Homework 3

Graded

Student

Boyuan Deng

Total Points

5 / 5 pts

Question 1

1 **1** / 1 pt

✔ **– 0 pts** Correct

Question 2

2 **1** / 1 pt

✔ **– 0 pts** Correct

Question 3

3 **1** / 1 pt

✔ **– 0 pts** Correct

Question 4

4 **1** / 1 pt

✔ **– 0 pts** Correct

Question 5

5 **1** / 1 pt

✔ **– 0 pts** Correct

No questions assigned to the following page.

PDE
Homework 3
Towards solving the wave equation in $\dim = 3$
Due Date: Feb 23

Saman H. Esfahani

Let $⨍_U$ denote the average integral over a domain U:

$$⨍_U := \frac{1}{\mathrm{vol}(U)}\int_U .$$

Suppose $u : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ is a C^2 solution of

$$u_t - \Delta u = 0, \quad ①$$

with given initial conditions f and g:

$$u(0,x) = f(x), \quad ② \; u_t(0,x) = g(x). \quad ③$$

For any fixed x, define
$U_x : [0,\infty)\times[0,\infty) \to \mathbb{R}$ and
$F_x, G_x : [0,\infty) \to \mathbb{R}$ by

U(t;r) dS(y)

$$U_x(t;r) := ⨍_{\partial B_r(x)} u(t;y)\,dy;$$

$$F_x(r) := ⨍_{\partial B_r(x)} f(y)\,dy;$$

and

$$G_x(r) := ⨍_{\partial B_r(x)} g(y)\,dy.$$

1- Then, **prove** that $U_x : [0,\infty)\times[0,\infty) \to \mathbb{R}$ solves the PDE Euler-possion-darboux eqn.

$$\frac{\partial^2}{\partial t^2}U_x - \underbrace{\frac{\partial^2}{\partial r^2}U_x - \frac{n-1}{r}\frac{\partial}{\partial r}U_x}_{\text{radial. part of }\Delta} = 0,$$

2- Moreover, **show** that it satisfies the initial conditions

t r

$$U_x(0,\cdot) = F_x \quad \text{and}$$

$$\frac{\partial}{\partial t}U_x(0,\cdot) = G_x.$$

3-Moreover, **prove that**

$$\lim_{r\to 0} U_x(t;r) = u(t;x) \quad \text{and}$$

$$\lim_{r\to 0}\frac{\partial}{\partial t}U_x(t;r) = \frac{\partial}{\partial t}u(t;x).$$

Now suppose $n = 3$. Define U_x, F_x, G_x :

$[0,\infty)\times[0,\infty) \to \mathbb{R}$ by

$$\tilde{U}_x = rU_x, \quad \tilde{F}_x = rF_x, \quad \text{and} \quad \tilde{G}_x = rG_x.$$

4- **Prove** the following relations:

IVP

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right)\tilde{U}_x = 0 \quad \text{on} \quad (0,\infty)\times(0,\infty),$$

$$\tilde{U}_x = \tilde{F}_x \text{ and } \frac{\partial}{\partial t}\tilde{U}_x = \tilde{G}_x \quad \text{on } \{t=0\}\times(0,\infty),$$

$$\tilde{U} = 0 \quad \text{on} \quad (0,\infty)\times\{r=0\}.$$

5- [Kirchhoff's formula.] Suppose $f \in C^3$ and $g \in C^2$. Following questions 1,2,3,4, **show** that we have the following explicit solution in dimension $n = 3$

$$u(x,t) = ⨍_{\partial B_t(x)} \left(f(y) + t\nabla_\nu f(y) + tg(y)\right) dS(y),$$

for orthonormal outward vector field ν.

Questions assigned to the following page: 2 and 1

1. For fix x,

$$U_x(t;r) := \fint_{\partial B_r(x)} u(t;y)\,dy$$

$$U(x;t,r) := \fint_{\partial B_r(x)} u(t;y)\,dS(y)$$

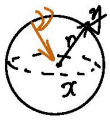
$$\frac{\partial U}{\partial r} = \fint_{\partial B_r(x)} Du(t,y)\,\frac{y-x}{r}\,dS(y)$$

$$= \fint_{\partial B_r(x)} \frac{\partial u}{\partial \nu}\,dS(y)$$

inward pointing unit normal

$$\overset{\lim_{r\to 0^+}}{=} \frac{r}{n}\fint_{B_r(x)} \Delta u(t,y)\,dy$$

$$= 0$$



$$\frac{\partial^2 U}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{r}{n}\fint_{B_r(x)} \Delta u(t,y)\,dy\right)$$

$$= \underbrace{\frac{1}{n}\fint_{B_r(x)} \Delta u(t,y)\,dy}_{A} + \frac{r}{n}\cdot\frac{\partial}{\partial r}\left(\fint_{B_r(x)} \Delta u(t,y)\,dy\right)$$

$$= A + \frac{r}{n}\frac{\partial}{\partial r}\left(\frac{n}{r}\fint_{\partial B_r(x)} \frac{\partial u}{\partial \nu}\,dS(y)\right)$$

$$\frac{\partial}{\partial y}\frac{\partial y}{\partial r}$$

$$= A + \frac{r}{n}\left[-\frac{n}{r^2}\fint_{\partial B_r(x)} \frac{\partial u}{\partial \nu}\,dS(y) + \frac{n}{r}\fint_{\partial B_r(x)} \frac{\partial}{\partial y}\frac{\overset{-1}{\cancel{\partial y}}\,\partial u}{\partial r\,\cancel{\partial \nu}}\,dS(y)\right]$$

$$= A - \frac{1}{r}\fint_{\partial B_r(x)} \frac{\partial u}{\partial \nu}\,dS(y) - \fint_{\partial B_r(x)} \frac{\partial}{\partial y}\frac{\partial u}{\partial r}\,dS(y)$$

$$= A - \frac{1}{n}\fint_{B_r(x)} \Delta u(t;y)\,dy + \fint_{\partial B_r(x)} \Delta u\,dS$$

$$= \fint_{\partial B_r(x)} \Delta u\,dS + \left(\frac{1}{n}-1\right)\fint_{B_r(x)} \Delta u\,dy$$

$$\lim_{r\to 0^+} U_{rr} = \frac{1}{n}\Delta u(x;t) \to 0$$

Since $U_r = \frac{r}{n}\fint_{B_r(x)} u_{tt}\,dy$

$$= \frac{1}{n\alpha(n)}\frac{1}{r^{n-1}}\int_{B_r(x)} u_{tt}\,dy$$

$$(r^{n-1}U_r)_r = \left(\frac{1}{n\alpha(n)}\int_{B_r(x)} u_{tt}\,dy\right)_r$$

$$= r^{n-1}\fint_{\partial B_r(x)} u_{tt}\,dS \overset{\lim_{r\to 0^+}}{=} r^{n-1}U_{tt} \to 0$$

Thus,

$$\frac{\partial^2}{\partial t^2}U_x - \frac{\partial^2}{\partial r^2}U_x - \frac{n-1}{r}\frac{\partial}{\partial r}U_x = 0$$

2. $U_x(t;r) := \fint_{\partial B_r(x)} u(t;y)\,dy$

$$= \fint_{\partial B_r(x)} \underbrace{u(0;y)}_{\text{from ②}}\,dy$$

$$= \fint_{\partial B_r(x)} f(y)\,dy$$

$$\overset{\text{by def}}{=} F_x$$

$$\frac{\partial}{\partial t}U_x(0,\cdot)$$

$$= \frac{\partial}{\partial t}\fint_{\partial B_r(x)} u(0,y)\,dy$$

$$= \fint_{\partial B_r(x)} \underbrace{u_t(0,y)}_{\text{from ③}}\,dy$$

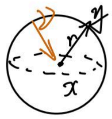
$$= \fint_{\partial B_r(x)} g(y)\,dy$$

$$\overset{\text{by def}}{=} G_x$$

3、By definition

$$\lim_{r\to 0} U_x(t;r) = \lim_{r\to 0} \fint_{\partial B_r(x)} u(t;y)\,dy$$

$$= u(t;x)$$



$$\lim_{r\to 0} \frac{\partial}{\partial t} U_x(t;r) = \lim_{r\to 0} U(x;r,t)$$

$$= \lim_{r\to 0} \frac{\partial}{\partial t} \fint_{\partial B_r(x)} u(t;y)\,dS(y)$$

$$= \lim_{r\to 0} \fint_{\partial B_r(x)} u_t(t;y)\,dS(y)$$

$$= u_t(t;x)$$

$$= \frac{\partial}{\partial t} u(t;x)$$

4. $n=3$

$\hat{U}_x = rU_x$, $\hat{F}_x = rF_x$, $\hat{G}_x = rG_x$

$\hat{U} = rU$ $\hat{F} = rF$, $\hat{G} = rG$

Since
$$\begin{cases} u_{tt} - \Delta u = 0 \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases}$$

$$\begin{cases} U := \fint_{\partial B_r(x)} u(t;y)\,dS(y) \\ F := \fint_{\partial B_r(x)} f(y)\,dS(y) \\ G = \fint_{\partial B_r(x)} g(y)\,dS(y) \end{cases}$$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right)\underbrace{\left[r\int_{\partial B_r(x)} u(t;y)\,dS(y)\right]}_{\hat{U}}$$

$$= \frac{\partial^2}{\partial t^2}\cdot r\int_{\partial B_r(x)} u(t;y)\,dS(y)$$

$$\quad - \frac{\partial^2}{\partial r^2}\cdot r\int_{\partial B_r(x)} u(t;y)\,dS(y)$$

$$= r\left(\int_{\partial B_r(x)} \overset{u_{tt} = u_{rr}}{u_{tt}}\,dS(y) - \int_{\partial B_r(x)} u_{rr}\,dS(y)\right)$$

$$= 0$$

$$\hat{U} = rU = r\int_{\partial B_r(x)} u(t;y)\,dS(y)$$

$$= r\int_{\partial B_r(x)} \underbrace{u(0;y)}_{f(y)}\,dS(y)$$

$$= r\int_{\partial B_r(x)} f(y)\,dS(y)$$

$$= rF$$

$$= \hat{F}$$

$$\frac{\partial}{\partial t}\hat{U} = \frac{\partial}{\partial t} r\int_{\partial B_r(x)} u(t;y)\,dS(y)$$

$$= r\int_{\partial B_r(x)} \overset{g(y)}{u_t(0,y)}\,dS(y)$$

$$= rG$$

$$= \hat{G}$$

$$\lim_{r\to 0^+} \hat{U} = rU = r\int_{\partial B_r(x)} u(t;y)\,dS(y)$$

$$= 0$$

Question assigned to the following page: 5

5. For the dimension $n=3$, from Q4 we have

$$\begin{cases} \check{U}_{tt} - \check{U}_{rr} = 0 & , \text{in } \mathbb{R}_+ \times (0,\infty) \\ \check{U} = \check{F} \, , \, \check{U}_t = \check{G} & , \text{on } \mathbb{R}_+ \times \{t=0\} \\ \check{U} = 0 & , \text{on } \{r=0\} \times (0 \times \infty) \end{cases}$$

$$\begin{aligned} \check{U}_{tt} &= r U_{tt} \\ &= r\left[U_{rr} + \frac{\overset{n}{3}-1}{r} U_r\right] \\ &= r U_{rr} + 2U_r = \check{U}_{rr} \end{aligned}$$

$$\check{F}_{rr}(0) = 0$$

By the d'Alembert's formula $u(x,t) = \begin{cases} \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(y)\,dy & \text{if } x \geq t \geq 0 \\ \frac{1}{2}[f(x+t) + f(t-x)] + \frac{1}{2}\int_{x-t}^{x+t} g(y)\,dy & \text{if } 0 \leq x \leq t \end{cases}$

For $0 \leq r \leq t$

$$\check{U} = \frac{1}{2}[\check{F}(r+t) - \check{F}(t-r)] + \frac{1}{2}\int_{-r+t}^{r+t} \check{G}(y)\,dy$$

$$u(x,t) = \lim_{r \to 0^+} U_x(r,t) = \lim_{r \to 0^+} U(x;r,t)$$

$$\begin{aligned} u(x,t) &= \lim_{r \to 0^+} \frac{\check{U}(x;r,t)}{r} \\ &= \lim_{r \to 0^+} \left[\frac{\check{F}(t+r) - \check{F}(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} \check{G}(y)\,dy\right] \\ &= \check{F}'(t) + \check{G}(t) \end{aligned}$$

$$\Rightarrow u(x,t) = \frac{\partial}{\partial t}\Big(t \underbrace{⨍_{\partial B_t(x)} f\,dS}\Big) + t⨍_{\partial B_t(x)} g\,dS\Big]$$

$$\begin{aligned} &= ⨍_{\partial B_1(0)} Df(x+tz)\cdot z\,dS(z) \\ &= ⨍_{\partial B_t(x)} Dg(y)\underbrace{\left(\frac{y-x}{t}\right)}_{\nu} dS(y) \end{aligned}$$

$$\begin{aligned} u(x,t) &= ⨍_{\partial B_t(x)}[t g(y) + f(y) + t \cdot D f(y)\cdot(y-x)]\,dS(y) \\ &= ⨍_{\partial B_t(x)}(t g(y) + f(y) + t \nabla_\nu f(y))\,dS(y) \end{aligned}$$