Chaitanya Kapoor
2020A3PS1219P
f20201219@pilani.bits-pilani.ac.in

Probability and Statistics

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1 Discrete probability

- **Expectation value**: For a discrete random variable X (this can also be replaced by a function $h(x)$) and probability mass function $p(x)$
$$E(x) = \mu_x = \sum_{x \in D} xp(x)$$
Some properties:
  1. $E(C) = C$
  2. $E(CX) = CE(X)$
  3. $E(CX + d) = CE(X) + d$
- **Variance**: Assume the discrete random variable X to have an expected value μ. Then
$$V(X) = \sum_D (x-\mu)^2 \cdot p(x) = E[(x-\mu)^2]$$
This can also be rewritten as $V(X) = \sigma^2 = E(X^2) - [E(X)]^2$.
For any arbitrary function $h(X)$ of the form $(aX + b)$, we have $V(h(X)) = a^2 V(X)$
- **Moment Generating Function (MGF):** For a discrete random variable X, $m_X(t) = E(e^{tX})$. Using this, we have the following
$$\left.\frac{d^r m_X(t)}{dt^r}\right|_{t=0} = E(X^r)$$
- **Bernoulli Distribution:** Consider a random variable X which takes Boolean values (0 and 1). Let the probability of obtaining the value 1 be p and 0 be $q = (1-p)$
  1. $f(x,p) = p^x(1-p)^{1-x}$
  2. $E(X) = p$
  3. $\mathrm{Var}(X) = pq$
  4. $m_X(t) = q + pe^t$
- **Binomial Distribution:** For a random variable X with parameters n and p (denoted by $X \sim (n,p)$)
  1. $m_X(t) = (q + pe^t)^n$
  2. $E(X) = np$
  3. $\mathrm{Var}(X) = npq$
  4. $p(x) = \binom{n}{x} p^x (1-p)^{n-x}$
- **Geometric Random Variable:** Let X denote the random variable representing the number of trials required to get the 1^{st} success.
  1. $g(n,p) = pq^{n-1}$
  2. $E(X) = \frac{1}{p}$ and $E(X^2) = \frac{1+q}{p^2}$
  3. $\mathrm{Var}(X) = \frac{q}{p^2}$
  4. $m_X(t) = \frac{pe^{-t}}{1-qe^t}$
- **Poisson distribution:** For a discrete random variable X with the parameter μ
  1. $p(x,\mu) = \frac{e^{-\mu} \cdot \mu^x}{x!}$
  2. $E(X) = \mathrm{Var}(X) = \mu$
  3. $m_X(t) = e^{\mu(e^t - 1)}$

Consider a binomial distribution $b(x; n, p)$ in the limit where $n \to \infty$ and $p \to 0$. In such a scenario, $b(x; n, p) \to p(x; \mu)$ where $\mu = np$

- **Poisson process:** Let $P_k(t)$ denote the probability that k events are observed during a particular time interval of length t, then
$$P_k(t) = e^{-\alpha t} \cdot \frac{(\alpha t)^k}{k!}$$
Where t is the random variable with the parameter $\mu = \alpha t$ and α specifies the rate of the process in question.

2 Continuous probability

Consider a function $f(x)$ to be a PDF of a continuous random variable x

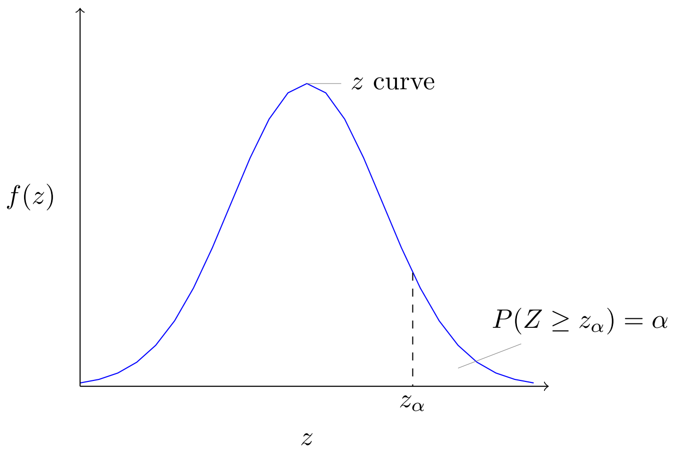
- $P(a \leq x \leq b) = \int_a^b f(x)dx$
- **CDF** $= F(x) = \int_{-\infty}^x f(x)dx$
- **Expectation value and MGF:**
$$E(x) = \int_{-\infty}^{\infty} xf(x)dx$$
$$E(H(x)) = \int_{-\infty}^{\infty} H(x)f(x)dx$$
$$m_x(t) = E(e^{xt}) = \int_{-\infty}^{\infty} e^{xt}f(x)dx$$
- **Moments, mean and variance:**
$$E(x^k) = \int_{-\infty}^{\infty} x^k f(x)dx = \left(\frac{d^k}{dx^k}\left(m_x(t)\right)\right)\bigg|_{t=0}$$
$$\mu = E(x) = \int_{-\infty}^{\infty} xf(x)dx$$
$$\sigma^2 = \left(\int_{-\infty}^{\infty} x^2 f(x)dx\right) - \left(\int_{-\infty}^{\infty} xf(x)dx\right)^2$$
- **Uniform distribution:**

  1. Probability density function:
$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}$$
  2. MGF, mean and variance:
$$m_x(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$$
$$\mu = \frac{a+b}{2}$$
$$\sigma^2 = \frac{(b-a)^2}{12}$$
- **Normal/Gaussian distribution:** It is denoted as $X \sim N(\mu, \sigma^2)$, where μ(mean) and σ^2 (variance) are the parameters of the distribution.

  1. Probability density function:
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$
  2. MGF, expectation:
$$m_x(t) = e^{\mu t + \frac{t^2\sigma^2}{2}}$$
$$E(x) = m_x'(t)\Big|_{t=0} = \mu$$

- **Standard Normal distribution:** Assume $z = \frac{x-\mu}{\sigma}$, then $E(z) = 0$ and $V(z) = 1$. In such a scenario, z is known as **the standard normal variate** which is denoted as $z \sim \mathcal{N}(0,1)$

  1. Probability density function:
  $$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$
  2. Cumulative density function:
  $$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-z^2/2} dz$$
  3. z_α for z critical values: z_α represents the $100(1-\alpha)$th percentile of the standard normal distribution. The values of z_α is often called as the z **critical values**.



- **Approximating Binomial distribution:** Let X be a random variable based on n trials with a success probability p. If the binomial histogram is not too skewed, then X has $\approx$ normal distribution with $\mu = np$ and $\sigma = \sqrt{npq}$ or
$$P(X \leq x) = B(x; n, p) \approx \Phi\left(\frac{x + 0.5 - np}{\sqrt{npq}}\right)$$

- **Exponential and Gamma distribution:**

  1. Exponential distribution:
  $$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
  Mean $(\mu) = \frac{1}{\lambda}$ and variance $(\sigma^2) = \frac{1}{\lambda^2}$
  2. Gamma function: For a parameter $\alpha > 0$, we define the **gamma function** as
  $$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$
  Some important properties:
     (a) $\Gamma(\alpha) = (\alpha - 1) \cdot \Gamma(\alpha - 1)$
     (b) $\Gamma(n) = (n-1)!$
     (c) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

  Gamma distribution:
  $$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
  Expectation: $E(x) = \mu = \alpha\beta$
  Variance: $V(x) = \sigma^2 = \alpha\beta^2$
  The gamma distribution of a random variable X with parameters α and β is denoted as:
  $$P(X \leq x) = F(x;\alpha,\beta) = F\left(\frac{x}{\beta};\alpha\right)$$

3. **Chi-Squared distribution:**
   For a positive integer ν (**degrees of freedom of the random variable** X), the PDF of this distribution takes the form

$$f(x;\nu) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\left(\frac{\nu}{2}-1\right)} e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

   This can be modelled as a gamma distribution with the parameters $\alpha = \frac{\nu}{2}$ and $\beta = 2$.