Basic solutions

Given a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ and a vector in its range $b \in \mathbb{R}^m$, form the linear system $Ax = b$.

A basis $\{e_1 ... e_n\}$ in the domain of A now gives many ways to express b as a combination of the image of the basis $\{Ae_i\}_i$:

$$b = \sum_{i \in B} \lambda_i A e_i$$

for many choices of m-sized $B \subset \{1 ... n\}$ and $\lambda_i, i \in B$.
Not every choice for B makes b expressible as a combination of $\{Ae_i\}_{i \in B}$, but at least one does.

For each such choice of B, we can rewrite $b = A \sum_{i \in B} \lambda_i e_i$ and denote

$$x_B \coloneqq \sum_{i \in B} \lambda_i e_i$$

so that $b = Ax_B$.

x_B is now well-defined for any choice B of (a subset of) basis vectors in the domain of A whose images form a basis of A's range. Such x_B are *special* solutions to the system $Ax = b$ and are called *basic* solutions.

Given B, those basis vectors e_i that form x_B are called *basic*, and the rest are called *non-basic*. In the sketches below, we see copies of the non-basic e_i attached to each basic solution e_i: