
**http://algs4.cs.princeton.edu**

# 2.2 Mergesort

- *mergesort*
- *bottom-up mergesort*
- *sorting complexity*
- *comparators*
- *stability*

# Two classic sorting algorithms: mergesort and quicksort

Critical components in the world's computational infrastructure.

- Full scientific understanding of their properties has enabled us to develop them into practical system sorts.
- Quicksort honored as one of top 10 algorithms of 20th century in science and engineering.

Mergesort. [this lecture]

…

Quicksort. [next lecture]

…


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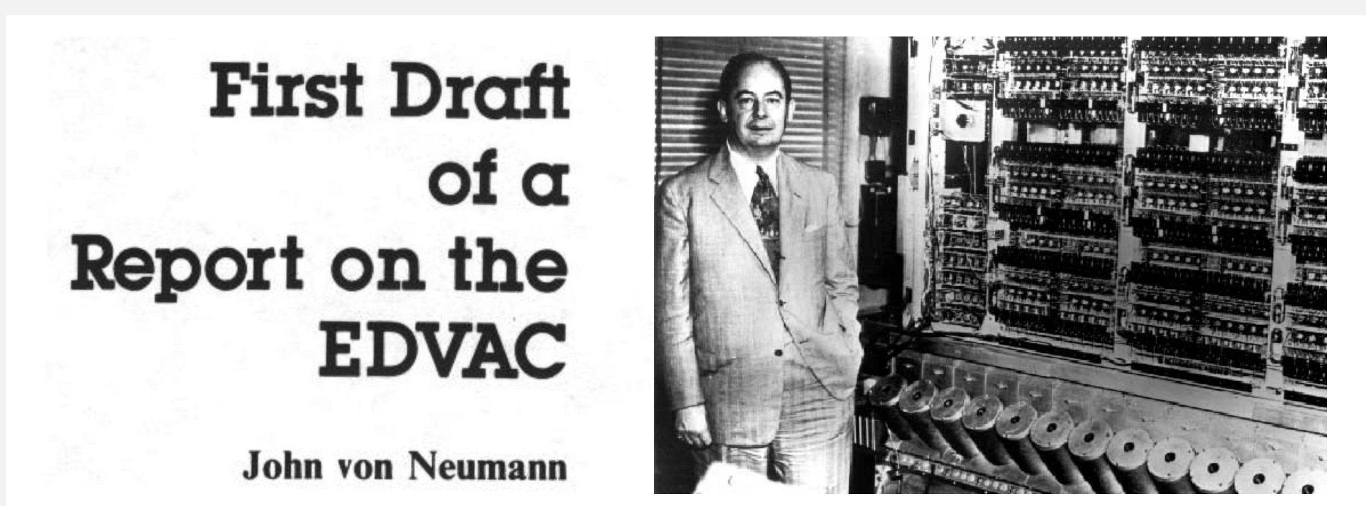
# 2.2 Mergesort

- *mergesort*
- *bottom-up mergesort*
- *sorting complexity*
- *comparators*
- *stability*

# Mergesort

Basic plan.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves.

# Abstract in-place merge demo

**Goal.** Given two sorted subarrays a[lo] to a[mid] and a[mid+1] to a[hi], replace with sorted subarray a[lo] to a[hi].

| | lo | | | | mid | mid+1 | | | | hi |
|---|---|---|---|---|---|---|---|---|---|---|
| a[] | E | E | G | M | R | A | C | E | R | T |
| | | | sorted | | | | | sorted | | |

# Abstract in-place merge demo

Goal. Given two sorted subarrays a[lo] to a[mid] and a[mid+1] to a[hi], replace with sorted subarray a[lo] to a[hi].

| | lo | | | | | | | | | hi |
|---|---|---|---|---|---|---|---|---|---|---|
| a[] | A | C | E | E | E | G | M | R | R | T |

**sorted**

# Merging:  Java implementation

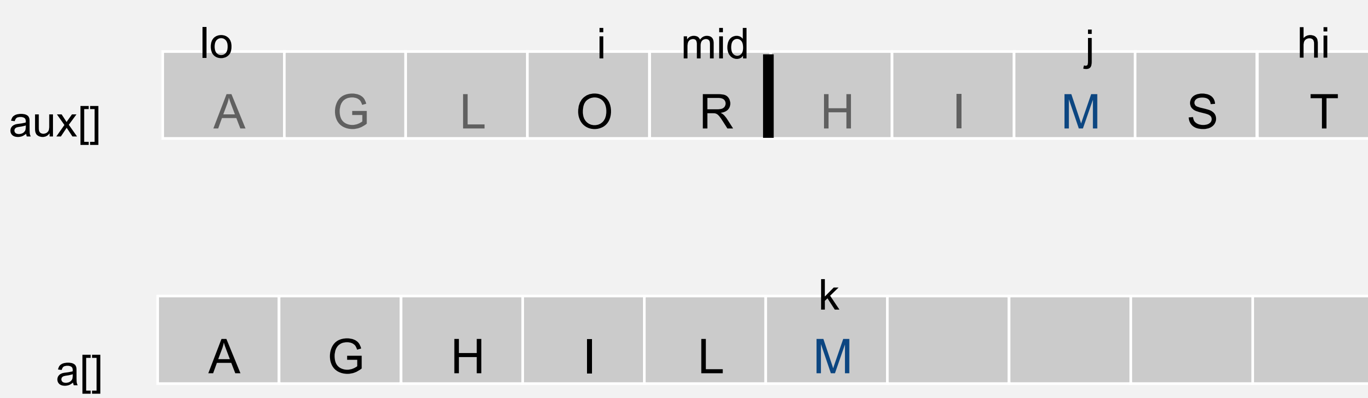
```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{

   for (int k = lo; k <= hi; k++)
      aux[k] = a[k];

   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)              a[k] = aux[j++];
      else if (j > hi)               a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                           a[k] = aux[i++];
   }
}
```

copy

merge



# Mergesort: Java implementation

```
public class Merge
{
   private static void merge(...)
   {  /* as before */  }

   private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort(a, aux, lo, mid);
      sort(a, aux, mid+1, hi);
      merge(a, aux, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {
      Comparable[] aux = new Comparable[a.length];
      sort(a, aux, 0, a.length - 1);
   }
}
```

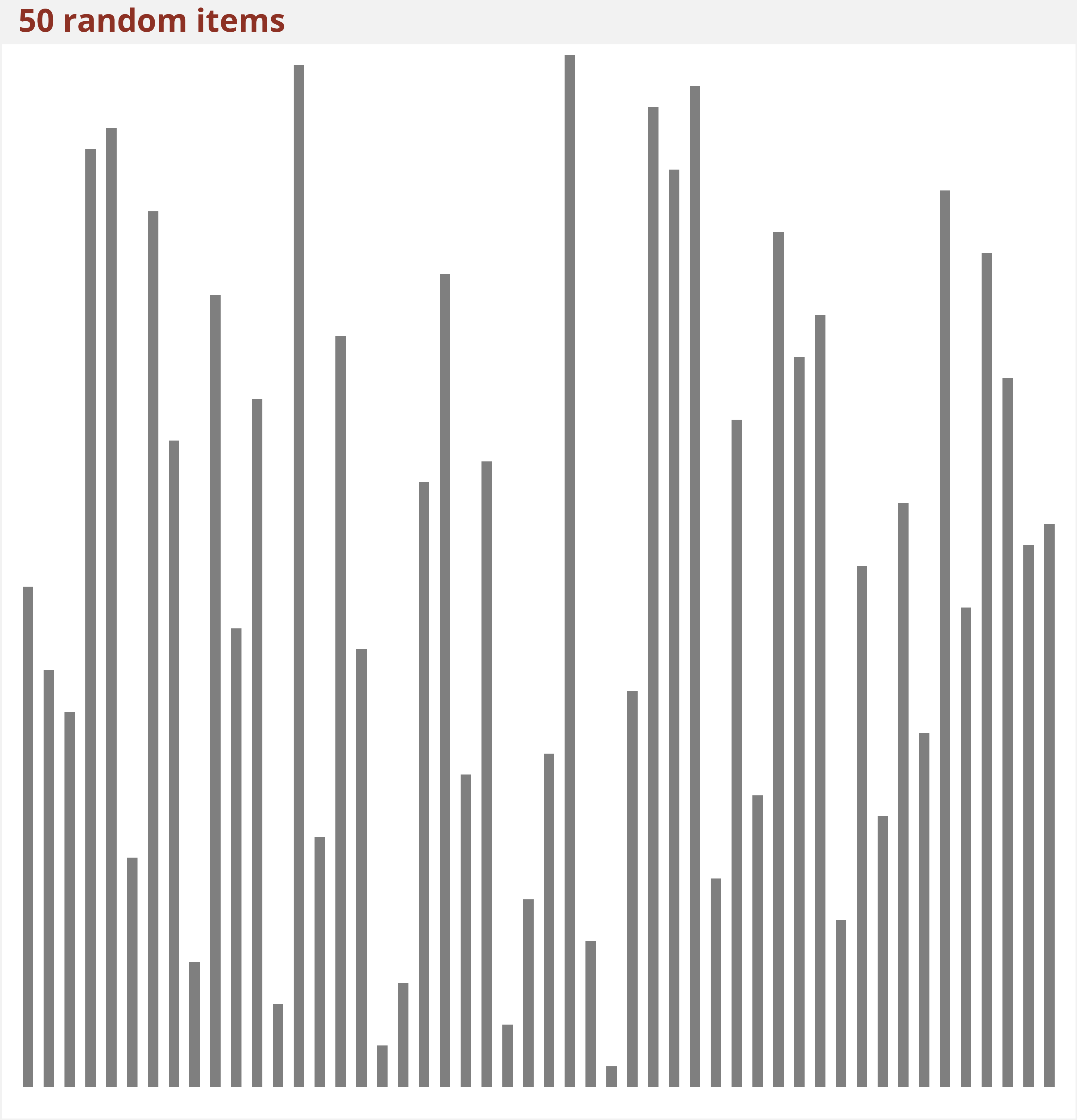
# Mergesort: trace

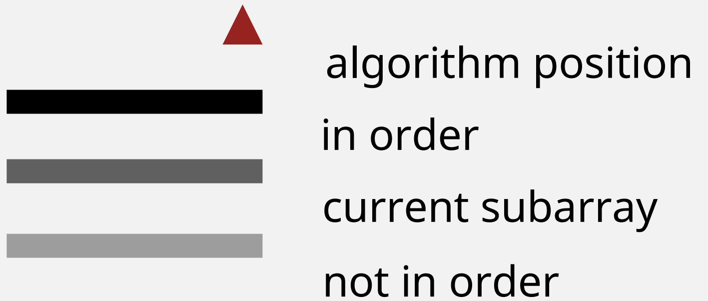
a[]

| lo hi | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | M | E | R | G | E | S | O | R | T | E | X | A | M | P | L | E |
| merge(a, aux, 0, 0, 1) | E | M | R | G | E | S | O | R | T | E | X | A | M | P | L | E |
| merge(a, aux, 2, 2, 3) | E | M | G | R | E | S | O | R | T | E | X | A | M | P | L | E |
| merge(a, aux, 0, 1, 3) | E | G | M | R | E | S | O | R | T | E | X | A | M | P | L | E |
| merge(a, aux, 4, 4, 5) | E | G | M | R | E | S | O | R | T | E | X | A | M | P | L | E |
| merge(a, aux, 6, 6, 7) | E | G | M | R | E | S | O | R | T | E | X | A | M | P | L | E |
| merge(a, aux, 4, 5, 7) | E | G | M | R | E | O | R | S | T | E | X | A | M | P | L | E |
| merge(a, aux, 0, 3, 7) | E | E | G | M | O | R | R | S | T | E | X | A | M | P | L | E |
| merge(a, aux, 8, 8, 9) | E | E | G | M | O | R | R | S | E | T | X | A | M | P | L | E |
| merge(a, aux, 10, 10, 11) | E | E | G | M | O | R | R | S | E | T | A | X | M | P | L | E |
| merge(a, aux, 8, 9, 11) | E | E | G | M | O | R | R | S | A | E | T | X | M | P | L | E |
| merge(a, aux, 12, 12, 13) | E | E | G | M | O | R | R | S | A | E | T | X | M | P | L | E |
| merge(a, aux, 14, 14, 15) | E | E | G | M | O | R | R | S | A | E | T | X | M | P | E | L |
| merge(a, aux, 12, 13, 15) | E | E | G | M | O | R | R | S | A | E | T | X | E | L | M | P |
| merge(a, aux, 8, 11, 15) | E | E | G | M | O | R | R | S | A | E | E | L | M | P | T | X |
| merge(a, aux, 0, 7, 15) | A | E | E | E | E | G | L | M | M | O | P | R | R | S | T | X |

result after recursive call

# Mergesort: animation

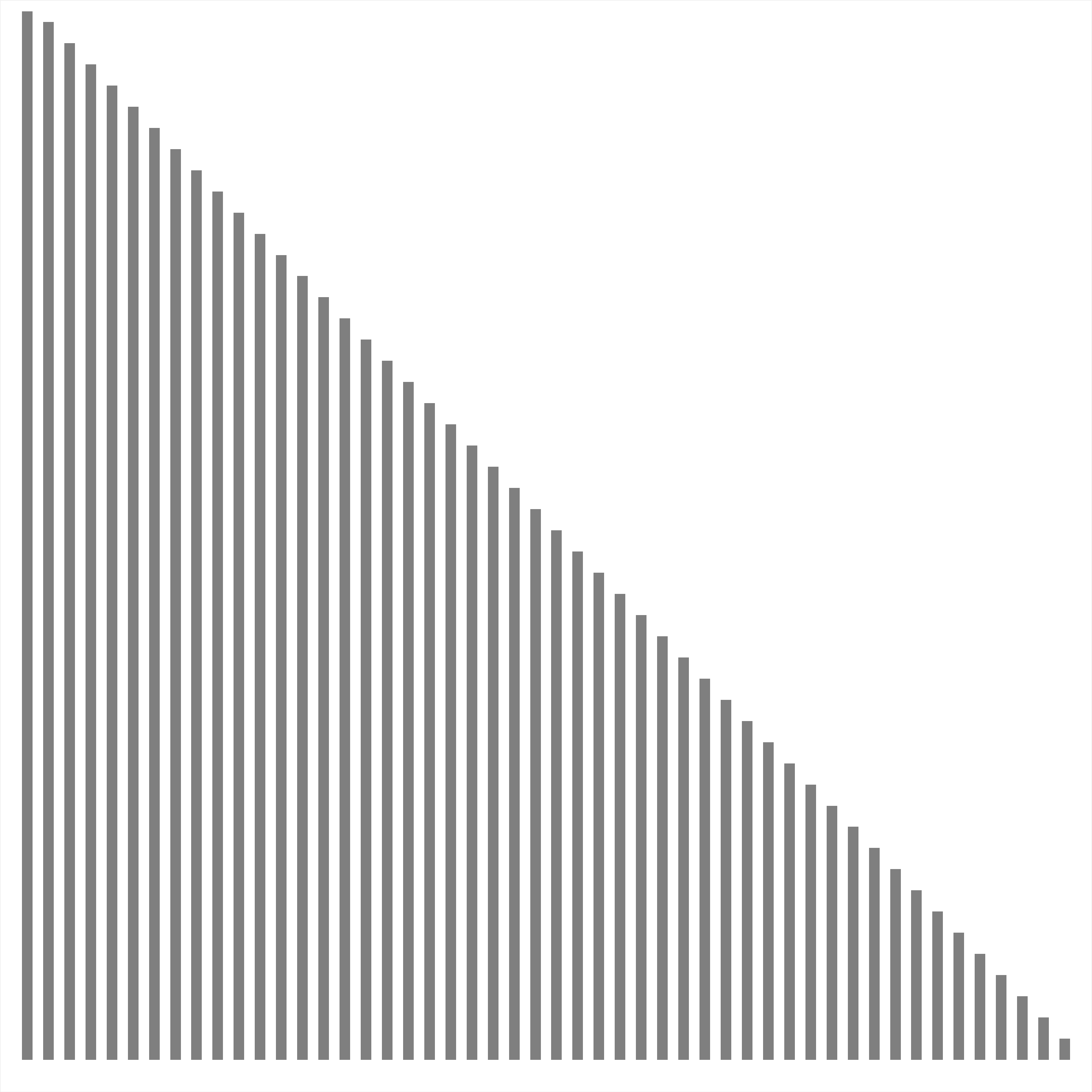


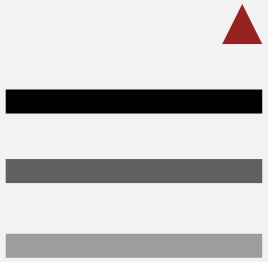




http://www.sorting-algorithms.com/merge-sort

# Mergesort: animation

**50 reverse-sorted items**






http://www.sorting-algorithms.com/merge-sort

# Mergesort: empirical analysis

Running time estimates:

- Laptop executes $10^8$ compares/second.
- Supercomputer executes $10^{12}$ compares/second.

| | insertion sort ($N^2$) | | | mergesort (N log N) | | |
|---|---|---|---|---|---|---|
| computer | thousand | million | billion | thousand | million | billion |
| home | instant | 2.8 hours | 317 years | instant | 1 second | 18 min |
| super | instant | 1 second | 1 week | instant | instant | instant |

Bottom line. Good algorithms are better than supercomputers.

# Mergesort: number of compares

Proposition. Mergesort uses $\le N \lg N$ compares to sort an array of length $N$.

Pf sketch. The number of compares $C(N)$ to mergesort an array of length $N$ satisfies the recurrence:

$$C(N) \;\le\; \underbrace{C(\lceil N/2 \rceil)}_{\text{left half}} \;+\; \underbrace{C(\lfloor N/2 \rfloor)}_{\text{right half}} \;+\; \underbrace{N}_{\text{merge}} \quad \text{for } N > 1, \text{ with } C(1) = 0.$$

We solve the recurrence when $N$ is a power of 2:

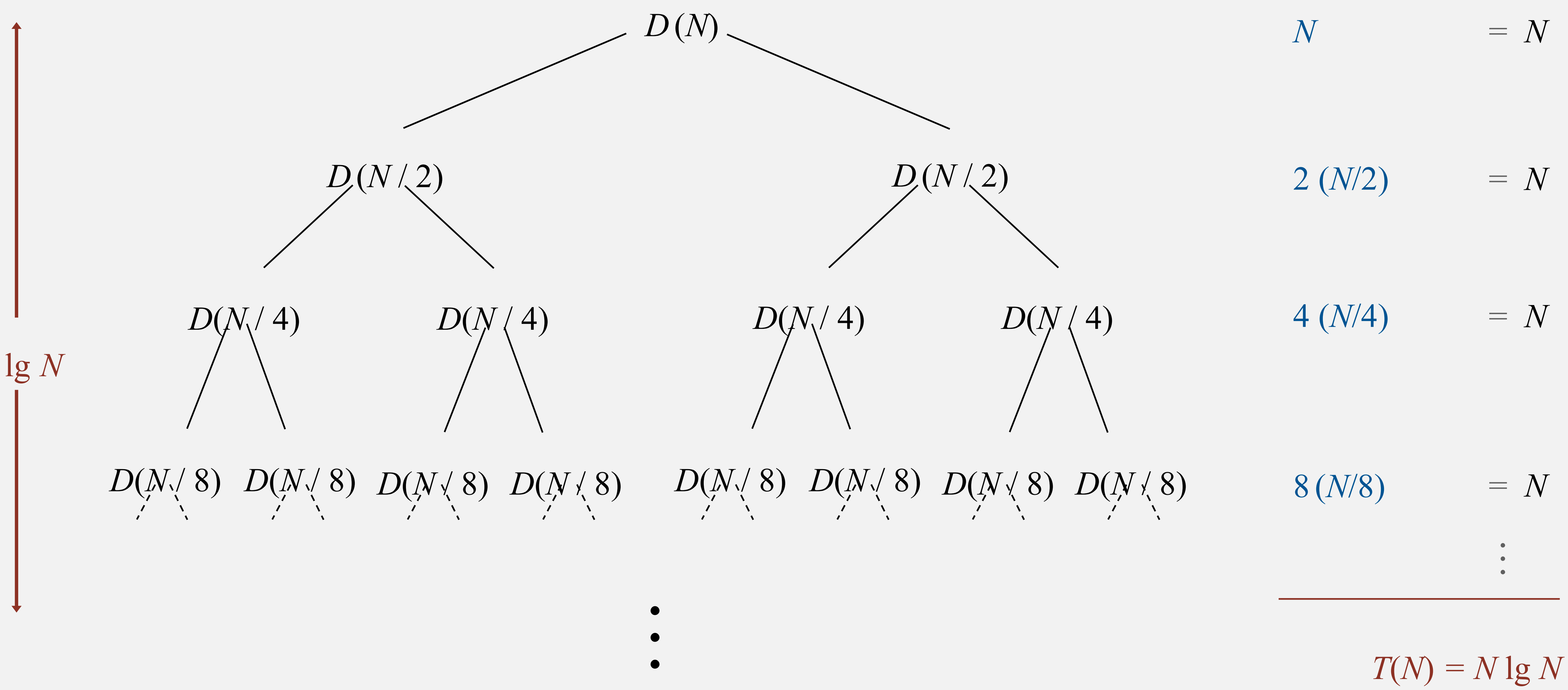
result holds for all N
(analysis cleaner in this case)

$$D(N) \;=\; 2\,D(N/2) \;+\; N, \text{ for } N > 1, \text{ with } D(1) = 0.$$

# Divide-and-conquer recurrence: proof by picture

**Proposition.** If $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

**Pf 1.** [assuming $N$ is a power of 2]



# Divide-and-conquer recurrence: proof by induction

**Proposition.** If $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

**Pf 2.** [assuming $N$ is a power of 2]

- Base case: $N = 1$.
- Inductive hypothesis: $D(N) = N \lg N$.
- Goal: show that $D(2N) = (2N) \lg (2N)$.

$$
\begin{aligned}
D(2N) &= 2\,D(N) + 2N && \text{given} \\
&= 2\,N \lg N + 2N && \text{inductive hypothesis} \\
&= 2\,N\,(\lg (2N) - 1) + 2N && \text{algebra} \\
&= 2\,N \lg (2N) && \text{QED}
\end{aligned}
$$

# Mergesort: number of array accesses

Proposition. Mergesort uses $\leq 6 N \lg N$ array accesses to sort an array of length $N$.

Pf sketch. The number of array accesses $A(N)$ satisfies the recurrence:

$$A(N) \leq A(\lceil N/2 \rceil) + A(\lfloor N/2 \rfloor) + 6N \text{ for } N > 1, \text{ with } A(1) = 0.$$

Key point. Any algorithm with the following structure takes $N \log N$ time:

```
public static void linearithmic(int N)
{
   if (N == 0) return;
   linearithmic(N/2);
   linearithmic(N/2);
   linear(N);
}
```

solve two problems of half the size

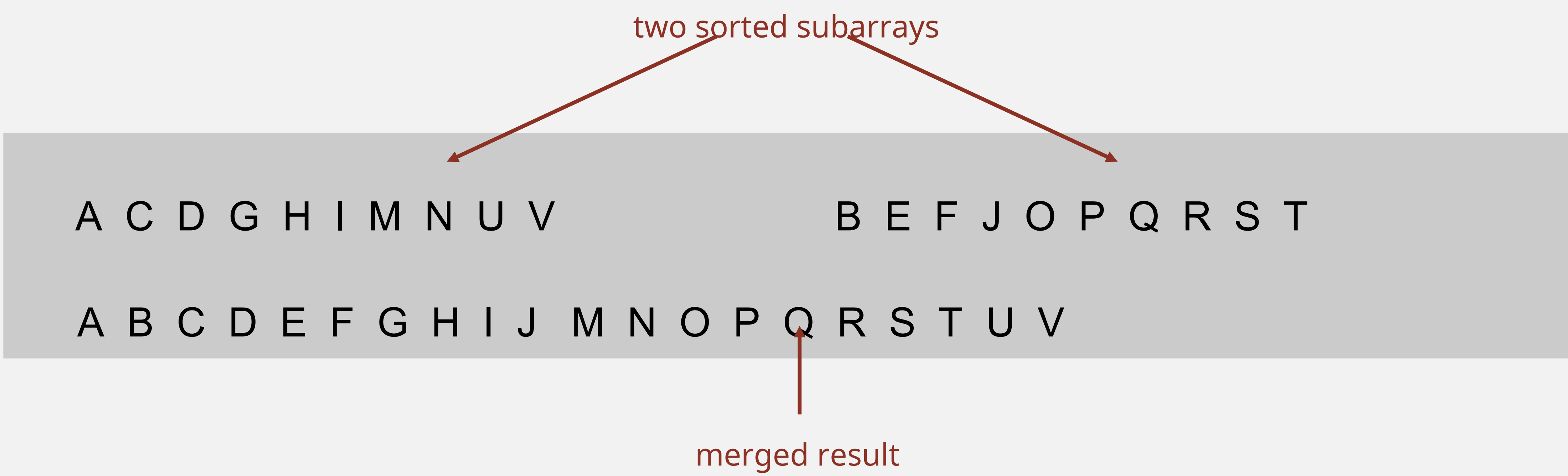
do a linear amount of work

Notable examples. FFT, hidden-line removal, Kendall-tau distance, …

# Mergesort analysis: memory

Proposition. Mergesort uses extra space proportional to $N$.

Pf. The array aux[] needs to be of length $N$ for the last merge.




Def. A sorting algorithm is in-place if it uses $\leq c \log N$ extra memory.

Ex. Insertion sort, selection sort, shellsort.

Challenge 1 (not hard). Use aux[] array of length $\sim \frac{1}{2} N$ instead of $N$.

Challenge 2 (very hard). In-place merge. [Kronrod 1969]

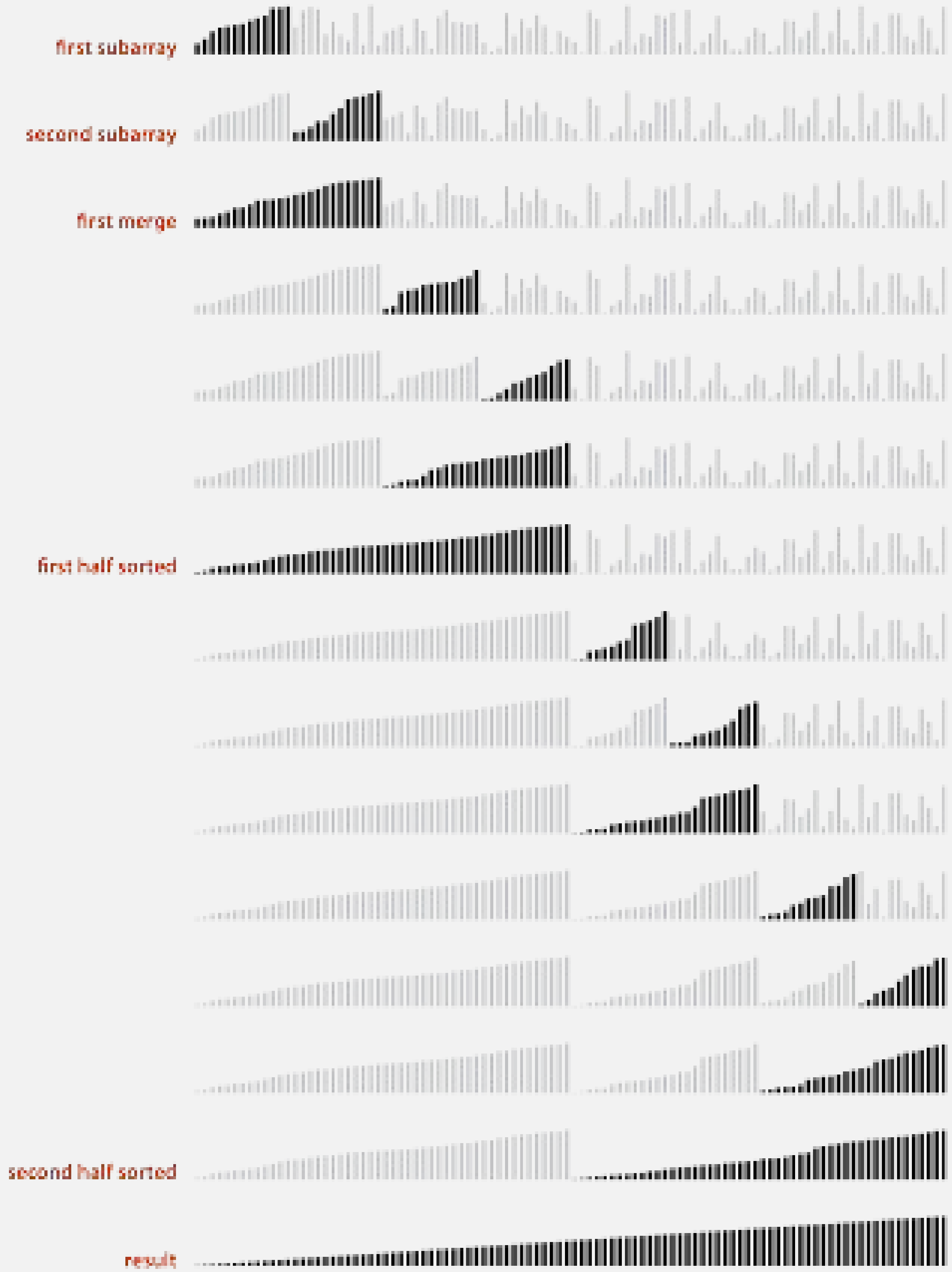
# Mergesort: practical improvements

## Use insertion sort for small subarrays.

- Mergesort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 10 items.

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo + CUTOFF - 1)
   {
      Insertion.sort(a, lo, hi);
      return;
   }
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

# Mergesort with cutoff to insertion sort: visualization



# Mergesort: practical improvements

## Stop if already sorted.

- Is largest item in first half ≤ smallest item in second half?
- Helps for partially-ordered arrays.

A B C D E F G H I J      M N O P Q R S T U V

A B C D E F G H I J M N O P Q R S T U V

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   if (!less(a[mid+1], a[mid])) return;
   merge(a, aux, lo, mid, hi);
}
```

# Mergesort: practical improvements

Eliminate the copy to the auxiliary array. Save time (but not space) by switching the role of the input and auxiliary array in each recursive call.

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)          aux[k] = a[j++];
      else if (j > hi)           aux[k] = a[i++];
      else if (less(a[j], a[i])) aux[k] = a[j++];
      else                       aux[k] = a[i++];
   }
}

private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (aux, a, lo, mid);
   sort (aux, a, mid+1, hi);
   merge(a, aux, lo, mid, hi);
```

merge from a[] to aux[]

assumes aux[] is initialize to a[] once, before recursive calls

switch roles of aux[] and a[]

# Java 6 system sort

Basic algorithm for sorting objects = mergesort.

- Cutoff to insertion sort = 7.
- Stop-if-already-sorted test.
- Eliminate-the-copy-to-the-auxiliary-array trick.

**Arrays.sort(a)**

**http://www.java2s.com/Open-Source/Java/6.0-JDK-Modules/j2me/java/util/Arrays.java.html**


http://algs4.cs.princeton.edu

# 2.2 Mergesort

- mergesort
- *bottom-up mergesort*
- sorting complexity
- comparators
- stability

# Bottom-up mergesort

Basic plan.

- Pass through array, merging subarrays of size 1.
- Repeat for subarrays of size 2, 4, 8, ....

| | a[i] | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| | M | E | R | G | E | S | O | R | T | E | X | A | M | P | L | E |
| sz = 1 | | | | | | | | | | | | | | | | |
| merge(a, aux, 0, 0, 1) | E | M | R | G | E | S | O | R | T | E | X | A | M | P | L | E |
| merge(a, aux, 2, 2, 3) | E | M | G | R | E | S | O | R | T | E | X | A | M | P | L | E |
| merge(a, aux, 4, 4, 5) | E | M | G | R | E | S | O | R | T | E | X | A | M | P | L | E |
| merge(a, aux, 6, 6, 7) | E | M | G | R | E | S | O | R | T | E | X | A | M | P | L | E |
| merge(a, aux, 8, 8, 9) | E | M | G | R | E | S | O | R | E | T | X | A | M | P | L | E |
| merge(a, aux, 10, 10, 11) | E | M | G | R | E | S | O | R | E | T | A | X | M | P | L | E |
| merge(a, aux, 12, 12, 13) | E | M | G | R | E | S | O | R | E | T | A | X | M | P | L | E |
| merge(a, aux, 14, 14, 15) | E | M | G | R | E | S | O | R | E | T | A | X | M | P | E | L |
| sz = 2 | | | | | | | | | | | | | | | | |
| merge(a, aux, 0, 1, 3) | E | G | M | R | E | S | O | R | E | T | A | X | M | P | E | L |
| merge(a, aux, 4, 5, 7) | E | G | M | R | E | O | R | S | E | T | A | X | M | P | E | L |
| merge(a, aux, 8, 9, 11) | E | G | M | R | E | O | R | S | A | E | T | X | M | P | E | L |
| merge(a, aux, 12, 13, 15) | E | G | M | R | E | O | R | S | A | E | T | X | E | L | M | P |
| sz = 4 | | | | | | | | | | | | | | | | |
| merge(a, aux, 0, 3, 7) | E | E | G | M | O | R | R | S | A | E | T | X | E | L | M | P |
| merge(a, aux, 8, 11, 15) | E | E | G | M | O | R | R | S | A | E | E | L | M | P | T | X |
| sz = 8 | | | | | | | | | | | | | | | | |
| merge(a, aux, 0, 7, 15) | A | E | E | E | E | G | L | M | M | O | P | R | R | S | T | X |

# Bottom-up mergesort: Java implementation

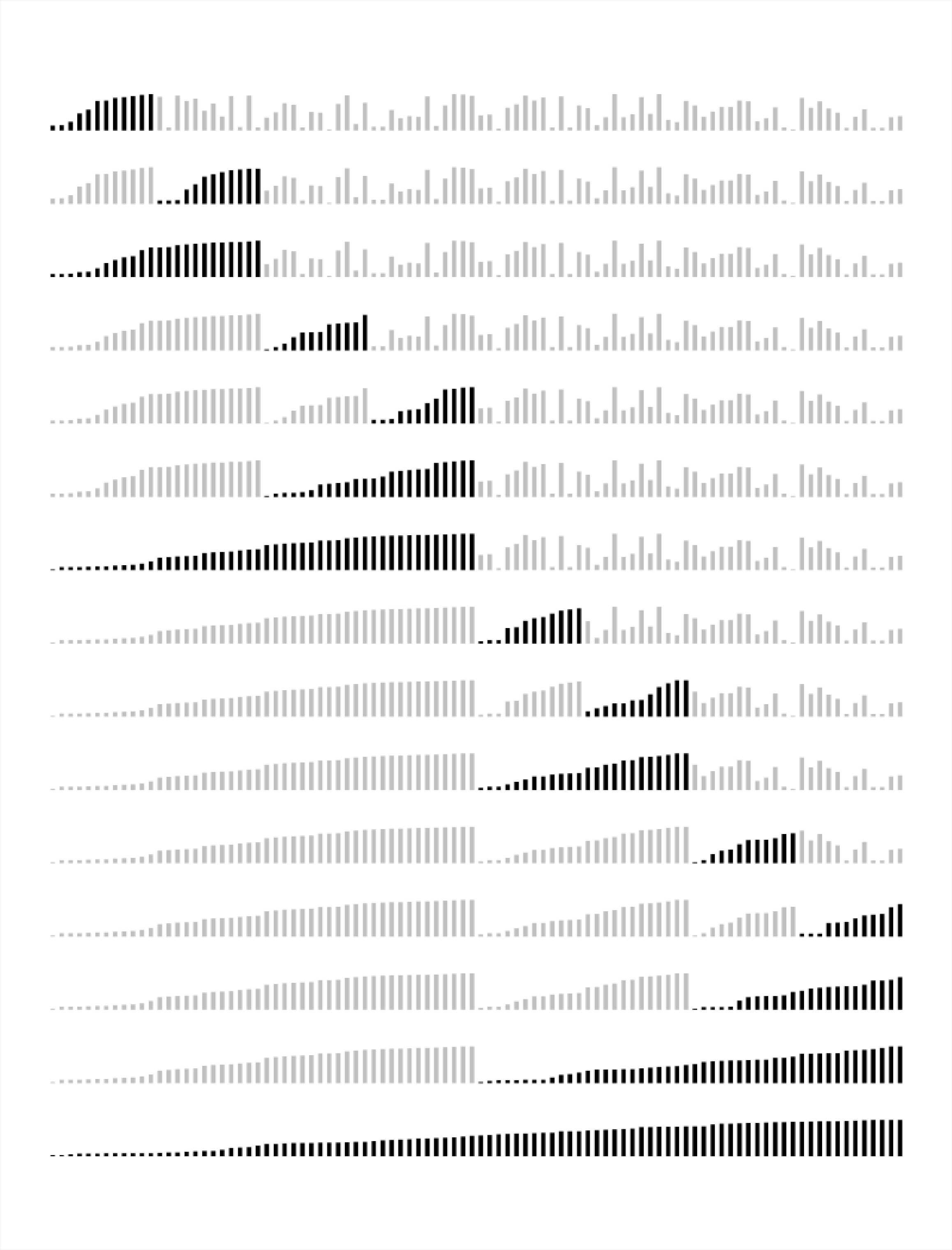
```
public class MergeBU
{
   private static void merge(...)
   {  /* as before */  }

   public static void sort(Comparable[] a)
   {
      int N = a.length;
      Comparable[] aux = new Comparable[N];
      for (int sz = 1; sz < N; sz = sz+sz)
         for (int lo = 0; lo < N-sz; lo += sz+sz)
            merge(a, aux, lo, lo+sz-1, Math.min(lo+sz+sz-1, N-1));
   }
}
```

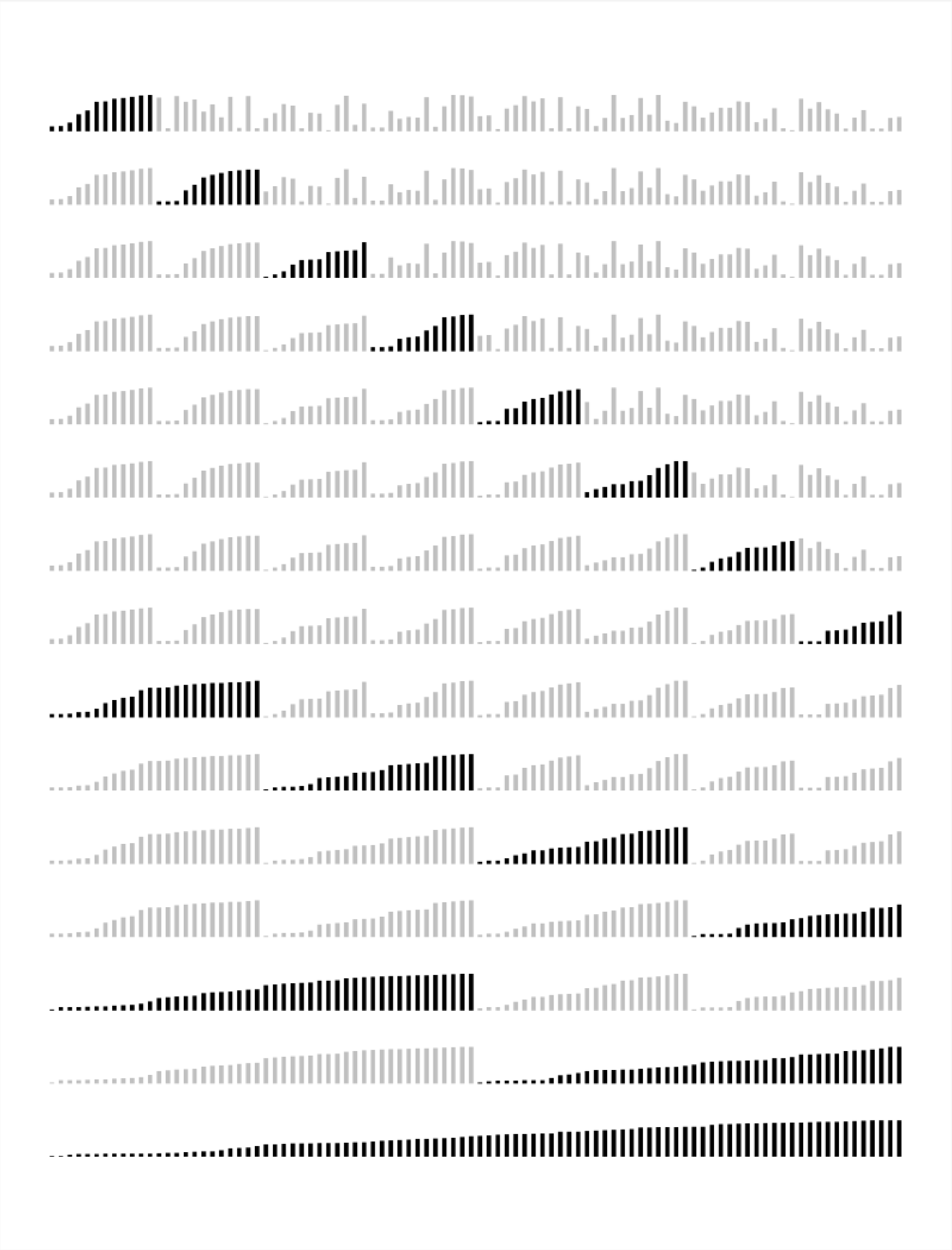
but about 10% slower than recursive,
top-down mergesort on typical systems

**Bottom line.** Simple and non-recursive version of mergesort.

# Mergesort: visualizations



top-down mergesort (cutoff = 12)



bottom-up mergesort (cutoff = 12)

# Natural mergesort

**Idea.** Exploit pre-existing order by identifying naturally-occurring runs.

**input**

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 10 | 16 | 3 | 4 | 23 | 9 | 13 | 2 | 7 | 8 | 12 | 14 |

**first run**

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 10 | 16 | 3 | 4 | 23 | 9 | 13 | 2 | 7 | 8 | 12 | 14 |

**second run**

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 10 | 16 | 3 | 4 | 23 | 9 | 13 | 2 | 7 | 8 | 12 | 14 |

**merge two runs**

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 4 | 5 | 10 | 16 | 23 | 9 | 13 | 2 | 7 | 8 | 12 | 14 |

**Tradeoff.** Fewer passes vs. extra compares per pass to identify runs.

# Timsort

- Natural mergesort.
- Use binary insertion sort to make initial runs (if needed).
- A few more clever optimizations.

**Tim Peters**

Intro

-----

This describes an adaptive, stable, natural mergesort, modestly called timsort (hey, I earned it <wink>). It has supernatural performance on many kinds of partially ordered arrays (less than lg(N!) comparisons needed, and as few as N-1), yet as fast as Python's previous highly tuned samplesort hybrid on random arrays.

In a nutshell, the main routine marches over the array once, left to right, alternately identifying the next run, then merging it into the previous runs "intelligently". Everything else is complication for speed, and some hard-won measure of memory efficiency.

...

Consequence. Linear time on many arrays with pre-existing order.

Now widely used. Python, Java 7, GNU Octave, Android, ….

# The Zen of Python


**http://www.python.org/dev/peps/pep-0020/**

**http://westmarch.sjsoft.com/2012/11/zen-of-python-poster/**


http://algs4.cs.princeton.edu

# 2.2 Mergesort

- *mergesort*
- *bottom-up mergesort*
- *sorting complexity*
- *comparators*
- *stability*

# Complexity of sorting

Computational complexity. Framework to study efficiency of algorithms for solving a particular problem $X$.

Model of computation. Allowable operations.

Cost model. Operation count(s).

Upper bound. Cost guarantee provided by some algorithm for $X$.

Lower bound. Proven limit on cost guarantee of all algorithms for $X$.

Optimal algorithm. Algorithm with best possible cost guarantee for $X$.

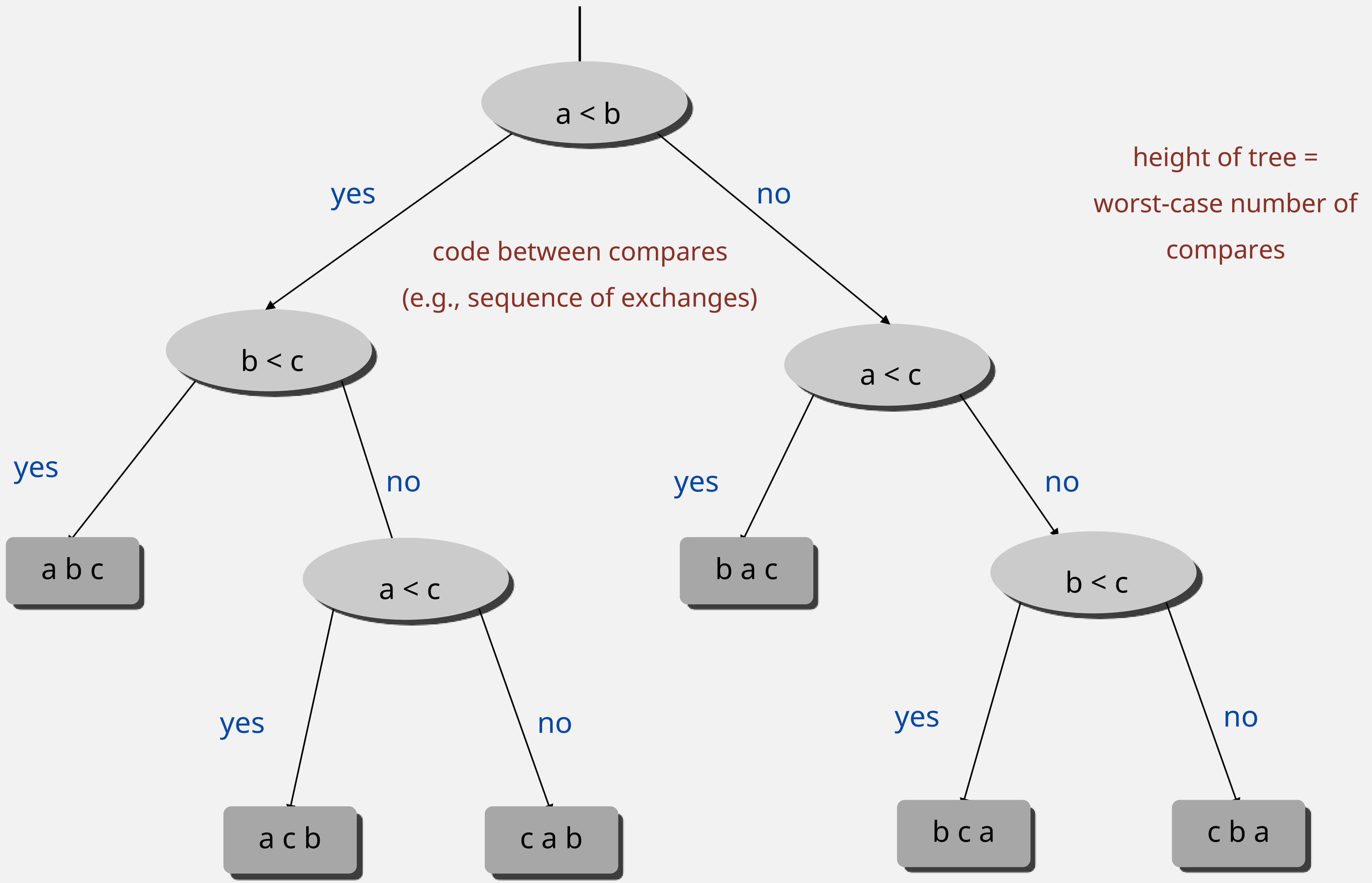
lower bound ~ upper bound

Example: sorting.

- Model of computation: decision tree.
- Cost model: # compares.
- Upper bound: $\sim N \lg N$ from mergesort.
- Lower bound:
- Optimal algorithm:

can access information only through compares (e.g., Java Comparable framework)

# Decision tree (for 3 distinct keys a, b, and c)

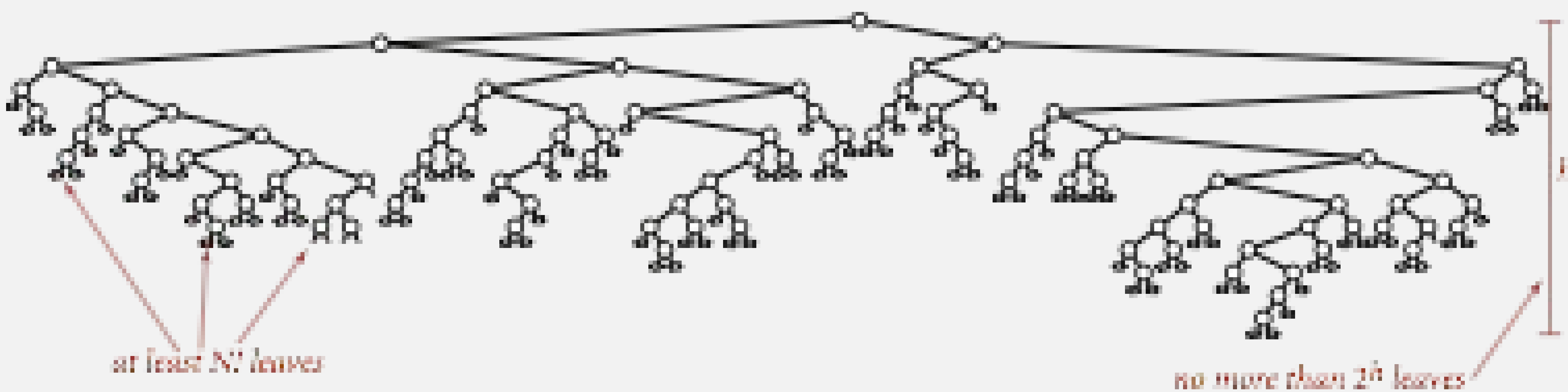



each leaf corresponds to one (and only one) ordering;
(at least) one leaf for each possible ordering

# Compare-based lower bound for sorting

**Proposition.** Any compare-based sorting algorithm must use at least $\lg(N!) \sim N \lg N$ compares in the worst-case.

**Pf.**
- Assume array consists of $N$ distinct values $a_1$ through $a_N$.
- Worst case dictated by **height** $h$ of decision tree.
- Binary tree of height $h$ has at most $2^h$ leaves.
- $N!$ different orderings $\Rightarrow$ at least $N!$ leaves.



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- Worst case dictated by height $h$ of decision tree.
- Binary tree of height $h$ has at most $2^h$ leaves.
- $N!$ different orderings $\Rightarrow$ at least $N!$ leaves.

$$2^h \geq \#\,\text{leaves} \geq N!$$

$$\Rightarrow\; h \geq \lg(N!) \sim N \lg N$$

Stirling's formula

# Complexity of sorting

Model of computation. Allowable operations.

Cost model. Operation count(s).

Upper bound. Cost guarantee provided by some algorithm for $X$.

Lower bound. Proven limit on cost guarantee of all algorithms for $X$.

Optimal algorithm. Algorithm with best possible cost guarantee for $X$.

Example: sorting.

- Model of computation: decision tree.
- Cost model: # compares.
- Upper bound: $\sim N \lg N$ from mergesort.
- Lower bound: $\sim N \lg N$.
- Optimal algorithm = mergesort.

First goal of algorithm design: optimal algorithms.

# Complexity results in context

Compares? Mergesort is optimal with respect to number compares.

Space? Mergesort is not optimal with respect to space usage.


Lessons. Use theory as a guide.

Ex. Design sorting algorithm that guarantees $\frac{1}{2} N \lg N$ compares?

Ex. Design sorting algorithm that is both time- and space-optimal?

# Complexity results in context (continued)

Lower bound may not hold if the algorithm can take advantage of:

- The initial order of the input.

  Ex: insert sort requires only a linear number of compares on partially-sorted arrays.

- The distribution of key values.

  Ex: 3-way quicksort requires only a linear number of compares on arrays with a constant number of distinct keys. [stay tuned]

- The representation of the keys.

  Ex: radix sort requires no key compares — it accesses the data via character/digit compares.


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# 2.2 Mergesort

- *mergesort*
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- *sorting complexity*
- ***comparators***
- *stability*

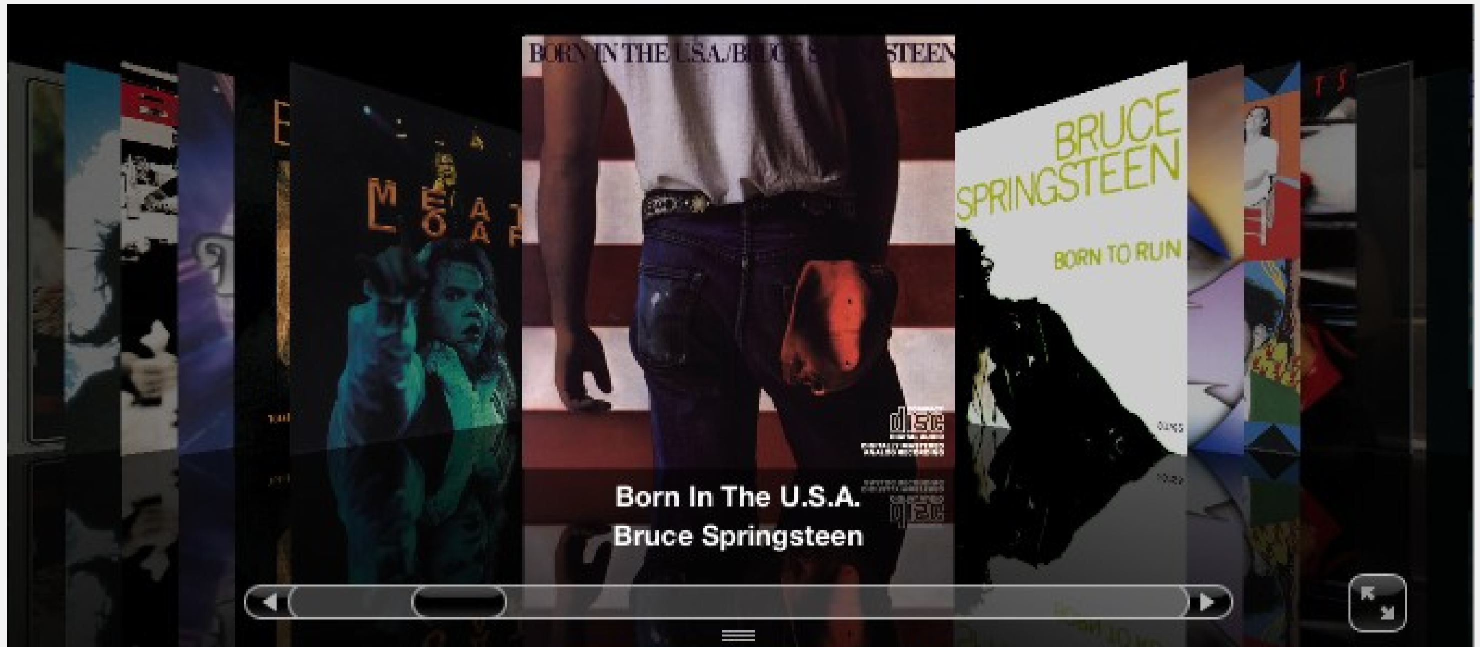
# Sort countries by gold medals

| NOC | Gold | Silver | Bronze | Total |
|---|---|---|---|---|
| United States (USA) | 46 | 29 | 29 | 104 |
| China (CHN)§ | 38 | 28 | 22 | 88 |
| Great Britain (GBR)* | 29 | 17 | 19 | 65 |
| Russia (RUS)§ | 24 | 25 | 32 | 81 |
| South Korea (KOR) | 13 | 8 | 7 | 28 |
| Germany (GER) | 11 | 19 | 14 | 44 |
| France (FRA) | 11 | 11 | 12 | 34 |
| Italy (ITA) | 8 | 9 | 11 | 28 |
| Hungary (HUN)§ | 8 | 4 | 6 | 18 |
| Australia (AUS) | 7 | 16 | 12 | 35 |

# Sort countries by total medals

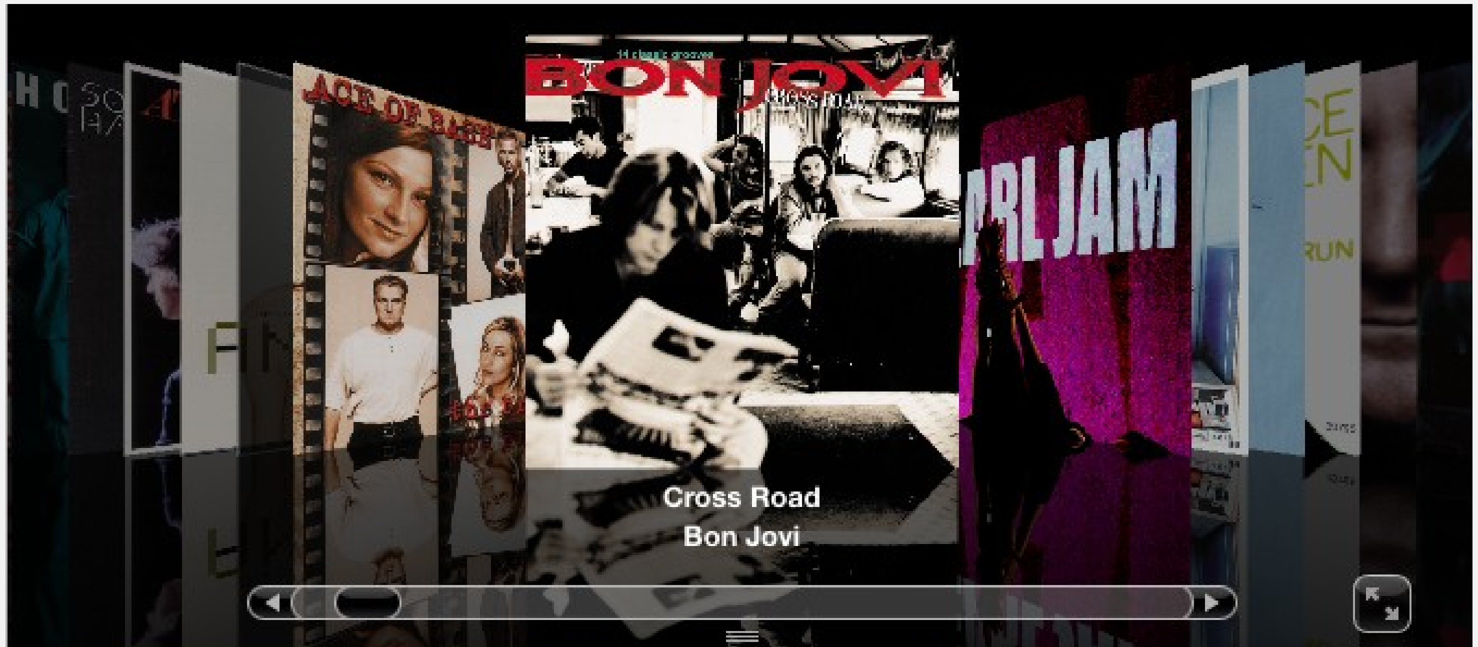
| NOC | Gold | Silver | Bronze | Total |
|---|---|---|---|---|
| United States (USA) | 46 | 29 | 29 | 104 |
| China (CHN)§ | 38 | 28 | 22 | 88 |
| Russia (RUS)§ | 24 | 25 | 32 | 81 |
| Great Britain (GBR)* | 29 | 17 | 19 | 65 |
| Germany (GER) | 11 | 19 | 14 | 44 |
| Japan (JPN) | 7 | 14 | 17 | 38 |
| Australia (AUS) | 7 | 16 | 12 | 35 |
| France (FRA) | 11 | 11 | 12 | 34 |
| South Korea (KOR) | 13 | 8 | 7 | 28 |
| Italy (ITA) | 8 | 9 | 11 | 28 |

# Sort music library by artist




| | Name | Artist ▲ | Time | Album |
|---|---|---|---|---|
| 12 | ☑ Let It Be | The Beatles | 4:03 | Let It Be |
| 13 | ☑ Take My Breath Away | BERLIN | 4:13 | Top Gun - Soundtrack |
| 14 | ☑ Circle Of Friends | Better Than Ezra | 3:27 | Empire Records |
| 15 | ☑ Dancing With Myself | Billy Idol | 4:43 | Don't Stop |
| 16 | ☑ Rebel Yell | Billy Idol | 4:49 | Rebel Yell |
| 17 | ☑ Piano Man | Billy Joel | 5:36 | Greatest Hits Vol. 1 |
| 18 | ☑ Pressure | Billy Joel | 3:16 | Greatest Hits, Vol. II (1978 - 1985) (Disc 2) |
| 19 | ☑ The Longest Time | Billy Joel | 3:36 | Greatest Hits, Vol. II (1978 - 1985) (Disc 2) |
| 20 | ☑ Atomic | Blondie | 3:50 | Atomic: The Very Best Of Blondie |
| 21 | ☑ Sunday Girl | Blondie | 3:15 | Atomic: The Very Best Of Blondie |
| 22 | ☑ Call Me | Blondie | 3:33 | Atomic: The Very Best Of Blondie |
| 23 | ☑ Dreaming | Blondie | 3:06 | Atomic: The Very Best Of Blondie |
| 24 | ☑ Hurricane | Bob Dylan | 8:32 | Desire |
| 25 | ☑ The Times They Are A-Changin' | Bob Dylan | 3:17 | Greatest Hits |
| 26 | ☑ Livin' On A Prayer | Bon Jovi | 4:11 | Cross Road |
| 27 | ☑ Beds Of Roses | Bon Jovi | 6:35 | Cross Road |
| 28 | ☑ Runaway | Bon Jovi | 3:53 | Cross Road |
| 29 | ☑ Rasputin (Extended Mix) | Boney M | 5:50 | Greatest Hits |
| 30 | ☑ Have You Ever Seen The Rain | Bonnie Tyler | 4:10 | Faster Than The Speed Of Night |
| 31 | ☑ Total Eclipse Of The Heart | Bonnie Tyler | 7:02 | Faster Than The Speed Of Night |
| 32 | ☑ Straight From The Heart | Bonnie Tyler | 3:41 | Faster Than The Speed Of Night |
| 33 | ☑ Holding Out For A Hero | Bonny Tyler | 5:49 | Meat Loaf And Friends |
| 34 | ☑ Dancing In The Dark | Bruce Springsteen | 4:05 | Born In The U.S.A. |
| 35 | ☑ Thunder Road | Bruce Springsteen | 4:51 | Born To Run |
| 36 | ☑ Born To Run | Bruce Springsteen | 4:30 | Born To Run |
| 37 | ☑ Jungleland | Bruce Springsteen | 9:34 | Born To Run |

# Sort music library by song name




| | Name | Artist | Time | Album |
|---|---|---|---|---|
| 1 | Alive | Pearl Jam | 5:41 | Ten |
| 2 | All Over The World | Pixies | 5:27 | Bossanova |
| 3 | All Through The Night | Cyndi Lauper | 4:30 | She's So Unusual |
| 4 | Allison Road | Gin Blossoms | 3:19 | New Miserable Experience |
| 5 | Ama, Ama, Ama Y Ensancha El ... | Extremoduro | 2:34 | Deltoya (1992) |
| 6 | And We Danced | Hooters | 3:50 | Nervous Night |
| 7 | As I Lay Me Down | Sophie B. Hawkins | 4:09 | Whaler |
| 8 | Atomic | Blondie | 3:50 | Atomic: The Very Best Of Blondie |
| 9 | Automatic Lover | Jay-Jay Johanson | 4:19 | Antenna |
| 10 | Baba O'Riley | The Who | 5:01 | Who's Better, Who's Best |
| 11 | Beautiful Life | Ace Of Base | 3:40 | The Bridge |
| 12 | Beds Of Roses | Bon Jovi | 6:35 | Cross Road |
| 13 | Black | Pearl Jam | 5:44 | Ten |
| 14 | Bleed American | Jimmy Eat World | 3:04 | Bleed American |
| 15 | Borderline | Madonna | 4:00 | The Immaculate Collection |
| 16 | Born To Run | Bruce Springsteen | 4:30 | Born To Run |
| 17 | Both Sides Of The Story | Phil Collins | 6:43 | Both Sides |
| 18 | Bouncing Around The Room | Phish | 4:09 | A Live One (Disc 1) |
| 19 | Boys Don't Cry | The Cure | 2:35 | Staring At The Sea: The Singles 1979-1985 |
| 20 | Brat | Green Day | 1:43 | Insomniac |
| 21 | Breakdown | Deerheart | 3:40 | Deerheart |
| 22 | Bring Me To Life (Kevin Roen Mix) | Evanescence Vs. Pa... | 9:48 | |
| 23 | Californication | Red Hot Chili Pepp... | 1:40 | |
| 24 | Call Me | Blondie | 3:33 | Atomic: The Very Best Of Blondie |
| 25 | Can't Get You Out Of My Head | Kylie Minogue | 3:50 | Fever |
| 26 | Celebration | Kool & The Gang | 3:45 | Time Life Music Sounds Of The Seventies – C |

# Comparable interface: review

Comparable interface: sort using a type's natural order.

```
public class Date implements Comparable<Date>
{
  private final int month, day, year;

  public Date(int m, int d, int y)
  {
    month = m;
    day   = d;
    year  = y;
  }
    ...
  public int compareTo(Date that)
  {
    if (this.year  < that.year ) return -1;
    if (this.year  > that.year ) return +1;
    if (this.month < that.month) return -1;
    if (this.month > that.month) return +1;
    if (this.day   < that.day  ) return -1;
    if (this.day   > that.day  ) return +1;
    return 0;
  }
```

natural order

# Comparator interface

Comparator interface: sort using an alternate order.

| public interface Comparator<Key> | | |
|---|---|---|
| int | compare(Key v, Key w) | *compare keys v and w* |

Required property. Must be a total order.

| string order | example |
|---|---|
| **natural order** | Now is the time |
| **case insensitive** | is Now the time |
| **Spanish language** | café cafetero cuarto churro nube ñoño |
| **British phone book** | McKinley Mackintosh |

pre-1994 order for digraphs ch and ll and rr

# Comparator interface: system sort

To use with Java system sort:

- Create Comparator object.
- Pass as second argument to Arrays.sort().

uses natural order

uses alternate order defined by Comparator<String> object

```
String[] a;
...
Arrays.sort(a);
...
Arrays.sort(a, String.CASE_INSENSITIVE_ORDER);
...
Arrays.sort(a, Collator.getInstance(new Locale("es")));
...
Arrays.sort(a, new BritishPhoneBookOrder());
...
```

Bottom line. Decouples the definition of the data type from the definition of what it means to compare two objects of that type.

# Comparator interface: using with our sorting libraries

To support comparators in our sort implementations:

- Use Object instead of Comparable.
- Pass Comparator to sort() and less() and use it in less().

**insertion sort using a Comparator**

```
public static void sort(Object[] a, Comparator comparator)
{
   int N = a.length;
   for (int i = 0; i < N; i++)
      for (int j = i; j > 0 && less(comparator, a[j], a[j-1]); j--)
         exch(a, j, j-1);
}

private static boolean less(Comparator c, Object v, Object w)
{  return c.compare(v, w) < 0;   }

private static void exch(Object[] a, int i, int j)
{  Object swap = a[i]; a[i] = a[j]; a[j] = swap;  }
```

# Comparator interface: implementing

To implement a comparator:

- Define a (nested) class that implements the `Comparator` interface.
- Implement the `compare()` method.

```
public class Student
{
   private final String name;
   private final int section;
   ...

   public static class ByName implements Comparator<Student>
   {
     public int compare(Student v, Student w)
      {  return v.name.compareTo(w.name);  }
   }

   public static class BySection implements Comparator<Student>
   {
      public int compare(Student v, Student w)
      {  return v.section - w.section;  }
```

this trick works here
since no danger of overflow

# Comparator interface: implementing

## To implement a comparator:

- Define a (nested) class that implements the `Comparator` interface.
- Implement the `compare()` method.

`Arrays.sort(a, new Student.ByName());`

| | | | | |
|---|---|---|---|---|
| Andrews | 3 | A | 664-480-0023 | 097 Little |
| Battle | 4 | C | 874-088-1212 | 121 Whitman |
| Chen | 3 | A | 991-878-4944 | 308 Blair |
| Fox | 3 | A | 884-232-5341 | 11 Dickinson |
| Furia | 1 | A | 766-093-9873 | 101 Brown |
| Gazsi | 4 | B | 766-093-9873 | 101 Brown |
| Kanaga | 3 | B | 898-122-9643 | 22 Brown |
| Rohde | 2 | A | 232-343-5555 | 343 Forbes |

`Arrays.sort(a, new Student.BySection());`

| | | | | |
|---|---|---|---|---|
| Furia | 1 | A | 766-093-9873 | 101 Brown |
| Rohde | 2 | A | 232-343-5555 | 343 Forbes |
| Andrews | 3 | A | 664-480-0023 | 097 Little |
| Chen | 3 | A | 991-878-4944 | 308 Blair |
| Fox | 3 | A | 884-232-5341 | 11 Dickinson |
| Kanaga | 3 | B | 898-122-9643 | 22 Brown |
| Battle | 4 | C | 874-088-1212 | 121 Whitman |
| Gazsi | 4 | B | 766-093-9873 | 101 Brown |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# 2.2 Mergesort

- *mergesort*
- *bottom-up mergesort*
- *sorting complexity*
- *comparators*
- *stability*

# Stability

A typical application. First, sort by name; then sort by section.

`Selection.sort(a, new Student.ByName());`

| | | | | |
|---|---|---|---|---|
| Andrews | 3 | A | 664-480-0023 | 097 Little |
| Battle | 4 | C | 874-088-1212 | 121 Whitman |
| Chen | 3 | A | 991-878-4944 | 308 Blair |
| Fox | 3 | A | 884-232-5341 | 11 Dickinson |
| Furia | 1 | A | 766-093-9873 | 101 Brown |
| Gazsi | 4 | B | 766-093-9873 | 101 Brown |
| Kanaga | 3 | B | 898-122-9643 | 22 Brown |
| Rohde | 2 | A | 232-343-5555 | 343 Forbes |

`Selection.sort(a, new Student.BySection());`

| | | | | |
|---|---|---|---|---|
| Furia | 1 | A | 766-093-9873 | 101 Brown |
| Rohde | 2 | A | 232-343-5555 | 343 Forbes |
| Chen | 3 | A | 991-878-4944 | 308 Blair |
| Fox | 3 | A | 884-232-5341 | 11 Dickinson |
| Andrews | 3 | A | 664-480-0023 | 097 Little |
| Kanaga | 3 | B | 898-122-9643 | 22 Brown |
| Gazsi | 4 | B | 766-093-9873 | 101 Brown |
| Battle | 4 | C | 874-088-1212 | 121 Whitman |

@#%&@! Students in section 3 no longer sorted by name.

A stable sort preserves the relative order of items with equal keys.

# Stability

**Q.** Which sorts are stable?

**A.** Need to check algorithm (and implementation).

| sorted by time | | sorted by location (not stable) | | sorted by location (stable) | |
|---|---|---|---|---|---|
| Chicago | 09:00:00 | Chicago | 09:25:52 | Chicago | 09:00:00 |
| Phoenix | 09:00:03 | Chicago | 09:03:13 | Chicago | 09:00:59 |
| Houston | 09:00:13 | Chicago | 09:21:05 | Chicago | 09:03:13 |
| Chicago | 09:00:59 | Chicago | 09:19:46 | Chicago | 09:19:32 |
| Houston | 09:01:10 | Chicago | 09:19:32 | Chicago | 09:19:46 |
| Chicago | 09:03:13 | Chicago | 09:00:00 | Chicago | 09:21:05 |
| Seattle | 09:10:11 | Chicago | 09:35:21 | Chicago | 09:25:52 |
| Seattle | 09:10:25 | Chicago | 09:00:59 | Chicago | 09:35:21 |
| Phoenix | 09:14:25 | Houston | 09:01:10 | Houston | 09:00:13 |
| Chicago | 09:19:32 | Houston | 09:00:13 | Houston | 09:01:10 |
| Chicago | 09:19:46 | Phoenix | 09:37:44 | Phoenix | 09:00:03 |
| Chicago | 09:21:05 | Phoenix | 09:00:03 | Phoenix | 09:14:25 |
| Seattle | 09:22:43 | Phoenix | 09:14:25 | Phoenix | 09:37:44 |
| Seattle | 09:22:54 | Seattle | 09:10:25 | Seattle | 09:10:11 |
| Chicago | 09:25:52 | Seattle | 09:36:14 | Seattle | 09:10:25 |
| Chicago | 09:35:21 | Seattle | 09:22:43 | Seattle | 09:22:43 |
| Seattle | 09:36:14 | Seattle | 09:10:11 | Seattle | 09:22:54 |
| Phoenix | 09:37:44 | Seattle | 09:22:54 | Seattle | 09:36:14 |

*no longer sorted by time*

*still sorted by time*

# Stability: insertion sort

Proposition. Insertion sort is stable.

```
public class Insertion
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
         for (int j = i; j > 0 && less(a[j], a[j-1]); j--)
            exch(a, j, j-1);
   }
}
```

| i | j | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|
| 0 | 0 | $B_1$ | $A_1$ | $A_2$ | $A_3$ | $B_2$ |
| 1 | 0 | $A_1$ | $B_1$ | $A_2$ | $A_3$ | $B_2$ |
| 2 | 1 | $A_1$ | $A_2$ | $B_1$ | $A_3$ | $B_2$ |
| 3 | 2 | $A_1$ | $A_2$ | $A_3$ | $B_1$ | $B_2$ |
| 4 | 4 | $A_1$ | $A_2$ | $A_3$ | $B_1$ | $B_2$ |
| | | $A_1$ | $A_2$ | $A_3$ | $B_1$ | $B_2$ |

Pf. Equal items never move past each other.

# Stability: selection sort

Proposition. Selection sort is not stable.

```
public class Selection
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
      {
         int min = i;
         for (int j = i+1; j < N; j++)
            if (less(a[j], a[min]))
               min = j;
         exch(a, i, min);
      }
   }
}
```

| i | min | 0 | 1 | 2 |
|---|---|---|---|---|
| 0 | 2 | $B_1$ | $B_2$ | A |
| 1 | 1 | A | $B_2$ | $B_1$ |
| 2 | 2 | A | $B_2$ | $B_1$ |
| | | A | $B_2$ | $B_1$ |

Pf by counterexample. Long-distance exchange can move one equal item past another one.

# Stability: shellsort

Proposition. Shellsort sort is not stable.

```
public class Shell
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      int h = 1;
      while (h < N/3) h = 3*h + 1;
      while (h >= 1)
      {
         for (int i = h; i < N; i++)
         {
            for (int j = i; j > h && less(a[j], a[j-h]); j -= h)
               exch(a, j, j-h);
         }
         h = h/3;
      }
   }
}
```

| h | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $A_1$ |
| 4 | $A_1$ | $B_2$ | $B_3$ | $B_4$ | $B_1$ |
| 1 | $A_1$ | $B_2$ | $B_3$ | $B_4$ | $B_1$ |
| | $A_1$ | $B_2$ | $B_3$ | $B_4$ | $B_1$ |

P

# Stability: mergesort

Proposition. Mergesort is stable.

```
public class Merge
{
   private static void merge(...)
   {  /* as before */  }

   private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort(a, aux, lo, mid);
      sort(a, aux, mid+1, hi);
      merge(a, aux, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {  /* as before */  }
}
```

Pf. Suffices to verify that merge operation is stable.

# Stability: mergesort

Proposition. Merge operation is stable.

```
private static void merge(...)
{
   for (int k = lo; k <= hi; k++)
      aux[k] = a[k];

   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)              a[k] = aux[j++];
      else if (j > hi)               a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                           a[k] = aux[i++];
   }
}
```

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $A_1$ | $A_2$ | $A_3$ | B | D | $A_4$ | $A_5$ | C | E | F | G |

Pf. Takes from left subarray if equal keys.

# Sorting summary

| | inplace? | stable? | best | average | worst | remarks |
|---|---|---|---|---|---|---|
| selection | ✓ | | $\frac{1}{2} N^2$ | $\frac{1}{2} N^2$ | $\frac{1}{2} N^2$ | $N$ exchanges |
| insertion | ✓ | ✓ | $N$ | $\frac{1}{4} N^2$ | $\frac{1}{2} N^2$ | use for small $N$ or partially ordered |
| shell | ✓ | | $N \log_3 N$ | ? | $c\, N^{3/2}$ | tight code; subquadratic |
| merge | | ✓ | $\frac{1}{2} N \lg N$ | $N \lg N$ | $N \lg N$ | $N \log N$ guarantee; stable |
| timsort | | ✓ | $N$ | $N \lg N$ | $N \lg N$ | improves mergesort when preexisting order |
| ? | ✓ | ✓ | $N$ | $N \lg N$ | $N \lg N$ | holy sorting grail |