

It is not a priori quite clear to the author why exactly the standard framework of measure-theoretic probability (and financial calculus, in particular) is defined as such. This article aims to derive it from common sense and not pull any insights from higher intelligence.

## 1. Available information

A fundamental concept underlying financial mathematics is the mathematical model of *available information* — the way we encode what we know and what we don't know about a real-world event or process.

### 1.1. Setup

Imagine a probability experiment that can be “run” and on which “observations” can be made.

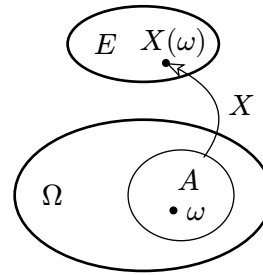
1. A single run fixes (determines) *everything* that can be observed about the experiment.
2. A *run* is commonly denoted by  $\omega$ , and the set of *all* runs is commonly denoted by  $\Omega$ .
3. Probabilities are assigned to sensible subsets of  $\Omega$ , forming a  $\sigma$ -algebra  $\mathcal{F} \subset \mathcal{P}(\Omega)$  and a probability measure  $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ .
4. In practice, we do not know exactly which run  $\omega \in \Omega$  occurred. Instead,
  - a. we can only know if the run is *among whole sets*  $A$  of runs;
  - b. we can usually only “look” at the experiment, and *observe* values of projections  $X : \Omega \rightarrow E$  onto other sets  $E$ .

#### 1.1.1. Events

Such a set of runs  $A$ , that we can know if  $\omega$  lies in, is called *an event* and is only allowed to be in  $\mathcal{F}$ . Thus it is in the domain of  $\mathbb{P}$  and has a well-defined *probability*  $\mathbb{P}(A)$ . The model does not really talk about whether a run is in a set not in  $\mathcal{F}$ .

#### 1.1.2. Random variables

If we observe a particular value  $e = X(\omega)$  through a projection  $X : \Omega \rightarrow E$ , then we know that the run that happened lies in  $X^{-1}(\{e\}) \subset \Omega$ . But we're only allowed to know if  $\omega$  is in  $\mathcal{F}$ -sets, so the only (sets of) values  $A \subset E$  (including the case  $A = \{e\}$ ) we are allowed to observe from  $X$  must have measurable preimages  $X^{-1}(A) \in \mathcal{F}$ .



When we observe  $e = X(\omega) \in E$ , we know that  $A = X^{-1}(e)$  happened, but not which  $\omega$  exactly.

Thus to model observables we use *measurable* functions  $X : \Omega \rightarrow E$  into measure spaces  $E$ , and we call  $X$  a *random variable*.

In this sense,  $\mathcal{F}$  is a restriction on the nature of all random variables, containing the information *that is allowed to be known* and revealed by any observables.

## 2. Experiments that happen in time

Imagine we're observing an experiment that is laid out as a process over time  $\mathbb{N}$ .

### 2.1. Atomicity of history

A single run  $\omega$  of the experiment contains the whole history of the process. That is, from a mathematical point of view, the future is predetermined.

As time evolves in the real world, though, we can only observe what is *currently* happening, and not the whole history, even though in theory it already “happened”. Many runs of the

experiment may have the same observable properties up to a particular time  $n \in \mathbb{N}$ , thus we only know that the run  $\omega$  we're observing lies in a (probably large) measurable subset of  $\Omega$  containing runs of the same observed history up to now but different futures.

For any  $n$ , let  $\mathcal{F}_n \subset \mathcal{F}$  contain all maximal sets of events that share a common observable property up to time  $n$  or earlier. That is,  $\mathcal{F}_n$  contains only, and all, sets of runs that a particular run can be determined to lie in, by observing it up to time  $n$ . Then trivially  $\mathcal{F}_n \subset \mathcal{F}_m$  for all  $n \leq m$ .

If two runs  $\omega$  and  $\omega'$  differ in no observable way up to time  $n$ , they are not  $\mathcal{F}_n$ -separable<sup>1</sup>.

Thus in order to model slowly-appearing history over  $\Omega$  (recall,  $\Omega$  is a set of atoms  $\omega$  containing entire individual histories) we introduce a nested sequence of  $\sigma$ -algebras in  $\mathcal{F}$

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_n \subset \dots$$

called a *filtration* over  $\Omega$ . It is a way to *lay out in time* the otherwise indivisible experiment runs  $\omega$  without actually chopping them in time pieces. So at each step  $n$  we're allowed to know only which event in  $\mathcal{F}_n$  has occurred and nothing more specific (like which run  $\omega$  has happened).

## 2.2. Random processes

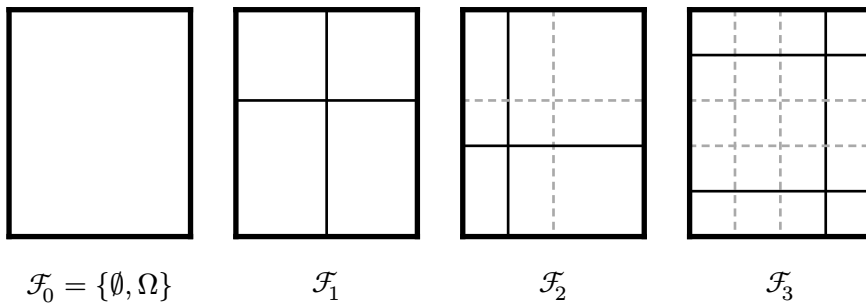
A sequence of real-world observations is then modelled by

- projections  $X_n : \Omega \rightarrow E$  that distinguish only properties observable up to time  $n$ ,
- i.e.  $X_n(\omega) \neq X_n(\omega')$  (with both values in disjoint measurable sets in  $E$ ) is only allowed if  $\omega$  and  $\omega'$  differ *in an observable way up to time  $n$* , so that  $X_n$  does not accidentally reveal information about  $\omega$  that is not supposed to be known at time  $n$ .
- That is, the preimages of observable (sets of) values in  $E$  must be maximal w.r.t. observables up to time  $n$ ,
- i.e.  $X_n^{-1}(A)$  must be in  $\mathcal{F}_n$  for all measurable  $A \subset E$ .

Or,  $X_n$  must be  $\mathcal{F}_n$ -measurable for all  $n$ . Such a sequence  $X_\bullet$  is called a *stochastic process*.

## 2.3. Filtration example diagram

Schematically we can imagine a filtration on a rectangular  $\Omega$  like this:



A run  $\omega$  is a point in the rectangle. Each of the four rectangles represents the same  $\Omega$  but with a different  $\sigma$ -algebra on it. The  $\sigma$ -algebras  $\mathcal{F}_n$  contain all rectangles outlined in each case (both dashed and non-dashed<sup>2</sup>):

1. In the beginning, we know nothing (except that the run *happened*), so no subrectangles.
2. In  $\mathcal{F}_1$ , the model allows to observe which of the four regions  $\omega$  lies in, but nothing more.

<sup>1</sup>I borrow that term from topology. “Two points  $x, y$  are *separated* by  $\mathcal{F}_n$ ” means that there are sets  $A, B \in \mathcal{F}_n$  that contain one but not the other:  $x \in A, y \notin B$  and  $x \in B, y \notin A$ .

<sup>2</sup>non-dashed borders indicate the newly introduced measurable sets that weren't measurable in the previous step

3. In  $\mathcal{F}_2$ , we're allowed to observe which of the 9 (smaller!) rectangles it is in, and so on.

## 2.4. Example

For an experiment with 3 consecutive coin flips we can put

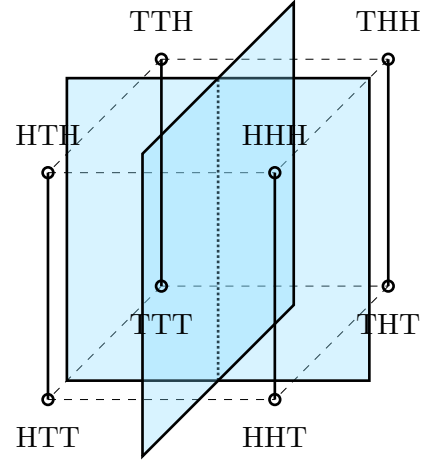
$$\Omega = \{\text{Heads, Tails}\}^3,$$

a discrete cube in 3D.

$\mathcal{F}_1$  contains only the front and back halves (apart from the whole cube and the empty set),  $\mathcal{F}_2$  is generated by the four vertical edges, and  $\mathcal{F}_3$  — by all 8 corners of the cube.

If  $X_i : \Omega \rightarrow \{\text{Heads, Tails}\}$  is the outcome of the  $i$ th coin flip, then  $X_i$  is allowed to (though does not necessarily have to) differ only on different sets from  $\mathcal{F}_i$ , which it does.

Again, keep in mind that the entire runs of the experiment are fully encoded in the actual corners. The random process  $X_1, X_2, X_3$  is only a sequence revealing smaller and smaller subsets in  $\Omega$  around the true  $\omega$ .



$\mathcal{F}_2$  is generated by the four vertical edges, i.e. the outcomes of the first two flips. Both  $X_1$  and  $X_2$  are  $\mathcal{F}_2$ -measurable, though neither of the two on its own separates the entire  $\mathcal{F}_2$ , only the vector  $(X_1, X_2)$  does.

## 3. Stopping times

Here we consider a property that is present up to some time point (potentially infinite) in the observed process (or equivalently, a property present only from some time point onward in the observed process).

Again, we model the time layout of the process by a filtration  $\mathcal{F}_\bullet$ . Then the presence of a property like above is modelled by a monotonic  $\mathcal{F}_\bullet$ -adapted boolean sequence  $P_\bullet$  (with values in  $2 = \{0, 1\}$  with the  $\mathcal{P}(2)$   $\sigma$ -algebra).

It is natural to ask now, what is the turning point? It is given by

$$T_X = \inf_{P_n=1} n = \inf \{n \in \mathbb{N} : P_n = 1\}$$

At any time  $n \in \mathbb{N}$ ,  $P_n$  tells us whether  $P$  has happened up to now. In terms of  $T_X$ , this is the event  $T_X \leq n$ , so it is  $\mathcal{F}_n$  measurable.

Conversely, for every  $\mathbb{N}$ -valued random variable  $T$  with  $\mathcal{F}_n$ -measurable  $T \leq n$  for every  $n$ , we can consider a monotone boolean sequence

$$X_n = \mathbb{1}_{\{T \leq n\}}$$

and then  $T$  is given from  $X_\bullet$  as the infimum above.

A *stopping time* for a filtration  $\mathcal{F}_\bullet$  is any  $\mathbb{N}$ -valued random variable for which

$$\forall n : \quad \{T \leq n\} \text{ is } \mathcal{F}_n\text{-measurable,}$$

i.e. it is the time at which an observed property (in the sense of a monotonic boolean sequence) “stops” holding.

## 4. Conditional expectation

If  $\mathcal{G} \subset \mathcal{F}$  are  $\sigma$ -algebras on  $\Omega$ , and  $X : \Omega \rightarrow E$  is a function,

then  $\mathcal{G}$ -measurability is a stronger restriction on  $X$  than  $\mathcal{F}$ -measurability,  
because  $X^{-1}(A) \in \mathcal{G}$  is a stronger condition than  $X^{-1}(A) \in \mathcal{F}$ .

That is, an event  $A \in E$  is allowed to reveal less information on the run  $\omega$  in the  $\mathcal{G}$  case than in the  $\mathcal{F}$  case, just because  $\mathcal{G}$  contains less information than  $\mathcal{F}$ .

Thus an  $\mathcal{F}$ -measurable  $X$  need not be  $\mathcal{G}$ -measurable.

TODO

## 5. Martingales

Recall that an  $\mathcal{F}_\bullet$ -martingale is an  $\mathcal{F}_\bullet$ -adapted process  $X_\bullet$  with zero- $\mathcal{F}_n$ -expectation increments  $\Delta X_{n+1} = X_{n+1} - X_n$ :

**Definition 5.1** (Martingale): An  $\mathcal{F}_\bullet$ -adapted process  $X_\bullet$  is called a **martingale** if

$$\forall n : \mathbb{E}[\Delta X_{n+1} | \mathcal{F}_n] = 0.$$

I prefer that phrasing of the definition because I find it more natural than the standard one. By writing

$$0 = \mathbb{E}[\Delta X_{n+1} | \mathcal{F}_n] = \mathbb{E}[X_{n+1} | \mathcal{F}_n] - \mathbb{E}[X_n | \mathcal{F}_n]$$

as  $\mathbb{E}[X_{n+1} | \mathcal{F}_n] = \mathbb{E}[X_n | \mathcal{F}_n]$  and applying  $\mathcal{F}_n$ -measurability to  $X_n$ , one gets equivalently

**Definition 5.2** (Martingale, standard): An  $\mathcal{F}_\bullet$ -adapted process  $X_\bullet$  is called a **martingale** if

$$\forall n : \mathbb{E}[X_{n+1} | \mathcal{F}_n] = X_n.$$

Note that we can always decompose a process into the sum of its increments up a given time, i.e.

$$\forall n : X_n = X_0 + \sum_{i=1}^n \Delta X_i.$$

Here again  $\Delta X_i$  denotes the “backwards” increment,  $\Delta X_{i+1} = X_{i+1} - X_i$ .

Now it is natural to ask, what happens if we scale the increments of a martingale, i.e. multiply each term in the above sum by some factor.

**Definition 5.3** (Martingale transform): For an  $\mathcal{F}_\bullet$ -martingale  $X_\bullet$  and a predictable  $H_\bullet$ , the martingale transform of  $X_\bullet$  by  $H_\bullet$  is denoted by  $(H \odot X)_\bullet$  and defined by

$$(H \odot X)_n := \sum_{i=1}^n H_i \Delta X_i.$$

Using this notion, a martingale can be characterized not only by having its *individual* increments-means vanish, but by having (only) the final expectation of all of its increments-transforms vanish:

**Proposition 5.1:** Let  $X_\bullet$  be  $\mathcal{F}_\bullet$ -adapted.

$X_\bullet$  is a martingale      if and only if      for any predictable  $H_\bullet$ ,  $\mathbb{E}[(H \odot X)_N] = 0$ .

## 6. Discrete-time market model

This section follows the first chapter of the book “Introduction to Stochastic Calculus Applied to Finance” (Lamberton & Lapeyre), whose explanations I found missing or unsatisfactory so I enhance the presentation with as much motivation as possible.

The market price vector is a stochastic process

$$S_i, \quad i = 0, \dots$$

with values in  $\mathbb{R}^{1+d}$  adapted to a filtration

$$\mathcal{F}_i, \quad i = 0, \dots$$

of the market state known at time  $i$ . We consider  $d \in \mathbb{N}$ ,  $\mathcal{F}_\bullet$  and  $S_\bullet$  fixed from now on.

The number  $1 + d$  reflects that we track  $d$  risky assets and one riskless asset (the time-value of money) in the first component of  $\mathbb{R}^{1+d}$ .

**Definition 6.1:** A *trading strategy* is any  $\mathbb{R}^{1+d}$ -valued *predictable* sequence

$$\phi_i, \quad i = 0, \dots$$

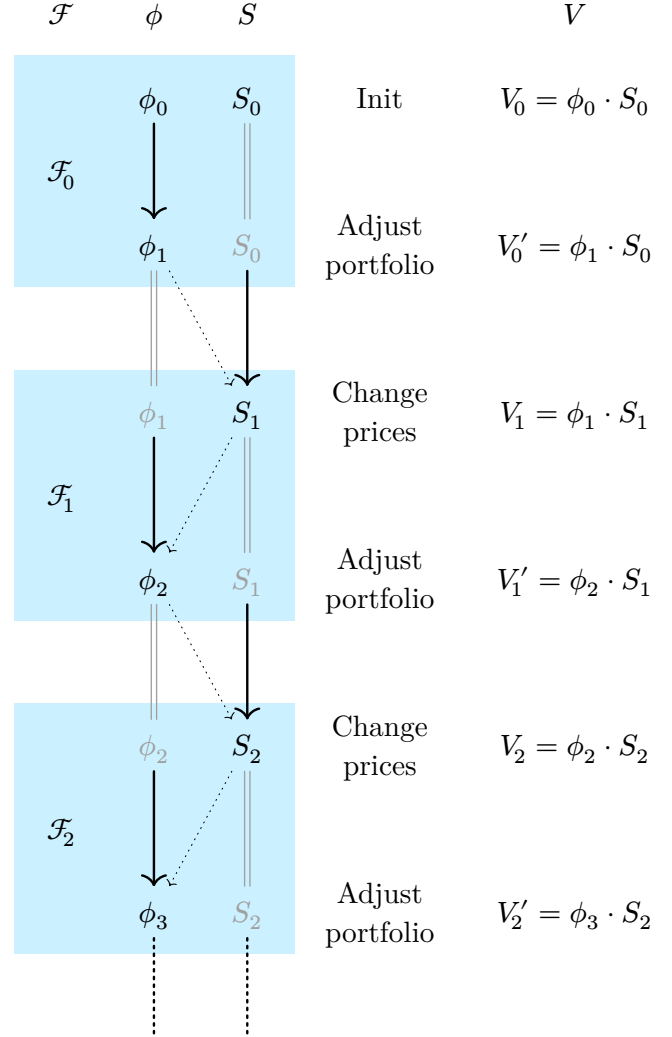
with respect to that filtration.

Predictable means that  $\phi_{i+1}$  is  $\mathcal{F}_i$ -measurable, i.e. the information used (revealed) by the *next* portfolio  $\phi_{i+1}$  is restricted to the information provided by the current market state  $\mathcal{F}_i$ , which is only guaranteed to contain the current stock prices (because  $S_i$  is  $\mathcal{F}_i$ -measurable) and past stock prices (because  $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ ).

**Definition 6.2:** The *value of the portfolio* defined by a strategy  $\phi_\bullet$  is

$$V_i(\phi) := \phi_i \cdot S_i.$$

Now a slight confusion may arise as to what is the time order of the quantities  $S_i$  and  $\phi_i$ . They both use the same index  $i$  so we might be under the impression that the time evolution is described by a sequence of pairs  $(\phi_i, S_i)$ . While not false, this may be deceiving when we define *self-financing* strategies. Consider the diagram on the right.



Here, in each timebox  $\mathcal{F}_i$ , first prices change and then the portfolio is adjusted, so we have an alternating sequence of price and portfolio changes.

The strategy  $\phi$  is then self-financing iff the portfolio value doesn't change within a time-box ( $V_i = V'_i$ ), but only responds to stock price changes  $S_i \rightarrow S_{i+1}$ .

Diagonal dotted arrows indicate the causal relationship between the time-evolution of the strategy and the stock prices in the process  $(\phi_0, S_0), (\phi_1, S_1), \dots$

**Notation 6.3:** Denote the *portfolio adjustment* (again, non-standard terminology) performed at time  $i$  by

$$\Delta\phi_i := \phi_{i+1} - \phi_i.$$

(consider the diagram above to convince yourself that  $\Delta\phi_i$  is  $\mathcal{F}_i$ -measurable)

It is a vector of the sells/buys of each stock performed at time  $i$ . That is,

$\Delta\phi_i^j$  is the amount of stock  $j$  that is sold (if  $\Delta\phi_i < 0$ ) or bought (if  $\Delta\phi_i > 0$ ) at time  $i$ .

**Notation 6.4:** Denote the *price changes* that happened at time  $n + 1$  by

$$\Delta S_{n+1} := S_{n+1} - S_n.$$

**Remark 6.5:**  $\Delta\phi_n$  was defined as the *forward* difference (between times  $n + 1$  and  $n$ ), while the  $\Delta S_n$  was defined as the *past* difference (between times  $n$  and  $n - 1$ ), so that both  $\Delta\phi_n$  and  $\Delta S_n$  are  $\mathcal{F}_n$ -measurable.

We also get a Leibniz rule for  $V_\bullet$  ( $dV = d\phi \cdot S + dS \cdot \phi$ ).

**Remark 6.6:** The two increments let us express the portfolio value with a recursive formula:

$$\underset{\text{next value}}{V_{n+1}(\phi)} = \underset{\text{prev value}}{V_n(\phi)} + \underset{\text{changes in strategy}}{\Delta\phi_n \cdot S_n} + \underset{\text{changes in prices}}{\phi_{n+1} \cdot \Delta S_n}.$$

That is, the next value is the previous value plus the changes from the strategy and the changes in the prices.

## 6.1. Self-financing strategies

When the second term in the sum on the right in the remark above,  $\Delta\phi_n \cdot S_n$ , vanishes for all  $n$ , we get a very important type of strategies. Denote the first part of recursive formula by

$$V'_n(\phi) = V_n(\phi) + \Delta\phi_n \cdot S_n,$$

so that

$$V_{n+1}(\phi) = V'_n(\phi) + \phi_{n+1} \cdot \Delta S_n.$$

Those two mutually-recursive formulas reflect the structure of the alternating sequence  $V_0, V'_0, V_1, V'_1, \dots$  in the diagram above, so

**Remark 6.1.1:**  $V'_n(\phi)$  is the *adjusted value* of the portfolio **after** the next strategy choices  $\phi_{n+1}$  are applied but **before** the prices have been updated.

Now the condition that the second term in the earlier recursive formula for  $V(\phi)$  vanishes for all  $n$  can be phrased simply as the equality  $V(\phi) = V'(\phi)$  (as processes). It is instructive to derive a direct expression for  $V'$ :

$$\begin{aligned} V'_n(\phi) &\equiv V_n(\phi) + \Delta\phi_n \cdot S_n, \\ &= \cancel{\phi_n \cdot S_n} + (\phi_{n+1} \cdot S_n - \cancel{\phi_n \cdot S_n}) \\ &= \phi_{n+1} \cdot S_n. \end{aligned}$$

**Definition 6.1.2:** A strategy  $\phi_i$  is *self-financing* if the value of the portfolio stays the same after the strategy adjustment:

$$V'_i(\phi) = V_i(\phi) \quad \text{for all } i,$$

i.e.

$$\phi_{i+1} \cdot S_i = \phi_i \cdot S_i \quad \text{for all } i.$$

Now we can get back to the condition on the vanishing second term in the recursive formula for  $V$ :

**Remark 6.1.3:** The self-financing condition  $V_i = V'_i$  is equivalent to

$$\Delta\phi_i \cdot S_i = 0,$$

i.e. all buys and sells cancel each other in value, i.e. no money is lost or needs to be brought in for the adjustment.

Then a self-financing strategy is one where the recursive formula for  $V$  has a vanishing second term, i.e.

$$V_{n+1}(\phi) = V_n(\phi) + \phi_{n+1} \cdot \Delta S_n.$$

We can apply the last formula iteratively and write

$$V_n(\phi) = V_0(\phi) + \sum_{i=1}^n \phi_i \cdot \Delta S_{i-1}$$

Note how close this expression is to Definition 5.3 — we could try to see it as a martingale transform under some conditions (though, the quantities  $\phi_n$  and  $S_n$  are vectors, so it would rather be a *sum* of the component-wise martingale transforms). However, even if  $V_0(\phi)$  were to vanish,  $\Delta S_n$  would generally not be a martingale because the riskless asset price process  $S_\bullet^0$  increases (maybe even deterministically). Instead, we can consider the discounted price process  $\tilde{S}_\bullet$  and the corresponding discounted portfolio value  $\tilde{V}_\bullet$ .



**Remark 6.1.4:** For a self-financing strategy with  $\tilde{V}_0(\phi) = 0$ , if each  $\tilde{S}_\bullet^j, j = 0, \dots, d$  were a martingale, then  $\tilde{V}_\bullet(\phi)$  would be a sum of the martingale transforms of the prices by the strategy,

$$\tilde{V}_\bullet(\phi) = \sum_j (\phi^j \odot \tilde{S}^j)_\bullet.$$

We'll get back to this observation in the next section.

It should be close to mind that even for a strategy that is not self-financing (i.e. some of the  $\Delta\phi_n \cdot S_n$  does not vanish), if we were to put the quantity  $\Delta\phi_n \cdot S_n$  into  $\phi_n^0$ , i.e. if we invest (borrow) the surplus (shortage) of sells-minus-buys into (from) the riskless asset, we'd get a self-financing strategy.

**Proposition 6.1.1:** Any  $\mathbb{R}^d$ -valued predictable sequence (a “strategy” only on the risky assets)

$$(\phi_n^1, \dots, \phi_n^d) \quad \text{for } n = 0, 1, \dots$$

is a restriction of a unique self-financing strategy ( $\mathbb{R}^{1+d}$ -valued)

$$(\phi_n^0, \phi_n^1, \dots, \phi_n^d) \quad \text{for } n = 0, 1, \dots$$

for any choice of an initial value  $V_0(\phi) \in \mathbb{R}$  (or a choice of any of  $\phi_n^0$  for  $n \geq 0$ ).

In other words, the self-financing strategies restricting to a given  $(\phi_n^1, \dots, \phi_n^d)_n$  are a one-parameter family indexed by  $V_0$ .

*Proof:* Fix  $\underline{\phi}_n = (\phi_n^1, \dots, \phi_n^d)$  for all  $n \geq 0$ . Any strategy  $\phi_\bullet$  that (on the risky assets) restricts to this process is determined by the choices of bank account amounts  $\varphi_\bullet^0$ .

The self-financing condition  $\Delta\phi_n \cdot S_n = 0$  imposes a restriction on  $\varphi_\bullet^0$ , though, and reads

$$\underbrace{(\phi_{n+1}^0 - \phi_n^0)} \cdot S_n^0 + \sum_{i=1}^d (\phi_n^i - \phi_{n-1}^i) \cdot S_n^i = 0 \quad \text{for all } n = 0, 1, \dots$$

Here a choice of any element of the sequence  $\phi_\bullet^0$  determines the rest, so the self-financing strategies restricting to  $\underline{\phi}_n$  are a one-parametric family indexed by (e.g.)  $\phi_0^0$ .

Since  $V_0 \equiv \phi_0^0 \cdot S_0 + \sum_{i=1}^d \phi_0^i \cdot S_0^i$  establishes a 1:1 relation between  $V_0$  and  $\phi_0^0$  (given  $\underline{\phi}_0$ ), we can also consider this parameter to be  $V_0$ .  $\square$

## 6.2. Arbitrage

**Definition 6.2.1** (Admissible strategy): A *self-financing* strategy is called **admissible** if  $V_n(\phi) \geq 0$  for all  $n$  (and all  $\omega \in \Omega$ ).

We fix an  $N \in \mathbb{N}$ , called the *horizon*. All indices  $n, i, j, k$  below vary between 0 and  $N$ .

The market is said to have an **arbitrage opportunity** if some admissible strategy with zero initial value delivers a strictly positive value on a non-null set.

That is, an arbitrage opportunity is a

$$\begin{array}{ccc} \text{zero-investment} & \text{risk-free} & \text{profit.} \\ (V_0 = 0) & (V_n \geq 0) & (V_N \geq 0) \\ & \text{for every } \omega \text{ and } n) & \text{on a non-null set)} \end{array}$$

A market without arbitrage opportunities is called viable.

**Definition 6.2.2** (Viable market): The market is called **viable** if every admissible strategy with  $V_0(\phi) = 0$  satisfies  $V_N(\phi) = 0$ .

As a silly (counter)example, if

$$\begin{array}{ll} \text{at } n = 0, & S_0^1 = S_0^0 \\ \text{at } n > 0, & S_n^1 = 2 \cdot S_n^0, \end{array}$$

i.e. the price of risky asset 1 equals the riskless at first but then becomes twice as much as the riskless one, then the constant strategy  $\phi_n = (-1, 1)$  has zero initial value and  $V_n = -S_n^0 + S_n^1 = S_n^0 > 0$ . Thus the market defined by those  $S_\bullet^i$  is not viable.

**Proposition 6.2.1** (Fundamental theorem of asset pricing):

A market is viable      if and only if      under some equivalent measure, the discounted price processes  $S_\bullet^k, k = 0, 1, \dots$  are martingales.

Writing out the definition of a viable market, and applying the characterization of martingales in Proposition 5.1, the proposition reads

<p>For any self-financing <math>\phi_\bullet</math>, if <math>\forall n : V_n(\phi) \geq 0</math> and <math>V_0(\phi) = 0</math>, then <math>V_N(\phi) = 0</math></p>	if and only if	<p>there exists <math>\mathbb{P}^*</math> equivalent to <math>\mathbb{P}</math>, such that for any predictable <math>H_\bullet</math>,</p> $\mathbb{E} \left[ \sum_{i=1}^N H_i \Delta S_i \right] = 0$
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### 6.3. Perfect hedging (attainable claims)

A **contingent claim** is a promise to pay some amount  $h$  at a *maturity time*  $T$ , the amount depending on the market state. We model it by a non-negative real random variable  $h$  that is only  $\mathcal{F}_T$ -measurable, so the amount might not be known in advance.

**Remark 6.3.1:** Consider a contingent claim  $h$  at a maturity time  $T$ .

If all admissible strategies  $\phi$  induce a portfolio value at  $T$  that is

- *less* than  $h$ , the contingent claim is essentially “free money” for the receiver;
- *greater* than  $h$ ,

**Definition 6.3.2:** An  $\mathbb{R}$ -valued random variable  $h$  is called **attainable** if there exists an admissible strategy  $\phi$  that results in the end in a portfolio of value exactly  $h$ :

$$V_N(\phi) = h.$$

**Remark 6.3.3:** If  $h$  is a constant  $h(\omega) = M \in \mathbb{R}_+$ , it is trivially attainable by the strategy with  $\phi_N^0 = M$  that is the constant zero on the risky assets (exists uniquely by the proposition above, and is obviously admissible).

Even if  $h$  is only *bounded* by a constant,  $h \leq M$ , we could attain the (higher) value  $M$  by the same approach. Thus attainability is *not* essentially about reaching *at least* some value — we are allowed to start with as much cash in advance as we need.

In the case of a finite  $\Omega$  (which is assumed in the book),  $h$  is bounded anyway. Attaining  $h$ , and not just  $M$ , is more difficult. If  $h$  is not  $\mathcal{F}_0$ -measurable, we cannot just prepare the cash in advance, because we are not allowed (by  $\mathcal{F}_0$ ) to know exactly how much we will need (even if know it won't exceed  $M$ ).

For example, if  $\mathcal{F}_0 = \{\emptyset, \Omega\}$ , the initial cash  $\phi_0^0$  is only allowed to be constant (by  $\mathcal{F}_0$ -measurability). Thus even in the bounded case we have to make use of the risky assets to achieve the precise target  $h$ , and we have to abide by  $\mathcal{F}_n$ -measurability all the way  $n = 0, \dots, N$ .

**Definition 6.3.4:** The market is called *complete* if every non-negative  $h$  is attainable.