Homework 1

Graded

Student

Boyuan Deng

Total Points

4.5 / 5 pts

Question 1

1 **1** / 1 pt

✔ **– 0 pts** Correct

Question 2

2 **1** / 1 pt

✔ **– 0 pts** Correct

Question 3

3 **1** / 1 pt

✔ **– 0 pts** Correct

Question 4

4 **1** / 1 pt

✔ **– 0 pts** Correct

Question 5

5 **0.5** / 1 pt

– 0 pts Correct

✔ **– 0.5 pts** Click here to replace this description.

It's true that u should decay at infinity, however, it's not trivial. You can use the mean value property to show that (look at the solutions on the webpage).

No questions assigned to the following page.

PDE
Homework 1
Due Date: Feb 9

Saman H. Esfahani

IVP

- 1- Find the explicit solution u to the following equation:

  directional movement in the spatial domain (x,t) growth/decay $\to e^c$

$$\begin{cases} u_t + b \cdot \nabla u + cu = 0, & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

  here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

- 2- Let $u : B_r(0) \subset \mathbb{R}^n \to \mathbb{R}$ be a solution to the following equation:

$$\begin{cases} -\Delta u = f, & \text{in } B_r^\circ(0), \\ u = g & \text{on } \partial B_r(0). \end{cases}$$

  Prove the following:

  origin   fundamental solution of the Laplace's equation

$$u(0) = \frac{1}{\text{vol}(\partial B_r(0)} \int_{\partial B_r(0)} (g dS) + \frac{1}{n(n-2)\text{vol}(B_1(0))} \int_{B_r(0)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) f dx.$$

- 3- Suppose $u : U \subset \mathbb{R}^n \to \mathbb{R}$ is harmonic. Show that $v = |\nabla u|^2$ is subharmonic; i.e., $-\Delta v \leq 0$.

- 4. Prove there exists a constant C such that for any function u, a solution of

$$\begin{cases} -\Delta u = f, & \text{in } B_r^\circ(0), \\ u = g & \text{on } \partial B_r(0). \end{cases}$$

  we have

$$\max_{B_1(0)} |u| \leq C\left(\max_{\partial B_1(0)} |g| + \max_{B_1(0)} |f|\right).$$

  Squre integrable

- 5. Suppose that $u \in L^2(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ and $\Delta u = 0$. Show $u = 0$.

  2nd continuity

  Liouville's thm

  u harmonic $C \bar{U}$

  u is a constant

Question assigned to the following page: 1

1. $\begin{cases} u_t + b \cdot \overset{(x,t)}{\nabla u} + cu = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$

$z: \mathbb{R} \to \mathbb{R}$ for $x \in \mathbb{R}^n$ & $t \in (0,\infty)$

$z(s) = u(x+bs,\ t+s)e^{cs}$

$$\frac{\partial z(s)}{\partial s} = \underbrace{\frac{\partial}{\partial s}(u(x+bs,t+s))}_{\frac{\partial u}{\partial x}\frac{\partial(x+bs)}{\partial s} + \frac{\partial u}{\partial t}\frac{\partial(t+s)}{\partial s}} e^{cs} + u(x+bs,t+s)\underbrace{\frac{\partial}{\partial s}e^{cs}}_{ce^{cs}}$$

$b \cdot D_x u(x+bs,t+s) + u_t(x+bs,t+s)$

$$= e^{cs}(b \cdot D_x u(x+bs,t+s) + u_t(x+bs,t+s)) + u(x+bs,t+s)\, c \cdot e^{cs}$$

$$= e^{cs}(u_t(x+bs,\ t+s) + b \cdot D_x u(x+bs,t+s) + c \cdot u(x+bs,t+s)) = 0$$

$s=0 \quad || \quad -t$

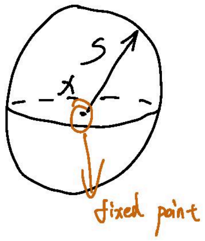
$\Downarrow \quad u = g \quad \text{on} \quad \mathbb{R}^n \times \{t=0\}$

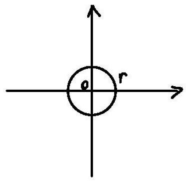
$$u(x,t) = \underbrace{z(0)}_{u(x+0,\,t+0)e^0 \,=\, u(x,t)} = \underbrace{z(-t)}_{u(x-bt,\,0)e^{-ct}}$$

Thus, the explicit solution would be $g(x-bt)e^{-ct}$

Question assigned to the following page: 2

2. Define that





$$\phi(s) := \frac{1}{n\alpha(n)s^{n-1}}\int_{\partial B_s(x)} u(y)\,dS(y)$$

$$= \frac{\int_{\partial B_s(x)} u(x+y s)\,dS(y)}{n\alpha(n)}$$

divergence thm

$$\phi'(s) = \frac{\int_{\partial B_s(x)} \nabla u(x+sy)\cdot y\,dS(y)}{n\alpha(n)}$$

$\Delta u = -f$, $x=0$

$$= \frac{\int_{B_s(x)} \nabla\cdot\nabla u(y)\,dy}{n\alpha(n)s^{n-1}} = -\frac{\int_{B_s(0)} f(y)\,dy}{n\alpha(n)s^{n-1}}$$

$$\phi(s)-\phi(\varepsilon)=\int_\varepsilon^r \phi'(s)\,ds \Rightarrow \underset{\varepsilon\to 0}{\phi(\varepsilon)} = \phi(s) - \int_\varepsilon^r \phi'(s)\,ds$$

$$= \frac{1}{n\alpha(n)s^{n-1}}\int_{\partial B_s(0)} g\,dS - \int_\varepsilon^r \phi'(s)\,ds$$

$$-\int_\varepsilon^r \phi'(s)\,ds = \int_\varepsilon^r \frac{1}{n\alpha(n)s^{n-1}}\int_{\partial B_s(0)} f(y)\,dy\,ds$$

$$= \frac{1}{n\alpha(n)}\int_\varepsilon^r \frac{1}{s^{n-1}}\int_{B_s(0)} f(y)\,dy\,ds$$

$$\frac{-1}{(n-2)s^{n-2}}\Big|_{s=\varepsilon}^{r} = \frac{1}{n-2}\left(\frac{1}{\varepsilon^{n-2}}-\frac{1}{r^{n-2}}\right)$$

$$= \frac{1}{n\alpha(n)}\left[\left(\frac{1}{2-n}\frac{1}{s^{n-2}}\int_{B_s(0)} f\,dy\right)\Big|_\varepsilon^r - \int_\varepsilon^r \frac{1}{2-n}\frac{1}{s^{n-2}}\int_{\partial B_s(0)} f\,dS\,ds\right]$$

$$= \frac{1}{n(n-2)\alpha(n)}\left[\int_\varepsilon^r \frac{1}{s^{n-2}}\int_{\partial B_s(0)} f\,dS\,ds - \frac{1}{r^{n-2}}\int_{B_s(0)} f\,dy + \frac{1}{\varepsilon^{n-2}}\int_{B_\varepsilon(0)} f\,dy\right]$$

$$= \frac{1}{n(n-2)\alpha(n)}\left[\int_\varepsilon^r \frac{1}{s^{n-2}}\int_{\partial B_s(0)} f\,dS\,ds + \overset{\star}{\frac{1}{\varepsilon^{n-2}}\int_{B_\varepsilon(0)} f\,dy} - \frac{1}{r^{n-2}}\int_{B_s(0)} f\,dy\right]$$

$$\Downarrow$$

$$\int_0^r ds\int_{\partial B_s(0)} \frac{1}{s^{n-2}} f\,dS = \int_{B_\varepsilon(0)} \frac{1}{|x|^{n-2}} f(x)\,dx$$

As $\varepsilon\to 0$, $\star \to \int_{B_\varepsilon(0)} \frac{1}{|x|^{n-2}} f(x)\,dx$

$$\lim_{\varepsilon\to 0} -\int_\varepsilon^r \phi'(s)\,ds = \frac{1}{n(n-2)\underset{\mathrm{vol}(B_1(0))}{\underset{\|}{\alpha(n)}}}\int_{B_r(0)}\left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) f\,dx$$

Thus , letting $\varepsilon\to 0$

$$u(0) = \phi(0) = \frac{1}{\mathrm{vol}(\partial B_r(0))}\int_{\partial B_r(0)} g\,dS + \frac{1}{n(n-2)\,\mathrm{vol}(B_1(0))}\int_{B_r(0)}\left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) f\,dx$$

Question assigned to the following page: 3

3、 $u: U \subset \mathbb{R}^n \to \mathbb{R}$ is harmonic

$$v = |\Delta u|^2 = \sum_{i=1}^{n} u_{x_i}^2 \quad \text{for } x_1, x_2 \dots x_n$$

$$\frac{\partial v}{\partial x_k} = 2\sum_{i=1}^{n} u_{x_i}\, u_{x_i x_k}$$

$$\frac{\partial^2 v}{\partial x_k \partial x_j} = 2\sum_{i=1}^{n}\left[\frac{\partial^2 u}{\partial x_i \partial x_k}\cdot\frac{\partial^2 u}{\partial x_i \partial x_{j \to k}} + \frac{\partial u}{\partial x_i}\,\frac{\partial^3 u}{\partial x_i x_k x_{j \to k}}\right]$$

$$\frac{\partial^2 v}{\partial x_k^2} = 2\sum_{i=1}^{n}\left[\underline{\frac{\partial^2 u}{\partial x_i \partial x_k}\cdot\frac{\partial^2 u}{\partial x_i \partial x_k}} + \frac{\partial u}{\partial x_i}\,\frac{\partial}{\partial x_i}\left(\frac{\partial^2 u}{\partial x_k^2}\right)\right]$$

$$= 2\sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial^2 u}{\partial x_i \partial x_k}\right)^2 + \frac{\partial u}{\partial x_i}\,\frac{\partial}{\partial x_i}\,\cancel{(\Delta u)}$$

harmonic $=0$

$$\geq 0$$

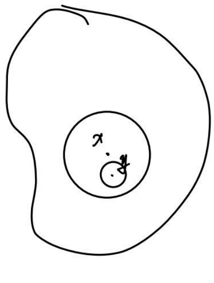
$$\therefore -\Delta v \leq 0$$

Question assigned to the following page: 4

4. Using Poisson's formula of ball

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n) r} \int_{\partial B_r(0)} \frac{g}{|x-y|^n} dS$$

Since integrating over the boundary, $y \in \partial B_r(0)$ $|x-y| \leq |x| + r$



$$u(x) = \frac{(r-|x|)(r+|x|)}{n\alpha(n) r} \int_{\partial B_r(0)} \frac{g}{|x-y|^n} dS$$

$$\leq \frac{(r-|x|)(r+|x|)}{n\alpha(n) r} \int_{\partial B_r(0)} \frac{g}{(|x|+r)^n} dS = \frac{r-|x|}{(|x|+r)^{n-1}} \frac{1}{n\alpha(n) r} \int_{\partial B_r(0)} g \, dS$$

$$= \frac{r-|x|}{(|x|+r)^{n-1}} \frac{r^{n-2}}{n\alpha(n) r^{n-1}} \int_{\partial B_r(0)} g \, dS$$

$$= \frac{r^{n-2}(r-|x|)}{(|x|+r)^{n-1}} u(0)$$

From 2 we know that

$$u(0) = \frac{1}{\mathrm{vol}(\partial B_r(0))} \int_{\partial B_r(0)} g \, dS + \frac{1}{n(n-2)\,\mathrm{vol}(B_1(0))} \int_{B_r(0)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) f \, dx$$

$$u(x) \leq \underbrace{\frac{r^{n-2}(r-|x|)}{(|x|+r)^{n-1}}}_{C:} \left[\frac{1}{\mathrm{vol}(\partial B_r(0))} \int_{\partial B_r(0)} g \, dS + \frac{1}{n(n-2)\,\mathrm{vol}(B_1(0))} \int_{B_r(0)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) f \, dx\right]$$

As u is harmonic on a bounded region

So $\max_{B_1(0)} |u| \leq C \left(\max_{\partial B_1(0)} |g| + \max_{B_1(0)} |f|\right)$

Question assigned to the following page: 5

5、 u is square integrable & 2nd continuity

? u is bounded

From Liouville thm, bc $u: \mathbb{R}^n \to \mathbb{R}$ be a bounded harmonic function ($\Delta u = 0$)

So that u is a constant

$u \in L^2(\mathbb{R}^n) \to \int_{\mathbb{R}^n} |u(x)|^2 dx < \infty$

u must decay at infinity $\to u \to 0$

Thus, $u = 0$