Moments of the univariate truncated normal distribution

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The PDF of the truncated normal distribution is[1]:

$$f(x;\mu,\sigma,a,b) = \begin{cases} \frac{1}{\sigma}\frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

where μ, σ are the mean and standard deviation of the untruncated normal, and

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}, \quad \Phi(x) = \frac{1}{2}\left(1 + \operatorname{erf}\left(x/\sqrt{2}\right)\right)$$

The mean and variance of this distribution are:

$$\begin{aligned} \langle x \rangle &= \mu + \frac{\phi(\alpha)-\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}\sigma \\ \operatorname{var} x &= \left[1 + \frac{\alpha\phi(\alpha)-\beta\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)} - \left(\frac{\phi(\alpha)-\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}\right)^2\right]\sigma^2 \end{aligned}$$

where

$$\alpha = \frac{a-\mu}{\sigma}, \quad \beta = \frac{b-\mu}{\sigma}$$

The numerical evaluation of $\frac{\phi(\alpha)-\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}$ and $\frac{\alpha\phi(\alpha)-\beta\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}$ can be problematic. We describe our approach below.

Numerical evaluation

Let us define the functions:

$$F_1(x,y) = \frac{\mathrm{e}^{-x^2}-\mathrm{e}^{-y^2}}{\operatorname{erf}(y)-\operatorname{erf}(x)}, \quad F_2(x,y) = \frac{x\mathrm{e}^{-x^2}-y\mathrm{e}^{-y^2}}{\operatorname{erf}(y)-\operatorname{erf}(x)} \tag{1}$$

The problem arises when x, y have the same sign and both are large in magnitude. Then $\operatorname{erf}(y) - \operatorname{erf}(x)$ is the difference of two numbers close to ± 1, but

[1] See https://en.wikipedia.org/wiki/Truncated_normal_distribution.

their difference is very small. Subtraction of two numbers that are both large relative to their difference is a classic source of floating-point errors (known as *catastrophic cancellation*).

Recall a few useful properties of the error and related functions:

$$\mathrm{erf}\,(-x) = \mathrm{erf}\,(x)\,, \quad \mathrm{erfc}\,(x) = 1 - \mathrm{erf}\,(x) = \mathrm{e}^{-x^2}\mathrm{erfcx}\,(x)$$

The functions $\mathrm{erf}\,(x)$, $\mathrm{erfc}\,(x)$, $\mathrm{erfcx}\,(x)$ are all available in numerical packages. Using these, we can make sure that the error function difference is never catastrophic. Let $\Delta = \mathrm{e}^{x^2-y^2}$. We propose:

$$\begin{array}{lll}
F_1\,(x,y) & F_2\,(x,y) & \\
= F_1\,(y,x)\,, & = F_2\,(y,x)\,, & \text{if } |x|>|y| \\
= P_1\,(x,y-x)\,, & = P_2\,(x,y-x)\,, & \text{if } |x-y|<10^{-7} \\
= \dfrac{1-\Delta}{\Delta\mathrm{erfcx}\,(-y)-\mathrm{erfcx}\,(-x)} & = \dfrac{x-y\Delta}{\Delta\mathrm{erfcx}\,(-y)-\mathrm{erfcx}\,(-x)} & \text{if } x,y\le 0 \\
= \dfrac{1-\Delta}{\mathrm{erfcx}\,(x)-\Delta\mathrm{erfcx}\,(y)} & = \dfrac{x-y\Delta}{\mathrm{erfcx}\,(x)-\Delta\mathrm{erfcx}\,(y)} & \text{if } x,y\ge 0 \\
= \dfrac{(1-\Delta)\,\mathrm{e}^{-x^2}}{\mathrm{erf}\,(y)-\mathrm{erf}\,(x)} & = \dfrac{(x-y\Delta)\,\mathrm{e}^{-x^2}}{\mathrm{erf}\,(y)-\mathrm{erf}\,(x)} & \text{otherwise}
\end{array}$$

When $x \approx y$, we employ a Taylor expansion of $F_i\,(x, x+\epsilon)$ in powers of $\epsilon = y - x$ to fourth-order,

$$P_1\,(x,\epsilon) = \sqrt{\pi}x + \frac{1}{2}\sqrt{\pi}\epsilon - \frac{1}{6}\sqrt{\pi}x\epsilon^2 - \frac{1}{12}\sqrt{\pi}\epsilon^3 + \frac{1}{90}\sqrt{\pi}x\left(x^2+1\right)\epsilon^4$$
$$P_2\,(x,\epsilon) = \frac{1}{2}\sqrt{\pi}\left(2x^2-1\right) + \sqrt{\pi}x\epsilon - \frac{1}{3}\sqrt{\pi}\left(x^2-1\right)\epsilon^2 - \frac{1}{3}\sqrt{\pi}x\epsilon^3 + \frac{1}{90}\sqrt{\pi}\left(2x^4+3x^2-8\right)\epsilon^4$$

Infinite arguments

The limit of Eq. (1) as $y \to \pm\infty$ is a well-defined function of x,

$$F_1\,(x,\pm\infty) = \frac{\pm 1}{\mathrm{erfcx}\,(\pm x)}, \qquad F_2\,(x,\pm\infty) = \frac{\pm x}{\mathrm{erfcx}\,(\pm x)} \tag{2}$$

We define $F_i\,(\infty,\pm\infty) = \lim_{x\to\infty} F_i\,(x,\pm\infty)$ and $F_i\,(-\infty,\pm\infty) = \lim_{x\to-\infty} F_i\,(x,\pm\infty)$. Consistent with these definitions, we use Eq. (2) to evaluate $F_1\,(x,\pm\infty)$ for all x and $F_2\,(x,\pm\infty)$ for all $-\infty < x < \infty$. Finally, $F_2\,(-\infty,\infty) = F_2\,(\infty,-\infty) = 0$, while $F_2\,(\infty,\infty) = F_2\,(-\infty,-\infty) = \infty$.

Mean and variance

From $F_i\,(x,y)$, we evaluate the mean and variance of the truncated normal distribution as follows:

$$\frac{\phi(\alpha)-\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}=\sqrt{\frac{2}{\pi}}F_1\left(\frac{\alpha}{\sqrt{2}},\frac{\beta}{\sqrt{2}}\right),\quad \frac{\alpha\phi(\alpha)-\beta\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}=\frac{2}{\sqrt{\pi}}F_2\left(\frac{\alpha}{\sqrt{2}},\frac{\beta}{\sqrt{2}}\right)$$

$$\frac{\alpha\phi(\alpha)-\beta\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}-\left(\frac{\phi(\alpha)-\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}\right)^2=\frac{2}{\sqrt{\pi}}F_2\left(\frac{\alpha}{\sqrt{2}},\frac{\beta}{\sqrt{2}}\right)-\frac{2}{\pi}\left[F_1\left(\frac{\alpha}{\sqrt{2}},\frac{\beta}{\sqrt{2}}\right)\right]^2$$