \int_U |\nabla u|^2\, dx$$

$$\begin{pmatrix} \ddots & & \\ & \lambda & \\ & & \ddots \end{pmatrix}$$

$$\Rightarrow \lambda \int_U |\nabla u|^2 dx \leq \int_U |fu|\, dx \leq \|f\|_{L^2(U)} \|u\|_{L^2(U)}$$

From Poincaré's inequality

$$(\|u\|_{L^2(U)})^2 \leq \underbrace{(C\|Du\|_{L^2(U)})^2}_{\lambda \int_U |\nabla u|^2 dx} \quad C^2 = \lambda$$

$$\|u\|^2_{L^2(U)} \leq \|f\|_{L^2(U)} \|u\|_{L^2(U)}$$

So that we have $\|u\|_{L^2(U)} \leq C \|f\|_{L^2(U)}$

$$\Rightarrow \|u\|_{L^\infty(U)} \leq C \|f\|_{L^\infty(U)}$$

C is derived from eigenvalue of A $\{a_{ij}\}$ and we have assumed that $a^{ij} \in L^\infty(U)$