Moments of the univariate truncated normal distribution

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The PDF of the truncated normal distribution is[1]:

$$f(x;\mu,\sigma,a,b)=\begin{cases}\frac{1}{\sigma}\frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)} & a\leq x\leq b\\ 0 & \text{otherwise}\end{cases}$$

where μ,σ are the mean and standard deviation of the untruncated normal, and

$$\phi(x)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2},\quad \Phi(x)=\frac{1}{2}\left(1+\operatorname{erf}\left(x/\sqrt{2}\right)\right)$$

The mean and variance of this distribution are:

$$\begin{aligned}\langle x\rangle &= \mu+\frac{\phi(\alpha)-\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}\sigma\\ \operatorname{var}x &= \left[1+\frac{\alpha\phi(\alpha)-\beta\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}-\left(\frac{\phi(\alpha)-\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}\right)^2\right]\sigma^2\end{aligned}$$

where

$$\alpha=\frac{a-\mu}{\sigma},\quad \beta=\frac{b-\mu}{\sigma}$$

The numerical evaluation of $\frac{\phi(\alpha)-\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}$ and $\frac{\alpha\phi(\alpha)-\beta\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}$ can be problematic. We describe our approach below.

Numerical evaluation

Let us define the functions:

$$F_1(x,y)=\frac{\mathrm{e}^{-x^2}-\mathrm{e}^{-y^2}}{\operatorname{erf}(y)-\operatorname{erf}(x)},\quad F_2(x,y)=\frac{x\mathrm{e}^{-x^2}-y\mathrm{e}^{-y^2}}{\operatorname{erf}(y)-\operatorname{erf}(x)}$$

The problem arises when x,y have the same sign and both are large in magnitude. Then $\operatorname{erf}(y)-\operatorname{erf}(x)$ is the difference of two numbers close to ± 1, but

[1] See https://en.wikipedia.org/wiki/Truncated_normal_distribution.

typically their difference itself is very small. Subtraction of two numbers that are both large relative to their difference is a classic source of floating-point errors (known as *catastrophic cancellation*).

Recall a few useful properties of the error and related functions:

$$\mathrm{erf}\left(-x\right)=\mathrm{erf}\left(x\right), \quad \mathrm{erfc}\left(x\right)=1-\mathrm{erf}\left(x\right)=\mathrm{e}^{-x^2}\mathrm{erfcx}\left(x\right)$$

The functions $\mathrm{erf}\left(x\right)$, $\mathrm{erfc}\left(x\right)$, $\mathrm{erfcx}\left(x\right)$ are all available in numerical packages. Using these, we can make sure that the error function difference is never catastrophic. Let $\epsilon=\mathrm{e}^{x^2-y^2}$. We propose:

$$\begin{array}{lll}
F_1\left(x,y\right) & F_2\left(x,y\right) & \\
=F_1\left(y,x\right), & =F_2\left(y,x\right), & \text{if } \left|x\right|>\left|y\right| \\
=P_1\left(x,y-x\right), & =P_2\left(x,y-x\right), & \text{if } \left|x-y\right|<10^{-7} \\
=\dfrac{1-\epsilon}{\epsilon\mathrm{erfcx}\left(-y\right)-\mathrm{erfcx}\left(-x\right)} & =\dfrac{x-y\epsilon}{\epsilon\mathrm{erfcx}\left(-y\right)-\mathrm{erfcx}\left(-x\right)} & \text{if } x,y\leq 0 \\
=\dfrac{1-\epsilon}{\mathrm{erfcx}\left(x\right)-\epsilon\mathrm{erfcx}\left(y\right)} & =\dfrac{x-y\epsilon}{\mathrm{erfcx}\left(x\right)-\epsilon\mathrm{erfcx}\left(y\right)} & \text{if } x,y\geq 0 \\
=\dfrac{\left(1-\epsilon\right)\mathrm{e}^{-x^2}}{\mathrm{erf}\left(y\right)-\mathrm{erf}\left(x\right)} & =\dfrac{\left(x-y\epsilon\right)\mathrm{e}^{-x^2}}{\mathrm{erf}\left(y\right)-\mathrm{erf}\left(x\right)} & \text{otherwise}
\end{array}$$

When $x\approx y$, we employ a Taylor expansion of $F_i\left(x,x+\epsilon\right)$ in powers of ϵ,

$$P_1\left(x,\epsilon\right)=\sqrt{\pi}x+\frac{1}{2}\sqrt{\pi}\epsilon-\frac{1}{6}\sqrt{\pi}x\epsilon^2-\frac{1}{12}\sqrt{\pi}\epsilon^3+\frac{1}{90}\sqrt{\pi}x\left(x^2+1\right)\epsilon^4$$

$$P_2\left(x,\epsilon\right)=\frac{1}{2}\sqrt{\pi}\left(2x^2-1\right)+\sqrt{\pi}x\epsilon-\frac{1}{3}\sqrt{\pi}\left(x^2-1\right)\epsilon^2-\frac{1}{3}\sqrt{\pi}x\epsilon^3+\frac{1}{90}\sqrt{\pi}\left(2x^4+3x^2-8\right)\epsilon^4$$

Mean and variance

Finally observe that,

$$\frac{\phi\left(\alpha\right)-\phi\left(\beta\right)}{\Phi\left(\beta\right)-\Phi\left(\alpha\right)}=\sqrt{\frac{2}{\pi}}F_1\left(\frac{\alpha}{\sqrt{2}},\frac{\beta}{\sqrt{2}}\right), \quad \frac{\alpha\phi\left(\alpha\right)-\beta\phi\left(\beta\right)}{\Phi\left(\beta\right)-\Phi\left(\alpha\right)}=\frac{2}{\sqrt{\pi}}F_2\left(\frac{\alpha}{\sqrt{2}},\frac{\beta}{\sqrt{2}}\right)$$

$$\frac{\alpha\phi\left(\alpha\right)-\beta\phi\left(\beta\right)}{\Phi\left(\beta\right)-\Phi\left(\alpha\right)}-\left(\frac{\phi\left(\alpha\right)-\phi\left(\beta\right)}{\Phi\left(\beta\right)-\Phi\left(\alpha\right)}\right)^2=\frac{2}{\sqrt{\pi}}F_2\left(\frac{\alpha}{\sqrt{2}},\frac{\beta}{\sqrt{2}}\right)-\frac{2}{\pi}\left[F_1\left(\frac{\alpha}{\sqrt{2}},\frac{\beta}{\sqrt{2}}\right)\right]^2$$