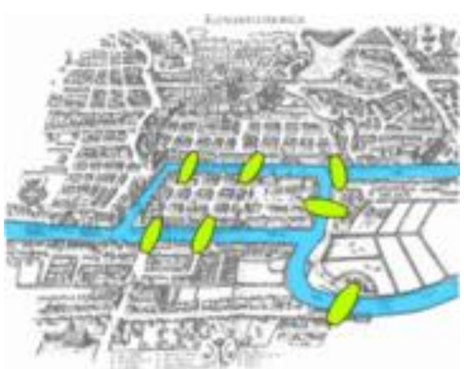
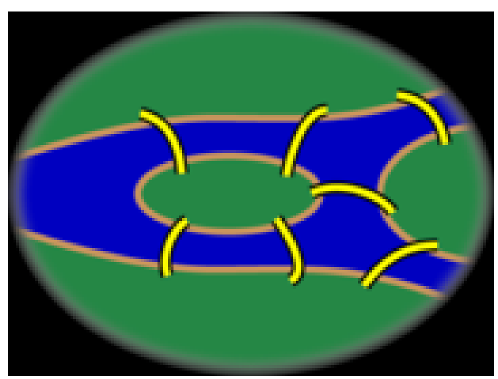
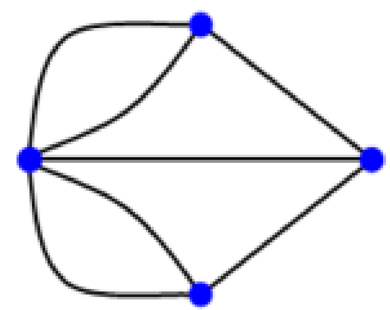
# INTRODUCTION TO **GRAPH THEORY**

# Graph Theory - History

Leonhard Euler's paper on "*Seven Bridges of Königsberg*" ,
published in 1736.







# Famous problems

- “The traveling salesman problem”
  - A traveling salesman is to visit a number of cities; how to plan the trip so every city is visited once and just once and the whole trip is as short as possible ?

# Famous problems

In 1852 Francis Guthrie posed the “four color problem” which asks if it is possible to color, using only four colors, any map of countries in such a way as to prevent two bordering countries from having the same color.

This problem, which was only solved a century later in 1976 by Kenneth Appel and Wolfgang Haken, can be considered the birth of graph theory.

# Examples

- Cost of wiring electronic components
- Shortest route between two cities.
- Shortest distance between all pairs of cities in a road atlas.
- Matching / Resource Allocation
- Task scheduling
- Visibility / Coverage

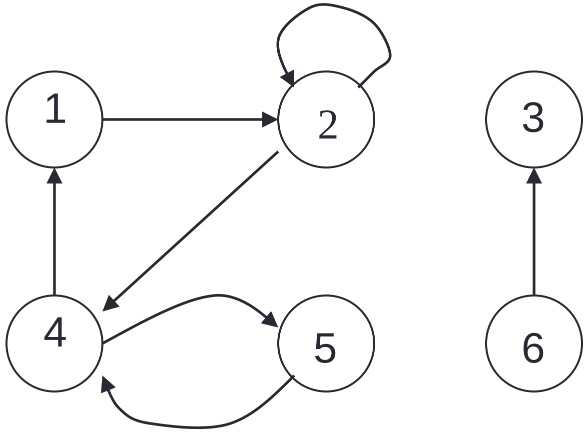
# Examples

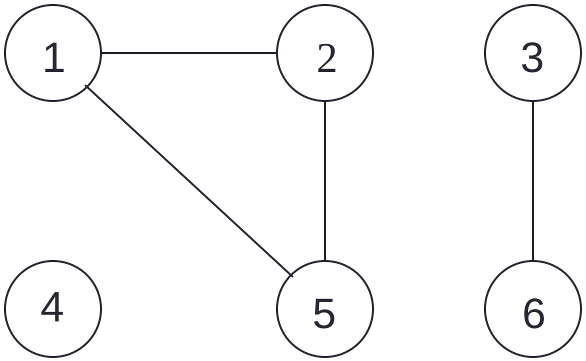
- Flow of material
  - liquid flowing through pipes
  - current through electrical networks
  - information through communication networks
  - parts through an assembly line
- In Operating systems to model resource handling (deadlock problems)
- In compilers for parsing and optimizing the code.

# BASICS

# What is a Graph?

- Informally a *graph* is a set of nodes joined by a set of lines or arrows.



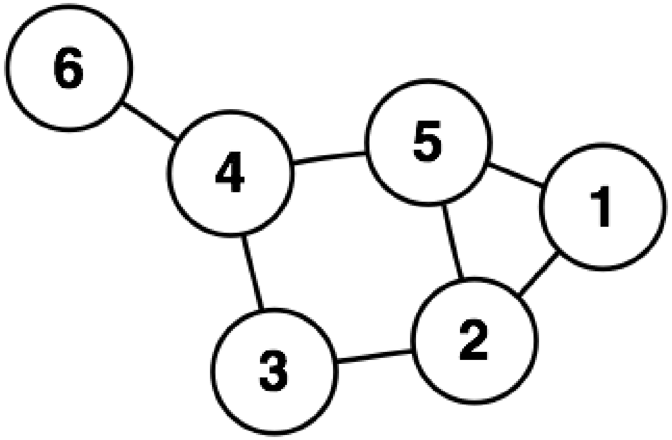
# Definition: Graph

- G is an ordered triple G:=(V, E, f)
  - V is a set of nodes, points, or vertices.
  - E is a set, whose elements are known as edges or lines.
  - f is a function
    - maps each element of E
    - to an unordered pair of vertices in V.

# Definitions

- Vertex
  - Basic Element
  - Drawn as a *node* or a *dot*.
  - V**ertex set** of *G* is usually denoted by *V*(*G*), or *V*
- Edge
  - A set of two elements
  - Drawn as a line connecting two vertices, called end vertices, or endpoints.
  - The edge set of G is usually denoted by E(G), or E.

# Example



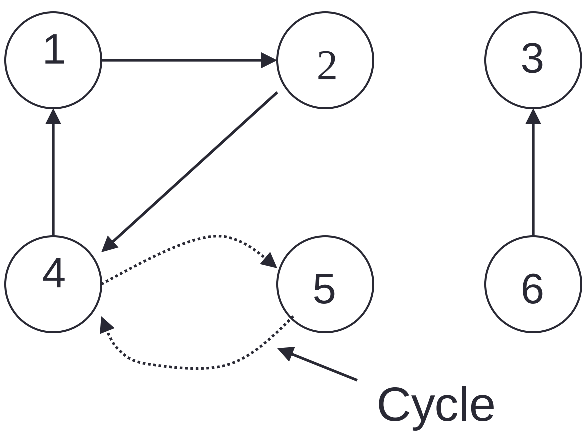

- V:={1,2,3,4,5,6}
- E:={{1,2},{1,5},{2,3},{2,5},{3,4},{4,5},{4,6}}

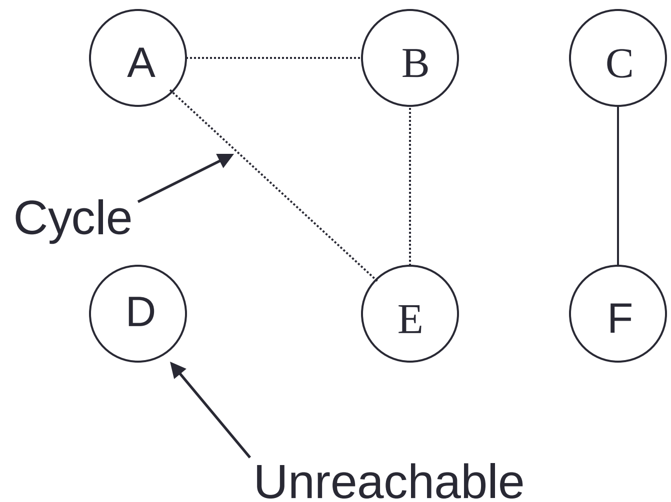
# Simple Graphs

*Simple graphs* are graphs without multiple edges or self-loops.

# Path

- A *path* is a sequence of vertices such that there is an edge from each vertex to its successor.
- A path is ***simple*** if each vertex is distinct.



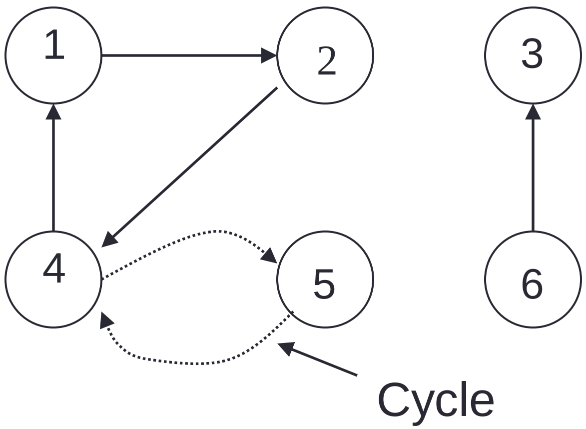

**Simple path from 1 to 5**
**= [ 1, 2, 4, 5 ]**

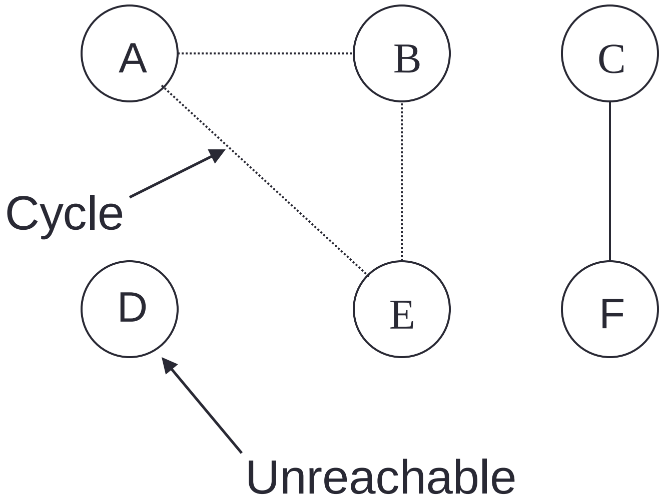
Our text's alternates the vertices and edges.

If there is path *p* from *u* to *v* then we say *v* is **reachable** from *u* via *p*.

# Cycle

- A path from a vertex to itself is called a ***cycle***.
- A graph is called ***cyclic*** if it contains a cycle;
  - otherwise it is called ***acyclic***



# Connectivity

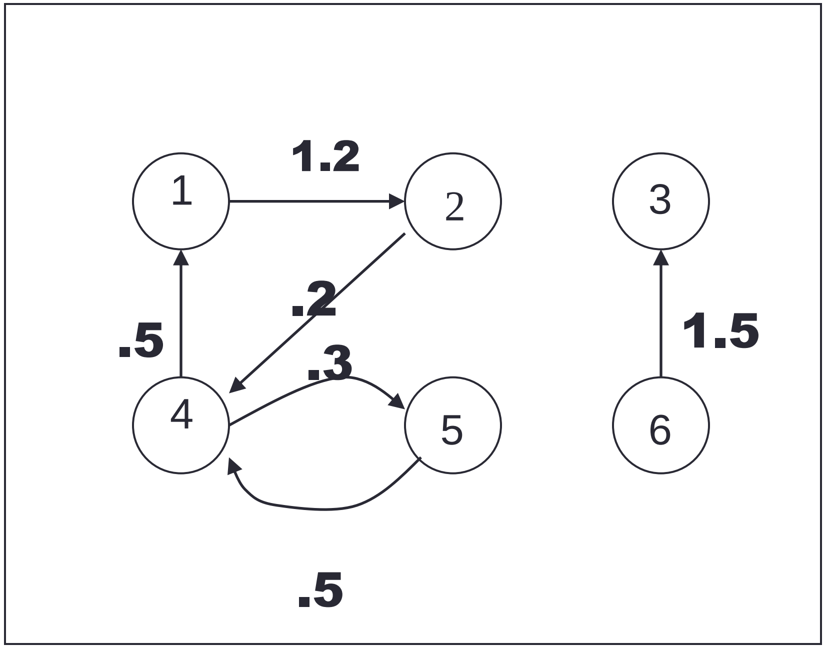
- is ***connected*** if
  - you can get from any node to any other by following a sequence of edges OR
  - any two nodes are connected by a path.
- A directed graph is ***strongly connected*** if there is a directed path from any node to any other node.

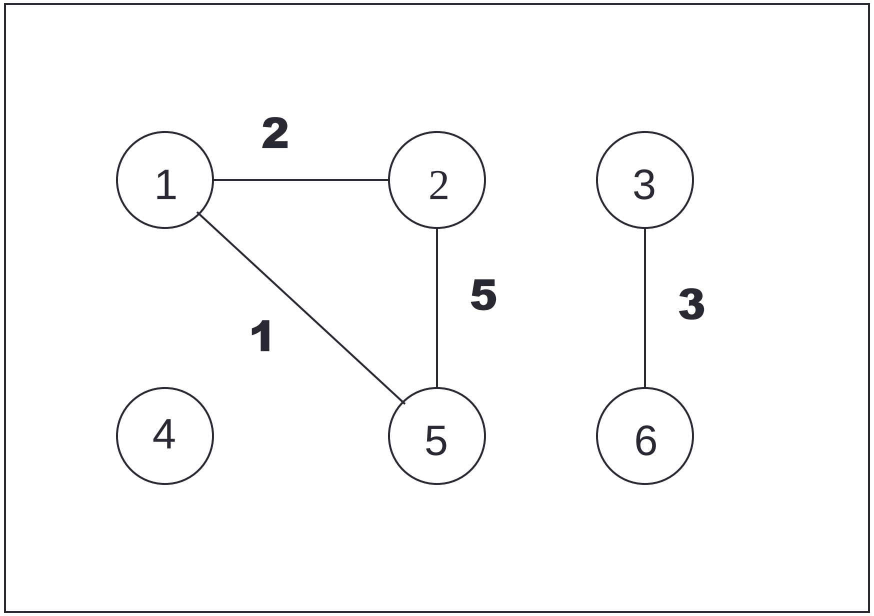
# Sparse/Dense

- A graph is ***sparse*** if $|E| \approx |V|$
- A graph is ***dense*** if $|E| \approx |V|^2$.

# A ***weighted graph***

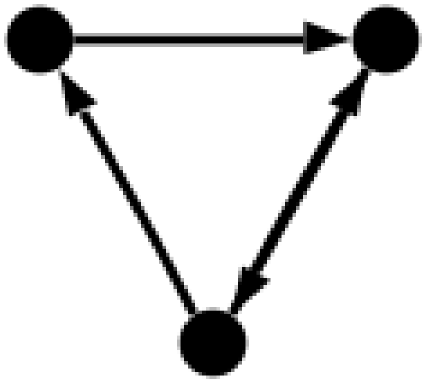
- is a graph for which each edge has an associated ***weight***, usually given by a ***weight function*** *w: E* $\rightarrow$ **R**.



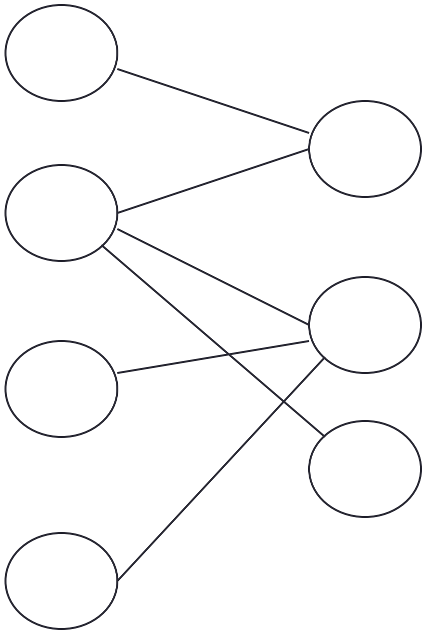
# Directed Graph (digraph)

- Edges have directions
  - An edge is an *ordered* pair of nodes



# Bipartite graph

- $V$ can be partitioned into 2 sets $V_1$ and $V_2$ such that $(u,v) \in E$ implies
  - either $u \in V_1$ and $v \in V_2$
  - OR $v \in V_1$ and $u \in V_2$.

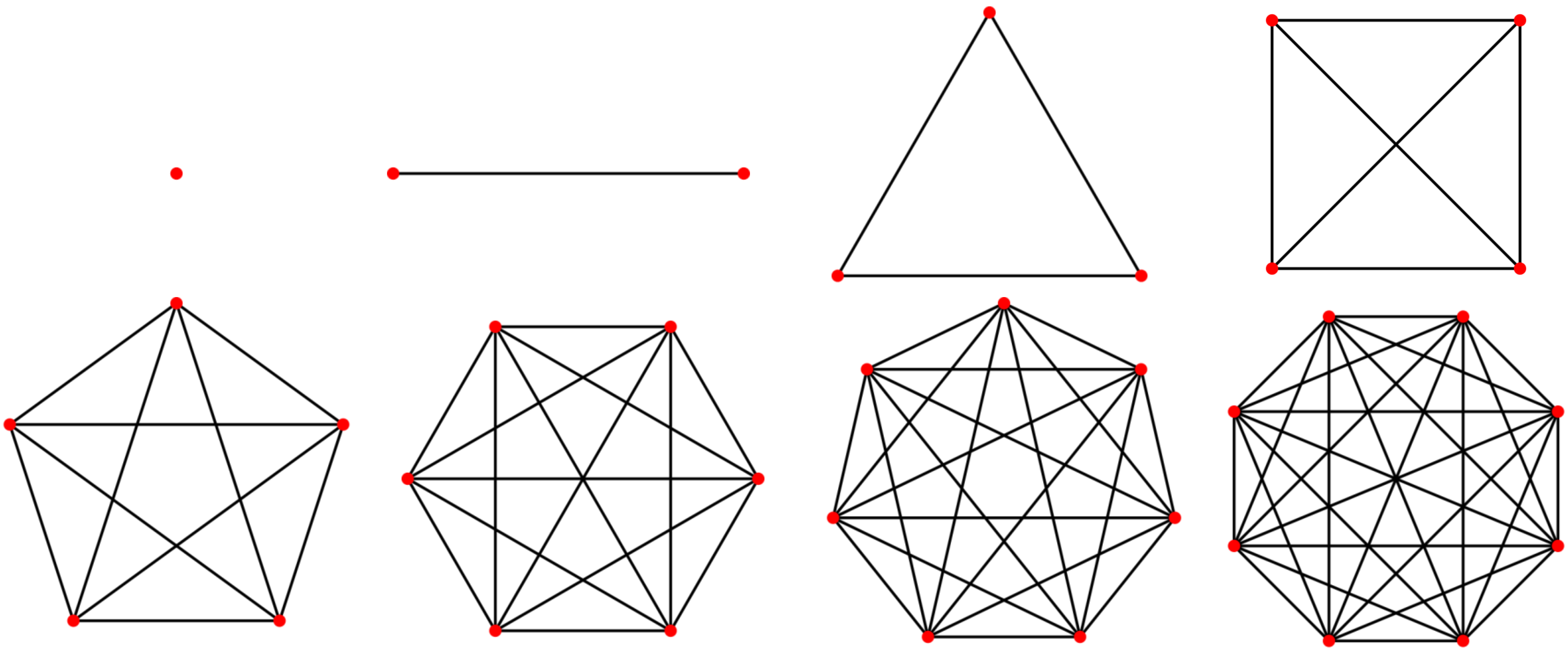


# Special Types

- Empty Graph / Edgeless graph
  - No edge
- Null graph
  - No nodes
  - Obviously no edge

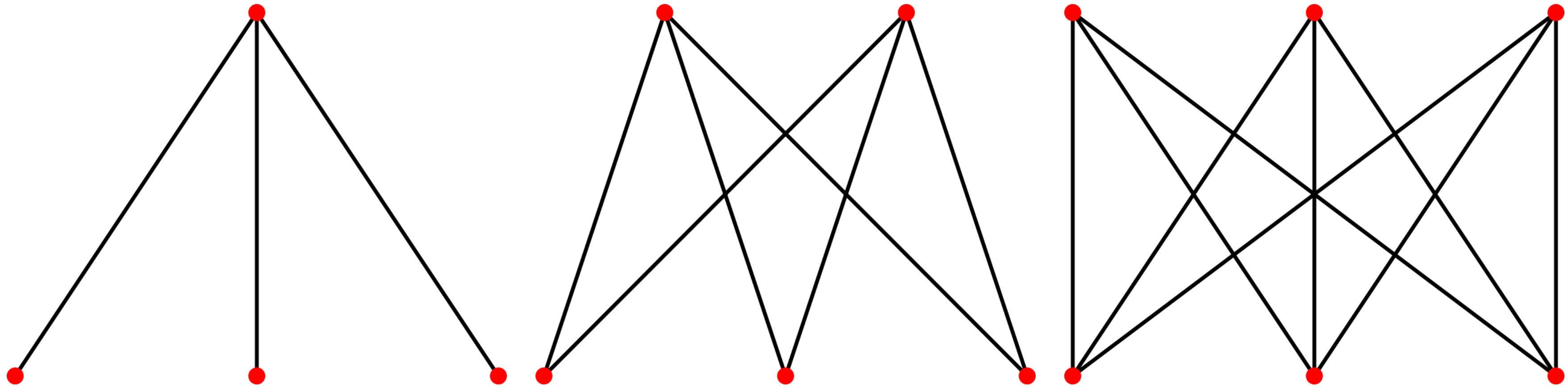
# Complete Graph

- Denoted $K_n$
- Every pair of vertices are adjacent
- Has n(n-1) edges

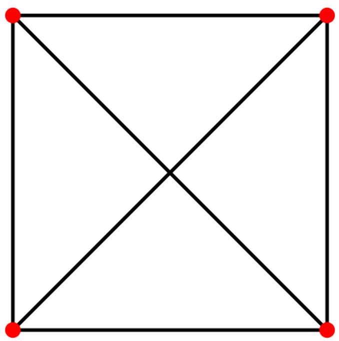


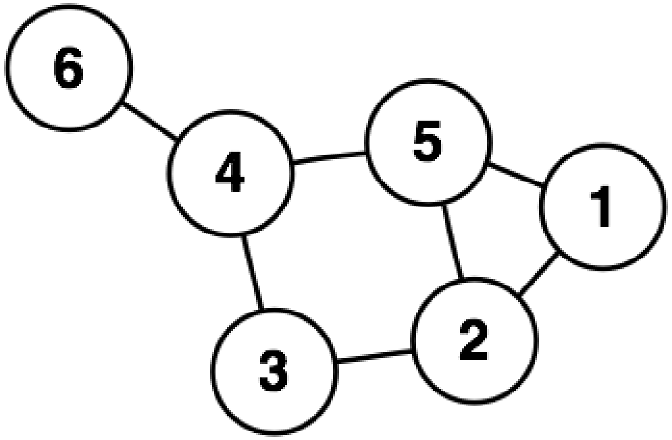
# Complete Bipartite Graph

- Bipartite Variation of Complete Graph
- Every node of one set is connected to every other node on the other set



# Planar Graph



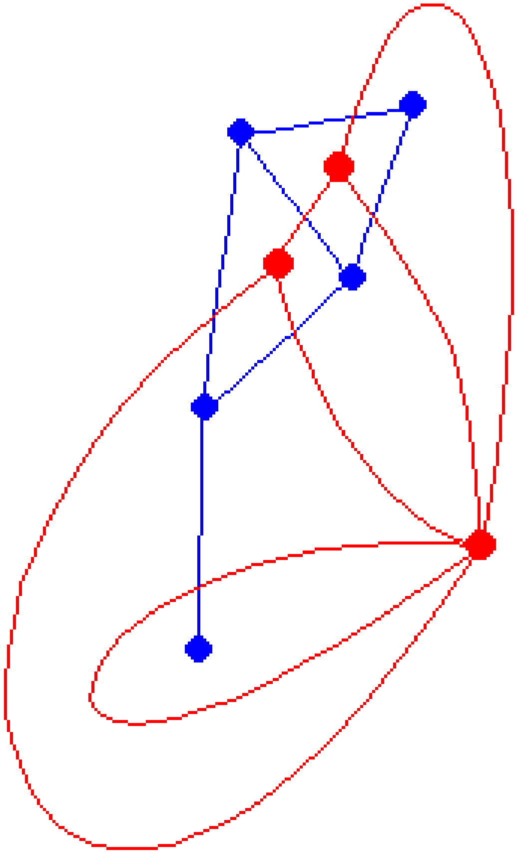



- Can be drawn on a plane such that no two edges intersect
- $K_4$ is the largest complete graph that is planar

# Dual Graph

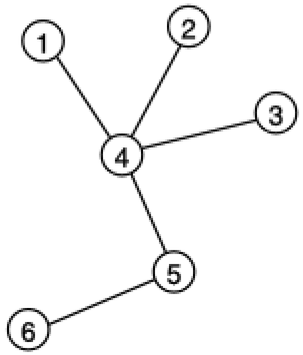
- Faces are considered as nodes
- Edges denote face adjacency

- Dual of dual is the original graph



# Tree

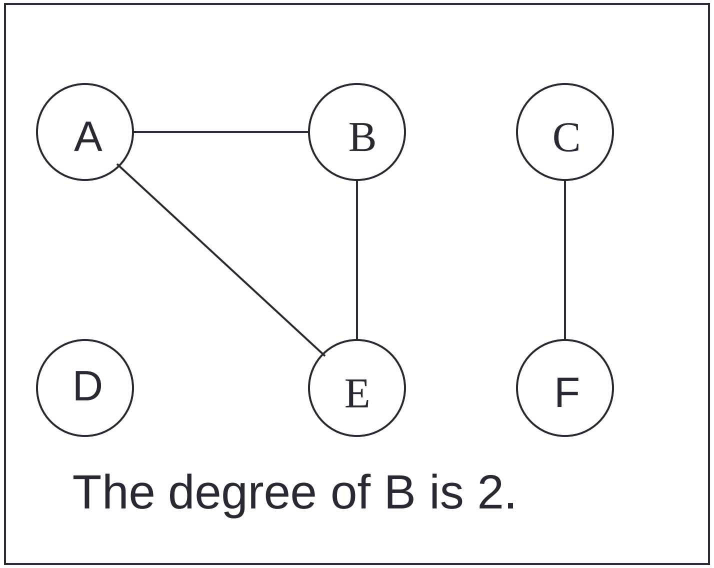
- Connected Acyclic Graph
- Two nodes have *exactly* one path between them



# Generalization: Hypergraph

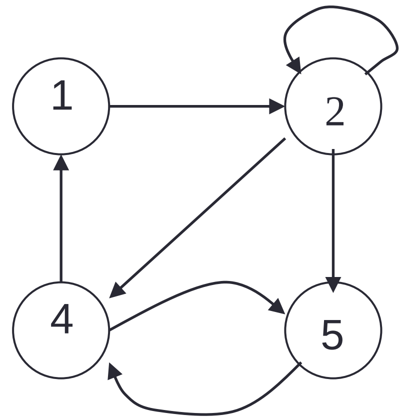
- Generalization of a graph,
  - edges can connect any number of vertices.
- Formally, an hypergraph is a pair (X,E) where
  - X is a set of elements, called nodes or vertices, and
  - E is a set of subsets of X, called hyperedges.
- Hyperedges are arbitrary sets of nodes,
  - contain an arbitrary number of nodes.

# Degree




- Number of edges incident on a node

# Degree (Directed Graphs)




The in degree of 2 is 2 and the out degree of 2 is 3.

- In degree: Number of edges entering
- Out degree: Number of edges leaving

- Degree = indegree + outdegree

# Degree: Simple Facts

- If $G$ is a digraph with $m$ edges, then

$$\sum \text{indeg}(v) = \sum \text{outdeg}(v) = m = |E|$$

- If $G$ is a graph with $m$ edges, then

$$\sum \text{deg}(v) = 2m = 2\,|E|$$

  - Number of Odd degree Nodes is even

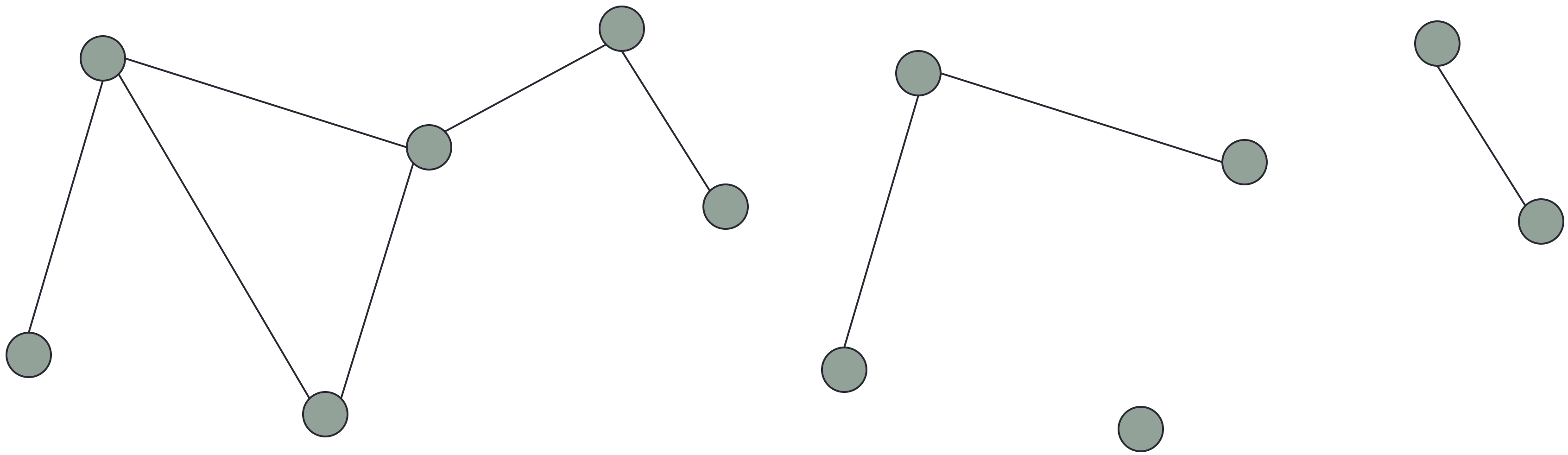
# SUBGRAPHS

# Subgraph

- Vertex and edge sets are subsets of those of G
  - a *supergraph* of a graph G is a graph that contains G as a subgraph.
- A graph G contains another graph H if some subgraph of G
  - is H or
  - is isomorphic to H.
- H is a *proper subgraph* if H!=G

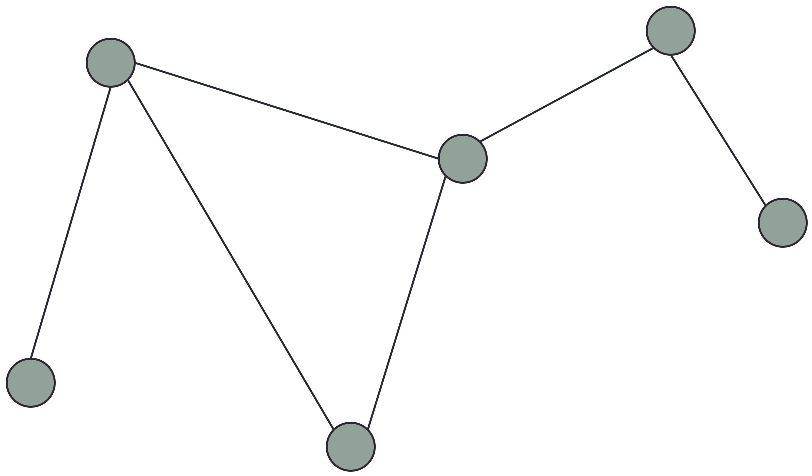
# Spanning subgraph

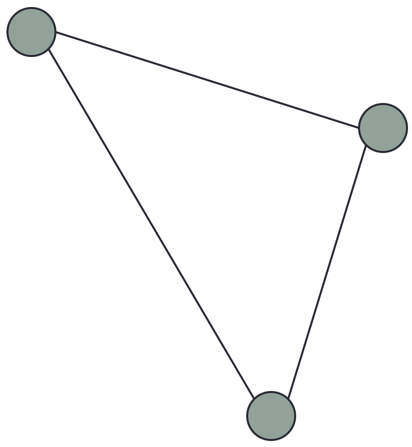
- Subgraph H has the same vertex set as G.
  - Possibly not all the edges
  - “H spans G”.



# Induced Subgraph

- For any pair of vertices x and y of H, xy is an edge of H if and only if xy is an edge of G.
  - H has the most edges that appear in G over the same vertex set.



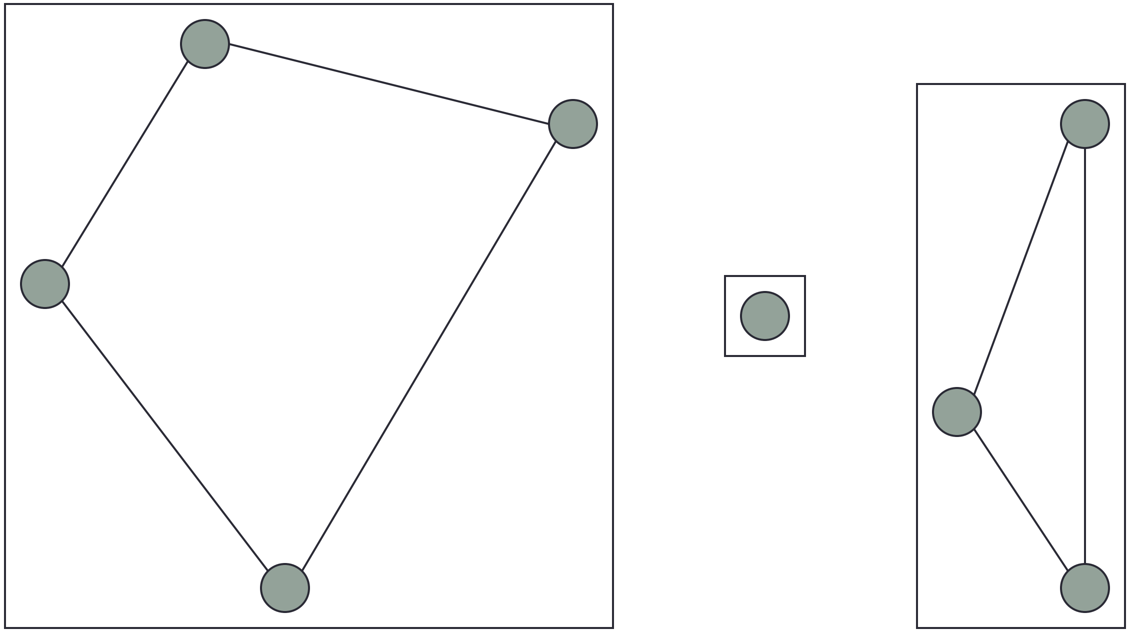


# Induced Subgraph (2)

- If H is chosen based on a vertex subset S of V(G), then H can be written as G[S]
  - “induced by S”
- A graph that does not contain H as an induced subgraph is said to be *H-free*

# Component

- Maximum Connected sub graph



# ISOMORPHISM

# Isomorphism

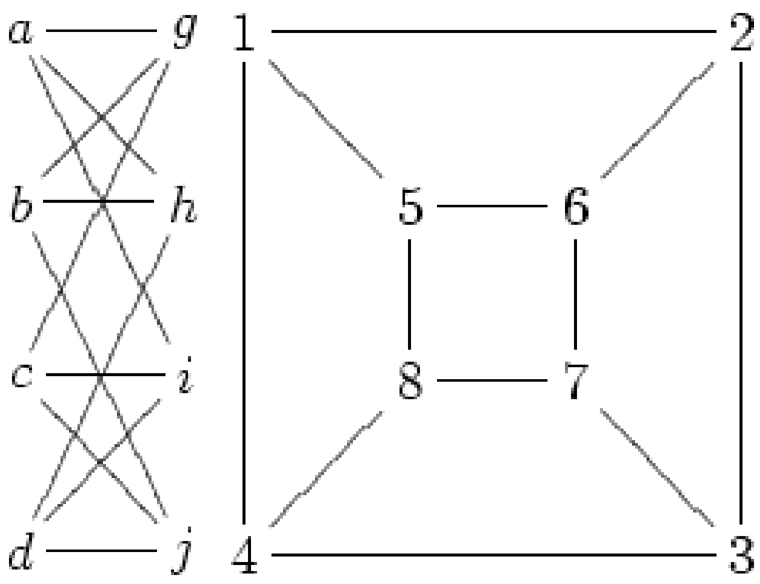
- Bijection, i.e., a one-to-one mapping:

  $f : V(G) \to V(H)$

  u and v from G are adjacent if and only if f(u) and f(v) are adjacent in H.
- If an isomorphism can be constructed between two graphs, then we say those graphs are ***isomorphic***.

# Isomorphism Problem

- Determining whether two graphs are isomorphic
- Although these graphs look very different, they are isomorphic; one isomorphism between them is




f(a) = 1  f(b) = 6  f(c) = 8  f(d) = 3

f(g) = 5  f(h) = 2  f(i) = 4  f(j) = 7

# GRAPH
# ABSTRACT DATA TYPE

# Graph ADT

- In computer science, a graph is an abstract data type (ADT)
- that consists of
  - a set of nodes and
  - a set of edges
    - establish relationships (connections) between the nodes.
- The graph ADT follows directly from the graph concept from mathematics.

# Representation (Matrix)

- Incidence Matrix
  - E x V
  - [edge, vertex] contains the edge's data
- Adjacency Matrix
  - V x V
  - Boolean values (adjacent or not)
  - Or Edge Weights

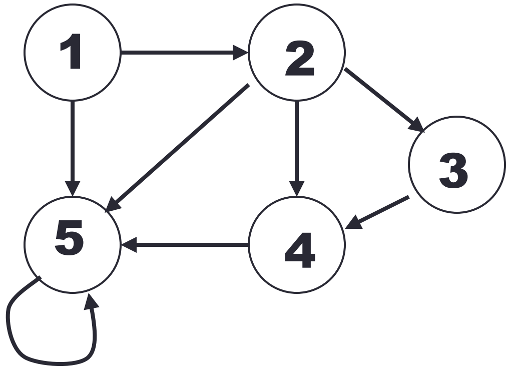
# Representation (List)

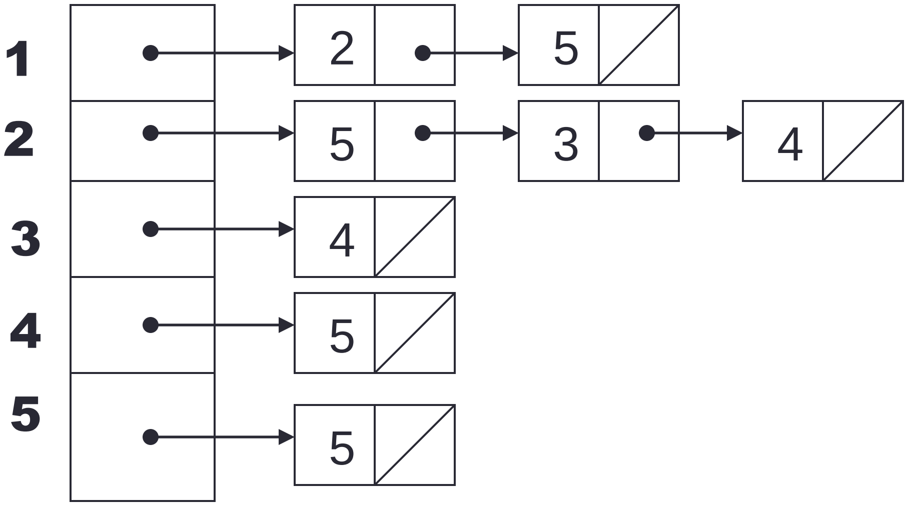
- Edge List
    - pairs (ordered if directed) of vertices
    - Optionally weight and other data
- Adjacency List

# Implementation of a Graph.

- ***Adjacency-list representation***
  - an array of $|V|$ lists, one for each vertex in $V$.
  - For each $u \in V$, $ADJ[u]$ points to all its adjacent vertices.

# Adjacency-list representation for a directed graph.




Variation: Can keep a second list of edges coming into a vertex.

# Adjacency lists

- Advantage:
  - Saves space for sparse graphs.  Most graphs are sparse.
  - Traverse all the edges that start at v, in $\theta$(degree(v))
- Disadvantage:
  - Check for existence of an edge (v, u) in worst case time $\theta$(degree(v))

# Adjacency List

- Storage
  - For a directed graph the number of items are

$$\sum_{v \in V} (\text{out-degree}(v)) = |E|$$

So we need $\Theta(V + E)$

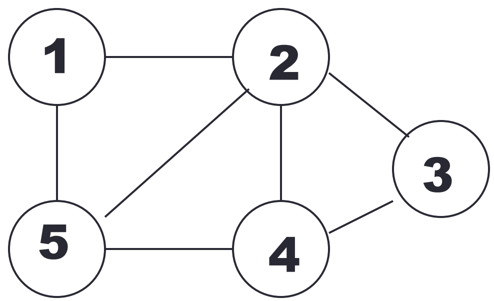
  - For undirected graph the number of items are

$$\sum_{v \in V} (\text{degree}(v)) = 2|E|$$

Also $\Theta(V + E)$

- Easy to modify to handle weighted graphs. How?

# *Adjacency matrix representation*



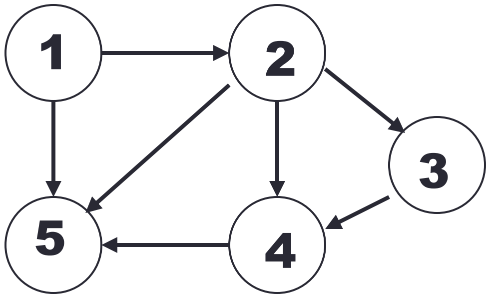

$$\begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 & 1 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 1 & 0 & 1 \\ 5 & 1 & 1 & 0 & 1 & 0 \\ \hline \end{array}$$

- $|V| \times |V|$ matrix $A = ( a_{ij} )$ such that

  $a_{ij} = 1$ if $(i, j) \in E$ and 0 otherwise.

  We arbitrarily uniquely assign the numbers $1, 2, \ldots, |V|$ to each vertex.

# Adjacency Matrix Representation for a Directed Graph




$$
\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 0 & 1 & 0 & 0 & 1 \\
2 & 0 & 0 & 1 & 1 & 1 \\
3 & 0 & 0 & 0 & 1 & 0 \\
4 & 0 & 0 & 0 & 0 & 1 \\
5 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

# Adjacency Matrix Representation

- Advantage:
  - Saves space for:
    - Dense graphs.
    - Small unweighted graphs using 1 bit per edge.
  - Check for existence of an edge in $\theta(1)$
- Disadvantage:
  - Traverse all the edges that start at v, in $\theta(|V|)$

# Adjacency Matrix Representation

- Storage
  - $\Theta(\mid V \mid^2)$ ( We usually just write, $\Theta( V^2)$ )
  - For undirected graphs you can save storage (only $1/2(V^2)$) by noticing the adjacency matrix of an undirected graph is symmetric. How?
- Easy to handle weighted graphs. How?

# GRAPH ALGORITHMS

# Graph Algorithms

- Shortest Path
  - Single Source
  - All pairs (Ex. Floyd Warshall)
- Network Flow
- Matching
  - Bipartite
  - Weighted
- Topological Ordering
- Strongly Connected

# Graph Algorithms

- Biconnected Component / Articulation Point
- Bridge
- Graph Coloring
- Euler Tour
- Hamiltonian Tour
- Clique
- Isomorphism
- Edge Cover
- Vertex Cover
- Visibility

# THANK YOU