Solution Manual

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for

Stochastic Processes, 2nd ed.

by

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2. The Possion Process

Ex. 2.1

$$\mathbb{P}\{N(h)=1\} = e^{-\lambda h}\lambda h = \lambda h + \lambda h(e^{-\lambda h}-1)$$

Since,

$$\lim_{h\to 0}\frac{\lambda h(e^{-\lambda h}-1)}{h} = 0$$

we have,

$$\mathbb{P}\{N(h)=1\} = \lambda h + o(h)$$

Similarly,

$$\mathbb{P}\{N(h)\geq 2\} = 1 - e^{-\lambda h}\lambda h - e^{-\lambda h} = o(h)$$

Ex. 2.2 (**a**)

$P_0(t+s) = 1-\lambda(t+s)-o(t+s) = (1-\lambda t - o(t))(1-\lambda s - o(s)) = P_0(t)P_0(s)$

(**b**)

$$\begin{aligned}
P_0(t) &= P_0\left(\lim_{n\to\infty}\sum_{i=1}^{n}\frac{t}{n}\right) = \lim_{n\to\infty}\left(P_0\left(\frac{t}{n}\right)\right)^n = \lim_{n\to\infty}\exp\left(n\log\left(P_0\left(\frac{t}{n}\right)\right)\right) \\
&= \lim_{n\to\infty}\exp(n\log(1-\lambda t/n + o(t/n))) \\
&= \lim_{n\to\infty}\exp\left(-n\left(\sum_{i=1}^{\infty}(\lambda t/n + o(t/n))^i\right)\right) \\
&= \lim_{n\to\infty}\exp\left(-\lambda t - \frac{to(t/n)}{t/n} - \left(\sum_{i=2}^{\infty}(\lambda t/n + o(t/n))^i\right)\right) \\
&= \exp(-\lambda t)
\end{aligned}$$

$$\mathbb{P}\{X_1 > t\} = P_0(t) = \exp(-\lambda t)$$

$$\begin{aligned}
\mathbb{P}\{X_2 > t | X_1 = s\} &= \mathbb{P}\{0 \text{ event in } (s, s+t] | X_1 = s\} \\
&= \mathbb{P}\{0 \text{ event in } (s, s+t]\} \quad (\because \text{ independent increments}) \\
&= P_0(t) \quad (\because \text{ stationarity}) \\
&= \exp(-\lambda t)
\end{aligned}$$

(**c**)

$$\begin{aligned}
\mathbb{P}\{N(t) \geq n\} = \mathbb{P}\{S_n \leq t\} &= \int_0^t \frac{\lambda^n x^{n-1} \exp(-\lambda x)}{(n-1)!} dx \\
&= -\frac{\exp(-\lambda t)(\lambda t)^{n-1}}{(n-1)!} - \int_0^t \frac{\lambda^{n-1} x^{n-2} \exp(-\lambda x)}{(n-2)!} dx \\
&. \\
&. \\
&= -\sum_{i=1}^{n-1} \frac{\exp(-\lambda t)(\lambda t)^i}{i!} - \int_0^t \lambda \exp(-\lambda x) dx \\
&= 1 - \sum_{i=0}^{n-1} \frac{\exp(-\lambda t)(\lambda t)^i}{i!}
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}\{N(t) = n\} &= \mathbb{P}\{N(t) \geq n\} - \mathbb{P}\{N(t) \geq n+1\} \\
&= \frac{\exp(-\lambda t)(\lambda t)^n}{n!}
\end{aligned}$$

Ex. 2.3

$$\begin{aligned}
\mathbb{P}\{N(s) = k | N(t) = n\} &= \frac{\mathbb{P}\{N(s) = k, N(t) = n\}}{\mathbb{P}\{N(t) = n\}} = \frac{\mathbb{P}\{N(s) = k, N(t-s) = n-k\}}{\mathbb{P}\{N(t) = n\}} \\
&= \frac{\exp(-\lambda s)(\lambda s)^k}{k!} \frac{\exp(-\lambda(t-s))(\lambda(t-s))^{t-s}}{(n-k)!} \frac{n!}{\exp(-\lambda t)(\lambda t)^n} \\
&= \binom{n}{k} (s/t)^k (1 - s/t)^{n-k}
\end{aligned}$$

Alternatively, given that $N(t) = n$, those n events have arrival times which are uniformly distributed over $(0, t)$ when considered as unordered random variables. Therefore, given $N(t) = n$ and $s < t$, $N(s)$ follows a binomial distribution with parameters n and $p = \frac{s}{t}$, which is the probability of a randomly chosen event (out of n events) to have an arrival time of less than or equal to s.

Ex. 2.4

$$\begin{aligned}
\mathbb{E}[N(t)N(t+s)] &= \mathbb{E}[N(t)(N(t+s) - N(t)) + N(t)^2] \\
&= \mathbb{E}[\mathbb{E}[N(t)(N(t+s) - N(t)) | N(t))] + \mathbb{E}[N(t)^2] \\
&= \mathbb{E}[\lambda s N(t)] + \lambda t + (\lambda t)^2 \qquad (\because N(t+s) - N(t) \perp N(t)) \\
&= \lambda^2 t(t+s) + \lambda t
\end{aligned}$$

Ex. 2.5

$$\begin{aligned}
\mathbb{P}\{N_1(t)+N_2(t)=n\} &= \sum_{k=0}^{\infty}\mathbb{P}\{N_1(t)+N_2(t)=n, N_1(t)=k\}\\
&= \sum_{k=0}^{n}\mathbb{P}\{N_1(t)+N_2(t)=n, N_1(t)=k\}\\
&= \sum_{k=0}^{n}\mathbb{P}\{N_2(t)=n-k, N_1(t)=k\}\\
&= \sum_{k=0}^{n}\mathbb{P}\{N_2(t)=n-k\}\mathbb{P}\{N_1(t)=k\} \qquad (\because N_1 \perp N_2)\\
&= \sum_{k=0}^{n}\frac{\exp(-\lambda_1 t)(\lambda_1 t)^k}{k!}\frac{\exp(-\lambda_2 t)(\lambda_2 t)^{n-k}}{(n-k)!}\\
&= \frac{\exp(-(\lambda_1+\lambda_2)t)t^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\lambda_1^k\lambda_2^{n-k}\\
&= \frac{\exp(-(\lambda_1+\lambda_2)t)((\lambda_1+\lambda_2)t)^n}{n!}
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}\{X_1^{(1)} < X_1^{(2)}\} &= \int_0^\infty \mathbb{P}\{X_1^{(1)} < X_1^{(2)}, X_1^{(2)} = t\}dt\\
&= \int_0^\infty \mathbb{P}\{X_1^{(1)} < t\}\mathbb{P}\{X_1^{(2)} = t\}dt\\
&= \int_0^\infty (1-\exp(-\lambda_1 t))\lambda_2\exp(-\lambda_2 t)dt\\
&= 1-\frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{\lambda_1}{\lambda_1+\lambda_2}
\end{aligned}$$

Ex. 2.6 The combined process $N(t)$ will have a rate $\mu_1+\mu_2$ (using **2.5**). Let S_N be the time when the machine fails where N represents the number of components failed by time S_N. Then, we require $\mathbb{E}S_N$ where,

$$\mathbb{E}S_N = \mathbb{E}[\mathbb{E}[S_N|N)] = \mathbb{E}\left[\frac{N}{\mu_1+\mu_2}\right] = \frac{\mathbb{E}N}{\mu_1+\mu_2}$$

Now, $\mathbb{E}N$ is given by,

$$\begin{aligned}
\mathbb{E}N &= \mathbb{E}[N|\text{last event is type-1 fail}]P(\text{last event is type-1 fail}) + \mathbb{E}[N|\text{last event is type-2 fail}]P(\text{last event i}\\
&= \sum_{k=n}^{n+m-1} k\binom{k-1}{n-1}\left(\frac{\mu_1}{\mu_1+\mu_2}\right)^n\left(\frac{\mu_2}{\mu_1+\mu_2}\right)^{k-n} + \sum_{k=m}^{m+n-1} k\binom{k-1}{m-1}\left(\frac{\mu_2}{\mu_1+\mu_2}\right)^m\left(\frac{\mu_1}{\mu_1+\mu_2}\right)^{k-m}
\end{aligned}$$

Ex. 2.7

$$\begin{aligned} f_{S_1,S_2,S_3}(s_1,s_2,s_3) &= f_{X_1,X_2,X_3}(s_1, s_2 - s_1, s_3 - s_2) \\ &= f_{X_1}(s_1) f_{X_2}(s_2 - s_1) f_{X_3}(s_3 - s_2) \quad (X_i \perp X_j) \\ &= \lambda \exp(-\lambda s_1) \lambda \exp(-\lambda(s_2 - s_1)) \lambda \exp(-\lambda(s_3 - s_2)) \\ &= \lambda^3 \exp(-\lambda s_3) \end{aligned}$$

Ex. 2.8 (**i**)

$$\begin{aligned} U_i &= \exp(-\lambda X_i) \\ \left|\frac{dU_i}{dX_i}\right| &= \lambda \exp(-\lambda X_i) \\ f_{X_i}(x) &= \lambda \exp(-\lambda x) \mathbb{I}(\exp(-\lambda x) \in (0,1)) \\ &= \lambda \exp(-\lambda x) \mathbb{I}(x \in (0,\infty)) \end{aligned}$$

(**ii**) Taking negative *log* of the inequality and dividing by λ gives,

$$\begin{aligned} \sum_{i=1}^{n} X_i \leq 1 < \sum_{i=1}^{n+1} X_i \\ S_n \leq 1 < S_{n+1} \end{aligned}$$

Thus n represents number of events till time 1 of a poisson process with rate λ. Therefore, $n = N(1)$ where $N(1)$ follows poisson distribution with mean $\lambda \cdot 1 = \lambda$.

Ex. 2.9 (**a**) Probability of winning equals the probability of exactly one event in $(s, T]$ which by stationarity of poisson process equals
$h(s) = \exp(-\lambda(T - s))\lambda(T - s)$.
(**b**)

$$\begin{aligned} \frac{dh(s)}{ds} = 0 \implies s = T - 1/\lambda \\ \left.\frac{d^2 h(s)}{ds^2}\right|_{s=T-1/\lambda} = -\lambda^2 e^{-1} < 0 \end{aligned}$$

(**c**)

$$h(T - 1/\lambda) = e^{-1}$$

Ex. 2.10 (**a**)

$$T = \begin{cases} X_1 + R, & X_1 \le s \\ s + W, & X_1 > s \end{cases}$$

$$\begin{aligned}
\mathbb{E}T &= \mathbb{E}[T|X_1 \le s]\mathbb{P}\{X_1 \le s\} + \mathbb{E}[T|X_1 > s]\mathbb{P}\{X_1 > s\} \\
&= \mathbb{E}[X_1 + R|X_1 \le s]\mathbb{P}\{X_1 \le s\} + \mathbb{E}[s + W|X_1 > s]\mathbb{P}\{X_1 > s\} \\
&= \mathbb{E}[X_1\mathbb{I}(X_1 \le s)] + R(1 - \exp(-\lambda s)) + (s + W)\exp(-\lambda s) \\
&= \frac{1 - \lambda s \exp(-\lambda s) - \exp(-\lambda s)}{\lambda} + R(1 - \exp(-\lambda s)) + (s + W)\exp(-\lambda s) \\
&= (R + 1/\lambda)(1 - \exp(-\lambda s)) + W\exp(-\lambda s)
\end{aligned}$$

(**b**) When $W < R + 1/\lambda$, minimum is achieved with $\exp(-\lambda s) = 1 \implies s = 0$. When $W > R + 1/\lambda$, minimum is achieved with $1 - \exp(-\lambda s) = 1 \implies s = \infty$. And, when $W = R + 1/\lambda$, then all values of s gives $\mathbb{E}T = W = R + 1/\lambda$.
(**c**) The expected time of arrival of bus is $\mathbb{E}[X_1] = 1/\lambda$. So, intuitively, if $W < R + 1/\lambda$, in order to minimize $\mathbb{E}T$, I will not wait at bus stop at all ($s = 0$), and reach home by walking. On the other hand, if $W > R + 1/\lambda$, I will wait for the bus to arrive indefinitely ($s = \infty$) (since the increase in time increases the likeliness of arrival of bus as $\lim_{t\to\infty} \mathbb{P}\{X_1 > t\} = 0$ and the expected arrival time is $1/\lambda$).

Ex. 2.11

$$W = \begin{cases} 0 & X_1 > T \\ W' + X_1 & X_1 \le T \end{cases}$$

Convince yourself that W' and W have the same distribution and hence the expectation. Also, note that W' is independent of X_1. Therefore,

$$\begin{aligned}
\mathbb{E}W &= \mathbb{E}[W|X_1 > T]\mathbb{P}\{X_1 > T\} + \mathbb{E}[W|X_1 \le T]\mathbb{P}\{X_1 \le T\} \\
&= 0 + \mathbb{E}[W' + X_1|X_1 \le T]\mathbb{P}\{X_1 \le T\} \\
&= \mathbb{E}[W']\mathbb{P}\{X_1 \le T\} + \mathbb{E}[X_1\mathbb{I}(X_1 \le T)]
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}W &= \mathbb{E}[W](1 - \exp(-\lambda T)) + \frac{1 - \lambda T\exp(-\lambda T) - \exp(-\lambda T)}{\lambda} \quad (\because \mathbb{E}W = \mathbb{E}W') \\
\mathbb{E}W &= \frac{\exp(\lambda T) - \lambda T - 1}{\lambda}
\end{aligned}$$

Ex. 2.12 Let type-1 events be those which are registered and type-2 events be those which are not registered. An event at arbitrary time s is type-1 event with probability $\mathbb{P}\{0 \text{ event in } [s - b, s)\} = \exp(-\lambda b)$.

(**a**) Since the probability of an event happening at an arbitrary time is classified as a type-1 event with a probability of $p = \exp(-\lambda b)$ which is independent of the time of happening of the event. Therefore, the first k events will be classified as type 1 event with probability $p^k = \exp(-\lambda kb)$. This can also be formally computed as follows:

$$\begin{aligned}
\mathbb{P}\{S_k^{(1)} < X_1^{(2)}\} &= \int_0^\infty \mathbb{P}\{X_1^{(2)} > t | S_k^{(1)} = t\} f_{S_k^{(1)}}(t)dt \\
&= \int_0^\infty \mathbb{P}\{X_1^{(2)} > t\} f_{S_k^{(1)}}(t)dt \quad (\because X_1^{(2)} \perp S_k^{(1)}) \\
&= \int_0^\infty \exp(-\lambda(1-p)t) \frac{(\lambda p)^k t^{k-1} \exp(-\lambda pt)}{(k-1)!} dt \\
&= p^k \int_0^\infty \frac{\lambda^k t^{k-1} \exp(-\lambda t)}{(k-1)!} dt \\
&= p^k \cdot 1 \\
&= \exp(-\lambda kb)
\end{aligned}$$

(**b**)

$$\mathbb{P}\{R(t) \geq n\} = \mathbb{P}\{N_1(t) \geq n\} = \sum_{k=n}^{\infty} \frac{\exp(-\lambda pt)(\lambda pt)^k}{k!}$$

Ex. 2.13 [verify] Let there be two types of events. Type-1 events cause failure with probability p and type-2 events do not cause failure.

$$\begin{aligned}
\mathbb{P}\{N = n | T = t\} &= \mathbb{P}\{N = n | \text{ first type-1 event occurs at } t\} \\
&= \mathbb{P}\{n-1 \text{ type-2 events occur before } t | \text{ first type-1 event occurs at } t\} \\
&= \mathbb{P}\{n-1 \text{ type-2 events occur before } t\} \quad (\because N_1(t) \perp N_2(t)) \\
&= \frac{\exp(-\lambda(1-p)t)(\lambda(1-p)t)^{n-1}}{(n-1)!}
\end{aligned}$$

Ex. 2.14 (**a**)

$$\mathbb{E}O_j = \mathbb{E}[\mathbb{E}[O_j | N_1, N_2, \ldots, N_{j-1}]] = \mathbb{E}\left[\sum_{i=1}^{j-1} P_{ij} N_i\right] = \sum_{i=1}^{j-1} P_{ij} \lambda_i$$

(**b**)

$$O_j \sim \text{Poisson}\left(\sum_{i=1}^{j-1} P_{ij} \lambda_i\right)$$

(**c**) $O_j \perp O_k$.

Ex. 2.15 (**a**) N_i follows negative binomial distribution with parameters n_i and P_i.
(**b**)

$$\begin{aligned}
\mathbb{P}\{N_i = n, N_j = n\} &= \mathbb{P}\{n \text{ flips with ith and jth sides } n_i \text{ and } n_j \text{ times.}\} \\
&= \mathbb{P}\{N_i = n, N_j = n | \text{ end with i}\}\mathbb{P}\{\text{end with i}\} + \\
&\quad \mathbb{P}\{N_i = n, N_j = n | \text{ end with j}\}\mathbb{P}\{\text{end with j}\} \\
&= \binom{n-1}{n_i - 1} P_i^{n_i} \binom{n - n_i}{n_j} P_j^{n_j} (1 - P_i - P_j)^{n - n_i - n_j} + \\
&\quad \binom{n-1}{n_j - 1} P_j^{n_j} \binom{n - n_j}{n_i} P_i^{n_i} (1 - P_i - P_j)^{n - n_i - n_j} \\
&= P_i^{n_i} P_j^{n_j} (1 - P - i - P_j)^{n - N - i - n_j} \frac{(n-1)!(n_i + n_j)}{n_i! n_j! (n - n_i - n_j)!} \\
&\neq \binom{n-1}{n_i - 1} P_i^{n_i} (1 - P_i)^{n - n_i} \binom{n-1}{n_j - 1} P_j^{n_j} (1 - P_j)^{n - n_j} \\
&= \mathbb{P}\{N_i = n\}\mathbb{P}\{N_j = n\}
\end{aligned}$$

So, N_i and N_j are dependent.
(**c**) Now, we have r independent poisson processes $N_i(t), i \in \{1, \ldots, r\}$, where $N_i(t)$ has a poisson distribution with mean $\lambda P_i t = P_i t$ (since $\lambda = 1$).

$$\mathbb{P}\{T > t\} = \prod_{i=1}^{r} \mathbb{P}\{S_{n_i}^{(i)} > t\}$$

where $S_{n_i}^{(i)} \sim \text{Gamma}(n_i, P_i)$.
(**d**) $T_i = S_{n_i}^{(i)}$ which are independent since the poisson processes are independent.
(**e**) $\mathbb{E}T = \int_0^\infty \mathbb{P}\{T > t\}$
(**f**)

$$T = \sum_{i=1}^{N} X_i \implies \mathbb{E}T = \mathbb{E}[\mathbb{E}[T|N]] = \frac{1}{\lambda}\mathbb{E}N = \mathbb{E}N$$

Ex. 2.16 Let N be the number of trials to be performed which follows Poisson(λ). Let O_i be the number of trials when ith outcome came up where the probability that a trial results in ith outcome is P_i. Then, O_i will follow Poisson(λP_i).

$$X_j = \sum_{i=1}^{n} \mathbb{I}(O_i = j)$$

$$\mathbb{E}X_j = \sum_{i=1}^{n} \mathbb{P}(O_i = j) = \sum_{i=1}^{n} \frac{\exp(-\lambda P_i)(\lambda P_i)^j}{j!}$$

$$\begin{aligned}\operatorname{Var} X_j &= \mathbb{E}X_j^2 - (\mathbb{E}X_j)^2 \\ &= \sum_{i=1}^{n} \frac{\exp(\lambda P_i)(\lambda P_i)^j}{j!} \left(1 - \frac{\exp(-\lambda P_i)(\lambda P_i)^j}{j!}\right)\end{aligned}$$

Ex. 2.17 (**a**)

$$\begin{aligned} f_{X(i)}(x) &= \mathbb{P}\{i-1 \text{ of the } X\text{'s} \le x, \text{ one } X \text{ equals } x, \text{ remaining } X\text{'s} > x\} \\ &= \binom{n}{i-1}\mathbb{P}\{X \le x\}^{i-1}\binom{n-(i-1)}{1}\mathbb{P}\{X = x\}\binom{n-i}{n-i}\mathbb{P}\{X > x\}^{n-i} \\ &= \frac{n!}{(i-1)!(n-i)!}F(x)^{i-1}f(x)\bar{F}(x)^{n-i}\end{aligned}$$

(**b**) Atleast i X's.
(**c**)

$$\begin{aligned}\mathbb{P}\{X_{(i)} \le x\} &= \sum_{k=i}^{n} \mathbb{P}\{k \text{ of the } X\text{'s are } \le x \text{ and remaining are } > x\} \\ &= \sum_{k=i}^{n} \binom{n}{k} F(x)^k \bar{F}(x)^{n-k}\end{aligned}$$

(**d**) Replace $y = F(x)$ and integrate (**a**).
(**e**) Given $N(t) = n$, for $i \le n$, S_i follows the distribution of ith order statistic of n random variables uniformly distributed in $(0, t)$. Therefore, $\mathbb{E}[S_i|N(t) = n] = \frac{i}{n+1}$ when $i \le n$. Given $N(t) = n$, for $i > n$, $\mathbb{E}[S_i|N(t) = n] = \mathbb{E}[S_i|S_i > t] = \mathbb{E}[S_i\mathbb{I}(S_i > t)]/\mathbb{P}\{S_i > t\}$ which equals,

$$\frac{\int_t^\infty x\frac{\lambda^i x^{i-1}\exp(-\lambda x)}{(i-1)!}}{\int_t^\infty \frac{\lambda^i x^{i-1}\exp(-\lambda x)}{(i-1)!}} = \frac{i}{\lambda}\frac{\bar{G}(t)}{\bar{F}(t)}, \text{ where } G \sim \text{Gamma}(i+1, \lambda), F \sim \text{Gamma}(i, \lambda)$$

Ex. 2.18

$$
\begin{aligned}
\mathbb{P}\{U_{(i)} = x | U_{(n)} = y\} &= \frac{\mathbb{P}\{U_{(i)} = x, U_{(n)} = y\}}{\mathbb{P}\{U_n = y\}} \mathbb{I}(x \leq y) \\
&= \frac{\frac{n!}{(i-1)!(n-i-1)!} f(x)f(y)F(x)^{i-1}(F(y) - F(x))^{n-i-1}}{\frac{n!}{(n-1)!} f(y)F(y)^{n-1}} \mathbb{I}(x \leq y) \\
&= \frac{n!}{(i-1)!(n-1-i)!} \frac{x^{i-1}(y-x)^{n-i-1}}{y^{n-1}} \mathbb{I}(x \leq y) \\
&= \frac{n!}{(i-1)!(n-1-i)!} \left(\frac{x}{y}\right)^{i-1} \left(1 - \frac{x}{y}\right)^{n-i-1} \mathbb{I}(x \leq y)
\end{aligned}
$$

Ex. 2.19 Type-j bus load arrival, where the number of customers in the bus equals j, follows a poisson process $N_j(t)$ having rate $\lambda\alpha_j$. Let the total number of customers arrived by time t is given by $N(t)$. Then,

$$N(t) = \sum_{j=1}^{\infty} jN_j(t)$$

Since, $N_j(t) \sim \text{Poisson}(\lambda\alpha_j t)$, $N(t)$ is a sum of poisson random variables and therefore $N(t) \sim \text{Poisson}(\gamma)$ where $\gamma = \lambda \sum_{j=1}^{\infty} j\alpha_j$.
Now, a randomly chosen customer who arrived at time s will be served by time t with probability $G(t-s)$. Let $\beta = \frac{1}{t}\int_0^t G(t-s)ds$, then the poisson process $N'(t)$ having rate $\gamma\beta$ corresponds to the number of customers served by time t. Clearly, $X(t) = N'(t)$.
(**a**)

$$\mathbb{E}X(t) = \gamma\beta t = \lambda\beta t \sum_{j=1}^{\infty} j\alpha_j$$

(**b**) $X(t) \sim \text{Poisson}(\lambda\beta t \sum_{j=1}^{\infty} j\alpha_j)$

Ex. 2.20 Let $p_i = \frac{1}{t}\int_0^t P_i(s)ds$. Then,

$$\begin{aligned}
\mathbb{P}\{N_i(t) = n_i, i \in \{1,\ldots,k\}\} &= \sum_m \mathbb{P}\{N_i(t) = n_i, i \in \{1,\ldots,k\} | N(t) = m\}\mathbb{P}\{N(t) = m\} \\
&= \mathbb{P}\left\{N_i(t) = n_i, i \in \{1,\ldots,k\} | N(t) = \sum_{j=1}^k n_j\right\}\mathbb{P}\left\{N(t) = \sum_{j=1}^k n_j\right\} \\
&= \frac{\left(\sum_{j=1}^k n_j\right)!}{\prod_{j=1}^k n_j!}\prod_{j=1}^k p_j^{n_j} \cdot \exp\left(-\lambda t\right)\frac{(\lambda t)^{\sum_{j=1}^k n_j}}{\left(\sum_{j=1}^k n_j\right)!} \\
&= \prod_{j=1}^k \exp(-\lambda p_j t)\frac{(\lambda p_j t)^{n_j}}{n_j!}
\end{aligned}$$

Therefore, $N_i(t) \perp N_j(t), i \neq j$ and $N_i(t) \sim \text{Poisson}(\lambda p_i t)$.

Ex. 2.21 We need to show that,

$$\int_0^s \alpha(s)ds = \mathbb{E}[\text{amount of time individual is in state } i \text{ during its first } t \text{ units in the system}]$$

Divide interval $(0,t]$ in n equal parts and let $h = t/n$. Then, the amount of time individual is in state i during its first t units in the system equals $\sum_{i=1}^n \mathbb{I}(\text{individual is in state } i \text{ during } ((i-1)h, ih])h$. Therefore,

$$\begin{aligned}
&\mathbb{E}[\text{amount of time individual is in state } i \text{ during its first } t \text{ units in the system}] = \\
&\quad \lim_{h\to 0}\mathbb{E}[\sum_{i=1}^n \mathbb{I}(\text{individual is in state } i \text{ during } ((i-1)h, ih])h] \\
&= \lim_{h\to 0}\sum_{i=1}^n \mathbb{P}\{\text{individual is in state } i \text{ during } ((i-1)h, ih]\}h \\
&= \int_0^t \alpha(s)ds
\end{aligned}$$

Ex. 2.22 A car entering at time s will be located in the interval (a,b) at time t when its velocity satisfies $a < V(t-s) < b \implies \frac{a}{t-s} < V < \frac{b}{t-s}$, the probability of which is $P(s) = F(b/(t-s)) - F(a/(t-s))$. Let $p = \frac{1}{t}\int_t^t P(s)ds$, then, the number of cars located in the interval (a,b) at time t will follow poisson distribution with mean λpt.

Ex. 2.23 (**a**) Using $\text{Var}[D(t)] = \text{Var}[\mathbb{E}[D(t)|N(t)]] + \mathbb{E}[\text{Var}[D(t)|N(t)]]$, we get,

$$
\begin{aligned}
\mathrm{Var}[\mathbb{E}[D(t)|N(t)]] &= \mathrm{Var}\left[\frac{N(t)}{\alpha t}(1-\exp(-\alpha t))\mathbb{E}[D]\right] \\
&= \frac{\lambda(1-\exp(-\alpha t))^2\mathbb{E}[D]^2}{\alpha^2 t} \\
\mathrm{Var}[D(t)|N(t)] &= \mathbb{E}[D]^2\exp(-2\alpha t)\,\mathrm{Var}\left[\sum_{i=1}^{N(t)}\exp(\alpha S_i)|N(t)\right] \\
&= \mathbb{E}[D]^2\exp(-2\alpha t)n\left(\frac{\exp(2\alpha t)-1}{2\alpha t} - \frac{(\exp(\alpha t)-1)^2}{\alpha^2 t^2}\right) \\
\mathbb{E}[\mathrm{Var}[D(t)|N(t)]] &= \mathbb{E}[D]^2\exp(-2\alpha t)\lambda\left(\frac{\exp(2\alpha t)-1}{2\alpha} - \frac{(\exp(\alpha t)-1)^2}{\alpha^2 t}\right) \\
\mathrm{Var}[D(t)] &= \frac{\mathbb{E}[D]^2\lambda(1-\exp(-2\alpha t))}{2\alpha}
\end{aligned}
$$

(**b**) Using property of independent increments of poisson process we have,

$$
\begin{aligned}
D(t+s) &= D(t)\exp(-\alpha s) + \sum_{i=N(t)+1}^{N(t+s)} D_i\exp(-\alpha(t+s-S_i)) \\
&= D(t)\exp(-\alpha s) + D'(s)\exp(-\alpha t)
\end{aligned}
$$

where $D'(s) \perp D(t)$ and $D'(s)$ follows the same distribution as $D(s)$. So,

$$
\begin{aligned}
\mathrm{Cov}(D(t), D(t+s)) &= \mathbb{E}[D(t)D(t+s)] - \mathbb{E}[D(t)]\mathbb{E}[D(t+s)] \\
&= \mathbb{E}[D(t)^2\exp(-\alpha s) + D(t)D'(s)\exp(-\alpha t)] \\
&\quad - \mathbb{E}[D(t)]^2\exp(-\alpha s) - \mathbb{E}[D(t)]\mathbb{E}[D'(s)]\exp(-\alpha t) \\
&= \mathrm{Var}[D(t)]\exp(-\alpha s)
\end{aligned}
$$

Ex. 2.24 Let the time taken T by a car to travel the highway of length L follow distribution G. Then, $\mathbb{P}(T \le t) = G(t) = \mathbb{P}(V \ge L/t) = \bar{F}(L/t)$. Let v be the speed of the car that enters the highway at time t. Then, the time taken by the car to travel the highway is $t_v = L/v$. Let s be the time a random car enters the highway and leaving after time T, then, the probability of an encounter with the car entering at time t is,

$$
P(s) = \begin{cases} \mathbb{P}\{T \ge t - s + t_v\} = \bar{G}(t-s+t_v), & s < t \\ \mathbb{P}\{T \le t + t_v - s\} = G(t-s+t_v), & t \le s < t + t_v \\ 0, & \text{otherwise} \end{cases}
$$

Then, expected number of encounters is given by,

$$\lambda\left(\int_0^t \bar{G}(t-s+t_v)ds + \int_t^{t+t_v} G(t-s+t_v)ds\right) = \lambda\left(1 - \int_{t_v}^{t+t_v} G(s)ds + \int_0^{t_v} G(s)ds\right)$$

The value of t_v minimizing the above, satisfies,

$$G(t_v) - G(t+t_v) + G(t_v) = 0 \implies G(t_v) = 1/2 \text{ as } t \to \infty$$

Thus, as $t \to \infty$,

$$\bar{F}(v) = 1/2 \implies v = F^{-1}(1/2)$$

Ex. 2.25

$$W = \sum_{i=1}^{N(t)} Y_i, \qquad Y_i \sim F_{S_i}$$
$$\mathbb{P}\{W \le w | N(t) = n\} = \mathbb{P}\left\{\sum_{i=1}^{n} Y_i \le w \middle| N(t) = n\right\}$$

Given $N(t) = n$, $S_1, S_2, \ldots, S_n$ are uniform$(0,t)$. Therefore, for all i,

$$\begin{aligned}\mathbb{P}\{Y_i \le y | N(t) = n\} &= \int_0^t \mathbb{P}\{Y_i \le y | N(t) = n, S_i = s\}\mathbb{P}\{S_i = s\}ds \\ &= \int_0^t F_s(y).\frac{1}{t}ds = \frac{1}{t}\int_0^t F_s(y)ds\end{aligned}$$

Therefore, W can be thought of as a compound poisson random variable, $\sum_{i=1}^{N} X_i$, where X_i are iid with distribution,

$$F(x) = \frac{1}{t}\int_0^t F_s(y)ds$$

and are also independent with N which follows poisson distribution with mean λt.

Ex. 2.26

$$f_{S_1,S_2,\ldots,S_n|S_n=t}(s_1,s_2,\ldots,s_n) = \begin{cases} f_{S_1,S_2,\ldots,S_n|S_n=t}(s_1,s_2,\ldots,t), & s_1 \le s_2 \le \ldots \le s_n = t \\ 0, & \text{otherwise}\end{cases}$$

$$
\begin{aligned}
f_{S_1,S_2,\ldots,S_n|S_n=t}(s_1,s_2,\ldots,t) &= \frac{f_{S_1,\ldots,S_n}(s_1,\ldots,t)}{f_{S_n}(t)} \\
&= \frac{f_{X_1,\ldots,X_n}(s_1,s_2-s_1,\ldots,t-s_{n-1})}{f_{S_n}(t)} \\
&= \frac{\lambda\exp(-\lambda s_1)\lambda\exp(-\lambda(s_2-s_1))\ldots\lambda\exp(-\lambda(t-s_{n-1}))}{\lambda^n t^{n-1}\exp(-\lambda t)/(n-1)!} \\
&= \frac{(n-1)!}{t^{n-1}}
\end{aligned}
$$

Ex. 2.28 First note that,

$$
\mathbb{E}\left[Y_1+\ldots+Y_k|\sum_{i=1}^{n}Y_i=y\right]=\mathbb{E}\left[Y_{j_1}+\ldots+Y_{j_k}|\sum_{i=1}^{n}Y_i=y\right]
$$

Taking every combination of k Y_i's, adding them, taking expectation and then using the linearity of expectation we get,

$$
\begin{aligned}
&\binom{n}{k}\mathbb{E}\left[Y_1+\ldots+Y_k|\sum_{i=1}^{n}Y_i=y\right]=\binom{n-1}{k-1}\mathbb{E}\left[Y_1+\ldots+Y_n|\sum_{i=1}^{n}Y_i=y\right]=\binom{n-1}{k-1}y \\
&\implies \mathbb{E}\left[Y_1+\ldots+Y_k|\sum_{i=1}^{n}Y_i=y\right]=\frac{ky}{n}
\end{aligned}
$$

Ex. 2.29 First we divide the interval $(t,t+s]$ into k equal subintervals and prove that the probability of greater than or equal to 2 events in any of those subintervals approaches 0 as k approaches ∞.

$$
\begin{aligned}
\mathbb{P}\{\geq 2 \text{ events in a subinterval}\} &= \cup_{i=1}^{k}\mathbb{P}\left\{\geq 2 \text{ events in } \left(t+\frac{(i-1)s}{k},t+\frac{is}{k}\right]\right\} \\
&\leq \sum_{i=1}^{k}\mathbb{P}\left\{\geq 2 \text{ events in } \left(t+\frac{(i-1)s}{k},t+\frac{is}{k}\right]\right\} \\
&= ko(s/k)=t\frac{o(s/k)}{s/k}\to 0 \text{ as } k\to\infty
\end{aligned}
$$

Let I_j be defined as,

$$
I_j=\begin{cases}1, & \text{an event in } \left(t+\frac{(i-1)s}{k},t+\frac{is}{k}\right] \\ 0, & \text{0 event in } \left(t+\frac{(i-1)s}{k},t+\frac{is}{k}\right]\end{cases}
$$

So, the number of events in $(t, t+s]$, by poisson approaximation of binomial distribution, follows a poisson distribution with mean,

$$\lim_{k\to\infty} \mathbb{E}\left[\sum_{j=1}^{k} I_j\right] = \lim_{k\to\infty} \sum_{j=1}^{k} \lambda\left(t + \frac{js}{k}\right)\frac{s}{k} = \int_t^{t+s} \lambda(x)dx = m(t+s) - m(t)$$

Ex. 2.31

$$\mathbb{P}\{N^*(t) = n\} = \mathbb{P}\{N(m^{-1}(t)) = n\} = \frac{\exp(-m(m^{-1}(t)))(m(m^{-1}(t)))^n}{n!} = \frac{\exp(-t)t^n}{n!}$$

Ex. 2.32(a) Let $t_1, t_2, \ldots, t_n$ be such that $0 < t_1 < t_2 < \ldots < t_n < t$ and Δ_i be such that $t_i + \Delta_i < t_{i+1}$, then,

$$\begin{aligned}
&\mathbb{P}\{t_i \le S_i \le t_i + h_i, i \in \{1, 2, \ldots, n\} | N(t) = n\} \\
&= \frac{e^{-m(t_1)}\left(\prod_{i=1}^{n} e^{-(m(t_i+\Delta t_i)-m(t_i))}(m(t_i + \Delta t_i) - m(t_i))\right) e^{-(m(t)-m(t_n+\Delta t_n))}}{e^{-m(t)}m(t)^n/n!} \\
&= \frac{n! \prod_{i=1}^{n} (m(t_i + \Delta t_i) - m(t_i))}{m(t)}
\end{aligned}$$

As $\Delta_i \to 0$,

$$f_{S_1,\ldots,S_{N(t)}|N(t)=n}(t_1, t_2, \ldots, t_n) = \frac{n! \prod_{i=1}^{n} \lambda(t_i)}{m(t)^n}$$

Therefore, the unordered set of arrival times has the same distribution as n iid random variables having distribution function,

$$F(x) = \begin{cases} m(x)/m(t) & x \le t \\ 1 & x > t \end{cases}$$

(**b**) Let $P(s)$ be the probability that a worker injured at time s is out of work at time t. Then,

$$P(s) = \bar{F}(t-s)$$

The two types of Poisson processes: $N_1(t)$ represents the number of workers out of work at time t and $N_2(t)$ represents the number of workers at work at time t. Now, a random worker injured before time t will be out of work at time t with probability p,

$$
\begin{aligned}
p &= \int_0^t \mathbb{P}\{\text{out of work at time } t | \text{injured at time } s\}\mathbb{P}\{\text{injured at time } s\}ds \\
&= \int_0^t P(s)\frac{\lambda(s)}{m(t)}ds \\
&= \frac{1}{m(t)}\int_0^t \bar{F}(t-s)\lambda(s)ds
\end{aligned}
$$

Finally,
$\mathbb{E}[X(t)] = \mathbb{E}[N_1(t)] = m(t)p = \int_0^t \bar{F}(t-s)\lambda(s)ds = \mathrm{Var}(N_1(t)) = \mathrm{Var}(X(t))$.

Ex. 2.33(a)

$$\mathbb{P}\{X > t\} = \mathbb{P}\{0 \text{ events in } \bar{B}_t(0)\} = \exp(-\lambda\pi t^2)$$

(**b**)

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}\{X > t\}dt = \int_0^\infty \exp(-\lambda\pi t^2)dt = \frac{1}{2\sqrt{\lambda}}$$

(**c**)

$$\mathbb{P}\{\pi R_1^2 > t\} = \mathbb{P}\{R_1 > \sqrt{t}/\sqrt{\pi}\} = \exp(-\lambda\pi t/\pi) = \exp(-\lambda t)$$

$$
\begin{aligned}
\mathbb{P}\{\pi R_2^2 - \pi R_1^2 > t | \pi R_1^2 = s\} &= \mathbb{P}\{0 \text{ event in } s/\pi < \|\vec{r}\|^2 \le (s+t)/\pi | \pi R_1^2 = s\} \\
&= \mathbb{P}\{0 \text{ event in } s/\pi < \|\vec{r}\|^2 \le (s+t)/\pi\} \text{ (non-overlapping regions)} \\
&= \exp(-\lambda(\pi(s+t)/\pi - \pi s/\pi)) = \exp(-\lambda t)
\end{aligned}
$$

Ex. 2.34

$$W = \sum_{i=1}^{N(t)} Y_i, \qquad Y_i \sim F_{S_i}$$

$$\mathbb{P}\{W \le w | N(t) = n\} = \mathbb{P}\left\{\sum_{i=1}^{n} Y_i \le w \middle| N(t) = n\right\}$$

Using **2.32**(**a**), for all i,

$$
\begin{aligned}
\mathbb{P}\{Y_i \le y | N(t) = n\} &= \int_0^t \mathbb{P}\{Y_i \le y | N(t) = n, S_i = s\}\mathbb{P}\{S_i = s\}ds \\
&= \int_0^t F_s(y).\frac{dm(s)}{m(t)} = \frac{1}{m(t)}\int_0^t F_s(y)dm(s)
\end{aligned}
$$

Therefore, W can be thought of as a compound poisson random variable, $\sum_{i=1}^{N} X_i$, where X_i are iid with distribution,

$$F(x) = \frac{1}{m(t)} \int_0^t F_s(y) dm(s)$$

and are also independent with N which follows poisson distribution with mean $\lambda(t)$.

Ex. 2.35(a)

$$\begin{aligned} N^*(t+s) - N^*(t) &= N(t+s+\tau) - N(t+\tau) \\ N^*(t) &= N(t+\tau) - N(\tau) \end{aligned}$$

$$N(t+\tau) - N(\tau) \perp N(t+s+\tau) - N(t+\tau) \implies N^*(t+s) - N^*(t) \perp N^*(t)$$

(**b**) Last implication is still valid.

3. Renewal Theory

3.1(**a**) True.

(**b**) True.

(**c**) If $F(0) = 0$, then true. If $F(0) > 0$, then false.

Ex. 3.2

$$N(\infty) + 1 = n \iff N(\infty) = n - 1 \iff X_i < \infty,\ \forall i < n \wedge X_n = \infty$$

$$\begin{aligned} \mathbb{P}\{N(\infty) + 1 = n\} &= \mathbb{P}\{X_i < \infty,\ \forall i < n \wedge X_n = \infty\} = \mathbb{P}\{X_n = \infty\} \prod_{i=1}^{n-1} \mathbb{P}\{X_i < \infty\} \\ &= F(\infty)^{n-1}(1 - F(\infty)) \end{aligned}$$

Therefore, $N(\infty) + 1$ is geometric with mean $1/(1 - F(\infty))$.

Ex. 3.3

$$\begin{aligned} \mathbb{P}\{X_{N(t)+1} \geq x\} &= \mathbb{P}\{X_{N(t)+1} \geq x | S_{N(t)} = 0\}\bar{F}(t) + \int_0^t \mathbb{P}\{X_{N(t)+1} \geq x | S_{N(t)} = y\}\bar{F}(t-y) dm(y) \\ &= \mathbb{P}\{X_1 \geq x | X_1 > t\}\bar{F}(t) + \int_0^t \mathbb{P}\{X \geq x | X > t - y\}\bar{F}(t-y) dm(y) \\ &= \mathbb{I}(x \leq t)(\bar{F}(t) + \int_0^{t-x} \bar{F}(t-y) dm(y) + \int_{t-x}^t dm(y)) + \mathbb{I}(x > t)\bar{F}(x)(1 + m(t)) \\ &= \mathbb{I}(x \leq t)(\mathbb{P}\{X \leq t - x\} + m(t) - m(t-x)) + \mathbb{I}(x > t)\bar{F}(x)(1 + m(t)) \end{aligned}$$

Ex. 3.4

$$\begin{aligned} m(t) &= \sum_{n=1}^{\infty} F_n(t) = F(t) + F(t) * \left(\sum_{n=1}^{\infty} F_n(t) \right) = F(t) + F(t) * m(t) \\ &= F(t) + \int_0^t m(t-x) dF(x) \end{aligned}$$

Ex. 3.5

$$\begin{aligned} m &= F + m * F \\ F &= m - m * F \\ F &= m - m_2 + m_2 * F \\ F(t) &= \sum_{n=1}^{\infty} (-1)^{n+1} m_n(t) \end{aligned}$$

Ex. 3.6

$$\forall s \le t,\ \mathbb{E}[N(s)|N(t)] = \frac{s}{t} N(t) \implies m(s) = \mathbb{E}[N(s)] = \mathbb{E}[\mathbb{E}[N(s)|N(t)]] = \frac{s}{t} m(t)$$

Therefore, $m(t) = kt$ where k is a positive constant.
Using **3.4**, we get,

$$\begin{aligned} kt &= F(t) + \int_0^t k(t-x) dF(x) \\ k &= \frac{dF(t)}{dt} + kF(t) \\ F(t) &= 1 - \exp(-kt) \end{aligned}$$

Hence, the interarrival times distribution is exponential and therefore $\{N(t), t \ge 0\}$ is a Poisson process.

Ex. 3.7 Using **3.4**, for $t \in [0,1]$, we get,

$$\begin{aligned} m(t) &= t + \int_0^t m(t-x) dt \\ \frac{dm(t)}{dt} &= 1 + m(t) \\ m(t) &= \exp(t) - 1 \end{aligned}$$

$$\mathbb{E}(N(1) + 1) = m(1) + 1 = e$$

Ex. 3.10 (**a**)

$$\lim_{m\to\infty} \frac{\sum\limits_{i=1}^{N_1+N_2+\ldots+N_m} X_i}{N_1+N_2+\ldots+N_m} = \mathbb{E}[X_1]$$

(**b**)

$$\lim_{m\to\infty} \frac{\sum\limits_{i=1}^{m} S_i}{m} . \frac{m}{\sum\limits_{i=1}^{m} N_i} = \frac{\mathbb{E}[S_1]}{\mathbb{E}[N_1]}$$

(**c**)

$$\mathbb{E}[X_1] = \frac{\mathbb{E}[S_1]}{\mathbb{E}[N_1]} \implies \mathbb{E}[S_1] = \mathbb{E}[X_1]\mathbb{E}[N_1]$$

Ex. 3.11 (**a**)

$$X_i = \begin{cases} 2 & w.p.\ 1/3 \\ 4 & w.p.\ 1/3 \\ 8 & w.p.\ 1/3 \end{cases}$$

$$N = \min\{n : X_n = 2\}$$

(**b**)

$$\mathbb{E}[T] = \mathbb{E}[X_1]\mathbb{E}[N] = \frac{14}{3}\frac{1}{1/3} = 14$$

(**c**)

$$\mathbb{E}\left[\sum_{i=1}^{N} X_i | N = n\right] = (4+8)\frac{1}{2}(n-1) + 2 = 6n - 4$$

$$\mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \frac{14n}{3}$$

(**d**)

$$\mathbb{E}\left[T\right] = \mathbb{E}[\mathbb{E}[T|N]] = \mathbb{E}[6N-4] = 6.3 - 4 = 14$$

Ex. 3.12

$h(t) = \mathbb{I}(t \le a)$

$$\lim_{t\to\infty} \int_0^t h(t-x)dm(x) = \lim_{t\to\infty} \int_{t-a}^t dm(x) = \lim_{t\to\infty} m(t) - m(t-a) = \lim_{t\to\infty} \frac{1}{\mu} \int_0^t h(x)dx = \frac{a}{\mu}$$

Ex. 3.13

$$\frac{\mathbb{E}[T_i]}{\mathbb{E}[\sum_{i=1}^n T_i]} = \frac{\mu_i}{\sum_{j=1}^n \mu_j}$$

Ex. 3.14 (**a**)

$$(t-x,t]$$

(**b**)

$$(t,t+x]$$

(**c**)

$$\mathbb{P}\{Y(t)>x\} = \mathbb{P}\{A(t+x)>x\}$$

(**d**)

$$\begin{aligned}\mathbb{P}\{Y(t)>y, A(t)>x\} &= \mathbb{P}\{\text{No event in } (t,t+y], \text{No event in } (t-x,t]\}\\ &= \mathbb{P}\{\text{No event in } (t-x,t+y]\}\\ &= \exp(-\lambda(x+y))\end{aligned}$$

Ex. 3.15 (**a**)

$$\mathbb{P}\{Y(t)>x|S_{N(t)}=t-s\} = \mathbb{P}\{X>x+s|X>s\} = \frac{\bar{F}(x+s)}{\bar{F}(x)}$$

(**b**)

$$\mathbb{P}\{Y(t)>x|A(t+x/2)=s\} = \begin{cases} 0 & s<x/2\\ \mathbb{P}\{X>x/2+s|X>s-x/2\} = \frac{\bar{F}(s+x/2)}{\bar{F}(s-x/2)} & s\geq x/2\end{cases}$$

(**c**)

$$\begin{aligned}\mathbb{P}\{Y(t)>x|A(t+x)>s\} &= \mathbb{P}\{\text{No event in } (t,t+x]|\text{No event in } (t+x-s,t+x]\}\\ &= \begin{cases} 1 & s\geq x\\ \frac{\mathbb{P}\{\text{No event in } (t,t+x]\}}{\mathbb{P}\{\text{No event in} (t+x-s,t+x]\}} = \frac{\exp(-\lambda x)}{\exp(-\lambda s)} = \exp(-\lambda(x-s)) & s<x\end{cases}\end{aligned}$$

(**d**)

$$\begin{aligned}\mathbb{P}\{Y(t)>x, A(t)>y\} &= \mathbb{P}\{Y(t>x), S_{N(t)}<t-y\}\\ &= \int_0^{t-y} \mathbb{P}\{Y(t)>x|S_{N(t)}=s\} dF_{S_{N(t)}}(s)\\ &= \int_0^{t-y} \frac{\bar{F}(x+t-s)}{\bar{F}(x)} \cdot \bar{F}(t-s) dm(s)\end{aligned}$$

(**e**)

$$\frac{A(t)}{t} = \frac{t - S_{N(t)}}{t} = 1 - \frac{S_{N(t)}}{N(t)} \cdot \frac{N(t)}{t} \to 1 - \mu \cdot \frac{1}{\mu} = 0$$

Ex. 3.16

$$\mathbb{E}[Y(t)] \to \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]} = \frac{n/\lambda^2 + n^2/\lambda^2}{2n/\lambda} = \frac{n+1}{2\lambda}$$

Ex. 3.17

$$g = h + g * F = h + F * (h + F * g) = \ldots = h + h * \sum_{n=1}^{\infty} F_n = h + h * m$$

(**a**)

$$\begin{aligned}\mathbb{P}\{\text{on at } t\} &= \mathbb{P}\{\text{on at } t | S_{N(t)} = 0\}\bar{F}(t) + \int_0^t \mathbb{P}\{\text{on at } t | S_{N(t)} = y\}\bar{F}(t-y)dm(y) \\ &= \bar{H}(t) + \int_0^t \bar{H}(t-y)dm(y) \to \frac{\mu_H}{\mu_F}\end{aligned}$$

(**b**)

$$\begin{aligned}\mathbb{E}[A(t)] &= \mathbb{E}[A(t)|S_{N(t)} = 0]\bar{F}(t) + \int_0^t \mathbb{E}[A(t)|S_{N(t)} = y]\bar{F}(t-y)dm(y) \\ &= t\bar{F}(t) + \int_0^t (t-y)\bar{F}(t-y)dm(y) \\ &\to \frac{\int_0^\infty t\bar{F}(t)dt}{\mu} = \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]}\end{aligned}$$

Ex. 3.19

$$\begin{aligned}\mathbb{P}\{S_{N(t)} \le s\} &= \sum_{n=0}^{\infty} \mathbb{P}\{S_n \le s, S_{n+1} > t\} \\ &= \mathbb{P}\{S_1 > t\} + \sum_{n=1}^{\infty} \int_0^\infty \mathbb{P}\{S_n \le s, S_{n+1} > t | S_n = y\}\mathbb{P}\{S_n = y\}dy \\ &= \bar{G}(t) + \sum_{n=1}^{\infty} \int_0^s \mathbb{P}\{S_{n+1} > t | S_n = y\}d(G * F_{n-1})(y) \\ &= \bar{G}(t) + \int_0^s \bar{F}(t-y)dm_D(y)\end{aligned}$$

Ex. 3.20 (**a**)

$$\mathbb{E}[T_{\cdot\to HHTHHTT}] = \mathbb{E}[T_{HHTHHTT\to HHTHHTT}] = 2^7$$

(**b**)

$$\mathbb{E}[T_{\cdot\to HHTT}] = 2^4$$
$$\mathbb{E}[T_{\cdot\to HTHT}] = \mathbb{E}[T_{\cdot\to HT}] + \mathbb{E}[T_{HT\to HTHT}] = 2^2 + 2^4$$

Ex. 3.21 $T_{\cdot\to WWWWWWW}$ is, by definition, stopping time. Using Wald's equation,
(**a**)

$$\mathbb{E}\left[\sum_{i=1}^{T_{\cdot\to WWWWWWW}} X_i\right] = \mathbb{E}[X_i]\mathbb{E}[T_{\cdot\to WWWWWWW}] = (2p-1)(\sum_{i=1}^{7} p^{-i})$$

(**b**)

$$\mathbb{E}\left[\sum_{i=1}^{T_{\cdot\to WWWWWWW}} Y_i\right] = \mathbb{E}[Y_i]\mathbb{E}[T_{\cdot\to WWWWWWW}] = p(\sum_{i=1}^{7} p^{-i})$$

Ex. 3.22

$$\mathbb{E}[N_A] = \mathbb{E}[N_{HH}] + p^{-4}q^{-2} = p^{-1} + p^{-2} + p^{-4}q^{-2}$$
$$\mathbb{E}[N_B] = p^{-2}q^{-3}$$
$$\mathbb{E}[N_{A|B}] = \mathbb{E}[N_A]$$
$$\mathbb{E}[N_{B|A}] = \mathbb{E}[N_{B|H}] = \mathbb{E}[N_B] - \mathbb{E}[N_H] = \mathbb{E}[N_B] - p^{-1}$$

$M = \min\{N_A, N_B\}$ and $a = \mathbb{P}\{A \text{ before } B\}$
(**a**)(**b**)

$$\mathbb{E}[N_A] = \mathbb{E}[N_A - M] + \mathbb{E}[M] = \mathbb{E}[N_A - M | M = N_B]a + \mathbb{E}[M]$$
$$70 = \mathbb{E}[N_{A|B}]a + \mathbb{E}[M] = (2 + 2^2 + 2^6)a + \mathbb{E}[M] = 70a + \mathbb{E}[M]$$
$$\mathbb{E}[N_B] = \mathbb{E}[N_{B|A}](1-a) + \mathbb{E}[M]$$
$$32 = 30(1-a) + \mathbb{E}[M]$$

So, $a = 0.68$ and $\mathbb{E}[M] = 22.4$

Ex. 3.23 Let A be the set of binary strings of length k where $1 \equiv H$ and $0 \equiv T$. Let σ be the binary string corresponding to first k flips of coin and F be the number of additional flips required to obtain the same pattern.

$$\mathbb{E}[F] = \sum_{a\in A} \mathbb{E}[F|\sigma = a]\mathbb{P}\{\sigma = a\} = \sum_{a\in A} \frac{1}{\mathbb{P}\{\sigma = a\}}\mathbb{P}\{\sigma = a\} = 2^k$$

Ex. 3.24 Let a renewal correspond to last 4 cards being of same suit. Let L denote the suit of the last renewal (i.e. of the 4 consecutive cards of same suit), N denote the suit of the card just after the last renewal. T be the time to get the first renewal i.e. the first time 4 consecutive cards of same suit appear. Let T' be the time between two renewals. Since,

$$\begin{aligned}\mathbb{E}[T'|L=i] &= \mathbb{E}[T'|L=i,N=i]\mathbb{P}\{N=i|L=i\} + \mathbb{E}[T'|L=i,N\neq i]\mathbb{P}\{N\neq i|L=i\} \\ &= 1\cdot 1/4 + \mathbb{E}[T]\cdot 3/4\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}[T'] &= \sum_{i=1}^{4}\mathbb{E}[T|L=i]\mathbb{P}\{L=i\} \\ &= 1/4 + \frac{3}{4}\mathbb{E}[T]\end{aligned}$$

Finally, using $\mathbb{E}[T'] = \lim_{n\to\infty}\mathbb{P}\{\text{renewal at } n\}^{-1} = 4^3$, we have, $\mathbb{E}[T] = 85$.

Ex. 3.25 (**a**)

$$m_D = G * \sum_{n=1}^{\infty} F_{n-1} = G + G * \sum_{n=1}^{\infty} F_n = G + G * m$$

(**b**)

$$\begin{aligned}\mathbb{E}[A_D(t)] &= t\bar{G}(t) + \int_0^t (t-y)\bar{F}(t-y)dm_D(y) \\ &\to 0 + \frac{\int_0^\infty x^2 dF(x)}{2\int_0^\infty x dF(x)} \quad \text{(by key-renewal theorem of delayed renewal process)}\end{aligned}$$

(**c**)

$$\mathbb{E}[X] - \int_0^t xg(x)dx = \int_t^\infty xg(x)dx \geq t\int_t^\infty g(x)dx = t\bar{G}(t)$$

$$0 \leq \lim_{t\to\infty} t\bar{G}(t) \leq \lim_{t\to\infty}\left(\mathbb{E}[X] - \int_0^t xg(x)dx\right) = 0$$

Ex. 3.26 The proof is similar to the proof of $m(t+a) - m(t) \to \frac{a}{\mathbb{E}[X]}$.

$$\mathbb{E}[R(t+a)] - \mathbb{E}[R(t)] \to a\lim_{t\to\infty}\frac{R(t)}{t} = a\frac{\mathbb{E}[R]}{\mathbb{E}[X]}$$

Ex. 3.27

$$
\begin{aligned}
\mathbb{E}[R_{N(t)+1}] &= \mathbb{E}[R_{N(t)+1}|S_N(t)=0]\bar{F}(t) + \int_0^t \mathbb{E}[R_{N(t)+1}|S_N(t)=y]\bar{F}(t-y)dm(y) \\
&= \mathbb{E}[R_1|X_1>t]\bar{F}(t) + \int_0^t \mathbb{E}[R|X>t-y]\bar{F}(t-y)dm(y) \\
&\to \frac{1}{\mu}\int_0^\infty \mathbb{E}[R|X>t]\bar{F}(t)dt \\
&= \frac{1}{\mu}\int_0^\infty \int_{-\infty}^\infty \int_t^\infty r dF_{R,X}(r,x)dt \\
&= \frac{1}{\mu}\int_0^\infty \int_{-\infty}^\infty \int_0^x dt r dF_{R,X}(r,x) \\
&= \frac{\mathbb{E}[RX]}{\mu}
\end{aligned}
$$

Ex. 3.28

$$
\begin{aligned}
N^* &= \sqrt{\frac{2K}{\mu c}} \\
\mathbb{E}[\text{cost}(N^*)] &= \sqrt{\frac{2Kc}{\mu}} - c/2
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[\text{cost}] &= \frac{c\mu\mathbb{E}[N(T)^2 - N(T)]/2 + K}{T} = \frac{c\mu^3T^2/2 + K}{T} \\
T^* &= \sqrt{\frac{2\mu K}{c}} \\
\mathbb{E}[T^*] &= \sqrt{\frac{2Kc}{\mu}}
\end{aligned}
$$

Ex. 3.29 (**a**)

$$
\begin{aligned}
\mathbb{E}[\text{cycle time}] &= \mathbb{E}[\min(A,X)] \\
\mathbb{E}[\text{reward in a cycle}] &= C_1\mathbb{P}\{X \geq A\} + (C_1+C_2)\mathbb{P}\{X < A\}
\end{aligned}
$$

(**b**)

$$
\begin{aligned}
\mathbb{E}[\text{cycle time}] &= \left(\frac{1}{\mathbb{P}\{X<A\}} - 1\right)A + \mathbb{E}[X|X<A] = \frac{\mathbb{E}[\min(A,X)]}{\mathbb{P}\{X<A\}} \\
\mathbb{E}[\text{reward in a cycle}] &= \left(\frac{1}{\mathbb{P}\{X<A\}} - 1\right)C_1 + C_1 + C_2 = \frac{C_1\mathbb{P}\{X\geq A\} + (C_1+C_2)\mathbb{P}\{X<A\}}{\mathbb{P}\{X<A\}}
\end{aligned}
$$

Ex. 3.30 Let T be the time to get m consecutive tails. Then, long run proportion of the number of heads is,

$$\frac{N_H(t)}{t} \to \frac{\mathbb{E}[\sum_{n=1}^{T} \mathbb{I}(X_n = H)]}{\mathbb{E}[T]} = \frac{\int_0^1 \sum_{k=1}^{m} p(1-p)^{-k} C p^{n-1}(1-p)^{m-1} dp}{\int_0^1 \sum_{k=1}^{m} (1-p)^{-k} C p^{n-1}(1-p)^{m-1} dp}$$

$$= 1 - \frac{\int_0^1 \sum_{k=1}^{m} (1-p)^{-(k-1)} C p^{n-1}(1-p)^{m-1} dp}{\int_0^1 \sum_{k=1}^{m} (1-p)^{-k} C p^{n-1}(1-p)^{m-1} dp} = 1 - \frac{\to \text{const} < \infty}{\to \infty} = 1$$

The denominator reaches infinity when $k = m$.