Homework 2

Graded

Student

Boyuan Deng

Total Points

7 / 7 pts

Question 1

1 **1** / 1 pt

✔ **− 0 pts** Correct

Question 2

2 **1** / 1 pt

✔ **− 0 pts** Correct

Question 3

3 **1** / 1 pt

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Question 4

4 **1** / 1 pt

✔ **− 0 pts** Correct

Question 5

5 **1** / 1 pt

✔ **− 0 pts** Correct

Question 6

6 **1** / 1 pt

✔ **− 0 pts** Correct

Question 7

7 **1** / 1 pt

✔ **− 0 pts** Correct

No questions assigned to the following page.

PDE
Homework 2
Due Date: Feb 16

Saman H. Esfahani

- 1- Find the function $u : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ a solution of the heat equation of the form

$$u(x,t) = \frac{1}{t^\alpha} v(\frac{|x|^2}{t}),$$

for a constant α and a function $v : \mathbb{R} \to \mathbb{R}$.

- 2- Prove the following Lemma.

Lemma 1. *The constant in the definition of the fundamental solution of the heat equation is chosen such that the total heat at any fixed time $t > 0$,*

$$\int_{x\in\mathbb{R}^n} \Phi(x,t)dx = 1,$$

where Φ is the fundamental solution to the heat equation

$$\Phi(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-|x|^2}{4t}}.$$

- 3. Prove the following theorem.

IVP

Theorem 1. *The function defined by $u = \Phi * g$ is smooth function on $\mathbb{R}^n \times \{t > 0\}$ and satisfies the homogeneous equation:*

$$\begin{cases} u_t - \Delta u = 0, & on \quad \mathbb{R}^n \times (0,\infty), \\ u = g, & on \quad \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

- 4. Prove the following theorem. non-homo

Theorem 2. *Suppose $f \in C^2(\mathbb{R}^n \times (0,\infty))$ and has compact support. Let u be the function defined by*

$$u(x,t) = \int_0^t \int_{y\in\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)dydt,$$

where Φ is the fundamental solution of the heat equation. Then, u is in C^2 and satisfies

$$\begin{aligned} u_t - \Delta u = f, &\quad on \quad \mathbb{R}^n \times (0,\infty), \\ u = 0, &\quad on \quad \mathbb{R}^n \times \{t = 0\}. \end{aligned}$$

- 5. Draw an approximate sketch of the heat ball with radius r. Explain your drawing.

- 6. Prove the following theorem.

Similar to 17 (a)

Theorem 3. *Let $u(x,t)$ be a solution of the heat equation. Then,*

MVP

$$u(x,t) = \frac{1}{4r^n} \int_{E(x,t,r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dyds,$$

for any $r > 0$.

7. Prove the following.

Theorem 4. *Suppose $u \in C^2(\overline{U}_T)$ is a solution to the heat equation, and U is a connected subset of $\mathbb{R}^n$. Then, if u has an interior point $(x_0,t_0) \in U_T$ which u is maximized there, then u is constant on U_T.*

Strong Max

Questions assigned to the following page: 2, 3, and 1

1. Dilation scaling $u(x,t)=\frac{1}{t^{\alpha}}v(\frac{x}{t})$

$u(x,t) \Rightarrow \lambda^{\alpha}(\lambda x, \lambda t)$ for all $\lambda > 0$, $x \in \mathbb{R}^n$, $t>0$

Setting $\lambda = t^{-1}$ $\quad v(y) := u(y,1)$ $\quad \frac{x}{t}$

$u_t - \Delta u = 0$

$\alpha t^{-(\alpha+1)} v(y) + t^{-(\alpha+1)} y \cdot Dv(y) + t^{-\alpha}\Delta v(y) = 0$

$\alpha v + y \cdot D_v + \Delta v = 0$

By guessing v to be radial $\Rightarrow v(y) = w(|y|)$

for $w: \mathbb{R} \to \mathbb{R}$

$a w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0$, for $r = |y|$

' indicates $\frac{d}{dr}$, set $\alpha = \frac{n}{2}$

$(r^{n-1}w')' + \frac{1}{2}(r^n w)' = 0$

$r^{n-1}w' + \frac{1}{2} r^n w = a$ constant

Assuming $\lim_{r\to\infty} w, w' = 0 \Rightarrow a = 0$, $w' = -\frac{1}{2} r w$

$w = b e^{-\frac{r^2}{4}}$ constant

$\Rightarrow \boxed{u = \frac{b}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}}$

2. $\int_{\mathbb{R}^n} \phi(x,t)\,dx = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}} dx$

Define: $z = \frac{x}{\sqrt{2t}}$

$= \frac{1}{(4\pi t)^{\frac{n}{2}}} \underbrace{(\sqrt{2t})^n}_{\text{Jocobian determinant}} \int_{\mathbb{R}^n} e^{-|z|^2} dz$

$= \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-|z|^2} dz$

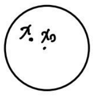
$= \frac{1}{\pi^{\frac{n}{2}}} \prod_{i=1}^{n} \underbrace{\int_{-\infty}^{\infty} e^{-z_i^2} dz_i}_{\sqrt{\pi}}$

$= 1$

3. The fundamental solution gives

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & (x \in \mathbb{R}^n, t>0) \\ 0 & (x \in \mathbb{R}^n, t<0) \end{cases}$$

$C^{\infty}(\mathbb{R}^n \times (0,\infty))$



$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = 0, & x \in \mathbb{R}^n, t>0 \quad \cdots ① \\ u(x,0) = g(x), & x \in \mathbb{R}^n \quad \cdots ② \end{cases}$$

$u(x,t) = \Phi(x,t) g = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\,dy$

$= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} - \frac{|x-y|^2}{4t} g(y)\,dy$

$u_t(x,t) - \Delta u(x,t) = \int_{\mathbb{R}^n} [\cancelto{0}{(\Phi_t - \Delta_x \Phi)}(x-y,t)] g(y)\,dy = 0$

Φ itself solves the heat eqn ①

Then we prove that

$\lim_{\substack{(x,t)\to(x^0,0) \\ x\in\mathbb{R}^n, t>0}} u(x,t) = g(x^0)$ for each point $x^0 \in \mathbb{R}^n$

Fix $x^0 \in \mathbb{R}^n$, $\varepsilon > 0$. Choose $\delta > 0$ such that

$|g(y) - g(x^0)| < \varepsilon$ if $|y - x^0| < \delta$, $y \in \mathbb{R}^n$

$|u(x,t) - g(x^0)| = \left|\int_{\mathbb{R}^n} \Phi(x-y,t)[g(y) - g(x^0)]\,dy\right|$

$\leq \int_{B_\delta(x^0)} \Phi(x-y,t)\,|g(y) - g(x^0)|\,dy$

$+ \int_{\mathbb{R}^n - B_\delta(x^0)} \Phi(x-y,t)\,|g(y) - g(x^0)|\,dy$

$=: I + J$

$I \leq \varepsilon \underbrace{\int_{\mathbb{R}^n} \Phi(x-y,t)\,dy}_{=1} = \varepsilon$

if $|x - x^0| \leq \frac{\delta}{2}$ & $|y - x^0| \geq \delta$

$\Rightarrow |y - x^0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}|y - x^0|$

$J \leq 2\|g\|_{L^\infty} \int_{\mathbb{R}^n - B_\delta(x_0)} \Phi(x-y,t)\,dy$

$\leq \frac{C}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n - B_\delta(x_0)} e^{-\frac{|x-y|^2}{4t}}\,dy$

$\leq \frac{C}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n - B_\delta(x_0)} e^{-\frac{|x-x^0|^2}{16t}}\,dy$

$= C \int_{\mathbb{R}^n - B_{\frac{\delta}{\sqrt{t}}}(x_0)} e^{-\frac{|z|^2}{16}}\,dz \to 0$, as $t \to 0^+$

Hence if $|x - x^0| < \frac{\delta}{2}$ & $t > 0$ is small enough

$|u(x,t) - g(x^0)| < 2\varepsilon$ satisfies eqn. ②

Questions assigned to the following page: 4 and 5

4. $u(x,t)=\int_0^t\int_{\mathbb{R}^n}\Phi(x-y,t-s)f(y,s)\,dy\,ds$

Change of variables

$$u(x,t)=\int_0^t\int_{\mathbb{R}^n}\Phi(y,s)f(x-y,t-s)\,dy\,ds$$

Since $f\in C^2(\mathbb{R}^n\times(0,\infty))$ has compact support

and $\Phi=\Phi(y,s)$ is smooth near $s=t>0$

$$u_t(x,t)=\int_0^t\int_{\mathbb{R}^n}\Phi(y,s)f_t(x-y,t-s)\,dy\,ds+\int_{\mathbb{R}^n}\Phi(y,t)f(x-y,0)\,dy$$

$$\Delta u(x,t)=u_{x_ix_j}(x,t)=\int_0^t\int_{\mathbb{R}^n}\Phi(y,s)f_{x_ix_j}(x-y,t-s)\,dy\,ds$$

$$u_t(x,t)-\Delta u(x,t)=\int_0^t\int_{\mathbb{R}^n}\Phi(y,s)\left(\frac{\partial}{\partial t}-\Delta_x\right)f(x-y,t-s)\,dy\,ds+\int_{\mathbb{R}^n}\Phi(y,t)f(x-y,0)\,dy$$

$$=\int_\varepsilon^t\int_{\mathbb{R}^n}\Phi(y,s)\left[\left(-\frac{\partial}{\partial s}-\Delta_y\right)f(x-y,t-s)\right]dy\,ds$$
$$+\int_0^\varepsilon\int_{\mathbb{R}^n}\Phi(y,s)\left[\left(-\frac{\partial}{\partial s}-\Delta_y\right)f(x-y,t-s)\right]dy\,ds$$
$$+\int_{\mathbb{R}^n}\Phi(y,t)f(x-y,0)\,dy$$

$$=:I_\varepsilon+J_\varepsilon+K$$

$$|J_\varepsilon|\le C(\|f_t\|_{L^\infty}+\|D^2f\|_{L^\infty})\int_0^\varepsilon\int_{\mathbb{R}^n}\Phi(y,s)\,dy\,ds\le\varepsilon C$$

Integrating by parts

$$I_\varepsilon=\int_\varepsilon^t\int_{\mathbb{R}^n}\left[\underbrace{\left(\frac{\partial}{\partial s}-\Delta_y\right)\Phi(y,s)}_{\to 0}\right]f(x-y,t-s)\,dy\,ds$$
$$+\int_{\mathbb{R}^n}\Phi(y,\varepsilon)f(x-y,t-\varepsilon)\,dy-\int_{\mathbb{R}^n}\Phi(y,t)f(x-y,0)\,dy$$

$$=\int_{\mathbb{R}^n}\Phi(y,\varepsilon)f(x-y,t-\varepsilon)\,dy-K$$

$$u_t(x,t)-\Delta u(x,t)=\lim_{\varepsilon\to0}\int_{\mathbb{R}^n}\Phi(y,\varepsilon)f(x-y,t-\varepsilon)\,dy$$
$$=f(x,t)\quad(x\in\mathbb{R}^n,\ t>0)$$

$$\|u(\cdot,t)\|_{L^\infty}\le t\|f\|_{L^\infty}\to 0$$

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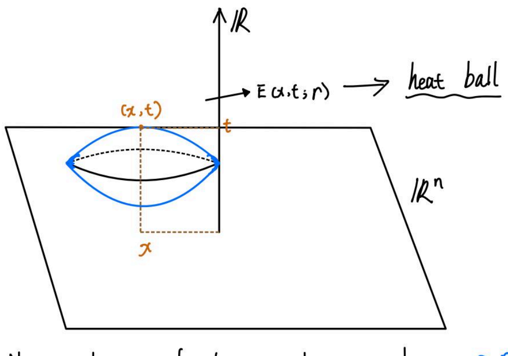
5. heat ball with radius r

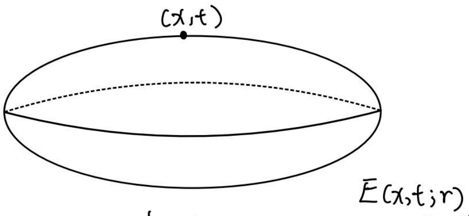


Notice that fixed x the spheres $\partial B(x,r)$ are level sets of the fundamental solution $\Phi(x-y)$ for Laplace's equation for fixed point (x,t) $\Rightarrow$ $\Phi(x-y,t-s)$ $\Rightarrow$ MVP

So define $E(x,t;r):=\{(y,s)\in\mathbb{R}^{n+1}\mid s\le t,\ \Phi(x-y,t-s)\ge\frac{1}{r^n}\}$

Question assigned to the following page: 6

6.



$$u(x,t)=\frac{1}{4r^n}\iint_{E(x,t;r)} u(y,s)\,\frac{|x-y|^2}{(t-s)^2}\,dy\,ds\ ,\quad \underline{s\le t}$$

$u(x,t)$ should not depend upon future times.

$$\xrightarrow{\text{Shift } x=0\ \&\ t=0}\quad \phi(r):=\frac{1}{r^n}\iint_{\underbrace{E(r)}_{E(0,0;r)}} u(y,s)\,\frac{|y|^2}{s^2}\,dy\,ds$$

$$=\iint_{E(1)} u(ry,r^2s)\,\frac{|y|^2}{s^2}\,dy\,ds$$

$$\phi'(r)=\iint_{E(1)}\sum_{i=1}^n y_i\,u_{y_i}\,\frac{|y|^2}{s^2}+2r\,u_s\,\frac{|y|^2}{s}\,dy\,ds$$

$$=\frac{1}{r^{n+1}}\iint_{E(r)}\sum_{i=1}^n y_i\,u_{y_i}\,\frac{|y|^2}{s^2}+2u_s\,\frac{|y|^2}{s}\,dy\,ds$$

$$=:A+B$$

$$\Downarrow\qquad \psi:=-\frac{n}{2}\log(-4\pi s)+\frac{|y|^2}{4s}+n\log r\qquad (\underbrace{\psi=0 \text{ on } \partial E(r)}_{\phi(y,-s)=r^{-n}})$$

$$B=\frac{1}{r^{n+1}}\iint_{E(r)} 4u_s\sum_{i=1}^n y_i\,\psi_{y_i}\,dy\,ds$$

$$\underset{\text{Stoke thm}}{=}-\frac{1}{r^{n+1}}\iint_{E(r)} 4n\,u_s\,\psi+4\sum_{i=1}^n u_{s y_i}\,y_i\,\psi\,dy\,ds$$

Integrating by parts with respect to s

$$B=\frac{1}{r^{n+1}}\iint_{E(r)} -4n\,u_s\,\psi+4\sum_{i=1}^n u_{y_i}\,y_i\,\psi\,dy\,ds$$

$$=\frac{1}{r^{n+1}}\iint_{E(r)} -4n\,u_s\,\psi+4\sum_{i=1}^n u_{y_i}\,y_i\left(-\frac{n}{2s}-\frac{|y|^2}{4s^2}\right)dy\,ds$$

$$=\frac{1}{r^{n+1}}\iint_{E(r)} -4n\,u_s\,\psi-\frac{2n}{s}\sum_{i=1}^n u_{y_i}\,y_i\,dy\,ds-A$$

$$\phi'(r)=A+B$$

$$=\frac{1}{r^{n+1}}\iint_{E(r)} -4n\,\Delta u\,\psi-\frac{2n}{s}\sum_{i=1}^n u_{y_i}\,y_i\,dy\,ds$$

$$=\sum_{i=1}^n\frac{1}{r^{n+1}}\iint_{E(r)} 4n\,u_{y_i}\,\psi_{y_i}-\frac{2n}{s}\,u_{y_i}\,y_i\,dy\,ds$$

$$=0$$

$$\xrightarrow{\phi\text{ is constant}}\quad \phi(r)=\lim_{t\to 0}\phi(t)=u(0,0)\Big(\underbrace{\lim_{t\to 0}\frac{1}{t^n}\iint_{E(t)}\frac{|y|^2}{s^2}\,dy\,ds}_{=\iint_{E(1)}\frac{|y|^2}{s^2}\,dy\,ds}\Big)=4u(0,0)$$

Question assigned to the following page: 7

7、Suppose $(x_0, t_0) \in U_T$ with $u(x_0, t_0) = M := \max_{\bar{U}_T} u$

$r>0$, sufficiently small

$E(x_0, t_0; r)$

$$M = u(x_0, t_0) = \frac{1}{4r^n} \iint_{E(x_0, t_0; r)} u(y, s) \frac{|x_0 - y|^2}{(t_0 - s)^2} \, dy \, ds \le M$$

$$1 = \frac{1}{4r^n} \iint_{E(x_0, t_0; r)} \frac{|x_0 - y|^2}{(t_0 - s)^2} \, dy \, ds$$

$\Rightarrow u(y, s) = M$ for all $(y, s) \in \bar{E}(x_0, t_0; r)$