Modular forms and L-functions

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CHAPTER 1

Preliminary analysis

1.1 Fourier transform

Let G be a locally compact group. Then $\hat{G}$ has the strucutre of a locally compact group and there is a canonical map $G \to \hat{\hat{G}}$.

Thm 1.1.1. *The canonical map $G \to \hat{\hat{G}}$ is an isomorphism.*

Proof. TODO. ∎

Definition 1.1.2. Let G be a topological group. A *Haar measure* is a left translation-invariant Borel measure on G satisfying some regularity conditions (e.g. being finite on compact sets).

Thm 1.1.3. *Let G be a locally compact abelian group. Then there is a Haar measure on G, unique up to scaling.*

Proof. TODO. ∎

Example 1.1.4. The Haar measure on $\mathbb{R}^\times_{>0}$ is $\mu(S) = \int_{\mathbb{R}_{>0}} 1_S(x)dx/x$.

Definition 1.1.5. Let G be a locally compact abelian group with a Haar measure dg, and let $f : G \to \mathbb{C}$ be a continuous L^1 function. The *Fourier transform* $\hat{f} : \hat{G} \to \mathbb{C}$ is given by

$$\hat{f}(\chi) = \int_G \chi(g)^{-1} f(g) dg. \tag{1.1}$$

Thm 1.1.6. *Given a locally compact abelian group G with a fixed Haar measure, there is some constant C such that for "suitable" $f : G \to \mathbb{C}$, we have*

$$\hat{\hat{f}}(g) = Cf(-g) \tag{1.2}$$

using the canonical isomorphism $G \to \hat{\hat{G}}$.

Thm 1.1.7. *(Poisson summation). Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\sum_{a\in\mathbb{Z}^n} f(a) = \sum_{b\in\mathbb{Z}^n} \hat{f}(b). \tag{1.3}$$

Proof. Let $g(x) = \sum_{a\in\mathbb{Z}^n} f(x+a)$ and consider its fourier transform as a function on $\mathbb{R}^n/\mathbb{Z}^n$. ■

1.2 Mellin transform and Γ-function

Definition 1.2.1. (Mellin transform). Let $f : \mathbb{R}_{>0} \to \mathbb{C}$ be a continuous function. Define

$$M(f, s) = \int_0^\infty y^s f(y) \frac{dy}{y} \tag{1.4}$$

whenever it converges.

Lemma 1.2.2. *Let $f : \mathbb{R}_{>0} \to \mathbb{C}$ be continuous. If (a) $y^N f(y) \to 0$ as $y \to \infty$ for all $N \in \mathbb{Z}$ and (b) there exists m such that $|y^m f(y)|$ is bounded as $y \to 0$, then $M(f, s)$ converges and is analytic on $\Re(s) > m$.*

Proof. For $0 < r < R < \infty$, the function

$$\int_r^R y^s f(y) \frac{dy}{y} \tag{1.5}$$

is analytic for all s. TODO. ■

Remark 1.2.3. When $G = \mathbb{R}_{>0}$, $\hat{G} = \{y \mapsto y^{i\sigma} : \sigma \in \mathbb{R}\} \cong \mathbb{R}$ so the Mellin transform is the analytic continuation of the fourier transform of f.

Proposition 1.2.4.

$$M(f(\alpha y), s) = \alpha^{-s} M(f, s) \tag{1.6}$$

for $\alpha > 0$.

Proof. Obvious. ■

1.2.1 The Γ-function

Definition 1.2.5. (Γ-function). The Γ-function is the Mellin transform of $f(y) = e^{-y}$.

This is analytic for $\Re(s) > 0$.

Proposition 1.2.6.

$$\Gamma(1) = 1, \quad s\Gamma(s) = \Gamma(s+1), s \neq 0. \tag{1.7}$$

Proof. First part obvious. For second part:

$$\Gamma(s) = \int_0^\infty e^{-y} y^{s-1} dy = \left[e^{-y} \frac{y^s}{s} \right]_0^\infty + \frac{1}{s} \int_0^\infty e^{-y} y^s dy = \frac{1}{s} \Gamma(s+1). \tag{1.8}$$

■

Iterating this result we get $\Gamma(s) = \left(\prod_{i=0}^N (s+i) \right)^{-1} \Gamma(s+N+1)$ and so we can extend Γ to a meromorphic function on $\mathbb{C}$ with

$$\operatorname{res}_{s=-N} \Gamma(s) = \frac{(-1)^N}{N!} \tag{1.9}$$

for $N \geq 0$.

Proposition 1.2.7.

$$\Gamma(s)^{-1} = e^{\gamma s} s \prod_{n \geq 1} \left(1 + \frac{s}{n} \right) e^{-s/n} \tag{1.10}$$

$$\pi^{1/2} \Gamma(2s) = 2^{2s-1} \Gamma(s) \Gamma\left(s + \frac{1}{2} \right) \tag{1.11}$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \tag{1.12}$$

for all $s \in \mathbb{C}$.

Remark 1.2.8. Merllin transforms are important because a lot of Dirichlet series are Mellin transforms. Consider the series $\sum_{n \geq 1} a_n / n^s$. We can write

$$\begin{aligned} (2\pi)^{-s} \Gamma(s) \sum_{n=1}^\infty \frac{a_n}{n^s} &= \sum_{n=1}^\infty a_n (2\pi n)^{-s} M(e^{-y}, s) \\ &= \sum_{n=1}^\infty M(a_n e^{-2\pi n y}, s) \\ &= M(f, s) \end{aligned} \tag{1.13}$$

where $f = \sum_{n=1}^\infty a_n e^{-2\pi n y}$.

1.3 Riemann ζ-function

Definition 1.3.1. (Riemann ζ-functions). The *Riemann ζ-function* is defined by

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \tag{1.14}$$

for $\Re(s) > 1$.

Proposition 1.3.2.

$$\zeta(s) = \prod_{p\ prime} \frac{1}{1-p^{-s}}. \tag{1.15}$$

Thm 1.3.3. *If $\Re(s) > 1$, then*

$$(2\pi)^{-s}\Gamma(s)\zeta(s) = \int_0^\infty \frac{y^s}{e^{2\pi y}-1}\frac{dy}{y} = M(f,s) \tag{1.16}$$

where

$$f(y) = \frac{1}{e^{2\pi y}-1}. \tag{1.17}$$

Proof. Let

$$f(y) = \frac{e^{-2\pi y}}{1-e^{-2\pi y}} = \sum_{n\geq 1} e^{-2\pi n y} \tag{1.18}$$

for $y > 0$. As $y \to 0$, $f(y) \sim 1/2\pi y$. So taking the Mellin transform of f gives the required result. ∎

Corollary 1.3.4. *$\zeta(s)$ extends meromorphically to $\mathbb{C}$ with a single pole located at $s = 1$ which is simple and has* $\operatorname{res}_{s=1}\zeta(s) = 1$.

Proof. Write

$$M(f,s) = M_0 + M_\infty = \left(\int_0^1 + \int_1^\infty\right)\frac{y^s}{e^{2\pi y}-1}\frac{dy}{y}. \tag{1.19}$$

M_∞ is holomorphic on $\mathbb{C}$. For fixed N, we can expand

$$f(y) = \sum_{n=-1}^{N-1} c_n y^n + y^N g_N(y) \tag{1.20}$$

for some $g \in C^\infty(\mathbb{R})$, since f has a simple pole at $s = 0$, and $c_{-1} = 1/2\pi$. Thus for $\Re(s) > 1$,

$$\begin{aligned} M_0 &= \sum_{n=-1}^{N-1} c_n \int_0^1 y^{n+s-1}dy + \int_0^1 y^{N+s-1}g_N(y)dy \\ &= \sum_{n=-1}^{N-1} \frac{c_n}{s+n} y^{n+s} + \int_0^1 g_N(y)y^{s+N-1}dy. \end{aligned} \tag{1.21}$$

This formula makes sense for $\Re(s) > -N$. Thus we may extend $(2\pi)^{-s}\Gamma(s)\zeta(s)$ to $\mathbb{C}$. This has at worst simple poles at $1, 0, -1, -2, \ldots$. But $\Gamma(s)$ has simple poles at $0, -1, -2, \ldots$ so $\zeta(s)$ must be analytic at $s = 0, -1, -2 \ldots$. Since $\Gamma(1) = 1$, $\operatorname{res}_{s=1}\zeta(s) = 1$. ∎

Definition 1.3.5. (Bernoulli numbers). The *Bernoulli numbers* are defined by

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}. \tag{1.22}$$

Corollary 1.3.6.

$$\zeta(0) = B_1, \quad \zeta(1-n) = -\frac{B_n}{n}, n > 1. \tag{1.23}$$

Proof. Keeping with the notation from the proof of the previous corollary, $(2\pi)^{-s}\Gamma(s)\zeta(s)$ has a simple pole at $s = 1-n$ with residue c_{n-1}. Thus

$$c_{n-1} = (2\pi)^{n-1}\frac{B_n}{n!}. \tag{1.24}$$

But also

$$\operatorname{res}_{s=1-n}(2\pi)^{-s}\Gamma(s)\zeta(s) = (2\pi)^{n-1}\zeta(1-n)\operatorname{res}_{s=1-n}\Gamma(s). \tag{1.25}$$

We know the residue of $\Gamma(s)$ and so plugging in its value and equating gives the result. ■

Definition 1.3.7. (Jacobi's ϑ-function).

$$\vartheta(s) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 s} \tag{1.26}$$

on $\Im(s) > 0$.

We then define $\Theta(y) = \vartheta(iy)$. Note that $\Theta(y) \to 1$ as $y \to \infty$.

Proposition 1.3.8.

$$M\left(\frac{\Theta(y)-1}{2}, \frac{s}{2}\right) = \pi^{-s/2}\gamma\left(\frac{s}{2}\right)\zeta(s). \tag{1.27}$$

Proof. Obvious. ■

Thm 1.3.9. *If $y > 0$, then*

$$\Theta\left(\frac{1}{y}\right) = y^{1/2}\Theta(y), \tag{1.28}$$

where we are taking the positive square root.

Proof. Let $g_t(x) = e^{-tx^2}$. Recall that $\hat{g}_t(y) = t^{-1/2}e^{-\pi y^2/t}$. By Poisson summation

$$\Theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = \sum_{n\in\mathbb{Z}} g_t(n) = \sum_{n\in\mathbb{Z}} \hat{g}_t(n) = t^{-1/2}\Theta(1/t). \tag{1.29}$$

■

Corollary 1.3.10.

$$\vartheta\left(-\frac{1}{z}\right) = \left(\frac{z}{i}\right)^{1/2} \vartheta(z), \tag{1.30}$$

where we are taking the branch of the square root that is positive on the positive real axis.

Proof. Follows from analytic continuation. ■

Notation 1.3.11.

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2) \tag{1.31}$$

$$\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s) \tag{1.32}$$

$$Z(s) = \Gamma_{\mathbb{R}}(s)\zeta(s). \tag{1.33}$$

Thm 1.3.12.

$$Z(s) = Z(1-s). \tag{1.34}$$

Moreover, $Z(s)$ is meromorphic with the poles located at $s = 0$ and $s = 1$.

Proof. Have

$$2Z(s) = M(\Theta(y) - 1, s/2) = \left(\int_0^1 + \int_1^\infty\right)(\Theta(y) - 1)y^{s/2})\frac{dy}{y}. \tag{1.35}$$

But

$$\begin{aligned}\int_0^1 (\Theta(y) - 1)y^{s/2}\frac{dy}{y} &= \int_0^1 (\Theta(y) - y^{-1/2})y^{s/2} + \int_0^1 y^{(s-1)/2} - y^{s/2})\frac{dy}{y} \\ &= \int_0^1 (\Theta(1/y) - 1)y^{(s-1)/2}\frac{dy}{y} + \frac{2}{s-1} - \frac{2}{s} \\ &= \int_1^\infty (\Theta(y) - 1)y^{(1-s)/2}\frac{dy}{y} + \frac{2}{s-1} - \frac{2}{s}.\end{aligned} \tag{1.36}$$

Thus

$$2Z(s) = \int_1^\infty (\Theta(y) - 1)(y^{s/2} + y^{(1-s)/2})\frac{dy}{y} + \frac{2}{s-1} - \frac{2}{s} = 2Z(1-s). \tag{1.37}$$

■

1.4 Dirichlet L-functions

Definition 1.4.1. (Dirichlet characters). Let $N \geq 1$. A *Dirichlet character mod N* is a character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$.

Definition 1.4.2. (Equivalent characters). We say $\chi_1 \in \widehat{(\mathbb{Z}/N_1\mathbb{Z})^\times}$ and $\chi_2 \in \widehat{(\mathbb{Z}/N_2\mathbb{Z})^\times}$ are *equivalent* if there exists an M such that $N_1, N_2 \mid M$ and

$$\begin{tikzcd} & (\mathbb{Z}/M\mathbb{Z})^\times \arrow[dl] \arrow[dr] & \\ (\mathbb{Z}/N_1\mathbb{Z})^\times \arrow[dr] & & (\mathbb{Z}/N_2\mathbb{Z})^\times \arrow[dl] \\ & \mathbb{C}^\times & \end{tikzcd} \tag{1.38}$$

commutes.

Note that the projection maps are surjective and so this defines an equivalence relation.

Definition 1.4.3. (Primitive character). We say $\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times}$ is *primitive* if there is no $M < N$ with $M \mid N$ such that χ factors through $(\mathbb{Z}/M\mathbb{Z})^\times$.

Lemma 1.4.4. *Let $\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times}$ and $M \mid N$. Then the following are equivalent*

1. *χ factors through $(\mathbb{Z}/M\mathbb{Z})^\times$*
2. *χ is constant on the fibers of $p : (\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/M\mathbb{Z})^\times$*
3. *$x, x + aM \in (\mathbb{Z}/N\mathbb{Z})^\times \implies \chi(x) = \chi(x + aM)$ for all $a \in \mathbb{Z}$.*

Proof. (2) $\Rightarrow$ (1). We can certainly define a function $\chi' : (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C}^\times$ so that χ factors through χ'. It remains to show that χ' is a homomorphism. Let $x, y \in (\mathbb{Z}/M\mathbb{Z})^\times$ and $a, b \in (\mathbb{Z}/N\mathbb{Z})^\times$ be preimages of x and y respectively. Then

$$\chi'(xy) = \chi'(p(a)p(b)) = \chi'(p(ab)) = \chi(ab) = \chi(a)\chi(b) = \chi'(x)\chi'(y) \tag{1.39}$$

so χ' is a homomorphism.

The remaining implications are obvious. ■

Corollary 1.4.5. *Let $\chi_1 \in \widehat{(\mathbb{Z}/N_1\mathbb{Z})^\times}$ and $\chi_2 \in \widehat{(\mathbb{Z}/N_2\mathbb{Z})^\times}$ be equivalent. Then there exists a $\chi \in \widehat{(\mathbb{Z}/d\mathbb{Z})^\times}$ where $d = (N_1, N_2)$, such that both χ_1 and χ_2 factor through χ.*

Proof. Write $d = aN_1 + bN_2$ for $a, b \in \mathbb{Z}$ and let M be such that $N_1, N_2 \mid M$ and

$$\begin{tikzcd} & (\mathbb{Z}/M\mathbb{Z})^\times \arrow[dl] \arrow[dr] & \\ (\mathbb{Z}/N_1\mathbb{Z})^\times \arrow[dr] & & (\mathbb{Z}/N_2\mathbb{Z})^\times \arrow[dl] \\ & \mathbb{C}^\times & \end{tikzcd} \tag{1.40}$$

commutes. Let $\psi \in \widehat{(\mathbb{Z}/M\mathbb{Z})}^\times$ be the vertical map. This map is constant on ■

Corollary 1.4.6. *If $\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})}^\times$ then there exists a unique $M \mid N$ and a primitive $\chi_* \in \widehat{(\mathbb{Z}/M\mathbb{Z})}^\times$ such that it is equivalent to χ.*

Proof. Obvious from previous corollary. ■

Definition 1.4.7. (Dirichlet L-series). Let $\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})}^\times$. The *Dirichlet L-series* of χ is

$$L(\chi, s) = \sum_{n \geq 1, (n,N)=1} \chi(n) n^{-s}. \tag{1.41}$$

Proposition 1.4.8.

$$L(\chi, s) = \prod_{p \nmid N} \frac{1}{1 - \chi(p) p^{-s}}. \tag{1.42}$$

Proposition 1.4.9. *Suppose $M \mid N$ and $\chi_N \in \widehat{(\mathbb{Z}/N\mathbb{Z})}^\times$ factors through $\chi_M \in \widehat{(\mathbb{Z}/M\mathbb{Z})}^\times$. Then*

$$L(\chi_M, s) = \prod_{p \nmid M, p \mid N} \frac{1}{1 - \chi_M(p) p^{-s}} L(\chi_N, s). \tag{1.43}$$

In particular,

$$\frac{L(\chi_M, s)}{L(\chi_N, s)} = \prod_{p \nmid M, p \mid N} \frac{1}{1 - \chi_M(p) p^{-s}} \tag{1.44}$$

is analytic and non-zero on $\Re(s) > 0$.

Thm 1.4.10.

1. *$L(\chi, s)$ has a meromorphic continuation to $\mathbb{C}$, which is analytic except for at worst a simple pole at $s = 1$.*
2. *If $\chi \neq \chi_0$ (the trivial character), then $L(\chi, s)$ is analytic everywhere. On the other hand, $L(\chi_0, s)$ has a simple pole with residue*

$$\frac{\phi(N)}{N} = \prod_{p \mid N} \left(1 - \frac{1}{p}\right). \tag{1.45}$$

Proof. Let $\phi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ be any N-periodic function, and let $L(\phi, s) = \sum_{n=1}^\infty \phi(n) n^{-s}$. Then

$$(2\pi)^{-s} \Gamma(s) L(\phi, s) = \sum_{n=1}^\infty \phi(n) M(e^{-2\pi y}, s) = M(f(y), s), \tag{1.46}$$

where $f(y) = \sum_{n=1}^\infty \phi(n) e^{-2\pi n y}$. A straightforward calculation shows that

$$f(y) = \sum_{n=1}^N \phi(n) \frac{e^{2\pi(N-n)y}}{e^{e\pi N y} - 1}. \tag{1.47}$$

Note that this is $\mathcal{O}(e^{-2\pi y})$ as $y \to \infty$. Write $M(f,s) = M_0(s) + M_\infty(s)$ as before. The second term is analytic for all $s \in \mathbb{C}$, and the first one is given by

$$M_0(s) = \sum_{n=1}^{N} \phi(n) \int_0^1 \frac{e^{2\pi(N-n)y}}{e^{2\pi Ny} - 1} y^s \frac{dy}{y}. \tag{1.48}$$

For any L we can write

$$\frac{e^{2\pi(N-n)y}}{e^{2\pi Ny} - 1} = \frac{1}{2\pi Ny} + \sum_{r=0}^{L-1} c_{r,n} y^r + y^L g_{L,n}(y) \tag{1.49}$$

for some $g_{L,n}(y) \in C^\infty[0,1]$. Hence

$$M_0(s) = \sum_{n=1}^{N} \phi(n) \left(\int_0^1 \frac{1}{2\pi Ny} y^s \frac{dy}{y} + \int_0^1 \sum_{r=0}^{L-1} c_{r,n} y^{r+s-1} dy \right) + G(s), \tag{1.50}$$

where G is a function that is analytic on $\Re(s) > -L$. Thus

$$(2\pi)^{-s}\Gamma(s)L(\phi,s) = \sum_{n=1}^{N} \phi(n) \left(\frac{1}{2\pi N(s-1)} + \frac{c_{0,n}}{s} + \cdots + \frac{c_{L-1,n}}{s+L-1} \right) + G(s). \tag{1.51}$$

The first part follows from setting

$$\phi(n) = \begin{cases} \chi(n) & \text{if } (n,N) = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1.52}$$

It also follows that

$$\operatorname{res}_{s=1} L(\chi,s) = \frac{1}{N} \sum_{n=1}^{N} \phi(n). \tag{1.53}$$

If $\chi \neq \chi_0$ this vanishes by orthogonality of characters. Otherwise it is equal to $|(\mathbb{Z}/N\mathbb{Z})^\times|/N = \phi(N)/N$. ■

lem:ord **Lemma 1.4.11.** *If $p \nmid N$ and T is any complex number, then*

$$\prod_{\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times}} (1 - \chi(p)T) = (1 - T^{f_p})^{\phi(N)/f_p} \tag{1.54}$$

where f_p is the order of p in $(\mathbb{Z}/N\mathbb{Z})^\times$.

Proof. Fix a p, write $f = f_p$, $G = (\mathbb{Z}/N\mathbb{Z})^\times$ and $H = \langle p \rangle \subseteq G$. The projection map $G \to G/H$ induces an injection $\widehat{G/H} \to \hat{G}$ and the restriction map $\hat{G} \to \hat{H}$ induces an injection (and hence an isomorphism)

$$\frac{\hat{G}}{\widehat{G/H}} \to \hat{H}. \tag{1.55}$$

Also $|\widehat{G/H}| = |G/H| = \phi(N)/f$. Thus

$$\prod_{\chi\in\hat{G}}(1-\chi(p)T) = \prod_{\chi\in\hat{H}}(1-\chi(p)T)^{\phi(N)/f} = \prod_{\zeta\in\mu_f}(1-\zeta T)^{\phi(N)/f} = (1-T^f)^{\phi(N)/f}. \tag{1.56}$$

■

lem:conv **Lemma 1.4.12.** *Let $D(s) = \sum_{n\geq 1} a_n n^{-s}$ be a Dirichlet series with real $a_n \geq 0$, and suppose that it is absolutely convergent for $\Re(s) > \sigma > 0$. Then if $D(s)$ can be analytically continued to an analytic function $\widetilde{D}$ on $\{\Re(s) > 0\}$, then the series converges for all real $s > 0$.*

Proof. Let $\rho > \sigma$. Then by the analytic continuation, we have a convergent Taylor series on $\{|s-\rho| < \rho\}$

$$D(s) = \sum_{k\geq 0}\frac{1}{k!}D^{(k)}(\rho)(s-\rho)^k. \tag{1.57}$$

Moreover, since $\rho > \sigma$, we can differentiate the Dirichlet series term by term to obtain the derivatives

$$D^{(k)}(\rho) = \sum_{n\geq 1} a_n(-\log n)^k n^{-\rho}. \tag{1.58}$$

Thus if $0 < x < \rho$,

$$D(x) = \sum_{k\geq 0}\frac{1}{k!}(\rho-x)^k\left(\sum_{n\geq 1} a_n(\log n)^k n^{-\rho}\right). \tag{1.59}$$

Since all the terms are non-negative, this must converge unconditionally and so rearranging we find

$$\begin{aligned} D(x) &= \sum_{n\geq 1} a_n n^{-\rho}\sum_{k\geq 0}\frac{1}{k!}(\rho-x)^k(\log n)^k \\ &= \sum_{n\geq 1} a_n n^{-\rho} e^{(\rho-x)\log n} = \sum_{n\geq 1} a_n n^{-x}. \end{aligned} \tag{1.60}$$

■

lem:div **Lemma 1.4.13.** *1.*

$$\sum_p p^{-x} \sim -\log(x-1) \tag{1.61}$$

as $x \to 1^+$.

2. If $\chi \neq \chi_0$ is a Dirichlet character mod N, then

$$\sum_{p\nmid N}\chi(p)p^{-x} \tag{1.62}$$

is bounded as $x \to 1^+$.

Proof. Let χ be a Dirichlet character mod N. Then

$$\log L(\chi, x) = \sum_{p \nmid N} -\log(1 - \chi(p)p^{-x}) = \sum_{p \nmid N} \sum_{r \geq 1} \frac{\chi(p)^r p^{-rx}}{r} \tag{1.63}$$

so

$$\begin{aligned} \left| \log L(\chi, x) - \sum_{p \nmid N} \chi(p) p^{-x} \right| &< \sum_{p \nmid N} \sum_{r \geq 2} p^{-rx} \\ &= \sum_{p \nmid N} \frac{p^{-2x}}{1 - p^{-x}} \\ &\leq \sum_{n \geq 1} \frac{n^{-2}}{1/2} \end{aligned} \tag{1.64}$$

which is a finite constant. When $\chi = \chi_0$ and $N = 1$, then $|\log \zeta(x) - \sum_p p^{-x}|$ is bounded as $x \to 1^+$. But $\zeta(s) = 1/(s-1) + \mathcal{O}(s)$ and so

$$\sum_p p^{-x} \sim -\log(x - 1). \tag{1.65}$$

When $\chi \neq \chi_0$, then $L(\chi, 1) \neq 0$ by the next theorem and so $\log L(\chi, x)$ is bounded as $x \to 1^+$ which implies the result. ∎

Thm 1.4.14. *If $\chi \neq \chi_0$ then $L(\chi, 1) \neq 0$.*

Proof. Let

$$\zeta_N(s) = \prod_{\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times}} L(\chi, s) = \prod_{p \nmid N} \prod_{\chi} (1 - \chi(p)p^{-s})^{-1} = L(\chi_0, s) \prod_{\chi \neq \chi_0} L(\chi, s) \tag{1.66}$$

for $\Re(s) > 1$. We know that $L(\chi_0, s)$ has a pole at $s = 1$ and is analytic elsewhere and so if any $L(\chi, 1) = 0$, then $\zeta_N(s)$ is analytic on $\Re(s) > 0$. Since the Dirichlet series for ζ_N has ≥ 0 coefficients, by lemma 1.4.12, it suffices to find some point on $\mathbb{R}_{>0}$ where the Dirichlet series for ζ_N does not converge. But by lemma 1.4.11

$$\begin{aligned} \zeta_N(s) &= \prod_{p \nmid N} (1 - p^{-f_p s})^{-\phi(N)/f_p} \\ &= \prod_{p \nmid N} (1 + p^{-f_p x} + p^{-2f_p x} + \cdots)^{\phi(N)/f_p} \\ &\geq \sum_{p \nmid N} p^{-\phi(N)x}. \end{aligned} \tag{1.67}$$

But $\sum_p p^{-1}$ diverges by lemma 1.4.13 and so the series for $\zeta_N(x)$ does not converge at $x = 1/\phi(N)$. ∎

Thm 1.4.15. *Let $a \in \mathbb{Z}$ be such that $(a, N) = 1$. Then there exists infinitely many primes $p \equiv a \pmod N$.*

Proof. It suffices to show that

$$\sum_{p\equiv \pmod N} p^{-x} \tag{1.68}$$

diverges as $x \to 1^+$. But when $(x, N) = 1$ we have

$$\sum_{\chi\in\widehat{(\mathbb{Z}/N\mathbb{Z})^\times}} \chi(x) = \begin{cases} \phi(N) & x \equiv 1 \pmod N \\ 0 & \text{otherwise} \end{cases} \tag{1.69}$$

by column orthogonality of the character table of $(\mathbb{Z}/N\mathbb{Z})^\times$. Thus

$$\sum_{p\equiv \pmod N} p^{-x} = \frac{1}{\phi(N)} \sum_{\chi\in(\mathbb{Z}/N\mathbb{Z})^\times} \chi(a)^{-1} \sum_p \chi(p) p^{-x}. \tag{1.70}$$

If $\chi = \chi_0$, then the sum is just

$$\sum_{p\nmid N} p^{-x} \sim -\log(x-1) \tag{1.71}$$

as $x \to 1^+$. By lemma 1.4.13 all the other sums are bounded and so the result follows. ∎

CHAPTER 2

The modular group

Definition 2.0.16.

$$\mathbf{GL}_2(\mathbb{R})^+ = \{\gamma \in \mathbf{GL}_2(\mathbb{R}) : \det \gamma > 0\} \tag{2.1}$$
$$\mathbf{PGL}_2(\mathbb{R})^+ = \mathbf{GL}_2(\mathbb{R})^+/\mathbb{R}^\times \cong \mathbf{PSL}_2(\mathbb{R}). \tag{2.2}$$

The latter is the group of Mobius transforms that map $\mathcal{H}$ to $\mathcal{H}$, and the stabiliser of i is $\mathbf{SO}(2)/\{\pm I\}$. In fact $\mathbf{PSL}_2(\mathbb{R})$ is the group of all holomorphic automorphisms of $\mathcal{H}$, and the subgroup $\mathbf{SO}(2) \subseteq \mathbf{SL}_2(\mathbb{R})$ is a maximal compact subgroup.

Thm 2.0.17. *The group* $\mathbf{SL}_2(\mathbb{R})$ *admits the Iwasawa decomposition*

$$\mathbf{SL}_2(\mathbb{R}) = KAN = NAK \tag{2.3}$$

where

$$K = \mathbf{SO}(2), \quad A = \left\{ \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \right\}, \quad N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}. \tag{2.4}$$

Proof. Let $A \in \mathbf{SL}_2(\mathbb{R})$. Then there exists a $B \in \mathbf{SO}(2)$ so that BA is upper triangular. The rest follows easily. ∎

Definition 2.0.18. For $\Gamma \leq \mathbf{SL}_2(\mathbb{R})$, we write $\bar{\Gamma}$ for the image in $\mathbf{PSL}_2(\mathbb{R})$.

Definition 2.0.19. The *modular group* is

$$\mathbf{PSL}_2(\mathbb{Z}) = \frac{\mathbf{SL}_2(\mathbb{Z})}{\{\pm I\}}. \tag{2.5}$$

Thm 2.0.20. *Let*

$$\mathcal{D} = \left\{ z \in \mathcal{H} : -\frac{1}{2} \leq \Re(z) \leq \frac{1}{2}, |z| > 1 \right\} \cup \{z \in \mathcal{H} : |z| = 1, \Re(s) \geq 0\}. \tag{2.6}$$

Then $\mathcal{D}$ *is a fundamental domain for the action of* $\mathbf{SL}_2(\mathbb{Z})$ *on* $\mathcal{H}$*. The stabiliser of* $z \in \mathcal{D}$ *in* $\mathbf{SL}_2(\mathbb{Z})$ *is trivial if* $z \neq i, \rho$*, and*

$$\bar{\Gamma}_i = \langle S \rangle \cong C_2, \quad \bar{\Gamma}_\rho = \langle TS \rangle \cong C_3. \tag{2.7}$$

Proposition 2.0.21. *The measure*

$$d\mu = \frac{dxdy}{y^2} \tag{2.8}$$

is invariant under $\mathbf{PSL}_2(\mathbb{R})$. *If* $\Gamma \subseteq \mathbf{PSL}_2(\mathbb{Z})$ *is of finite index then* $\mu(\Gamma\backslash\mathcal{H})$.

Proof. Note that

$$\frac{dx \wedge dy}{y^2} = \frac{idz \wedge d\bar{z}}{2(\Im(z))^2}. \tag{2.9}$$

It is then straightforward to see that it is invariant under $\mathbf{PSL}_2(\mathbb{R})$. For the last part it suffices to show that $\mu(\mathcal{D})$ is finite. A straightforward calculation shows that

$$\mu(\mathcal{D}) = \int_{\mathcal{D}} \frac{dxdy}{y^2} = \frac{\pi}{3}. \tag{2.10}$$

■

Definition 2.0.22. (Principal congruence subgroup). For $N \geq 1$, the *principal congruence subgroup* of level N is

$$\Gamma(N) = \ker(\mathbf{SL}_2(\mathbb{Z}) \to \mathbf{SL}_2(\mathbb{Z}/N\mathbb{Z})). \tag{2.11}$$

Any $\Gamma \subseteq \mathbf{SL}_2(\mathbb{Z})$ containing some $\Gamma(N)$ is called a *congruence subgroup*, and its *level* is the smallest N such that $\Gamma \supseteq \Gamma(N)$. $\Gamma_0(N)$ is defined as the preimage of upper triangular matrices, and $\Gamma_1(N)$ the preimage of strictly upper triangular matrices.

CHAPTER 3

Modular forms of level 1

3.1 Basic definitions

Definition 3.1.1. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in \mathbf{GL}_2(\mathbb{R})^+, z \in \mathcal{H}, \tag{3.1}$$

and $f : \mathcal{H} \to \mathbb{C}$ be any function. We write $j(\gamma, z) = cz + d$. We define the *slash operator* to be

$$(f \underset{k}{|} \gamma)(z) = (\det \gamma)^{k/2} j(\gamma, z)^{-k} f(\gamma(z)). \tag{3.2}$$

Remark 3.1.2. It will be helpful to note that

$$\gamma \begin{pmatrix} z \\ 1 \end{pmatrix} = j(\gamma, z) \begin{pmatrix} \gamma z \\ 1 \end{pmatrix}. \tag{3.3}$$

Proposition 3.1.3. *1. $j(\gamma\delta, z) = j(\gamma, \delta z) j(\delta, z)$*

2. $j(\gamma^{-1}, z) = j(\gamma, \gamma^{-1}(s))^{-1}$

3. $\gamma : \phi \mapsto g \underset{k}{|} \gamma$ is a right action of $\mathbf{GL}_2(\mathbb{R})^+$ on functions on $\mathcal{H}$.

Proof. 1. We have

$$j(\gamma\delta, z) \begin{pmatrix} \gamma\delta z \\ 1 \end{pmatrix} = \gamma\delta \begin{pmatrix} z \\ 1 \end{pmatrix} = j(\delta, z)\gamma \begin{pmatrix} \delta z \\ 1 \end{pmatrix} = j(\delta, z) j(\gamma, \delta z) \begin{pmatrix} \gamma\delta z \\ 1 \end{pmatrix}. \tag{3.4}$$

2. Set $\delta = \gamma^{-1}$.

3. Follows from (i).

∎

CHAPTER 4

Hecke operators

Thm 4.0.4. *Let $G = \mathbf{GL}_2(\mathbb{Q})$ and $\Gamma \subseteq \mathbf{SL}_2(\mathbb{Z})$ be a subgroup of finite index. Then for all $g \in G$, $|\Gamma : \Gamma \cap g^{-1}\Gamma| < \infty$.*

Proof. Consider the case $\Gamma = \mathbf{SL}_2(\mathbb{Z})$, and

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}). \tag{4.1}$$

We claim that if $\det g = \pm N$, $N \geq$ then $g^{-1}\Gamma g \supseteq \Gamma(N)$ from which the theorem follows. To do this it suffices to show that if $\gamma \in \Gamma(N)$ then $g\gamma g^{-1}$ has integer entries. But

$$\pm N g\gamma g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \gamma \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \equiv NI \equiv 0 \pmod N \tag{4.2}$$

so $g\gamma g^{-1}$ has integer entries.

Now, in general if $g' \in \mathbf{GL}_2(\mathbb{Q})$ then we can write $g' \frac{1}{M} g$ for g with integer entries, and conjugating by g' and g yields the same result.

The general case follows from the fact that if $|G : H| < \infty$ and $|G : H'| < \infty$ then $|G : H \cap H'| < \infty$ and the following diagram

$$\begin{tikzcd}
 & \Gamma(1) \arrow[dl, dash, "<\infty"'] \arrow[drr, dash, "<\infty"] & & & & g^{-1}\Gamma(1)g \arrow[dll, dash, "<\infty"'] \arrow[dr, dash, "<\infty"] & \\
\Gamma \arrow[drr, dash] & & & \Gamma(1) \cap g^{-1}\Gamma(1)g \arrow[dl, dash] \arrow[dr, dash] & & & g^{-1}\Gamma g \arrow[dll, dash] \\
 & & \Gamma \cap g^{-1}\Gamma(1)g \arrow[dr, dash] & & \Gamma(1) \cap g^{-1}\Gamma g \arrow[dl, dash] & & \\
 & & & \Gamma \cap g^{-1}\Gamma g. & & &
\end{tikzcd} \tag{4.3}$$

■

Notation 4.0.5. Let G be a group, M a $\mathbb{Z}G$-module and $\Gamma \leq G$. For $g \in G$ and $m \in M^\Gamma$, we write

$$m|[\Gamma g \Gamma] = \sum_{i=1}^{n} m g_i, \tag{4.4}$$

where

$$\Gamma g \Gamma = \coprod_{i=1}^{n} \Gamma g_i. \tag{4.5}$$

Proposition 4.0.6. *1. $m|[\Gamma g \Gamma]$ depends only on $\Gamma g \Gamma$.*

2. $m|[\Gamma g \Gamma] \in M^\Gamma$.

Proof. Obvious. ■

Thm 4.0.7. *There is a product on $\mathcal{H}(G, \Gamma)$, the free abelian group generated by double cosets, making it into an associative ring, the Hecke algebra of (G, Γ), with unit $[\Gamma e \Gamma] = [\Gamma]$, such that for every G-module M, we have M^Γ is a right $\mathcal{H}(G, \Gamma)$-module by the operation $|$.*

Proof. There is an obvious bijection $\Theta : \mathcal{H}(G, \Gamma) \to M$ where $M = \mathbb{Z}[\Gamma \backslash G]^\Gamma$. We can view $\Gamma g \Gamma$ as living in M^Γ. We then define

$$[\Gamma g \Gamma] \cdot [\Gamma h \Gamma] = \Theta^{-1}((\Gamma g \Gamma)|[\Gamma h \Gamma]). \tag{4.6}$$

This is clearly well defined. If we write

$$\Gamma g \Gamma = \coprod \Gamma g_i \tag{4.7}$$

$$\Gamma h \Gamma = \coprod \Gamma h_j \tag{4.8}$$

then

$$[\Gamma g \Gamma] \cdot [\Gamma h \Gamma] = \Theta^{-1}\left(\sum_{i,j} [\Gamma g_i h_j]\right). \tag{4.9}$$ eq:mult

This then is clearly associative and has unit $[\Gamma e \Gamma]$. The rest follows in a staight-forward manner. ■

Remark 4.0.8.

$$[\Gamma g \Gamma] \cdot [\Gamma h \Gamma] = \sum_{k \in S} \sigma(k) [\Gamma k \Gamma] \tag{4.10}$$

where S is a set of double coset representatives and $\sigma(k)$ is the number of pairs (i, j) such that $\Gamma g_i h_j = \Gamma k$. This follows immediately form equation 4.9.

4.1 Hecke operators on modular forms

Let $G = \mathbf{GL}_2(\mathbb{Q})^+$ and $\Gamma = \mathbf{SL}_2(\mathbb{Z})$. We wish to understand the single and double cosets.

We start by understanding the cosets of $\Gamma \subseteq \mathbf{GL}_2(\mathbb{Z})^+$. There is a bijection

$$\begin{aligned} \left\{ \begin{matrix} \text{Cosets } \Gamma\gamma \text{ such that} \\ \gamma \in \mathbf{Mat}_2(\mathbb{Z}),\ \det\gamma = n \end{matrix} \right\} &\leftrightarrow \left\{ \begin{matrix} \text{Subgroups } \Lambda \subseteq \mathbb{Z}^2 \\ \text{of index } n \end{matrix} \right\} \\ \Gamma\gamma \quad &\rightarrow \quad \text{row span of } \gamma \\ \text{oriented generators of } \Lambda \quad &\leftarrow \quad \Lambda \end{aligned} \tag{4.11}$$

This follows essentially from Pick's theorem.

We want to use this bijection to pick coset representatives. Fix a $\Lambda \subseteq \mathbb{Z}^2$ of index n. Let $d \in \mathbb{N}$ be such that $\Lambda \cap \mathbb{Z}e_2 = (d\mathbb{Z})e_2$. Let $\lambda = ae_1 + be_2 \in \Lambda$ with $a > 0$ minimal. There is a unique such λ with $0 \le b < d$. It is clear from this construction that $\Lambda = \langle ae_1 + be_2, de_2 \rangle$. In fact this gives us a new bijection

$$\left\{ \begin{matrix} \text{Subgroups } \Lambda \subseteq \mathbb{Z}^2 \\ \text{of index } n \end{matrix} \right\} \leftrightarrow \Pi_n \tag{4.12}$$

where

$$\Pi_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathbf{Mat}_2(\mathbb{Z}) : a, d \ge 1, ad = n, 0 \le b < d \right\}. \tag{4.13}$$

It follows that

$$\{\gamma \in \mathbf{Mat}_2(\mathbb{Z}) : \det\gamma = n\} = \coprod_{\gamma \in \Pi_n} \Gamma\gamma. \tag{4.14}$$

Proposition 4.1.1. *1. Let $\gamma \in \mathbf{Mat}_2(\mathbb{Z})$ and $\det\gamma = n \ge 1$. Then*

$$\Gamma\gamma\Gamma = \Gamma \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix} \Gamma \tag{4.15}$$

for unique $n_1, n_2 \ge 1$ and $n_2 \mid n_1, n_1 n_2 = n$.

2.

$$\{\gamma \in \mathbf{Mat}_2(\mathbb{Z}) : \det\gamma = n\} = \coprod \Gamma \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix} \Gamma \tag{4.16}$$

where the sum is over $n_1, n_2 \ge 1, n_2 \mid n_1, n_1 n_2 = n$.

3. Let γ, n_1, n_2 be as above, if $d \ge 1$, then

$$\Gamma(d^{-1}\gamma)\Gamma = \Gamma \begin{pmatrix} n_1/d & 0 \\ 0 & n_2/d \end{pmatrix} \Gamma. \tag{4.17}$$

Proof. Smith normal form (though some extra work needs to be done). ■

Corollary 4.1.2. *The set*

$$\left\{\Gamma\begin{pmatrix} r_1 & 0 \\ 0 & r_2\end{pmatrix}\Gamma : r_1, r_2 \in \mathbb{Q}_{>0}, r_1/r_2 \in \mathbb{Z}\right\} \tag{4.18}$$

is a $\mathbb{Z}$ basis for $\mathcal{H}(G,\Gamma)$.

Proof. It clearly contains all double cosets. To see that the double cosets are distinct multiply by a constant to clear denominators and the result follows. ∎

Notation 4.1.3. For n_1, n_2 with $n_1 \mid n_2$ we define

$$T(n_1, n_2) = \left[\Gamma\begin{pmatrix} n_1 & 0 \\ 0 & n_2\end{pmatrix}\Gamma\right]. \tag{4.19}$$

From T we define $R(n) = T(n,n)$ and $T(n) = \sum_{1 \geq n_1 \mid n_2, n_1 n_2 = n} T(n_1, n_2)$.

Lemma 4.1.4. *Let G be an abelian group of size nm with $(n,m) = 1$. Then there exists a unique subgroup $H \leq G$ of size n.*

Proof. Since $\operatorname{ord}(g_1 + g_2) \mid \operatorname{ord}(g_1)\operatorname{ord}(g_2)$ it follows that all elements coprime to m form a subgroup of G. Call this subgroup H. By Sylow's theorems, $n \mid |H|$. By Cauchy's theorem, $|H| \mid n$. Thus we have existence. Uniqueness follows since if H' is another such subgroup, then by construction $H' \leq H$. Since they have the same order they must be equal. ∎

Corollary 4.1.5. *Let $\Lambda \leq \mathbb{Z}^2$ be a subgroup of index nm with $(m,n) = 1$. Then there exists a unique $\Lambda \leq \Lambda' \leq \mathbb{Z}^2$ such that $|\Lambda' : \Lambda| = n$.*

cor:lat **Corollary 4.1.6.** *Let L_k denote the set of subgroups of $\mathbb{Z}^2$ of index k. If $(n,m) = 1$ then*

$$L_{mn} = \coprod_{\Lambda \in L_n} \{\textit{subgroups of } \Lambda \textit{ of index } m\} \tag{4.20}$$

Thm 4.1.7. 1. *$R(nm) = R(n)R(m)$ for all $n, m \geq 1$ and $R(n)$ lies in the center of $\mathcal{H}(G,\Gamma)$ for all $n \geq 1$.*

2. *$T(nm) = T(n)T(m)$ whenever $(m,n) = 1$.*

3. *$T(p)T(p^r) = T(p^{r+1}) + pR(p)T(p^{r-1})$ for $r \geq 1$.*

Proof. 1. Since

$$\Gamma\begin{pmatrix} n & 0 \\ 0 & n\end{pmatrix}\Gamma = \Gamma\begin{pmatrix} n & 0 \\ 0 & n\end{pmatrix} \tag{4.21}$$

we have

$$\left[\Gamma\begin{pmatrix} n & 0 \\ 0 & n\end{pmatrix}\Gamma\right][\Gamma\gamma\Gamma] = \left[\Gamma\begin{pmatrix} n & 0 \\ 0 & n\end{pmatrix}\gamma\Gamma\right] \tag{4.22}$$

for all γ. The result follows.

2. We have
$$\Theta(T(m)T(n)) = \sum_{\delta\in\Pi_m,\gamma\in\Pi_n} [\Gamma\delta\gamma] \tag{4.23}$$
and
$$\Theta(T(mn)) = \sum_{\epsilon\in\Pi_{mn}} [\Gamma\epsilon]. \tag{4.24}$$
But $\{\Gamma\delta\gamma : \delta \in \Pi_m\}$ corresponds to subgroups of $\mathbb{Z}^2\gamma$ of index m and so the result follows from corollary 4.1.6.

3. We have
$$\Theta(T(p^r)T(p)) = \sum_{\delta\in\Pi_{p^r},\gamma\in\Pi_p} [\Gamma\delta\gamma]. \tag{4.25}$$
As before $\{\Gamma\delta\gamma : \delta \in \Pi_{p^r}\}$ corresponds to subgroups of $\mathbb{Z}^2\gamma$ of index p^r. Note that all index p subgroups of $\mathbb{Z}^2$ contain $p\mathbb{Z}^2$ and so if Λ is an index p^{r+1} subgroup of $\mathbb{Z}^2$ contained in $p\mathbb{Z}^2 \subset$ then it is contained in all $p+1$ index p subgroups of $\mathbb{Z}^2$. If $p\mathbb{Z}^2$ does not contain Λ then $|\mathbb{Z}^2 : \Lambda + p\mathbb{Z}^2| = p$. Using the fact that $p\mathbb{Z}^2$ is contained in all index p subgroups it is straightforward to show that $\Lambda + p\mathbb{Z}^2$ is the unique index p subgroup of $\mathbb{Z}^2$ containing Λ. The result follows.

■

Corollary 4.1.8. *$\mathcal{H}(G,\Gamma)$ is commutative and is generated by $\{T(p), R(p), R(p)^{-1} :$ p prime$\}$.*

Proof. It suffices to show that given set generates $\mathcal{H}(G,\Gamma)$. It is clear that $T(n_1,n_2), R(p), R(p)^{-1}$ generate $\mathcal{H}(G,\Gamma)$. But we know
$$T(n_1,n_2) = R(n_2)T(n_1/n_2, 1) \tag{4.26}$$
so we only need to show that we can obatin the $T(n,1)$. Note that when n is squarefree $T(n) = T(n,1)$ so this case is ok. The remaining cases follow from induction. ■