

Determinants

Dhruv Kohli

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- How to test invertibility of a matrix?
- How to compute volume of a box in n - dimensions?
- Any explicit formula for the solution of $Ax = b$?
- Any explicit formula for pivots of A ?
- What is the dependence of $A^{-1}b$ on each element of b ?

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- 1 Determinant is defined only for square matrices.
 - 2 $\det A = 0 \iff A$ is singular.
 - 3 $\det A$ = volume of a box in n -dimensional space.
 - 4 $\det A = \pm(\text{product of pivots})$ where the sign depends on number of row exchanges in elimination. Even number of exchanges implies positive sign.
- **The simple things about the determinant are not the explicit formulas, but the properties it possesses.**

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1 $\det I = 1$.

2 Determinant changes sign when two rows are exchanged because determinant of a permutation matrix P is ± 1 . If the number of row exchanges required to bring P to I is even then $\det P = 1$ else -1 .

3 Determinant depends linearly on a row. Proof by determinant computing determinant along that row.

$$\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

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- 4 If two rows of A are equal then $\det A = 0$. Proof: use 2.
- 5 Subtracting a multiple of one row from another leaves the same determinant. Proof: use 3 and 4.
- 6 If A has a zero row, then $\det A = 0$. Proof: use 5 and 4.
- 7 If A is triangular then $\det A = \text{product of diagonal entries}$. Proof: use 5 to derive diagonal matrix, then use 3 and 1.
- 8 $\det A = \pm(\text{product of pivots})$, $\det A = 0 \iff A$ is singular. Proof: elimination leads to U which has pivots on the diagonal. For singular matrices one of the row will be zero. Then use 7.
- 9 $\det AB = \det A \det B$. Proof: $A = P_1^T L_1 U_1$, $B = P_2^T L_2 U_2$.
- 10 $\det A = \det A^T$. Proof: $A = P^T L U$, $A^T = U^T L^T P$ and $\det P^T P = \det I = 1$. This means - we can exchange rows by columns in above results.¹

¹Singular case separately for 7,8,9,10

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- 1 If A is invertible then $PA = LDU$, $\det P = \pm 1$ and product rule gives $\det A = \pm \det L \det D \det U = \pm (\text{product of pivots})$
- 2 Suppose $A_{n \times n}$ is split into n^n terms by applying property 3 to each row in the following way -

$$\begin{vmatrix} a+0 & 0+b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

Among n^n terms only $n!$ terms will be non-zero when the non-zero terms are in different columns otherwise there will be atleast one column of 0s making determinant 0.

The $n!$ terms correspond to $n!$ permutations of $(1, \dots, n)$ which gives another formula for determinant:

$$\det A = \sum_{\text{all } P\text{'s}} a_{1\alpha} a_{2\beta} \dots a_{n\gamma} \det P$$

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- Consider the terms involving a_{11} . This means $\alpha = 1$. This leaves some permutation (β, \dots, γ) of resulting columns $(2, \dots, n)$. We collect all those terms as C_{11} which is the determinant of the submatrix formed by deleting row 1 and column 1.

$$C_{11} = \sum_{\text{all } P\text{'s s.t. } P_{11}=1} a_{2\beta} \dots a_{n\gamma} \det P$$

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

M_{ij} is called a minor (smaller determinant) which is obtained by computing the determinant of the matrix when i th row and j th column are deleted.

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Result 1 - Computation of inverse A^{-1}

$$A^{-1} = C^T / \det A \implies AC^T = \det A I$$

Proof - Hints:

$$(AC^T)_{ij} = \sum_{k=1}^n A_{ik} C_{jk} = \det A \mathbb{I}(i = j)$$

Note that when $i \neq j$, $(AC^T)_{ij}$ represents determinant of the matrix A with i th row copied into j th row (2 rows are equal).

Result 2 - Solution of $Ax = b$

$x_j = \det B_j / \det A$ where B_j is A with b in j th column.

Proof - Hints:

$$(A^{-1}b)_j = \left(\frac{C^T}{\det A} b \right)_j = \frac{\sum_{k=1}^n C_{kj} b_k}{\det A} = \frac{\det B_j}{\det A}$$

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Result 3 - Volume of a box

Volume of a box whose edges are in rows of A equals $\det A$.

Proof - Hints: If edges are \perp and are of length l_1, l_2, \dots, l_n ,

$$AA^T = \text{diag}(l_i^2) \implies \det(AA^T) = \det(A)^2 = \prod_{i=1}^n l_i^2$$

Sign of $\det A$ will indicate whether the edges form a RH-set of coord. $x - y - z$ or a LH-set $y - x - z$. If the edges are \nparallel then with row ops. it can be made \perp by reducing matrix to RREF. Det. is invariant to row ops, so vol. stays same.

Result 4 - Formula of pivots

$d_k = \det A_k / \det A_{k-1}$, A_k is left submatrix of A of order k .

$$A_k = L_k D_k U_k \implies \det A_k = \prod_{i=1}^k d_i \implies d_k = \frac{\det A_k}{\det A_{k-1}}$$

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Result 5

Elimination can be completed without row exchanges i.e. $P = I$ and $A = LU$ if and only if the leading submatrices A_1, A_2, \dots, A_n are all non-singular.

Proof - Hints: Follows from result 4.

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Gilbert Strang. *Linear algebra and its applications*.
Belmont, CA: Thomson, Brooks/Cole, 2006.