Homework 5

● Graded

Student

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Total Points

4 / 4 pts

Question 1

1 **1** / 1 pt

✔ **– 0 pts** Correct

Question 2

2 **1** / 1 pt

✔ **– 0 pts** Correct

Question 3

3 **1** / 1 pt

✔ **– 0 pts** Correct

Question 4

4 **1** / 1 pt

✔ **– 0 pts** Correct

No questions assigned to the following page.

Partial differential equations
Homework 5
Due date: March 8

Saman H. Esfahani

1- Use the mean value property to prove that the zeros of a harmonic function

$$u : U \subset \mathbb{R}^n \to \mathbb{R},$$

are never isolated.

2- Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain, $c > 0$ and $T > 0$ constants, given functions $f \in C(\Omega \times [0, T])$, $g \in C(\partial\Omega \times [0, T])$, and $\varphi \in C(\Omega)$. Use the energy method

$$E(u)(t) := \int_{\Omega\times\{t\}} u^2(x, t)dx,$$

to prove the uniqueness of the solution to the initial boundary value problem:

$$\begin{aligned} u_t - \overset{>0}{c}\Delta u &= \overset{\text{const}}{f} && \text{in } \Omega \times (0, T], \\ u &= g && \text{on } \partial\Omega \times (0, T], \\ u &= \varphi && \text{on } \Omega \times \{0\}. \end{aligned}$$

3- The following fully non-linear PDE

$$v_{tt}v_{xx} - 2v_{xt}v_{xt} + v_t^2 v_{xx} = 0 \qquad (1)$$

$v_{xx}(v_{tt} + v_t^2) = 2v_{xt}^2$

is a special case of the so-called Monge-Ampère equation. Here, you will reduce this PDE to an equivalent first-order equation and then solve it.

(a) The equation (1) is equivalent to a simpler equation for a new function $u = v_t/v_x$. Find this equation for the function u.

(b) For the given initial conditions

$$v(x, 0) = 1 + 2e^{3x},$$
$$v_t(x, 0) = 4e^{3x},$$

on $-\infty < x < \infty$, find the corresponding initial conditions for u.

(c) Solve the equation for u using the method of characteristics.

(d) Find v the solution to the equation (1).

4- The Legendre transform of a C^2 convex function $f : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$f^*(p) = \sup_{x\in\mathbb{R}^n} \{x \cdot p - f(x)\}.$$

(a) Prove $f^*(p)$ is a convex function.

(b) Let $H(p) = \frac{1}{r}|p|^r$, for some $1 < r < \infty$. Let s be the number where

$$\frac{1}{r} + \frac{1}{s} = 1.$$

Let $L(v) = \frac{1}{s}|v|^s$. We want to show $H^* = L$. To do so, first prove:

$$H^*(v) \geq L(v).$$

Hint: Do this using the fact that the supremum of a function is greater ~~than the~~ value of the function obtained at a specific point.

(c) Prove with the use of the Young's inequality.

$$H^*(v) \leq L(v).$$

Theorem 1. *Let $u \in C^2(U)$ be a solution to a non-linear first-order PDE:*

$$F(\nabla u(x), u(x), x) = 0.$$

Let $p(s) = \nabla u(x(s))$ and $z(s) = u(x(s))$. Then,

$$x'(s) = \nabla_p F,$$
$$p'(s) = -\nabla_x F - \nabla_z F p,$$
$$z'(s) = p \cdot \nabla_p F.$$

Theorem 2 (Young's inequality)**.**

If $\dfrac{1}{r} + \dfrac{1}{s} = 1$ *and* r, s *are positive, then* $ab \leq \dfrac{a^r}{r} + \dfrac{b^s}{s}$.

4. $f^*(p) = \sup_{x\in\mathbb{R}^n} \{x p - f(x)\}$

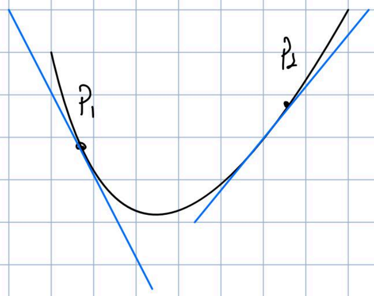
e.g. $\frac{\partial u}{\partial x}$

(a) Prove $f^*(p)$ is a convex function?

Use Jensen inequality:

Assume f^* is a convex function, $\lambda \in [0,1]$, $p_1, p_2 \in \mathbb{R}^n$

$f^*(\lambda p_1 + (1-\lambda)p_2) \le \lambda f^*(p_1) + (1-\lambda) f^*(p_2)$



By the defination

$$f^*(p) = \sup_{x\in\mathbb{R}^n} \{x(\lambda p_1 + (1-\lambda)p_2) - f(x)\}$$

$$= \sup_{x\in\mathbb{R}^n} \{x(\lambda p_1 + (1-\lambda)p_2) - f(x)\}$$

$$= \sup_{x\in\mathbb{R}^n} \{\lambda (xp_1) + (1-\lambda)(xp_2) - f(x)\}$$

Since $f(x)$ is independent from λ

$$= \sup_{x\in\mathbb{R}^n} \{\lambda (xp_1 - f(x)) + (1-\lambda)(xp_2 - f(x))\}$$

$\lambda f^*(p_1) \quad + (1-\lambda) f^*(p_2)$

$$\le \lambda \sup_{x\in\mathbb{R}^n}\{xp_1 - f(x)\} + (1-\lambda) \sup_{x\in\mathbb{R}^n} (xp_2 - f(x))\}$$

So that $f(x)$ is convex

(b) $H(p) = \frac{1}{r}|p|^r$, $r \in (1,\infty)$, s satisfies $\frac{1}{r} + \frac{1}{s} = 1$

$L(v) = \frac{1}{s}|v|^s$, $H^* = L$

? $H^* = L$

$\underline{H(p) + L(v) = \frac{|p|^r}{r} + \frac{|v|^s}{s} \ge pv} \Rightarrow H(p) \ge pv - L(v)$

$L(v) \ge pv - H(p)$

$= \sup(pv - H(p)) = H^*(v)$ // This also prove (c)

$H^*(v) = \sup\{pv - H(p)\} = \sup\{pv - \sup\{pr - L(r)\}\}$

$= \sup[\inf(p(v-r) + L(r))]$

$= \sup(L(v)) \Rightarrow H^*(v) \ge L(v)$, we have also proved that $L(v) \ge H^*(v)$

So that $H^*(v) = L(v)$

convex function

$g \le \lambda g(x_1) + (1-\lambda) g(x_2)$

$L(r) \ge L(v) + (r-v)\underset{DL(v)}{\underline{p}}$

(c) $H^*(v) - L(v) = \sup_{x\in\mathbb{R}^n}\{x\cdot v - H(x)\} = \sup_{x\in\mathbb{R}^n}\{x\cdot v - \frac{1}{r}|x|^r\} - \frac{1}{s}|v|^s$

$$\ge x\cdot v - \underline{\left(\frac{|x|^r}{r} + \frac{|v|^s}{s}\right)} \le 0$$

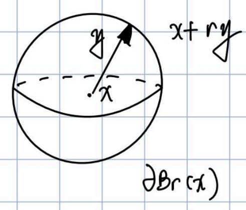
since $\frac{1}{r} + \frac{1}{s} = 1$, Young's inequality $\ge xv$

$H^*(v) \le L(v)$

Question assigned to the following page: 1

1. $u: U \subset \mathbb{R}^n \to \mathbb{R}$

$\Delta u = 0$



Assume there is an isolated zero x_0

for some $r>0$, there is $u \neq 0$ on the $\partial B_r(x_0)$

while $u(x_0) = 0$

Since we have that $u(x) = ⨍_{\partial B_r(x)} u(x+ry) \, dS$

$\Rightarrow u(x_0) = ⨍_{\partial B_r(x_0)} u(x_0 + ry) \, dS$

Given that $u(x_0) = 0$, so the right hand side should equal 0

This violated the sense that there is $u \neq 0$ on the $\partial B_r(x_0)$

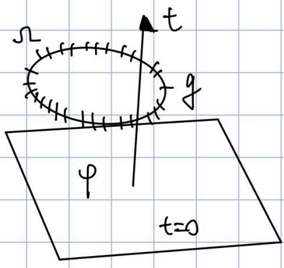
So the zeros of harmonic functions are never isolated

Question assigned to the following page: 2

2. There exists at most one function $u \in C^2(\bar{\Omega})$

Assume there is another solution $\hat{u}$

Define $w := u - \hat{u}$ To prove $w \equiv 0$?



$$\begin{cases} u_t - c\Delta u = f & \text{in } \Omega \times (0,T] \\ u = g & \text{on } \partial\Omega \times (0,T] \\ u = \varphi & \text{on } \Omega \times \{0\} \end{cases} \Rightarrow \begin{cases} w_t - c\Delta w = 0 \\ w = 0 \\ w = 0 \end{cases}$$ (Become homo)

$E(u)(t) := \int_{\Omega \times \{t\}} w^2(x,t)\,dx$

Assume that $w \neq 0$

Then $E(w)(t) = \int_{\Omega \times \{t\}} w^2(x,t)\,dx = 0 \cdots$ ①

$\frac{d}{dt} \leftarrow \dot{E}(w)(t) = 2\int_{\Omega \times \{t\}} w \overset{c\Delta w}{w_t}\,dx$

$= 2c\int w\,w_{xx}\,dx$

$= 2c\,\overset{0}{\cancel{w}}w_x - \int_{\Omega \times \{t\}} |w_x|^2\,dx \leq 0$

Since its derivative $[0,T]$ is negative The value of $E(u,t)$ should decrease

From ① we have that $E(u,t) = 0$, if it decrease among domain

It is conflict to the fact that $E(w)(t) \geq 0$

So there is at most one function $u \in C^2(\bar{\Omega})$

Question assigned to the following page: 3

3、 $v_{tt}v_{xx} - 2v_{xt}v_{xt} + v_t^2 v_{xx} = 0 \cdots ①$

$u = \frac{v_t}{v_x}$?

(a) $u_t = \frac{d(v_t v_x^{-1})}{dt} = \frac{v_{tt}v_x - v_t v_{xt}}{v_x^2} \cdots ②$

$u_x = \frac{d(v_t v_x^{-1})}{dx} = \frac{v_{tx}v_x - v_t v_{xx}}{v_x^2} \cdots ③$

Since $v_t = u v_x$

$v_{tt} = u_t v_x + u v_{xt} \cdots ⑤$

$v_{tx} = u_x v_x + u v_{xx} \cdots ⑥$

⑥ → ⑤ $v_{tt} = u_t v_x + u u_x v_x + u^2 v_{xx}$

By looking at ②&③, change ① to

$v_{tt}v_{xx} - v_{xt}v_{xt} = v_{xt}v_{xt} - v_t^2 v_{xx}$

② → $\frac{\partial}{\partial x}(u_t v_x^2) = \frac{\partial}{\partial x}(v_{tt}v_x - v_t v_{xt})$ Since v is a function of (x,t) → v_{xxt} & $v_{ttx} = \frac{\partial}{\partial x}(c)$ a constant

$= \cancel{v_{ttx}}v_x + v_{tt}v_{xx} - v_{tx}v_{xt} - \cancel{v_t}v_{xxt}$

$u_{tx}v_x^2 + 2u_t v_x v_{xx}$ $= LHS$

③ → $u_x v_x^2 = v_{tx}v_x - v_t v_{xx}$

$\frac{\partial}{\partial t}(u_x v_x^2) = \cancel{v_{txt}}v_x + v_{tx}v_{xt} - v_{tt}v_{xx} - \cancel{v_t}v_{xxt}$

$u_{tx}v_x^2 + 2u_x v_x v_{xx} + v_{tt}v_{xx} - v_t^2 v_{xx} = RHS$

$\cancel{u_{tx}v_x^2} + 2u_t v_x \cancel{v_{xx}} = \cancel{u_{tx}v_x^2} + 2u_x v_x \cancel{v_{xx}} + v_{tt}\cancel{v_{xx}} - v_t^2 \cancel{v_{xx}}$

$\Rightarrow 2v_x(u_t - u_x) = v_{tt} - v_t^2$

$2v_x(u_t - u_x) = u_t v_x + u u_x v_x + u^2 v_{xx} - u^2 v_x^2$

$u_t - 2u_x = u \cdot u_x + u^2 \frac{v_{xx}}{v_x} - u^2 v_x$

$u_t - 2u_x =$

Question assigned to the following page: 3

(b) IC: $\begin{cases} v(x,0) = 1+2e^{3x} \cdots ③ \\ v_t(x,0) = 4e^{3x} \cdots ④ \end{cases}$

$$v_x(x,0) = 6e^{3x}$$

Since $u = \frac{v_t}{v_x}$

From ③ & ④

$$u(x,0) = \frac{4e^{3x}}{6e^{3x}} = \frac{2}{3}$$ a constant

(c) $v_{xx}(u_t - uu_x + u^2 v_x) - 2u_x^2 v_x = 0$

$$F(Du, u, x) = 0$$

$$\Rightarrow F(\overset{u_t}{P_1}, \overset{u_x}{P_2}, \overset{u}{z}, \overset{v_x}{x_1}, \overset{v_{xx}}{x_2}) = 0$$

$\Rightarrow$

$$\begin{cases} D_x F = \\ D_z F = \\ D_p F = \end{cases}$$

$$\begin{cases} \dot{p}(s) = -D_x F - D_z F \cdot p \\ \dot{z}(s) = D_p F \cdot p \\ \dot{x}(s) = D_p F \end{cases}$$

$$\begin{cases} \dot{p}(s) = \\ \dot{z}(s) = \\ \dot{x}(s) = \end{cases}$$

(d)