Q1) use RK method for systems to approximate the solutions of the following systems of first-order differential equations and compare the results to actual solutions.

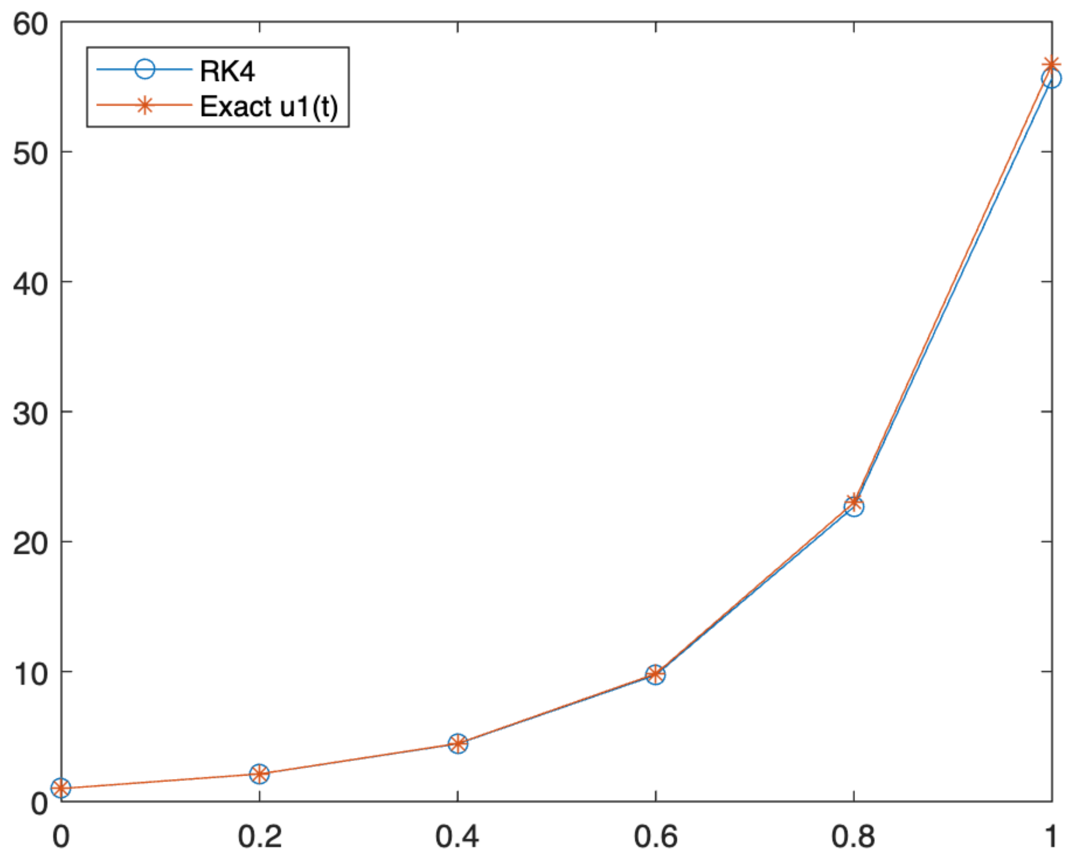
a) $\begin{cases} u_1' = 3u_1 + 2u_2 - (2t^2+1)e^{2t} & u_1(0) = 1 \\ u_2' = 4u_1 + u_2 + (t^2+2t-4)e^{2t} & u_2(0) = 1 \end{cases}$ $0 \le t \le 1$ $h = 0.2$

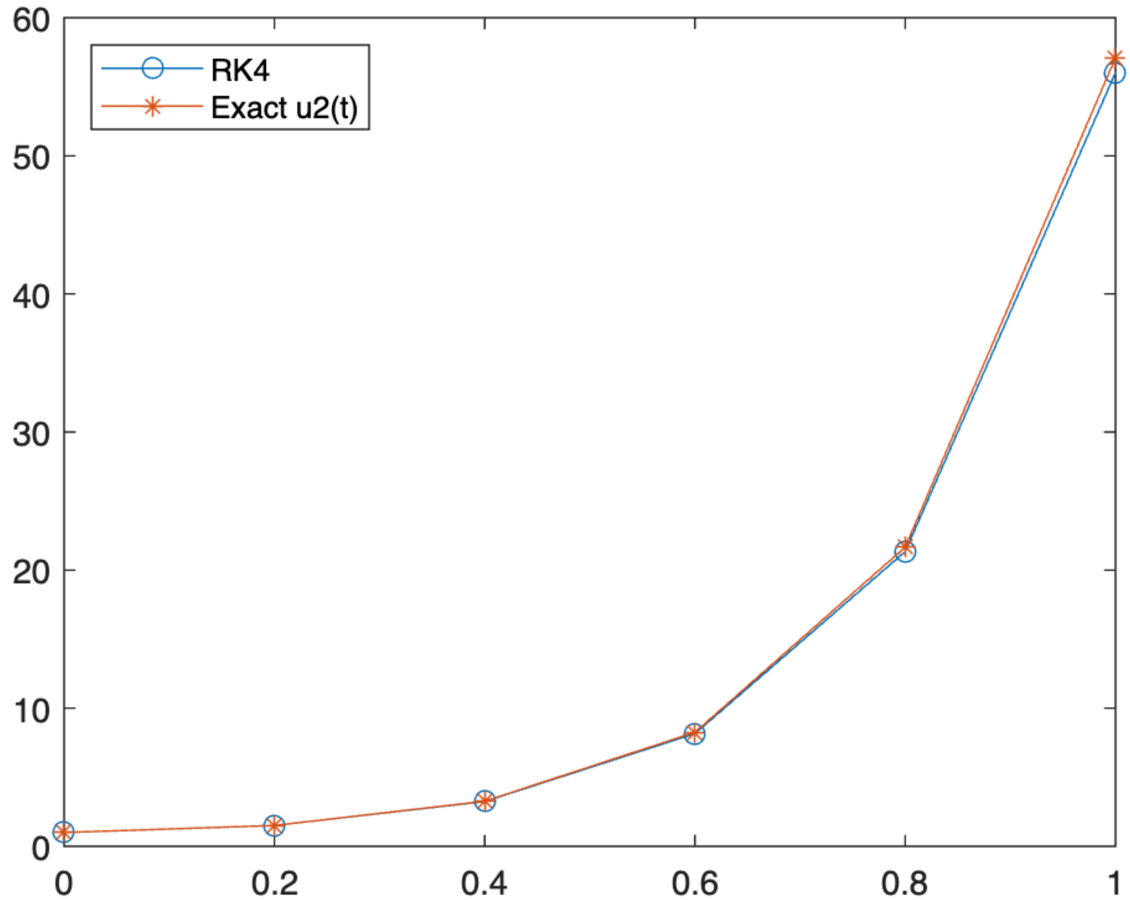
* code in separate file *

The output is shown below.

```
>> fivept9_q1a
Approximate values
w1(i): 1.0000    2.1204    4.4412    9.7391   22.6766   55.6612
w2(i): 1.0000    1.5070    3.2422    8.1634   21.3435   56.0305

Exact solution:  u1(ti)        for ti = 0, 0.2, 0.4, 0.6, 0.8, 1.0 respectively
    1.0000    2.1250    4.4651    9.8324   23.0026   56.7375

Absolute error: |u1(t_i) - w1(i)|    for ti = 0, 0.2, 0.4, 0.6, 0.8, 1.0 respectively
         0    0.0046    0.0239    0.0932    0.3261    1.0763

Exact solution:  u2(ti)        for ti = 0, 0.2, 0.4, 0.6, 0.8, 1.0 respectively
    1.0000    1.5116    3.2660    8.2563   21.6689   57.1054

Absolute error: |u2(t_i) - w2(i)|    for ti = 0, 0.2, 0.4, 0.6, 0.8, 1.0 respectively
         0    0.0046    0.0237    0.0929    0.3253    1.0749
```

Plots





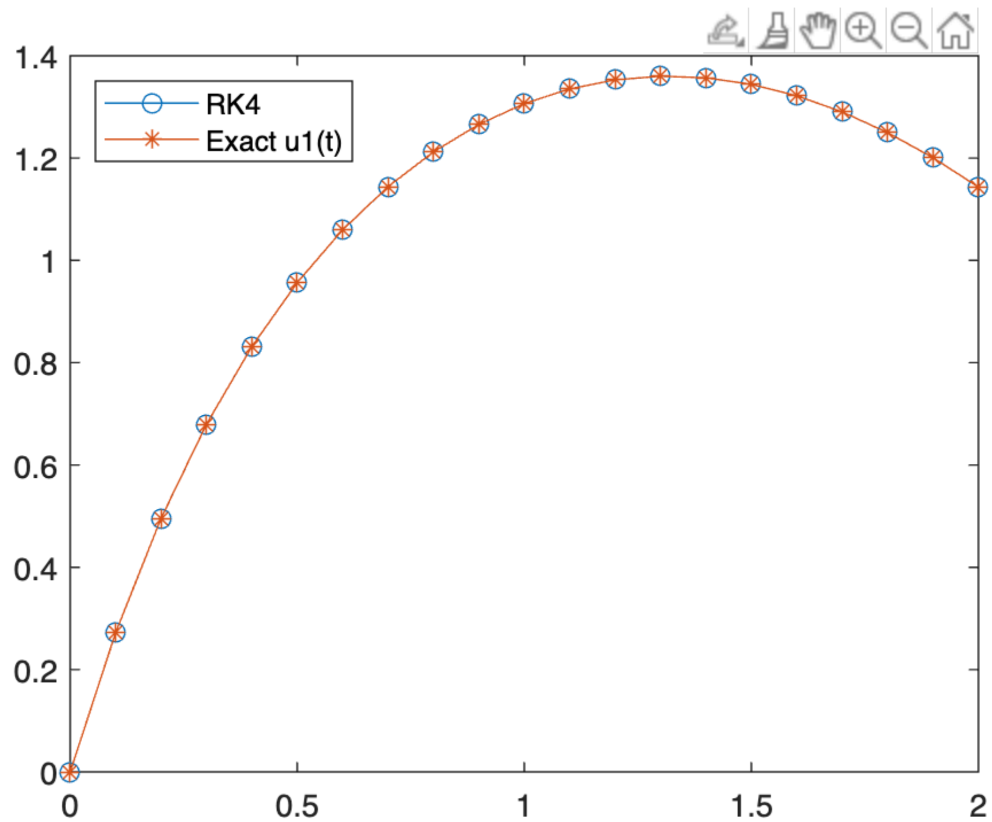
RK4
Exact u2(t)
60
50
40
30
20
10
0
0
0.2
0.4
0.6
0.8
1

Q1(b) $\begin{cases} u_1' = -4u_1 - 2u_2 + \cos(t) + 4\sin t \\ u_2' = 3u_1 + u_2 - 3\sin(t) \end{cases}$ $u_1(0) = 0$, $u_2(0) = -1$, $0 \le t \le 2$, $h = 0.1$

For u_1':

t_i	Approximate values $w_1(t_i)$	Exact solution: $u1(t_i)$	Absolute error: |u1(t_i) − w1(i)| 1.0e-04 *
0.0	0	0	0
0.10	0.272041370942637	0.272046746562784	0.053756201460753
0.20	0.495481688508659	0.495490744879746	0.090563710875013
0.30	0.679521862491442	0.679533375836722	0.115133452806004
0.40	0.831387406854316	0.831400506145486	0.130992911702155
0.50	0.956713899934873	0.956727975686585	0.140757517125278
0.60	1.059862687469840	1.059877321758684	0.146342888438511
0.70	1.144179453888847	1.144194366937297	0.149130484501381
0.80	1.212205973485076	1.212220983144655	0.150096595790128
0.90	1.265853461411425	1.265868452665509	0.149912540836628
1.00	1.306544398449604	1.306559300677556	0.149022279514188
1.10	1.335328440498373	1.335343210732927	0.147702345536516
1.20	1.352976992416689	1.352991603212806	0.146107961167541
1.30	1.360060184219202	1.360074615056550	0.144308373484225
1.40	1.357009301241291	1.357023532621237	0.142313799462723
1.50	1.344167160580019	1.344181170165186	0.140095851677291
1.60	1.321828470783140	1.321842231074084	0.137602909431322
1.70	1.290271841479358	1.290285318637286	0.134771579276549
1.80	1.249784808913138	1.249797962426783	0.131535136449479
1.90	1.200682999456321	1.200695782420353	0.127829640328425
2.00	1.143324355695280	1.143336715521439	0.123598261587698

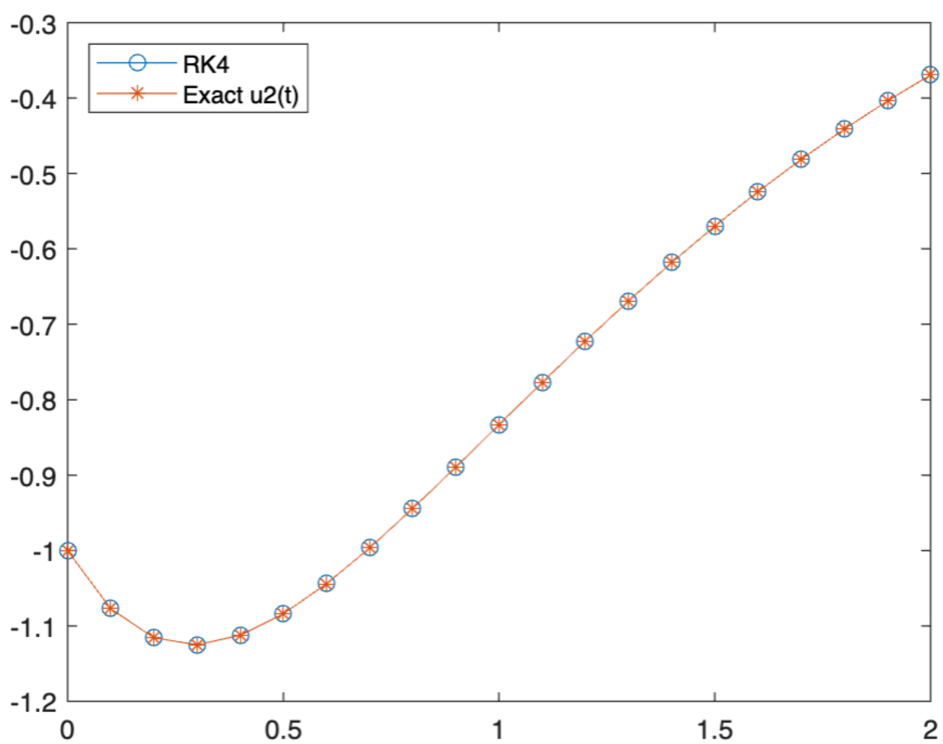
Plot



For u_2':

t_i	Approximate values ($w_2(t_i)$)	Exact solution: u2(t_i)	Absolute error: |u2(t_i) - w2(i)| 1.0e-04 *
0.0	-1.000000000000000	-1.000000000000000	0
0.10	-1.077045486579260	-1.077050747951915	0.052613726548856
0.20	-1.115543329049486	-1.115552167162667	0.088381131806425
0.30	-1.124820189852668	-1.124831389857101	0.112000044327765
0.40	-1.112289511111745	-1.112302209872475	0.126987607294016
0.50	-1.083819501666444	-1.083833096795016	0.135951285713531
0.60	-1.044032404380886	-1.044046484457675	0.140800767896021
0.70	-0.996547692160667	-0.996561983491016	0.142913303483594
0.80	-0.944179530121116	-0.944193856362354	0.143262412383161
0.90	-0.889096950996212	-0.889111202778624	0.142517824118782
1.00	-0.832953644755312	-0.832967757041102	0.141122857894027
1.10	-0.776992998955190	-0.777006934369571	0.139354143802173
1.20	-0.722132992403113	-0.722146729157781	0.137367546680922
1.30	-0.669034699340328	-0.669048222673370	0.135233330418316
1.40	-0.618157470279495	-0.618170766574383	0.132962948879989
1.50	-0.569803290776147	-0.569816343709562	0.130529334144702
1.60	-0.524152357812641	-0.524165146027234	0.127882145929403
1.70	-0.481291536325010	-0.481304032237552	0.124959125412660
1.80	-0.441237050325756	-0.441249219770174	0.121694444185638
1.90	-0.403952511481491	-0.403964313955574	0.118024740827227
2.00	-0.369363182594154	-0.369374571932370	0.113893382154284

Plot



Q3a) Use RK Systems Algorithm to approximate the solutions of the following higher order differential eqns and compare the results to the actual solutions.

$$\begin{cases} y'' - 2y' + y = te^t - t & 0 \le t \le 1 \quad h = 0.1 \\ y(0) = 0 \quad y'(0) = 0 \end{cases}$$

Converting to first order equations:

Let $u_1 = y$

$u_2 = y'$

Then,

$$\begin{cases} u_1' = u_2 & 0 \le t \le 1 \quad h = 0.1 \\ u_2' = 2u_2 - u_1 + te^t - t \\ u_1(0) = 0, \quad u_1'(0) = 0 \end{cases}$$

Solving the above system of equations, we get:

Approximate values

t_i	$w_1(t_i)$	$w_2(t_i)$
0.0	0	0
0.1	0.000008972441866	0.000363919772210
0.2	0.000153519452906	0.003178822155658
0.3	0.000834268349890	0.011719085346594
0.4	0.002832054800946	0.030353253368885
0.5	0.007429677753379	0.064798388161892
0.6	0.016561489531559	0.122423744846457
0.7	0.032996170727492	0.212612330129245
0.8	0.060558962737314	0.347190281155790
0.9	0.104400702010302	0.540935589204793
1.0	0.171322241743389	0.812179520049142

Exact solution: $y(t_i)$

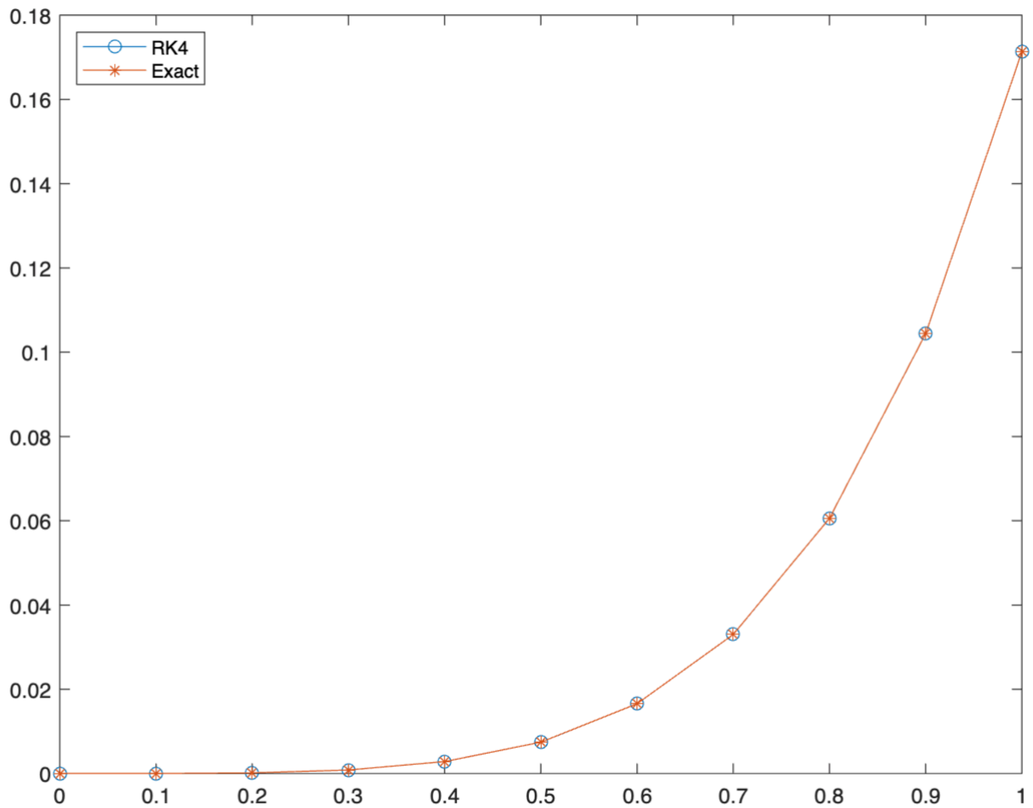
t_i	$y(t_i)$
0.0	0
0.10	0.000008939496743
0.20	0.000153501699186
0.30	0.000834337513298
0.40	0.002832313000873
0.50	0.007430265856445
0.60	0.016562597360771
0.70	0.032998049488682
0.80	0.060561940088985
0.90	0.104405200278214
1.00	0.171328799868887

Absolute error: |y(t_i) - w1(i)|

1.0e-05 *

0
0.003294512250546
0.001775372034490
0.006916340794375
0.025819992742612
0.058810306578436
0.110782921152255
0.187876119006614
0.297735167145013
0.449826791230901
0.655812549735235

Plots



RK4
Exact
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Q3b)

$$\begin{cases} t^2 y'' - 2ty' + 2y = t^3 \ln(t) & 1 \le t \le 2 \quad h = 0.1 \\ y(1) = 1, \quad y'(1) = 0 \end{cases}$$

Converting into a system of first-order equations, let

let $u_1 = y$, $u_2 = y'$

$u_1' = u_2$

$t^2 u_2' - 2t u_2 + 2u_1 = t^3 \ln(t)$

$$\Rightarrow u_2' = \frac{2t u_2 - 2u_1 + t^3 \ln(t)}{t^2} =$$

$$= \frac{2}{t} u_2 - \frac{2}{t^2} u_1 + t \ln(t)$$

$$\therefore \begin{cases} u_1' = u_2 & 1 \le t \le 2 \quad h = 0.1 \\ u_2' = \frac{2}{t} u_2 - \frac{2}{t^2} u_1 + t \ln(t) \\ u_1(1) = 1 \quad u_2(1) = 0 \end{cases}$$

Approximate values

t_i	$w_1(t_i)$	$w_2(t_i)$
1.00	1.000000000000000	0
1.10	0.990178177403956	-0.194513070021503
1.20	0.961524371865859	-0.376187257250166
1.30	0.915455017906110	-0.542408996255207
1.40	0.853637134398139	-0.690774486413968
1.50	0.777968968549979	-0.819058483116066
1.60	0.690563420521865	-0.925189578232002
1.70	0.593733688810971	-1.007230285955757
1.80	0.489980717565894	-1.063360763400368
1.90	0.381982126201937	-1.091865331540112
2.00	0.272582373136931	-1.091121188666557

Exact solution: $y(t_i)$

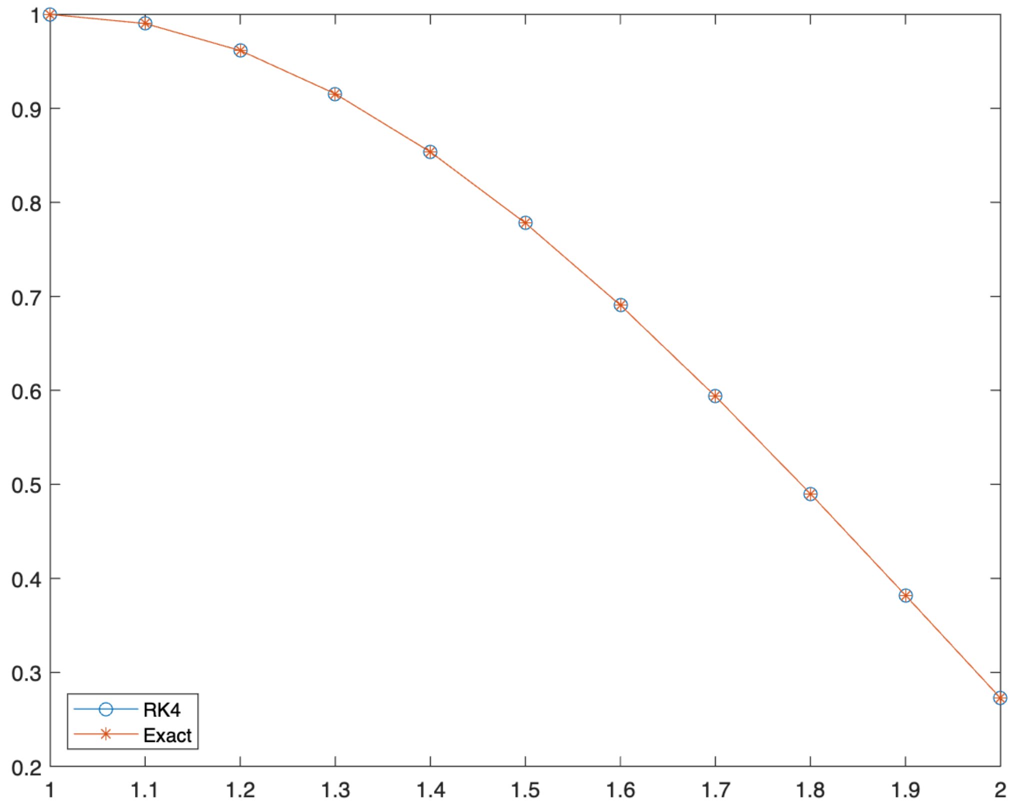
t_i	
1.00	1.000000000000000
1.10	0.990178924659778
1.20	0.961525825069977
1.30	0.915457144517539
1.40	0.853639908644304
1.50	0.777972369932527
1.60	0.690567432695266
1.70	0.593738298734221
1.80	0.489985914854579
1.90	0.381987902628227
2.00	0.272588722239782

Absolute error: |y(t_i) - w1(i)|

1.0e-05 *

0
0.074725582244195
0.145320411837346
0.212661142895421
0.277424616512700
0.340138254828482
0.401217340073590
0.460992324990972
0.519728868420000
0.577642628996067
0.634910285024493

Plots



Ch 11.1 Q 2a)

BVP $\begin{cases} y'' = y' + 2y + \cos x & 0 \le x \le \frac{\pi}{2} \\ y(0) = -0.3, \quad y(\pi/2) = -0.1 \end{cases}$ $\quad h = \pi/4$

This is in the form:

$$\begin{cases} y''(x) = p(x)\,y' + q(x)\,y + r(x) & a \le x \le b \\ y(a) = \alpha \qquad y(b) = \beta \end{cases}$$

For this case: $p(x) = 1$
$q(x) = 2$
$r(x) = \cos(x)$
$\alpha = -0.3$
$\beta = -0.1$

We know that a solution for an eqn in this form can be written as:

$$y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} \cdot y_2(x)$$

where $y_1(x)$ is the solution for IVP

$$(*) \begin{cases} y_1'' = p(x)\,y_1' + q(x)\,y_1 + r(x) & a \le x \le b \\ y_1(a) = \alpha, \quad y_1'(a) = 0 \end{cases}$$

and $y_2(x)$ is the solution for IVP

$$(**) \begin{cases} y_2'' = p(x)\,y_2' + q(x)\,y_2 & a \le x \le b \\ y_2(a) = 0 \qquad y_2'(a) = 1 \end{cases}$$

Let $u_1 = y_1$ and $u_2 = y_1'$. Then, rewriting $(*)$ we get:

$u_1'(x) = u_2$
$u_2'(x) = p(x)\,y_1' + q(x)\,y_1 + r(x)$
$= p(x)u_2 + q(x)\,u_1 + r(x)$
$= 1 \cdot u_2 + 2 \cdot u_1 + \cos(x)$ // sub $p(x), q(x), r(x)$

$y_1(a) = \alpha \Rightarrow u_1(a) = \alpha \Rightarrow u_1(0) = -0.3$
$y_1'(a) = 0 \Rightarrow u_2(a) = 0$

$\therefore$ Our first system of equations:

$$Ⓐ \begin{cases} u_1' = u_2 & 0 \le x \le \pi/2 \\ u_2' = u_2 + 2u_1 + \cos(x) \\ u_1(0) = -0.3, \quad u_2(0) = 0 \end{cases}$$

Letting $v_1 = y_2$ and $v_2 = y_2'$, we rewrite $(**)$ to get:

$v_1'(x) = v_2$
$v_2'(x) = p(x)y_2' + q(x)\,y_2$
$= p(x)\,v_2 + q(x)\,v_1 \quad = 1 \cdot v_2 + 2 \cdot v_1$ // sub $p(x), q(x)$

$y_2(a) = 0 \Rightarrow v_1(0) = 0$
$y_2'(a) = 1 \Rightarrow v_2(0) = 1$

So our second system of IVPs is the following:

$$Ⓑ \begin{cases} v_1' = v_2 & 0 \le x \le \pi/2 \\ v_2' = v_2 + 2v_1 \\ v_1(0) = 0, \quad v_2(0) = 1 \end{cases}$$

Using RK4, we solve (A). We get:

Note: t and x represent some variable

```
>> elevenpt1_q2
 First row: approximations for u1(t), second row: approximations for u2(t):
   -0.3000   -0.1440    0.6146
         0    0.4631    1.7476
```

Since we set $u_1 = y_1$, the first row yields, $w_{1,i}$, which are approximations for $y_1(x)$.

Similarly, using RK4, we solve (B). We get:

```
First row: approximations for v1(t), second row: approximations for v2(t):
         0    1.4153    7.3063
    1.0000    3.2888   14.8225
```

Since we set $v_1 = y_2$, the first row yields, $w_{2,i}$, which are approximations for $y_2(x)$

Using $w_{1,i}$ and $w_{2,i}$,

$$w_i = w_{1,i} + \frac{(-0.1) - w_{1,\pi/2}}{w_{2,\pi/2}} \cdot w_{2,i} \qquad \text{for all } i$$

// from: $y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} \cdot y_2(x)$

We get the following results for w_i and $y(t_i)$:

```
w_i : approximations of y(t)
  -0.300000000000000  -0.282452219299731  -0.100000000000000
```

for $t_i = 0.0, \pi/4, \pi/2$ respectively

```
Exact solution, y(ti)
  -0.300000000000000  -0.282842712474619  -0.100000000000000
```

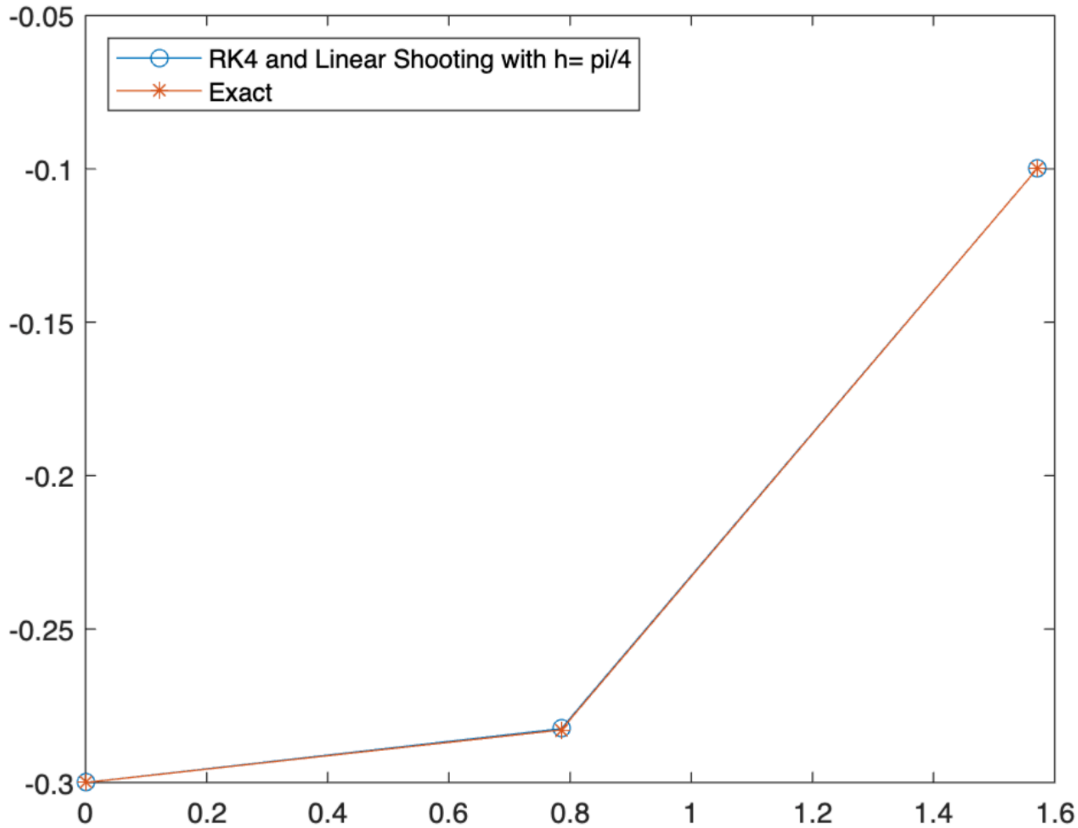
for $t_i = 0.0, \pi/4, \pi/2$ respectively

The absolute error:

```
Absolute error, | y(ti) - w_i |
   1.0e-03 *

   0.000000000000056   0.390493174887985   0.000000000000056
```

for $t_i = 0.0, \pi/4, \pi/2$ respectively

Plot



RK4 and Linear Shooting with h= pi/4
Exact
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6

11.1.2b) Repeating the above for $h = \pi/8$, we get:

```
>> elevenpt1_q2b
 First row: approximations for u1(t), second row: approximations for u2(t):
  -0.300000000000000  -0.265015820806410  -0.138394842307134   0.132162811997705   0.658835039819961
                   0   0.190724667733125   0.475823035650440   0.948282025156352   1.838342659061927

 First row: approximations for v1(t), second row: approximations for v2(t):
                   0   0.505039415729032   1.447305628868730   3.400528284739367   7.604129778664369
   1.000000000000000   1.685383741634555   3.350647979445936   7.109020406869258  15.416229048875353
```

```
w_i : approximations of y(t)
  -0.300000000000000  -0.315414961313018  -0.282825073836818  -0.207184370581857  -0.100000000000000
```

for $t_i = 0.0, \pi/8, \pi/4, 3\pi/8, \pi/2$ respectively

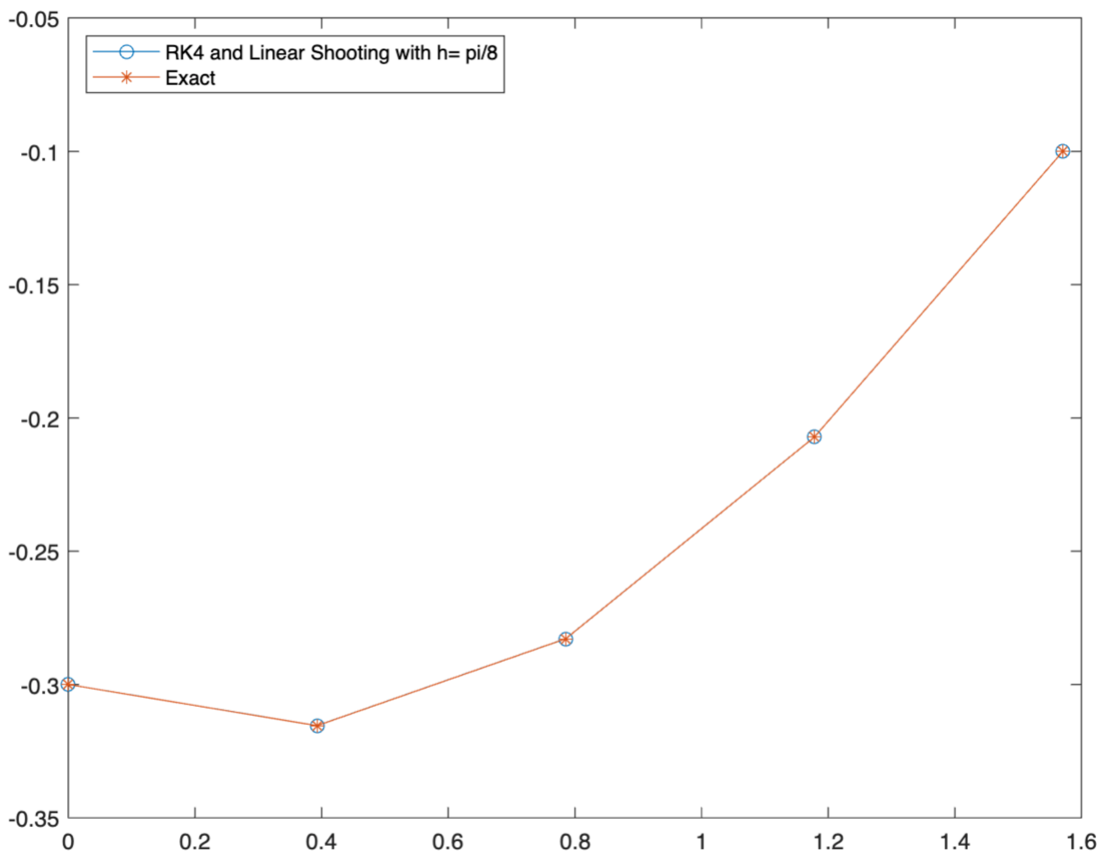
```
Exact solution, y(ti)
  -0.300000000000000  -0.315432202989895  -0.282842712474619  -0.207192982960656  -0.100000000000000
```

for $t_i = 0.0, \pi/8, \pi/4, 3\pi/8, \pi/2$ respectively

```
Absolute error, | y(ti) - w_i |
   1.0e-04 *

   0.000000000000555   0.172416768768890   0.176386378015203   0.086123787981784   0.000000000000555
```

for $t_i = 0.0, \pi/8, \pi/4, 3\pi/8, \pi/2$ respectively

Plot:



Q 11.2.1

Use Non-linear Shooting Algorithm with h=0.5 to approximate the solution to the BVP:

$$y'' = -(y')^2 - y + \ln(x) \qquad 1 \le x \le 2 \qquad y(1) = 0 \quad , \; y(2) = \ln 2$$

Using the secant method and arbitrarily chosen initial guesses $t_1 = 0.2$, $t_2 = 0.4$,

```
>> elevenpt2_q1
For iter = 6, the approximations for y(t_i), w_i , is :
                    0   0.405498418269510   0.693145709600929
```

for t_i = 1.0, 1.5, 2 respectively

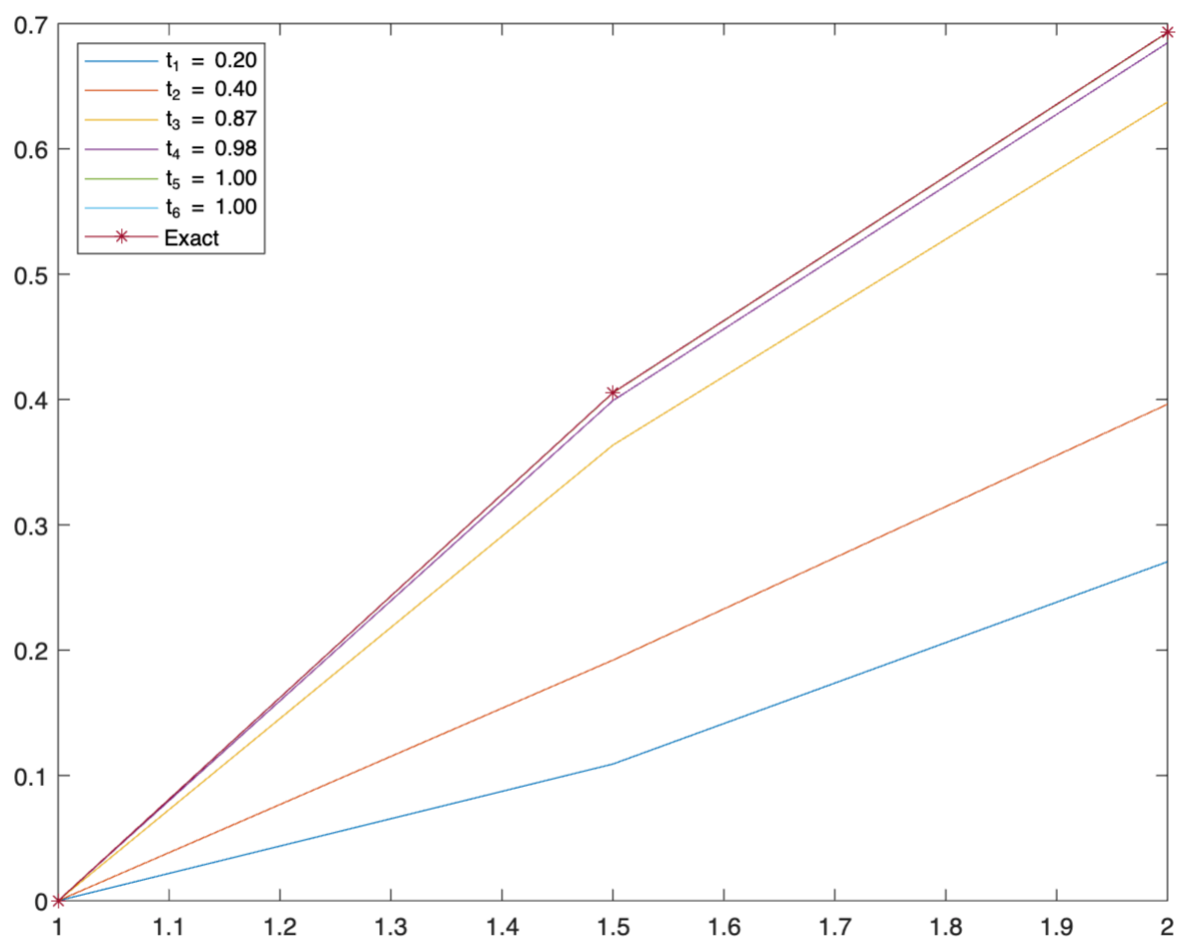
```
Exact solution y(ti):
                    0   0.405465108108164   0.693147180559945
```

for t_i = 1.0, 1.5, 2 respectively

```
Absolute error, | y(ti) - w_i |
   1.0e-04 *

                    0   0.333101613457232   0.014709590164808
```

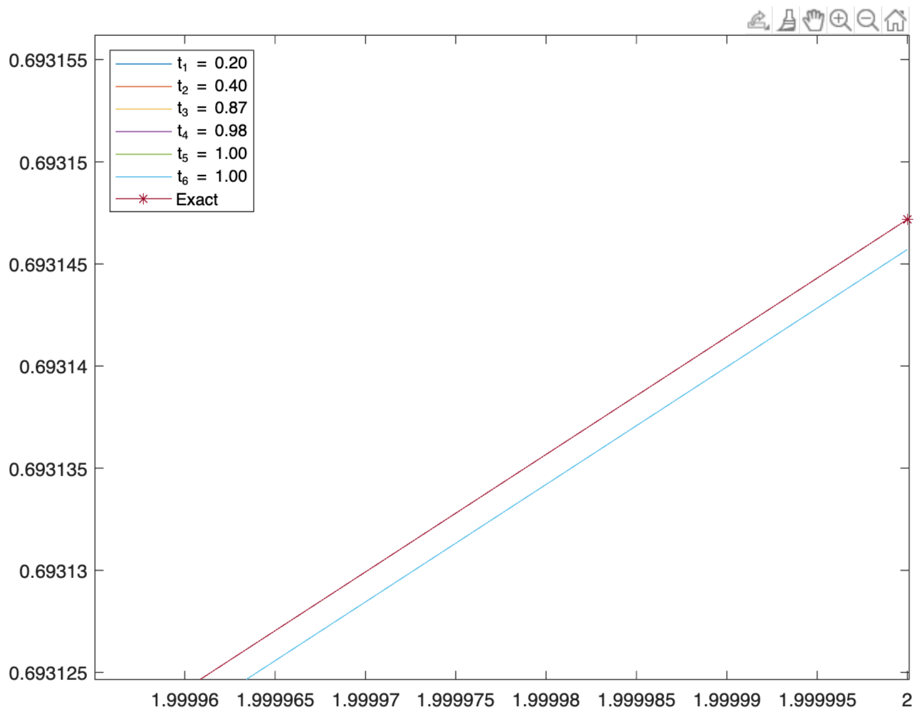
for t_i = 1.0, 1.5, 2 respectively

Plot:



Plot (zoomed in):

You can see that the exact solution is very close to the t_6 plot, meaning after 6 iterations we get very close to the real solution.



I repeated the approximation by conducting 10 iterations. From the results below, you can see that the t_{10} plot is closer to the exact solution than the t_6 plot. In fact, the plots are so close that even zooming in at max MATLAB capacity still doesn't enable you to differentiate the t_{10} and exact solution plot.

```
>> elevenpt2_q1
For iter = 10, the approximations for y(t_i), w_i , is :
                   0   0.405499535352558   0.693147180559945
```

for t_i = 1.0, 1.5, 2 respectively

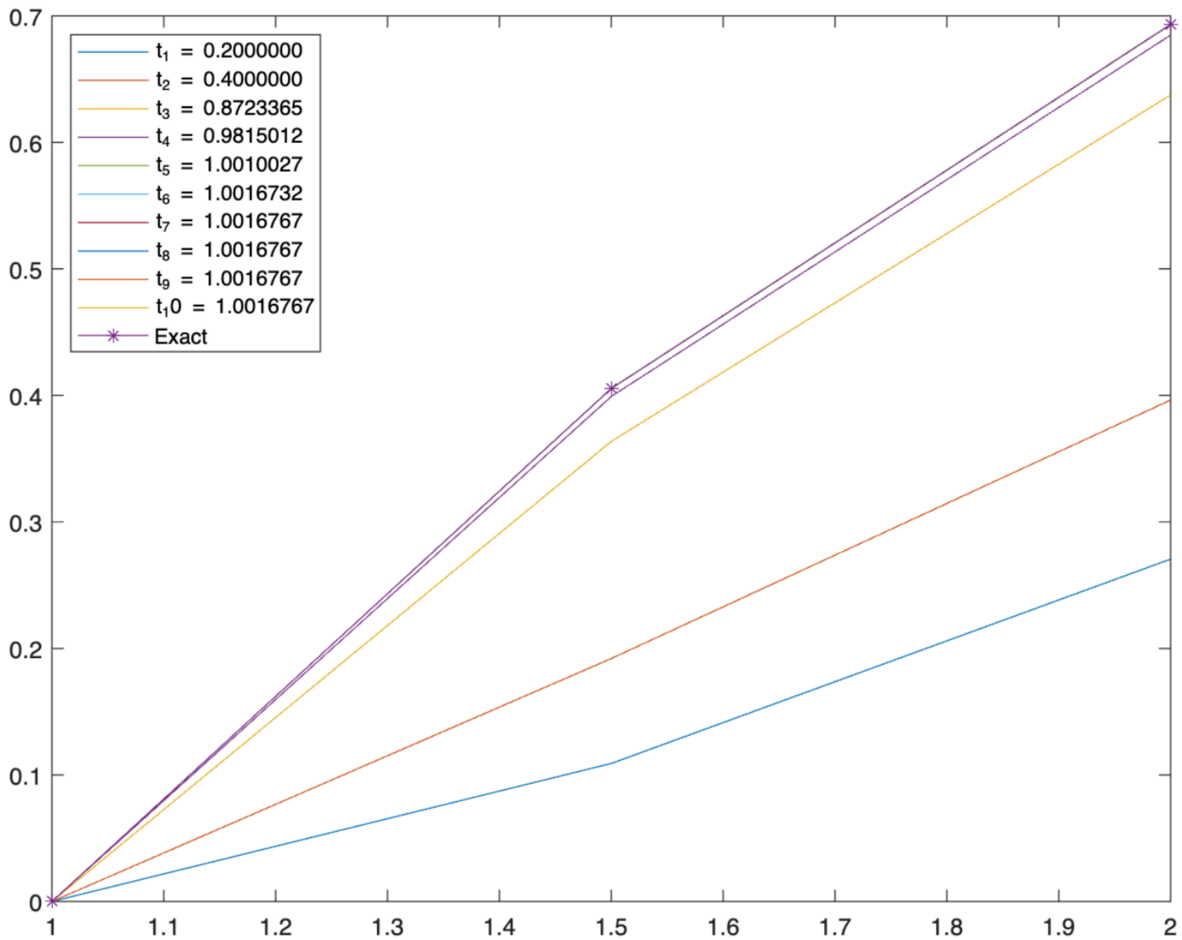
```
Exact solution y(ti):
                   0   0.405465108108164   0.693147180559945
```

for t_i = 1.0, 1.5, 2 respectively

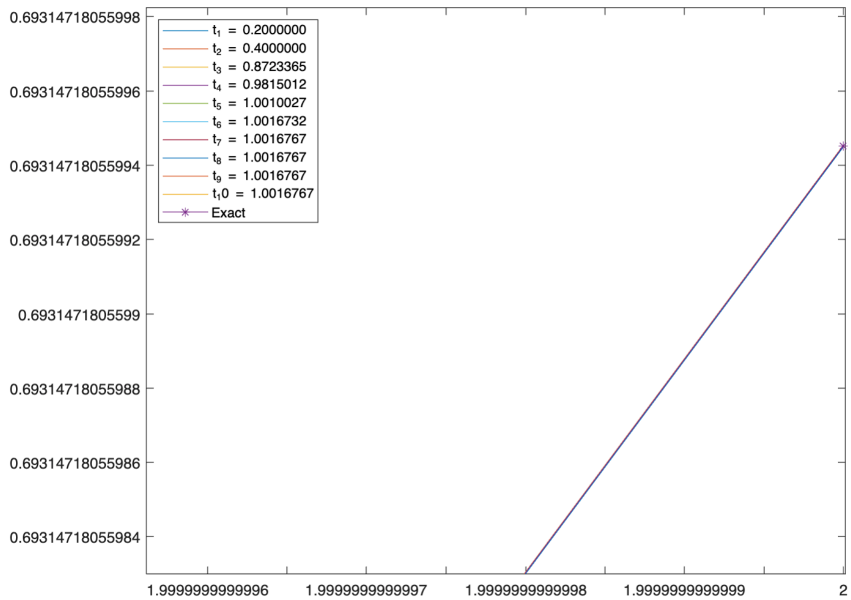
```
Absolute error, | y(ti) - w_i |
   1.0e-04 *

                   0   0.344272443938420                   0
```

for t_i = 1.0, 1.5, 2 respectively

Plot



Zoomed in at max capacity



Q 11.2.2

$y'' = 2y^3 \quad , \quad -1 \le x \le 0, \quad y(-1) = \frac{1}{2} \;, \; y(0) = \frac{1}{3} \;, \; h = 0.25$

Results:

```
>> elevenpt2_q2
For iter = 10, the approximations for y(t_i), w_i , is :
   0.500000000000000   0.444446336863980   0.400001939685661   0.363637538375934   0.333333333333333

Exact solution y(ti):
   0.500000000000000   0.444444444444444   0.400000000000000   0.363636363636364   0.333333333333333

Absolute error, | y(ti) - w_i |
   1.0e-05 *

                   0   0.189241953585384   0.193968566075675   0.117473956995351   0.000000000005551
```

for $t_i = -1.00, -0.75, -0.50, -0.25, 0$ respectively

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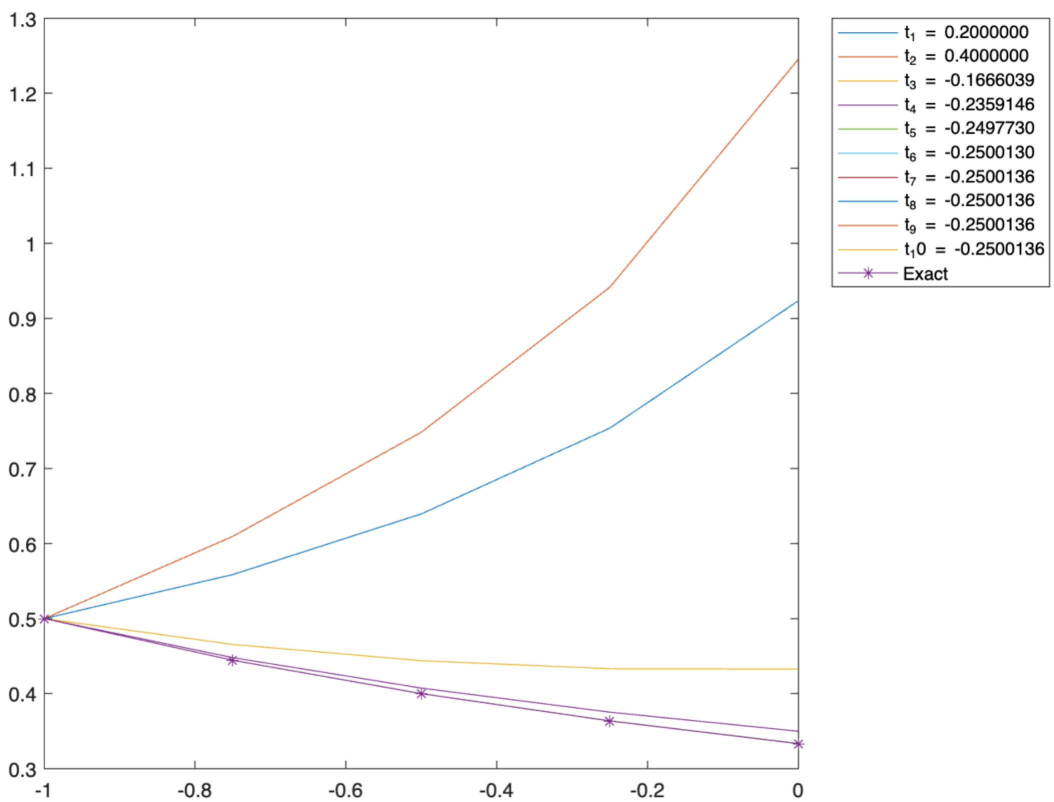

Plot (zoomed in):

How close the plots for t_{10} and the exact solution are is only apparent when incredibly zoomed in.

As you can see from the plot below, even zooming in so close only shows the t_7 and t_{10} plot.