

Foreign exchange futures: pricing and arbitrage

Here we derive how foreign exchange futures can be priced based on interest rates, and thus how an arbitrage strategy can be applied if the price is inappropriate.

Then we show how the same approach can be used to *recreate* futures in markets where they are not readily available.

Setup

For ease of notation we assume two currencies £, and \$, related by a (time-dependent) exchange rate α_t :

$$\begin{aligned}\pounds_t &= \alpha_t \$ \\ \$ &= \frac{1}{\alpha_t} \pounds_t\end{aligned}$$

We are given

- the interest rates in both currencies for a particular time period, r_\pounds and $r_\$$
- the exchange rate now, α_0 (the rate at the end of the period, α_1 , is unknown)
- a dealing limit of $\pounds_0 X$

and we are interested in the potential profit of borrowing from the lower interest currency and investing into the higher interest currency.

Strategies

At time zero, we can either UK-borrow $\pounds X$, convert to \$ and US-invest that $\$ \alpha_0 X$, or do the converse – US-borrow and UK-invest. To talk about the two cases uniformly, we introduce a parameter $\varepsilon = +1$ or $\varepsilon = -1$.

In the table below, a trade of positive value means we *borrow* that amount, and conversely, a negative value traded denotes an *investment*.

	£	\$	
Trade now	$\pounds \varepsilon X$	$-\$ \alpha_0 \varepsilon X$	At time zero, we trade $\pounds \varepsilon X$ and $-\$ \alpha_0 \varepsilon X$, so $\varepsilon = +1$ means we borrow $\pounds X$, convert to $\$ \alpha_0 X$ and invest in dollars; and conversely for $\varepsilon = -1$.
Trade in 1 year	$-\pounds \varepsilon X e^{r_\pounds}$	$\$ \alpha_0 \varepsilon X e^{r_\$}$	In one year, we (must) trade the interest-accumulated values in the opposite direction – repay the loan and receive on the investment.

Profit

The difference in the trades in one year symbolically is

$$\Delta \equiv \$ \alpha_0 \varepsilon X e^{r_\$} - \pounds \varepsilon X e^{r_\pounds}.$$

This value only has meaning if we can convert between future dollars and pounds at some rate α_1 . Put $\$ = \frac{1}{\alpha_1} \pounds$ to get a difference expressed solely in pounds of

$$\Delta = \pounds \varepsilon \left(\frac{\alpha_0}{\alpha_1} e^{r_\$} - e^{r_\pounds} \right) X.$$

If this expression does not vanish, the strategy results in a profit/loss, so is an arbitrage opportunity (as we started with zero initial wealth).

Δ vanishes only if

$$\frac{\alpha_0}{\alpha_1} = \frac{e^{r_{\pounds}}}{e^{r_{\$}}} \quad \text{i.e.} \quad \alpha_1 = \alpha_0 \frac{e^{r_{\$}}}{e^{r_{\pounds}}}.$$

If a dealer trades foreign exchange futures at any other rate α_1 that results in $\Delta \neq 0$ (thus allowing us to actually perform the currency exchange between the traded values in 1 year), we can set

$$\varepsilon = \text{sign}\left(\frac{\alpha_0}{\alpha_1} - \frac{e^{r_{\pounds}}}{e^{r_{\$}}}\right)$$

and ensure profit by following the resulting strategy:

When a dealer provides a price that is	
too high $\alpha_1 > \alpha_0 e^{r_{\$}-r_{\pounds}}$	too low $\alpha_1 < \alpha_0 e^{r_{\$}-r_{\pounds}}$
Set $\varepsilon = -1$	Set $\varepsilon = +1$
<p>Borrow $\pounds X$ $\xRightarrow{\text{Convert}}$ Invest $\\$ \alpha_0 X$</p> <p>$\Downarrow$ \Downarrow</p> <p>Repay $\pounds e^{r_{\pounds}} X$ $\xleftarrow{\text{Convert}}$ Receive $\\$ e^{r_{\\$}} X$</p> <p>\Downarrow</p> <p>Profit $(\pounds e^{r_{\pounds}} - \\$ e^{r_{\\$}})X$</p>	<p>Invest $\pounds X$ $\xleftarrow{\text{Convert}}$ Borrow $\\$ \alpha_0 X$</p> <p>$\Downarrow$ \Downarrow</p> <p>Receive $\pounds e^{r_{\pounds}} X$ $\xRightarrow{\text{Convert}}$ Repay $\\$ e^{r_{\\$}} X$</p> <p>\Downarrow</p> <p>Profit $(\\$ e^{r_{\\$}} - \pounds e^{r_{\pounds}})X$</p>
In both cases, profit comes from <i>the higher amount of converted currency</i> , not from the higher interest rate! ($r_{\pounds} > r_{\$}$ or $r_{\$} > r_{\pounds}$)	

It is remarkable that the choice on which of the two strategies to apply depends on α_1 , but not on which of the rates $r_{\pounds}, r_{\$}$ is higher. As a function of α_1 ,

$$\varepsilon_{r_{\pounds}, r_{\$}}(\alpha_1) = \text{sign}\left(\frac{\alpha_0}{\alpha_1} - \frac{e^{r_{\pounds}}}{e^{r_{\$}}}\right)$$

is always monotonic (non-increasing), and e.g. even interchanging $r_{\pounds}, r_{\$}$ will only move the threshold at which it changes sign. So for sufficiently large deviations of α_1 from the theoretical rate (when $\Delta = 0$), we might end up borrowing from the high-interest currency (expensive) and invest in the low-interest currency (cheap), which is counter-intuitive.

Hedging foreign exchange with money market

In currencies where foreign exchange futures are not as developed, we can use the same procedure above to recreate a future manually and hedge a foreign exchange with it.

Setup

We want to convert a future amount between two currencies at a known rate, where

- the *foreign* currency amount is specified,

- while the *home* currency amount is unknown, because the future exchange rate is unknown, and that is the risk.

so that the total amount spent or received in the home currency is known *now*.

We could also consider the converse: a fixed *home* currency amount and an unknown foreign one, but that is uninteresting because here we only care about *our* currency and there would be nothing to hedge.

Still, there are two possible scenarios, depending on the direction of the conversion:

Want to convert:	home \mapsto foreign	foreign \mapsto home
Risk is:	foreign is strong, home is weak	foreign is weak, foreign is strong
...in which case we will:	have to use too much of the local currency to cover the foreign amount	receive too little in the local currency after the conversion
This role is played by	an importer who has to pay for foreign goods	an exporter who will be payed for exported goods

Say we want to convert the foreign future amount $\$1A$ into local pounds (we play a British importer). Interest rates for the period til the exchange are r_{\pounds} and $r_{\$}$, and the current exchange rate is α_0 .

Strategy

Again, we borrow in one currency and invest in the other, but now we also have the future amount $\$1A$ that we want to turn into an α_1 -independent pound value in the future.

Copying directly the table from earlier:

	\pounds	$\$$
Trade now	$\pounds \varepsilon X$	$-\$ \alpha_0 \varepsilon X$
Trade in 1 year	$-\pounds \varepsilon X e^{r_{\pounds}}$	$\$ \alpha_0 \varepsilon X e^{r_{\$}}$

In the previous section, the future total included gaining from the investment and repaying the loan; this time we also have the $\$1A$ to convert, so the future total is

$$\Delta = \underbrace{-\pounds \varepsilon X e^{r_{\pounds}}}_{\text{pounds result}} + \underbrace{\$ \alpha_0 \varepsilon X e^{r_{\$}}}_{\text{dollar result}} + \underbrace{\$ A}_{\text{amount to convert}}.$$

Now we want the local currency value of Δ to be independent of the future exchange rate. That means that we want the dollar parts to cancel out, i.e. we want that

$$\$ \alpha_0 \varepsilon X e^{r_{\$}} + \$ A = 0,$$

i.e. the dollar result of the borrow-invest strategy should completely compensate the amount A that is to be converted. So we want to *pay* dollars, meaning *now* we have to borrow them.

The last equation immediately implies that $\varepsilon = -1$ and $X = \frac{1}{\alpha_0 e^{r_{\$}}} A$. Looking back at the table, that means that

1. *now* we have to *borrow* dollars having a future value of $\$A$
2. convert the borrowed to pounds and invest the pounds,
3. *then*, repay the dollar loan with the future amount $\$A$ that will be available, thus getting rid of all dollars,
4. and earn on the pounds investment at the known pounds interest rate.