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Maths F111 Formulae

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Miscellaneous coordinate systems:

- Polar coordinates:
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$
- Cylindrical coordinates: $r \geq 0, 0 \leq \theta \leq 2\pi, -\infty < z < \infty$
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$
- Spherical coordinates: $\rho =$ Distance from origin, $\phi =$ Angle with **positive** $z-$axis, $\theta =$ Angle with x axis
$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$

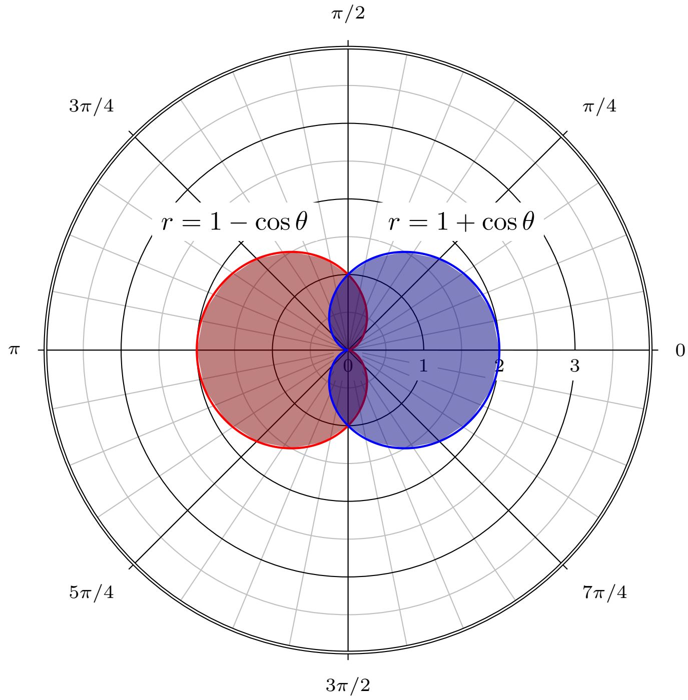
1 Vector valued functions

- Parametric equation of a line passing through (x_0, y_0, z_0) and parallel to $a\hat{i} + b\hat{j} + c\hat{k}$:
$$\begin{cases} x(t) = x_0 + ta \\ y(t) = y_0 + tb \\ z(t) = z_0 + tc \end{cases}$$
- Arc length of a curve for $a \leq x \leq b$:
$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$
Arc length parameter $\vec{s}(t) = \int_{t_0}^t |v(\tau)| d\tau$
- Unit tangent vector $\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{d\vec{r}}{d\vec{s}}$, $\vec{s}$ being the arc length parameter
- Curvature (κ):
$$\kappa = \left|\frac{d\overrightarrow{T}}{ds}\right| = \left|\frac{d\overrightarrow{T}/dt}{ds/dt}\right| = \frac{1}{|\vec{v}|}\left|\frac{d\overrightarrow{T}}{dt}\right|$$
Unit normal ($\overrightarrow{N}$):
$$\overrightarrow{N} = \frac{\left(\frac{d\overrightarrow{T}}{ds}\right)}{\left|\frac{d\overrightarrow{T}}{ds}\right|} = \frac{1}{\kappa}\left(\frac{d\overrightarrow{T}}{ds}\right)$$
Curvature can also be written in terms of v and a as $\frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3}$
- $\vec{a}(t) = \frac{d|v|}{dt}\overrightarrow{T} + \kappa|\vec{v}|^2\overrightarrow{N}$
- Binormal vector ($\overrightarrow{B}$) $= \overrightarrow{T} \times \overrightarrow{N}$
$\overrightarrow{T}$ and $\overrightarrow{N}$: Osculating plane
$\overrightarrow{B}$ and $\overrightarrow{N}$: Normal plane
$\overrightarrow{B}$ and $\overrightarrow{T}$: Rectifying plane
- Torsion $\tau = -\frac{d\overrightarrow{B}}{ds}\cdot\overrightarrow{N}$

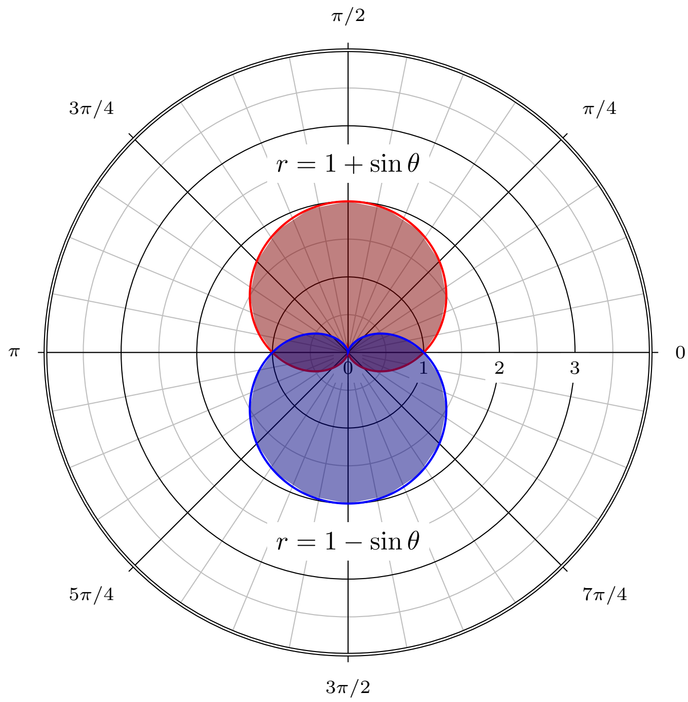
2 Polar coordinates

- Important polar curves:

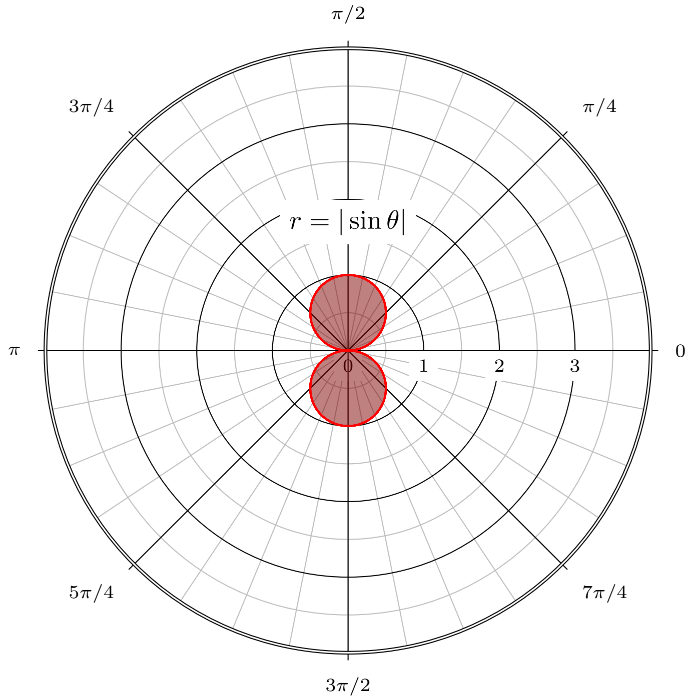
  1. $r = 1 - \cos\theta$, $r = 1 + \cos\theta$:



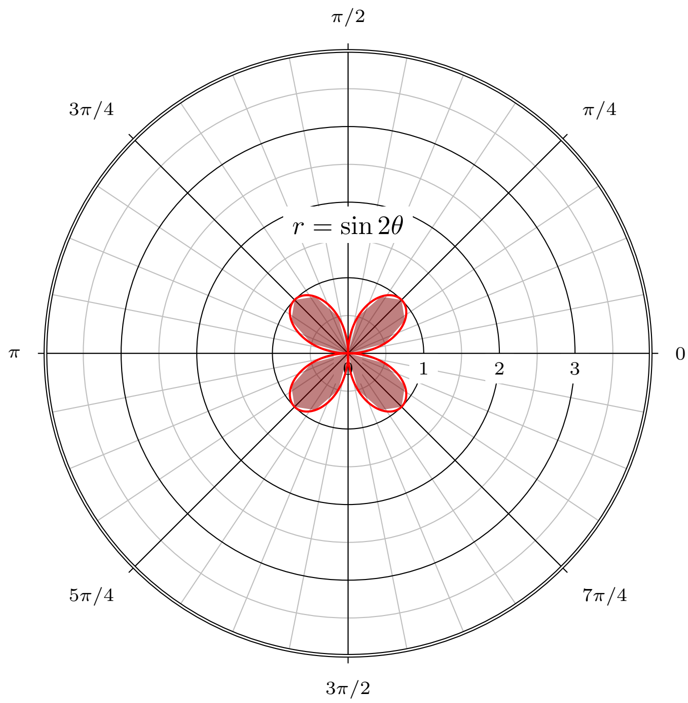
  2. $r = 1 + \sin\theta, r = 1 - \sin\theta$:



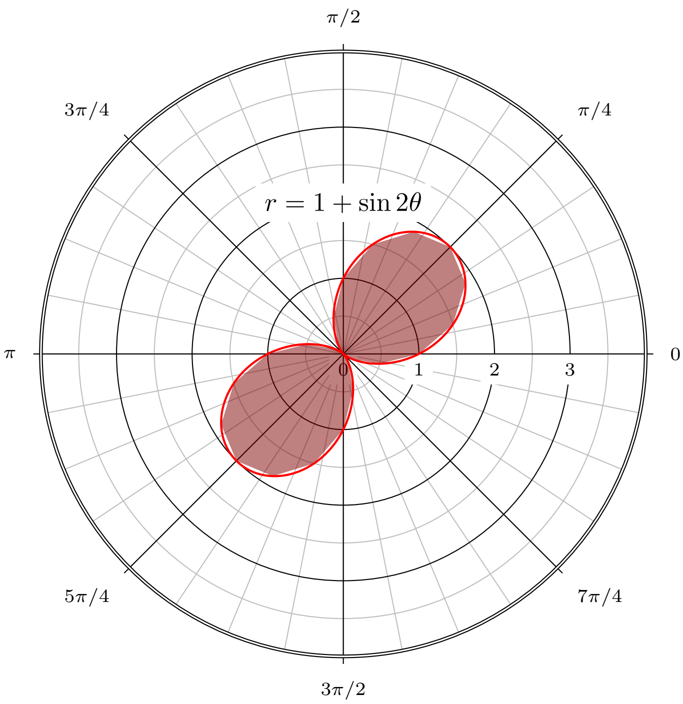
3. $r = |\sin\theta|$:



4. $r = \sin 2\theta$:



5. $r = 1 + \sin 2\theta$:



- Area enclosed by a polar curve:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

  Length of a polar curve:

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

3 Partial derivatives

- From the 1^{st} principles, the partial derivative of a function $f(x, y)$ with respect to x is given as:

$$\frac{\partial f}{\partial x} = f_x = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

  A similar idea is followed for calculating $\frac{\partial f}{\partial y}$

- Implicit differentiation: Let $f(x, y) = c$. After a bit of work, we get the following

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

- Higher order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = \lim_{h \to 0} \frac{f_x(x + h, y) - f_x(x, y)}{h}$$
$$\frac{\partial^2 f}{\partial y \partial x} = f_{xy} = \lim_{k \to 0} \frac{f_x(x, y + k) - f_x(x, y)}{k}$$

- Directional derivative : $\left(\frac{df}{dt}\right)_{\hat{n}} = \nabla f \cdot \hat{n}$

- Angle between 2 surfaces : $\theta = \cos^{-1}\left(\frac{\nabla f \cdot \nabla g}{|\nabla f||\nabla g|}\right)$

- Equation of tangent plane at (x_0, y_0, z_0): $(x - x_0)f_x + (y - y_0)f_y + (z - z_0)f_z = 0$
  Equation of normal at (x_0, y_0, z_0):

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$$

- Linearization of $f(x,y)$ at (x_0,y_0):

$$f(x,y) \approx f(x_0,y_0) + (x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0)$$

- Extrema: At the critical points: $f_x(x_0,y_0)=0; f_y(x_0,y_0)=0$
If at (x_0,y_0):
  1. $f_{xx}f_{yy} - f_{xy}^2 > 0; f_{xx} < 0 \implies$ **Local max**
  2. $f_{xx}f_{yy} - f_{xy}^2 > 0; f_{xx} > 0 \implies$ **Local min**
  3. $f_{xx}f_{yy} - f_{xy}^2 < 0 \implies$ **Saddle point**
  4. $f_{xx}f_{yy} - f_{xy}^2 = 0 \implies$ **Further investigation**

- Lagrange multipliers: For a function $f(x,y,z)$ subject to the constraints $g_1(x,y,z)$ and $g_2(x,y,z)$, we have:

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

4 Multiple integrals

- Double integrals to compute volume over a region R under a surface $f(x,y)$:

$$V = \iint\limits_R f(x,y)dA$$

For computing the area of a closed bounded region R:

$$A = \iint\limits_R dA$$

$$\text{Average value of } f \text{ over a region } R = \frac{1}{\text{Area of R}} \iint\limits_R f dA$$

- Volume in polar coordinates:

$$V = \iint\limits_R f(r,\theta) r dr d\theta$$

- Triple integrals in cylindrical coordinates:

$$I = \iiint\limits_R f(r,\theta,z)dV = \iiint\limits_R f(r,\theta,z) dz r dr d\theta$$

Triple integrals in spherical coordinates:

$$I = \iiint\limits_R f(\rho,\phi,\theta)dV = \iiint\limits_R f(\rho,\phi,\theta)\rho^2 \sin\phi d\rho d\phi d\theta$$

- Substitutions in iterated integrals:
  1. Double integrals: For a coordinate transformation $x = g(u,v)$ and $y = h(u,v)$, we define the **Jacobian** as $J(u,v) = \frac{\partial(x,y)}{\partial(u,v)}$:

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

  The new integral becomes:

$$\iint\limits_R f(x,y)dxdy = \iint\limits_R f(g(u,v),h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$$

  2. Triple integrals: Following a similar set of steps as above, we introduce a new transformation for the z coordinate as $z = k(u,v,w)$. The Jacobian changes to a 3×3 determinant in this case to give the new integral:

$$\iiint\limits_D f(x,y,z)dxdydx = \iiint\limits_G h(u,v,w)|J(u,v,w)|dudvdw$$

  Where $h(u,v,w)$ is the transform of $f(x,y,z)$

5 Integrals and vector fields

- Line integral of f over a curve C:

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))|\vec{v}(t)|dt$$

Line integral over a vector field $\overrightarrow{F}$: $\int_c \overrightarrow{F} \cdot d\vec{r}$

- Work done by a force $\overrightarrow{F}(x,y,z)$ over a smooth curve C with the parametrization $\vec{r}(t)$:

$$W = \int_C \overrightarrow{F} \cdot \overrightarrow{T} ds$$

- Flux of a vector field $F = M(x,y)\hat{\imath} + N(x,y)\hat{\jmath}$ with the **outward normal** $\vec{n} = \overrightarrow{T} \times \vec{k}$:

$$\int_C \overrightarrow{F} \cdot \vec{n} dx = \int_C (My'(t) - Nx'(t))dt$$

- Circulation of a vector field $F = M(x,y,z)\hat{\imath} + N(x,y,z)\hat{\jmath} + P(x,y,z)\hat{k}$:

$$\oint_C \overrightarrow{F} \cdot d\vec{r} = \oint Mdx + Ndy + Pdz = \int_C (Mx'(t) + Ny'(t) + Pz'(t))dt$$

- If $\overrightarrow{F}$ is conservative $\Leftrightarrow \overrightarrow{F} = \nabla f$
$\oint_C \overrightarrow{F} \cdot d\vec{r} = 0$ over every closed curve if $\overrightarrow{F}$ is a conservative field

- If $\overrightarrow{F}$ is conservative, then curl$(\overrightarrow{F}) = 0$, which is given as:

$$\text{curl}(F) = \nabla \times \overrightarrow{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = 0$$

- Green's theorem:

  1. Circulation-curl/tangential form:

$$\oint_C \overrightarrow{F} \cdot \overrightarrow{T} ds = \oint_C Mdx + Ndy = \iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

  2. Flux-divergence/normal form:

$$\oint_C \overrightarrow{F} \cdot \hat{n} ds = \oint_C Mdx - Ndy = \iint\limits_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

- Surface area of $f(x,y,z) = c$ over a region R with the unit normal vector $\vec{p}$ is given as:

$$A = \iint\limits_S d\sigma = \iint\limits_R \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|} dA$$

For a surface of the form $f(x,y,z) = z - m(x,y)$ with the projeted region R in the XY plane:

$$A = \iint\limits_S d\sigma = \iint\limits_R \sqrt{1 + m_x^2 + m_y^2} dA$$

- A surface $\vec{r}(t)$ parametrized as $\vec{r}(u,v)$ has the surface area $A = \iint\limits_S d\sigma = \iint\limits_R |\vec{r_u} \times \vec{r_v}| dudv$

- Surface integral of $g(x,y,z)$ over the projected region S of the function $f(x,y,z) = c$ is:

$$\iint\limits_R g(x,y,z) \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|} dA$$

Similarly, the surface integral of a vector field F over a surface S is $\iint\limits_C \overrightarrow{F} \cdot \hat{n} d\sigma$ and the flux over the region is $\pm \iint\limits_R \frac{F \cdot \nabla f}{|\nabla f \cdot p|} dA$. The sign of flux depends on **the choice of the normal vector**

- Stoke's theorem:
$$\oint_C \overrightarrow{F} \cdot d\vec{r} = \iint\limits_S (\nabla \times \overrightarrow{F}) \cdot \hat{n} d\sigma = \iint\limits_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$
This integral can be taken over a region R to give:
$$\oint_C \overrightarrow{F} \cdot d\vec{r} = \iint\limits_R \frac{(\nabla \times \overrightarrow{F}) \cdot \nabla f}{|\nabla f \cdot \hat{p}|} dA$$
For a surface parametrized as $r(u, v)$, the integral takes the form:
$$\oint_C \overrightarrow{F} \cdot d\vec{r} = \iint\limits_R (\nabla \times \overrightarrow{F}) \cdot (\vec{r_u} \times \vec{r_v}) dudv$$

- Divergence theorem: The divergence of a vector field $F = M\hat{\imath} + N\hat{\jmath} + P\hat{k}$ is given as:
$$\text{div}(\overrightarrow{F}) = \nabla \cdot \overrightarrow{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$
The flux of a vector fielf $\overrightarrow{F}$ over a surface S in the direction of the outward normal $\hat{n}$ is calculated by the integral:
$$\iint\limits_S \overrightarrow{F} \cdot \hat{n} d\sigma = \iiint\limits_D (\nabla \cdot \overrightarrow{F}) dV$$

6 Infinite series

For the sake of brevity, I define two series: $\sum a_n = a_n$ and $\sum b_n = b_n$

- Integral test: If $f(x)$ is $+ve$, continuous and decreasing $\forall x \geq N$ for some N, then a_n and $\int_N^\infty f(x)dx$ converge or diverge together
- $p-$series: $\sum \frac{1}{n^p}$ is **convergent** for $p > 1$ and **divergent** for $p \leq 1$
- Direct comparison test:
  1. If $a_n \leq b_n \forall n \geq N$ and b_n is convergent, then a_n is also convergent
  2. If $a_n \geq b_n \forall n \geq N$ and b_n is divergent, then a_n is also divergent
- Limit comparison test: a_n, b_n are $+ve$ $\forall n \geq N$, then:
  1. $\lim\limits_{n\to\infty} \left(\frac{a_n}{b_n}\right) = c$ which is *finite and nonzero*, then a_n and b_n **converge or diverge together**
  2. $\lim\limits_{n\to\infty} \left(\frac{a_n}{b_n}\right) = 0$,then a_n **converges** and b_n **diverges**
  3. $\lim\limits_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \infty$, then a_n **diverges** and b_n **converges**
- Ratio test: If $\lim\limits_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = r$, then:
  1. $r < 1 \implies a_n$ is **convergent**
  2. $r > 1 \implies a_n$ is **divergent**
  3. $r = 1 \implies$ **inconclusive**
- Root test: If $\lim\limits_{n\to\infty} (a_n)^{1/n} = r$, then:
  1. $r < 1 \implies a_n$ is **convergent**
  2. $r > 1 \implies a_n$ is **divergent**
  3. $r = 1 \implies$ **inconclusive**
- Lebinitz's test: An alternating series $\sum(-1)^{n+1}u_n$ converges if:

1. $u_n \geq u_{n+1} \forall n \geq N$ for some N
2. $\lim_{n\to\infty} u_n = 0$

- Radius of convergence R of a power series $\sum a_n(x-a)^n$ is given by the relation:

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} |a_n|^{1/n}$$

- Taylor series for a function $f(x)$ about the point $x = a$ is given by:

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

  The **Maclaurin series** are the Taylor series generated at $a = 0$