N-body simulation setup

what do we want to solve?

- we need to solve the equation of motion for each particle, i.e., find their positions at any time.
- For this, we have to solve $\vec{F} = m \cdot \vec{a}$
  - → with $\vec{a}$ we can calculate $\vec{v}$: $v = v_0 + a \cdot \Delta t$
  - → with $\vec{v}$ we can calculate $\vec{x}$: $x = x_0 + v \Delta t$
  - → where Δt is chosen
  - → alternatively, the position can be found with $x = x_0 + \frac{1}{2} a \Delta t^2$
- The detail is that $\vec{F} = -m \vec{\nabla} \phi$ (→ set $m = 1$)

  so the problem actually is to find ϕ ...

- To find ϕ, recall that the field caused by the particle masses is given by the poisson equation:

$$\vec{\nabla}^2 \phi = \underbrace{4\pi G}_{\text{set } = 1} \rho$$

Given ρ, we can calculate:

$$\frac{\partial \phi}{\partial x} = \frac{\phi(x + \Delta x) - \phi(x - \Delta x)}{2 \Delta x}$$

or use np.gradient(ϕ).

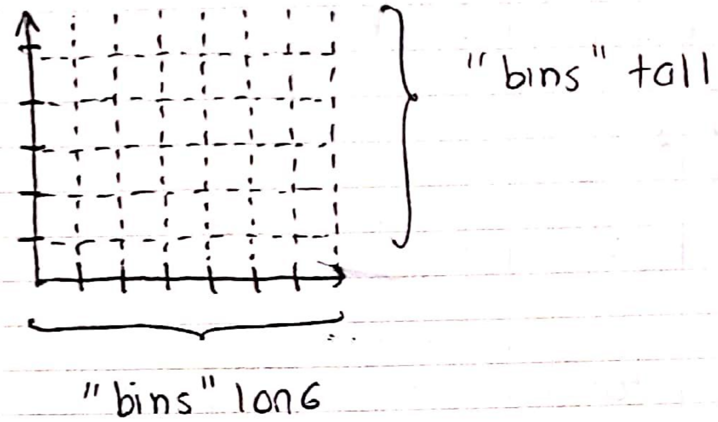
And do the same for the y-coordinate.

The next step is to find ρ, and here is where the coding comes!

- To find ρ :

1) set the mesh $\Rightarrow$ use np.histogram 2D

   → use the "bins" argument to set the division of the grid



   → it receives the arrays with the positions $\Rightarrow x, y$

   → it returns the "histogrammed" values of x and y (i.e., it averages all the positions ~~in~~ within one box) in <u>one</u> array

2) The density will be the sum of all elements in the resulting array from part 1), divided by the dimension of the grid :

$$\rho = \frac{\sum \text{elements in array}}{\text{bins}^2}$$

<u>when do we use FFT ?</u>

- we have ρ and we want to solve $\nabla^2 \phi = \rho$
- The trick is to use Green's functions

  → if we have an operator $\mathcal{L}$ acting over the Green's function y :

$$\mathcal{L}\, y = f$$

in our case: $\nabla^2 \quad \phi \quad \rho$

  → the solution to this equation is $y = \underset{\downarrow}{G} \otimes \rho$

Given depending on the operator $\mathcal{L}$

→ when the operator is $\mathcal{L} = \partial_x^2 + \partial_y^2$

the corresponding function is $\frac{1}{2\pi} \ln\left(\sqrt{x^2+y^2}\right) = G$

use the middle points of the grid

- so we need to calculate $\phi = G \otimes \rho$

and here is where we use FFT :

→ $\phi = F^{-1}(\hat{G} * \hat{\rho})$

with $\hat{G} = F(G)$ and $\hat{\rho} = F(\rho)$

This will allow us to calculate ϕ faster.

To summarize

1) set initial positions
2) set the mesh and its dimensions
3) calculate ρ
4) using the middle point of each box, calculate G
5) Take the FT of ρ and G, multiply them
6) Take the IFT of step 5). This is ϕ
7) Calculate the gradient of ϕ. Because $m=1$, the result of this step is $\vec{F}$.
8) Because $m=1$, $\vec{F}$ of step 7) is equal to $\vec{a}$
9) calculate the new velocity of the particle
10) use result in 9) to calculate the new position
11) update all positions and repeat.