

## 1 Discrete probability

- **Expectation value:** For a discrete random variable  $X$  (this can also be replaced by a function  $h(x)$ ) and probability mass function  $p(x)$

$$E(x) = \mu_x = \sum_{x \in D} xp(x)$$

Some properties:

1.  $E(C) = C$
2.  $E(CX) = CE(X)$
3.  $E(CX + d) = CE(X) + d$

- **Variance:** Assume the discrete random variable  $X$  to have an expected value  $\mu$ . Then

$$V(X) = \sum_D (x - \mu)^2 \cdot p(x) = E[(x - \mu)^2]$$

This can also be rewritten as  $V(X) = \sigma^2 = E(X^2) - [E(X)]^2$ .

For any arbitrary function  $h(X)$  of the form  $(aX + b)$ , we have  $V(h(X)) = a^2V(X)$

- **Moment Generating Function (MGF):** For a discrete random variable  $X$ ,  $m_X(t) = E(e^{tX})$ . Using this, we have the following

$$\left. \frac{d^r m_X(t)}{dt^r} \right|_{t=0} = E(X^r)$$

- **Bernoulli Distribution:** Consider a random variable  $X$  which takes Boolean values (0 and 1). Let the probability of obtaining the value 1 be  $p$  and 0 be  $q = (1 - p)$

1.  $f(x, p) = p^x(1 - p)^{1-x}$
2.  $E(X) = p$
3.  $\text{Var}(X) = pq$
4.  $m_X(t) = q + pe^t$

- **Binomial Distribution:** For a random variable  $X$  with parameters  $n$  and  $p$  (denoted by  $X \sim (n, p)$ )

1.  $m_X(t) = (q + pe^t)^n$
2.  $E(X) = np$
3.  $\text{Var}(X) = npq$
4.  $p(x) = \binom{n}{x} p^x (1 - p)^{n-x}$

- **Geometric Random Variable:** Let  $X$  denote the random variable representing the number of trials required to get the 1<sup>st</sup> success.

1.  $g(n, p) = pq^{n-1}$
2.  $E(X) = \frac{1}{p}$  and  $E(X^2) = \frac{1+q}{p^2}$
3.  $\text{Var}(X) = \frac{q}{p^2}$
4.  $m_X(t) = \frac{pe^{-t}}{1 - qe^t}$

- **Poisson distribution:** For a discrete random variable  $X$  with the parameter  $\mu$

1.  $p(x, \mu) = \frac{e^{-\mu} \cdot \mu^x}{x!}$
2.  $E(X) = \text{Var}(X) = \mu$
3.  $m_X(t) = e^{\mu(e^t - 1)}$

Consider a binomial distribution  $b(x; n, p)$  in the limit where  $n \rightarrow \infty$  and  $p \rightarrow 0$ . In such a scenario,  $b(x; n, p) \rightarrow p(x; \mu)$  where  $\mu = np$

- **Poisson process:** Let  $P_k(t)$  denote the probability that  $k$  events are observed during a particular time interval of length  $t$ , then

$$P_k(t) = e^{-\alpha t} \cdot \frac{(\alpha t)^k}{k!}$$

Where  $t$  is the random variable with the parameter  $\mu = \alpha t$  and  $\alpha$  specifies the rate of the process in question.

## 2 Continuous probability

Consider a function  $f(x)$  to be a PDF of a continuous random variable  $x$

- $P(a \leq x \leq b) = \int_a^b f(x)dx$
- **CDF** =  $F(x) = \int_{-\infty}^x f(x)dx$
- **Expectation value and MGF:**

$$E(x) = \int_{-\infty}^{\infty} xf(x)dx$$

$$E(H(x)) = \int_{-\infty}^{\infty} H(x)f(x)dx$$

$$m_x(t) = E(e^{xt}) = \int_{-\infty}^{\infty} e^{xt}f(x)dx$$

- **Moments, mean and variance:**

$$E(x^k) = \int_{-\infty}^{\infty} x^k f(x)dx = \left( \frac{d^k}{dx^k} (m_x(t)) \right) \Big|_{t=0}$$

$$\mu = E(x) = \int_{-\infty}^{\infty} xf(x)dx$$

$$\sigma^2 = \left( \int_{-\infty}^{\infty} x^2 f(x)dx \right) - \left( \int_{-\infty}^{\infty} xf(x)dx \right)^2$$

- **Uniform distribution:**

1. Probability density function:

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}$$

2. MGF, mean and variance:

$$m_x(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$$

$$\mu = \frac{a+b}{2}$$

$$\sigma^2 = \frac{(b-a)^2}{12}$$

- **Normal/Gaussian distribution:** It is denoted as  $X \sim N(\mu, \sigma^2)$ , where  $\mu$ (mean) and  $\sigma^2$  (variance) are the parameters of the distribution.

1. Probability density function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

2. MGF, expectation:

$$m_x(t) = e^{\mu t + \frac{t^2 \sigma^2}{2}}$$

$$E(x) = m'_x(t) \Big|_{t=0} = \mu$$

- **Standard Normal distribution:** Assume  $z = \frac{x-\mu}{\sigma}$ , then  $E(z) = 0$  and  $V(z) = 1$ . In such a scenario,  $z$  is known as **the standard normal variate** which is denoted as  $z \sim \mathcal{N}(0, 1)$

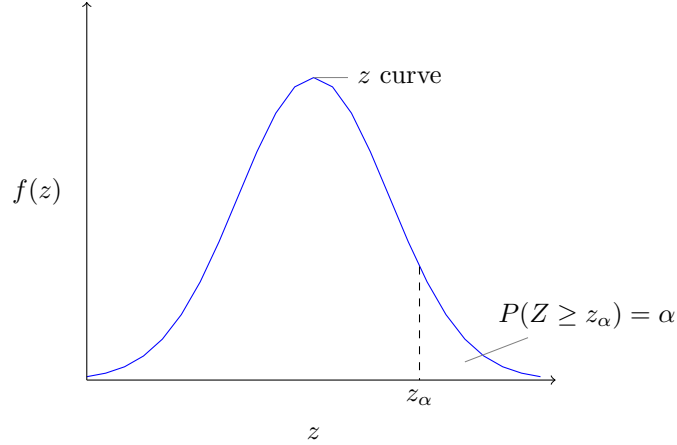
1. Probability density function:

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

2. Cumulative density function:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-z^2/2} dz$$

3.  $z_\alpha$  for  $z$  critical values:  $z_\alpha$  represents the  $100(1-\alpha)$ th percentile of the standard normal distribution. The values of  $z_\alpha$  is often called as the  **$z$  critical values**.



- **Approximating Binomial distribution:** Let  $X$  be a random variable based on  $n$  trials with a success probability  $p$ . If the binomial histogram is not too skewed, then  $X$  has  $\approx$  normal distribution with  $\mu = np$  and  $\sigma = \sqrt{npq}$  or

$$P(X \leq x) = B(x; n, p) \approx \Phi\left(\frac{x + 0.5 - np}{\sqrt{npq}}\right)$$

- **Exponential and Gamma distribution:**

1. Exponential distribution:

$$f(x; \lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Mean ( $\mu$ ) =  $\frac{1}{\lambda}$  and variance ( $\sigma^2$ ) =  $\frac{1}{\lambda^2}$

2. Gamma function: For a parameter  $\alpha > 0$ , we define the **gamma function** as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

Some important properties:

- (a)  $\Gamma(\alpha) = (\alpha - 1) \cdot \Gamma(\alpha - 1)$
- (b)  $\Gamma(n) = (n - 1)!$
- (c)  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Gamma distribution:

$$f(x; \alpha, \beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

**Expectation:**  $E(x) = \mu = \alpha\beta$

**Variance:**  $V(x) = \sigma^2 = \alpha\beta^2$

The gamma distribution of a random variable  $X$  with parameters  $\alpha$  and  $\beta$  is denoted as:

$$P(X \leq x) = F(x; \alpha, \beta) = F\left(\frac{x}{\beta}; \alpha\right)$$

### 3. Chi-Squared distribution:

For a positive integer  $\nu$  (**degrees of freedom of the random variable  $X$** ), the PDF of this distribution takes the form

$$f(x; \nu) = \begin{cases} \frac{1}{2^{\nu/2} \Gamma(\nu/2)} x^{(\frac{\nu}{2}-1)} e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

This can be modelled as a gamma distribution with the parameters  $\alpha = \frac{\nu}{2}$  and  $\beta = 2$ .