Modular forms and L-functions

Emile T. Okada

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CHAPTER 1

Preliminary analysis

1.1 Fourier transform

Let G be a locally compact group. Then $\hat{G}$ has the strucutre of a locally compact group and there is a canonical map $G \to \hat{\hat{G}}$.

Thm 1.1.1. *The canonical map $G \to \hat{\hat{G}}$ is an isomorphism.*

Proof. TODO. ■

Definition 1.1.2. Let G be a topological group. A *Haar measure* is a left translation-invariant Borel measure on G satisfying some regularity conditions (e.g. being finite on compact sets).

Thm 1.1.3. *Let G be a locally compact abelian group. Then there is a Haar measure on G, unique up to scaling.*

Proof. TODO. ■

Example 1.1.4. The Haar measure on $\mathbb{R}^{\times}_{>0}$ is $\mu(S) = \int_{\mathbb{R}_{>0}} 1_S(x)dx/x$.

Definition 1.1.5. Let G be a locally compact abelian group with a Haar measure dg, and let $f : G \to \mathbb{C}$ be a continuous L^1 function. The *Fourier transform* $\hat{f} : \hat{G} \to \mathbb{C}$ is given by

$$\hat{f}(\chi) = \int_G \chi(g)^{-1} f(g) dg. \tag{1.1}$$

Thm 1.1.6. *Given a locally compact abelian group G with a fixed Haar measure, there is some constant C such that for "suitable" $f : G \to \mathbb{C}$, we have*

$$\hat{\hat{f}}(g) = Cf(-g) \tag{1.2}$$

using the canonical isomorphism $G \to \hat{\hat{G}}$.

Thm 1.1.7. *(Poisson summation). Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\sum_{a \in \mathbb{Z}^n} f(a) = \sum_{b \in \mathbb{Z}^n} \hat{f}(b). \tag{1.3}$$

Proof. Let $g(x) = \sum_{a \in \mathbb{Z}^n} f(x + a)$ and consider its fourier transform as a function on $\mathbb{R}^n / \mathbb{Z}^n$. ∎

1.2 Mellin transform and Γ-function

Definition 1.2.1. (Mellin transform). Let $f : \mathbb{R}_{>0} \to \mathbb{C}$ be a continuous function. Define

$$M(f, s) = \int_0^\infty y^s f(y) \frac{dy}{y} \tag{1.4}$$

whenever it converges.

Lemma 1.2.2. *Let $f : \mathbb{R}_{>0} \to \mathbb{C}$ be continuous. If (a) $y^N f(y) \to 0$ as $y \to \infty$ for all $N \in \mathbb{Z}$ and (b) there exists m such that $|y^m f(y)|$ is bounded as $y \to 0$, then $M(f, s)$ converges and is analytic on $\Re(s) > m$.*

Proof. For $0 < r < R < \infty$, the function

$$\int_r^R y^s f(y) \frac{dy}{y} \tag{1.5}$$

is analytic for all s. TODO. ∎

Remark 1.2.3. When $G = \mathbb{R}_{>0}$, $\hat{G} = \{y \mapsto y^{i\sigma} : \sigma \in \mathbb{R}\} \cong \mathbb{R}$ so the Mellin transform is the analytic continuation of the fourier transform of f.

Proposition 1.2.4.

$$M(f(\alpha y), s) = \alpha^{-s} M(f, s) \tag{1.6}$$

for $\alpha > 0$.

Proof. Obvious. ∎

1.2.1 The Γ-function

Definition 1.2.5. (Γ-function). The Γ-function is the Mellin transform of $f(y) = e^{-y}$.

This is analytic for $\Re(s) > 0$.

Proposition 1.2.6.

$$\Gamma(1) = 1, \quad s\Gamma(s) = \Gamma(s+1), s \neq 0. \tag{1.7}$$

Proof. First part obvious. For second part:

$$\Gamma(s) = \int_0^\infty e^{-y} y^{s-1} dy = \left[e^{-y} \frac{y^s}{s} \right]_0^\infty + \frac{1}{s} \int_0^\infty e^{-y} y^s dy = \frac{1}{s} \Gamma(s+1). \quad (1.8)$$

∎

Iterating this result we get $\Gamma(s) = \left(\prod_{i=0}^N (s+i) \right)^{-1} \Gamma(s+N+1)$ and so we can extend Γ to a meromorphic function on $\mathbb{C}$ with

$$\operatorname{res}_{s=-N} \Gamma(s) = \frac{(-1)^N}{N!} \quad (1.9)$$

for $N \geq 0$.

Proposition 1.2.7.

$$\Gamma(s)^{-1} = e^{\gamma s} s \prod_{n \geq 1} \left(1 + \frac{s}{n} \right) e^{-s/n} \quad (1.10)$$

$$\pi^{1/2} \Gamma(2s) = 2^{2s-1} \Gamma(s) \Gamma \left(s + \frac{1}{2} \right) \quad (1.11)$$

$$\Gamma(s) \Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \quad (1.12)$$

for all $s \in \mathbb{C}$.

Remark 1.2.8. Merllin transforms are important because a lot of Dirichlet series are Mellin transforms. Consider the series $\sum_{n \geq 1} a_n / n^s$. We can write

$$(2\pi)^{-s} \Gamma(s) \sum_{n=1}^\infty \frac{a_n}{n^s} = \sum_{n=1}^\infty a_n (2\pi n)^{-s} M(e^{-y}, s)$$

$$= \sum_{n=1}^\infty M(a_n e^{-2\pi n y}, s)$$

$$= M(f, s) \quad (1.13)$$

where $f = \sum_{n=1}^\infty a_n e^{-2\pi n y}$.

1.3 Riemann ζ-function

Definition 1.3.1. (Riemann ζ-functions). The *Riemann ζ-function* is defined by

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \quad (1.14)$$

for $\Re(s) > 1$.

Proposition 1.3.2.

$$\zeta(s) = \prod_{p \ prime} \frac{1}{1 - p^{-s}}. \tag{1.15}$$

Thm 1.3.3. *If $\Re(s) > 1$, then*

$$(2\pi)^{-s}\Gamma(s)\zeta(s) = \int_0^\infty \frac{y^s}{e^{2\pi y} - 1} \frac{dy}{y} = M(f, s) \tag{1.16}$$

where

$$f(y) = \frac{1}{e^{2\pi y} - 1}. \tag{1.17}$$

Proof. Let

$$f(y) = \frac{e^{-2\pi y}}{1 - e^{-2\pi y}} = \sum_{n \geq 1} e^{-2\pi n y} \tag{1.18}$$

for $y > 0$. As $y \to 0$, $f(y) \sim 1/2\pi y$. So taking the Mellin transform of f gives the required result. ∎

Corollary 1.3.4. *$\zeta(s)$ extends meromorphically to $\mathbb{C}$ with a single pole located at $s = 1$ which is simple and has $\text{res}_{s=1} \zeta(s) = 1$.*

Proof. Write

$$M(f, s) = M_0 + M_\infty = \left(\int_0^1 + \int_1^\infty \right) \frac{y^s}{e^{2\pi y} - 1} \frac{dy}{y}. \tag{1.19}$$

M_∞ is holomorphic on $\mathbb{C}$. For fixed N, we can expand

$$f(y) = \sum_{n=-1}^{N-1} c_n y^n + y^N g_N(y) \tag{1.20}$$

for some $g \in C^\infty(\mathbb{R})$, since f has a simple pole at $s = 0$, and $c_{-1} = 1/2\pi$. Thus for $\Re(s) > 1$,

$$M_0 = \sum_{n=-1}^{N-1} c_n \int_0^1 y^{n+s-1} dy + \int_0^1 y^{N+s-1} g_N(y) dy$$

$$= \sum_{n=-1}^{N-1} \frac{c_n}{s+n} y^{n+s} + \int_0^1 g_N(y) y^{s+N-1} dy. \tag{1.21}$$

This formula makes sense for $\Re(s) > -N$. Thus we may extend $(2\pi)^{-s}\Gamma(s)\zeta(s)$ to $\mathbb{C}$. This has at worst simple poles at $1, 0, -1, -2, \dots$. But $\Gamma(s)$ has simple poles at $0, -1, -2, \dots$ so $\zeta(s)$ must be analytic at $s = 0, -1, -2 \dots$. Since $\Gamma(1) = 1$, $\text{res}_{s=1} \zeta(s) = 1$. ∎

Definition 1.3.5. (Bernoulli numbers). The *Bernoulli numbers* are defined by

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}. \tag{1.22}$$

Corollary 1.3.6.

$$\zeta(0) = B_1, \quad \zeta(1-n) = -\frac{B_n}{n}, n > 1. \tag{1.23}$$

Proof. Keeping with the notation from the proof of the previous corollary, $(2\pi)^{-s}\Gamma(s)\zeta(s)$ has a simple pole at $s = 1 - n$ with residue c_{n-1}. Thus

$$c_{n-1} = (2\pi)^{n-1} \frac{B_n}{n!}. \tag{1.24}$$

But also

$$\mathrm{res}_{s=1-n}(2\pi)^{-s}\Gamma(s)\zeta(s) = (2\pi)^{n-1}\zeta(1-n)\,\mathrm{res}_{s=1-n}\Gamma(s). \tag{1.25}$$

We know the residue of $\Gamma(s)$ and so plugging in its value and equating gives the result. $\blacksquare$

Definition 1.3.7. (Jacobi's ϑ-function).

$$\vartheta(s) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 s} \tag{1.26}$$

on $\Im(s) > 0$.

We then define $\Theta(y) = \vartheta(iy)$. Note that $\Theta(y) \to 1$ as $y \to \infty$.

Proposition 1.3.8.

$$M\left(\frac{\Theta(y) - 1}{2}, \frac{s}{2}\right) = \pi^{-s/2}\gamma\left(\frac{s}{2}\right)\zeta(s). \tag{1.27}$$

Proof. Obvious. $\blacksquare$

Thm 1.3.9. *If $y > 0$, then*

$$\Theta\left(\frac{1}{y}\right) = y^{1/2}\Theta(y), \tag{1.28}$$

where we are taking the positive square root.

Proof. Let $g_t(x) = e^{-tx^2}$. Recall that $\hat{g}_t(y) = t^{-1/2}e^{-\pi y^2/t}$. By Poisson summation

$$\Theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = \sum_{n \in \mathbb{Z}} g_t(n) = \sum_{n \in \mathbb{Z}} \hat{g}_t(n) = t^{-1/2}\Theta(1/t). \tag{1.29}$$

$\blacksquare$

Corollary 1.3.10.

$$\vartheta\left(-\frac{1}{z}\right) = \left(\frac{z}{i}\right)^{1/2}\vartheta(z), \tag{1.30}$$

where we are taking the branch of the square root that is positive on the positive real axis.

Proof. Follows from analytic continuation. ∎

Notation 1.3.11.

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2) \tag{1.31}$$

$$\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s) \tag{1.32}$$

$$Z(s) = \Gamma_{\mathbb{R}}(s)\zeta(s). \tag{1.33}$$

Thm 1.3.12.

$$Z(s) = Z(1-s). \tag{1.34}$$

Moreover, $Z(s)$ is meromorphic with the poles located at $s = 0$ and $s = 1$.

Proof. Have

$$2Z(s) = M(\Theta(y) - 1, s/2) = \left(\int_0^1 + \int_1^\infty\right)(\Theta(y) - 1)y^{s/2}\frac{dy}{y}. \tag{1.35}$$

But

$$\int_0^1 (\Theta(y) - 1)y^{s/2}\frac{dy}{y} = \int_0^1 (\Theta(y) - y^{-1/2})y^{s/2} + \int_0^1 y^{(s-1)/2} - y^{s/2})\frac{dy}{y}$$

$$= \int_0^1 (\Theta(1/y) - 1)y^{(s-1)/2}\frac{dy}{y} + \frac{2}{s-1} - \frac{2}{s}$$

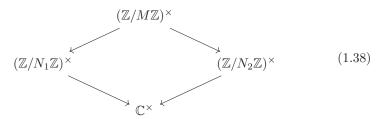$$= \int_1^\infty (\Theta(y) - 1)y^{(1-s)/2}\frac{dy}{y} + \frac{2}{s-1} - \frac{2}{s}. \tag{1.36}$$

Thus

$$2Z(s) = \int_1^\infty (\Theta(y) - 1)(y^{s/2} + y^{(1-s)/2})\frac{dy}{y} + \frac{2}{s-1} - \frac{2}{s} = 2Z(1-s). \tag{1.37}$$

∎

1.4 Dirichlet L-functions

Definition 1.4.1. (Dirichlet characters). Let $N \geq 1$. A *Dirichlet character mod N* is a character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$.

Definition 1.4.2. (Equivalent characters). We say $\chi_1 \in (\widehat{\mathbb{Z}/N_1\mathbb{Z}})^\times$ and $\chi_2 \in (\widehat{\mathbb{Z}/N_2\mathbb{Z}})^\times$ are *equivalent* if there exists an M such that $N_1, N_2 \mid M$ and

$$
\begin{array}{ccc}
 & (\mathbb{Z}/M\mathbb{Z})^\times & \\
\swarrow & & \searrow \\
(\mathbb{Z}/N_1\mathbb{Z})^\times & & (\mathbb{Z}/N_2\mathbb{Z})^\times \\
\searrow & & \swarrow \\
 & \mathbb{C}^\times &
\end{array}
\tag{1.38}
$$

commutes.

Note that the projection maps are surjective and so this defines an equivalence relation.

Definition 1.4.3. (Primitive character). We say $\chi \in (\widehat{\mathbb{Z}/N\mathbb{Z}})^\times$ is *primitive* if there is no $M < N$ with $M \mid N$ such that χ factors through $(\mathbb{Z}/M\mathbb{Z})^\times$.

Lemma 1.4.4. *Let $\chi \in (\widehat{\mathbb{Z}/N\mathbb{Z}})^\times$ and $M \mid N$. Then the following are equivalent*

1. *χ factors through $(\mathbb{Z}/M\mathbb{Z})^\times$*

2. *χ is constant on the fibers of $p : (\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/M\mathbb{Z})^\times$*

3. *$x, x + aM \in (\mathbb{Z}/N\mathbb{Z})^\times \implies \chi(x) = \chi(x + aM)$ for all $a \in \mathbb{Z}$.*

Proof. $(2) \Rightarrow (1)$. We can certainly define a function $\chi' : (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C}^\times$ so that χ factors through χ'. It remains to show that χ' is a homomorphism. Let $x, y \in (\mathbb{Z}/M\mathbb{Z})^\times$ and $a, b \in (\mathbb{Z}/N\mathbb{Z})^\times$ be preimages of x and y respectively. Then
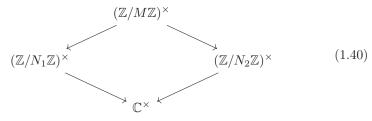
$$
\chi'(xy) = \chi'(p(a)p(b)) = \chi'(p(ab)) = \chi(ab) = \chi(a)\chi(b) = \chi'(x)\chi'(y) \tag{1.39}
$$

so χ' is a homomorphism.

The remaining implications are obvious. ∎

Corollary 1.4.5. *Let $\chi_1 \in (\widehat{\mathbb{Z}/N_1\mathbb{Z}})^\times$ and $\chi_2 \in (\widehat{\mathbb{Z}/N_2\mathbb{Z}})^\times$ be equivalent. Then there exists a $\chi \in (\widehat{\mathbb{Z}/d\mathbb{Z}})^\times$ where $d = (N_1, N_2)$, such that both χ_1 and χ_2 factor through χ.*

Proof. Write $d = aN_1 + bN_2$ for $a, b \in \mathbb{Z}$ and let M be such that $N_1, N_2 \mid M$ and

$$
\begin{array}{ccc}
 & (\mathbb{Z}/M\mathbb{Z})^\times & \\
\swarrow & & \searrow \\
(\mathbb{Z}/N_1\mathbb{Z})^\times & & (\mathbb{Z}/N_2\mathbb{Z})^\times \\
\searrow & & \swarrow \\
 & \mathbb{C}^\times &
\end{array}
\tag{1.40}
$$

commutes. Let $\psi \in (\widehat{\mathbb{Z}/M\mathbb{Z}})^\times$ be the vertical map. This map is constant on ∎

Corollary 1.4.6. *If $\chi \in (\widehat{\mathbb{Z}/N\mathbb{Z}})^\times$ then there exists a unique $M \mid N$ and a primitive $\chi_* \in (\widehat{\mathbb{Z}/M\mathbb{Z}})^\times$ such that it is equivalent to χ.*

Proof. Obvious from previous corollary. ∎

Definition 1.4.7. (Dirichlet L-series). Let $\chi \in (\widehat{\mathbb{Z}/N\mathbb{Z}})^\times$. The *Dirichlet L-series* of χ is

$$L(\chi, s) = \sum_{n \geq 1, (n,N)=1} \chi(n) n^{-s}. \tag{1.41}$$

Proposition 1.4.8.

$$L(\chi, s) = \prod_{p \nmid N} \frac{1}{1 - \chi(p)p^{-s}}. \tag{1.42}$$

Proposition 1.4.9. *Suppose $M \mid N$ and $\chi_N \in (\widehat{\mathbb{Z}/N\mathbb{Z}})^\times$ factors through $\chi_M \in (\widehat{\mathbb{Z}/M\mathbb{Z}})^\times$. Then*

$$L(\chi_M, s) = \prod_{p \nmid M, p \mid N} \frac{1}{1 - \chi_M(p)p^{-s}} L(\chi_N, s). \tag{1.43}$$

In particular,

$$\frac{L(\chi_M, s)}{L(\chi_N, s)} = \prod_{p \nmid M, p \mid N} \frac{1}{1 - \chi_M(p)p^{-s}} \tag{1.44}$$

is analytic and non-zero on $\Re(s) > 0$.

Thm 1.4.10. *1. $L(\chi, s)$ has a meromorphic continuation to $\mathbb{C}$, which is analytic except for at worst a simple pole at $s = 1$.*

2. If $\chi \neq \chi_0$ (the trivial character), then $L(\chi, s)$ is analytic everywhere. On the other hand, $L(\chi_0, s)$ has a simple pole with residue

$$\frac{\phi(N)}{N} = \prod_{p \mid N} \left(1 - \frac{1}{p} \right). \tag{1.45}$$

Proof. Let $\phi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ be any N-periodic function, and let $L(\phi, s) = \sum_{n=1}^\infty \phi(n) n^{-s}$. Then

$$(2\pi)^{-s} \Gamma(s) L(\phi, s) = \sum_{n=1}^\infty \phi(n) M(e^{-2\pi y}, s) = M(f(y), s), \tag{1.46}$$

where $f(y) = \sum_{n=1}^\infty \phi(n) e^{-2\pi n y}$. A straightforward calculation shows that

$$f(y) = \sum_{n=1}^N \phi(n) \frac{e^{2\pi(N-n)y}}{e^{e\pi N y} - 1}. \tag{1.47}$$

Note that this is $\mathcal{O}(e^{-2\pi y})$ as $y \to \infty$. Write $M(f,s) = M_0(s) + M_\infty(s)$ as before. The second term is analytic for all $s \in \mathbb{C}$, and the first one is given by

$$M_0(s) = \sum_{n=1}^{N} \phi(n) \int_0^1 \frac{e^{2\pi(N-n)y}}{e^{2\pi Ny} - 1} y^s \frac{dy}{y}. \tag{1.48}$$

For any L we can write

$$\frac{e^{2\pi(N-n)y}}{e^{2\pi Ny} - 1} = \frac{1}{2\pi Ny} + \sum_{r=0}^{L-1} c_{r,n} y^r + y^L g_{L,n}(y) \tag{1.49}$$

for some $g_{L,n}(y) \in C^\infty[0,1]$. Hence

$$M_0(s) = \sum_{n=1}^{N} \phi(n) \left(\int_0^1 \frac{1}{2\pi Ny} y^s \frac{dy}{y} + \int_0^1 \sum_{r=0}^{L-1} c_{r,n} y^{r+s-1} dy \right) + G(s), \tag{1.50}$$

where G is a function that is analytic on $\Re(s) > -L$. Thus

$$(2\pi)^{-s} \Gamma(s) L(\phi,s) = \sum_{n=1}^{N} \phi(n) \left(\frac{1}{2\pi N(s-1)} + \frac{c_{0,n}}{s} + \cdots + \frac{c_{L-1,n}}{s+L-1} \right) + G(s). \tag{1.51}$$

The first part follows from setting

$$\phi(n) = \begin{cases} \chi(n) & \text{if } (n,N) = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1.52}$$

It also follows that

$$\mathrm{res}_{s=1} L(\chi,s) = \frac{1}{N} \sum_{n=1}^{N} \phi(n). \tag{1.53}$$

If $\chi \neq \chi_0$ this vanishes by orthogonality of characters. Otherwise it is equal to $|(\mathbb{Z}/N\mathbb{Z})^\times|/N = \phi(N)/N$. $\blacksquare$

lem:ord **Lemma 1.4.11.** *If $p \nmid N$ and T is any complex number, then*

$$\prod_{\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times}} (1 - \chi(p)T) = (1 - T^{f_p})^{\phi(N)/f_p} \tag{1.54}$$

where f_p is the order of p in $(\mathbb{Z}/N\mathbb{Z})^\times$.

Proof. Fix a p, write $f = f_p$, $G = (\mathbb{Z}/N\mathbb{Z})^\times$ and $H = \langle p \rangle \subseteq G$. The projection map $G \to G/H$ induces an injection $\widehat{G/H} \to \hat{G}$ and the restriction map $\hat{G} \to \hat{H}$ induces an injection (and hence an isomorphism)

$$\frac{\hat{G}}{\widehat{G/H}} \to \hat{H}. \tag{1.55}$$

Also $\widehat{|G/H|} = |G/H| = \phi(N)/f$. Thus

$$\prod_{\chi \in \hat{G}} (1-\chi(p)T) = \prod_{\chi \in \hat{H}} (1-\chi(p)T)^{\phi(N)/f} = \prod_{\zeta \in \mu_f} (1-\zeta T)^{\phi(N)/f} = (1-T^f)^{\phi(N)/f}. \tag{1.56}$$

∎

lem:conv   **Lemma 1.4.12.** *Let $D(s) = \sum_{n \geq 1} a_n n^{-s}$ be a Dirichlet series with real $a_n \geq 0$, and suppose that it is absolutely convergent for $\Re(s) > \sigma > 0$. Then if $D(s)$ can be analytically continued to an analytic function $\widetilde{D}$ on $\{\Re(s) > 0\}$, then the series converges for all real $s > 0$.*

Proof. Let $\rho > \sigma$. Then by the analytic continuation, we have a convergent Taylor series on $\{|s - \rho| < \rho\}$

$$D(s) = \sum_{k \geq 0} \frac{1}{k!} D^{(k)}(\rho)(s-\rho)^k. \tag{1.57}$$

Moreover, since $\rho > \sigma$, we can differentiate the Dirichlet series term by term to obtain the derivatives

$$D^{(k)}(\rho) = \sum_{n \geq 1} a_n (-\log n)^k n^{-\rho}. \tag{1.58}$$

Thus if $0 < x < \rho$,

$$D(x) = \sum_{k \geq 0} \frac{1}{k!} (\rho - x)^k \left(\sum_{n \geq 1} a_n (\log n)^k n^{-\rho} \right). \tag{1.59}$$

Since all the terms are non-negative, this must converge unconditionally and so rearranging we find

$$D(x) = \sum_{n \geq 1} a_n n^{-\rho} \sum_{k \geq 0} \frac{1}{k!} (\rho - x)^k (\log n)^k$$

$$= \sum_{n \geq 1} a_n n^{-\rho} e^{(\rho-x)\log n} = \sum_{n \geq 1} a_n n^{-x}. \tag{1.60}$$

∎

lem:div   **Lemma 1.4.13.**   *1.*

$$\sum_p p^{-x} \sim -\log(x-1) \tag{1.61}$$

     as $x \to 1^+$.

   2. If $\chi \neq \chi_0$ is a Dirichlet character mod N, then

$$\sum_{p \nmid N} \chi(p) p^{-x} \tag{1.62}$$

     is bounded as $x \to 1^+$.

Proof. Let χ be a Dirichlet character mod N. Then

$$\log L(\chi, x) = \sum_{p \nmid N} -\log(1 - \chi(p)p^{-x}) = \sum_{p \nmid N} \sum_{r \geq 1} \frac{\chi(p)^r p^{-rx}}{r} \tag{1.63}$$

so

$$\left| \log L(\chi, x) - \sum_{p \nmid N} \chi(p)p^{-x} \right| < \sum_{p \nmid N} \sum_{r \geq 2} p^{-rx}$$

$$= \sum_{p \nmid N} \frac{p^{-2x}}{1 - p^{-x}}$$

$$\leq \sum_{n \geq 1} \frac{n^{-2}}{1/2} \tag{1.64}$$

which is a finite constant. When $\chi = \chi_0$ and $N = 1$, then $|\log \zeta(x) - \sum_p p^{-x}|$ is bounded as $x \to 1^+$. But $\zeta(s) = 1/(s-1) + \mathcal{O}(s)$ and so

$$\sum_p p^{-x} \sim -\log(x - 1). \tag{1.65}$$

When $\chi \neq \chi_0$, then $L(\chi, 1) \neq 0$ by the next theorem and so $\log L(\chi, x)$ is bounded as $x \to 1^+$ which implies the result. $\blacksquare$

Thm 1.4.14. *If $\chi \neq \chi_0$ then $L(\chi, 1) \neq 0$.*

Proof. Let

$$\zeta_N(s) = \prod_{\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times}} L(\chi, s) = \prod_{p \nmid N} \prod_{\chi} (1 - \chi(p)p^{-s})^{-1} = L(\chi_0, s) \prod_{\chi \neq \chi_0} L(\chi, s) \tag{1.66}$$

for $\Re(s) > 1$. We know that $L(\chi_0, s)$ has a pole at $s = 1$ and is analytic elsewhere and so if any $L(\chi, 1) = 0$, then $\zeta_N(s)$ is analytic on $\Re(s) > 0$. Since the Dirichlet series for ζ_N has ≥ 0 coefficients, by lemma 1.4.12, it suffices to find some point on $\mathbb{R}_{>0}$ where the Dirichlet series for ζ_N does not converge. But by lemma 1.4.11

$$\zeta_N(s) = \prod_{p \nmid N} (1 - p^{-f_p s})^{-\phi(N)/f_p}$$

$$= \prod_{p \nmid N} (1 + p^{-f_p x} + p^{-2f_p x} + \cdots)^{\phi(N)/f_p}$$

$$\geq \sum_{p \nmid N} p^{-\phi(N)x}. \tag{1.67}$$

But $\sum_p p^{-1}$ diverges by lemma 1.4.13 and so the series for $\zeta_N(x)$ does not converge at $x = 1/\phi(N)$. $\blacksquare$

Thm 1.4.15. *Let $a \in \mathbb{Z}$ be such that $(a, N) = 1$. Then there exists infinitely many primes $p \equiv a \pmod{N}$.*

Proof. It suffices to show that

$$\sum_{p \equiv \ (\text{mod } N)} p^{-x} \tag{1.68}$$

diverges as $x \to 1^+$. But when $(x, N) = 1$ we have

$$\sum_{\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times}} \chi(x) = \begin{cases} \phi(N) & x \equiv 1 \pmod{N} \\ 0 & \text{otherwise} \end{cases} \tag{1.69}$$

by column orthogonality of the character table of $(\mathbb{Z}/N\mathbb{Z})^\times$. Thus

$$\sum_{p \equiv \ (\text{mod } N)} p^{-x} = \frac{1}{\phi(N)} \sum_{\chi \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a)^{-1} \sum_p \chi(p) p^{-x}. \tag{1.70}$$

If $\chi = \chi_0$, then the sum is just

$$\sum_{p \nmid N} p^{-x} \sim -\log(x - 1) \tag{1.71}$$

as $x \to 1^+$. By lemma 1.4.13 all the other sums are bounded and so the result follows. ∎

CHAPTER 2

The modular group

Definition 2.0.1.

$$\mathbf{GL}_2(\mathbb{R})^+ = \{\gamma \in \mathbf{GL}_2(\mathbb{R}) : \det \gamma > 0\} \tag{2.1}$$

$$\mathbf{PGL}_2(\mathbb{R})^+ = \mathbf{GL}_2(\mathbb{R})^+/\mathbb{R}^\times \cong \mathbf{PSL}_2(\mathbb{R}). \tag{2.2}$$

The latter is the group of Mobius transforms that map $\mathcal{H}$ to $\mathcal{H}$, and the stabiliser of i is $\mathbf{SO}(2)/\{\pm I\}$. In fact $\mathbf{PSL}_2(\mathbb{R})$ is the group of all holomorphic automorphisms of $\mathcal{H}$, and the subgroup $\mathbf{SO}(2) \subseteq \mathbf{SL}_2(\mathbb{R})$ is a maximal compact subgroup.

Thm 2.0.2. *The group* $\mathbf{SL}_2(\mathbb{R})$ *admits the Iwasawa decomposition*

$$\mathbf{SL}_2(\mathbb{R}) = KAN = NAK \tag{2.3}$$

where

$$K = \mathbf{SO}(2), \quad A = \left\{ \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \right\}, \quad N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}. \tag{2.4}$$

Proof. Let $A \in \mathbf{SL}_2(\mathbb{R})$. Then there exists a $B \in \mathbf{SO}(2)$ so that BA is upper triangular. The rest follows easily. ∎

Definition 2.0.3. For $\Gamma \leq \mathbf{SL}_2(\mathbb{R})$, we write $\bar{\Gamma}$ for the image in $\mathbf{PSL}_2(\mathbb{R})$.

Definition 2.0.4. The *modular group* is

$$\mathbf{PSL}_2(\mathbb{Z}) = \frac{\mathbf{SL}_2(\mathbb{Z})}{\{\pm I\}}. \tag{2.5}$$

Thm 2.0.5. *Let*

$$\mathcal{D} = \left\{ z \in \mathcal{H} : -\frac{1}{2} \leq \Re(z) \leq \frac{1}{2}, |z| > 1 \right\} \cup \left\{ z \in \mathcal{H} : |z| = 1, \Re(s) \geq 0 \right\}. \tag{2.6}$$

Then $\mathcal{D}$ *is a fundamental domain for the action of* $\mathbf{SL}_2(\mathbb{Z})$ *on* $\mathcal{H}$. *The stabiliser of* $z \in \mathcal{D}$ *in* $\mathbf{SL}_2(\mathbb{Z})$ *is trivial if* $z \neq i, \rho$, *and*

$$\bar{\Gamma}_i = \langle S \rangle \cong C_2, \quad \bar{\Gamma}_\rho = \langle TS \rangle \cong C_3. \tag{2.7}$$

Proposition 2.0.6. *The measure*

$$d\mu = \frac{dxdy}{y^2} \tag{2.8}$$

is invariant under $\mathbf{PSL}_2(\mathbb{R})$. *If* $\Gamma \subseteq \mathbf{PSL}_2(\mathbb{Z})$ *is of finite index then* $\mu(\Gamma \backslash \mathcal{H})$.

Proof. Note that

$$\frac{dx \wedge dy}{y^2} = \frac{idz \wedge d\bar{z}}{2(\Im(z))^2}. \tag{2.9}$$

It is then straightforward to see that it is invariant under $\mathbf{PSL}_2(\mathbb{R})$. For the last part it suffices to show that $\mu(\mathcal{D})$ is finite. A straightforward calculation shows that

$$\mu(\mathcal{D}) = \int_{\mathcal{D}} \frac{dxdy}{y^2} = \frac{\pi}{3}. \tag{2.10}$$

■

Definition 2.0.7. (Principal congruence subgroup). For $N \geq 1$, the *principal congruence subgroup* of level N is

$$\Gamma(N) = \ker(\mathbf{SL}_2(\mathbb{Z}) \to \mathbf{SL}_2(\mathbb{Z}/N\mathbb{Z})). \tag{2.11}$$

Any $\Gamma \subseteq \mathbf{SL}_2(\mathbb{Z})$ containing some $\Gamma(N)$ is called a *congruence subgroup*, and its *level* is the smallest N such that $\Gamma \supseteq \Gamma(N)$. $\Gamma_0(N)$ is defined as the preimage of upper triangular matrices, and $\Gamma_1(N)$ the preimage of strictly upper triangular matrices.

CHAPTER 3

Modular forms of level 1

3.1 Basic definitions

Definition 3.1.1. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in \mathbf{GL}_2(\mathbb{R})^+, z \in \mathcal{H}, \tag{3.1}$$

and $f : \mathcal{H} \to \mathbb{C}$ be any function. We write $j(\gamma, z) = cz + d$. We define the *slash operator* to be

$$(f\underset{k}{|}\gamma)(z) = (\det \gamma)^{k/2} j(\gamma, z)^{-k} f(\gamma(z)). \tag{3.2}$$

Remark 3.1.2. It will be helpful to note that

$$\gamma \begin{pmatrix} z \\ 1 \end{pmatrix} = j(\gamma, z) \begin{pmatrix} \gamma z \\ 1 \end{pmatrix}. \tag{3.3}$$

Proposition 3.1.3. *1.* $j(\gamma\delta, z) = j(\gamma, \delta z)j(\delta, z)$

2. $j(\gamma^{-1}, z) = j(\gamma, \gamma^{-1}(s))^{-1}$

3. $\gamma : g \mapsto g\underset{k}{|}\gamma$ *is a right action of* $\mathbf{GL}_2(\mathbb{R})^+$ *on functions on* $\mathcal{H}$.

Proof. 1. We have

$$j(\gamma\delta, z) \begin{pmatrix} \gamma\delta z \\ 1 \end{pmatrix} = \gamma\delta \begin{pmatrix} z \\ 1 \end{pmatrix} = j(\delta, z)\gamma \begin{pmatrix} \delta z \\ 1 \end{pmatrix} = j(\delta, z)j(\gamma, \delta z) \begin{pmatrix} \gamma\delta z \\ 1 \end{pmatrix}. \tag{3.4}$$

2. Set $\delta = \gamma^{-1}$.

3. Follows from (i).

■

CHAPTER 4

Hecke operators

Thm 4.0.1. *Let $G = \mathbf{GL}_2(\mathbb{Q})$ and $\Gamma \subseteq \mathbf{SL}_2(\mathbb{Z})$ be a subgroup of finite index. Then for all $g \in G$, $|\Gamma : \Gamma \cap g^{-1}\Gamma| < \infty$.*

Proof. Consider the case $\Gamma = \mathbf{SL}_2(\mathbb{Z})$, and

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}). \tag{4.1}$$

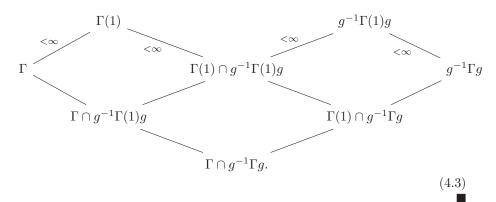We claim that if $\det g = \pm N$, $N \geq$ then $g^{-1}\Gamma g \supseteq \Gamma(N)$ from which the theorem follows. To do this it suffices to show that if $\gamma \in \Gamma(N)$ then $g\gamma g^{-1}$ has integer entries. But

$$\pm N g \gamma g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \gamma \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \equiv NI \equiv 0 \pmod{N} \tag{4.2}$$

so $g\gamma g^{-1}$ has integer entries.

Now, in general if $g' \in \mathbf{GL}_2(\mathbb{Q})$ then we can write $g'\frac{1}{M}g$ for g with integer entries, and conjugating by g' and g yields the same result.

The general case follows from the fact that if $|G : H| < \infty$ and $|G : H'| < \infty$ then $|G : H \cap H'| < \infty$ and the following diagram

$$\tag{4.3}$$

∎

Notation 4.0.2. Let G be a group, M a $\mathbb{Z}G$-module and $\Gamma \leq G$. For $g \in G$ and $m \in M^{\Gamma}$, we write

$$m|[\Gamma g \Gamma] = \sum_{i=1}^{n} m g_i, \qquad (4.4)$$

where

$$\Gamma g \Gamma = \coprod_{i=1}^{n} \Gamma g_i. \qquad (4.5)$$

Proposition 4.0.3. *1. $m|[\Gamma g \Gamma]$ depends only on $\Gamma g \Gamma$.*

2. $m|[\Gamma g \Gamma] \in M^{\Gamma}$.

Proof. Obvious. ∎

Thm 4.0.4. *There is a product on $\mathcal{H}(G, \Gamma)$, the free abelian group generated by double cosets, making it into an associative ring, the Hecke algebra of (G, Γ), with unit $[\Gamma e \Gamma] = [\Gamma]$, such that for every G-module M, we have M^{Γ} is a right $\mathcal{H}(G, \Gamma)$-module by the operation $|$.*

Proof. There is an obvious bijection $\Theta : \mathcal{H}(G, \Gamma) \to M$ where $M = \mathbb{Z}[\Gamma \backslash G]^{\Gamma}$. We can view $\Gamma g \Gamma$ as living in M^{Γ}. We then define

$$[\Gamma g \Gamma] \cdot [\Gamma h \Gamma] = \Theta^{-1}((\Gamma g \Gamma)|[\Gamma h \Gamma]). \qquad (4.6)$$

This is clearly well defined. If we write

$$\Gamma g \Gamma = \coprod \Gamma g_i \qquad (4.7)$$

$$\Gamma h \Gamma = \coprod \Gamma h_j \qquad (4.8)$$

then

$$[\Gamma g \Gamma] \cdot [\Gamma h \Gamma] = \Theta^{-1}\left(\sum_{i,j} [\Gamma g_i h_j]\right). \qquad (4.9) \quad \boxed{\texttt{eq:mult}}$$

This then is clearly associative and has unit $[\Gamma e \Gamma]$. The rest follows in a staightforward manner. ∎

Remark 4.0.5.

$$[\Gamma g \Gamma] \cdot [\Gamma h \Gamma] = \sum_{k \in S} \sigma(k)[\Gamma k \Gamma] \qquad (4.10)$$

where S is a set of double coset representatives and $\sigma(k)$ is the number of pairs (i, j) such that $\Gamma g_i h_j = \Gamma k$. This follows immediately form equation 4.9.

4.1 Hecke operators on modular forms

Let $G = \mathbf{GL}_2(\mathbb{Q})^+$ and $\Gamma = \mathbf{SL}_2(\mathbb{Z})$. We wish to understand the single and double cosets.

We start by understanding the cosets of $\Gamma \subseteq \mathbf{GL}_2(\mathbb{Z})^+$. There is a bijection

$$\left\{ \begin{array}{c} \text{Cosets } \Gamma\gamma \text{ such that} \\ \gamma \in \mathbf{Mat}_2(\mathbb{Z}), \det\gamma = n \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Subgroups } \Lambda \subseteq \mathbb{Z}^2 \\ \text{of index } n \end{array} \right\} \tag{4.11}$$
$$\Gamma\gamma \quad \rightarrow \quad \text{row span of } \gamma$$
$$\text{oriented generators of } \Lambda \quad \leftarrow \quad \Lambda$$

This follows essentially from Pick's theorem.

We want to use this bijection to pick coset representatives. Fix a $\Lambda \subseteq \mathbb{Z}^2$ of index n. Let $d \in \mathbb{N}$ be such that $\Lambda \cap \mathbb{Z}e_2 = (d\mathbb{Z})e_2$. Let $\lambda = ae_1 + be_2 \in \Lambda$ with $a > 0$ minimal. There is a unique such λ with $0 \le b < d$. It is clear from this construction that $\Lambda = \langle ae_1 + be_2, de_2 \rangle$. In fact this gives us a new bijection

$$\left\{ \begin{array}{c} \text{Subgroups } \Lambda \subseteq \mathbb{Z}^2 \\ \text{of index } n \end{array} \right\} \leftrightarrow \Pi_n \tag{4.12}$$

where

$$\Pi_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathbf{Mat}_2(\mathbb{Z}) : a, d \ge 1, ad = n, 0 \le b < d \right\}. \tag{4.13}$$

It follows that

$$\{\gamma \in \mathbf{Mat}_2(\mathbb{Z}) : \det\gamma = n\} = \coprod_{\gamma \in \Pi_n} \Gamma\gamma. \tag{4.14}$$

Proposition 4.1.1. *1. Let $\gamma \in \mathbf{Mat}_2(\mathbb{Z})$ and $\det\gamma = n \ge 1$. Then*

$$\Gamma\gamma\Gamma = \Gamma \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix} \Gamma \tag{4.15}$$

for unique $n_1, n_2 \ge 1$ and $n_2 \mid n_1, n_1 n_2 = n$.

2.

$$\{\gamma \in \mathbf{Mat}_2(\mathbb{Z}) : \det\gamma = n\} = \coprod \Gamma \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix} \Gamma \tag{4.16}$$

where the sum is over $n_1, n_2 \ge 1, n_2 \mid n_1, n_1 n_2 = n$.

3. Let γ, n_1, n_2 be as above, if $d \ge 1$, then

$$\Gamma(d^{-1}\gamma)\Gamma = \Gamma \begin{pmatrix} n_1/d & 0 \\ 0 & n_2/d \end{pmatrix} \Gamma. \tag{4.17}$$

Proof. Smith normal form (though some extra work needs to be done). ∎

Corollary 4.1.2. *The set*

$$\left\{ \Gamma \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \Gamma : r_1, r_2 \in \mathbb{Q}_{>0}, r_1/r_2 \in \mathbb{Z} \right\} \tag{4.18}$$

is a $\mathbb{Z}$ basis for $\mathcal{H}(G, \Gamma)$.

Proof. It clearly contains all double cosets. To see that the double cosets are distinct multiply by a constant to clear denominators and the result follows. ∎

Notation 4.1.3. For n_1, n_2 with $n_1 \mid n_2$ we define

$$T(n_1, n_2) = \left[\Gamma \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix} \Gamma \right]. \tag{4.19}$$

From T we define $R(n) = T(n, n)$ and $T(n) = \sum_{1 \geq n_1 \mid n_2, n_1 n_2 = n} T(n_1, n_2)$.

Lemma 4.1.4. *Let G be an abelian group of size nm with $(n, m) = 1$. Then there exists a unique subgroup $H \leq G$ of size n.*

Proof. Since $\operatorname{ord}(g_1 + g_2) \mid \operatorname{ord}(g_1) \operatorname{ord}(g_2)$ it follows that all elements coprime to m form a subgroup of G. Call this subgroup H. By Sylow's theorems, $n \mid |H|$. By Cauchy's theorem, $|H| \mid n$. Thus we have existence. Uniqueness follows since if H' is another such subgroup, then by construction $H' \leq H$. Since they have the same order they must be equal. ∎

Corollary 4.1.5. *Let $\Lambda \leq \mathbb{Z}^2$ be a subgroup of index nm with $(m, n) = 1$. Then there exists a unique $\Lambda \leq \Lambda' \leq \mathbb{Z}^2$ such that $|\Lambda' : \Lambda| = n$.*

cor:lat **Corollary 4.1.6.** *Let L_k denote the set of subgroups of $\mathbb{Z}^2$ of index k. If $(n, m) = 1$ then*

$$L_{mn} = \coprod_{\Lambda \in L_n} \{\text{subgroups of } \Lambda \text{ of index } m\} \tag{4.20}$$

Thm 4.1.7.   *1. $R(nm) = R(n)R(m)$ for all $n, m \geq 1$ and $R(n)$ lies in the center of $\mathcal{H}(G, \Gamma)$ for all $n \geq 1$.*

   2. $T(nm) = T(n)T(m)$ whenever $(m, n) = 1$.

   3. $T(p)T(p^r) = T(p^{r+1}) + pR(p)T(p^{r-1})$ for $r \geq 1$.

Proof.   1. Since

$$\Gamma \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \Gamma = \Gamma \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \tag{4.21}$$

we have

$$\left[\Gamma \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \Gamma \right] [\Gamma \gamma \Gamma] = \left[\Gamma \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \gamma \Gamma \right] \tag{4.22}$$

for all γ. The result follows.

2. We have

$$\Theta(T(m)T(n)) = \sum_{\delta \in \Pi_m, \gamma \in \Pi_n} [\Gamma \delta \gamma] \tag{4.23}$$

and

$$\Theta(T(mn)) = \sum_{\epsilon \in \Pi_{mn}} [\Gamma \epsilon]. \tag{4.24}$$

But $\{\Gamma \delta \gamma : \delta \in \Pi_m\}$ corresponds to subgroups of $\mathbb{Z}^2 \gamma$ of index m and so the result follows from corollary 4.1.6.

3. We have

$$\Theta(T(p^r)T(p)) = \sum_{\delta \in \Pi_{p^r}, \gamma \in \Pi_p} [\Gamma \delta \gamma]. \tag{4.25}$$

As before $\{\Gamma \delta \gamma : \delta \in \Pi_{p^r}\}$ corresponds to subgroups of $\mathbb{Z}^2 \gamma$ of index p^r. Note that all index p subgroups of $\mathbb{Z}^2$ contain $p\mathbb{Z}^2$ and so if Λ is an index p^{r+1} subgroup of $\mathbb{Z}^2$ contained in $p\mathbb{Z}^2 \subset$ then it is contained in all $p + 1$ index p subgroups of $\mathbb{Z}^2$. If $p\mathbb{Z}^2$ does not contain Λ then $|\mathbb{Z}^2 : \Lambda + p\mathbb{Z}^2| = p$. Using the fact that $p\mathbb{Z}^2$ is contained in all index p subgroups it is straightforward to show that $\Lambda + p\mathbb{Z}^2$ is the unique index p subgroup of $\mathbb{Z}^2$ containing Λ. The result follows. ∎

Corollary 4.1.8. $\mathcal{H}(G, \Gamma)$ *is commutative and is generated by*

$$\{T(p), R(p), R(p)^{-1} : p \text{ prime}\}. \tag{4.26}$$

Proof. It suffices to show that given set generates $\mathcal{H}(G, \Gamma)$. It is clear that $T(n_1, n_2), R(p), R(p)^{-1}$ generate $\mathcal{H}(G, \Gamma)$. But we know

$$T(n_1, n_2) = R(n_2)T(n_1/n_2, 1) \tag{4.27}$$

so we only need to show that we can obatin the $T(n, 1)$. Note that when n is squarefree $T(n) = T(n, 1)$ so this case is ok. The remaining cases follow from induction. ∎

Now that we understand the Hecke algebra $\mathcal{H}(G, \Gamma)$ fairly well, let us try to understand how it acts on various spaces. Let

$$V_k = \{\text{all functions } f : \mathcal{H} \to \mathbb{C}\}. \tag{4.28}$$

This has a right action by $\mathbf{GL}_2(\mathbb{Q})$ given by the slash operator, and $M_k \subseteq V_k^\Gamma$. Thus M_k is naturally a right $\mathcal{H}(G, \Gamma)$-module. Note that

$$f \underset{k}{\big|} R(n) = f \tag{4.29}$$

for all $n \geq 1$ and so the $R(n)$ act trivially.

Definition 4.1.9. Let $T_n = T_n^k : V_k^\Gamma \to V_k^\Gamma$ be the map defined by

$$f \mapsto n^{k/2-1} f \underset{k}{|} T(n). \tag{4.30}$$

Proposition 4.1.10.   *1. $T_{mn}^k = T_m^k \circ T_n^k$ if $(m,n) = 1$, and*

$$T_{p^{r+1}}^k = T_{p^r}^k T_p^k - p^{k-1} T_{p^{r-1}}^k. \tag{4.31}$$

2. If $f \in M_k$ then $T_n f \in M_k$. Similarly for S_k.

3. We have

$$a_n(T_m f) = \sum_{1 \leq d | (m,n)} d^{k-1} a_{mn/d^2}(f). \tag{4.32}$$

In particular

$$a_0(T_m f) = \sigma_{k-1}(m) a_0(f). \tag{4.33}$$

Proof.   1. Clear from earlier work.

2. Sufficient to see that $T_n f$ is homolorphic on $\mathcal{H}$ and at infinity. But this is clear. For S_k see next part.

3. If $r \in \mathbb{Z}$ then

$$q^r \underset{k}{|} T(m) = m^{k/2-1} \sum_{\gamma \in \Pi_m} q^r \underset{k}{|} \gamma$$

$$= m^{k-1} d^{-k} \sum_{d|m, 0 \leq b < d} \exp\left(2\pi i r \left(\frac{m}{d^2} z + \frac{b}{d}\right)\right). \tag{4.34}$$

But

$$\sum_{0 \leq b < d} \exp\left(2\pi i \frac{r}{d} b\right) = \begin{cases} d & \text{if } d \mid r \\ 0 & \text{otherwise} \end{cases}. \tag{4.35}$$

Thus

$$q^r \underset{k}{|} T(m) = \left(\frac{m}{d}\right)^{k-1} \sum_{d|m,r} q^{rm/d^2} \tag{4.36}$$

and so

$$
\begin{aligned}
T_m f &= \sum_{r \geq 0} a_r(f) \sum_{d|m,r} \left(\frac{m}{d}\right)^{k-1} q^{rm/d^2} \\
&= \sum_{r \geq 0} \sum_{d|m} [d \mid r] a_r(f) \left(\frac{m}{d}\right)^{k-1} q^{rm/d^2} \\
&= \sum_{s|m} \sum_{r \geq 0} [m/s \mid r] a_r(f) s^{k-1} q^{rs^2/m} \\
&= \sum_{s|m} \sum_{l \geq 0} a_{ml/s}(f) s^{k-1} q^{ls} \\
&= \sum_{n \geq 0} \left(\sum_{s|(m,n)} a_{mn/s^2}(f) s^{k-1} \right) q^n.
\end{aligned}
\tag{4.37}
$$

∎

Corollary 4.1.11. *Let $f \in M_k$ be such that $T_m f = \lambda f$ for some $m \geq 1$ and $\lambda \in \mathbb{C}$. Then*

1. *for every n with $(n,m) = 1$, we have $a_{nm}(f) = \lambda a_n(f)$.*

2. *if $a_0(f) \neq 0$, then $\lambda = \sigma_{k-1}(m)$.*

Corollary 4.1.12. *Let $0 \neq f \in M_k$, and $k \geq 4$ with $T_m f = \lambda_m f$ for all $m \geq 1$. Then*

1. *if $f \in S_k$, then $a_1(f) \neq 0$ and*

$$
f = a_1(f) \sum_{n \geq 1} \lambda_n q^n.
\tag{4.38}
$$

2. *If $f \notin S_k$, then $f = a_0(f) E_k$.*

Proof. 1. Apply the previous corollary with $n = 1$.

2. Since $a_0(f) \neq 0$, we know from the previous corollary that $a_n(f) = \sigma_{k-1}(n) a_1(f)$. Thus we have

$$
f = a_0(f) + a_1(f) \sum_{n \geq 1} \sigma_{k-1}(n) q^n = A + B E_k.
\tag{4.39}
$$

But since f and E_k are modular forms, and $k \neq 0$, we know $A = 0$.

∎

Definition 4.1.13. (Hecke eigenform). Let $f \in S_k \smallsetminus \{0\}$. Then f is a *Hecke eigenform* if for all $n \geq 1$, we have

$$
T_n f = \lambda_n f
\tag{4.40}
$$

for some $\lambda_n \in \mathbb{C}$. It is *normalised* if $a_1(f) = 1$.

Thm 4.1.14. *There exists a basis for S_k consisting of normalised Hecke eigenforms.*

We prove this soon.

CHAPTER 5

L-functions of eigenforms

Given a function f with a q expansion

$$f = \sum_{n \geq 1} a_n q^n \in S_k(\Gamma(1)), \tag{5.1}$$

we can form a Dirichlet series

$$L(f, s) = \sum_{n \geq 1} a_n n^{-s}. \tag{5.2}$$

Lemma 5.0.1. *If*

$$f = \sum_{n \geq 1} a_n q^n \in S_k(\Gamma(1)), \tag{5.3}$$

then $|a_n| \ll n^{k/2}$.

Proof. If $f \in S_k$ then $y^{k/2}|f|$ is bounded on $\mathcal{H}$. Also

$$a_n(f) = \frac{1}{2\pi} \int_{|q|=r} q^{-n} \widetilde{f}(q) \frac{dq}{q} \tag{5.4}$$

for $r \in (0, 1)$. Thus for any y,

$$|a_n(f)| = \left| \int_0^1 e^{-2\pi i n(x+iy)} f(x + iy) dx \right|$$
$$\leq e^{2\pi n y} \sup_{0 \leq x \leq 1} |f(x + iy)| \ll e^{2\pi n y} y^{-k/2}. \tag{5.5}$$

Setting $y = 1/n$ gives the result. $\blacksquare$

Corollary 5.0.2. *If* $f \in S_k(\Gamma(1))$, *then* $L(f, s)$ *converges absolutely for* $\Re(s) > k/2 + 1$.

Proposition 5.0.3. *Suppose* f *is a normalised eigenform. Then*

$$L(f, s) = \prod_{p \; prime} \frac{1}{1 - a_p p^{-s} + p^{k-1-2s}}. \tag{5.6}$$

Proof. We have

$$(1 - a_p p^{-s} + p^{k-1-2s})(1 + a_p p^{-s} + a_{p^2} p^{-2s} + \cdots)$$
$$= 1 + \sum_{r \geq 2} (a_{p^r} + p^{k-1} a_{p^{r-2}} - a_p a_{p^{r-1}}) p^{-rs}. \quad (5.7)$$

Since f is an eigenform the right hand side is just equal to 1. But also $a_{mn} = a_n a_m$ when $(m, n) = 1$ and so

$$L(f, s) = \prod_p (1 + a_p p^{-s} + a_{p^2} p^{-2s} + \cdots) = \prod_{p \text{ prime}} \frac{1}{1 - a_p p^{-s} + p^{k-1-2s}}. \quad (5.8)$$

Convergence follows in a similar manner as the proof for $\zeta(s)$. ∎

CHAPTER 6

Modular forms for subgroups of $\mathbf{SL}_2(\mathbb{Z})$

Definition 6.0.1. (Cusps). The *cusps* of Γ (or $\bar{\Gamma}$) are the orbits of Γ on $\mathbb{P}^1(\mathbb{Q})$.

Definition 6.0.2. (Width of a cusp). Let $\alpha \in \mathbb{P}^1(\mathbb{Q})$ be a representation of a cusp of Γ. Let $g \in \Gamma(1)$ be such that $g(\infty) = \alpha$. Then $\gamma(\alpha) = \alpha$ iff $g^{-1}\gamma g(\infty) = \infty$. So

$$g^{-1}\bar{\Gamma}_\alpha g = \overline{g^{-1}\Gamma g} \tag{6.1}$$