Moments of the truncated bivariate Normal distribution

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Abstract

Miscellaneous notes about the computation of the moments of a truncated bivariate normal distribution.

1 Introduction

We consider a standardized normal bivariate distribution of the form:

$$\mathcal{N}(\mathbf{x};\rho)=\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{H(\mathbf{x})}{2(1-\rho^2)}\right),\quad H(\mathbf{x})=x_1^2+x_2^2-2\rho x_1 x_2$$

where $-1<\rho<1$ and $\mathbf{x}=(x_1,x_2)$. The "standardization" means that $\langle x_i\rangle_{\mathcal{N}}=0$ and $\langle x_i^2\rangle_{\mathcal{N}}=1$ for $i=1,2$, while $\rho=\langle x_1 x_2\rangle_{\mathcal{N}}$. Any normal bivariate distribution can be brought into this form by a translation and scaling of the variables x_1,x_2. Note that $\mathcal{H}(\mathbf{x})$ is concave everywhere because the Hessian

$$\nabla^2 H=\begin{bmatrix}2 & -2\rho\\ -2\rho & 2\end{bmatrix}$$

has the positive eigenvalues $2(1\pm\rho)$.

The levelsets of $\mathcal{N}(\mathbf{x};\rho)$ are the ellipses $H(\mathbf{x})=c$ for some constant $c\geq 0$. The axes of this ellipse are the lines $x_1\pm x_2=0$ with semi-lenghts $\sqrt{c/(1\pm\rho)}$, respectively (Fig. 1). Let $\mathfrak{f}=\sqrt{|\rho|\,c/(1-\rho^2)}$. The focci lie at a distance $\sqrt{2}\mathfrak{f}$ from the origin. If $\rho<0$ the focci are at the coordinates $x_1=-x_2=\pm\mathfrak{f}$, and if $\rho>0$ the focci are at the coordinates $x_1=x_2=\pm\mathfrak{f}$. The ellipse $H(\mathbf{x})=c$ is the locus of points whose distances to the focci add up to $2\sqrt{c/(1-|\rho|)}$.

Define a rectangular domain $\mathcal{D}$, with vertices at $\mathbf{a}=(a_1,a_2)$ and $\mathbf{b}=(b_1,b_2)$,

$$\mathcal{D}(\mathbf{a},\mathbf{b})=\left\{\mathbf{x}\in\mathbb{R}^2:\mathbf{a}\leq\mathbf{x}\leq\mathbf{b}\right\}$$

where the vector inequalities are component-wise and we assume that $\mathbf{a}\leq\mathbf{b}$. The bivariate density truncated to $\mathcal{D}$ is given by:

$$\mathcal{P}(\mathbf{x};\mathbf{a},\mathbf{b},\rho)=\frac{1}{Z(\mathbf{a},\mathbf{b},\rho)}\begin{cases}\mathcal{N}(\mathbf{x};\rho) & \mathbf{a}\leq\mathbf{x}\leq\mathbf{b}\\ 0 & \text{otherwise}\end{cases}$$

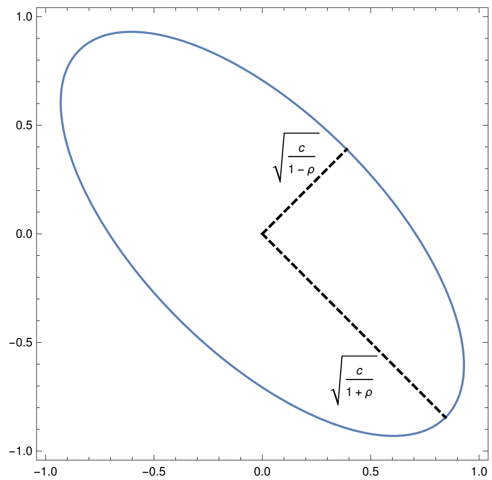

Figure 1: The levelsets $H(\mathbf{x}) = c$ are ellipses. The curve shown has $c = 0.5$, $\rho = -0.65$.

where Z is a normalization constant, given by:

$$Z(\mathbf{a}, \mathbf{b}, \rho) = \int_{a_1}^{b_1} \mathrm{d}x_1 \int_{a_2}^{b_2} \mathrm{d}x_2 \, \mathcal{N}(\mathbf{x}; \rho)$$

In this note we want to compute the moments:

$$\langle x_i \rangle_{\mathcal{P}} = \int_{a_1}^{b_1} \mathrm{d}x_1 \int_{a_2}^{b_2} \mathrm{d}x_2 \mathcal{P}(\mathbf{x}; \mathbf{a}, \mathbf{b}, \rho)\, x_i, \quad i = 1, 2$$

$$\langle x_i^2 \rangle_{\mathcal{P}} = \int_{a_1}^{b_1} \mathrm{d}x_1 \int_{a_2}^{b_2} \mathrm{d}x_2 \mathcal{P}(\mathbf{x}; \mathbf{a}, \mathbf{b}, \rho)\, x_i^2, \quad i = 1, 2$$

$$\langle x_1 x_2 \rangle_{\mathcal{P}} = \int_{a_1}^{b_1} \mathrm{d}x_1 \int_{a_2}^{b_2} \mathrm{d}x_2 \mathcal{P}(\mathbf{x}; \mathbf{a}, \mathbf{b}, \rho)\, x_1 x_2$$

2 General parameters

Consider now the general truncated Gaussian with a positive semi-definite covariance matrix Σ and means μ. The change of variables:

$$\xi_i = \frac{x_i - \mu_i}{\sqrt{\Sigma_{ii}}}$$

leads to the previously considered distribution, with non-truncated moments given by:

$$\langle \xi_i \rangle_{\mathcal{N}} = 0, \quad \langle \xi_i^2 \rangle_{\mathcal{N}} = 1, \quad i = 1, 2$$

$$\rho = \langle \xi_1 \xi_2 \rangle_{\mathcal{N}} = \frac{\Sigma_{12}}{\sqrt{\Sigma_{11} \Sigma_{22}}}$$

If the original variables x_i are bounded by $a_i \leq x_i \leq b_i$ then the ξ_i are bounded by $\alpha_i \leq \xi_i \leq \beta_i$, where

$$\alpha_i = \frac{a_i - \mu_i}{\sqrt{\Sigma_{ii}}}, \qquad \beta_i = \frac{b_i - \mu_i}{\sqrt{\Sigma_{ii}}}$$

Using the methods described below, we can compute the truncated moments of ξ_i. Then:

$$\begin{aligned}
\langle x_1^m x_2^n \rangle_{\mathcal{P}} &= \left\langle \left(\sqrt{\Sigma_{11}} \xi_1 + \mu_1 \right)^m \left(\sqrt{\Sigma_{11}} \xi_2 + \mu_2 \right)^n \right\rangle_{\mathcal{P}} \\
&= \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \mu_1^{m-i} \mu_2^{n-j} \Sigma_{11}^{i/2} \Sigma_{22}^{j/2} \left\langle \xi_1^i \xi_2^j \right\rangle_{\mathcal{P}}
\end{aligned}$$

Therefore it suffices to consider the truncated normal distribution with $\mu_i = 0$ and $\Sigma_{ii} = 1$, $\Sigma_{12} = \Sigma_{21} = \rho$.