Eigenvalues and Eigenvectors

Dhruv Kohli

Department of Mathematics
Indian Institute of Technology, Guwahati

Outline

Introduction

- $Ax = \lambda x$ is a nonlinear equation. Given λ, it becomes linear in x.
- Solving $Ax = \lambda x \iff (A - \lambda I)x = 0$ is to find x in $N(A - \lambda I)$ where λ is chosen so that $A - \lambda I$ has a nullspace. $x = 0$ is always a solution, but is not interesting.
- $N(A - \lambda I)$ must contain non-zero vector. It must be singular i.e. λ is an eval. of $A \iff det(A - \lambda I) = 0$ which is the characterisitc equation. Each λ is associated with an evec. x.
- Examples - Diagonal and triangular matrices have evals. on their diagonal and evals. of Projection matrices are 1 and 0 (Why?).
- Geometrically, we find λ and x s.t. $Ax \parallel x$.

Introduction contd.

Result 1

A matrix $A_{n\times n}$ has n evals. $\{\lambda_i\}_{i=1}^{n}$ where,

$$\sum_{i=1}^{n}\lambda_i = tr(A), \quad \prod_{i=1}^{n}\lambda_i = det(A)$$

Proof - Hints:

$$A-\lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & . & a_{1n} \\ a_{21} & a_{22}-\lambda & . & a_{2n} \\ . & . & . & . \\ a_{n1} & a_{n2} & . & a_{nn}-\lambda \end{bmatrix}$$

Σ evals. $= (-1)^{n-1}\times$ coefficient of λ^{n-1} in $det(A-\lambda I)=0$ which equals $tr(A)$. $\prod$ evals. $=$ constant term in $det(A-\lambda I)$. Or put $\lambda=0$ in $det(A-\lambda I) = \prod_{i=1}^{n}(\lambda_1-\lambda)$ (Why ?).

Diagonalization of a Matrix

Result 2

Suppose $A_{n\times n}$ has n linearly independent vectors which are placed in the columns of a matrix S, then $S^{-1}AS$ is a diagonal matrix Λ whose diagonal entries are evals. of A.

Proof - Hints:

Note that S is invertible.

$AS = [Ax_1, \ldots, Ax_n] = [\lambda_1 x_1, \ldots, \lambda_n x_n] = S \; diag(\lambda_1, \ldots, \lambda_n)$

Therefore, $AS = S\Lambda$. Also, note that $A = S\Lambda S^{-1}$.

Result 3

If $A_{n\times n}$ has n distinct evals. then n evecs. are linearly independent.

Proof - Hints: For $n = 2$,

$$0 = c_1x_1 + c_2x_2 \Rightarrow A0 = 0 = c_1\lambda_1x_1 + c_2\lambda_2x_2$$

Diagonalization of a Matrix contd.

Result 4

The evals. of A^k are $\{\lambda_i^k\}_{i=1}^n$ and every evec. of A is an evec. of A^k. If S diagonalizes A, then, it also diagonalizes A^k.

Proof - Hints: Let λ_i be an eval. of A and x_i be the associated evec. Then, $A^k x_i = A^{k-1}\lambda_i x_i \ldots = \lambda_i^k x_i$. If S diagonalizes A, then, $S^{-1}A^k S = S^{-1}ASS^{-1}A^{k-1}S \ldots = \Lambda^k$.

- If A is invertible, this rule also applies to its inverse.
- Analogy of this rule to product of two different matrices does not hold (construct an example) unless their evecs. are same.

Diagonalization of a Matrix contd.

Result 5

Diagonalizable matrices share the same evec. matrix S if and only if $AB = BA$ i.e. they commute.

Proof - Hints:

$(\implies)$ $AB = S^{-1}\Lambda_1 SS^{-1}\Lambda_2 S = S^{-1}\Lambda_1\Lambda_2 S = BA$

$(\impliedby)$ Let x be evec of A, then $ABx = BAx = \lambda Bx$, therefore, Bx is an evec. of A. If we assume that all evals. of A are distinct, then all eigenspaces are one-dimensional, and since x and Bx are evec. of A with same eval. λ, Bx must be a multiple of x. So, x is evec. of B (try to prove when evals. are repeated).

Complex Matrices

- $\bar{x}^T y$ is different from $\bar{y}^T x$.
- A Hermitian is $A^H = \bar{A}^T$ and A is said to be Hermitian if $A = A^H$ and it contains real diagonal entries and the off-diagonal entries are mirror images across main diagonal.
- Inner product of x and y is $x^H y$. Orthogonal vectors have $x^H y = 0$.
- The squared length of x is $x^H x = \sum_{i=1}^{n} |x_i|^2$.
- $(AB)^H = B^H A^H$.

Complex Matrices contd.

Result 6

If $A = A^H$, then for all complex real vectors x, the number $x^H Ax$ is real and therefore, evals. of A are real.
Proof - Hints: $(x^H Ax)^H = x^H A^H x = x^H Ax$, therefore, $x^H Ax$ is real. Let λ be an eval. of A which is possibly complex. Then, $Ax = \lambda x \Rightarrow x^H Ax = \lambda x^H x \Rightarrow \lambda = \frac{x^H Ax}{x^H x}$. The denominator is real by definition and the numerator is real because A is Hermitian, therefore, λ is real.

Result 7

Two evecs. of a real symmetric matrix or a Hermitian matrix, if they come from different evals., are orthogonal to one another.
Proof - Hints: Let x and y be evecs. associated with different evals. Then, $x^H \lambda_2 y = x^H Ay = x^H A^H y = (Ax)^H y = \lambda_1 x^H y$, therefore, $x^H y(\lambda_1 - \lambda_2) = 0$.

Complex Matrices contd.

- When A is Hermitian, the diagonalizing matrix can be chosen so that the columns are orthonormal.
- If A is real-symmetric then evecs. are also real.
- Spectral Theorem: A real symmetric matrix can be factored into $Q\Lambda Q^T$ where columns of Q are orthonormal evecs and evals in Λ ($Q^{-1} = Q^T$). Also, $A = Q\Lambda Q^T$ which can be written as the combinations one dimensional projections onto line through evec x_i, i.e. $\sum_{i=1}^{n} \lambda_i x_i x_i^T$.
- Surely, if the eigenvalues of a symmetric matrix are distinct then $A = Q\Lambda Q^T$, but, even if the symmetric matrix has repeated evals., it still has a complete set of orthonormal evecs. [We will see soon.]
- Complex matrix with orthonormal columns is called Unitary matrix. $U^H U = I, UU^H = I$ and $U^H = U^{-1}$.

Complex Matrices contd.

Result 8

Unitary matrix preserve distances, have evals. with absolute value of 1 and evecs. corresponding to different evals. are orthogonal.

Proof - Hints:

$$(Ux)^H(Uy) = x^H U^H Uy = x^H y, (Ux)^H(Ux) = x^H x$$

$$Ux = \lambda x \Rightarrow (Ux)^H(Ux) = (\lambda x)^H(\lambda x) \Rightarrow |\lambda| = 1$$

$$x^H y = (Ux)^H Uy = \lambda_1^H \lambda_2 x^H y \Rightarrow x^H y(\lambda_1^H \lambda_2 - 1) = 0$$

Since $\lambda_1 \neq \lambda_2$ and $|\lambda_1| = |\lambda_2| = 1$ $\lambda_1^H \lambda_2 \neq 1$, therefore, $x^H y = 0$.

Complex Matrices contd.

Result 9

Skew-Hermitian matrix have $K^H = -K$. If A is Hermitian then $K = iA$ is skew-Hermitian and eigenvalues of a skew-Hermitian matrix are imaginary.
Proof - Hints: $K^H = A^H(-i) = -iA^H = -iA = -K$. Note that $x^H Kx$ is imaginary $\because (x^H Kx)^H = x^H K^H x = -x^H Kx$. Therefore, $\lambda = \frac{x^H Kx}{x^H x}$ has imaginary numerator and real denominator.

- Diagonal entries of K are imaginary (allowing zero).
- Evecs. of skew-Hermitian are still orthogonal and K can be decomposed into $K = U\Lambda U^H$ with unitary U.

Complex Matrices contd.

- A Hermitian or symmetric matrix can be compared to a real number (evals. are real).
- A Unitary matrix can be compared to a number on unit circle i.e. a complex number of absolute value 1 (evals. have absolute value of 1).
- A skew-Hermitian matric can be compared with pure imaginary numbers (evals. are imaginary).
- Normal matrices can be compared with all complex numbers (evals. are of form $a + ib$).
- A nonnormal matrix without orthogonal evecs. belong to none of these classes and is outstide the whole analogy.

Similarity Transformation

- The matrices A and $M^{-1}AM$ are similar. Going from one to the other is similarity transformation.
- These combination $M^{-1}AM$ arise upon change of variables in differential or difference equation.

$$\frac{du}{dt} = Au \xrightarrow{u=Mv} M\frac{dv}{dt} = AMv \text{ or } \frac{dv}{dt} = M^{-1}AMv$$

Result 10

Suppose $B = M^{-1}AM$ then both A and B have same evals. Every evec. x of A corresponds to evec. $M^{-1}x$ of B.
Proof - Hints: $Ax = \lambda x \Rightarrow MBM^{-1}x = \lambda x \Rightarrow BM^{-1}\lambda M^{-1}x$.
Alternatively, $det(A - \lambda I) = det(MBM^{-1} - \lambda MM^{-1}) = det(M)det(B - \lambda I)det(M^{-1}) = det(B - \lambda I)$

- Every $M^{-1}AM$ has same number of independent evecs. as A (each evec. is multiplied with M^{-1}).

Similarity Transformation contd.

- Every linear transformation is represented by a matrix. The matrix depends on the choice of basis. If we change the basis by M we change the matrix A to a similar matrix B.
- **Similar Matrices represent the same transformation T with respect to different bases.**

Change of Basis = Similarity Transformation

The matrices A and B which represent the same linear transformation with respect to different bases are similar:

$$\begin{aligned}[T]_{V \text{ to } v} &= [I]_{v \text{ to } V}[T]_{v \text{ to } v}[I]_{V \text{ to } v} \\ B \quad &= M^{-1} \quad\quad A \quad\quad M\end{aligned}$$

Similarity Transformation contd.

- The aim is to find an M which can most simplify A i.e. $M^{-1}AM$ becomes diagonal which is equivalent to finding evecs. of A and fill in the columns of M with them. The algebraist says the same thing in the language of Linear Transformation: Choose a basis consisting of evecs.
- $M^{-1}AM$ do not arise in solving $Ax = b$. There we multiply A on LHS by a matrix that subtracts a multiple of one row from another. Such a transformation preserved null space and row space but normally changes eigenvalues.
- Eigenvalues are calculated by a sequence of similarities. The matrix goes gradually towards a triangular form, and the eigenvalues gradually appear on the diagonal.

Similarity Transformation contd.

Result 11 - Schur's Lemma

There is a unitary matrix $M = U$ s.t $U^{-1}AU = T$ is triangular. The eigenvalues of A appear along the diagonal of this similar matrix T.

Proof - Hints: A will have atleast one eval. and therefore has atleast one **unit** evec. Put this evec. in the first column of a matrix U_1 and fill the rest of the matrix such that U_1 becomes unitary.

$$U_1A = U_1\begin{bmatrix} \lambda_1 & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & \dots & \vdots \\ 0 & * & \dots & * \end{bmatrix} \implies U_1^{-1}AU_1 = \begin{bmatrix} \lambda_1 & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & \dots & \vdots \\ 0 & * & \dots & * \end{bmatrix}$$

Similarity Transformation contd.

Result 11 contd.

Now, work recursively with RHS matrix with first column and first row removed. Let M_2 be the unitary matrix corresponding to the submatrix then,

$$U_2 = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & & & \\ 0 & & M_2 & \\ 0 & & & \end{bmatrix} \implies U_2^{-1}U_1^{-1}AU_1U_2 = \begin{bmatrix} \lambda_1 & * & . & * \\ 0 & \lambda_2 & . & * \\ \vdots & \vdots & . & \vdots \\ 0 & 0 & . & * \end{bmatrix}$$

The product $U = U_1U_2U_3\dots$ is still a unitary matrix.

This result applies to all matrices, with no assumption that A is diagonalizable. This can also be used to prove that the powers A^k approach zero when all $|\lambda_i| < 1$, and the exponentials e^{At} approach 0 when all $Re\lambda_i < 0$ - even without full set of evecs.

Similarity Transformation contd.

Result 12 - Spectral Theorem

Every real symmetric matrix can be diagonalized by an orthogonal matrix and every hermitian matrix can be diagonalized with unitary matrix.

$$(real) \quad Q^{-1}AQ = \Lambda or A = Q\Lambda Q^T$$

$$(complex) \quad U^{-1}AU = \Lambda or A = U\Lambda U^H$$

Proof - Hints: From Schur's lemma, $U^{-1}AU = T$. Since $A = A^H$, $T = T^H$. Therefore, T is diagonal and equals Λ.

Every Hermitian matrix with k different evals. has a spectral decomposition into $A = \sum_{i=1} \lambda_i P_i$, where P_i is the porjection onto the eigenspace for λ_i. Since there is a full set of evecs., the projection add up to the identity and since the eigenspaces are orthogonal, $P_j P_i = 0$.

Similarity Transformation contd.

- For which matrices $T = \Lambda$? Hermitian, skew-Hermitian and unitary matrices are in this class. They correspond to numbers on real axis, imaginary axis and the unit circle.
- The whole class contain matrices (corresponding to all complex numbers) which are called normal.

Result 13

Matrix N is normal if it commutes with N^H : $NN^H = N^H N$. For such matrices and no others $T = U^{-1}NU$ is diagonal Λ. Normal matrices are exactly those that have a complete set of orthonormal evecs.
Proof - Hints: If N is normal then T is normal. A triangular normal matrix is a diagonal matrix.

Bibliography

Gilbert Strang. *Linear algebra and its applications*. Belmont, CA: Thomson, Brooks/Cole, 2006.