Homework 6

● Graded

Student

Boyuan Deng

Total Points

10 / 10 pts

Question 1

1 **1** / 1 pt

✔ **– 0 pts** Correct

Question 2

2 **1** / 1 pt

✔ **– 0 pts** Correct

Question 3

3 **1** / 1 pt

✔ **– 0 pts** Correct

Question 4

4 **1** / 1 pt

✔ **– 0 pts** Correct

Question 5

5 **1** / 1 pt

✔ **– 0 pts** Correct

Question 6

6 **1** / 1 pt

✔ **– 0 pts** Correct

Question 7

7 **1** / 1 pt

✔ **– 0 pts** Correct

Question 8

8 **1** / 1 pt

✔ **– 0 pts** Correct

Question 9

9 **1** / 1 pt

✔ **− 0 pts** Correct

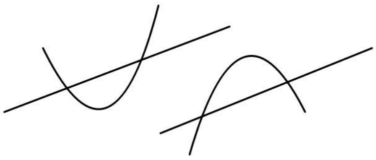
Question 10

10 **1** / 1 pt

✔ **− 0 pts** Correct

No questions assigned to the following page.

PDE
Homework 6
Due Date: March 29



Saman H. Esfahani

$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$

- 1. Let U be an open subset in $\mathbb{R}^n$ and $0 < \beta < \gamma \leq 1$. Prove the interpolation inequality:

$$\|u\|_{C^{0,\gamma}(U)} \leq \|u\|_{C^{0,\beta}(U)}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C^{0,1}(U)}^{\frac{\gamma-\beta}{1-\beta}}.$$

- 2. Let $p, q \in [1, \infty]$ with $1/p + 1/q = 1$. Then for all $f \in L^p(U)$ and $g \in L^q(U)$, we have $fg \in L^1$ and

$$\|fg\|_{L^1(U)} \leq \|f\|_{L^p(U)} \|g\|_{L^q(U)}.$$

- 3- Suppose a continuous function $u : U \subset \mathbb{R}^n \to \mathbb{R}$ is weakly differentiable with weak partial derivatives $\partial_{x_1} u, \ldots, \partial_{x_n} u$ which are continuous. Show that $u \in C^1(U)$. Local Lipschitz

Banach space

- 4- Show that $L^p(U)$ is a vector space and $\|\cdot\|_{L^p(U)}$ is a norm on $L^p(U)$ for any p, where $1 \leq p \leq \infty$.

5- Prove the following. Let I be a bounded open interval in $\mathbb{R}$. Let $u \in W^{1,p}(I)$. Then $u \in W_0^{1,p}(I)$ if and only if $u = 0$ on ∂I.
continuous extension

6- Let $1 \leq p < q < r \leq \infty$, then show that on any space

$$L^p \cap L^r \subset L^q.$$

7- ~~Let~~ $U \subset \mathbb{R}^n$ be an open subset. Is $C_0^\infty(U)$ dense in $L^2(U)$ with respect to the L^2-norm? (Prove or disprove)

compact

8- Let $U \subset \mathbb{R}^n$ be a connected open subset. Show that if $u \in C^{0,\gamma}(U)$ ~~with exponent $\gamma >$~~ 1, then u is constant.

9- Let $U \subset \mathbb{R}^n$ be a convex domain. Show that if $u \in C^1(U)$ and $D^\alpha u$ is bounded on U, for each $|\alpha| = 1$, then $u \in C^{0,1}(U)$.

10- The Rellich-Kondrachov Compactness Theorem says that $W^{1,p}(U)$ is compactly embedded into $L^q(U)$ for every $1 \leq q < p^*$. What about the case where $q = p^*$? Prove or find a counterexample.

Questions assigned to the following page: 1 and 2

1. $U \subseteq \mathbb{R}^n$, $0 < \beta < \gamma \le 1$

$$\|u\|_{C^{0,\gamma}(U)} \le \|u\|^{\frac{1-\gamma}{1-\beta}}_{C^{0,\beta}(U)} \|u\|^{\frac{\gamma-\beta}{1-\beta}}_{C^{0,1}(U)}$$

$$\frac{1-\gamma}{1-\beta} + \frac{\gamma-\beta}{1-\beta} = 1$$

$$\|u\|_{C^{0,\gamma}(U)} = \sup_{x\in U} |u(x)| + \sup_{\substack{x,y\in U \\ x\neq y}} \left\{ \frac{|u(x)-u(y)|}{|x-y|^\gamma} \right\}$$

$$(1-t)\beta + t = \gamma$$

$$|u| = |u|^{1-t} |u|^t$$

$$\frac{|u(x)-u(y)|}{|x-y|^\gamma} = \left(\frac{|u(x)-u(y)|}{|x-y|^\beta}\right)^{1-t} \left(\frac{|u(x)-u(y)|}{|x-y|}\right)^t$$

$$\le [u]_\beta^{1-t} [u]_1^t$$

$$\|u\|_\gamma \le \underbrace{|u|_\infty^{1-t} |u|_\infty^t}_{\text{max norm}} + \underbrace{[u]_\beta^{1-t} [u]_1^t}_{\text{Hölder constant}}$$

Motivation:

$$(|u|_\infty + [u]_\beta)^{1-t} \rightarrow \|u\|_\beta^{1-t}$$

$$(|u|_\infty + [u]_1)^t \rightarrow \|u\|_1^t$$

$$= (|u|_\infty + [u]_\beta)\left[\left(\frac{|u|_\infty}{|u|_\infty + [u]_\beta}\right)^{1-t}\left(\frac{|u|_\infty}{|u|_\infty}\right)^t\right.$$

$$\left. + \left(\frac{[u]_\beta}{|u|_\infty + [u]_\beta}\right)^{1-t}\left(\frac{[u]_1}{[u]_\beta}\right)^t\right]$$

$$\le (|u|_\infty + [u]_\beta)\left(\frac{|u|_\infty + [u]_1}{|u|_\infty + [u]_\beta}\right)^t$$

$$= \underbrace{(|u|_\infty + [u]_\beta)^{1-t}}_{\Downarrow \ \|u\|_\beta^{1-t}} (|u|_\infty + [u]_1)^t \quad \|u\|_1^t$$

We then prove that

$$\|u\|_{C^{0,\gamma}(U)} \le \|u\|^{\frac{1-\gamma}{1-\beta}}_{C^{0,\beta}(U)} \|u\|^{\frac{\gamma-\beta}{1-\beta}}_{C^{0,1}(U)}$$

2. $p, q \in [1,\infty]$ $\quad \frac{1}{p} + \frac{1}{q} = 1$

$f \in L^p(U)$, $g \in L^q(U)$, $fg \in L^1(U)$

From Young's Inequality $ab \le \frac{a^p}{p} + \frac{b^q}{q}$

Want to prove

$$\|fg\|_{L^1(U)} \le \|f\|_{L^p(U)} \|g\|_{L^q(U)}$$

$$\frac{\|fg\|_{L^1(U)}}{\|f\|_{L^p(U)}\|g\|_{L^q(U)}} \le 1 \quad \frac{1}{p} + \frac{1}{q}$$

$$\|f\|_p = \left(\sum_{i=1}^n |f_i|^p\right)^{1/p} \qquad \|fg\|_1 = \sum_{i,k=1}^n |f_i g_k| = \int f(x) g(x)\,dx \cdots ①$$

Just like: $\|f\|_p^p = \sum_{i=1}^n |f_i|^p = \int |f(x)|^p dx \cdots ②$

$$\|g\|_q^q = \sum_{i=1}^n |g_i|^q = \int |g(x)|^q dx \cdots ③$$

Define $a = \frac{|f(x)|}{\|f(x)\|_p}$ $\quad b = \frac{|g(x)|}{\|g(x)\|_q}$

$= \|fg\|_1$

$$\int \frac{|f(x)|}{\|f(x)\|_p} \cdot \frac{|g(x)|}{\|g(x)\|_q} = \int \frac{\cancel{|f(x)|^p}}{p\,\cancel{\|f(x)\|_p^p}} + \int \frac{\cancel{|g(x)|^q}}{q\,\cancel{\|g(x)\|_q^q}}$$

② ③ ③

$$\frac{1}{p} + \frac{1}{q} = 1$$

Then we prove

$$\|fg\|_{L^1(U)} \le \|f\|_{L^p(U)} \|g\|_{L^q(U)}$$

Questions assigned to the following page: 3 and 4

3. $u: U \subset \mathbb{R}^n \to \mathbb{R}$ weakly differentiable $\partial_{x_1} u \cdots \partial_{x_n}$

if $u \in C^1(U)$ local Lipschitz ?

First we assume $u \in C^1(U)$

$\Rightarrow \partial_{x_i} u := v$ since weak partial derivative is continuous

Assume there is a infinitely differentiable function $\phi \in C_c^\infty(U)$, since $u \in C^1(U)$, $\phi \in C_c^\infty(U)$:

$$\int_U u \phi_{x_i} dx = -\int_U u_{x_i} \phi \, dx + \int_{\partial U} \cancel{u\phi}^{\,0} \nu^i ds$$

ϕ is vanishing on boundary

$$\int_U u \phi_{x_i} dx = -\int_U \underbrace{u_{x_i}}_{v} \phi \, dx$$

$$\Downarrow$$

$$\int_U u \phi_{x_i} dx = -\int_U v \phi \, dx \quad \cdots \star$$

The LHS might work in some cases even if $u \notin C^1(U)$, but the RHS only has meaning when $u \in C^1(U)$, where $\partial_{x_i} u = v$. For higher order derivative k, RHS only works when $u \in C^k(U)$

4. By defination
$$\|x\|_{L^p(U)} = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$

$\boxed{\text{When } p = 0}$

$$\Rightarrow \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} = \left(|x_1|^0 + |x_2|^0 + \cdots + |x_n|^0\right)^{1/0}$$

However, if we scale x_i by $\lambda \in C$
$\|\lambda x\| \neq |\lambda| \, \|x\|$, which implies non-homogeneous

$\boxed{\text{When } 0 < p < 1}$

To prove triangle inequality, we first prove Hölder inequality
Let's define $q \in [1, \infty)$
However, $\frac{1}{p} + \frac{1}{q} \neq 1$ since $\frac{1}{p} > 1$, $\frac{1}{q} > 0$

$\boxed{\text{When } p \geq 1}$

$$\|x\|_{L^p(U)} = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \geq 0 \checkmark \cdots ①$$

$$\|\lambda x\|_{L^p(U)} = \left(\sum_{i=1}^n |\lambda x_i|^p\right)^{1/p} = \left(\sum_{i=1}^n |\lambda|^p |x_i|^p\right)^{1/p} = |\lambda| \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \checkmark \cdots ②$$

$$\overset{①}{\Longrightarrow} \|x\|_{L^p(U)} \geq 0, \ \|x\|_{L^p(U)} = 0 \text{ iff } x = 0 \checkmark \cdots ③$$

Since $p \geq 1$, Hölder inequality $\Rightarrow$

$$\sum_{i=1}^n |a_i b_i| \leq \left(\sum_{i=1}^n |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |b_i|^p\right)^{\frac{1}{p}}$$

$$\sum_{i=1}^n |a_i + b_i|^p = \sum_{i=1}^n |a_i + b_i| \, |a_i + b_i|^{p-1}$$

$$\leq \sum_{i=1}^n |a_i| \, |a_i + b_i|^{p-1} + \sum_{i=1}^n |b_i| \, |a_i + b_i|^{p-1}$$

$q := \frac{p}{p-1} > 1$

$$\leq \left(\sum_{i=1}^n |a_i|^p\right)^{1/p} \left(\sum_{i=1}^n |a_i + b_i|^{q(p-1)}\right)^{1/q} + \left(\sum_{i=1}^n |b_i|^p\right)^{1/p} \left(\sum_{i=1}^n |a_i + b_i|^{\overset{p}{q(p-1)}}\right)^{1/q}$$

$$= \left[\left(\sum_{i=1}^n |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^n |b_i|^p\right)^{1/p}\right] \left(\sum_{i=1}^n |a_i + b_i|^p\right)^{1/q}$$

$$\frac{\sum_{i=1}^n |a_i + b_i|^p}{\left(\sum_{i=1}^n |a_i + b_i|^p\right)^{1/q}} \leq \left(\sum_{i=1}^n |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^n |b_i|^p\right)^{1/p}$$

$$\left(\sum_{i=1}^n |a_i + b_i|^p\right)^{1 - \frac{1}{q} = \frac{1}{p}} \leq \left(\sum_{i=1}^n |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^n |b_i|^p\right)^{1/p}$$

$$\left(\sum_{i=1}^n |a_i + b_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^n |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^n |b_i|^p\right)^{1/p}$$

we then prove triangle inequality

So $L^p(U)$ is a vector space and norm $\|\cdot\|_{L^p(U)}$ is a norm for any $1 \leq p \leq \infty$

Questions assigned to the following page: 6 and 5

5. Trace zero function P273

I interval $\to \partial I$?

$\partial I \longrightarrow$ continuous extension (vanishes on the boundary)

By Trace defination, we have
$T: W^{1,p}(I) \longrightarrow L^p(\partial I)$

Since $u \in W^{1,p}(I)$, and $\partial I \in C'$
Then we have
$$Tu = 0 \quad \text{on } \partial I$$

$$\begin{cases} u \in W^{1,p}(\mathbb{R}_+), \text{ with compact support in } \mathbb{R}^+ \\ Tu = 0 \quad \text{on } \partial\mathbb{R}_+ \end{cases}$$

Since $Tu=0$, $\exists\ u_m \in C'(\bar{\mathbb{R}}_+)$ s.t.
$$\begin{cases} u_m \to u & \text{in } W^{1,p}(\mathbb{R}_+) \\ Tu_m = u_m|_{\partial\mathbb{R}} \to 0 & \text{in } L^p(\partial\mathbb{R}^+) \end{cases}$$

x' is on the boundary

If $x' \in \partial\mathbb{R}^+$, $x_n \geq 0$, we have
$$\int |u_m(x', x_n)| \leq \int |u_m(x', 0)| + \int_0^{x_n} |u_{m,x_n}(x', t)|\,dt$$
$$\Downarrow$$
$$\int_{\partial\mathbb{R}} |u(x', x_n)|^p dx' \leq C x_n^{p-1} \int_0^{x_n} \int_{\partial\mathbb{R}} |Du|^p dx' dt \quad \cdots ①$$

$\xi \in C^\infty(\mathbb{R}_+)$, $\xi \equiv 1 \quad [0,1]$
$\xi \equiv 0 \quad \mathbb{R}^+ - [0,2]$ vanish on the boundary

$$\begin{cases} \xi_m(x) := \xi(m x_n) \\ w_m := u(x)(1-\xi_m) \end{cases} \Rightarrow \begin{cases} w_{m,x_n} = u_{x_n}(1-\xi_m) - m u \xi' \\ D_{x'} w_m = D_{x'} u (1-\xi_m) \end{cases}$$
$$\Downarrow ①$$
$$\Rightarrow \int_{\mathbb{R}} |Dw_m - Du|^p dx \leq C \int_{\mathbb{R}} |\xi_m|^p |Du|^p dx + C m^p \int_0^{\frac{2}{m}} |\xi_m|^p \int_{\partial\mathbb{R}} |u|^p dx' dt$$
$$=: A + B$$

$A \to 0$ as $m \to \infty$
$$B \leq C m^p \int_0^{\frac{2}{m}} t^{p-1} dt \int_0^{\frac{2}{m}} \int_{\partial\mathbb{R}} |Du|^p dx' dt$$
$$\leq C \int_0^{\frac{2}{m}} \int_{\partial\mathbb{R}} |Du|^p dx' dx_n \to 0 \quad \text{as } m \to \infty$$

$w_m \to u$ in $W^{1,p}(\mathbb{R})$

Hence $u \in W_0^{1,p}(I)$

6. Assume that $u \in L^p(U) \cap L^r(U) \cdots ①$

We want to prove
$$\|u\|_{L^q(U)} \leq \|u\|_{L^p(U)}^{\theta} \|u\|_{L^r(U)}^{1-\theta}$$

Since $1 \leq p < q < r \leq \infty$, by convexity
$$\Rightarrow \frac{1}{q} = \frac{\theta}{p} + \frac{(1-\theta)}{r} \cdots ②$$
$$\int_U |u|^q dx = \int_U |u|^{\theta q} |u|^{(1-\theta)q} dx$$
$$\leq \left(\int_U |u|^{\theta q \frac{p}{\theta q}} dx\right)^{\frac{\theta q}{p}} \left(\int_U |u|^{(1-\theta)q \frac{r}{(1-\theta)q}} dx\right)^{\frac{(1-\theta)q}{r}}$$

From ① $\quad 1 = \frac{\theta q}{p} + \frac{(1-\theta)q}{r}$

$\Rightarrow$ which implies Hölder inequality

For the RHS, we could get that $u \in L^q(U)$

With assumption ①

$$L^p(U) \cap L^r(U) \in L^q(U)$$

Questions assigned to the following page: 7 and 8

7. $C_0^\infty(U)$ dense in $L^2(U)$, $U \subset \mathbb{R}^n$

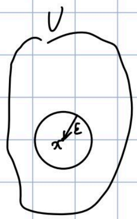
$\rightarrow C_0^\infty(U) \cap L^2(U) \subseteq L^2(U)$

Motivation :

converge in $L^2(U)$, compactness

Define $\eta \in C_0^\infty(\mathbb{R}^n)$, $\rightarrow \eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right)$ dimension

$f \in C_0^\infty(\mathbb{R}^n) \xrightarrow{V \subset\subset U}$



Goal: $u_m^\varepsilon \to u_m$ in $L^2(V)$ as $\varepsilon \to 0$

uniformly in set m

From $f_\varepsilon = f * \eta_\varepsilon$ fix (x)

$$|f^\varepsilon(x) - f(x)|^2 = \frac{1}{\varepsilon^{2n}}\int_{B_\varepsilon(x)} \eta^2\left(\frac{x-z}{\varepsilon}\right)(f(z)-f(x))^2 \, dz$$

$y = \frac{x-z}{\varepsilon}$ change of variable

$$= \int_{B_1(0)} \eta^2(y)\big(\underbrace{f(x-\varepsilon y)}_{t=1} - \underbrace{f(x)}_{t=0}\big)^2 \, dy$$

line segmentation

$$= \int_{B_1(0)} \eta^2(y)\left(\int_0^1 \frac{d}{dt}(f(x-\varepsilon t y))\,dt\right)^2 dy$$

$$= \varepsilon^2 \int_{B_1(0)} \eta^2(y) \left(\int_0^1 Df(x-\varepsilon t y)\cdot y \, dt\right)^2 dy$$

Integrate both side

$$\Rightarrow \int_V |f^\varepsilon(x) - f(x)|^2 dx \le \varepsilon^2 \int_{B_1(0)} \eta^2(y) \int_0^1 \int_V |Df_m(x-\varepsilon t y)|^2 \, dx\, dt\, dy$$

$$\le \varepsilon^2 \int_V |Df_m(z)|^2 dz$$

By taking limit $\varepsilon \to 0$

$$\|f^\varepsilon(x) - f(x)\|_{L^2(V)} \to 0$$

∴ So that $C_0^\infty(U)$ is dense in $L^2(U)$

8. $u \in C^{0,\gamma}(U)$, $U \subset \mathbb{R}^n$

For 2 points $x > y$

$$|u(x) - u(y)| \le C|x-y|^\gamma$$

$$\underbrace{\lim_{x\to y}\left|\frac{u(x)-u(y)}{x-y}\right|}_{\Downarrow\ u'(s)} \le \underbrace{\lim_{x\to y} C|x-y|^{\gamma-1}}_{\to 0} \qquad \gamma > 1$$

This implies that $u'(s) = 0$ everywhere

So that u is a constant

Questions assigned to the following page: 9 and 10

9. $u \in C^1(U)$, $D^\alpha u$ is bounded on U for $|\alpha|=1$, then $u \in C^{0,1}(U)$

$U \subset \mathbb{R}^n$ a convex domain $\Rightarrow$ for $0 \le \lambda \le 1$, $\lambda x + (1-\lambda) y$ also in U

Since $D^\alpha u$ is bounded , then $\exists C > 0$, $\forall x \in U$

for $|\alpha|=1$, $|D^\alpha u(x)| \le C \cdots ①$

By mean value thm:

$D^\alpha u(x) \Rightarrow$

$$|u(y) - u(x)| = |\nabla u(z)(y-x)|$$

$$\le \underbrace{|\nabla u(z)|}_{\text{from } ①} \, |y-x|$$

$$\le C\sqrt{n}\, |y-x|$$ Local Lipschitz

$\mathbb{R}^n$

$\Downarrow$

This implies Hölder continuous with exponent $\gamma = 1$ which $u \in C^{0,1}(U)$

10. The thm assume that $U \subset \mathbb{R}^n$ (bounded) , ∂U is C^1 , $1 \le p < n$

$W^{1,p}(U) \subset\subset L^q(U)$

Assume that $u \in W^{1,p}(U)$, this implies that there is a subsequence u_i converges in $L^q(U)$

$(|u_i|^q + |u_{i+1}|^q + \cdots)^{1/q}$

$\Rightarrow u_i^\varepsilon \to u_i$ in $L^q(V)$

compact support in some bounded open set $V \subset \mathbb{R}^n$

$u^\varepsilon : \eta_\varepsilon * u$, $0 < \varepsilon \le 1$

inlet vector

$$|u_i^\varepsilon - u_i| = \frac{1}{\varepsilon^n} \int_{B_\varepsilon(x)} \eta\left(\frac{x-z}{\varepsilon}\right)(u_i(z) - u_i(x))\, dz$$

$\Downarrow$

subsequence

$u_i^\varepsilon(x) = u\left(\frac{x}{\varepsilon}\right)$

$p^* = q \Rightarrow$ same power

$W^{1,p}$: $\left(\sum \int_U |D^\alpha u|^p\right)^{1/p}$ L^q: $\left(\sum_{i=1}^{n} |u|^q\right)^{1/q}$

$$\|u_i^\varepsilon\| = \varepsilon^{\frac{n}{q}} \|u\|_q$$

$$\|Du_i^\varepsilon\| = \varepsilon^{\frac{n-p}{p}} \|D^{|\alpha|} u\|_p$$ $|\alpha| \le p$

$\Downarrow$

$\varepsilon^{\underbrace{f(p^*)}_{\text{function of } p^*}} u\left(\frac{x}{\varepsilon}\right) \in W^{1,p}$

Non-zero

Non-zero norm in L^{p^*} $\to$ no convergence in $L^q(V)$