Foreign exchange futures

Now we look at how foreign exchange futures can be priced, and thus how an arbitrage strategy can be applied if the price is inappropriate.

Setup

For ease of notation we assume two currencies £, and $, related by a (time-dependent) exchange rate α:

$$\pounds = \alpha\$$$
$$\$ = \frac{1}{\alpha}\pounds$$

We are given
- the interest rates in both currencies for a particular time period, $r_\pounds$ and $r_\$$
- the exchange rate now, α_0 (the rate at the end of the period, α_1, is unknown)
- a dealing limit of $\pounds X$

and we are interested in the potential profit of borrowing from the lower interest currency and investing into the higher interest currency.

Strategies

At time zero, we can either UK-borrow $\pounds X$, convert to \$ and US-invest that $\$\alpha X$, or do the converse – US-borrow and UK-invest. To talk about the two cases uniformly, we introduce a parameter $\varepsilon = +1$ or $\varepsilon = -1$.

In the table below, a trade of positive value means we *borrow* that amount, and conversely, a negative value traded denotes an *investment*.

	£	\$
Trade now	$\pounds\varepsilon X$	$-\$\alpha_0\varepsilon X$
Trade in 1 year	$-\pounds\varepsilon X e^{r_\pounds}$	$\$\alpha_0\varepsilon X e^{r_\$}$

At time zero, we trade $\pounds\varepsilon X$ and $-\$\alpha_0\varepsilon X$, so $\varepsilon = +1$ means we borrow $\pounds X$, convert to $\$\alpha_0 X$ and invest in dollars; and conversely for $\varepsilon = -1$.

In one year, we (must) trade the interest-accumulated values in the opposite direction – repay the loan and receive on the investment.

Profit

The difference in the trades in one year symbolically is

$$\Delta \equiv \$\alpha_0\varepsilon X e^{r_\$} - \pounds\varepsilon X e^{r_\pounds}.$$

This value only has meaning if we can can convert between future dollars and pounds at some rate α_1. Put $\$ = \frac{1}{\alpha_1}\pounds$ to get a difference expressed solely in pounds of

$$\Delta = \pounds\varepsilon\left(\frac{\alpha_0}{\alpha_1}e^{r_\$} - e^{r_\pounds}\right)X.$$

If this expression does not vanish, the strategy results in a profit/loss, so is an arbitrage opportunity (as we started with zero initial wealth).

Δ vanishes only if

$$\frac{\alpha_0}{\alpha_1} = \frac{e^{r_\pounds}}{e^{r_\$}} \qquad \text{i.e.} \quad \alpha_1 = \alpha_0\frac{e^{r_\$}}{e^{r_\pounds}}.$$

If a dealer trades foreign exchange futures at any other rate α_1 that results in $\Delta \neq 0$ (thus allowing us to actually perform the currency exchange between the traded values in 1 year), we can set

$$\varepsilon = \operatorname{sign}\left(\frac{\alpha_0}{\alpha_1} - \frac{e^{r_\pounds}}{e^{r_\$}}\right)$$

and ensure profit by following the resulting strategy:

<table>
<tr><td colspan="2">When a dealer provides a price that is</td></tr>
<tr><td>too high
$\alpha_1 > \alpha_0 e^{r_\$ - r_\pounds}$</td><td>too low
$\alpha_1 < \alpha_0 e^{r_\$ - r_\pounds}$</td></tr>
<tr><td>Set $\varepsilon = -1$</td><td>Set $\varepsilon = +1$</td></tr>
<tr><td>

$$\begin{tikzcd}
\text{Borrow } \pounds X \arrow[rr, Rightarrow, "\text{Convert}"] \arrow[d, Rightarrow] & & \text{Invest } \$\alpha_0 X \arrow[d, Rightarrow] \\
\text{Repay } \pounds e^{r_\pounds} X & {} \arrow[d, Rrightarrow] & \text{Receive } \$e^{r_\$} X \arrow[ll, Rightarrow, "\text{Convert}"'] \\
& \text{Profit } (\pounds e^{r_\pounds} - \$e^{r_\$})X &
\end{tikzcd}$$

</td><td>

$$\begin{tikzcd}
\text{Invest } \pounds X \arrow[d, Rightarrow] & & \text{Borrow } \$\alpha_0 X \arrow[ll, Rightarrow, "\text{Convert}"'] \arrow[d, Rightarrow] \\
\text{Receive } \pounds e^{r_\pounds} X \arrow[rr, Rightarrow, "\text{Convert}"] & {} \arrow[d, Rrightarrow] & \text{Repay } \$e^{r_\$} X \\
& \text{Profit } (\$e^{r_\$} - \pounds e^{r_\pounds})X &
\end{tikzcd}$$

</td></tr>
<tr><td colspan="2">In both cases, profit comes from the higher amount of converted currency,
not from the higher interest rate! ($r_\pounds > r_\$$ or $r_\$ > r_\pounds$)</td></tr>
</table>

It is remarkable that the choice on which of the two strategies to apply depends on α_1, but not on which of the rates $r_\pounds$, $r_\$$ is higher. As a function of α_1,

$$\varepsilon_{r_\pounds, r_\$}(\alpha_1) = \operatorname{sign}\left(\frac{\alpha_0}{\alpha_1} - \frac{e^{r_\pounds}}{e^{r_\$}}\right)$$

is always monotonic (non-increasing), and e.g. even interchanging $r_\pounds, r_\$$ will only move the threshold at which it changes sign. So for sufficiently large deviations of α_1 from the theoretical rate (when $\Delta = 0$), we might end up borrowing from the high-interest currency (expensive) and invest in the low-interest currency (cheap), which is counter-intuitive.