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Oscillations

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1 Simple harmonic oscillators

- Simple pendulum:
  - Differential equation $\rightarrow \ddot{\theta} + \frac{g}{L}\theta = 0$.
  - Time period $\rightarrow 2\pi\sqrt{\frac{L}{g}}$
- Physical pendulum:
  - Differential equation $\rightarrow \ddot{\theta} + \frac{mgL}{2I}\theta = 0$.
  - Time period $\rightarrow 2\pi\sqrt{\frac{I_a}{mgL}}$
    Where L is the distance between the axis of rotation and the center of mass of the body, and I_a is moment of inertia about the **axis of rotation**. This equation can also be rewritten in terms of the radius of gyration of the body k:
    $$T = 2\pi\sqrt{\frac{(k^2 + L^2)}{gL}}$$
- Spring-mass system:
  - Equation of motion $\rightarrow x(t) = A\cos\omega t$ and $v(t) = -A\omega\sin\omega t$
  - Kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\sin^2\omega t = \frac{1}{2}m\omega^2A^2(1-\cos^2\omega t) = \frac{1}{2}m\omega^2(A^2 - x^2)$
  - Potential energy $= \frac{1}{2}kx^2$
  - Total energy $= \frac{1}{2}m\omega^2A^2$

2 Damped harmonic motion

Generally, in addition to the spring force kx, we must take into account a **damping force** which is proportional to the velocity of the oscillator, ie- $F_{\text{damp}} = -b\vec{v} = -b\dot{x}$. Hence, the equation of motion of the system takes the form

$$m\ddot{x} = -kx - b\dot{x}$$
$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$
$$\therefore \ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

Where $\gamma = \frac{b}{m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$. The solution of this characteristic differential equation is

$$x(t) = e^{-\frac{\gamma}{2}t}(C_1e^{qt} + C_2e^{-qt})$$
$$\text{Where } q = \sqrt{\frac{\gamma^2}{4} - \omega_0^2} = \omega$$

Depending on the relation between γ and ω_0, we characterize the types of damped motion.

1. Over-damped : $\frac{\gamma^2}{4} > \omega_0$ and $x(t) = e^{-\frac{\gamma}{2}t}(C_1e^{qt} + C_2e^{-qt})$
2. Critically damped : $\frac{\gamma^2}{4} = \omega_0$ and $x(t) = (C_1 + C_2t)e^{-\frac{\gamma}{2}t}$
3. Over damped : $\frac{\gamma^2}{4} < \omega_0$ and $x(t) = Ae^{-\frac{\gamma}{2}t}\cos qt$

The energy of a damped system also decays with time, and can be described with the help of the parameter γ, ie: $\boxed{E = E_0e^{-\gamma t}}$. In this regard, a new parameter called the **Quality factor (Q-value)** is introduced. It can be defined in any two of the following ways

$$Q = \frac{\omega}{\gamma}$$
$$= 2\pi\left(\frac{\text{Energy at } t}{\text{Energy loss per cycle}}\right)$$

3 Forced oscillations

A free or damped oscillation with a harmonic driving force ($F = F_0 \cos \omega t$) has the differential equation

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$$

We try a solution of the form $x = Ae^{i(\omega t - \delta)}$. After a series of steps, we arrive at the following expressions:

$$A(\omega) = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}}$$
$$\delta(\omega) = \arctan\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right)$$

The cases of **resonant frequency** are of particular interest to us (when $\omega = \omega_0$). At this condition,

$$A(\omega_0) = A_0 = \frac{F_0 Q}{m\omega^2}$$
$$\tan \delta(\omega_0) = \infty$$

Another case of interest is when the angular frequency is *just less than* the resonant frequency. This is known as **amplitude resonance** and the particular frequency is denoted by ω_m. The frequency and amplitude respectively are:

$$\omega_m = \omega_0 \sqrt{\left(1 - \frac{1}{2Q^2}\right)} \approx \omega_0 \left(1 - \frac{1}{4Q^2}\right)$$
$$A(\omega_m) = \frac{F_0}{m}\left(\frac{Q}{\sqrt{1 - \frac{1}{4Q^2}}}\right)$$

Next, we take into consideration a **forced oscillation without the action of a damping force**. The equation of motion for such a body is

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$$

Which has the solution

$$x(t) = A_0 \cos(\omega t + \varphi) + \frac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega t$$

All the symbols having their usual meaning.
A set of results for terms relating to power are listed below:

1. **Average power** $(\bar{P}) = \frac{F_0}{2m\omega_0 Q} \frac{1}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}$
2. **Maximum power** $(\bar{P}_{\max}) = \frac{F_0^2 Q}{2m\omega_0}$
3. **Half power frequency** $\omega = \omega_0 \left(1 \pm \frac{\omega_0}{2Q}\right) = \omega_0 \left(1 \pm \frac{\gamma}{2}\right)$
4. **Bandwidth** $(\Delta\omega) = \gamma = \frac{\omega_0}{Q}$ which represents the **full width at half maximum** (FWHM)