Homework 8

Graded

Student

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Total Points

2 / 3 pts

Question 1

1 — **1** / 1 pt

✔ **– 0 pts** Correct

Question 2

2 — **1** / 1 pt

✔ **– 0 pts** Correct

Question 3

3 — **0** / 1 pt

– 0 pts Correct

✔ **– 1 pt** Incorrect

No questions assigned to the following page.

PDE
Homework 8
Due Date: April 26

Saman H. Esfahani

2nd parabolic PDE

- 1. prove there is at most one smooth solution to the heat equation with the following initial and boundary conditions.

$$u_t - \Delta u = f, \quad \text{in } U \times [0, T],$$
$$\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial U \times [0, T],$$
$$u = g, \quad \text{on } U \times \{0\},$$

for given functions f, g and where ν is the outward orthonormal vector field to ∂U.

- 2. prove there is at most one smooth solution to the beam equation with the following initial and boundary conditions.

$L^2 u$

$$u_{tt} + u_{xxxx} = f, \quad \text{in } (0, 1) \times [0, T]$$
$$u = u_x = 0,$$
$$\text{on } (\{0\} \times [0, T]) \cup (\{1\} \times [0, T]),$$
$$u = g, \; u_t = h, \quad \text{on } [0, 1] \times \{0\},$$

for given functions g, h.

- 3. In Galerkin's method, the functions w_i, which are defined as the solutions to the Dirichlet problem $-\Delta w_i = \lambda_i w_i$ can be understood as both an orthogonal basis for $L^2(U)$ and also $H_0^1(U)$. By investigating the rigorous definition of the basis for infinite-dimensional spaces, explain how that's possible.

Question assigned to the following page: 1

1. Define $[u(t)](x)$ / $[f(t)](x)$, $v \in H_0^1(U)$

$u \in L^2(0,T; H_0^1(U))$, with $u_t \in L^2(0,T; H^{-1}(U))$

$$\int_{U\times\{t\}} u_t v\,dx + B[u,v;t] = \int_{U\times\{t\}} fv \quad \text{for all test functions}$$

$$\langle u_t, v\rangle + B[u,v;t] = (f,v)$$

Since $H_0^1(U)$ is a vector space $\Rightarrow$ countable orthonormal basis $w_1, w_2 \cdots$

smooth function

Which solves $-\Delta w_k = \lambda_k w_k$ on U

$E_N :=$ the space generated by $w_1, \cdots, w_N = \langle w_1, \cdots w_N\rangle$

$\Rightarrow E_1 \subset E_2 \subset \cdots \subset E_N \subset \cdots \subset H_0^1(U)$

Define the projection map $P_N : H_0^1(U) \to E_N$

$$P_N\Big(\sum_i c_i w_i\Big) = \sum_{i=1}^{N} c_i w_i$$

let u_N denote the solution to the restriction of the parabolic PDE to E_N

$$(u_N)_t + P_N(\overset{\Delta}{\underline{L}}(u_N)) = P_N(f)$$

Assume that f & g are in $H_0^1(U)$ $\Rightarrow$

$f = \sum_i f_i(t) w_i$, $g = \sum_i g_i w_i$, $u_N = \sum_i (c_i^N(t)) w_i$

$$B[u,v;t] = \sum_{i,j} \int_{U\times\{t\}} a_{ij}(t,x)\, u_{x_i} v_{x_j}\,dx + \sum_i \int_{U\times\{t\}} b_i(t,x)\, u_{x_i} v\,dx + \int_{U\times\{t\}} c(t,x)\, uv\,dx$$

Substituting into the PDE $\Rightarrow$ A system of ODE

$$c_i^N(t)' + \sum_{l=1}^{N} e_{i,l}(t)\, c_l^N(t) = f_i(t)$$

The initial condition then become

$$c_i^N(0) = g_i$$

$\Rightarrow$ we could easily find $u_N = \sum_{i=1}^{N} c_i^N(t) w_i$

$\Rightarrow$ We then try to prove its uniqueness

Assume u_1 & u_2 to be two weak solutions

$u := u_1 - u_2$

Goal: Show $u = 0$ everywhere on $U \times [0,T]$

Question assigned to the following page: 1

$(u_t, v)_{L^2(U\times\{t\})} + B[u, v; t] = 0$

Let $v = u$ on each time slice

$\Rightarrow \underbrace{(u_t, u)_{L^2(U\times\{t\})}}_{\frac{1}{2}\partial_t \|u\|^2_{L^2}} + B[u, u; t] = 0$

Since for every t since L is uniformly elliptic

$$B[u, u; t] \geq \beta \|u\|^2_{H_0^1(U)} - \gamma \|u\|^2_{L^2(U)} \geq -\gamma \|u\|^2_{L^2(U)}$$

Let $h(t) = \|u\|^2_{L^2(U\times\{t\})}$

$$h'(t) \leq 2\gamma h(t)$$

The Grönwall's inequality says

$$h(t) \leq h(0) \int_0^t (-2\gamma) = 0$$

Since $h(0) = 0 \Leftarrow u = 0$ when $t = 0$

Suppose f & g are smooth bounded functions. Suppose the coefficients defining L are also smooth & bounded. Then, the unique weak solution u is smooth.

Question assigned to the following page: 2

2.

$$u_{tt} + u_{xxxx} = f \quad \text{in } (0,1) \times [0,T]$$

$$u = u_x = 0 \quad \text{on } (\{0\} \times [0,T]) \cup \{1\} \times [0,T])$$

$$u = \overset{H_0^1(U)}{g}, \; u_t = \overset{L^2(U)}{\underline{h}} \quad \text{on } [0,1] \times \{0\}$$

According to the energy estimates, we see that the sequence $\{u_m\}_{m=1}^{\infty}$ is bounded in $L^2(0,T; H_0^1(U))$, $\{u_m'\}_{m=1}^{\infty}$ is bounded in $L^2(0,T; L^2(U))$ and $\{u_m''\}_{m=1}^{\infty}$ is bounded in $L^2(0,T; H^{-1}(U))$, $u'' \in L^2(0,T; H^{-1}(U))$

Next fix an integer N and choose a function $v \in C^1([0,T]; H_0^1(U))$ of the form $v(t) = \sum_{k=1}^{N} C_k^N(t) w_k$

$$\langle u'', v \rangle + B[u, v, t] = (f, v)$$

$\Downarrow$ Integrating by parts twice with t

$$\int_0^T (v'', u) + B[u, v, t] = \int_0^T (f, v)\, dt - (u(0), v'(0)) + \langle u'(0), v(0) \rangle$$

$\Downarrow$

$$\int_0^T (v'', u^N) + B[u_N, v, t] = \int_0^T (f, v)\, dt - (\underbrace{u_m(0)}_{g}, v'(0)) + (\underbrace{u_m'(0)}_{h}, v(0))$$

Since $v(0)$, $v'(0)$ are arbitrary. Hence u is a weak solution

Assume there are 2 weak solutions $u_1, u_2 \in H_0^1(U)$ and $u = u_1 - u_2 = 0$ with $f \equiv g \equiv h \equiv 0$

To verify this, fix $0 \le s \le T$ and set

$$v(t) := \begin{cases} \int_t^s u(\tau)\, d\tau & \text{if } 0 \le t \le s \\ 0 & \text{if } s \le t \le T \end{cases}$$

$v(t) \in H_0^1(U)$ for each $0 \le t \le T$

$$\int_0^s \underline{\langle u'', v \rangle} + \underline{B[u, v; t]}\, dt = 0$$

Since $u'(0) = v(s) = 0$

Question assigned to the following page: 2

Integrating by parts the first term

$$\int_0^s -(u', v') + B[Lu, v; t]\,dt = 0$$

Now $v' = -u$ $(0 \le t \le s)$, and so

$$\int_0^s \langle u', u\rangle - B[Lv', v; t]\,dt = 0$$

$$\int_0^s \frac{d}{dt}\left(\frac{1}{2}\|u\|^2_{L^2(U)} - \frac{1}{2}B[Lv, v; t]\right)dt = -\int_0^s C[Lu, v; t] + D[v, v; t]\,dt$$

For $u, v \in H_0^1(U)$ Hence

$$\frac{1}{2}\|u(s)\|^2_{L^2(U)} + \frac{1}{2}B[v(0), v(0); t] = -\int_0^s C[Lu, v; t] + D[v, v; t]\,dt$$

$$\|u(s)\|^2_{L^2(U)} + \|v(0)\|^2_{H_0^1(U)} \le C\left(\int_0^s \|v\|^2_{H_0^1(U)} + \|Lu\|^2_{L^2(U)}\,dt + \|v(0)\|^2_{L^2(U)}\right)$$

Now Let's write $w(t) := \int_0^t u(\tau)\,d\tau$ $(0 \le t \le T)$

$$\|u(s)\|^2_{L^2(U)} + \|w(s)\|^2_{H_0^1(U)} \le C\left(\int_0^s \|w(t) - w(s)\|^2_{H_0^1(U)} + \|Lu\|^2_{L^2(U)}\,dt + \|w(s)\|^2_{L^2(U)}\right)$$

However $\|w(t) - w(s)\|^2_{H_0^1(U)} \le 2\|w(t)\|^2_{H_0^1(U)} + 2\|w(s)\|^2_{H_0^1(U)}$

And $\|w(s)\|_{L^2(U)} \le \int_0^s \|u(t)\|_{L^2(U)}\,dt$

$$\|u(s)\|^2_{L^2(U)} + (1 - 2sC_1)\|w(s)\|^2_{H_0^1(U)} \le C_1\int_0^s \|w\|^2_{H_0^1(U)} + \|Lu\|^2_{L^2(U)}\,dt$$

Choose T_1 so small that

$$1 - 2T_1C_1 \ge \frac{1}{2}$$

Then if $0 \le s \le T_1$, we have

$$\|u(s)\|^2_{L^2(U)} + \|w(s)\|^2_{H_0^1(U)} \le C\int_0^s \|Lu\|^2_{L^2(U)} + \|w\|^2_{H_0^1(U)}\,dt$$

$\xrightarrow[\text{inequality}]{\text{Gronwall's}}$ implies $u \equiv 0$ on $[0, T_1]$

Question assigned to the following page: 3

3、By looking at the basis for infinite-dimensional spaces

We know that S_1: any linear subset of a vector space V

If S_1 spans V then S_1 is an orthonormal basis

Let's first assume that W_i is an orthonormal basis of $H_0^1(U)$

where $H_0^1(U) \overset{\text{bounded}}{\subseteq} H^1(U) = W^{1,2}(U)$

$$2 \downarrow L^2(U)$$

Say $V \subset L^2(U) \cap H_0^1(U)$

if w_i is an orthonormal basis of $H_0^1(U)$ but not $L^2(U)$

We could do $S_2 = W \cup \{v_i\}$ as a larger linearly independent set

$\Rightarrow S_1 \subset S_2 \subset S_3 \cdots$

Clearly the Union of all S_i is a linearly independent set, since any finite linear combination of the elements of the union must involve elements from one of sets S_i.