Smaller project in third week

We study finite difference methods in one dimension with different boundary conditions. We look at a rather simple equation where exact solutions are known:

$$y''(x) - y(x) = 0$$

on the interval $[0, 1]$. We know well that the general solution is given by

$$y_{\text{general}}(x) = c_1 e^x + c_2 e^{-x}$$

and the constants c_1 and c_2 are to be determined by the boundary conditions if the problem is well-defined.

1. We divide the interval $[0, 1]$ in $n - 1$ subintervals of length h with

$$x_1 = 0 < x_2 < \cdots < x_{n-1} < x_n = 1$$

and we let y_i be the approximate solution at x_i. Discretize the equation at the inner points $x_2, \cdots, x_{n-1}$ by using

$$f''(x) \simeq \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}$$

You have to construct a total of $n - 2$ equations for the variables $y_2, \cdots, y_{n-1}$.

2. We first consider Dirichlet boundary equations, that is conditions on y itself in the endpoints:

$$y(0) = 1 \qquad y(1) = -1$$

Note that the values of y_1 and y_n are known (and have to be added in somewhere). Solve the linear system for $y_2, \cdots, y_{n-1}$, plot the solution and compare to the actual solution

$$y_2 = -0.5820e^x + 1.5820e^{-x}$$

3. We now consider Neumann boundary equations, that is conditions on y' at the endpoints:

$$y'(0) = 0 \qquad y'(1) = 1$$

The equations from question 1 are still valid, but we also have to construct two extra equations for y_1 and y_n. Use the discretizations

$$f'(x) \simeq \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h}$$

and

$$f'(x) \simeq \frac{-3f(x) + 4f(x-h) - f(x-2h)}{-2h}$$

Solve the linear system for $y_1, \cdots, y_n$, plot the solution and compare to the actual solution

$$y_3 = 0.4255e^{x} + 0.4255e^{-x}$$

4. Same problem with Robin boundary conditions (mix of y and y'):

$$y'(0) = y(0) + 1 \qquad y'(1) = -y(1)$$

The actual solution is

$$y_4 = -0.5e^{-x}$$