Moments of the univariate truncated normal distribution

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The PDF of the truncated normal distribution is[1]:

$$f\left(x;\mu,\sigma,a,b\right)=\begin{cases}\frac{1}{\sigma}\frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)} & a\leq x\leq b\\ 0 & \text{otherwise}\end{cases}$$

where μ,σ are the mean and standard deviation of the untruncated normal, and

$$\phi\left(x\right)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^{2}/2},\quad\Phi\left(x\right)=\frac{1}{2}\left(1+\mathrm{erf}\left(x/\sqrt{2}\right)\right)$$

$$\left(x-a\right)^{2}=x^{2}-2ax+a^{2}$$

The mean and variance of this distribution are:

$$\begin{aligned}\left\langle x\right\rangle &=\mu+\frac{\phi\left(\alpha\right)-\phi\left(\beta\right)}{\Phi\left(\beta\right)-\Phi\left(\alpha\right)}\sigma\\ \mathrm{var}\,x &=\left[1+\frac{\alpha\phi\left(\alpha\right)-\beta\phi\left(\beta\right)}{\Phi\left(\beta\right)-\Phi\left(\alpha\right)}-\left(\frac{\phi\left(\alpha\right)-\phi\left(\beta\right)}{\Phi\left(\beta\right)-\Phi\left(\alpha\right)}\right)^{2}\right]\sigma^{2}\end{aligned}$$

where

$$\alpha=\frac{a-\mu}{\sigma},\quad\beta=\frac{b-\mu}{\sigma}$$

The numerical evaluation of $\frac{\phi(\alpha)-\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}$ and $\frac{\alpha\phi(\alpha)-\beta\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}$ can be problematic. We describe our approach below.

Numerical evaluation

Let us define the functions:

$$F_{1}\left(x,y\right)=\frac{\mathrm{e}^{-x^{2}}-\mathrm{e}^{-y^{2}}}{\mathrm{erf}\left(y\right)-\mathrm{erf}\left(x\right)},\quad F_{2}\left(x,y\right)=\frac{x\mathrm{e}^{-x^{2}}-y\mathrm{e}^{-y^{2}}}{\mathrm{erf}\left(y\right)-\mathrm{erf}\left(x\right)}$$

[1] See https://en.wikipedia.org/wiki/Truncated_normal_distribution.

The problem arises when x, y have the same sign and both are large in magnitude. Then $\mathrm{erf}(y) - \mathrm{erf}(x)$ is the difference of two numbers close to ± 1, but typically their difference itself is very small. Subtraction of two numbers that are both large relative to their difference is a classic source of floating-point errors (known as *catastrophic cancellation*).

Recall a few useful properties of the error and related functions:

$$\mathrm{erf}(-x) = \mathrm{erf}(x), \quad \mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \mathrm{e}^{-x^2}\mathrm{erfcx}(x)$$

The functions $\mathrm{erf}(x)$, $\mathrm{erfc}(x)$, $\mathrm{erfcx}(x)$ are all available in numerical packages. Using these, we can make sure that the error function difference is never catastrophic. Let $\epsilon = \mathrm{e}^{x^2-y^2}$. We propose:

$$\begin{array}{lll}
F_1(x,y) & F_2(x,y) & \\
= F_1(y,x), & = F_2(y,x), & \text{if } |x| > |y| \\
= P_1(x, y-x), & = P_2(x, y-x), & \text{if } |x-y| < 10^{-7} \\
= \dfrac{1-\epsilon}{\epsilon \mathrm{erfcx}(-y) - \mathrm{erfcx}(-x)} & = \dfrac{x - y\epsilon}{\epsilon \mathrm{erfcx}(-y) - \mathrm{erfcx}(-x)} & \text{if } x, y \le 0 \\
= \dfrac{1-\epsilon}{\mathrm{erfcx}(x) - \epsilon \mathrm{erfcx}(y)} & = \dfrac{x - y\epsilon}{\mathrm{erfcx}(x) - \epsilon \mathrm{erfcx}(y)} & \text{if } x, y \ge 0 \\
= \dfrac{(1-\epsilon)\mathrm{e}^{-x^2}}{\mathrm{erf}(y) - \mathrm{erf}(x)} & = \dfrac{(x - y\epsilon)\mathrm{e}^{-x^2}}{\mathrm{erf}(y) - \mathrm{erf}(x)} & \text{otherwise}
\end{array}$$

When $x \approx y$, we employ a Taylor expansion of $F_i(x, x+\epsilon)$ in powers of ϵ,

$$P_1(x,\epsilon) = \sqrt{\pi}x + \frac{1}{2}\sqrt{\pi}\epsilon - \frac{1}{6}\sqrt{\pi}x\epsilon^2 - \frac{1}{12}\sqrt{\pi}\epsilon^3 + \frac{1}{90}\sqrt{\pi}x\left(x^2+1\right)\epsilon^4$$
$$P_2(x,\epsilon) = \frac{1}{2}\sqrt{\pi}\left(2x^2-1\right) + \sqrt{\pi}x\epsilon - \frac{1}{3}\sqrt{\pi}\left(x^2-1\right)\epsilon^2 - \frac{1}{3}\sqrt{\pi}x\epsilon^3 + \frac{1}{90}\sqrt{\pi}\left(2x^4+3x^2-8\right)\epsilon^4$$

Mean and variance

Finally observe that,

$$\frac{\phi(\alpha) - \phi(\beta)}{\Phi(\beta) - \Phi(\alpha)} = \sqrt{\frac{2}{\pi}} F_1\left(\frac{\alpha}{\sqrt{2}}, \frac{\beta}{\sqrt{2}}\right), \quad \frac{\alpha\phi(\alpha) - \beta\phi(\beta)}{\Phi(\beta) - \Phi(\alpha)} = \frac{2}{\sqrt{\pi}} F_2\left(\frac{\alpha}{\sqrt{2}}, \frac{\beta}{\sqrt{2}}\right)$$

$$\frac{\alpha\phi(\alpha) - \beta\phi(\beta)}{\Phi(\beta) - \Phi(\alpha)} - \left(\frac{\phi(\alpha) - \phi(\beta)}{\Phi(\beta) - \Phi(\alpha)}\right)^2 = \frac{2}{\sqrt{\pi}} F_2\left(\frac{\alpha}{\sqrt{2}}, \frac{\beta}{\sqrt{2}}\right) - \frac{2}{\pi}\left[F_1\left(\frac{\alpha}{\sqrt{2}}, \frac{\beta}{\sqrt{2}}\right)\right]^2$$