Solution Manual

prepared by

Dhruv Kohli

for

Introduction to Topological Manifolds, 2nd ed.

by

John M. Lee

2. Topological Spaces

Ex. 2.4
(**a**) ($\implies$) For all $x \in M$ and every $r > 0$, $B_r^d(x)$ is open ball in M with respect to d. Both d and d' generate the same topology on M which implies that $B_r^d(x)$ must be open with respect to d'. Therefore, $\exists\, r_1 > 0$ s.t. $B_{r_1}^{d'}(x) \subseteq B_r^d(x)$. By symmetry, $\exists\, r_2 > 0$ s.t. $B_{r_2}^d(x) \subseteq B_r^{d'}(x)$.

($\impliedby$) Let $A \subseteq M$ be open in M with respect to d. Then, $\forall x \in A$, $\exists\, r > 0$ s.t. $B_r^d(x) \subseteq A$. Also, $\exists\, r_1 > 0$ s.t. $B_{r_1}^{d'}(x) \subseteq B_r^d(x)$. Therefore, $\forall x \in A$, $\exists\, r_1 > 0$ s.t.$B_{r_1}^{d'}(x) \subseteq A$. Hence, A is also open in M with respect to d'. Similarly, every open subset of M with respect to d' is also open with repect to d. Hence, d and d' generate same topology on M.

(**b**) $\forall x \in M, \forall r > 0$ and for $r_1 = rc > 0$ and $r_2 = \frac{r}{c} > 0$, $B_{r_1}^{d'}(x) = B_r^d(x)$ and $B_r^{d'}(x) = B_{r_2}^d(x)$. Then use (**a**).

(**c**)

$$d'(x, y) \leq d(x, y) \leq \sqrt{n} d'(x, y)$$

$\forall x \in M, \forall r > 0$ s.t. for $r_1 = \frac{r}{\sqrt{n}} > 0$ and $r_2 = r > 0$, $B_{r_1}^{d'}(x) \subseteq B_r^d(x)$ and $B_r^{d'}(x) \subseteq B_{r_2}^d(x)$. Then use (**a**).

(**d**) $\forall x \in X, B_{0.5}^d(x) = \{x\}$. Therefore, every subset of X is open with respect to d. Then, d generates discrete topology on X.

(**e**) $\forall x \in \mathbb{Z}, B_{0.5}^d(x) = \{x\} = B_{0.5}^{d'}(x)$.

Ex. 2.5

$$\mathcal{T} = \{U \subseteq Y \text{ and } U \text{ is open in } X\}$$

(i) $U = \phi$ and $U = Y \in \mathcal{T}$.

(ii) $U_1, \dots, U_n \in \mathcal{T} \implies U_i \subseteq Y$ and U_i is open in $X \implies \cap_{i=1}^n U_i \subseteq Y$ and $\cap_{i=1}^n U_i$ is open in X by definition.

(iii) $\forall \alpha \in A, U_\alpha \in \mathcal{T} \implies \forall \alpha \in A, U_\alpha \subseteq Y$ and $\forall \alpha \in A, U_\alpha$ is open in X $\implies \cup_{\alpha \in A} U_\alpha \subseteq Y$ and $\cup_{\alpha \in A} U_\alpha$ is open in X by definition.

Ex. 2.6
(i) $\phi \in \mathcal{T}_\alpha$ and $X \in \mathcal{T}_\alpha \implies \phi \in \cap_{\alpha \in A} \mathcal{T}_\alpha$ and $X \in \cap_{\alpha \in A} \mathcal{T}_\alpha$.

(ii) $U_1, \dots, U_n \in \cap_{\alpha \in A} \mathcal{T}_\alpha \implies \forall i, U_i \in \mathcal{T}_\alpha \implies \cap_{i=1}^n U_i \in \mathcal{T}_\alpha \implies \cap_{i=1}^n U_i \in \cap_{\alpha \in A} \mathcal{T}_\alpha$.

(iii) $\forall \beta \in B, U_\beta \in \cap_{\alpha \in A} \mathcal{T}_\alpha \implies \forall \beta \in B, U_\beta \in \mathcal{T}_\alpha \implies \cup_{\beta \in B} U_\beta \in \mathcal{T}_\alpha \implies \cup_{\beta \in B} U_\beta \in \cap_{\alpha \in A} \mathcal{T}_\alpha$.

Ex. 2.9
(**a**) ($\implies$) Suppose $p \in \operatorname{Int} A$. Then by definition of $\operatorname{Int} A$, $\exists\ C \subseteq A$ and C is open in X s.t. $p \in C$. ($\impliedby$) Suppose C is a neighbourhood (open in X) of a point p s.t. $C \subseteq A$. Then by definition of $\operatorname{Int} A$, $C \subseteq \operatorname{Int} A$. Hence, $p \in C \subseteq \operatorname{Int} A \implies p \in \operatorname{Int} A$.

(**b**) First note that $\operatorname{Ext} A = X \setminus \bar{A} = \bigcup\{X \setminus B$ where $B \supseteq A$ and B is closed in $X\}$ which can further be simplified as $\operatorname{Ext} A = \bigcup\{D$ where $X \setminus D \subseteq X \setminus A$ and D is open in $X\}$. Now, use a similar argument as in (**a**).

(**c**) Suppose $p \in \partial A$, then, $p \notin \operatorname{Int} A \cup \operatorname{Ext} A$ which implies that $\nexists\ C$ neighbourhood (open in X) of p s.t. $C \subseteq A$ or $X \setminus C \subseteq X \setminus A$ which further implies that every neighbourhood of p contains both a point of A and a point of $X \setminus A$. ($\impliedby$) Suppose every neighbourhood of $p \in X$ contains both a point of A and a point of $X \setminus A$, then, by definition of $\operatorname{Int} A$ and $\operatorname{Ext} A$, $p \notin \operatorname{Int} A \cup \operatorname{Ext} A$, which implies that $p \in X \setminus \operatorname{Int} A \cup \operatorname{Ext} A \equiv p \in \partial A$.

(**d**) Negate (**b**).

(**e**) First note that X is the disjoint union of $\operatorname{Int} A$, ∂A and $\operatorname{Ext} A$. Using (**a**), (**b**) and (**c**), conclude that $p \in \operatorname{Int} A \cup \partial A \iff$ every neighbourhood of p has a point in A. Using (**d**), conclude that $\bar{A} = \operatorname{Int} A \cup \partial A$. Using $\operatorname{Int} A \subseteq A \subseteq \operatorname{Int} A \cup \partial A \implies A \cup \partial A = \operatorname{Int} A \cup \partial A$, conclude that $\bar{A} = A \cup \partial A = \operatorname{Int} A \cup \partial A$.

(**f**) Use (**a**), (**b**), $\operatorname{Ext} A = X \setminus \bar{A}$, $\partial A = X \setminus \operatorname{Int} A \cup \operatorname{Ext} A$, the fact that union of two open sets is open and the complement of a closed (open) set is open (closed).

(**g**) and (**h**) follows from above derived results.

Ex. 2.10
($\implies$) Note that $\bar{A}$ contains all limit points (using **2.9**(**b**) and **2.9**(**d**)) and if A is closed then by using **2.9**(**h**), $A = \bar{A}$. ($\impliedby$) Suppose $p \in \partial A$, then, p can either be an isolated point or a limit point. If p is isolated then $p \in A$ by definition. Since A contains all its limit points, therefore, if p is a limit point then also $p \in A$. Hence, the boundary ∂A is contained in A. Using **2.9**(**h**) conclude that A is closed.

Ex. 2.11
($\implies$) If $\bar{A} = X$, then, by using **2.9**(**d**), $\forall x \in X$, every neighbourhood of x

contains a point in A. Suppose B be any non-empty open subset of X and let $y \in B \subseteq X$ then B is a neighbourhood of y, hence, contains a point in A.
$(\Longleftarrow)$ $\forall x \in X$, every neighbourhood of x is an open subset of X (by definition of neighbourhood). Since every open subset of X contains a point in A, therefore, every neighbourhood of x contains a point in A and by using **2.9(d)** $x \in \bar{A}$. Hence, $\bar{A} = X$.

Ex. 2.12
Neighbourhood of $x \in X \equiv B_r^d(x)$ for some $r > 0$. Every neighbourhood of $x \equiv \forall r > 0, B_r^d(x)$.

Ex. 2.13
$\forall x \in X$, $\{x\}$ is a neighbourhood of x. Therefore, by definition of convergence of sequence, $\exists N \in \mathbb{N}$ s.t. $\forall i \geq N, x_i \in \{x\}$. In other words, $\exists N \in \mathbb{N}$ s.t. $\forall i \geq N, x_i = x$. Therefore, for every sequence (x_i) converging to $x \in X$, $x_i = x$ for all but finitely many i.

Ex. 2.14
By definition of convergence of sequence, for every neighbourhood U of $x \in X$, $\exists N \in \mathbb{N}$ s.t. $\forall i \geq N, x_i \in U$ where x_i is a point in A. In other words, every neighbourhood of $x \in X$ contains a point in A and by using **2.9(d)**, $x \in \bar{A}$.

Ex. 2.16
Method (i) $(\Longrightarrow)$ Let $A \subseteq Y$ be closed in Y. Then $Y \setminus A \subseteq Y$ will be open in Y. Since f is a continuous function, $f^{-1}(Y \setminus A)$ is open in X. Note that $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$, which implies that $X \setminus f^{-1}(A)$ is open in X, hence, $f^{-1}(A)$ is closed in X. $(\Longleftarrow)$ Let $A \subseteq Y$ be open in Y. Then $Y \setminus A \subseteq Y$ will be closed in Y and $f^{-1}(Y \setminus A)$ is closed in X. By proposition, $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$ is closed in X, hence, $f^{-1}(A)$ is open in X. Therefore, by definition of continuous function, f is continuous.

Method (ii) $(\Longrightarrow)$ Let $A \subseteq Y$ be closed in Y. Consider a sequence (x_i) where $x_i \in f^{-1}(A)$ converging to $x \in X$. Define a new sequence (y_i) where $y_i = f(x_i) \in A$. Since f is continuous, the sequence (y_i) converges to $y = f(x)$. Since A is closed, by using **2.14**, $y = f(x) \in A$ which implies $x \in f^{-1}(A)$. Again, by using **2.14**, $f^{-1}(A)$ is closed. $(\Longleftarrow)$ Proof of converse is same as in (i).

Ex. 2.18
(a) The constant map is given by $f(x) = y$ where $y \in Y$. Consider $U \subseteq Y$ s.t. U is open in Y. If $y \in U$, then $f^{-1}(U) = X$ where X is open in X. If $y \notin U$, then $f^{-1}(y) = \phi$ where ϕ is again open in X. Therefore, the preimage of every open subset of Y is open in X and thus, by definition of continuous function, f is continuous.

(**b**) The identity map is given by $\mathrm{Id}_X(x) = x$ where $x \in X$. Let $U \subseteq X$ be open in X. Then, $\mathrm{Id}_X^{-1}(U) = U$. Conclude that Id_X is continuous using definition of continuous function.

[verify] (**c**) Let $U \subseteq X$ be open in X. The restriction of f to U is given by $f\big|_U : U \to Y$. Let $A \subseteq Y$ be open in Y, then,
$f\big|_U^{-1}(A) = \{x \in U : f(x) \in A\} = f^{-1}(A) \cap U$. Since, f is continuous, $f^{-1}(A)$ is open in X and therefore, $f^{-1}(A) \cap U$ is open in X (and is open in U with respect to subspace topology on U).

Ex. 2.20
(*i*) $X \approx X$ because Id_X is a continuous bijective function with continuous inverse.

(*ii*) Suppose $X \approx Y$ with f as the homeomorphism from X to Y. Then, $f^{-1} : Y \to X$ is a continuous bijective function with continuous inverse $((f^{-1})^{-1} = f)$ and thus, is a homeomorphism from Y to X. Therefore, $Y \approx X$.

(*iii*) Suppose $X \approx Y$ with respect to f, $Y \approx Z$ with respect to g then $g \circ f : X \to Z$ is a continuous bijective function with continuous inverse $((g \circ f)^{-1} = f^{-1} \circ g^{-1})$ because f^{-1} and g^{-1} are continuous. Thus, $g \circ f$ is a homeomorphism from X to Z. Therfore, $X \approx Z$.

Ex. 2.21
$(\implies)$ f is a homeomorphism from X_1 to X_2 then f and f^{-1} are continuous. Let $U \subseteq X_1$ be open in X_1, then the preimage of U in f^{-1}, $f(U)$, will be an open subset of X_2. Similarly, let $U \subseteq X_2$ be open in X_2, then the preimage of U in f, $f^{-1}(U)$, will be an open subset of X_1. In other words, if $V = f^{-1}(U)$ then $f(V) \subseteq X_2$ being open in X_2 implies that $V \subseteq X_1$ is open in X_1. $(\impliedby)$ The condition $U \in \mathcal{T}_1 \iff f(U) \in \mathcal{T}_2$ which is equivalent to $f^{-1}(U) \in \mathcal{T}_1 \iff U \in \mathcal{T}_2$ implies, by definition of continuous function, that f and f^{-1} are continuous. Since f is already bijective, implies that f is a homeomorphism from X_1 to X_2.

Ex. 2.22
$U \subseteq X$ is open in X and f is a homeomorphism from X to Y. Continuity of f^{-1} implies $f(U)$ is open in Y. Since f is bijective from X to Y implies that $f\big|_U$ is bijective from $U \subseteq X$ to $f(U) \subseteq Y$. Let $V \subseteq f(U)$ be open in $f(U)$ (with respect to subspace topology on $f(U)$) then
$f\big|_U^{-1}(V) = \{x \in U : f(x) \in V\} = f^{-1}(V) \cap U$. Since f is continuous, $V \subseteq f(U) \subseteq Y$ is open in Y and f is continuous implies that $f^{-1}(V) \subseteq U \subseteq X$ is open in X, thus, intersection of $f^{-1}(V)$ and U is open in X (and in U with

respect to subspace topology on U) which implies that $f\big|_U$ is continuous. Now, let $A \subset U$ (with respect to subspace topology on U) be open in U then $f\big|_U(A) = \{f(x) \in f(U) : x \in A\} = f(A) \cap f(U)$ which is open in Y (and in $f(U)$) by a similar argument, which implies that $f\big|_U^{-1}$ is continuous. So, $f\big|_U$ is a continuous bijective function from U to $f(U)$ which has continuous inverse. Hence, $f\big|_U$ is a homeomorphism from U to $f(U)$.

Ex. 2.23
Note that the identity function in the question is different from the identity function defined from $(X, \mathcal{T})$ to $(X, \mathcal{T})$ which is always continuous (and in fact, is a homeomorphism from X to itself).

($\Longrightarrow$) Let $U \in \mathcal{T}_2$. Since Id_X is continuous, preimage of U in Id_X, $\mathrm{Id}_X^{-1}(U) = U$, must blie in $\mathcal{T}_1$ i.e. $U \in \mathcal{T}_1$. Therefore, $\mathcal{T}_2 \subseteq \mathcal{T}_1$, making $\mathcal{T}_1$ finer than $\mathcal{T}_2$. ($\Longleftarrow$) Let $U \in \mathcal{T}_2$, then, $U = \mathrm{Id}_X^{-1}(U) \in \mathcal{T}_1$. By definition of continuous function, Id_X is continuous.

For Id_X (which is already a bijective map) to be a homeomorphism from $(X, \mathcal{T}_1)$ to $(X, \mathcal{T}_2)$, Id_X and Id_X^{-1} must be continuous which is the case if and only if $\mathcal{T}_2 \subseteq \mathcal{T}_1$ and $\mathcal{T}_1 \subseteq \mathcal{T}_2$, respectively. Thus, Id_X and Id_X^{-1} are continuous (and hence, Id_X is a homeomorphism from $(X, \mathcal{T}_1)$ to $(X, \mathcal{T}_2)$) if and only if $\mathcal{T}_1 = \mathcal{T}_2$.

Ex. 2.27

$$\varphi(x, y, z) = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}} = (x', y', z') \text{ where } \max\{|x|, |y|, |z|\} = 1$$

$$\max\{|x|, |y|, |z|\} = 1 \implies \max\{|x'|, |y'|, |z'|\} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

$$\therefore \ \varphi^{-1}(x', y', z') = \frac{(x', y', z')}{\max\{|x'|, |y'|, |z'|\}}$$

Ex. 2.28
Define $s(x) : [0, 1) \to \mathbb{S}^1$ as $s(x) = e^{2\pi i x}$ and its inverse as $x(s) = \frac{\log(s)}{2\pi i}$. Observe that $\mathrm{Re}(s(x)) = \cos(2\pi x)$ and $\mathrm{Im}(s(x)) = \sin(2\pi x)$ are continuous functions of $x \in [0, 1)$ making $s(x)$ a continuous function of $x \in [0, 1)$. However, $x(s)$ is discontinuous at $s = 1 + 0i$. Note that $x(1 + 0^- i)$ will be close to 1, while $x(1 + 0i) = 0$.

Ex. 2.29
(**a**) $\implies$ (**b**) and (**a**) $\implies$ (**c**): Since f is a homeomorphism, f^{-1} is continuous. By the definition of continuous function, let $U \subseteq X$ be open in X,

then, $(f^{-1})^{-1}(U) = f(U)$ will be open in Y making f an open map. Similarly, use **2.16** to conclude that f is a closed map.

(**b**) $\implies$ (**a**) Since f is an open map, by definition of continuous function, f^{-1} is continuous. Therefore, f is continuous and bijective with continuous inverse, hence, f is a homeomorphism from X to Y.

(**c**) $\implies$ (**a**) Use **2.16** and an argument similar to (**b**) $\implies$ (**a**).

Ex. 2.32
(**a**) Let $f : X \to Y$ be a homeomorphism from X to Y. Let $x \in X$ and $U \subseteq X$ be a neighbourhood of x, then, $f(U)$ is open subset of Y because f^{-1} is continuous. By using **2.22**, $f\big|_U : U \to f(U)$ is a homeomorphism from U to $f(U)$, thus, a local homeomorphism.

(**b**) (Continuity): Let $U \subseteq Y$ be open in Y. We must show that $f^{-1}(U)$ is open. Let $x \in f^{-1}(U)$. Then, by definition of local homeomorphism, $\exists\ V_x \subseteq X$ which is a neighbourhood of x s.t. $f(V_x)$ is open and $f\big|_{V_x} : V_x \to f(V_x)$ is a homeomorphism. Since U and $f(V_x)$ are open in Y, then, so is $U \cap f(V_x)$ is open in Y. Since, $f\big|_{V_x}$ is continuous,
$f\big|_{V_x}^{-1}(U \cap f(V_x)) = \{x \in V_x : f(x) \in U \cap f(V_x)\} = V_x \cap f^{-1}(U)$ is open in X. But $V_x \cap f^{-1}(U)$ is a neighbourhood of x contained in $f^{-1}(U)$ and because $x \in f^{-1}(U)$ is arbitrary, therefore, $f^{-1}(U) = \cup_{x \in f^{-1}(U)}(V_x \cap f^{-1}(U))$ is open in X. Hence, f is continuous. (Open): Let $A \subseteq X$ be open in X. By the defintion of local homeomorphism, for every $x \in A$, $\exists U_x \subseteq X$ which is a neighbourhood of x in X s.t. $f(U_x)$ is open in Y and $f\big|_{U_x} : U_x \to f(U_x)$ is a homeomorphism. Since , $U_x \cap A$ is open in U_x, therefore, $f(U_x \cap A)$ is open in $f(U_x)$ and thus in Y. Finally, $A = \cup_{x \in A} U_x \cap A$ and $f(\cup_{x \in A} U_x \cap A)$ is open in Y, so is $f(A)$.

(**c**) Bijective local homeomorphism is bijective, continuous and open, thus, homeomorphism by (**2.29**).

Ex. 2.33
Let (y_i) be any sequence in Y which converges to some $y \in Y$. The only neighbourhood of y is Y itself and since, $\forall i \geq 1, y_i \in Y$, y can take any value in Y. Thus, every sequence in Y converges to every point of Y.

Ex. 2.35
Let $f^{-1}(0) = \{p\}$ for some $p \in X$. Let $q \in X$ s.t. $q \neq p$ and $f(q) = a \neq 0$. Then, $f^{-1}((-a/2, a/2))$ is a neighbourhood of p and $f^{-1}((3a/2, 4a/2))$ is a neighbourhood of q s.t. they are disjoint. Note that no point of X can lie in both neighbourhoods.

Ex. 2.38
Since the finite set X has Hausdorff topology, every finite subset of X is closed and its complement is open. Therefore, every subset of X is both closed and open. Therefore, the topology on X is discrete.

Ex. 2.40
$(\implies)$ Let $U \subseteq X$ be open, then, $\forall p \in U, \exists\ C \subseteq U$ s.t. C is open in X and $p \in C$. By definition of basis, $C = \cup_{\alpha \in A} B_\alpha$. Since $p \in C$, $\exists\ B \in \{B_\alpha : \alpha \in A\}$ s.t. $p \in B \subseteq C \subseteq U$. $(\impliedby)$ The proof of converse follows directly from the definition of open set.

Ex. 2.42
We must show that the an element of $\mathcal{B}$ is an open subset of X and every open subset of X is the union of some collection of elements of $\mathcal{B}$.

(**a**) Let $p \in C_s(x)$, then, define $s^* = \min\limits_{i=1}^{n}(\min(|x_i + s/2 - p_i|, |p_i - (x_i - s/2)|))$ and conclude that $C_{s^*}(p)$ is a neighbourhood of p contained in $C_s(x)$. Therefore, $C_s(x)$ is open in X. Let A be an open subset of $\mathbb{R}^n$. Then, A is a union of open balls contained in it. If $B_r(p)$ is such a ball, then, $C_{\sqrt{2}r}(p) \subseteq B_r(p)$. Therefore, $A = \cup_{x \in A} B_{r_x}(x) = \cup_{x \in A} C_{\sqrt{2}r_x}(x)$. Thus, A is a union of open cubes. Hence, $\mathcal{B}_1$ is a basis for the Euclidean topology on $\mathbb{R}^n$.

(**b**) First, note that we can always find a rational number between two irrational numbers and a rational number between a rational and an irrational number. Here, is a sketch of proof. Let m and n are two irrational numbers s.t. $m > n > 0$. Define $r = m - n$, then, by Archimedes property, we can find a t such that $\frac{1}{r} < t$. Therefore, $rt > 1 \implies mt > nt + 1$ and we can find $p \in \mathbb{N}$ s.t. $mt > p > nt \implies m > \frac{p}{t} > n$. Now, let $B_r(x)$ be an open ball with rational r and x has rational coordinates. By definition, it is open. Let A be an open subset of $\mathbb{R}^n$ and for some arbitrary $y \in A$, let $B_s(y) \subseteq A$ be an arbitrary open ball containing y. We must find a ball with rational radius and coordinates s.t. it contains y and is contained in or equal to $B_s(y)$. If y and s are rational then take $B_{r_y}(x_y) = B_s(y)$. If s and y are irrational (workout the case when one of them is rational in a similar manner), we find a rational x_y s.t. $x_y \in B_{s/2}(y)$ and a rational r_y s.t. $|x_y - y| < r_y < s/2$. Define x_y s.t $x_{y_i} \in (y_i, y_i + s/2)$ is rational and define r_y s.t. $r_y \in (|x_y - y|, s/2)$ is rational (this is possible based on the argument in beginning). Based on this construction, $B_{r_y}(x_y)$ contains y and is contained in $B_s(y)$. Finally, $A = \cup_{y \in A} B_s(y) = \cup_{y \in A} B_{r_y}(x_y)$. Therefore, $\mathcal{B}_2$ is a basis.

Ex. 2.45
(**i**) By property 1 of basis, $B \subseteq X$, therefore, $\cup_{B \in \mathcal{B}} B \subseteq X$. By property 2 of basis, since X is open in X, $X = \cup_{\alpha \in A} B_\alpha \subseteq \cup_{B \in \mathcal{B}} B$. Therefore, $X = \cup_{B \in \mathcal{B}} B$.

(**ii**) $B_1, B_2 \in \mathcal{B}$, then $B_1 \cap B_2$ is open subset of X. Then $B_1 \cap B_2$ satisfy the basis criterion with respect to $\mathcal{B}$ i.e. for every $x \in B_1 \cap B_2, \exists\ B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subseteq B_1 \cap B_2$.

Ex. 2.51
Let $\{B_\alpha, \alpha \in A\}$ be the countable basis. Form a subset D of X in the following manner - Take any one x_α from B_α and put it in D. Then, $D = \{x_\alpha, \alpha \in A\}$ is a countable dense subset of X because, for every $x \in X$, and for every neighbourhood of x, there exist a collection of basis, the union of which forms the neighbourhood and thus, every neighbourhood of x has a point in D making x to be in closure of D. Thus, $\bar{D} = X$.

Ex. 2.54
($\Longrightarrow$) Let M be a 0-manifold. Let $p \in M$, then, $\exists$ neighbourhood U of p s.t. U is homeomorphic to a single point. This can only be the case when $U = \{p\}$. Adding or removing an element to U makes sure that there is no bijection from U to a sinlge point. Since p was arbitrary, for every point p in M, $\{p\}$ is an open subset of M. Since M is second countable, therefore, countably many points p exist in M. Using the the properties of a topology, arbitrary union of single the point sets $\{p\}$ are also open, making M to be a countable discrete space. ($\Longleftarrow$) Let M be a countable discrete space, then it is locally Euclidean of dimension 0, since every point p has a neighbourhood $\{p\}$ which is homeomorphic to single point. It is also second countable, since the basis is the collection of all single point sets $\{p\}$ in M. Finally, M is Hausdorff because $\{p_1\} \cap \{p_2\} = \phi$ when $p_1 \neq p_2$, where $\{p_1\}$ and $\{p_2\}$ are neighbourhoods of p_1 and p_2. Therefore, M is a 0-manifold.

3. New Spaces from Old

Ex. 3.1
(i) $V = \phi$ gives $U = \phi$ and $V = X$ gives $U = S$.

(ii) Let $(U_i)_{i=1}^n$ be open subsets of S, then, $\exists$ $(V_i)_{i=1}^n$ which are open subsets of X s.t. $U_i = S \cap V_i$. Since $\cap_{i=1}^n V_i$ is open in X,
$\cap_{i=1}^n U_i = \cap_{i=1}^n (S \cap V_i) = S \cap (\cap_{i=1}^n V_i)$ is open in S.

(iii) Let $U_\alpha, \alpha \in A$ be open subsets of S, then, $\exists$ $V_\alpha, \alpha \in A$ which are open subsets of X s.t. $U_\alpha = S \cap V_\alpha$. Since $\cup_{\alpha \in A} V_\alpha$ is open in X,
$\cup_{\alpha \in A} U_\alpha = \cup_{\alpha \in A} S \cap V_\alpha = S \cap (\cup_{\alpha \in A} V_\alpha)$ is open in S.

Ex. 3.2
$(\Longrightarrow)$ Let $B \subseteq S$ be closed in S. Then $S \setminus B$ will be open in S. Therefore, $\exists$ $V \subseteq X$ s.t. V is open in X and $S \setminus B = S \cap V$. Then,
$B = S \setminus (S \cap V) = S \cap (X \setminus S \cup X \setminus V) = S \cap X \setminus V$, where $X \setminus V$ is closed in X. $(\Longleftarrow)$ Let $B = S \cap V$ where V is closed in X. Then, $S \setminus B = S \cap (X \setminus V)$, where $X \setminus V$ is open in X. Thus, $S \setminus B$ is open in S and hence, B is closed in S.

Ex. 3.3

Ex. 3.6
(**a**) Since U is open in S, $U = S \cap V$ where V is open in X. Because, S is also open in X and U is the intersection of two open subsets of X, hence, U is open in X. Similarly, using **3.2**, U is closed in S, then, $U = S \cap V$ where V is closed in X. Since, S is closed in X and U is the intersection of two closed subsets of X, hence, U is closed in X.

(**b**) Since $U \subseteq S$, $U = S \cap U$. By definition of subspace topology, if U is open in X then U is open in S and by using **3.2**, if U is closed in X, then U is closed in S.

Ex. 3.7
(**a**) Let $p \in S$ s.t. $p \in$ closure of U in S. Therefore, every relative neighbourhood of p contains a point in U. Let V be an arbitrary neighbourhood of p in X. Then, $S \cap V$ is a relative neighbourhood of p which contains a point in U. Since, $S \cap V \subseteq V$, V contains a point in U. Since, V is arbitrary neighbourhood of p in X which contains a point in U, $p \in \bar{U}$, and hence, $p \in \bar{U} \cap S$. Thus, closure of U in $S \subseteq \bar{U} \cap S$.

Now, let $p \in \bar{U} \cap S$. Then, $p \in S$ and every neighbourhood of p in X contains a point in U. Let A be an arbitrary relative neighbourhood of p, then, $A = S \cap V$ where V is open in X. Note that $p \in A$ implies that $p \in V$ and therefore, V is a neighbourhood of p in X. Since, $U \subseteq S$ and V contains a

point in U, therefore, $A = S \cap V$ contains a point in U. Since, A was arbitrary, $p \in$ closure of U in S. Thus, $\bar{U} \cap S \subseteq$ closure of U in S.
(**b**) Let $p \in \operatorname{Int} U \cap S$, then, $p \in S$ and $\exists\ V \subseteq U$ s.t. V is open in X and $p \in V$. Therefore, $p \in S \cap V$. Since $V \subseteq U$ and V is open in X, $S \cap V \subseteq U$ and is open in S. Therefore, $p \in$ interior of U in S. Thus, $\operatorname{Int} U \cap S \subseteq$ interior of U in S.

Following example shows that interior of U in $S \nsubseteq \operatorname{Int} U \cap S$: Consider $S = [0, 2] \subseteq \mathbb{R}$. Let $U = [0, 1)$. Then U is relatively open in S (because $U = S \cap (-1, 1)$) and therefore the interior of U in S is U itself. But, $\operatorname{Int} U = (0, 1)$ and $\operatorname{Int} U \cap S = (0, 1)$. Now, $0 \in$ interior of U in S but $0 \notin \operatorname{Int} U \cap S$.

Ex. 3.12
(**c**) ($\Longrightarrow$) Let $p_i \to p$ in S. Then, for every relative neighbourhood U of p, $\exists\ N \in \mathbb{N}$ s.t. $\forall i \geq N, p_i \in U$. Let V be an arbitrary neighbourhood of p in X. Since, $S \cap V$ is a relative neighbourhood of p in S, $\exists\ N \in \mathbb{N}$ s.t. $\forall i \geq N, p_i \in S \cap V \subseteq V$, implies, $\exists\ N \in \mathbb{N}$ s.t. $\forall i \geq N, p_i \in V$. Since, V is arbitrary, $p_i \to p$ in X. ($\Longleftarrow$) Let $p_i \to p$ in X. Then, for every neighbourhood V of p, $\exists\ N \in \mathbb{N}$ s.t. $\forall i \geq N, p_i \in V$. But $p_i \in S$, therefore, for every neighbourhood V of p, $\exists\ N \in \mathbb{N}$ s.t. $\forall i \geq N, p_i \in S \cap V$. Let U be a relative neighbourhood of p, then, $\exists\ V \subseteq X$ open in X s.t. $U = S \cap V$. Also, $p \in U$ implies $p \in V$ and therefore, V is a neighbourhood of p in X. By above argument, $\exists\ N \in \mathbb{N}$ s.t. $\forall i \geq N, p_i \in U$. Since, U was arbitrary, $p_i \to p$ in S.

(**d**) Let $p_1, p_2 \in S \subseteq X$. Since X is Hausdorff, $\exists\ U_1$ and U_2 neighbourhood of p_1 and p_2 in X s.t. $U_1 \cap U_2 = \phi$. Define relative neighbourhoods of p_1 and p_2 as $S \cap U_1$ and $S \cap U_2$, respectively. Then, $S \cap U_1 \cap S \cap U_2 = S \cap (U_1 \cap U_2) = S \cap \phi = \phi$. Therefore, S is also Hausdorff.

(**e**) Let $p \in S \subseteq X$. Since X is first countable, there exists a countable collection of neighbourhoods of p in X, $\mathcal{B}_p$, such that for every neighbourhood V of p in X, $\exists\ B \in \mathcal{B}_p$ s.t. $B \subseteq V$. Define a new collection of relative neighbourhoods of p in S as $\mathcal{B}_{S_p} = \{S \cap B : B \in \mathcal{B}_p\}$. Consider an arbitrary relative neighbourhood U of p in S. Then, $\exists\ V \subseteq X$, a neighbourhood of p in X s.t. $U = S \cap V$. Since, $\exists B \in \mathcal{B}_p$ s.t. $B \subseteq V$, therefore, $S \cap B \subseteq S \cap V = U$ where $S \cap B \in \mathcal{B}_{S_p}$. Since U and p are arbitrary, we conclude that for every $p \in S$, there exists a collection of relative neighbourhood of p in S, $\mathcal{B}_{S_p}$ s.t. for every relative neighbourhood U of p, there exists $B \in \mathcal{B}_{S_p}$ s.t. $B \subseteq U$. Finally, note that $|\mathcal{B}| = |\mathcal{B}_{S_p}|$, therefore, S is first countable.

(**f**) Let $\mathcal{B}$ be the countable set of basis for X and $\mathcal{B}_S$ be the basis for S. Using (**b**), $|\mathcal{B}_S| = |\mathcal{B}|$, therefore, $\mathcal{B}_S$ is countable and hence, S is second countable.

Ex. 3.13
$\eta_S : S \hookrightarrow X$ be the inclusion map from S to X.

(i) Injective: $\eta_S(x_1) = \eta_S(x_2) \implies x_1 = x_2$.

(ii) Continuous: Let $A \subseteq X$ be open in X, then, $\eta_S^{-1}(A) = S \cap A$ which is open in S with respect to subspace topology on S.

(iii) Homeomorphism onto its image: $\eta_S' : S \to \eta_S(S)$ where $\eta_S(S) = S$ is nothing but Id_S which is a homeomorphism from S with subspace topology to itself with same topology.

Ex. 3.17
Let $S = [0, 1)$ and $\eta_S : S \hookrightarrow \mathbb{R}$ be an inclusion map. Note that S is both open and closed in S but $\eta_S(S) = [0, 1)$ is neither open nor closed in $\mathbb{R}$. Therefore, η_S is neither an open nor a closed map but it is still a topological embedding using **3.13**.

Ex. 3.19
Image of a surjective map is same as the codomain. Therefore, by definition of topological embedding, a surjective topological embedding is a homeomorphism.

Ex. 3.25
(i) $\cup_{B \in \mathcal{B}} B = \cup_{U_i \subseteq X_i \text{ is open in } X_i} (U_1, \ldots, U_n) = (X_1, \ldots, X_n)$.

(ii) Let $(A_1, \ldots, A_n)$ be open in $(X_1, X_2, \ldots, X_n)$ then note that $(A_1, \ldots, A_n)$ is already in $\mathcal{B}$.

Ex. 3.26

Ex. 3.29
Let U be open in X_i. Then $\pi_i^{-1}(U) = (X_1, \ldots, X_{i-1}, U, X_{i+1}, \ldots, X_n)$. Since, X_j is open in X_j and U is open in X_i, $\pi_i^{-1}(U)$ is open in $(X_1, \ldots, X_n)$, π_i is continuous.

Ex. 3.32
(**a**) The basis of the three topologies are same.

(**b**) Injective: $f(x) = f(x') \implies (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x', x_{i+1}, \ldots, x_n) \implies x = x'$. Continuous: Let $U = (U_1, \ldots, U_n)$ be open in $(X_1, X_2, \ldots, X_n)$. Then $f^{-1}(U) = U_i$ is open in X_i by definition. Continuous and injective onto image follows from Corollary 3.10. Surjective onto image implies bijective onto image. Let U_i be open in X_i, then, $f(U_i) = (X_1, \ldots, X_{i-1}, U_i, X_{i+1}, \ldots, X_n)$ is open in $(X_1, X_2, \ldots, X_n)$.

(**c**) Let $U = (U_1, \dots, U_n)$ be open in $(X_1, X_2, \dots, X_n)$. Then $\pi_i(U) = U_i$ is open in X_i, hence, π_i is an open map.

(**d**) Let $(p_1, \dots, p_n) \in (U_1, \dots, U_n)$, where U_i is open in X_i, then, $p_i \in U_i$ and by basis criterion, $\exists\ B_i \in \mathcal{B}_i$ s.t. $p_i \in B_i \subseteq U_i$. Therefore, $(p_1, \dots, p_n) \in (B_1, \dots, B_n) \subseteq (U_1, \dots, U_n)$ and $(U_1, \dots, U_n)$ satisfies basis criterion with respect to basis $\{(B_1, \dots, B_n) : B_i \in \mathcal{B}_i\}$

(**e**) Product topology basis: $\{(U_1, \dots, U_n)$ where U_i is open in subspace S_i i.e. $\exists\ V_i$ open in X_i s.t. $U_i = S_i \cap V_i\}$. Subspace topology basis: $\{(U_1, \dots, U_n) : (U_1, \dots, U_n) = (S_1, \dots, S_n) \cap (V_1, \dots, V_n)$ for V_i open in $X_i\}$. Here, also, $U_i = S_i \cap V_i$.

(**f**) Let $p = (p_1, \dots, p_n)$ and $p' = (p'_1, \dots, p'_n)$ are points in $(X_1, \dots, X_n)$. Since, X_i is Hausdorff, $\exists\ U_i$ and U'_i neighbourhood of p_1 and p'_1 s.t. $U_i \cap U'_i = \phi$. Define neighbourhoods of p and p' as $(U_1, \dots, U_n)$ and $(U'_1, \dots, U'_n)$, then, their intersection is $(U_1 \cap U'_1, \dots, U_n \cap U'_n) = (\phi, \dots, \phi) = \phi$. Therefore, $(X_1, \dots, X_n)$ is Hausdorff.

(**g**) Define a collection of neighbourhoods of $p = (p_1, \dots, p_n)$ as $\mathcal{B}_p = \{(B_1, \dots, B_n) : B_i \in \mathcal{B}_{p_i}\}$. Since $\mathcal{B}_{p_i}$ is countable, then, so is $\mathcal{B}_p$ because $|\mathcal{B}_p| = \prod_{i=1}^n |\mathcal{B}_{p_i}|$.

(**h**) From (**d**), $|\mathcal{B}| = \prod_{i=1}^n |\mathcal{B}_i|$. Since $|\mathcal{B}_i|$ is countable and n is finite, then, so is $|\mathcal{B}|$. Therefore, $(X_1, \dots, X_n)$ is second countable.

Ex. 3.34

Ex. 3.40
(**i**) ϕ and $\sqcup_{\alpha \in A} X_\alpha$ are open.

(**ii**) Let $(U_i)_{i=1}^n$ be open in $\sqcup_{\alpha \in A} X_\alpha$, then, $U_i = \sqcup_{\alpha \in A} U_{i_\alpha}$ where U_{i_α} is open in X_α. Since $\cap_{i=1}^n U_{i_\alpha}$ is open in X_α, therefore, $\cap_{i=1}^n U_i = \cap_{i=1}^n \sqcup_{\alpha \in A} U_{i_\alpha} = \sqcup_{\alpha \in A} \cap_{i=1}^n U_{i_\alpha}$ is open in $\sqcup_{\alpha \in A} X_\alpha$.

(**iii**) Let $(U_\beta)_{\beta \in B}$ be open in $\sqcup_{\alpha \in A} X_\alpha$, then, $U_\beta = \sqcup_{\alpha \in A} U_{\beta_\alpha}$ where U_{β_α} is open in X_α. Since $\cup_{\beta \in B} U_{\beta_\alpha}$ is open in X_α, therefore, $\cup_{\beta \in B} U_\beta = \cup_{\beta \in B} \sqcup_{\alpha \in A} U_{\beta_\alpha} = \sqcup_{\alpha \in A} \cup_{\beta \in B} U_{\beta_\alpha}$ is open in $\sqcup_{\alpha \in A} X_\alpha$.

Ex. 3.43
(**a**) ($\implies$) Let $U = \sqcup_{\alpha \in A} U_\alpha$, where U_α is the intersection of U with X_α, be a closed subset of $\sqcup_{\alpha \in A} X_\alpha$, then, $\sqcup_{\alpha \in A} X_\alpha \setminus \sqcup_{\alpha \in A} U_\alpha = \sqcup_{\alpha \in A} X_\alpha \setminus U_\alpha$ is open in $\sqcup_{\alpha \in A} X_\alpha$. Therefore, $X_\alpha \setminus U_\alpha$ is open in X_α, implying that, U_α is closed in X_α. ($\impliedby$) Let $U = \sqcup_{\alpha \in A} U_\alpha \subseteq \sqcup_{\alpha \in A} X_\alpha$ where U_α is the intersection of U with X_α which is closed in X_α. Then,

$\sqcup_{\alpha\in A}X_\alpha \setminus U = \sqcup_{\alpha\in A}X_\alpha \setminus \sqcup_{\alpha\in A}U_\alpha = \sqcup_{\alpha\in A}X_\alpha \setminus U_\alpha$, the intersection of which with X_α is $X_\alpha \setminus U_\alpha$ which is open in X_α. Therefore, $\sqcup_{\alpha\in A}X_\alpha \setminus U$ is open in $\sqcup_{\alpha\in A}X_\alpha$, hence, U is closed in $\sqcup_{\alpha\in A}X_\alpha$.

(**b**) (Injective): $\eta_\alpha(x_1) = \eta_\alpha(x_2) \implies x_1 = x_2$. (Continuous): Let $U = \sqcup_{\alpha\in A}U_\alpha$ be open subset of $\sqcup_{\alpha\in A}X_\alpha$, then, U_α is open subset of X_α. Since, $\eta_\alpha^{-1}(U) = U_\alpha$ which is open in X_α, therefore, η_α is continuous. (Open map): Let U_α be open in X_α, then $\eta_\alpha(U_\alpha) = (U_\alpha, \alpha)$, the intersection of which with X_α is U_α which is open X_α and the intersection with $X_{\alpha'}, \alpha' \neq \alpha$ is ϕ which is again open in $X_{\alpha'}$. Therefore, $\eta_\alpha(U_\alpha)$ is open in $\sqcup_{\alpha\in A}X_\alpha$ and thus, η_α is an open map. (Closed map): Proceed in a similar manner as for (Open map). By proposition (**3**.**16**), η_α is a topological embedding.

(**c**) Let $x_1 = (p_1, \alpha_1)$ and $x_2 = (p_2, \alpha_2)$ are point in $\sqcup_{\alpha\in A}X_\alpha$. If $\alpha_1 \neq \alpha_2$, then $X_{\alpha_1} = (X_{\alpha_1}, \alpha_1)$ and $X_{\alpha_2} = (X_{\alpha_2}, \alpha_2)$ are open neighbourhoods containing x_1 and x_2 with empty intersection. If $\alpha_1 = \alpha_2$, then, since X_α is Hausdorff, $\exists\ U_1$ and U_2, neighbourhoods of p_1 and p_2 in X_α s.t. $U_1 \cap U_2 = \phi$, we define neighbourhoods $V_1 = (U_1, \alpha_1)$ and $V_2 = (U_2, \alpha_1)$ in $\sqcup_{\alpha\in A}X_\alpha$ whose intersection is $(U_1 \cap U_2, \alpha_1) = (\phi, \alpha_1) = \phi$.

(**d**) Let $\mathcal{B}_{\alpha_p}$ be the countable collection of neighbourhoods for $p \in X_\alpha$ s.t. for every neighbourhood of p, $\exists\ B_\alpha \in \mathcal{B}_{\alpha_p}$ s.t. B_α is contained in the neighbourhood. Then, $(\mathcal{B}_{\alpha_p}, \alpha)$ is the countable collection of neighbourhood of (p, α) in $\sqcup_{\alpha\in A}X_\alpha$ s.t. for every neighbourhood of (p, α), $\exists\ (B_\alpha, \alpha) \in (\mathcal{B}_{\alpha_p}, \alpha)$ s.t. (B_α, α) is contained in the neighbourhood.

(**e**) Let $\mathcal{B}_\alpha$ be the basis of X_α, then $\mathcal{B} = \sqcup_{\alpha\in A}\mathcal{B}_\alpha$ is the basis of $\sqcup_{\alpha\in A}X_\alpha$ where $|\mathcal{B}| = \sum_{\alpha\in A} |\mathcal{B}_\alpha|$ which is countable if $\mathcal{B}_\alpha$ is countable and A is countable.

Ex. 3.44
($\implies$) If $\sqcup_{\alpha\in A}X_\alpha$ is an n-manifold, then it is second countable. By using **3**.**43**(**e**), we have $\sum_{\alpha\in A} \mathcal{B}_\alpha$ is countable. We are given that $\mathcal{B}_\alpha$ is countable and conclude that A shouble be countable. ($\impliedby$) Converse follows directly from **3**.**43**(**e**), (**d**) and the fact that (p, α) has a neighbourhood which is homeomorphic to an open subset of $\mathbb{R}^n$ because p has a neighbourhood in X_α which is homeomorphic to an open subset of $\mathbb{R}^n$ and $(X_\alpha, \alpha) \approx X_\alpha$.

Ex. 3.45
An element of (X, Y) is (x, y) for some $x \in X$ and $y \in Y$ and an element of $\sqcup_{y\in Y}X$ is (x, y) where $x \in X$ and $y \in Y$. So, the two spaces are same. Let U be an open subset of X, then (U, y) is an open subset of (X, Y). By definition of disjoint topology, (U, y) is open in $\sqcup_{y\in Y}X$ because the intersection of it, with X is U which is open in X. Converse follows in a similar manner.

Ex. 3.46

(i) $q^{-1}(\phi) = \phi$ and $q^{-1}(Y) = X$ because q is surjective.

(ii) Let $(V_i)_{i=1}^n$ be open in Y, then, $\forall i \in \{1, \ldots, n\}, q^{-1}(V_i)$ is open in X. Since, $q^{-1}(\cap_{i=1}^n V_i) = \cap_{i=1}^n q^{-1}(V_i)$ which is open in X, therefore, $\cap_{i=1}^n V_i$ is open in Y.

(iii) Let $(V_\alpha)_{\alpha \in A}$ be open in Y, then, $\forall \alpha \in A, q^{-1}(V_\alpha)$ is open in X. Since, $q^{-1}(\cup_{\alpha \in A} V_\alpha) = \cup_{\alpha \in A} q^{-1}(V_\alpha)$ is open in X, therefore, $\cup_{\alpha \in A} V_\alpha$ is open in Y.

Ex. 3.55
Let $(X_\alpha)_{\alpha \in A}$ be a collection of Hausdorff spaces. Let p be the point where all the base points $(p_\alpha)_{\alpha \in A}$ collapse to form wedge sum $\bigvee_{\alpha \in A} X_\alpha$. Let p_1 and p_2 be two distinct points in $\bigvee_{\alpha \in A} X_\alpha$.

If $p_1 \neq p$ and $p_2 \neq p$, then two cases arise - (i) $p_1, p_2 \in X_\alpha$, then, since X_α is Hausdorff, $\exists\ U_1, U_2$ neighbourhoods of p_1 and p_2 such that $U_1 \cap U_2 = \phi$, (ii) $p_1 \in X_\alpha$ and $p_2 \in X_\beta$, then, let U_1 be a neighbourhood of p_1 which does not contain p (which certainly exist because X_α is Hausdorff). Similarly, let U_2 be the neighbourhood of p_2 which does not contain p. Then, $U_1 \subseteq X_\alpha$ and $U_2 \subseteq X_\beta$ where $p \notin U_1$ and $p \notin U_2$, therefore, $U_1 \cap U_2 = \phi$.

If one of $p_i = p$, then use argument in (ii), and finally, conclude that $\bigvee_{\alpha \in A} X_\alpha$ is Hausdorff.

Ex. 3.59
$(\mathbf{a}) \implies (\mathbf{b}), (\mathbf{c}), (\mathbf{d})$ Since U is saturated, $\exists\ V \subseteq Y$ s.t. $U = q^{-1}(V)$. Then, $q(U) = V$ and therefore, $U = q^{-1}(q(U))$. Also, $V = \cup_{y \in V}\{y\}$, thus, $U = q^{-1}(\cup_{y \in V}\{y\}) = \cup_{y \in V} q^{-1}(y)$. Let $x \in U$ and x' be any arbitrary point in X s.t. $q(x) = q(x')$. Since $q(x) \in V$, then $q(x') \in V$, implies that, $x' \in q^{-1}(V) = U$.

$(\mathbf{b}) \implies (\mathbf{a})$ Take $V = q(U)$.

$(\mathbf{c}) \implies (\mathbf{a})$ $U = \cup_{y \in V} q^{-1}(y) = q^{-1}(\cup_{y \in V}\{y\}) = q^{-1}(V)$.

$(\mathbf{d}) \implies (\mathbf{a})$ Let $q(U) = V$, then, $U \subseteq q^{-1}(V)$. We show that $q^{-1}(V) \subseteq U$. Let $x' \in q^{-1}(V)$, then, $q(x') \in V$. Since, $V = q(U), \exists\ x \in U$ s.t. $q(x) \in V$ and $q(x) = q(x')$. By the given condition, $x' \in U$, therefore, $q^{-1}(V) \subseteq U$. Hence, $U = q^{-1}(V)$.

Ex. 3.61
$(\implies)$ Let $U \subseteq X$ s.t. U is saturated and open in X, then, $\exists\ V \subseteq Y$ s.t. $U = q^{-1}(V)$. Given that $q^{-1}(V)$ is open, by definition of quotient map, V is open in Y. Similarly, let $U \subseteq X$ s.t. U is saturated and closed in X, then, $\exists\ V \subseteq Y$ s.t. $U = q^{-1}(V)$. Given that $X \setminus q^{-1}(V) = q^{-1}(Y)$ is open, by

surjectivity of quotient map, $X \setminus q^{-1}(V) = q^{-1}(Y) \setminus q^{-1}(V) = q^{-1}(Y \setminus V)$ and by definition of quotient map, $Y \setminus V$ is open in Y, thus, V is closed in Y.
($\Longleftarrow$) Let $U \subseteq Y$ be open in Y, then $q^{-1}(U)$ is open in X due to continuity of q. Now, let $U = q^{-1}(V)$ be open in X for some $V \subseteq Y$. Since, U is saturated and open, by the proposition, $q(U) = V$ is open subset of Y, therefore, q is a quotient map. OR Let $U = q^{-1}(V)$ be open in X, then, $X \setminus U = X \setminus q^{-1}(V)$ is closed in X. Using surjectivity of q, $X \setminus q^{-1}(V) = q^{-1}(Y) \setminus q^{-1}(V) = q^{-1}(Y \setminus V)$. Given that $q^{-1}(Y \setminus V)$ is closed in X, by proposition, $Y \setminus V$ is closed subset of Y and therefore, V is open subset of Y. Hence, q is a quotient map.

Ex. 3.63
(**a**) Let $q_i : X_i \to X_{i+1}$ be a quotient map for all $i \in \{1, \ldots, n\}$. Then, $q : X_1 \to X_{n+1}$ be their composition given by $q = q_n \circ \ldots \circ q_1$. Let U be open subset of X_{n+1}, then $q^{-1}(U) = q_1^{-1}(q_2^{-1}(\ldots(q_n^{-1}(U))\ldots))$ is open subset of X_1 by iteratively applying the definition of quotient map. Similarly, let $q^{-1}(U) = q_1^{-1}(q_2^{-1}(\ldots(q_n^{-1}(U))\ldots))$ be open subset of X_1 for some U in X_{n+1}. Using definition of quotient map q_1, we have $q_1(q^{-1}(U)) = q_2^{-1}(\ldots(q_n^{-1}(U))\ldots)$ is open in X_2. Similarly, applying the defintion of quotient maps $q_2, \ldots, q_n$ in an iterative fashion, we get, U is open in X_{n+1}.

(**b**) Injective quotient map, q, is bijective. Continuity of q follows from the preimage of any open subset of Y being open in X. Injectivity of q ensures that $\forall V \subseteq X, \exists\, U \subseteq Y$ s.t. $V = q^{-1}(U)$. Let $V = q^{-1}(U)$ be open in X, then, by using defintion of quotient map, $q(V) = U$ is open in Y. Thus, q^{-1} is continuous and q is a homeomorphism.

(**c**) ($\Longrightarrow$) Let $K \subseteq Y$ be closed in Y, then, $Y \setminus K$ is open in Y. By definition of quotient map, $q^{-1}(Y \setminus K)$ is open in X. By surjectivity of q, $q^{-1}(Y \setminus K) = q^{-1}(Y) \setminus q^{-1}(K) = X \setminus q^{-1}(K)$ which is open in X, therefore, $q^{-1}(K)$ is closed in X. ($\Longleftarrow$) Let $q^{-1}(K)$ be closed in X for some $K \subseteq Y$, then, $X \setminus q^{-1}(K)$ is open in X. By surjectivity of q, $X \setminus q^{-1}(K) = q^{-1}(Y) \setminus q^{-1}(K) = q^{-1}(Y \setminus K)$ which is open in X. By definition of q, $Y \setminus K$ is open in Y, therefore, $K \subseteq Y$ is closed in Y.

(**d**) Let $U \subseteq X$ be saturated and open in X. Let $V \subseteq q(U)$, then, $q\big|_U^{-1}(V) = U \cap q^{-1}(V) \subseteq U$. Note that $q\big|_U^{-1}(V)$ open in U, implies that $U \cap q^{-1}(V)$ is open in U i.e. $U \cap q^{-1}(V) = U \cap A$ for some open A in X. If U would not have been saturated, we wouldn't be able to say anything (open or closed) about $q^{-1}(V)$, and therefore, couldn't conclude that V is open. However, U is saturated, therefore, $q^{-1}(V) \subseteq U$ and $U \cap q^{-1}(V) = q^{-1}(V)$ is open. Using the definition of q, V is open in Y. Since $V \subseteq q(U)$ where $q(U)$ is open in Y, V is open in $q(U)$. Now, let $V \subseteq q(U)$ open in $q(U)$, therefore, $V = q(U) \cap A$ where A is open in Y. Using definition of q, $q^{-1}(A)$ is open in

X and $q\big|_U^{-1}(V) = U \cap q^{-1}(A)$ is then open in U. Also, $q\big|_U$ is surjective by definition, therefore, is a quotient map. Proceed similarly if U is closed saturated subset of X.

(**e**) Let U be open subset of $\sqcup_\alpha Y_\alpha$, then, $U = \sqcup_\alpha U_\alpha$ where $U_\alpha = U \cap Y_\alpha$ is open in Y_α and $q^{-1}(U) = \sqcup_\alpha q_\alpha^{-1}(U_\alpha) \subseteq \sqcup_\alpha X_\alpha$. By definition of q_α, $q^{-1}(U) \cap X_\alpha = q_\alpha^{-1}(U_\alpha)$ is open subset of X_α, therefore, $q^{-1}(U)$ is an open subset of $\sqcup_\alpha X_\alpha$. Let U be a subset of $\sqcup_\alpha Y_\alpha$, then, $U = \sqcup_\alpha U_\alpha$ where $U_\alpha = U \cap Y_\alpha \subseteq Y_\alpha$. Let $q^{-1}(U) = \sqcup_\alpha q_\alpha^{-1}(U_\alpha) \subseteq \sqcup_\alpha X_\alpha$ be open in $\sqcup_\alpha X_\alpha$, then $q^{-1}(U) \cap X_\alpha = q_\alpha^{-1} U_\alpha$ is open in X_α. By the definition of q_α, $U_\alpha = U \cap Y_\alpha$ is open in Y_α, making U to be open in $\sqcup_\alpha Y_\alpha$. Finally, surjectivity of q follows by observing that $y \in Y_\alpha \xleftarrow{q_\alpha} x \in X_\alpha \iff (y, \alpha) \in \sqcup_\alpha Y_\alpha \xleftarrow{q} (x, \alpha) \in \sqcup_\alpha X_\alpha$ Thus, q is a quotient map.

Ex. 3.72

Let Y_q be the set with quotient topology and Y_g be the same set with different topology satisfying the characteristic property of quotient topology. Let $\mathrm{Id}_{\mathrm{qg}} : Y_q \to Y_g$ and $\mathrm{Id}_{gq} : Y_g \to Y_q$. Note that $\mathrm{Id}_{qg} = \mathrm{Id}_{gq}^{-1}$. Using the characteristic property, we have, Id_{gq} is continuous because $\mathrm{Id}_{gq} \circ q = q$ is continuous and Id_{qg} is continuous because $\mathrm{Id}_{qg} \circ q = q$ is continuous. Therefore, Id_{qg} is a continuous bijective map from Y_q to Y_g with continuous inverse, hence, Id_{qg} is a homeomorphism from Y_q to Y_g. Thus, Y_g has same topology as Y_q which is the quotient topology.

Ex. 3.83

Ex. 3.85

4. Connectedness and Compactness

Ex. 4.3
Suppose $Y = \{[x_\alpha] : \alpha \in A\}$ be the set of equivalence classes where $|A| > 1$ and $\forall \alpha \in A, [x_\alpha]$ is open. Let q be the quotient map corresponding to the equivalence relation, then, $q^{-1}([x_\alpha])$ is open subset of X. Since $q^{-1}(Y) = X$, define $U_1 = [x_1]$ and $U_2 = \{[x_\beta] : \beta \in A - \{1\}\}$. Note that both U_1 and U_2 are open in Y, so are $q^{-1}(U_1)$ and $q^{-1}(U_2)$ in X by defintion of quotient map. Now, $q^{-1}(U_1) \cap q^{-1}(U_2) = \phi$ and $q^{-1}(U_1) \cup q^{-1}(U_2) = q^{-1}(Y) = X$ implies that X is disconnected, reaching a contradiction. Hence, $|A| = 1$ and there is only one equivalence class, namely X itself.

Ex. 4.4
($\Longrightarrow$) Let X be disconnected, then, $\exists\, U_1, U_2 \subseteq X$ which are non-empty open subsets of X s.t. $U_1 \cap U_2 = \phi$ and $U_1 \cup U_2 = X$. Define a function $f : X \to \{0, 1\}$ as

$$f(x) = \begin{cases} 0 & x \in U_1 \\ 1 & x \in U_2 \end{cases}$$

Then, f is a non-constant function which is continuous because the preimage of open subsets $\phi, \{0\}, \{1\}$ and $\{0, 1\}$ of $\{0, 1\}$ are ϕ, U_1, U_2 and X respectively, which are open in X. ($\Longleftarrow$) Let the given function be $g : X \to \{0, 1\}$, then, define $U_1 = g^{-1}(\{0\})$ and $U_2 = g^{-1}(\{1\})$ (both must be non-empty other wise function is constant) and note that U_1 and U_2 are preimages of open subsets of $\{0, 1\}$ in a continuous function, hence, are open subsets of X with $U_1 \cap U_2 = \phi$ and $U_1 \cup U_2 = f^{-1}(\{0, 1\}) = X$ implying that X is disconnected.

Ex. 4.5
($\Longrightarrow$) Follows from definition of disconnected topological space. ($\Longleftarrow$) Let $f : X \to \sqcup_{\alpha \in A} V_\alpha$, where $|A| \geq 2$, be a homeomorphism. Define $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(\sqcup_{\alpha \in A - \{1\}} V_\alpha)$, then U_1 and U_2 are open in X because they are preimages of open subsets of $\sqcup_{\alpha \in A} V_\alpha$ in a continuous function, with $U_1 \cap U_2 = \phi$ and $U_1 \cup U_2 = X$, implying that X is disconnected.

Ex. 4.10
For the sake of argument, let M_U and M_L represent the same connected manifold M with nonempty boundary where U and L imply that they are homeomorphic to upper half space and lower half space, respectively. Let $D(M)$ be disconnected. Then, $\exists\, U, V \neq \phi$ such that U and V are open in $D(M)$, $U \cap V = \phi$ and $U \cup V = D(M)$. Since both M_U and M_L are closed connected subsets of $D(M)$, using **4.9(a)**, $M_U \subseteq U$ or $M_U \subseteq V$ and $M_L \subseteq U$ or $M_L \subseteq V$. If both M_U and M_L are subsets of U, then, $D(M) = M_U \cup M_L \subseteq U$, which contradicts that $V \neq \phi$. By symmetry, if M_U and M_L are subsets of V, then contradicts that $U \neq \phi$. Finally, if $M_U \subseteq U$ and $M_L \subseteq V$, then, $dM_U = dM_L = M_U \cap M_L \subseteq U \cap V$, contradicting

$U \cap V = \phi$. Therefore, our assumption that $D(M)$ is disconnected is wrong, hence, $D(M)$ is connected.

Ex. 4.14
(**a**) Let X be a path connected space, therefore, $\forall p, q \in X$, $\exists\ f_{p,q} : I \to X$ s.t. $f_{p,q}$ is continuous, $f_{p,q}(0) = p$ and $f_{p,q}(1) = q$. Let $g : X \to g(X)$ be continuous. Then, $\forall a, b \in g(X)$, define $h : I \to g(X)$ as $h = g \circ f_{p',q'}$ for some $p' \in g^{-1}(\{a\})$ and $q' \in g^{-1}(\{b\})$. Then, h is continuous because it is a composition of continuous maps, $h(0) = g(f_{p',q'}(0)) = g(p') = a$ and $h(1) = g(f_{p',q'}(0)) = g(q') = b$. Therefore, h is a path in $g(X)$ from a to b. Since a and b were arbitrary, $g(X)$ is path-connected.

(**b**) Let $p, q \in \cup_{\alpha \in A} B_\alpha$ be arbitrary whera a is a common point of the path-conencted subspaces. If $p, q \in B_\beta$ for some $\beta \in A$, then, since B_α is path-connected, there is a path in B_α from p to q, hence a path in $\cup_{\alpha \in A} B_\alpha$ from p to q. If $p \in B_1$ and $q \in B_2$, then define a path in $\cup_{\alpha \in A} B_\alpha$ from p to q as $h : I \to \cup_{\alpha \in A} B_\alpha$ given by,

$$h(u) = \begin{cases} f_{p,a}(2u) & 0 < u \leq 0.5 \\ g_{a,q}(2u-1) & 0.5 < u \leq 1 \end{cases}$$

Note that h is continuous at $u = 0.5$, hence, continuous in I, $h(0) = f_{p,a}(0) = p$ and $h(1) = g_{a,q}(1) = q$. Since p and q were arbitrary, $\cup_{\alpha \in A} B_\alpha$ is path-connected.

(**c**) Let $p = (p_1, \ldots, p_n), q = (q_1, \ldots, q_n) \in (X_1, \ldots, X_n)$, then $f_{p_1,q_1} \times \ldots \times f_{p_n,q_n}$ is the required path from p to q.

(**d**) Use the fact that quotient map is continuous and surjective and argument in (**a**).

Ex. 4.22
(**a**) We must show that path components are disjoint and their union is X. Let U and V be distinct path components of X. Suppose $x \in U \cap V$, then by **4.13**(**b**) $U \cup V$ is path-connected. By maximality of U and V we get $U \cup V = U = V$, hence, U and V are not distinct, a contradiction. Therefore, $U \cap V = \phi$. Now, let $x \in X$, then $\{x\}$ is a path-connected subset of X containing x. Let B_x be the set of all path-connected subsets containing x, then, their union is path-connected and it certainly is maximal, so it is a path-component containing x. Since x was arbitrary, therefore, union of path-components is X.

(**b**) A path-connected subset is connected. Therefore, every path-component which is a path-connected subset, is also a connected subset of X, hence is contained in a single component. Path components are disjoint as proved in (**a**). Let U be a component and $x \in U$. Then, there is a path component which contains x (from (**a**)), which itself is contained U, therefore, a component is disjoint union of path components.

(**c**) Since components cover X and from (**b**), path-components cover X. Let A be a path-connected subset of X, then it has a point common with some path component B. Using 4.13(b), $A \cup B$ is path-connected. By maximality of B, $A \cup B = B$, therefore A is contained in B.

Ex. 4.24
Using **4.8** and **4.13**(**a**) every space homeomorphic to a (path-)connected space is (path-)connected. Consider a manifold M with or without boundary. Since, every basis B of M is homeomorphic to an open subset of $\mathbb{R}^n$ or an open subset of $\mathbf{H}^n$ which are (path-)connected, therefore, B is (path-)connected. So, M is locally connected and locally path-connected.

Ex. 4.28
($\Longrightarrow$) Let $\mathcal{U}_X$ be an open cover of A containing open subsets of X whose union contains A. Define a cover $\mathcal{U}_A$ as $\mathcal{U}_A = \{A \cap U : U \in \mathcal{U}_X\}$ which contains open subsets of A whose union is A. Since A is compact in the subspace topology, then, there is a finite subcover i.e. $\exists\ V_1, \ldots, V_k \in \mathcal{U}_A$ s.t. $\cup_{i=1}^k V_i = A$. Note that $V_i = A \cap U_i$, therefore, the corresponding U_i's form a finite subcover of $\mathcal{U}_X$ containing A. ($\Longleftarrow$) Let $\mathcal{U}_A$ be an open cover containing open subsets of A whose union is A. Then, for each $U_\alpha \in \mathcal{U}_A$, $\exists\ V_\alpha$ which is an open subset of X, s.t., $U_\alpha = A \cap V_\alpha$. The collection of all V_α's form an open cover of A containing open subsets of X whose union contains A. So, $\mathcal{U}_X$ has a finite subcover, i.e., $\exists V_1, V_2, \ldots, V_k$ s.t. $A \subseteq \cup_{i=1}^k V_k$. The collection of corresponding U_i's where $U_i = A \cap V_i$ is a finite subcover of A containing open subsets of A whose union is A.

Ex. 4.29
Let $(A_i)_{i=1}^n$ be finitely many compact subsets of of X. Let $\mathcal{U}_{A_i}$ be an open cover containing open subsets of A_i whose union is A_i. Then, $\cup_{i=1}^n \mathcal{U}_{A_i}$ is an open cover of $\cup_{i=1}^n A_i$. Since, A_i's are compact, there exists finite subcovers, i.e., $\exists\ (U_{A_{i_j}})_{j=1}^{k_i} \in \mathcal{U}_{A_i}$ whose union is A_i. Then, a collection of these finite subcovers is a subcover of $\cup_{i=1}^n \mathcal{U}_{A_i}$. Since this collection is finite, therefore, using **4.28**, $\cup_{i=1}^n A_i$ is compact.

Ex. 4.37
Let q be the quotient map from $M \sqcup M$ to $D(M) = M \cup_h M$. Since, M is compact, $M \sqcup M$ is compact. Using **4.36**(**d**), $D(M)$ is compact.

Ex. 4.38
Suppose $\cap_n F_n = \phi$, then $\cup_n X \setminus F_n = X$. Since F_i is closed, therefore, $X \setminus F_i$ is open and $\{X \setminus F_n : n \in \mathbb{N}\}$ is an open cover of X. Since X is compact, there

exists a finite subcover, $\{X \setminus F_{n_i} : i \in \{1, 2, \dots, k\}\}$. Since, $F_i \supseteq F_{i+1}$, therefore, $X \setminus F_i \subseteq X \setminus F_{i+1}$ and we get $X \setminus F_{n_k} = X$, which implies $F_{n_k} = \phi$ which is a contradiction (because $F_i \neq \phi$). So, $\cap_n F_n \neq \phi$.

Alternatively,
Note that (using **4.36(a)**) F_i is compact. Let $\cup_{n\geq 1} F_n = \phi$, then, $\cup_{n\geq 2} X \setminus F_n \supseteq F_1$, therfore, $\{X \setminus F_i : i \geq 2\}$ is an open cover of F_1. So, it has a finite subcover, say, $\{X \setminus F_{k_i} : i \in \{1, 2, \dots, m\}\}$ where $F_{k_i} \supseteq F_{k_{i+1}}$. Therefore, $F_1 \subseteq \cup_{i=1}^m X \setminus F_{k_i} \subseteq X \setminus F_{k_m}$. So, $F_1 \cap F_{k_m} = \phi$, but $F_{k_m} \subseteq F_1$, which means, $F_1 \cap F_{k_m} = F_{k_m} = \phi$. This contradicts the fact that F_{k_m} is non empty. Therefore, $\cup_{n\geq 1} F_n \neq \phi$.

Ex. 4.49
(**4.46**) Let (p_k) be an arbitrary bounded sequence in $\mathbb{R}^n$. Then, $\exists\ M > 0$ s.t. $p_k \in [-M, M]^n$ for all k.

- $[-M, M]^n$ is a closed and bounded subset of $\mathbb{R}^n \implies$ it is compact.
- Compactness $\implies$ Limit point compactness.
- For first countable Hausdorff spaces, limit point compactness $\implies$ Sequential compactness.

Note that $\mathbb{R}^n$, being a metric space (equipped with some metric (*)), is first countable and Hausdorff, and so is its subset $[-M, M]^n$ in the subspace topology. By above arguments, $[-M, M]^n$ is sequentially compact. Hence, by the definition of sequential compactness, the sequence (p_k) has a subsequence which converges to a point in $[-M, M]^n$.

(*) Same metric which is being used to evaluate convergence.
A direct argument based on the following results is also possible:
- For metric spaces, compactness, limit point compactness and sequential compactness are all equivalent properties. - Subset of a metric space is a metric subspace with metric inherited from the original space.

(**4.47**) ($\implies$) Let A be a subset of $\mathbb{R}^n$ which is a complete metric space and x be a limit point of A. Then, $\exists$ a Cauchy sequence (x_k) s.t. $x_k \in A$ and $x_k \to x$. Since, A is complete, $x \in A$. Therefore, A contains all of its limit points, hence is closed. ($\impliedby$) Let A be closed in $\mathbb{R}^n$ and (x_k) be a Cauchy sequence in s.t. $x_k \in A$. Since, a Cauchy sequence is bounded, (x_k) is bounded and hence, by **4.46**, has a convergent subsequence. A Cauchy sequence with convergent subsequence is convergent. Therefore, (x_k) converges to say x, where x is a limit point of A. Since, A is closed, $x \in A$. Therefore, A is a complete metric space. Finally, $\mathbb{R}^n$ is closed in $\mathbb{R}^n$, therefore, is a complete metric space.

(**4.48**) Let X be a compact metric space and (x_k) be a Cauchy sequence s.t. $x_k \in X$. By **4.45**, X is sequentially compact, therefore, (x_k) has a convergent subsequence. A Cauchy sequence with a convergent subsequence is convergent (to some point in X). Therefore, X is complete.

Ex. 4.58
$A = \mathbb{S}^n \setminus \{0, 0, \ldots, 0, 1\}$ is an open subset of $\mathbb{S}^n$ and is homeomorphic to $\mathbb{B}^n$. The closure of A is given by $\bar{A} = \mathbb{S}^n$ but $\bar{A} \not\approx \bar{\mathbb{B}}^n$.

Ex. 4.61
Clearly, $\phi_i^{-1}(B_r(x))$ is an open subset of X because ϕ_i is continuous. Now, let $p \in U_i$ be mapped to $x \in \hat{U_i}$ where x is irrational. Since $\hat{U_i}$ is open, $\exists\ r(x) > 0$ s.t. $B_{r(x)}(x) \subseteq \hat{U_i}$. Now, even if $r(x)$ is irrational, $\exists\ x'$ and r' s.t. both x' and r' are rational and $x \in B_{r'}(x')$. And therefore, $\phi_i^{-1}(B_{r'}(x'))$ which is an element of the basis, contains x. Finally, we conclude that
$U_i = \cup_{x \in \hat{U_i}} \phi^{-1} B_r(x)$ where r and x are rational.

Ex. 4.67
Let $X_1, X_2, \ldots, X_n$ be locally compact spaces and $(X_1, \ldots, X_n)$ be the corresponding product space. Let $p = (p_1, \ldots, p_n) \in (X_1, \ldots, X_n)$, then, for each i, $\exists\ U_i$ which is open in X_i such that there is V_i which is compact in X_i and $p_i \in U_i \subseteq V_i$. Then, $(U_1, \ldots, U_n)$ is a neighbourhood of p and is open in $(X_1, \ldots, X_n)$. Since, finite product of compact spaces is compact, $(V_1, \ldots, V_n)$ is compact in $(X_1, \ldots, X_n)$. Also, $p \in (U_1, \ldots, U_n) \subseteq (V_1, \ldots, V_n)$, therefore, $(X_1, \ldots, X_n)$ is locally compact.

Ex. 4.70
Let X be a Baire space and A be a meager subset. Then, $A = \cup_{\alpha \in A} U_\alpha$ where U_α is nowhere dense. Note that $U_\alpha \subseteq \bar{U}_\alpha$, therefore, $X \setminus U_\alpha \supseteq X \setminus \bar{U}_\alpha$ and $X \setminus A \supseteq \cap_{\alpha \in A} X \setminus \bar{U}_\alpha$. Since, X is a Baire space, $\cap_{\alpha \in A} X \setminus \bar{U}_\alpha$ is dense. So, $X \setminus A$ is dense, hence, A has dense complement.

Ex. 4.73
Let $x \in X$, then choose $A \in \mathcal{A}$ such that $x \in A$. Since A intersects only finitely many other sets in $\mathcal{A}$, X is locally finite.

Ex. 4.78
Let X be a compact Hausdorff space. Let A and B be disjoint closed subsets of X. Then, by **4.36(a)**, A and B are compact. Finally, by **4.34**, there are disjoint open subsets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$. Therefore, X is normal.

Ex. 4.79
Let X be a normal space and A be a closed subspace of X. Let U_1 and U_2 be disjoint closed subset of in A. Then, U_1 and U_2 are disjoint and closed in X (by **3.5(a)**). Since, X is normal, $\exists$ disjoint open subsets $V_1, V_2 \subseteq X$ such that

$U_1 \subseteq V_1$ and $U_2 \subseteq V_2$. Then, $A \cap V_1$ and $A \cap V_2$ are disjoint and open in A such that $U_1 \subseteq A \cap V_1$ and $U_2 \subseteq A \cap V_2$. Therefore, A is normal.