# Moments of the univariate truncated normal distributions

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The PDF of the truncated normal distribution is[1]:

$$f\left(x;\mu,\sigma,a,b\right)=\begin{cases}\frac{1}{\sigma}\frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)} & a\leq x\leq b\\ 0 & \text{otherwise}\end{cases}$$

where $\mu,\sigma$ are the mean and standard deviation of the untruncated normal, and

$$\phi\left(x\right)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^{2}/2},\quad\Phi\left(x\right)=\frac{1}{2}\left(1+\mathrm{erf}\left(x/\sqrt{2}\right)\right)$$

The mean and variance of this distribution are:

$$\begin{aligned}\left\langle x\right\rangle &=\mu+\frac{\phi\left(\alpha\right)-\phi\left(\beta\right)}{\Phi\left(\beta\right)-\Phi\left(\alpha\right)}\sigma\\ \operatorname{var}x &=\left[1+\frac{\alpha\phi\left(\alpha\right)-\beta\phi\left(\beta\right)}{\Phi\left(\beta\right)-\Phi\left(\alpha\right)}-\left(\frac{\phi\left(\alpha\right)-\phi\left(\beta\right)}{\Phi\left(\beta\right)-\Phi\left(\alpha\right)}\right)^{2}\right]\sigma^{2}\end{aligned}$$

where

$$\alpha=\frac{a-\mu}{\sigma},\quad\beta=\frac{b-\mu}{\sigma}$$

The numerical evaluation of $\frac{\phi(\alpha)-\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}$ and $\frac{\alpha\phi(\alpha)-\beta\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}$ can be problematic. We describe our approach below.

## Numerical evaluation

Let us define the functions:

$$F_{1}\left(x,y\right)=\frac{\mathrm{e}^{-x^{2}}-\mathrm{e}^{-y^{2}}}{\mathrm{erf}\left(y\right)-\mathrm{erf}\left(x\right)},\quad F_{2}\left(x,y\right)=\frac{x\mathrm{e}^{-x^{2}}-y\mathrm{e}^{-y^{2}}}{\mathrm{erf}\left(y\right)-\mathrm{erf}\left(x\right)}$$

Observe that $F_{i}\left(x,y\right)=-F_{i}\left(y,x\right)$, $i=1,2$. Therefore without loss of generality we assume that $x\leq y$. The problem arises when $x,y$ have the same sign

[1] See https://en.wikipedia.org/wiki/Truncated_normal_distribution.

and are both large in absolute value. Then $\mathrm{erf}\left(y\right)-\mathrm{erf}\left(x\right)$ is the difference of two numbers close to $\pm 1$, but typically their difference itself is very small. Subtraction of two numbers that are both large relative to their difference is a classic source of floating-point errors (knwon as *catastrophic cancellation*). To deal with it, we propose to use the following equivalent expressions,

$$F_1\left(x,y\right)=\begin{cases}-F_1\left(y,x\right) & x>y\\ P_1\left(x,y-x\right) & y-x<10^{-5}\\ \mathrm{e}^{-x^2}\frac{1-\mathrm{e}^{x^2-y^2}}{\mathrm{erf}(y)-\mathrm{erf}(x)} & x\le 0\le y\\ \mathrm{e}^{-x^2}\frac{1-\mathrm{e}^{x^2-y^2}}{\mathrm{erfc}(x)-\mathrm{erfc}(y)} & x<y<0\\ \frac{1-\mathrm{e}^{x^2-y^2}}{\mathrm{erfcx}(x)-\mathrm{e}^{x^2-y^2}\mathrm{erfcx}(y)} & 0<x<y\end{cases}$$

$$F_2\left(x,y\right)=\begin{cases}-F_2\left(y,x\right) & x>y\\ P_2\left(x,y-x\right) & y-x<10^{-5}\\ \mathrm{e}^{-x^2}\frac{x-y\mathrm{e}^{x^2-y^2}}{\mathrm{erf}(y)-\mathrm{erf}(x)} & x\le 0\le y\\ \mathrm{e}^{-x^2}\frac{x-y\mathrm{e}^{x^2-y^2}}{\mathrm{erfc}(x)-\mathrm{erfc}(y)} & x<y<0\\ \frac{x-y\mathrm{e}^{x^2-y^2}}{\mathrm{erfcx}(x)-\mathrm{e}^{x^2-y^2}\mathrm{erfcx}(y)} & 0<x<y\end{cases}$$

The functions $\mathrm{erf}\left(x\right)$, $\mathrm{erfc}\left(x\right)$, $\mathrm{erfcx}\left(x\right)$ are all available in numerical packages. When $x\approx y$, we employ a Taylor expansion of $F_i\left(x,x+\epsilon\right)$ in powers of $\epsilon$,

$$P_1\left(x,\epsilon\right)=\sqrt{\pi}x+\frac{1}{2}\sqrt{\pi}\epsilon-\frac{1}{6}\sqrt{\pi}x\epsilon^2-\frac{1}{12}\sqrt{\pi}\epsilon^3+\frac{1}{90}\sqrt{\pi}x\left(x^2+1\right)\epsilon^4$$
$$P_2\left(x,\epsilon\right)=\frac{1}{2}\sqrt{\pi}\left(2x^2-1\right)+\sqrt{\pi}x\epsilon-\frac{1}{3}\sqrt{\pi}\left(x^2-1\right)\epsilon^2-\frac{1}{3}\sqrt{\pi}x\epsilon^3+\frac{1}{90}\sqrt{\pi}\left(2x^4+3x^2-8\right)\epsilon^4$$