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Math 151B, Hw 1
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Q 1(a) Show each of the following IVPs has a unique solution and find the solution.

$$\begin{cases} y' = y\cos t & 0 \le t \le 1 \\ y(0) = 1 \end{cases}$$

Proof:

We define domain $D \subset \mathbb{R}^2$ as the following:
domain $D = \{(t,y) \mid 0 \le t \le 1, -\infty < y < \infty\}$

Sub-claim: domain $D \subset \mathbb{R}^2$ is a convex set

Proof:

- suppose (t_1, y_1) and (t_2, y_2) belong in $D = \{(t,y) \mid 0 \le t \le 1, -\infty < y < \infty\}$

$\Rightarrow \quad 0 \le t_1 \le 1, \quad 0 \le t_2 \le 1$

- choose $\lambda \in [0,1]$

If,

$$0 \le t_1 \le 1$$

$$\Rightarrow 0 \cdot (1-\lambda) \le (1-\lambda)t_1 \le (1-\lambda) \cdot 1$$

$$\Leftrightarrow \quad 0 \le (1-\lambda)t_1 \le (1-\lambda) \qquad (1)$$

If,

$$0 \le t_2 \le 1$$

$$\Rightarrow \lambda \cdot 0 \le \lambda \cdot t_2 \le \lambda \cdot 1$$

$$\Leftrightarrow \quad 0 \le \lambda \cdot t_2 \le \lambda \qquad (2)$$

By (1) and (2),

$$0 + 0 \le (1-\lambda)t_1 + \lambda t_2 \le (1-\lambda) + \lambda$$

$$\Leftrightarrow \quad 0 \le (1-\lambda)t_1 + \lambda t_2 \le 1 \qquad (3)$$

If,

$$-\infty < y_1 < \infty$$

$$\Rightarrow -\infty(1-\lambda) < (1-\lambda)y_1 < \infty(1-\lambda) \qquad (4)$$

If, $\quad -\infty < y_2 < \infty$

$$\Rightarrow -\infty(\lambda) < \lambda y_2 < \lambda(\infty) \qquad (5)$$

Thus, by (4) and (5)

$$(1-\lambda)(-\infty) + \lambda(-\infty) < (1-\lambda)y_1 + \lambda y_2 < (1-\lambda)\infty + \lambda(\infty)$$

$$\Rightarrow \quad -\infty < (1-\lambda)y_1 + \lambda y_2 < \infty \qquad (5)$$

Thus, by (3) and (5)

$\{((1-\lambda)t_1 + \lambda t_2, (1-\lambda)y_1 + \lambda y_2)\} \in D, \quad \forall \lambda \in [0,1], \quad \forall (t_1, y_1), (t_2, y_2) \in D$

$\therefore$ domain D is a convex set. ■

$y' = y\cos t$

$f(t,y) = y\cos t$

$$\left|\frac{\partial f}{\partial y}(t,y)\right| = \left|\frac{\partial}{\partial y}(y\cos t)\right| = |\cos t|$$

$\forall\, t \in \mathbb{R},\quad |\cos t| \leq 1$

$\Rightarrow \forall\, (t,y) \in D \subset \mathbb{R}^2,\ |\cos t| \leq 1$ and $1 > 0$.

$\therefore$ Since $D \subset \mathbb{R}^2$ convex set and $1 > 0$ s.t.

$$\left|\frac{\partial f}{\partial y}(t,y)\right| \leq 1 \qquad \forall\, (t,y) \in D$$

so by Thm from class, f is Lipschitz continuous in y with Lipschitz constant 1.

Since $f(t,y)$ is Lipschitz continuous in y on D $\Rightarrow$ $f(t,y)$ is continuous in y on D

Thus, f meets the conditions in Thm 5.4 in textbook,
and by Thm 5.4, the IVP

$$y'(t) = y\cos t \qquad 0 \leq t \leq 1, \qquad y(0) = 1$$

has a unique solution $y(t)$ for $0 \leq t \leq 1$.

Solution: $y' = y\cos t$

$\Rightarrow \dfrac{dy}{dt} = y\cos t$

$\Rightarrow \dfrac{1}{y}\,dy = \cos t\,dt$

$\Rightarrow \int \frac{1}{y}\,dy = \int \cos t\,dt$

$\Rightarrow \ln(y) = \sin(t) + C,$

$\Leftrightarrow \ln(y) = \sin(t) + \ln(C)$

$\Leftrightarrow \ln(y) - \ln(C) = \sin(t)$

$\ln\left(\dfrac{y}{C}\right) = \sin t$

$e^{\sin t} = \dfrac{y}{C}$

$$y = Ce^{\sin t}$$

$y(0) = 1 \quad \Rightarrow \quad 1 = Ce^{\sin(0)}$

$\Rightarrow C = 1$

$\therefore$ Exact solution: $\boxed{y = e^{\sin t}}$

Q3a). Does f satisfy the Lipschitz condition in y on $D = \{(t,y) \mid 0 \le t \le 1, -\infty < y < \infty\}$

· Can Thm 5.6 be used to show that the IVP $\begin{cases} y' = f(t,y) & 0 \le t \le 1 \\ y(0) = 1 \end{cases}$ is well-posed?

$$f(t,y) = t^2 y + 1$$

· Choose $(t, y_1), (t, y_2) \in D$.

$$|f(t,y_1) - f(t,y_2)| = |(t^2 y_1 + 1) - (t^2 y_2 + 1)|$$

$$= |t^2 y_1 - t^2 y_2|$$

$$= |t^2 (y_1 - y_2)|$$

$$= |t^2| \, |y_1 - y_2|$$

$$= t^2 |y_1 - y_2| \le 1 \cdot |y_1 - y_2| \qquad \text{since } 0 \le t \le 1$$

Thus, f satisfies the Lipschitz condition on D with Lipschitz constant $L = 1$. (1)

Since f satisfies the Lipschitz condition in y on D, and $f(t,y)$ is a polynomial function and we know that polynomial functions are continuous everywhere (2)

$\Rightarrow f(t,y)$ is continuous on D.

$\therefore$ By (1) and (2), f satisfies the conditions of Thm 5.6.

By applying Thm 5.6, the IVP

$$y' = t^2 y + 1 \quad 0 \le t \le 1, \quad y(0) = 1$$

is well-posed.

5.2

Q 1a) Use Euler's method to approximate solutions for the following IVP.

$y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0 \quad$ with $h = 0.5$

$N = \frac{b-a}{h}$ (# of steps)

$w_0 = \alpha$

$w_{i+1} = w_i + h \cdot f(t_i, w_i)$

$$N = \frac{1-0}{0.5} = 2 \text{ steps}$$

$w_0 = y(0) = 0$

$w_1 = w_0 + h \cdot f(t_0, w_0) = w_0 + h \cdot (t_0 e^{3t_0} - 2w_0)$

$= 0 + 0.5(0 \cdot e^{3 \cdot 0} - 2 \cdot 0)$

$$= 0 + 0.5(0)$$
$$= 0$$

$$\boxed{w_1 = 0}$$

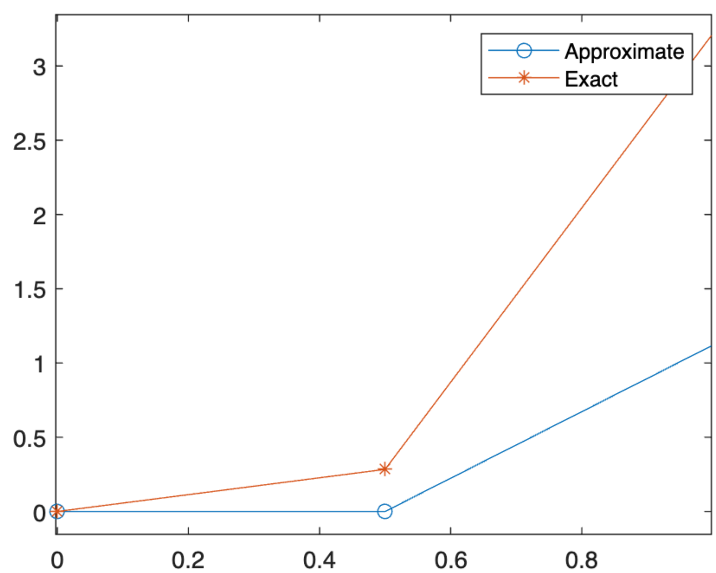
$t_1 = t_0 + h = 0 + 0.5 = 0.5$

$w_2 = w_{1+1} = w_1 + h \cdot f(t_1, w_1)$

$$= w_1 + h \cdot \left(t_1 e^{3t_1} - 2w_1\right)$$
$$= 0 + 0.5\left(0.5 e^{3(0.5)} - 2 \cdot 0\right)$$
$$= 0.5\left(0.5 e^{1.5}\right)$$

$$= (0.5^2)\, e^{1.5}$$

$$\boxed{w_2 = 0.5^2 e^{1.5} = 1.120422}$$



Q 3a) The actual solution to the IVP is given here. Compare the actual error at each step to the error bound.

$$y(t) = \frac{1}{5} t e^{3t} - \frac{1}{25} e^{3t} + \frac{1}{25} e^{-2t}$$

$$y''(t) = \frac{d}{dt}\, y'(t) = \frac{d}{dt}\left(f(t,y)\right)$$

$$= \left(\frac{\partial}{\partial t} f(t,y) + \frac{\partial}{\partial y} f(t,y) \cdot \frac{\partial y}{\partial t}\right)$$

$i = 1,\ t_i = t_1 = 0.5,\ w_i = w_1 = 0$

$i = 2,\ t_i = t_2 = 1,\ w_i = w_2 =$

Note:

$$|y''(t)| = \left| \frac{\partial}{\partial t} f(t,y) + \frac{\partial}{\partial y} f(t,y) \cdot \frac{\partial y}{\partial t} \right|$$

$$|y'(t)| = \left| \frac{1}{5}\left(e^{3t} + 3te^{3t}\right) - \frac{3}{25} e^{3t} - \frac{2}{25} e^{-2t} + 0 \cdot \frac{\partial y}{\partial t} \right|$$

$$\Rightarrow |y''(t)| = \left| \frac{3}{5} e^{3t} + \frac{3}{5}\left(e^{3t} + 3te^{3t}\right) - \frac{9}{25} e^{3t} + \frac{4}{25} e^{-2t} + 0 \cdot \frac{\partial y}{\partial t} \right|$$

$$= \left| \frac{3}{5}e^{3t} + \frac{3}{5}e^{3t} + \frac{9}{5}te^{3t} - \frac{9}{25}e^{3t} + \frac{4}{25}e^{-2t} \right|$$

$$= \left| e^{3t}\left(\frac{6}{5} + \frac{9}{5}t - \frac{9}{25}\right) + \frac{4}{25}e^{-2t} \right|$$

$$= \left| e^{3t}\left(\frac{21}{25} + \frac{9}{5}t\right) + \frac{4}{25}e^{-2t} \right|$$

$$\leq \left| e^{3}\left(\frac{21}{25} + \frac{9}{5}(1)\right) + \frac{4}{25}e^{-2} \right|$$

$$= \frac{66}{25}e^{3} + \frac{4}{25}e^{-2} = M \qquad t \in [0,1]$$

We know that $D = \{(t,y) \mid 0 \leq t \leq 1,\ -\infty < y < \infty\}$ is a convex set from proof in 5.2 Q 1a).

· choose $(t,y) \in D$.

$$\left| \frac{\partial}{\partial y}(f(t,y)) \right| = \left| \frac{\partial}{\partial y}(te^{3t} - 2y) \right| = |-2| = 2$$

$$= \left| \frac{\partial}{\partial y} f(t,y) \right| \leq 2 \qquad \forall (t,y) \in D$$

so by theorem, f is Lipschitz continuous in y on D with $L = 2$.

Since $f(t,y)$ is a sum of two continuous functions in $\mathbb{R} \Rightarrow f$ is continuous on $D \subset \mathbb{R}^2$

Thus, f meets all conditions for Thm 5.4 and by Thm 5.4, the IVP

$$\begin{cases} y'(t) = te^{3t} - 2y & 0 \leq t \leq 1 \\ y(0) = 0 \end{cases}$$

has a unique solution.

$\therefore$ By thm 5.9, the sequence $\{w_i\}_{i=0}^{2}$ generated by Euler's method satisfies $|y(t_i) - w_i| \leq \frac{h.M}{2L}\left(e^{L(t_i - \alpha)} - 1\right)$

$\therefore$ Error bound for $i=1$

$$|y(t_1) - w_1| \leq \frac{h.M}{2L}\left(e^{L(t_1 - 0)} - 1\right)$$

$$= \frac{(0.5)\left(\frac{66}{25}e^{3} + \frac{4}{25}e^{-2}\right)}{(2)(2)} \left(e^{2(0.5-0)} - 1\right)$$

$$= 11.3938$$

Error bound for $i=2$:

$$|y(t_2) - w_2| \leq \frac{(0.5)\left(\frac{66}{25}e^{3} + \frac{4}{25}e^{-2}\right)}{(2)(2)} \left(e^{2(1-0)} - 1\right)$$

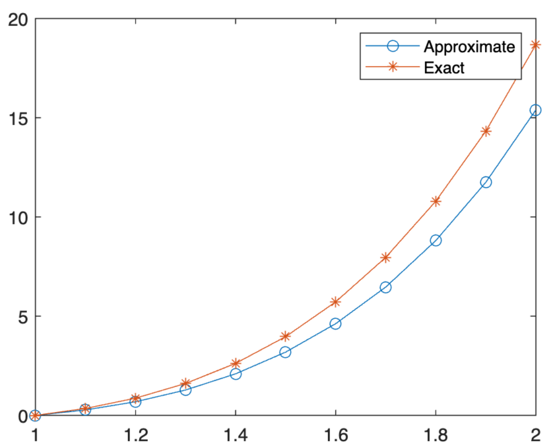
$$= 42.3654$$

<u>Exact errors</u>

$$|y(t_1) - w_1| = |y(0.5) - 0| = 0.28367 \leq 11.3938$$

$$|y(t_2) - w_2| = |y(1) - 0.5^2 e^{1.5}| = 2.09868 \leq 42.3654$$

Q9a)



Q9b) $y(1.04)$

Linear interpolation between two points (t_0, w_0) and (t_1, w_1)

$$y - y_1 = m(x - x_1) \qquad (t_0, w_0) = (1, 0) \qquad (t_1, w_1) = (1.1, 0.271828) \text{ // from plot}$$

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$y - w_0 = \left(\frac{w_1 - w_0}{t_1 - t_0}\right)(t - t_0)$$

$$y - 0 = \left(\frac{0.271828 - 0}{1.1 - 1}\right)(t - 1)$$

$$\Rightarrow y_{approx}(t) = 2.71828(t-1)$$

$$\Rightarrow y_{approx}(1.04) = 2.71828(1.04-1) = 0.1087312$$ //approximate

Actual: $y_{actual}(1.04) = (1.04)^2(e^{1.04} - e) = 0.119987$

$$|y_{actual}(1.04) - y_{approx}(1.04)| = 0.0112558$$

$y(1.55)$

$(1.5, 3.18745)$, $(1.6, 4.62082)$

$$y - 3.18745 = \left(\frac{4.62082 - 3.18745}{1.6 - 1.5}\right)(t-1.5)$$

$$\Rightarrow y_{approx}(t) = 14.3337(t-1.5) + 3.18745$$

$$\Rightarrow y_{approx}(1.55) = 14.3337(1.55-1.5) + 3.18745$$
$$= 3.904135$$

$$y_{act}(1.55) = (1.55^2)(e^{1.55} - e) = 4.78864$$

$$|y_{act}(1.55) - y_{approx}(1.55)| = 0.884505$$

$y(1.97)$

$(1.9, 11.748)$ $(2.0, 15.3982)$

$$y - 11.748 = \left(\frac{15.3982 - 11.748}{2.0 - 1.9}\right)(t-1.9)$$

$$y_{app}(t) = 36.502(t-1.9) + 11.748$$

$$y_{app}(1.97) = 14.30314$$

$$y_{act}(1.97) = 1.97^2(e^{1.97} - e) = 17.2793$$

$$|y_{act}(1.97) - y_{app}(1.97)| = 2.97616$$

Error bound

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left|e^{L(t_i - a)} - 1\right| = 0.1$$

$$\left|\frac{\partial}{\partial y} f(t,y)\right| = \left|\frac{\partial}{\partial y}\left(\frac{2}{t} y + t^2 e^t\right)\right|$$

$$= \left|\frac{2}{t}\right|$$

$$\leq 2 \qquad \text{for } t \in [1,2]$$

We know $D = \{(t,y) \mid 1 \leq t \leq 2, -\infty < y < \infty\}$ convex set

$\therefore$ by theorem, f is Lipschitz continuous in y on D with $\underline{L=2}$

$$|y''(t)| = \left|\frac{\partial}{\partial t} f(t,y) + \frac{\partial}{\partial y} f(t,y) \cdot \frac{\partial y}{\partial t}\right|$$

$$= \frac{d}{dt}\left| t^2 e^t + 2te^t - 2te + 0 \cdot \frac{\partial y}{\partial t}\right|$$

$$= \left|2t e^t + t^2 e^t + 2(e^t + te^t) - 2e + 0 \cdot \frac{\partial y}{\partial t}\right|$$

$$= \left| t^2 e^t + 4te^t + 2e^t - 2e \right|$$

$$\leq \left| (2)^2 e^2 + 4(2) e^2 + 2e^2 - 2e \right|$$

$$= 98.01022172\,8 \qquad = M$$

$$|y(t_i) - w_i| \leq \frac{hM}{2L} \left| e^{L(t_i - a)} - 1\right| = 0.1$$

$$= h\left(\frac{98.010221728}{2(2)}\right)\left| e^{2(2-1)} - 1\right| = 0.1$$

$$\Rightarrow \underline{\underline{h = 0.00063878}}$$

5.3 Q1a)

Taylor's method of order 2 to approximate solutions for the IVP:

$$\begin{cases} y'(t) = te^{3t} - 2y & 0 \le t \le 1 \\ y(0) = 0 \end{cases} \qquad h = 0.5$$

$$f'(t,y) = \frac{d}{dt}\left(y'(t)\right) = \frac{d}{dt}\left(te^{3t} - 2y\right) = 3te^{3t} + e^{3t} - 2y'$$

$$= 3te^{3t} + e^{3t} - 2(te^{3t} - 2y)$$

$$= te^{3t} + e^{3t} + 4y$$

$w_0 = \alpha$

$$w_{i+1} = w_i + h \cdot f(t_i, w_i) + \frac{h^2}{2} f'(t_i, w_i)$$

$t_0 = 0$, $w_0 = 0$

$t_i = h \cdot i \Rightarrow t_1 = 0.5$

$$w_1 = w_{0+1} = w_0 + h \cdot \left(t_0 e^{3t_0} - 2w_0\right) + \frac{h^2}{2}\left(t_0 e^{3t_0} + e^{3t_0} + 4w_0\right)$$

$$= 0 + 0.5\left(0 \cdot e^{3 \cdot 0} - 2 \cdot 0\right) + \frac{0.5^2}{2}\left(0 \cdot e^{3 \cdot 0} + e^{3 \cdot 0} + 4 \cdot 0\right)$$

$$= \frac{0.5^2}{2} \cdot (1)$$

$$= \frac{0.5^2}{2} = \underline{\underline{0.125}}$$

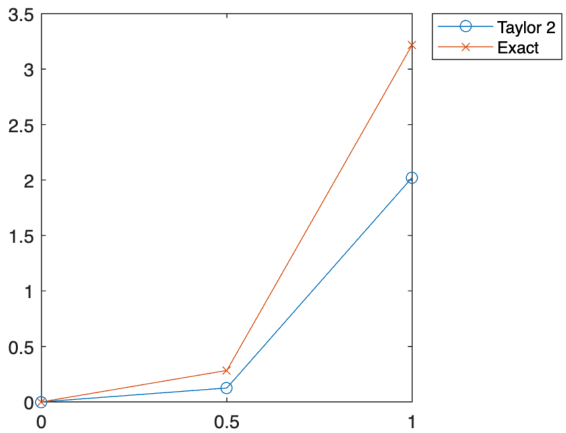
$t_2 = 0.5(2) = 1$

$$w_2 = w_1 + h\left(t_1 e^{3t_1} - 2w_1\right) + \frac{h^2}{2}\left(t_1 e^{3t_1} + e^{3t_1} + 4w_1\right)$$

$$= 0.125 + 0.5\left(0.5e^{3(0.5)} - 2(0.125)\right) + \frac{0.5^2}{2}\left(0.5e^{3(0.5)} + e^{3(0.5)} + 4(0.125)\right)$$

$$= 2.02324.$$

$$\boxed{\begin{array}{ll} t_0 = 0 & w_0 = 0 \\ t_1 = 0.5 & w_1 = 0.125 \\ t_2 = 1 & w_2 = 2.02324 \end{array}}$$



Q3a) Repeat with Taylor's method of order 4.

$$f''(t,y) = \frac{d}{dt}\left(te^{3t} + e^{3t} + 4y\right)$$

$$= 3te^{3t} + e^{3t} + 3e^{3t} + 4y'$$

$$= 3te^{3t} + 4e^{3t} + 4\left(te^{3t} - 2y\right)$$

$$= 3te^{3t} + 4e^{3t} + 4te^{3t} - 8y$$

$$= 7te^{3t} + 4e^{3t} - 8y$$

$$f'''(t,y) = \frac{d}{dt}\left(7te^{3t} + 4e^{3t} - 8y\right)$$

$$= \frac{d}{dt} \quad (7t+4)e^{3t} - 8y$$

$$= 3e^{3t}(7t+4) + 7e^{3t} - 8y'$$

$$= 3e^{3t}(7t+4) + 7e^{3t} - 8\left(te^{3t} - 2y\right)$$

$$= 3e^{3t}(7t+4) + 7e^{3t} - 8te^{3t} + 16y$$

$$= e^{3t}\left(21t + 12 + 7 - 8t\right) + 16y$$

$$= e^{3t}(13t + 19) + 16y$$

$t_0 = 0 \quad w_0 = 0$

$t_1 = 0.5 \quad w_1 = 0.257812$

$t_2 = 1 \quad w_2 =$

$$w_1 = w_0 + h \cdot f(t_0, w_0) + \frac{h^2}{2} f'(t_0, w_0) + \frac{h^3}{3!} f''(t_0, w_0) + \frac{h^4}{4!} f'''(t_0, w_0)$$

$$w_1 = 0.125 + \frac{h^3}{3!} f''(t_0, w_0) + \frac{h^4}{4!} f'''(t_0, w_0)$$

$$w_1 = 0.125 + \frac{0.5^3}{6}\left[(7(0)+4)e^{3\cdot 0} - 8\cdot 0\right] + \frac{0.5^4}{24}\left[((13)(0)+19)e^{3\cdot 0} + 16\cdot 0\right]$$

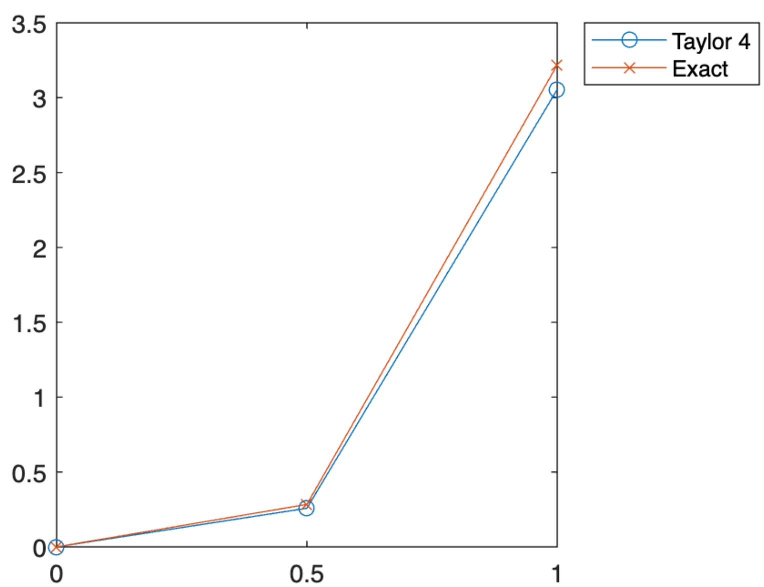
$$w_1 = 0.125 + \frac{0.5^3}{6}(4) + \frac{0.5^4}{24}(19)$$

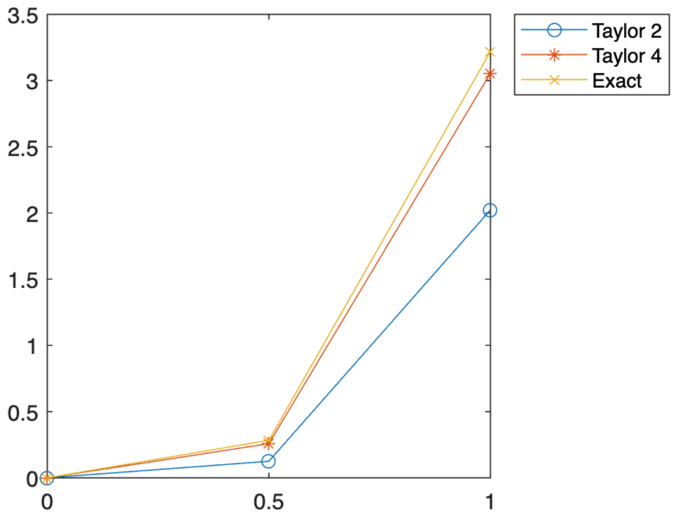
$$= 0.257812$$

$$w_2 = 0.2578 + \frac{0.5^3}{6}\left[(7(0.5)+4)e^{3\cdot(0.5)} - (8)(0.257812)\right]$$

$$+ \frac{0.5^4}{24}\left[(13(0.5)+19)e^{3\cdot(0.5)} + 16(0.257812)\right]$$

$$= 3.05529$$





Taylor 2
Taylor 4
Exact
3.5
3
2.5
2
1.5
1
0.5
0
0
0.5
1