

Miscellaneous coordinate systems:

- Polar coordinates:

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

- Cylindrical coordinates:  $r \geq 0, 0 \leq \theta \leq 2\pi, -\infty < z < \infty$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}$$

- Spherical coordinates:  $\rho$  = Distance from origin,  $\phi$  = Angle with **positive**  $z$ -axis,  $\theta$  = Angle with  $x$  axis

$$\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases}$$

## 1 Vector valued functions

- Parametric equation of a line passing through  $(x_0, y_0, z_0)$  and parallel to  $a\hat{i} + b\hat{j} + c\hat{k}$ :

$$\begin{cases} x(t) = x_0 + ta \\ y(t) = y_0 + tb \\ z(t) = z_0 + tc \end{cases}$$

- Arc length of a curve for  $a \leq x \leq b$ :

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

Arc length parameter  $\vec{s}(t) = \int_{t_0}^t |\vec{v}(\tau)| d\tau$

- Unit tangent vector  $\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{d\vec{r}}{ds}$ ,  $s$  being the arc length parameter
- Curvature ( $\kappa$ ):

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| = \left| \frac{d\vec{T}/dt}{ds/dt} \right| = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$$

Unit normal ( $\vec{N}$ ):

$$\vec{N} = \frac{\left( \frac{d\vec{T}}{ds} \right)}{\left| \frac{d\vec{T}}{ds} \right|} = \frac{1}{\kappa} \left( \frac{d\vec{T}}{ds} \right)$$

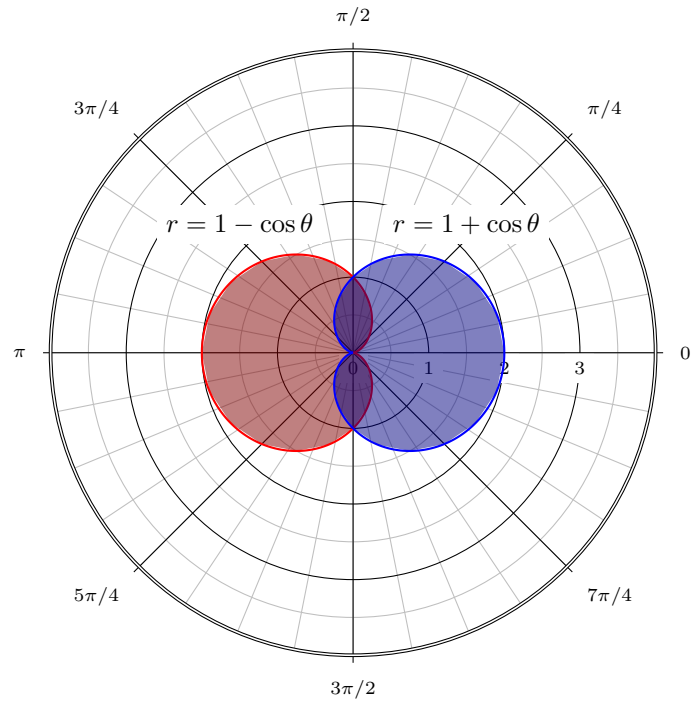
Curvature can also be written in terms of  $v$  and  $a$  as  $\frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$

- $\vec{a}(t) = \frac{d|v|}{dt} \vec{T} + \kappa |v|^2 \vec{N}$
- Binormal vector ( $\vec{B}$ ) =  $\vec{T} \times \vec{N}$   
 $\vec{T}$  and  $\vec{N}$ : Osculating plane  
 $\vec{B}$  and  $\vec{N}$ : Normal plane  
 $\vec{B}$  and  $\vec{T}$ : Rectifying plane
- Torsion  $\tau = -\frac{d\vec{B}}{ds} \cdot \vec{N}$

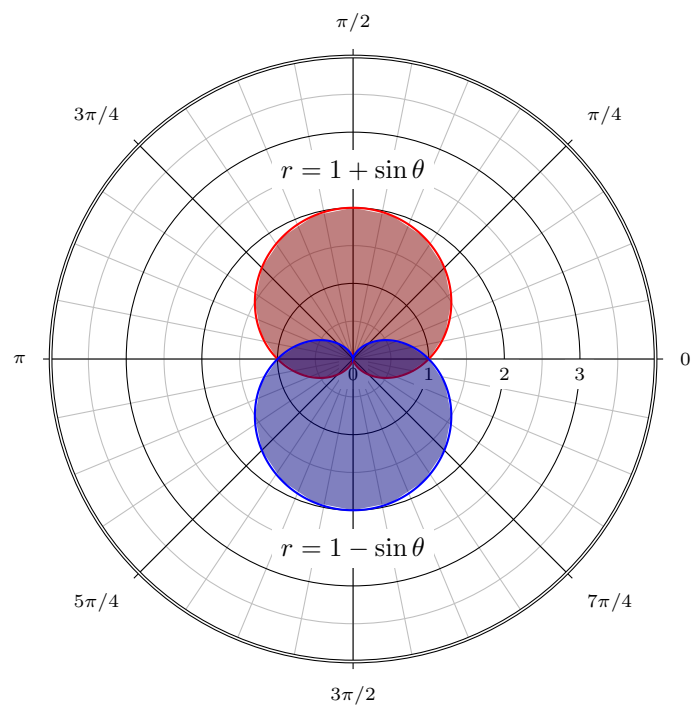
## 2 Polar coordinates

- Important polar curves:

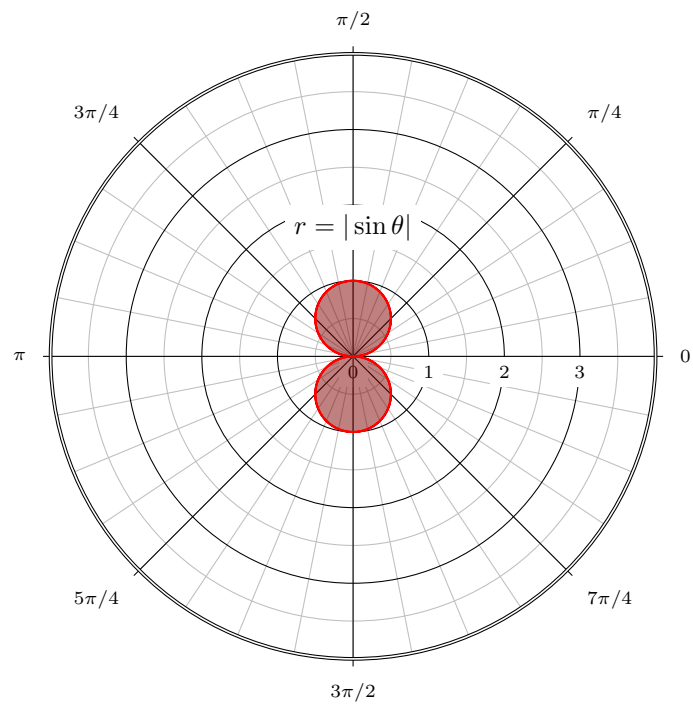
1.  $r = 1 - \cos \theta$ ,  $r = 1 + \cos \theta$ :



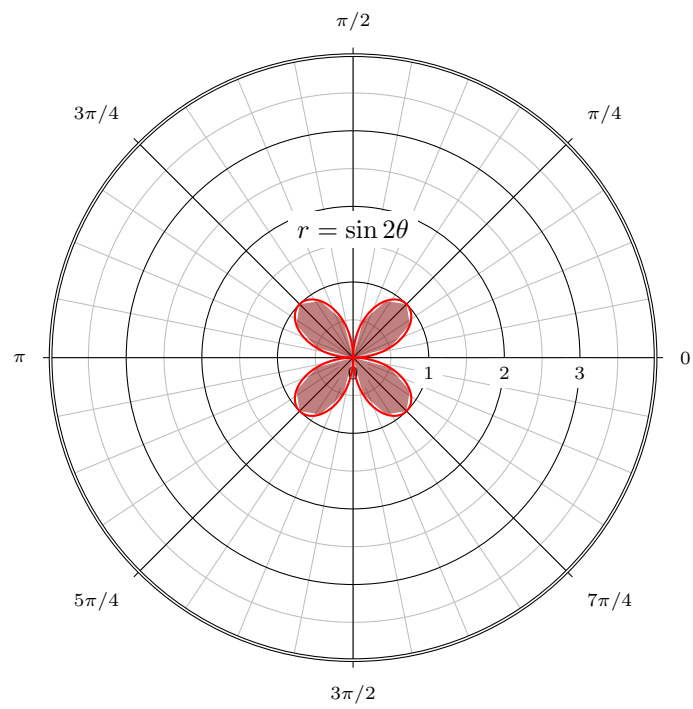
2.  $r = 1 + \sin \theta$ ,  $r = 1 - \sin \theta$ :



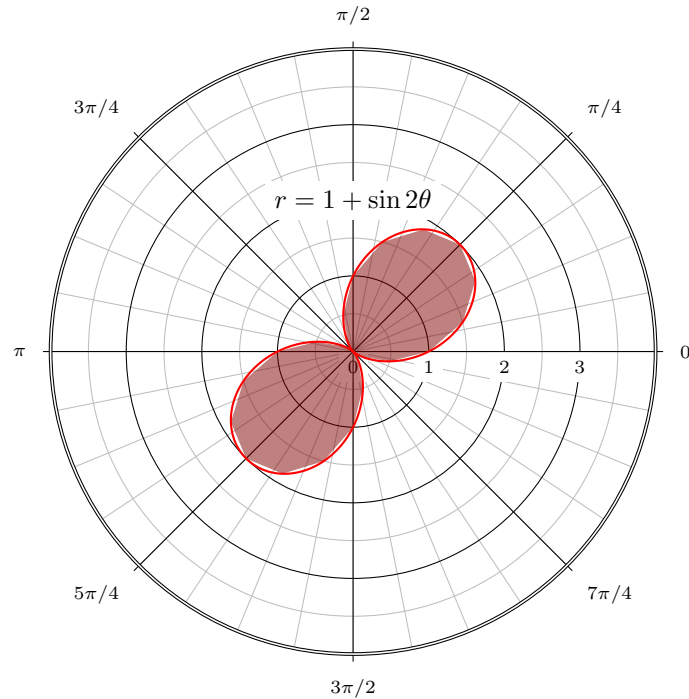
3.  $r = |\sin \theta|$ :



4.  $r = \sin 2\theta$ :



5.  $r = 1 + \sin 2\theta$ :



- Area enclosed by a polar curve:

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$$

Length of a polar curve:

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

### 3 Partial derivatives

- From the 1<sup>st</sup> principles, the partial derivative of a function  $f(x, y)$  with respect to  $x$  is given as:

$$\frac{\partial f}{\partial x} = f_x = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

A similar idea is followed for calculating  $\frac{\partial f}{\partial y}$

- Implicit differentiation: Let  $f(x, y) = c$ . After a bit of work, we get the following

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

- Higher order partial derivatives:

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= f_{xx} = \lim_{h \rightarrow 0} \frac{f_x(x+h, y) - f_x(x, y)}{h} \\ \frac{\partial^2 f}{\partial y \partial x} &= f_{xy} = \lim_{k \rightarrow 0} \frac{f_x(x, y+k) - f_x(x, y)}{k} \end{aligned}$$

- Directional derivative :  $\left(\frac{df}{dt}\right)_{\hat{n}} = \nabla f \cdot \hat{n}$
- Angle between 2 surfaces :  $\theta = \cos^{-1} \left( \frac{\nabla f \cdot \nabla g}{|\nabla f| |\nabla g|} \right)$
- Equation of tangent plane at  $(x_0, y_0, z_0)$ :  $(x - x_0)f_x + (y - y_0)f_y + (z - z_0)f_z = 0$   
Equation of normal at  $(x_0, y_0, z_0)$ :

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$$

- Linearization of  $f(x, y)$  at  $(x_0, y_0)$ :

$$f(x, y) \approx f(x_0, y_0) + (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0)$$

- Extrema: At the critical points:  $f_x(x_0, y_0) = 0$ ;  $f_y(x_0, y_0) = 0$   
If at  $(x_0, y_0)$ :

1.  $f_{xx}f_{yy} - f_{xy}^2 > 0$ ;  $f_{xx} < 0 \implies$  **Local max**
2.  $f_{xx}f_{yy} - f_{xy}^2 > 0$ ;  $f_{xx} > 0 \implies$  **Local min**
3.  $f_{xx}f_{yy} - f_{xy}^2 < 0 \implies$  **Saddle point**
4.  $f_{xx}f_{yy} - f_{xy}^2 = 0 \implies$  **Further investigation**

- Lagrange multipliers: For a function  $f(x, y, z)$  subject to the constraints  $g_1(x, y, z)$  and  $g_2(x, y, z)$ , we have:

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

## 4 Multiple integrals

- Double integrals to compute volume over a region  $R$  under a surface  $f(x, y)$ :

$$V = \iint_R f(x, y) dA$$

For computing the area of a closed bounded region  $R$ :

$$A = \iint_R dA$$

$$\text{Average value of } f \text{ over a region } R = \frac{1}{\text{Area of } R} \iint_R f dA$$

- Volume in polar coordinates:

$$V = \iint_R f(r, \theta) r dr d\theta$$

- Triple integrals in cylindrical coordinates:

$$I = \iiint_R f(r, \theta, z) dV = \iiint_R f(r, \theta, z) dz r dr d\theta$$

Triple integrals in spherical coordinates:

$$I = \iiint_R f(\rho, \phi, \theta) dV = \iiint_R f(\rho, \phi, \theta) \rho^2 \sin \phi d\rho d\phi d\theta$$

- Substitutions in iterated integrals:

1. Double integrals: For a coordinate transformation  $x = g(u, v)$  and  $y = h(u, v)$ , we define the **Jacobian** as  $J(u, v) = \frac{\partial(x, y)}{\partial(u, v)}$ :

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

The new integral becomes:

$$\iint_R f(x, y) dx dy = \iint_R f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

2. Triple integrals: Following a similar set of steps as above, we introduce a new transformation for the  $z$  coordinate as  $z = k(u, v, w)$ . The Jacobian changes to a  $3 \times 3$  determinant in this case to give the new integral:

$$\iiint_D f(x, y, z) dx dy dz = \iiint_G h(u, v, w) |J(u, v, w)| du dv dw$$

Where  $h(u, v, w)$  is the transform of  $f(x, y, z)$

## 5 Integrals and vector fields

- Line integral of  $f$  over a curve  $C$ :

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) |\vec{v}(t)| dt$$

Line integral over a vector field  $\vec{F}$ :  $\int_C \vec{F} \cdot d\vec{r}$

- Work done by a force  $\vec{F}(x, y, z)$  over a smooth curve  $C$  with the parametrization  $\vec{r}(t)$ :

$$W = \int_C \vec{F} \cdot \vec{T} ds$$

- Flux of a vector field  $F = M(x, y)\hat{i} + N(x, y)\hat{j}$  with the **outward normal**  $\vec{n} = \vec{T} \times \vec{k}$ :

$$\int_C \vec{F} \cdot \vec{n} dx = \int_C (My'(t) - Nx'(t)) dt$$

- Circulation of a vector field  $F = M(x, y, z)\hat{i} + N(x, y, z)\hat{j} + P(x, y, z)\hat{k}$ :

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C Mdx + Ndy + Pdz = \int_C (Mx'(t) + Ny'(t) + Pz'(t)) dt$$

- If  $\vec{F}$  is conservative  $\Leftrightarrow \vec{F} = \nabla f$   
 $\oint_C \vec{F} \cdot d\vec{r} = 0$  over every closed curve if  $\vec{F}$  is a conservative field
- If  $\vec{F}$  is conservative, then  $\text{curl}(\vec{F}) = 0$ , which is given as:

$$\text{curl}(F) = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = 0$$

- Green's theorem:

1. Circulation-curl/tangential form:

$$\oint_C \vec{F} \cdot \vec{T} ds = \oint_C Mdx + Ndy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

2. Flux-divergence/normal form:

$$\oint_C \vec{F} \cdot \hat{n} ds = \oint_C Mdx - Ndy = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

- Surface area of  $f(x, y, z) = c$  over a region  $R$  with the unit normal vector  $\vec{p}$  is given as:

$$A = \iint_S d\sigma = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|} dA$$

For a surface of the form  $f(x, y, z) = z - m(x, y)$  with the projected region  $R$  in the  $XY$  plane:

$$A = \iint_S d\sigma = \iint_R \sqrt{1 + m_x^2 + m_y^2} dA$$

- A surface  $\vec{r}(t)$  parametrized as  $\vec{r}(u, v)$  has the surface area  $A = \iint_S d\sigma = \iint_R |\vec{r}_u \times \vec{r}_v| du dv$
- Surface integral of  $g(x, y, z)$  over the projected region  $S$  of the function  $f(x, y, z) = c$  is:

$$\iint_R g(x, y, z) \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|} dA$$

Similarly, the surface integral of a vector field  $F$  over a surface  $S$  is  $\iint_C \vec{F} \cdot \hat{n} d\sigma$  and the flux over the region is  $\pm \iint_R \frac{\vec{F} \cdot \nabla f}{|\nabla f \cdot \vec{p}|} dA$ . The sign of flux depends on **the choice of the normal vector**

- Stoke's theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} d\sigma = \iint_S \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

This integral can be taken over a region  $R$  to give:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \frac{(\nabla \times \vec{F}) \cdot \nabla f}{|\nabla f \cdot \hat{p}|} dA$$

For a surface parametrized as  $r(u, v)$ , the integral takes the form:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\nabla \times \vec{F}) \cdot (\vec{r}_u \times \vec{r}_v) du dv$$

- Divergence theorem: The divergence of a vector field  $F = M\hat{i} + N\hat{j} + P\hat{k}$  is given as:

$$\text{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

The flux of a vector field  $\vec{F}$  over a surface  $S$  in the direction of the outward normal  $\hat{n}$  is calculated by the integral:

$$\iint_S \vec{F} \cdot \hat{n} d\sigma = \iiint_D (\nabla \cdot \vec{F}) dV$$

## 6 Infinite series

For the sake of brevity, I define two series:  $\sum a_n = a_n$  and  $\sum b_n = b_n$

- Integral test: If  $f(x)$  is *ve*, continuous and decreasing  $\forall x \geq N$  for some  $N$ , then  $a_n$  and  $\int_N^\infty f(x) dx$  converge or diverge together
- $p$ -series:  $\sum \frac{1}{n^p}$  is **convergent** for  $p > 1$  and **divergent** for  $p \leq 1$
- Direct comparison test:
  1. If  $a_n \leq b_n \forall n \geq N$  and  $b_n$  is convergent, then  $a_n$  is also convergent
  2. If  $a_n \geq b_n \forall n \geq N$  and  $b_n$  is divergent, then  $a_n$  is also divergent
- Limit comparison test:  $a_n, b_n$  are *ve*  $\forall n \geq N$ , then:
  1.  $\lim_{n \rightarrow \infty} \left( \frac{a_n}{b_n} \right) = c$  which is *finite and nonzero*, then  $a_n$  and  $b_n$  **converge or diverge together**
  2.  $\lim_{n \rightarrow \infty} \left( \frac{a_n}{b_n} \right) = 0$ , and if  $b_n$  **converges** then  $a_n$  **converges**
  3.  $\lim_{n \rightarrow \infty} \left( \frac{a_n}{b_n} \right) = \infty$ , and if  $b_n$  **diverges** then  $a_n$  **diverges**
- Ratio test: If  $\lim_{n \rightarrow \infty} \left( \frac{a_{n+1}}{a_n} \right) = r$ , then:
  1.  $r < 1 \implies a_n$  is **convergent**
  2.  $r > 1 \implies a_n$  is **divergent**
  3.  $r = 1 \implies$  **inconclusive**
- Root test: If  $\lim_{n \rightarrow \infty} (a_n)^{1/n} = r$ , then:
  1.  $r < 1 \implies a_n$  is **convergent**
  2.  $r > 1 \implies a_n$  is **divergent**
  3.  $r = 1 \implies$  **inconclusive**
- Leibnitz's test: An alternating series  $\sum (-1)^{n+1} u_n$  converges if:

1.  $u_n \geq u_{n+1} \forall n \geq N$  for some  $N$
  2.  $\lim_{n \rightarrow \infty} u_n = 0$
- Radius of convergence  $R$  of a power series  $\sum a_n(x-a)^n$  is given by the relation:

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} |a_n|^{1/n}$$

- Taylor series for a function  $f(x)$  about the point  $x = a$  is given by:

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x-a)^n = f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots$$

The **Maclaurin series** are the Taylor series generated at  $a = 0$