COMP9418: Advanced Topics in Statistical Machine Learning

MAP Inference

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Introduction

- In this lecture, we study algorithm to compute queries of the form
  - MAP: maximum a posteriori hypothesis
  - MPE: maximum a posteriori explanation
- In these queries, we are interested in finding the most probable instantiations of a subset of variables
- We discuss variations of the Variable Elimination algorithm to compute MAP and MPE queries

Introduction: Example

- Consider a Bayesian network on the right
  - It concerns a population of 55% males and 45% females
  - They can suffer of a medical condition C that is more likely in males
  - There are two diagnosis tests for C, T_1 and T_2
  - T_2 is more effective on females
  - Both tests are equally effective on males

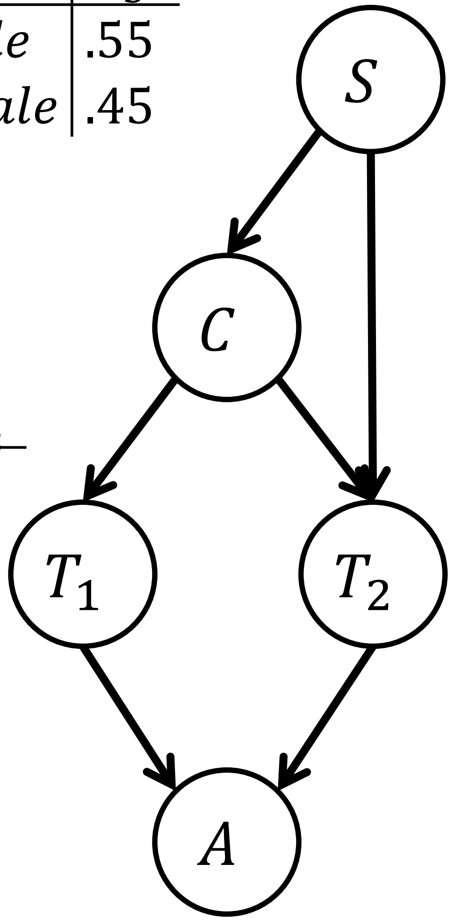
S	C	$\Theta_{S\|C}$
$male$	yes	.05
$male$	no	.95
$female$	yes	.01
$female$	no	.99

C	T_1	$\Theta_{T_1\|C}$
yes	ve	.80
yes	$\overline{ve}$	.20
no	ve	.20
no	$\overline{ve}$	.80

S	Θ_S
$male$	.55
$female$	.45

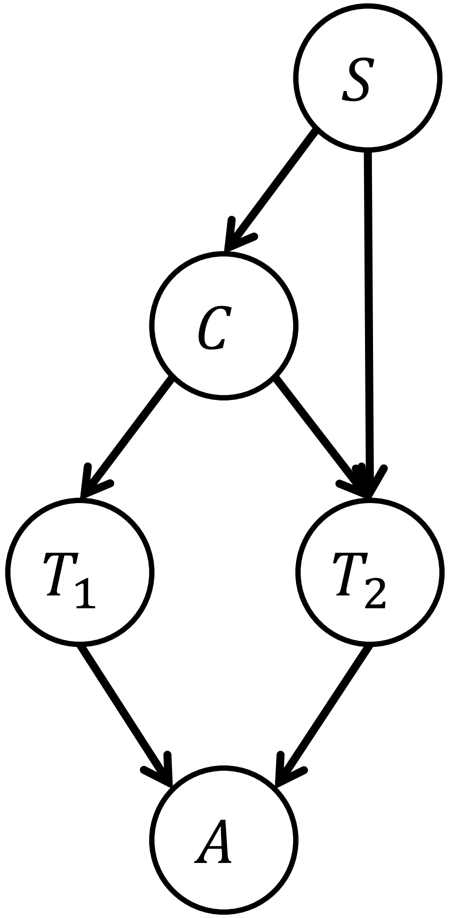
S	C	T_2	$\Theta_{T_2\|C,S}$
$male$	yes	ve	.80
$male$	yes	$\overline{ve}$	.20
$male$	no	ve	.20
$male$	no	$\overline{ve}$	.80
$female$	yes	ve	.95
$female$	yes	$\overline{ve}$	.05
$female$	no	ve	.05
$female$	no	$\overline{ve}$	.95

T_1	T_2	A	$\Theta_{A\|T_1,T_2}$
ve	ve	yes	1
ve	ve	no	0
ve	$\overline{ve}$	yes	0
ve	$\overline{ve}$	no	1
$\overline{ve}$	ve	yes	0
$\overline{ve}$	ve	no	1
$\overline{ve}$	$\overline{ve}$	yes	1
$\overline{ve}$	$\overline{ve}$	no	0



Introduction: Example

- We can partition this population in four groups
  - Males and females, with or without the condition
- Suppose a person takes both tests with the same results
  - Leads to the evidence $A = yes$
- What is the most likely group this individual belongs?
  - This is an example of MAP instantiation
  - The most likely instantiation of S and C given $A = yes$
  - In this query, S and C are *MAP variables*
- The answer for this example is
  - $S = male$ and $C = no$ with posterior probability of $\sim 49.3\%$



MAP and Inference

- Variable and factor elimination algorithms can compute MAP instantiations
  - They are efficient with small number of MAP variables
  - We compute the posterior marginal over MAP variables and select the instantiation with maximal probability
- However, this approach is exponential in the number of MAP variables
- Our objective in this lecture is to present algorithms for MAP instantiations
  - Not necessarily exponential in the number of MAP variables

MAP and MPE

- MPE is a special case of MAP when MAP variables contain all unobserved network variables
  - In the previous example, it would result in 16 groups
  - Males and females, with or without the condition and the four possible outcomes for the two tests
- This is the MAP instantiation for S, C, T_1 and T_2
  - The answer is $S = female, C = no, T_1 = \overline{ve}, T_2 = \overline{ve}$
  - With posterior probability $\sim 47\%$

- This case of MAP is known as *MPE instantiation*
  - MPE instantiations are much easier to compute than MAP
  - That is why they have their own name
- MPE is not the answer for MAP
  - MPE projection on variables S and C is $S = female, C = no$
  - But the MAP answer of previous slides is $S = male$ and $C = no$
- Although this technique is sometimes used as an approximation for MAP

Computing MPE

- Given a network
  - The *MPE probability* for the variables $\boldsymbol{Q}$ of a network given evidence $\boldsymbol{e}$ is defined as

$$MPE_P(\boldsymbol{e}) \stackrel{\text{def}}{=} \max_{\boldsymbol{q}} P(\boldsymbol{q}, \boldsymbol{e})$$

- There may be several instantiations $\boldsymbol{q}$ with maximal probability
  - Each of them is an *MPE instantiation*
  - The set of such instantiations is defined as

$$MPE(\boldsymbol{e}) \stackrel{\text{def}}{=} \underset{\boldsymbol{q}}{argmax}\, P(\boldsymbol{q}, \boldsymbol{e})$$

- MPE instantiations can be characterized as instantiations $\boldsymbol{q}$ that maximize the posterior distribution

$$MPE(\boldsymbol{e}) \stackrel{\text{def}}{=} \underset{\boldsymbol{q}}{argmax}\, P(\boldsymbol{q}|\boldsymbol{e})$$

  - Since $P(\boldsymbol{q}|\boldsymbol{e}) = \frac{P(\boldsymbol{q},\boldsymbol{e})}{P(\boldsymbol{e})}$
  - $P(\boldsymbol{e})$ is independent of the instantiation $\boldsymbol{q}$

Computing MPE by Variable Elimination

- Returning to our example

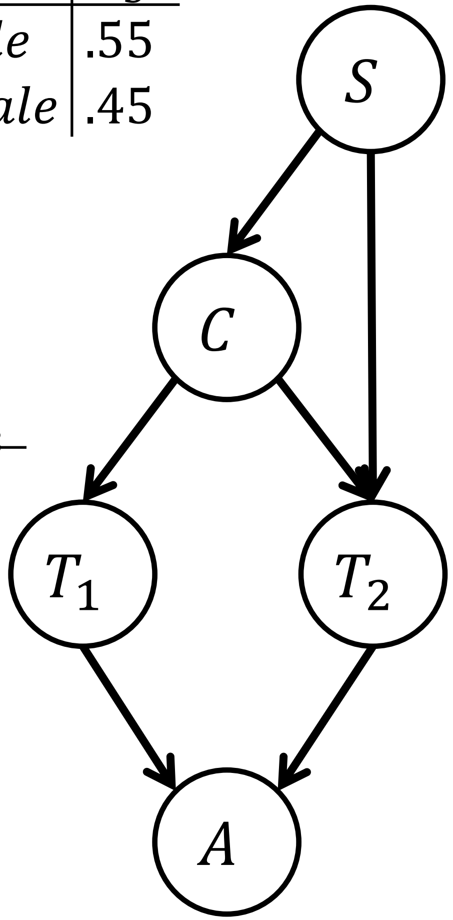
S	C	$\Theta_{S\|C}$
$male$	yes	$.05$
$male$	no	$.95$
$female$	yes	$.01$
$female$	no	$.99$

C	T_1	$\Theta_{T_1\|C}$
yes	ve	$.80$
yes	$\overline{ve}$	$.20$
no	ve	$.20$
no	$\overline{ve}$	$.80$

S	Θ_S
$male$	$.55$
$female$	$.45$

S	C	T_2	$\Theta_{T_2\|C,S}$
$male$	yes	ve	$.80$
$male$	yes	$\overline{ve}$	$.20$
$male$	no	ve	$.20$
$male$	no	$\overline{ve}$	$.80$
$female$	yes	ve	$.95$
$female$	yes	$\overline{ve}$	$.05$
$female$	no	ve	$.05$
$female$	no	$\overline{ve}$	$.95$

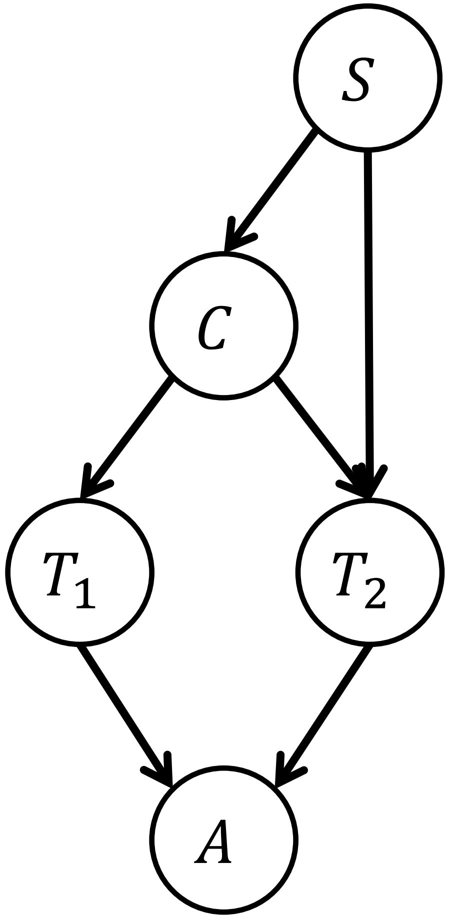
T_1	T_2	A	$\Theta_{A\|T_1,T_2}$
ve	ve	yes	1
ve	ve	no	0
ve	$\overline{ve}$	yes	0
ve	$\overline{ve}$	no	1
$\overline{ve}$	ve	yes	0
$\overline{ve}$	ve	no	1
$\overline{ve}$	$\overline{ve}$	yes	1
$\overline{ve}$	$\overline{ve}$	no	0



Computing MPE by Variable Elimination

- Returning to our example
  - We can compute the joint probability for this Bayesian network
  - (Even rows omitted since they have zero probabilities)
  - The MPE instantiation (assuming no evidence) is given in row 31
  - MPE probability (MPE_P) is .338580

	S	C	T_1	T_2	A	$P(.)$
1	$male$	yes	ve	ve	yes	.017600
3	$male$	yes	ve	$\overline{ve}$	no	.004400
5	$male$	yes	$\overline{ve}$	ve	no	.004400
7	$male$	yes	$\overline{ve}$	$\overline{ve}$	yes	.001100
9	$male$	no	ve	ve	yes	.020900
11	$male$	no	ve	$\overline{ve}$	no	.083600
13	$male$	no	$\overline{ve}$	ve	no	.083600
15	$male$	no	$\overline{ve}$	$\overline{ve}$	yes	.334400
17	$female$	yes	ve	ve	yes	.003420
19	$female$	yes	ve	$\overline{ve}$	no	.000180
21	$female$	yes	$\overline{ve}$	ve	no	.000855
23	$female$	yes	$\overline{ve}$	$\overline{ve}$	yes	.000045
25	$female$	no	ve	ve	yes	.004455
27	$female$	no	ve	$\overline{ve}$	no	.084645
29	$female$	no	$\overline{ve}$	ve	no	.017820
31	$female$	no	$\overline{ve}$	$\overline{ve}$	yes	.338580



Computing MPE by Variable Elimination

- We can compute MPE_p using Variable Elimination
  - However, when eliminating a variable, we maximize out instead of summing it out
- To maximize out a variable B from a factor $\phi(A, B, C)$, we produce another factor over remaining variables A and C
  - By merging all rows that agree on the values of these remaining variables
  - As we merge rows, we drop reference to the maximized variable and assign to the resulting row the maximum probability associated with the merged rows

A	B	C	$\phi(A, B, C)$
0	0	0	7
0	0	1	4.5
0	1	0	.2
0	1	1	2
1	0	0	3
1	0	1	.5
1	1	0	1.2
1	1	1	3

A	C	$\max_B \phi(A, C)$
0	0	7
0	1	4.5
1	0	3
1	1	3

Computing MPE by Variable Elimination

- The result of maximizing out variable B from factor ϕ is
  - Another factor, $\max_B \phi$ that does not mention B
  - The new factor agrees with the old factor on the MPE probability
- We can continue to maximize out $\max_B \phi$ until we get the trivial factor
  - The probability assigned to this factor is the MPE probability
  - This method can be extended to provide the MPE instantiation (more later)
- Maximization is commutative
  - Allow us to refer to maximizing out a set of variables without specifying the order
  - Also, $\max_X \phi_1 \phi_2 = \phi_1 \max_X \phi_2$ if variable X appears only in ϕ_2

MPE VE: Algorithm

$\boldsymbol{Q} \leftarrow$ variables in the network

$\pi \leftarrow$ elimination order of variables $\boldsymbol{Q}$

$\boldsymbol{S} \leftarrow \{\phi^{\boldsymbol{e}}: \phi$ is a factor of the network$\}$

for $i = 1$ to $|\boldsymbol{Q}|$ **do**

$\sigma_i \leftarrow \prod_k \phi_k$, where ϕ_k belongs to $\boldsymbol{S}$ and mentions variable $\pi(i)$

$\tau_i \leftarrow \max_{\pi(i)} \sigma_i$

replace all factors ϕ_k in $\boldsymbol{S}$ by factor τ_i

return trivial factor $\prod_{\tau \in S} \tau$

Notes:

- All factors are eliminated leading to a trivial factor
- $\phi^{\boldsymbol{e}}$ is a factor with the rows of factor ϕ that match the evidence $\boldsymbol{e}$
- Pruning should eliminate edges only since all variables are relevant to the answer
- This algorithm has the same complexity as VE, i.e., the time and space complexity are $O(n \exp(w))$ for n variables and an elimination width w

Computing MPE: Example

- Returning to our example
  - Let us run MPE VE on this example
  - With the elimination order $\pi = S, C, A, T_1, T_2$

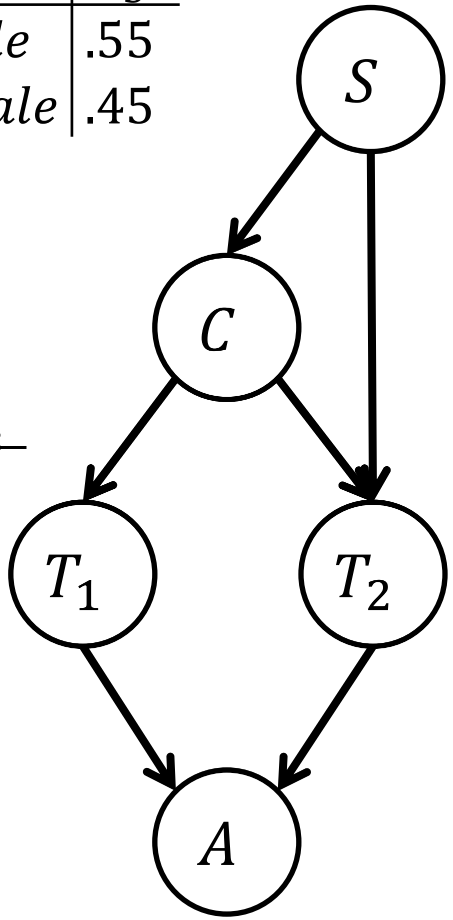
S	C	$\Theta_{S\|C}$
$male$	yes	$.05$
$male$	no	$.95$
$female$	yes	$.01$
$female$	no	$.99$

C	T_1	$\Theta_{T_1\|C}$
yes	ve	$.80$
yes	$\overline{ve}$	$.20$
no	ve	$.20$
no	$\overline{ve}$	$.80$

S	Θ_S
$male$	$.55$
$female$	$.45$

S	C	T_2	$\Theta_{T_2\|C,S}$
$male$	yes	ve	$.80$
$male$	yes	$\overline{ve}$	$.20$
$male$	no	ve	$.20$
$male$	no	$\overline{ve}$	$.80$
$female$	yes	ve	$.95$
$female$	yes	$\overline{ve}$	$.05$
$female$	no	ve	$.05$
$female$	no	$\overline{ve}$	$.95$

T_1	T_2	A	$\Theta_{A\|T_1,T_2}$
ve	ve	yes	1
ve	ve	no	0
ve	$\overline{ve}$	yes	0
ve	$\overline{ve}$	no	1
$\overline{ve}$	ve	yes	0
$\overline{ve}$	ve	no	1
$\overline{ve}$	$\overline{ve}$	yes	1
$\overline{ve}$	$\overline{ve}$	no	0



Computing MPE: Example

- Returning to our example
  - Let us run MPE VE on this example
  - With the elimination order $\pi = S, C, A, T_1, T_2$

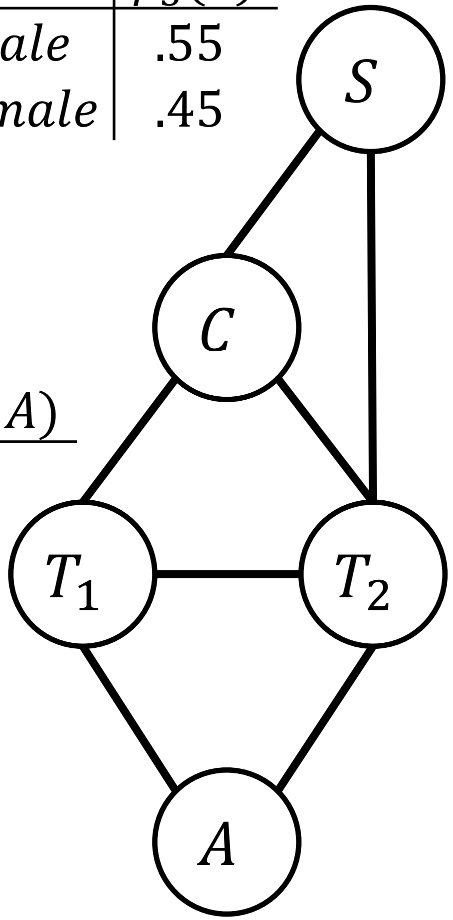
S	C	$\phi_1(S,C)$
$male$	yes	.05
$male$	no	.95
$female$	yes	.01
$female$	no	.99

C	T_1	$\phi_2(T_1,C)$
yes	ve	.80
yes	$\overline{ve}$	.20
no	ve	.20
no	$\overline{ve}$	.80

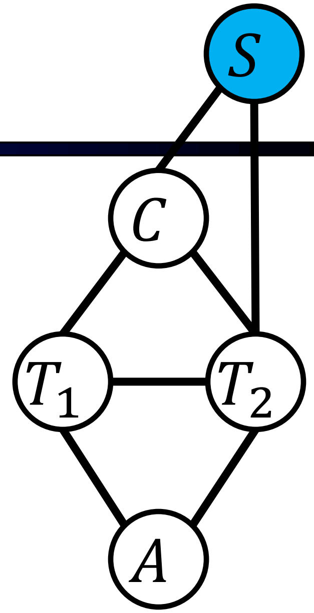
S	$\phi_3(S)$
$male$	.55
$female$	.45

S	C	T_2	$\phi_4(T_2,C,S)$
$male$	yes	ve	.80
$male$	yes	$\overline{ve}$	.20
$male$	no	ve	.20
$male$	no	$\overline{ve}$	.80
$female$	yes	ve	.95
$female$	yes	$\overline{ve}$	.05
$female$	no	ve	.05
$female$	no	$\overline{ve}$	.95

T_1	T_2	A	$\phi_5(T_1,T_2,A)$
ve	ve	yes	1
ve	ve	no	0
ve	$\overline{ve}$	yes	0
ve	$\overline{ve}$	no	1
$\overline{ve}$	ve	yes	0
$\overline{ve}$	ve	no	1
$\overline{ve}$	$\overline{ve}$	yes	1
$\overline{ve}$	$\overline{ve}$	no	0



Computing MPE: Example



- Returning to our example
  - Let us run MPE VE on this example
  - With the elimination order $\pi = S, C, A, T_1, T_2$

S	C	$\phi_1(S,C)$
$male$	yes	.05
$male$	no	.95
$female$	yes	.01
$female$	no	.99

C	T_1	$\phi_2(T_1,C)$
yes	ve	.80
yes	$\overline{ve}$	.20
no	ve	.20
no	$\overline{ve}$	.80

S	$\phi_3(S)$
$male$	.55
$female$	.45

$\times$

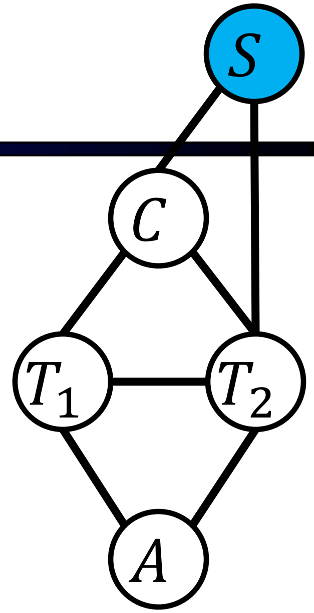
S	C	T_2	$\phi_4(T_2,C,S)$
$male$	yes	ve	.80
$male$	yes	$\overline{ve}$	.20
$male$	no	ve	.20
$male$	no	$\overline{ve}$	.80
$female$	yes	ve	.95
$female$	yes	$\overline{ve}$	.05
$female$	no	ve	.05
$female$	no	$\overline{ve}$	.95

$\approx$

S	C	T_2	$\sigma_1(T_2,S,C)$
$male$	yes	ve	.440
$male$	yes	$\overline{ve}$	.110
$male$	no	ve	.110
$male$	no	$\overline{ve}$	.440
$female$	yes	ve	.428
$female$	yes	$\overline{ve}$	.023
$female$	no	ve	.023
$female$	no	$\overline{ve}$	.428

T_1	T_2	A	$\phi_5(T_1,T_2,A)$
ve	ve	yes	1
ve	ve	no	0
ve	$\overline{ve}$	yes	0
ve	$\overline{ve}$	no	1
$\overline{ve}$	ve	yes	0
$\overline{ve}$	ve	no	1
$\overline{ve}$	$\overline{ve}$	yes	1
$\overline{ve}$	$\overline{ve}$	no	0

Computing MPE: Example



- Returning to our example
  - Let us run MPE VE on this example
  - And use the elimination order $\pi = S, C, A, T_1, T_2$

C	T_1	$\phi_2(T_1, C)$
yes	ve	.80
yes	$\overline{ve}$	.20
no	ve	.20
no	$\overline{ve}$	.80

S	C	$\phi_1(S, C)$
$male$	yes	.05
$male$	no	.95
$female$	yes	.01
$female$	no	.99

$\times$

S	C	T_2	$\sigma_1(T_2, S, C)$
$male$	yes	ve	.440
$male$	yes	$\overline{ve}$	.110
$male$	no	ve	.110
$male$	no	$\overline{ve}$	.440
$female$	yes	ve	.428
$female$	yes	$\overline{ve}$	.023
$female$	no	ve	.023
$female$	no	$\overline{ve}$	.428

$\approx$

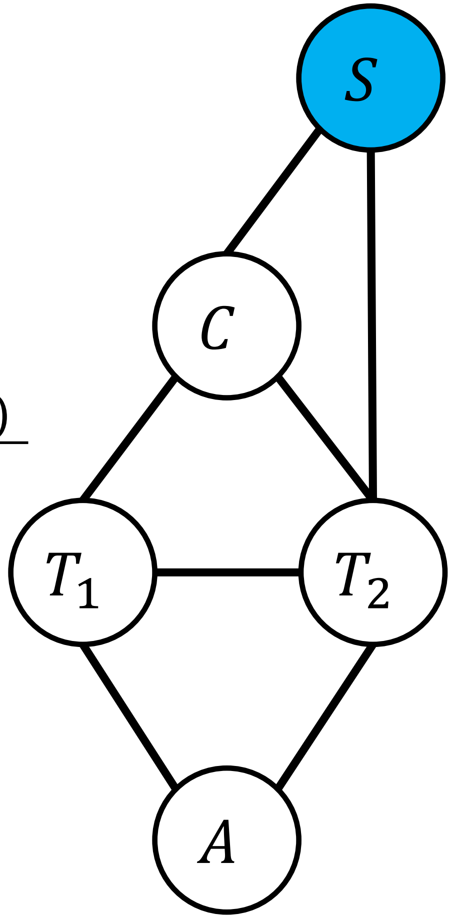
S	C	T_2	$\sigma_2(T_2, S, C)$
$male$	yes	ve	.0220
$male$	yes	$\overline{ve}$	.0055
$male$	no	ve	.1045
$male$	no	$\overline{ve}$	.4180
$female$	yes	ve	.0043
$female$	yes	$\overline{ve}$	.0002
$female$	no	ve	.0228
$female$	no	$\overline{ve}$	.4237

T_1	T_2	A	$\phi_5(T_1, T_2, A)$
ve	ve	yes	1
ve	ve	no	0
ve	$\overline{ve}$	yes	0
ve	$\overline{ve}$	no	1
$\overline{ve}$	ve	yes	0
$\overline{ve}$	ve	no	1
$\overline{ve}$	$\overline{ve}$	yes	1
$\overline{ve}$	$\overline{ve}$	no	0

Computing MPE: Example

- Returning to our example
  - Let us run MPE VE on this example
  - And use the elimination order $\pi = S, C, A, T_1, T_2$

C	T_1	$\phi_2(T_1, C)$
yes	ve	.80
yes	$\overline{ve}$	.20
no	ve	.20
no	$\overline{ve}$	.80



S	C	T_2	$\sigma_2(T_2, S, C)$
$male$	yes	ve	.0220
$male$	yes	$\overline{ve}$	.0055
$male$	no	ve	.1045
$male$	no	$\overline{ve}$	.4180
$female$	yes	ve	.0043
$female$	yes	$\overline{ve}$	.0002
$female$	no	ve	.0228
$female$	no	$\overline{ve}$	.4237

C	T_2	$\tau_2(T_2, C)$
yes	ve	.0220
yes	$\overline{ve}$	.0055
no	ve	.1045
no	$\overline{ve}$	.4237

T_1	T_2	A	$\phi_5(T_1, T_2, A)$
ve	ve	yes	1
ve	ve	no	0
ve	$\overline{ve}$	yes	0
ve	$\overline{ve}$	no	1
$\overline{ve}$	ve	yes	0
$\overline{ve}$	ve	no	1
$\overline{ve}$	$\overline{ve}$	yes	1
$\overline{ve}$	$\overline{ve}$	no	0

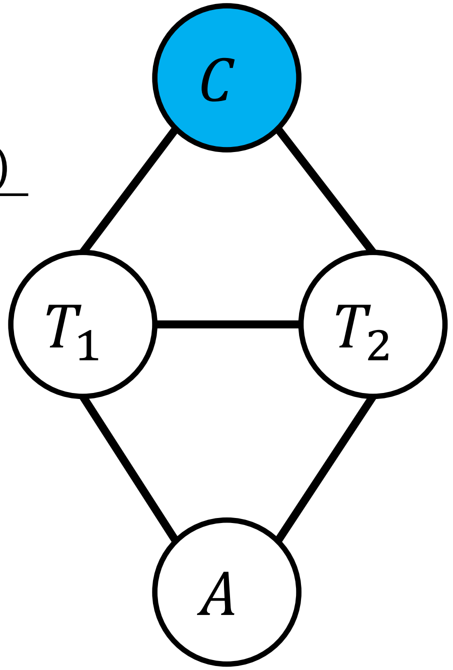
Computing MPE: Example

- Returning to our example
  - Let us run MPE VE on this example
  - And use the elimination order $\pi = S, C, A, T_1, T_2$

C	T_2	$\tau_2(T_2, C)$
yes	ve	.0220
yes	$\overline{ve}$	.0055
no	ve	.1045
no	$\overline{ve}$	.4237

C	T_1	$\phi_2(T_1, C)$
yes	ve	.80
yes	$\overline{ve}$	.20
no	ve	.20
no	$\overline{ve}$	.80

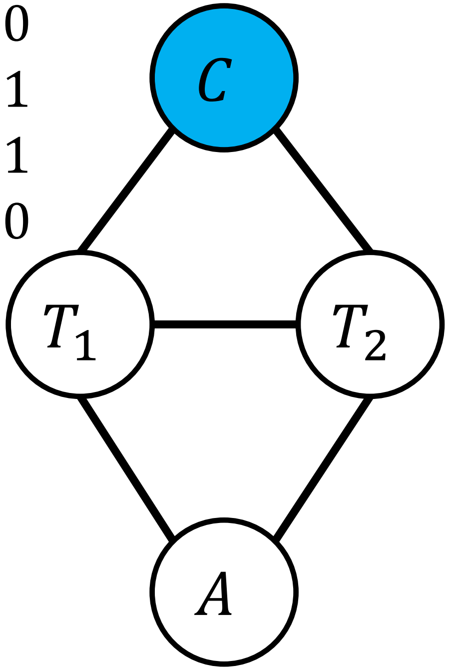
T_1	T_2	A	$\phi_5(T_1, T_2, A)$
ve	ve	yes	1
ve	ve	no	0
ve	$\overline{ve}$	yes	0
ve	$\overline{ve}$	no	1
$\overline{ve}$	ve	yes	0
$\overline{ve}$	ve	no	1
$\overline{ve}$	$\overline{ve}$	yes	1
$\overline{ve}$	$\overline{ve}$	no	0



Computing MPE: Example

- Returning to our example
  - Let us run MPE VE on this example
  - And use the elimination order $\pi = S, C, A, T_1, T_2$

T_1	T_2	A	$\phi_5(T_1, T_2, A)$
ve	ve	yes	1
ve	ve	no	0
ve	$\overline{ve}$	yes	0
ve	$\overline{ve}$	no	1
$\overline{ve}$	ve	yes	0
$\overline{ve}$	ve	no	1
$\overline{ve}$	$\overline{ve}$	yes	1
$\overline{ve}$	$\overline{ve}$	no	0



C	T_2	$\tau_2(T_2, C)$
yes	ve	.0220
yes	$\overline{ve}$	.0055
no	ve	.1045
no	$\overline{ve}$	.4237

$\times$

C	T_1	$\phi_2(T_1, C)$
yes	ve	.80
yes	$\overline{ve}$	.20
no	ve	.20
no	$\overline{ve}$	.80

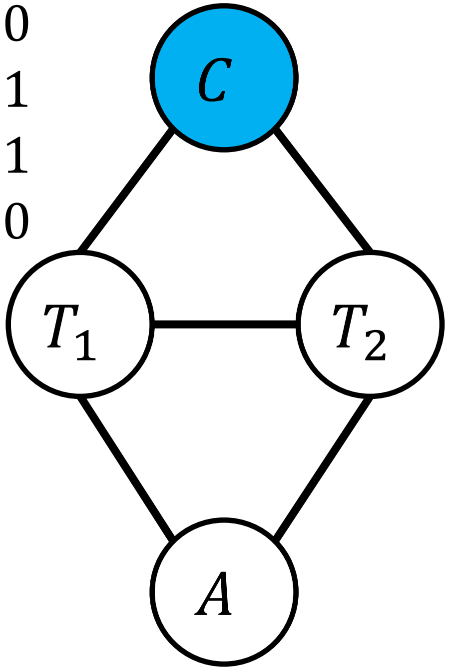
$\approx$

C	T_2	T_1	$\phi_3(C, T_1, T_2)$
yes	ve	ve	.0176
yes	ve	$\overline{ve}$	.0044
yes	$\overline{ve}$	ve	.0044
yes	$\overline{ve}$	$\overline{ve}$	.0011
no	ve	ve	.0209
no	ve	$\overline{ve}$	.0836
no	$\overline{ve}$	ve	.0847
no	$\overline{ve}$	$\overline{ve}$	.3390

Computing MPE: Example

- Returning to our example
  - Let us run MPE VE on this example
  - And use the elimination order $\pi = S, C, A, T_1, T_2$

T_1	T_2	A	$\phi_5(T_1, T_2, A)$
ve	ve	yes	1
ve	ve	no	0
ve	$\overline{ve}$	yes	0
ve	$\overline{ve}$	no	1
$\overline{ve}$	ve	yes	0
$\overline{ve}$	ve	no	1
$\overline{ve}$	$\overline{ve}$	yes	1
$\overline{ve}$	$\overline{ve}$	no	0



C	T_2	T_1	$\sigma_3(C, T_1, T_2)$
yes	ve	ve	.0176
yes	ve	$\overline{ve}$	.0044
yes	$\overline{ve}$	ve	.0044
yes	$\overline{ve}$	$\overline{ve}$	.0011
no	ve	ve	.0209
no	ve	$\overline{ve}$	.0836
no	$\overline{ve}$	ve	.0847
no	$\overline{ve}$	$\overline{ve}$	.3390

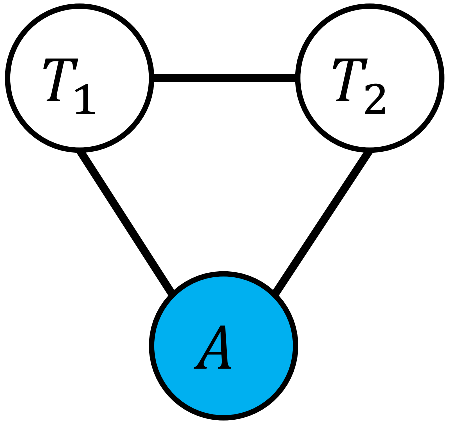
T_2	T_1	$\tau_3(T_2, T_1)$
ve	ve	.0209
ve	$\overline{ve}$	.0836
$\overline{ve}$	ve	.0847
$\overline{ve}$	$\overline{ve}$	.3390

Computing MPE: Example

- Returning to our example
  - Let us run MPE VE on this example
  - And use the elimination order $\pi = S, C, A, T_1, T_2$

T_2	T_1	$\tau_3(T_2, T_1)$
ve	ve	.0209
ve	$\overline{ve}$	.0836
$\overline{ve}$	ve	.0847
$\overline{ve}$	$\overline{ve}$	.3390

T_1	T_2	A	$\phi_5(T_1, T_2, A)$
ve	ve	yes	1
ve	ve	no	0
ve	$\overline{ve}$	yes	0
ve	$\overline{ve}$	no	1
$\overline{ve}$	ve	yes	0
$\overline{ve}$	ve	no	1
$\overline{ve}$	$\overline{ve}$	yes	1
$\overline{ve}$	$\overline{ve}$	no	0



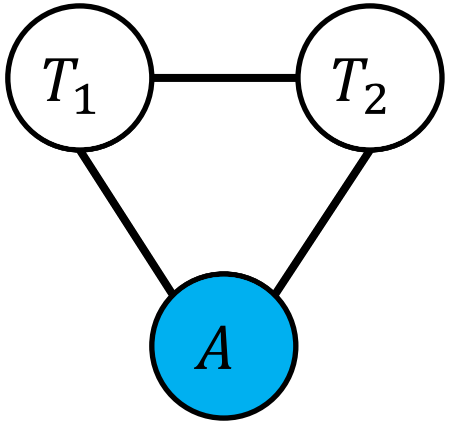
Computing MPE: Example

- Returning to our example
  - Let us run MPE VE on this example
  - And use the elimination order $\pi = S, C, A, T_1, T_2$

T_2	T_1	$\tau_3(T_2, T_1)$
ve	ve	.0209
ve	$\overline{ve}$	.0836
$\overline{ve}$	ve	.0847
$\overline{ve}$	$\overline{ve}$	.3390

T_1	T_2	A	$\phi_5(T_1, T_2, A)$
ve	ve	yes	1
ve	ve	no	0
ve	$\overline{ve}$	yes	0
ve	$\overline{ve}$	no	1
$\overline{ve}$	ve	yes	0
$\overline{ve}$	ve	no	1
$\overline{ve}$	$\overline{ve}$	yes	1
$\overline{ve}$	$\overline{ve}$	no	0

T_1	T_2	$\tau_4(T_1, T_2)$
ve	ve	1
ve	$\overline{ve}$	1
$\overline{ve}$	ve	1
$\overline{ve}$	$\overline{ve}$	1

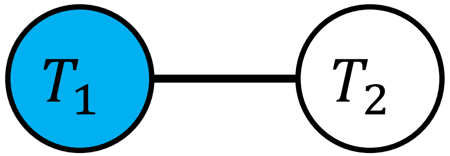


Computing MPE: Example

- Returning to our example
  - Let us run MPE VE on this example
  - And use the elimination order $\pi = S, C, A, T_1, T_2$

T_1	T_2	$\tau_4(T_1, T_2)$
ve	ve	1
ve	$\overline{ve}$	1
$\overline{ve}$	ve	1
$\overline{ve}$	$\overline{ve}$	1

T_2	T_1	$\tau_3(T_2, T_1)$
ve	ve	.0209
ve	$\overline{ve}$	.0836
$\overline{ve}$	ve	.0847
$\overline{ve}$	$\overline{ve}$	.3390



Computing MPE: Example

- Returning to our example
  - Let us run MPE VE on this example
  - And use the elimination order $\pi = S, C, A, T_1, T_2$

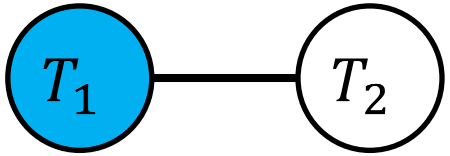
T_1	T_2	$\tau_4(T_1, T_2)$
ve	ve	1
ve	$\overline{ve}$	1
$\overline{ve}$	ve	1
$\overline{ve}$	$\overline{ve}$	1

$\times$

T_2	T_1	$\tau_3(T_2, T_1)$
ve	ve	.0209
ve	$\overline{ve}$	.0836
$\overline{ve}$	ve	.0847
$\overline{ve}$	$\overline{ve}$	.3390

$=$

T_1	T_2	$\sigma_5(T_1, T_2)$
ve	ve	.0209
ve	$\overline{ve}$	.0847
$\overline{ve}$	ve	.0836
$\overline{ve}$	$\overline{ve}$	.3390

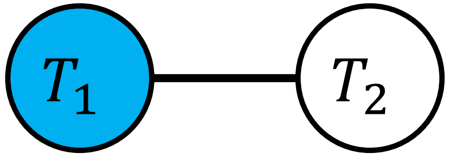


Computing MPE: Example

- Returning to our example
  - Let us run MPE VE on this example
  - And use the elimination order $\pi = S, C, A, T_1, T_2$

T_1	T_2	$\sigma_5(T_1, T_2)$
ve	ve	.0209
ve	$\overline{ve}$	.0847
$\overline{ve}$	ve	.0836
$\overline{ve}$	$\overline{ve}$	.3390

T_2	$\tau_5(T_2)$
ve	.0836
$\overline{ve}$	.3390



Computing MPE: Example

- Returning to our example
  - Let us run MPE VE on this example
  - And use the elimination order $\pi = S, C, A, T_1, T_2$

T_2	$\tau_5(T_2)$
ve	.0836
$\overline{ve}$	.3390

T_2

Computing MPE: Example

- Returning to our example
  - Let us run MPE VE on this example
  - And use the elimination order $\pi = S, C, A, T_1, T_2$
  - $MPE_P \approx 0.3390$
- We can also be interested in the MPE instantiation
  - However, we lost this piece of information during the elimination

Recovering MPE Instantiation

- We can modify the previous algorithm to compute the MPE instantiation
  - In addition to the MPE probability
- The idea is to use *extended factors*
  - It assigns to each instantiation a number and an instantiation
- We use $\phi[x]$ to denote the instantiation
  - While continuing to use $\phi(x)$ for denoting the number
  - The instantiation $\phi[x]$ is used to record the MPE instantiation as it is being constructed

S	C	T_2	$\phi(.)$
$male$	yes	ve	$.0220$
$male$	yes	$\overline{ve}$	$.0055$
$male$	no	ve	$.1045$
$male$	no	$\overline{ve}$	$.4180$
$female$	yes	ve	$.0043$
$female$	yes	$\overline{ve}$	$.0002$
$female$	no	ve	$.0228$
$female$	no	$\overline{ve}$	$.4237$

C	T_2	$\phi(.)$	$\phi[.]$
yes	ve	$.0220$	$male$
yes	$\overline{ve}$	$.0055$	$male$
no	ve	$.1045$	$male$
no	$\overline{ve}$	$.4237$	$female$

Computing MPE Instantiation: Example

- Returning to our example
  - Let us run MPE VE on this example, but now computing the MPE instantiation with evidence $A = yes$

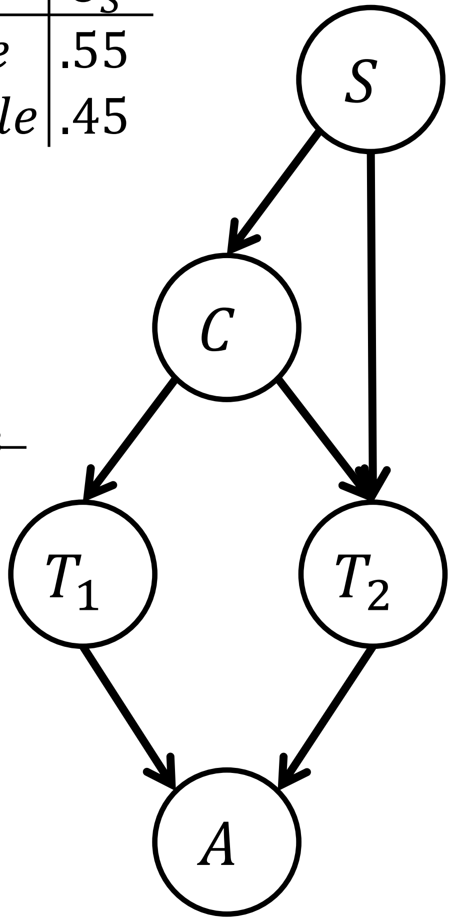
S	C	$\Theta_{C\|S}$
$male$	yes	.05
$male$	no	.95
$female$	yes	.01
$female$	no	.99

C	T_1	$\Theta_{T_1\|C}$
yes	ve	.80
yes	$\overline{ve}$	.20
no	ve	.20
no	$\overline{ve}$	.80

S	Θ_S
$male$	.55
$female$	.45

S	C	T_2	$\Theta_{T_2\|C,S}$
$male$	yes	ve	.80
$male$	yes	$\overline{ve}$	.20
$male$	no	ve	.20
$male$	no	$\overline{ve}$	.80
$female$	yes	ve	.95
$female$	yes	$\overline{ve}$	.05
$female$	no	ve	.05
$female$	no	$\overline{ve}$	.95

T_1	T_2	A	$\Theta_{A\|T_1,T_2}$
ve	ve	yes	1
ve	ve	no	0
ve	$\overline{ve}$	yes	0
ve	$\overline{ve}$	no	1
$\overline{ve}$	ve	yes	0
$\overline{ve}$	ve	no	1
$\overline{ve}$	$\overline{ve}$	yes	1
$\overline{ve}$	$\overline{ve}$	no	0



Computing MPE Instantiation: Example

- Returning to our example
  - Let us run MPE VE on this example, but now computing the MPE instantiation with evidence $A = yes$

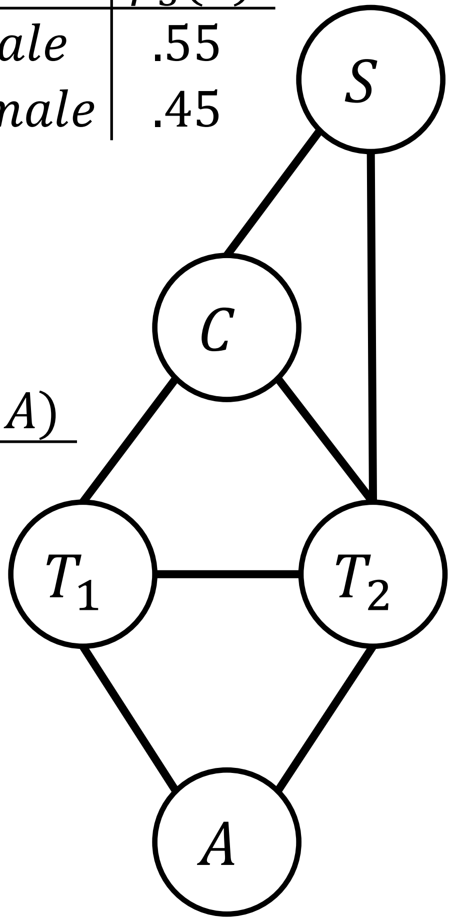
S	C	$\phi_1(S,C)$
$male$	yes	.05
$male$	no	.95
$female$	yes	.01
$female$	no	.99

C	T_1	$\phi_2(T_1,C)$
yes	ve	.80
yes	$\overline{ve}$	.20
no	ve	.20
no	$\overline{ve}$	.80

S	$\phi_3(S)$
$male$	.55
$female$	.45

S	C	T_2	$\phi_4(T_2,C,S)$
$male$	yes	ve	.80
$male$	yes	$\overline{ve}$	.20
$male$	no	ve	.20
$male$	no	$\overline{ve}$	.80
$female$	yes	ve	.95
$female$	yes	$\overline{ve}$	.05
$female$	no	ve	.05
$female$	no	$\overline{ve}$	.95

T_1	T_2	A	$\phi_5(T_1,T_2,A)$
ve	ve	yes	1
ve	ve	no	0
ve	$\overline{ve}$	yes	0
ve	$\overline{ve}$	no	1
$\overline{ve}$	ve	yes	0
$\overline{ve}$	ve	no	1
$\overline{ve}$	$\overline{ve}$	yes	1
$\overline{ve}$	$\overline{ve}$	no	0



Computing MPE Instantiation: Example

- Returning to our example
  - Let us run MPE VE on this example, but now computing the MPE instantiation with evidence $A = yes$

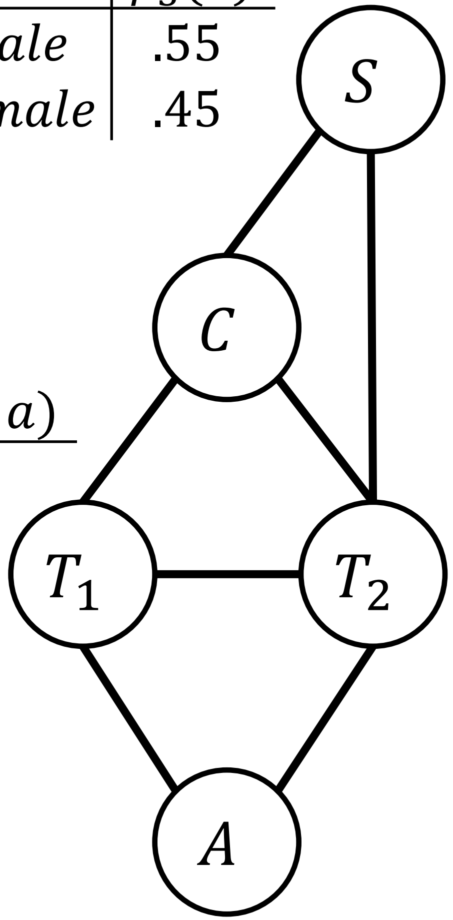
S	C	$\phi_1(S,C)$
$male$	yes	$.05$
$male$	no	$.95$
$female$	yes	$.01$
$female$	no	$.99$

C	T_1	$\phi_2(T_1,C)$
yes	ve	$.80$
yes	$\overline{ve}$	$.20$
no	ve	$.20$
no	$\overline{ve}$	$.80$

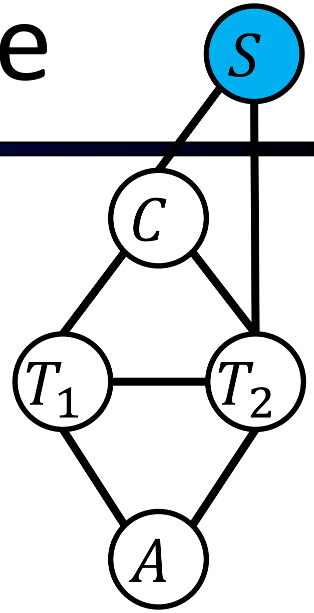
S	$\phi_3(S)$
$male$	$.55$
$female$	$.45$

S	C	T_2	$\phi_4(T_2,C,S)$
$male$	yes	ve	$.80$
$male$	yes	$\overline{ve}$	$.20$
$male$	no	ve	$.20$
$male$	no	$\overline{ve}$	$.80$
$female$	yes	ve	$.95$
$female$	yes	$\overline{ve}$	$.05$
$female$	no	ve	$.05$
$female$	no	$\overline{ve}$	$.95$

T_1	T_2	a	$\phi_5(T_1,T_2,a)$
ve	ve	yes	1
ve	$\overline{ve}$	yes	0
$\overline{ve}$	ve	yes	0
$\overline{ve}$	$\overline{ve}$	yes	1



Computing MPE Instantiation: Example



- Returning to our example
  - Let us run MPE VE on this example, but now computing the MPE instantiation with evidence $A = yes$

S	C	$\phi_1(S,C)$
$male$	yes	.05
$male$	no	.95
$female$	yes	.01
$female$	no	.99

C	T_1	$\phi_2(T_1,C)$
yes	ve	.80
yes	$\overline{ve}$	.20
no	ve	.20
no	$\overline{ve}$	.80

S	$\phi_3(S)$
$male$	.55
$female$	.45

$\times$

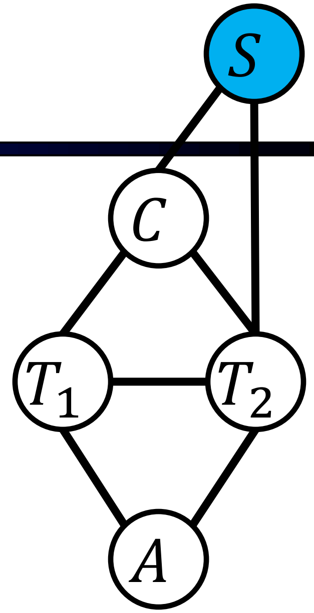
S	C	T_2	$\phi_4(T_2,C,S)$
$male$	yes	ve	.80
$male$	yes	$\overline{ve}$	.20
$male$	no	ve	.20
$male$	no	$\overline{ve}$	.80
$female$	yes	ve	.95
$female$	yes	$\overline{ve}$	.05
$female$	no	ve	.05
$female$	no	$\overline{ve}$	.95

$\approx$

S	C	T_2	$\sigma_1(T_2,S,C)$
$male$	yes	ve	.440
$male$	yes	$\overline{ve}$	.110
$male$	no	ve	.110
$male$	no	$\overline{ve}$	.440
$female$	yes	ve	.428
$female$	yes	$\overline{ve}$	.023
$female$	no	ve	.023
$female$	no	$\overline{ve}$	.428

T_1	T_2	a	$\phi_5(T_1,T_2,a)$
ve	ve	yes	1
ve	$\overline{ve}$	yes	0
$\overline{ve}$	ve	yes	0
$\overline{ve}$	$\overline{ve}$	yes	1

Computing MPE: Example



- Returning to our example
  - Let us run MPE VE on this example, but now computing the MPE instantiation with evidence $A = yes$

C	T_1	$\phi_2(T_1, C)$
yes	ve	.80
yes	$\overline{ve}$	.20
no	ve	.20
no	$\overline{ve}$	.80

S	C	$\phi_1(S, C)$
$male$	yes	.05
$male$	no	.95
$female$	yes	.01
$female$	no	.99

$\times$

S	C	T_2	$\sigma_1(T_2, S, C)$
$male$	yes	ve	.440
$male$	yes	$\overline{ve}$	.110
$male$	no	ve	.110
$male$	no	$\overline{ve}$	.440
$female$	yes	ve	.428
$female$	yes	$\overline{ve}$	.023
$female$	no	ve	.023
$female$	no	$\overline{ve}$	.428

$\approx$

S	C	T_2	$\sigma_2(T_2, S, C)$
$male$	yes	ve	.0220
$male$	yes	$\overline{ve}$	.0055
$male$	no	ve	.1045
$male$	no	$\overline{ve}$	.4180
$female$	yes	ve	.0043
$female$	yes	$\overline{ve}$	.0002
$female$	no	ve	.0228
$female$	no	$\overline{ve}$	.4237

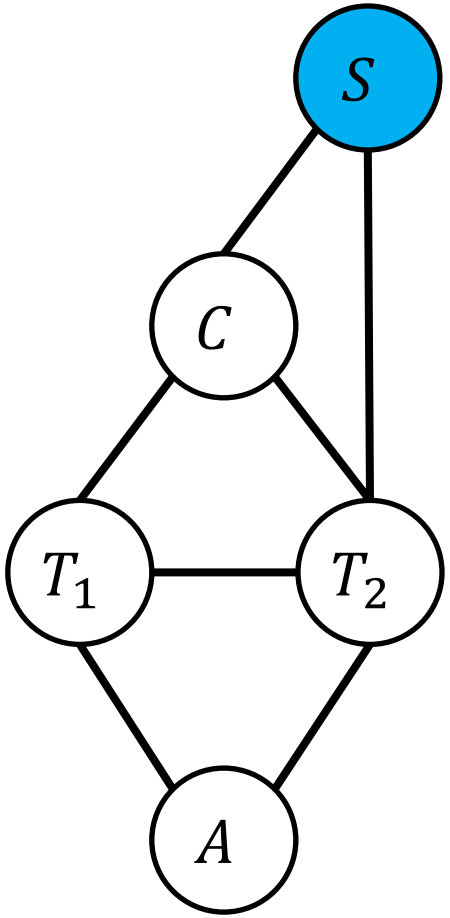
T_1	T_2	a	$\phi_5(T_1, T_2, a)$
ve	ve	yes	1
ve	$\overline{ve}$	yes	0
$\overline{ve}$	ve	yes	0
$\overline{ve}$	$\overline{ve}$	yes	1

Computing MPE: Example

- Returning to our example
  - Let us run MPE VE on this example, but now computing the MPE instantiation with evidence $A = yes$

T_1	T_2	a	$\phi_5(T_1, T_2, a)$
ve	ve	yes	1
ve	$\overline{ve}$	yes	0
$\overline{ve}$	ve	yes	0
$\overline{ve}$	$\overline{ve}$	yes	1

C	T_1	$\phi_2(T_1, C)$
yes	ve	.80
yes	$\overline{ve}$	.20
no	ve	.20
no	$\overline{ve}$	.80



S	C	T_2	$\sigma_2(T_2, S, C)$
$male$	yes	ve	.0220
$male$	yes	$\overline{ve}$	.0055
$male$	no	ve	.1045
$male$	no	$\overline{ve}$	.4180
$female$	yes	ve	.0043
$female$	yes	$\overline{ve}$	.0002
$female$	no	ve	.0228
$female$	no	$\overline{ve}$	.4237

C	T_2	$\tau_2(T_2, C)$	
yes	ve	.0220	$male$
yes	$\overline{ve}$	.0055	$male$
no	ve	.1045	$male$
no	$\overline{ve}$	.4237	$female$

Computing MPE Instantiation: Example

- Returning to our example
  - Let us run MPE VE on this example, but now computing the MPE instantiation with evidence $A = yes$

T_1	T_2	a	$\phi_5(T_1, T_2, a)$
ve	ve	yes	1
ve	$\overline{ve}$	yes	0
$\overline{ve}$	ve	yes	0
$\overline{ve}$	$\overline{ve}$	yes	1

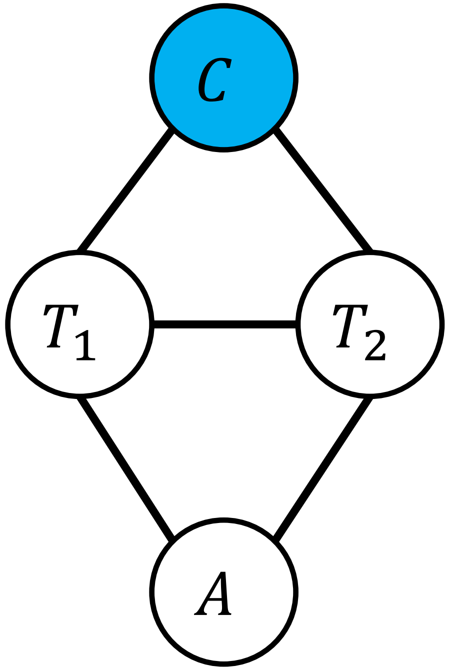
C	T_2	$\tau_2(T_2, C)$	
yes	ve	$.0220$	$male$
yes	$\overline{ve}$	$.0055$	$male$
no	ve	$.1045$	$male$
no	$\overline{ve}$	$.4237$	$female$

$\times$

C	T_1	$\phi_2(T_1, C)$
yes	ve	$.80$
yes	$\overline{ve}$	$.20$
no	ve	$.20$
no	$\overline{ve}$	$.80$

$\approx$

C	T_2	T_1	$\sigma_3(\mathrm{C}, T_1, T_2)$	
yes	ve	ve	$.0176$	$male$
yes	ve	$\overline{ve}$	$.0044$	$male$
yes	$\overline{ve}$	ve	$.0044$	$male$
yes	$\overline{ve}$	$\overline{ve}$	$.0011$	$male$
no	ve	ve	$.0209$	$male$
no	ve	$\overline{ve}$	$.0836$	$male$
no	$\overline{ve}$	ve	$.0847$	$female$
no	$\overline{ve}$	$\overline{ve}$	$.3390$	$female$



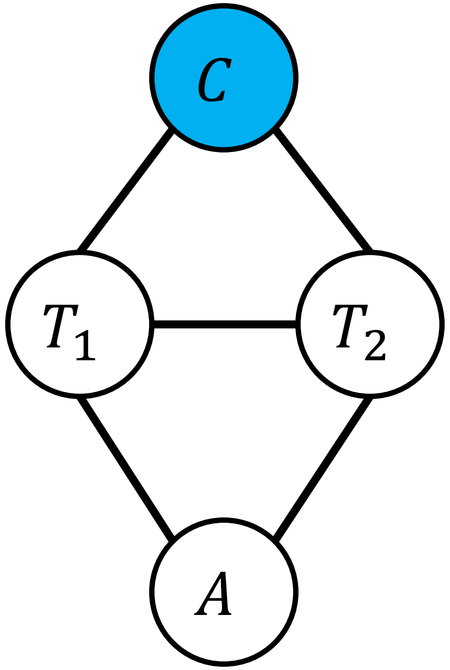
Computing MPE Instantiation: Example

- Returning to our example
  - Let us run MPE VE on this example, but now computing the MPE instantiation with evidence $A = yes$

T_1	T_2	a	$\phi_5(T_1, T_2, a)$
ve	ve	yes	1
ve	$\overline{ve}$	yes	0
$\overline{ve}$	ve	yes	0
$\overline{ve}$	$\overline{ve}$	yes	1

C	T_2	T_1	$\sigma_3(C, T_1, T_2)$	
yes	ve	ve	.0176	$male$
yes	ve	$\overline{ve}$	.0044	$male$
yes	$\overline{ve}$	ve	.0044	$male$
yes	$\overline{ve}$	$\overline{ve}$	.0011	$male$
no	ve	ve	.0209	$male$
no	ve	$\overline{ve}$	.0836	$male$
no	$\overline{ve}$	ve	.0847	$female$
no	$\overline{ve}$	$\overline{ve}$	.3390	$female$

T_2	T_1	$\tau_3(T_2, T_1)$	
ve	ve	.0209	$male, no$
ve	$\overline{ve}$	.0836	$male, no$
$\overline{ve}$	ve	.0847	$female, no$
$\overline{ve}$	$\overline{ve}$	.3390	$female, no$



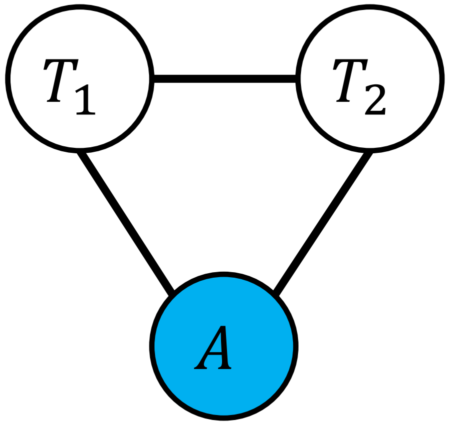
Computing MPE Instantiation: Example

- Returning to our example
  - Let us run MPE VE on this example, but now computing the MPE instantiation with evidence $A = yes$

T_2	T_1	$\tau_3(T_2, T_1)$	
ve	ve	$.0209$	$male, no$
ve	$\overline{ve}$	$.0836$	$male, no$
$\overline{ve}$	ve	$.0847$	$female, no$
$\overline{ve}$	$\overline{ve}$	$.3390$	$female, no$

T_1	T_2	a	$\phi_5(T_1, T_2, a)$
ve	ve	yes	1
ve	$\overline{ve}$	yes	0
$\overline{ve}$	ve	yes	0
$\overline{ve}$	$\overline{ve}$	yes	1

T_1	T_2	$\tau_4(T_1, T_2)$
ve	ve	1
ve	$\overline{ve}$	0
$\overline{ve}$	ve	0
$\overline{ve}$	$\overline{ve}$	1



Computing MPE Instantiation: Example

- Returning to our example
  - Let us run MPE VE on this example, but now computing the MPE instantiation with evidence $A = yes$

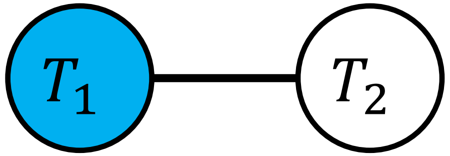
T_2	T_1	$\tau_3(T_2,T_1)$	
ve	ve	$.0209$	$male, no$
ve	$\overline{ve}$	$.0836$	$male, no$
$\overline{ve}$	ve	$.0847$	$female, no$
$\overline{ve}$	$\overline{ve}$	$.3390$	$female, no$

$\times$

T_1	T_2	$\tau_4(T_1,T_2)$
ve	ve	1
ve	$\overline{ve}$	0
$\overline{ve}$	ve	0
$\overline{ve}$	$\overline{ve}$	1

$\approx$

T_1	T_2	$\sigma_5(T_1,T_2)$	
ve	ve	$.0209$	$male, no$
ve	$\overline{ve}$	0	$female, no$
$\overline{ve}$	ve	0	$male, no$
$\overline{ve}$	$\overline{ve}$	$.3390$	$female, no$

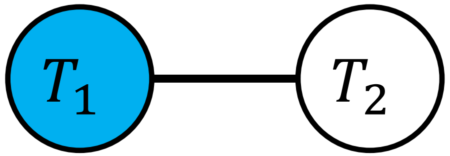


Computing MPE Instantiation: Example

- Returning to our example
  - Let us run MPE VE on this example, but now computing the MPE instantiation with evidence $A = yes$

T_1	T_2	$\sigma_5(T_1, T_2)$	
ve	ve	$.0209$	$male, no$
ve	$\overline{ve}$	0	$female, no$
$\overline{ve}$	ve	0	$male, no$
$\overline{ve}$	$\overline{ve}$	$.3390$	$female, no$

T_2	$\tau_5(T_2)$	
ve	$.0290$	$male, no, ve$
$\overline{ve}$	$.3390$	$female, no, \overline{ve}$



Computing MPE Instantiation: Example

- Returning to our example
  - Let us run MPE VE on this example, but now computing the MPE instantiation with evidence $A = yes$

T_2	$\tau_5(T_2)$	
ve	$.0290$	$male, no, ve$
$\overline{ve}$	$.3390$	$female, no, \overline{ve}$

τ_6	
.3390	$female, no, \overline{ve}, \overline{ve}$

T_2

Computing MPE Instantiation: Example

- Returning to our example
  - Let us run MPE VE on this example, but now computing the MPE instantiation with evidence $A = yes$
- Since $P(\mathbf{e}) = P(A = yes) = .7205$
  - $MPE_p(\mathbf{Q}|\mathbf{e}) = 47\%$

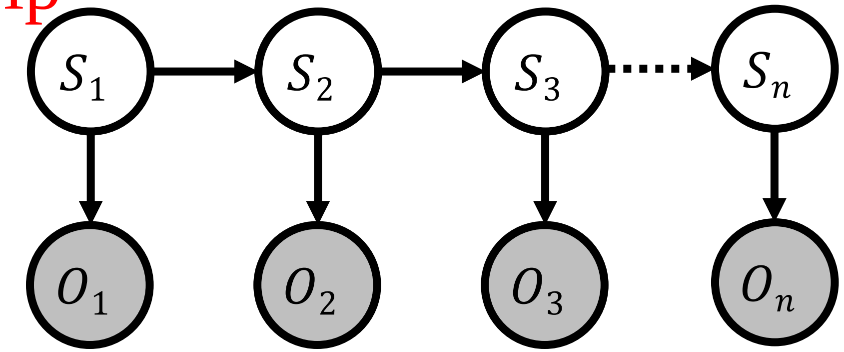
T_2	$\tau_5(T_2)$	
ve	.0290	$male, no, ve$
$\overline{ve}$	.3390	$female, no, \overline{ve}$

τ_6	
.3390	$female, no, \overline{ve}, \overline{ve}$

T_2

MPE and HMM

- We have seen MPE queries in the context of HMM
  - If we apply the MPE algorithm with elimination order $\pi = O_1, S_1, O_2, S_2, \ldots, O_n, S_n$, we obtain the Viterbi algorithm
  - If we apply the VE algorithm with same order, we obtain the Forward algorithm
- This elimination order has width = 1 for HMMs
  - Therefore both algorithms have linear time and space complexity



Computing MAP

- Given a network
  - The *MAP probability* for the variables $\boldsymbol{M}$ given evidence $\boldsymbol{e}$ is defined as

$$MAP_P(\boldsymbol{M}, \boldsymbol{e}) \stackrel{\text{def}}{=} \max_{m} P(\boldsymbol{m}, \boldsymbol{e})$$

- There may be several instantiations $\boldsymbol{m}$ with maximal probability

  - Each of them is a MAP instantiation
  - The set of such instantiations is defined as

$$MAP(\boldsymbol{M}, \boldsymbol{e}) \stackrel{\text{def}}{=} argmax_{m}\, P(\boldsymbol{m}, \boldsymbol{e})$$

- MPE instantiations can be characterized as instantiations m that maximize the posterior distribution $P(m|e)$

$$MAP(\boldsymbol{M}, \boldsymbol{e}) \stackrel{\text{def}}{=} argmax_{q}\, P(\boldsymbol{q}|\boldsymbol{e})$$

  - Since $P(\boldsymbol{m}|\boldsymbol{e}) = \frac{P(\boldsymbol{m},\boldsymbol{e})}{P(\boldsymbol{e})}$
  - $P(\boldsymbol{e})$ is independent of the instantiation $\boldsymbol{m}$

Computing MAP by Variable Elimination

- We can compute the MAP probability $MAP_P(\boldsymbol{M}, \boldsymbol{e})$ using the VE algorithm
  - First, summing out all non-MAP variables: computes the marginal $P(\boldsymbol{M}, \boldsymbol{e})$ in factored form
  - Second, maximizing out MAP variables $\boldsymbol{M}$: solve MPE problem over the resulting marginal

- The resulting algorithm can be thought of a combination of MPE and VE algorithms

- We can use extended factors just as when computing an MPE instantiation

MAP VE: Algorithm

$\boldsymbol{Q} \leftarrow$ variables in the network

$\pi \leftarrow$ elimination order of variables $\boldsymbol{Q}$ in which variables $\boldsymbol{M}$ appear last

$\boldsymbol{S} \leftarrow \{\phi^{\boldsymbol{e}} : \phi$ is a factor of the network$\}$

for $i = 1$ **to** length of order π **do**

$\sigma_i \leftarrow \prod_k \phi_k$, where ϕ_k belongs to $\boldsymbol{S}$ and mentions variable $\pi(i)$

if $\pi(i) \in \boldsymbol{M}$ **then**

$\tau_i \leftarrow \max_{\pi(i)} \sigma_i$

else

$\tau_i \leftarrow \sum_{\pi(i)} \sigma_i$

replace all factors ϕ_k in $\boldsymbol{S}$ by factor τ_i

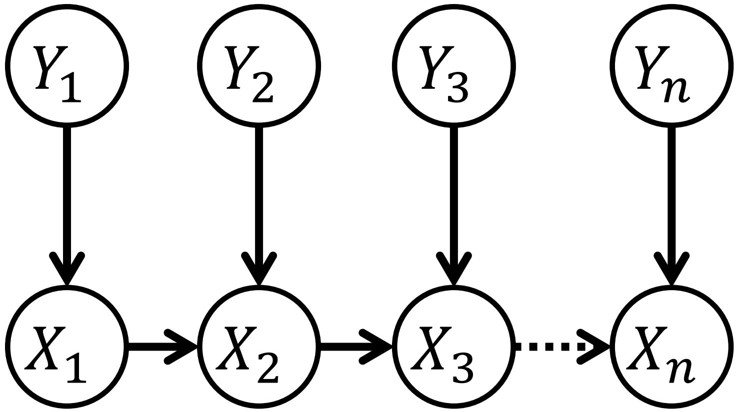
return trivial factor $\prod_{\tau \in S} \tau$

Notes:

- If the network is a Bayesian network, you can prune nodes and edges
- The elimination is special in the sense the MAP variables appear last
- The algorithm perform both types of elimination: maximizing-out MAP variables and summing-out non-MAP variables

MAP VE Complexity

- Given n variables and an elimination order of width w
  - The time and space complexity of MAP is $O(n\exp(w))$
  - Like MPE VE algorithm
- However, MAP variable order is constrained
  - It requires MAP variables to be last in the order
  - This means that we may not be able to use a good ordering because low-width orders do not satisfy this constraint
- For example, the polytree structure on the right
  - It has treewidth of 2 since it has at most two parents per node
  - If we want to compute MAP for variables $\boldsymbol{M} = \{Y_1, \ldots, Y_n\}$, any order that $\boldsymbol{M}$ comes last has $width \geq n$
  - Therefore, MPE VE is linear, but MAP VE is exponential in this case



MAP VE Complexity

- In general, we cannot use arbitrary elimination orders
  - We cannot interleave variables that are summing out with those maximizing out
  - Maximization does not commute with summation

$$\left[\sum_X \max_Y f\right](\boldsymbol{z}) \geq \left[\max_Y \sum_X f\right](\boldsymbol{z})$$

for all instantiations $\boldsymbol{z}$

- The complexity of MAP VE is at best exponential in the constrained treewidth
  - A variable order π is **M**-constrained iff variables **M** appear last in the order π.
  - The ***M**-constrained treewidth* of a graph is the width of its best **M**-constrained variable order

- Computing MAP is therefore more difficult than computing MPE in the context of VE

Computing MAP Instantiation: Example

- Let us run MAP VE with evidence $A = yes$ and MAP variables S and C, and elimination order $\pi = A, T_1, T_2, S, C$

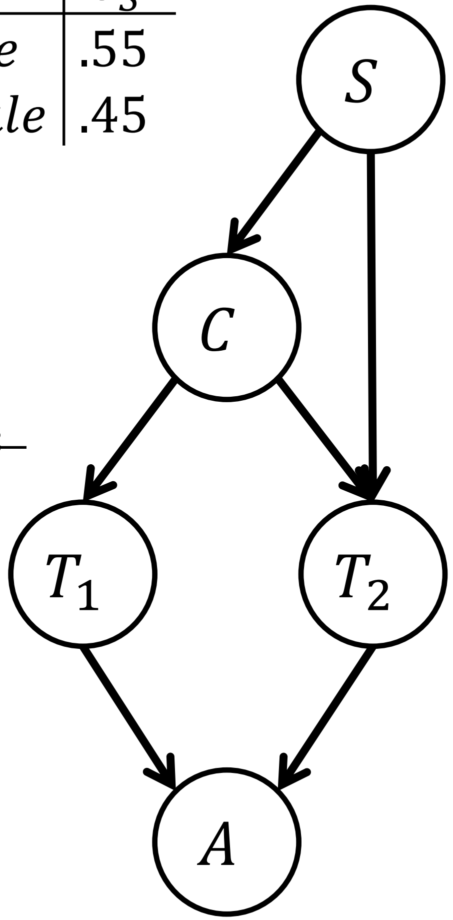
S	C	$\Theta_{C\|S}$
$male$	yes	.05
$male$	no	.95
$female$	yes	.01
$female$	no	.99

C	T_1	$\Theta_{T_1\|C}$
yes	ve	.80
yes	$\overline{ve}$	.20
no	ve	.20
no	$\overline{ve}$	.80

S	Θ_S
$male$	.55
$female$	.45

S	C	T_2	$\Theta_{T_2\|C,S}$
$male$	yes	ve	.80
$male$	yes	$\overline{ve}$	.20
$male$	no	ve	.20
$male$	no	$\overline{ve}$	.80
$female$	yes	ve	.95
$female$	yes	$\overline{ve}$	.05
$female$	no	ve	.05
$female$	no	$\overline{ve}$	.95

T_1	T_2	A	$\Theta_{A\|T_1,T_2}$
ve	ve	yes	1
ve	ve	no	0
ve	$\overline{ve}$	yes	0
ve	$\overline{ve}$	no	1
$\overline{ve}$	ve	yes	0
$\overline{ve}$	ve	no	1
$\overline{ve}$	$\overline{ve}$	yes	1
$\overline{ve}$	$\overline{ve}$	no	0



Computing MAP Instantiation: Example

- Let us run MAP VE with evidence $A = yes$ and MAP variables S and C, and elimination order $\pi = A, T_1, T_2, S, C$

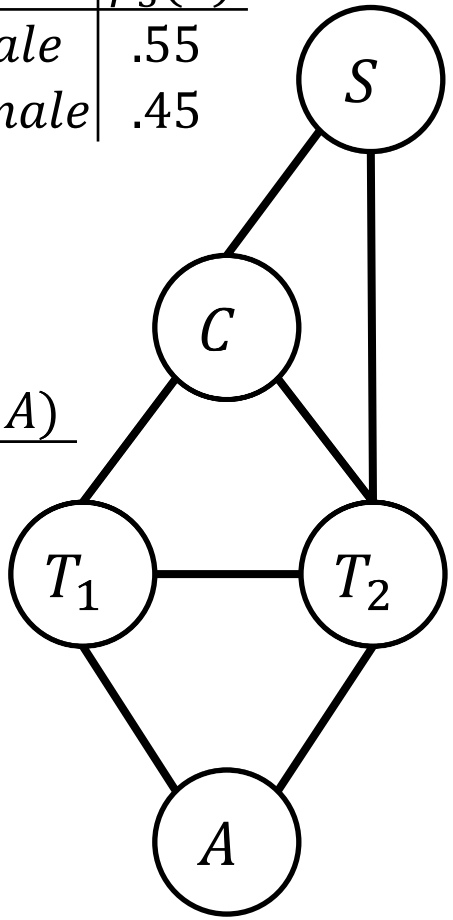
S	C	$\phi_1(S,C)$
$male$	yes	$.05$
$male$	no	$.95$
$female$	yes	$.01$
$female$	no	$.99$

C	T_1	$\phi_2(T_1,C)$
yes	ve	$.80$
yes	$\overline{ve}$	$.20$
no	ve	$.20$
no	$\overline{ve}$	$.80$

S	$\phi_3(S)$
$male$	$.55$
$female$	$.45$

S	C	T_2	$\phi_4(T_2,C,S)$
$male$	yes	ve	$.80$
$male$	yes	$\overline{ve}$	$.20$
$male$	no	ve	$.20$
$male$	no	$\overline{ve}$	$.80$
$female$	yes	ve	$.95$
$female$	yes	$\overline{ve}$	$.05$
$female$	no	ve	$.05$
$female$	no	$\overline{ve}$	$.95$

T_1	T_2	A	$\phi_5(T_1,T_2,A)$
ve	ve	yes	1
ve	ve	no	0
ve	$\overline{ve}$	yes	0
ve	$\overline{ve}$	no	1
$\overline{ve}$	ve	yes	0
$\overline{ve}$	ve	no	1
$\overline{ve}$	$\overline{ve}$	yes	1
$\overline{ve}$	$\overline{ve}$	no	0



Computing MAP Instantiation: Example

- Let us run MAP VE with evidence $A = yes$ and MAP variables S and C, and elimination order $\pi = A, T_1, T_2, S, C$

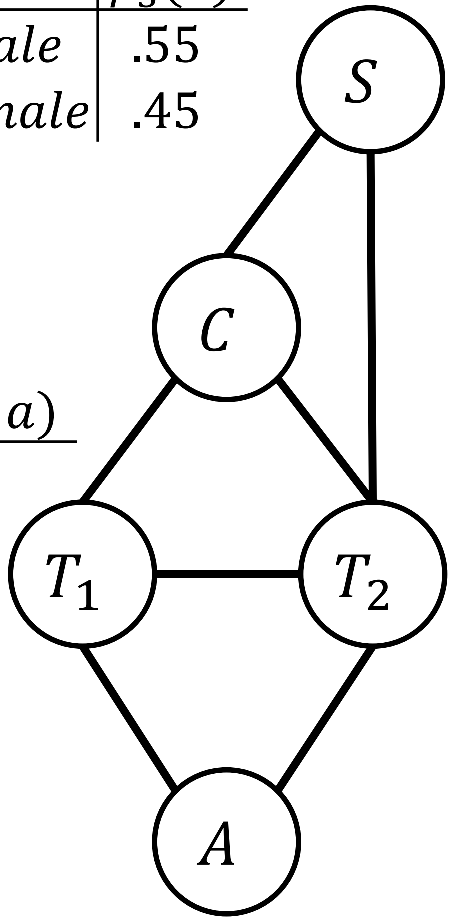
S	C	$\phi_1(S, C)$
$male$	yes	.05
$male$	no	.95
$female$	yes	.01
$female$	no	.99

C	T_1	$\phi_2(T_1, C)$
yes	ve	.80
yes	$\overline{ve}$	.20
no	ve	.20
no	$\overline{ve}$	.80

S	$\phi_3(S)$
$male$	.55
$female$	.45

S	C	T_2	$\phi_4(T_2, C, S)$
$male$	yes	ve	.80
$male$	yes	$\overline{ve}$	.20
$male$	no	ve	.20
$male$	no	$\overline{ve}$	.80
$female$	yes	ve	.95
$female$	yes	$\overline{ve}$	.05
$female$	no	ve	.05
$female$	no	$\overline{ve}$	.95

T_1	T_2	a	$\phi_5(T_1, T_2, a)$
ve	ve	yes	1
ve	$\overline{ve}$	yes	0
$\overline{ve}$	ve	yes	0
$\overline{ve}$	$\overline{ve}$	yes	1

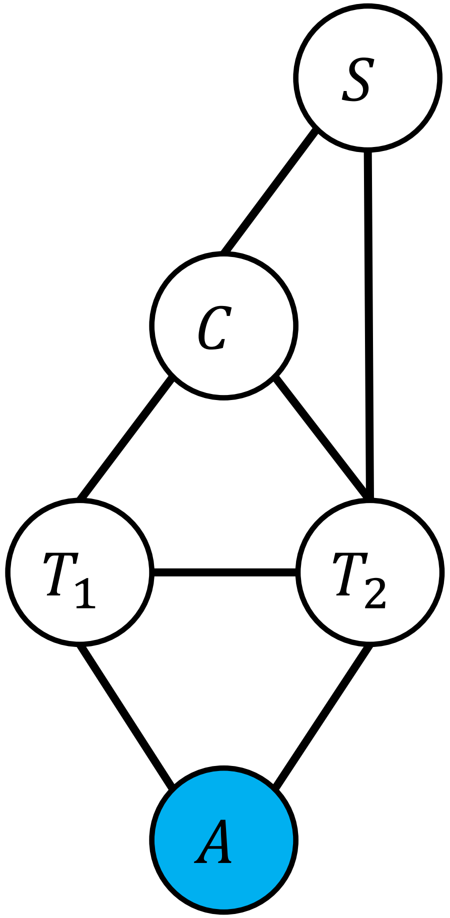


Computing MAP Instantiation: Example

- Let us run MAP VE with evidence $A = yes$ and MAP variables S and C, and elimination order $\pi = A, T_1, T_2, S, C$

T_1	T_2	a	$\phi_5(T_1, T_2, a)$
ve	ve	yes	1
ve	$\overline{ve}$	yes	0
$\overline{ve}$	ve	yes	0
$\overline{ve}$	$\overline{ve}$	yes	1

T_1	T_2	$\tau_1(T_1, T_2)$
ve	ve	1
ve	$\overline{ve}$	0
$\overline{ve}$	ve	0
$\overline{ve}$	$\overline{ve}$	1



Computing MAP Instantiation: Example

- Let us run MAP VE with evidence $A = yes$ and MAP variables S and C, and elimination order $\pi = A, T_1, T_2, S, C$

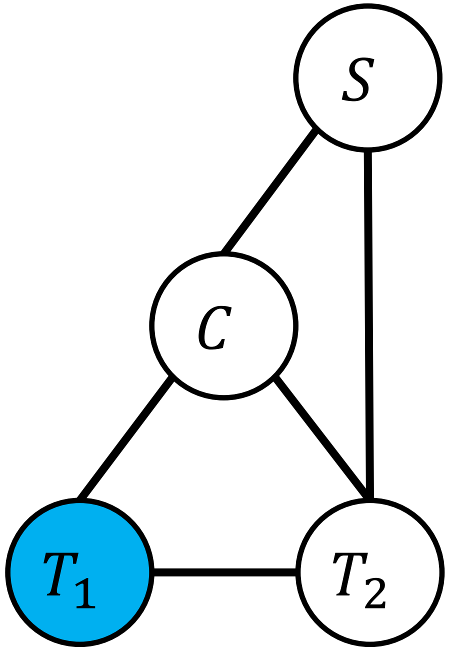
T_1	T_2	$\tau_1(T_1, T_2)$
ve	ve	1
ve	$\overline{ve}$	0
$\overline{ve}$	ve	0
$\overline{ve}$	$\overline{ve}$	1

$\times$

C	T_1	$\phi_2(T_1, C)$
yes	ve	.80
yes	$\overline{ve}$	.20
no	ve	.20
no	$\overline{ve}$	.80

$=$

C	T_1	T_2	$\sigma_2(T_1, T_2, C)$
yes	ve	ve	.80
yes	ve	$\overline{ve}$	0
yes	$\overline{ve}$	ve	0
yes	$\overline{ve}$	$\overline{ve}$	.20
no	ve	ve	.20
no	ve	$\overline{ve}$	0
no	$\overline{ve}$	ve	0
no	$\overline{ve}$	$\overline{ve}$	.80

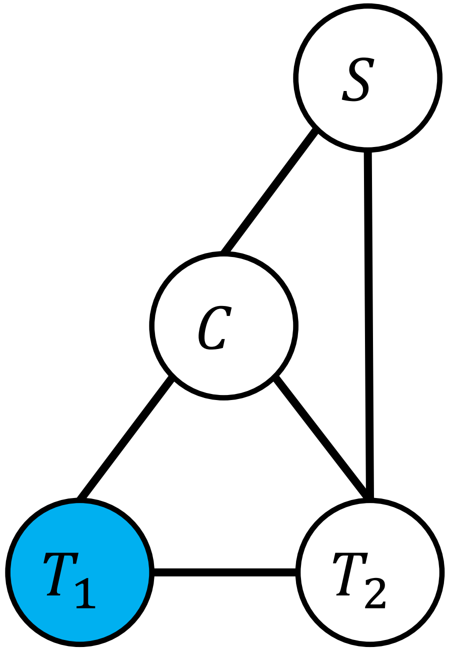


Computing MAP Instantiation: Example

- Let us run MAP VE with evidence $A = yes$ and MAP variables S and C, and elimination order $\pi = A, T_1, T_2, S, C$

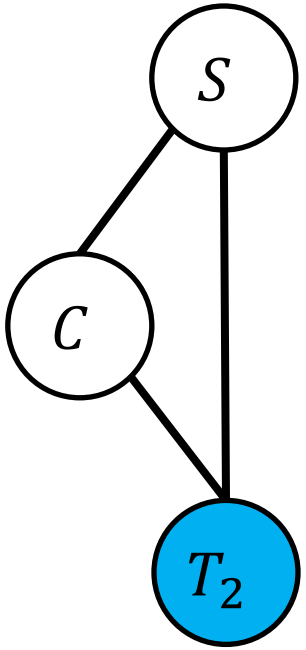
C	T_1	T_2	$\sigma_2(T_1, T_2, C)$
yes	ve	ve	.80
yes	ve	$\overline{ve}$	0
yes	$\overline{ve}$	ve	0
yes	$\overline{ve}$	$\overline{ve}$	.20
no	ve	ve	.20
no	ve	$\overline{ve}$	0
no	$\overline{ve}$	ve	0
no	$\overline{ve}$	$\overline{ve}$	.80

C	T_2	$\tau_2(T_2, C)$
yes	ve	.80
yes	$\overline{ve}$	.20
no	ve	.20
no	$\overline{ve}$	.80



Computing MAP Instantiation: Example

- Let us run MAP VE with evidence $A = yes$ and MAP variables S and C, and elimination order $\pi = A, T_1, T_2, S, C$



C	T_2	$\tau_2(T_2, C)$
yes	ve	.80
yes	$\overline{ve}$	.20
no	ve	.20
no	$\overline{ve}$	.80

$\times$

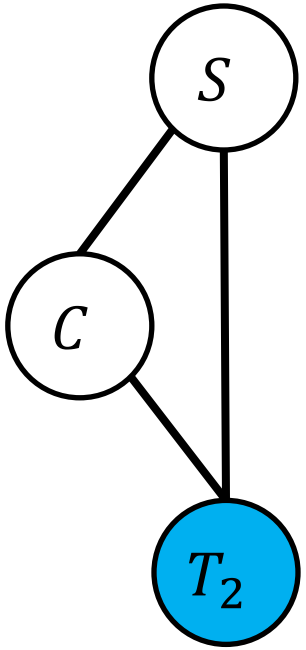
S	C	T_2	$\phi_4(T_2, C, S)$
$male$	yes	ve	.80
$male$	yes	$\overline{ve}$	.20
$male$	no	ve	.20
$male$	no	$\overline{ve}$	.80
$female$	yes	ve	.95
$female$	yes	$\overline{ve}$	.05
$female$	no	ve	.05
$female$	no	$\overline{ve}$	.95

$=$

S	C	T_2	$\sigma_3(T_2, S, C)$
$male$	yes	ve	.64
$male$	yes	$\overline{ve}$	.04
$male$	no	ve	.04
$male$	no	$\overline{ve}$	.64
$female$	yes	ve	.76
$female$	yes	$\overline{ve}$	.01
$female$	no	ve	.01
$female$	no	$\overline{ve}$	.76

Computing MAP Instantiation: Example

- Let us run MAP VE with evidence $A = yes$ and MAP variables S and C, and elimination order $\pi = A, T_1, T_2, S, C$

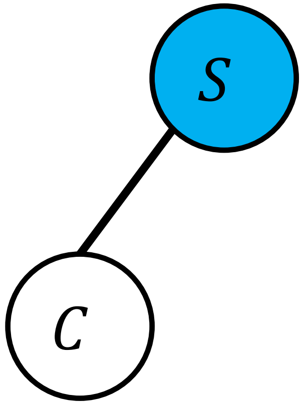


S	C	T_2	$\sigma_3(T_2, S, C)$
$male$	yes	ve	.64
$male$	yes	$\overline{ve}$	.04
$male$	no	ve	.04
$male$	no	$\overline{ve}$	.64
$female$	yes	ve	.76
$female$	yes	$\overline{ve}$	.01
$female$	no	ve	.01
$female$	no	$\overline{ve}$	.76

S	C	$\tau_3(C, S)$
$male$	yes	.68
$male$	no	.68
$female$	yes	.77
$female$	no	.77

Computing MAP Instantiation: Example

- Let us run MAP VE with evidence $A = yes$ and MAP variables S and C, and elimination order $\pi = A, T_1, T_2, S, C$



S	C	$\tau_3(C,S)$
$male$	yes	.68
$male$	no	.68
$female$	yes	.77
$female$	no	.77

$\times$

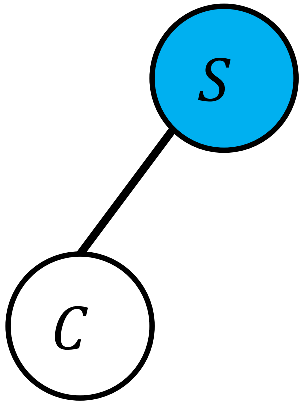
S	$\phi_3(S)$
$male$	.55
$female$	.45

$\approx$

S	C	$\sigma_4(C,S)$
$male$	yes	.37
$male$	no	.37
$female$	yes	.35
$female$	no	.35

Computing MAP Instantiation: Example

- Let us run MAP VE with evidence $A = yes$ and MAP variables S and C, and elimination order $\pi = A, T_1, T_2, S, C$



S	C	$\sigma_4(C,S)$
$male$	yes	.37
$male$	no	.37
$female$	yes	.35
$female$	no	.35

$\times$

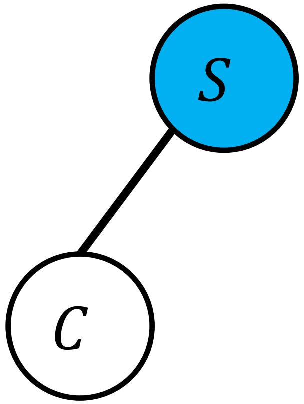
S	C	$\phi_1(S,C)$
$male$	yes	.05
$male$	no	.95
$female$	yes	.01
$female$	no	.99

$\approx$

S	C	$\sigma_5(C,S)$
$male$	yes	.018
$male$	no	.351
$female$	yes	.004
$female$	no	.347

Computing MAP Instantiation: Example

- Let us run MAP VE with evidence $A = yes$ and MAP variables S and C, and elimination order $\pi = A, T_1, T_2, S, C$

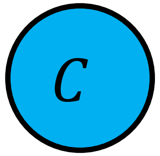


S	C	$\sigma_5(C,S)$
$male$	yes	$.018$
$male$	no	$.351$
$female$	yes	$.004$
$female$	no	$.347$

C	$\tau_5(C)$	
yes	$.018$	$male$
no	$.351$	$male$

Computing MAP Instantiation: Example

- Let us run MAP VE with evidence $A = yes$ and MAP variables S and C, and elimination order $\pi = A, T_1, T_2, S, C$
- Since $P(\boldsymbol{e}) = P(A = yes) = .7205$
  - $MAP_p(S, C|\boldsymbol{e}) \approx 49\%$



C	$\tau_5(C)$	
yes	$.018$	$male$
no	$.351$	$male$

τ_6	
$.351$	$male, no$

Conclusion

- MPE and MAP queries can be answered with simple adaptations of the VE algorithm
  - We introduced an elimination operation with max
- In MPE, we operate only with multiplication and maximization operations
  - We can easily replace probabilities with log-probabilities
  - This creates numerically stable algorithms, since multiplying several small numbers can lead to underflows
- Task
  - Read chapter 10 (except Sections 2.2, 2.3, 3.2 and 3.3) of the textbook