Derivation of Backpropagation in Convolutional Neural Network (CNN)

Zhifei Zhang

University of Tennessee, Knoxvill, TN

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Abstract— Derivation of backpropagation in convolutional neural network (CNN) is conducted based on an example with two convolutional layers. The step-by-step derivation is helpful for beginners. First, the feedforward procedure is claimed, and then the backpropagation is derived based on the example.
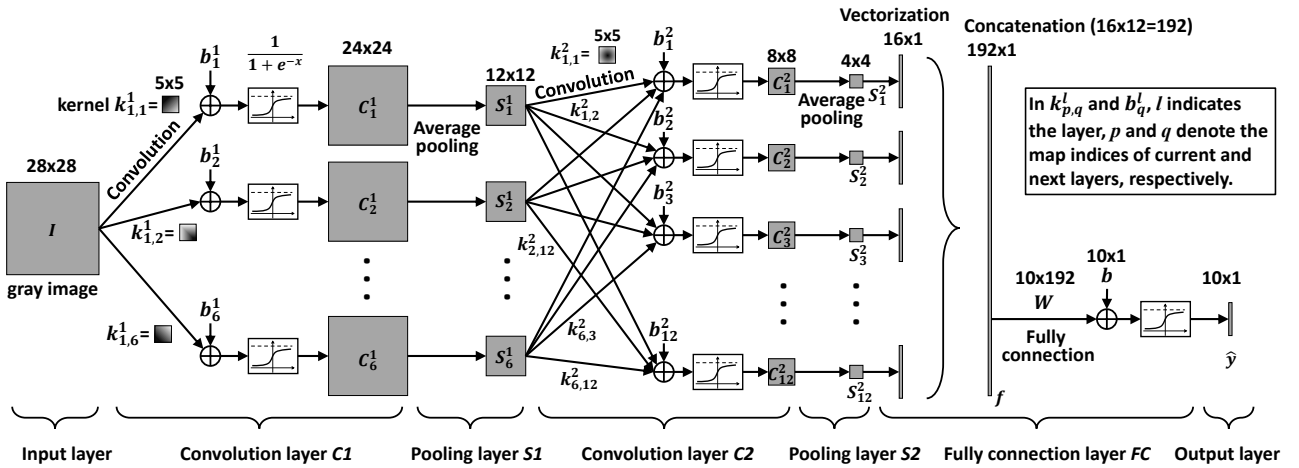
1 Feedforward

Figure 1: The structure of CNN example that will be discussed in this paper. It is exactly the same to the structure used in the demo of Matlab DeepLearnToolbox [1]. All later derivation will use the same notations in this figure.

1.1 Initialization of Parameters

The parameters are:

- **C1 layer**, $k_{1,p}^1$ (size 5×5) and b_p^1 (size 1×1), $p = 1, 2, \cdots 6$

- **C2 layer**, $k_{p,q}^2$ (size 5×5) and b_q^2 (size 1×1), $q = 1, 2, \cdots 12$

- **FC layer**, W (size 10×192) and b (size 10×1)

All bias, b_p^1, b_q^2, and b, are initialize to zero. The others are drew randomly from a uniform distribution defined based on the kernel size and number of input and output maps on corresponding layers [2] (see section 4.6 of [2]).

$$k_{1,p}^1 \sim U\left(\pm\sqrt{\frac{6}{(1+6) \times 5^2}}\right) \tag{1}$$

$$k_{p,q}^2 \sim U\left(\pm\sqrt{\frac{6}{(6+12) \times 5^2}}\right) \tag{2}$$

$$W \sim U\left(\pm\sqrt{\frac{6}{192+10}}\right) \tag{3}$$

where $U(\pm x)$ denotes a uniform distribution with upper and lower bounds of $\pm x$. Totally, the number of parameters is $(5 \times 5 + 1) \times 6 + (5 \times 5 \times 6 + 1) \times 12 + 10 \times 192 + 10 = 3898$.

1.2 Convolution Layer C1

$$C_p^1 = \sigma(I * k_{1,p}^1 + b_p^1), \text{ where } \sigma(x) = \frac{1}{1 + \exp^{-x}} \tag{4}$$

$$C_p^1(i,j) = \sigma\left(\sum_{u=-2}^{2}\sum_{v=-2}^{2} I(i-u, j-v) \cdot k_{1,p}^1(u,v) + b_p^1\right) \tag{5}$$

where $p = 1, 2, \cdots, 6$ because there are 6 feature maps on C1 layer, $*$ denotes the convolution, and i, j are row and column indices of the feature map. Only keeping those parts of the convolution that are computed without the zero-padded edges, the size of C_p^1 is 24×24, rather than 28×28 like I.

1.3 Pooling Layer S1

$$S_p^1(i,j) = \frac{1}{4}\sum_{u=0}^{1}\sum_{v=0}^{1} C_p^1(2i-u, 2j-v), \ i,j = 1, 2, \cdots, 12 \tag{6}$$

1.4 Convolution Layer C2

$$C_q^2 = \sigma\left(\sum_{p=1}^{6} S_p^1 * k_{p,q}^2 + b_q^2\right) \tag{7}$$

$$C_q^2(i,j) = \sigma\left(\sum_{p=1}^{6}\sum_{u=-2}^{2}\sum_{v=-2}^{2} S_p^1(i-u, j-v) \cdot k_{p,q}^2(u,v) + b_q^2\right) \tag{8}$$

where $q = 1, 2, \cdots, 12$ because there are 12 feature maps on C2 layer. Only keeping those parts of the convolution that are computed without the zero-padded edges, the size of C_q^2 is 8×8, rather than 12×12 like S_p^1.

1.5 Pooling Layer S2

$$S_q^2(i,j) = \frac{1}{4}\sum_{u=0}^{1}\sum_{v=0}^{1} C_q^2(2i-u, 2j-v), \ i,j = 1, 2, \cdots, 4 \tag{9}$$

1.6 Vectorization and Concatenation

Each S_q^2 is a 4×4 matrix, and there are 12 such matrices on the S2 layer. First, each S_q^2 is vectorized by column scan, then all 12 vectors are concatenated to form a long vector with the length of $4 \times 4 \times 12 = 192$. We denote this process by

$$f = F\left(\{S_q^2\}_{q=1,2,\cdots,12}\right),\tag{10}$$

and the reverse process is

$$\{S_q^2\}_{q=1,2,\cdots,12} = F^{-1}(f).\tag{11}$$

1.7 Fully Connection Layer FC

$$\hat{y} = \sigma(W \times f + b)\tag{12}$$

1.8 Loss Function

Assuming the true label is y, the loss function is express by

$$L = \frac{1}{2}\sum_{i=1}^{10}\left(\hat{y}(i) - y(i)\right)^2\tag{13}$$

2 Backpropagation

In the backpropagation, we'll update the parameters from the back to start, namely W and b, $k_{p,q}^2$ and b_q^2, $k_{1,p}^1$ and b_p^1.

2.1 ΔW (size 10×192)

$$\Delta W(i,j) = \frac{\partial L}{\partial W(i,j)}\tag{14}$$

$$= \frac{\partial L}{\partial \hat{y}(i)} \cdot \frac{\partial \hat{y}(i)}{\partial W(i,j)}\tag{15}$$

$$= (\hat{y}(i) - y(i)) \cdot \frac{\partial}{\partial W(i,j)}\sigma\left(\sum_{j=1}^{192} W(i,j) \times f(j) + b(i)\right)\tag{16}$$

$$= (\hat{y}(i) - y(i)) \cdot \hat{y}(i)(1 - \hat{y}(i)) \cdot f(j)\tag{17}$$

Let $\Delta \hat{y}(i) = (\hat{y}(i) - y(i)) \cdot \hat{y}(i)(1 - \hat{y}(i))$, whose size is 10×1, then

$$\Delta W(i,j) = \Delta \hat{y}(i) \cdot f(j)\tag{18}$$

$$\implies \Delta W = \Delta \hat{y} \times f^T\tag{19}$$

2.2 Δb (size 10×1)

$$\Delta b(i) = \frac{\partial L}{\partial b(i)} \tag{20}$$

$$= \frac{\partial L}{\partial \hat{y}(i)} \cdot \frac{\partial \hat{y}(i)}{\partial b(i)} \tag{21}$$

$$= (\hat{y}(i) - y(i)) \cdot \frac{\partial}{\partial b(i)} \sigma \left(\sum_{j=1}^{192} W(i,j) \times f(j) + b(i) \right) \tag{22}$$

$$= (\hat{y}(i) - y(i)) \cdot \hat{y}(i)(1 - \hat{y}(i)) \tag{23}$$

$$\implies \Delta b = \Delta \hat{y} \tag{24}$$

2.3 $\Delta k_{p,q}^2$ (size 5×5)

Because of concatenation, vectorization, and pooling, we need to compute the back-propagation error ΔC_q^2 on C2 layer before calculating $\Delta k_{p,q}^2$.

$$\Delta f(j) = \frac{\partial L}{\partial f} \tag{25}$$

$$= \sum_{i=1}^{10} \frac{\partial L}{\partial \hat{y}(i)} \cdot \frac{\partial \hat{y}(i)}{\partial f(j)} \tag{26}$$

$$= (\hat{y}(i) - y(i)) \cdot \frac{\partial}{\partial f(j)} \sigma \left(\sum_{j=1}^{192} W(i,j) \times f(j) + b(i) \right) \tag{27}$$

$$= \sum_{i=1}^{10} (\hat{y}(i) - y(i)) \cdot \hat{y}(i)(1 - \hat{y}(i)) \cdot W(i,j) \tag{28}$$

$$= \sum_{i=1}^{10} \Delta \hat{y}(i) \cdot W(i,j) \tag{29}$$

$$\implies \Delta f = W^T \times \Delta \hat{y} \tag{30}$$

From section 1.6, we reshape the long error vector Δf (size 192×1) by

$$\{\Delta S_q^2\}_{q=1,2,\cdots,12} = F^{-1}(\Delta f), \tag{31}$$

which gets the error on S2 layer (twelve 4×4 error maps). Because there is no parameters on S2 layer, we do not need to do any derivative stuff. Then, upsampling is performed to obtain the error on C2 layer.

$$\Delta C_q^2(i,j) = \frac{1}{4} \Delta S_q^2 \left(\lceil i/2 \rceil, \lceil j/2 \rceil \right), \ i,j = 1, 2, \cdots, 8 \tag{32}$$

where $\lceil \cdot \rceil$ denotes the ceiling function. Note that the size of ΔS_q^2 and ΔC_q^2 are 4×4 and 8×8, respectively. Now, we are ready to derive $\Delta k_{p,q}^2$.

$$\Delta k_{p,q}^2(u,v) = \frac{\partial L}{\partial k_{p,q}^2(u,v)} \tag{33}$$

$$= \sum_{i=1}^{8} \sum_{j=1}^{8} \frac{\partial L}{\partial C_q^2(i,j)} \cdot \frac{\partial C_q^2(i,j)}{\partial k_{p,q}^2(u,v)} \tag{34}$$

$$= \sum_{i=1}^{8} \sum_{j=1}^{8} \Delta C_q^2(i,j) \cdot \frac{\partial}{\partial k_{p,q}^2(u,v)} \sigma \left(\sum_{p=1}^{6} \sum_{u=-2}^{2} \sum_{v=-2}^{2} S_p^1(i-u,j-v) \cdot k_{p,q}^2(u,v) + b_q^2 \right) \tag{35}$$

$$= \sum_{i=1}^{8} \sum_{j=1}^{8} \Delta C_q^2(i,j) \cdot C_q^2(i,j) \left(1 - C_q^2(i,j) \right) \cdot S_p^1(i-u,j-v) \tag{36}$$

Let

$$\Delta C_{q,\sigma}^2(i,j) = \Delta C_q^2(i,j) \cdot C_q^2(i,j) \left(1 - C_q^2(i,j) \right), \tag{37}$$

which is actually the error before sigmoid function on C2 layer. Therefore,

$$C_{q,\sigma}^2(i,j) = \sum_{p=1}^{6} \sum_{u=-2}^{2} \sum_{v=-2}^{2} S_p^1(i-u,j-v) \cdot k_{p,q}^2(u,v) + b_q^2 \tag{38}$$

Rotating S_p^1 180 degrees, we get $S_{p,rot180}^1$, thus $S_{p,rot180}^1(u-i,v-j) = S_p^1(i-u,j-v)$. Therefore, $\Delta k_{p,q}^2$ can be expressed by

$$\Delta k_{p,q}^2(u,v) = \sum_{i=1}^{8} \sum_{j=1}^{8} S_{p,rot180}^1(u-i,v-j) \cdot \Delta C_{q,\sigma}^2(i,j) \tag{39}$$

$$\implies \Delta k_{p,q}^2 = S_{p,rot180}^1 * \Delta C_{q,\sigma}^2 \tag{40}$$

2.4 Δb_q^2 (size 1×1)

$$\Delta b_q^2 = \frac{\partial L}{\partial b_q^2} \tag{41}$$

$$= \sum_{i=1}^{8} \sum_{j=1}^{8} \frac{\partial L}{\partial C_q^2(i,j)} \cdot \frac{\partial C_q^2(i,j)}{\partial b_q^2} \tag{42}$$

$$= \sum_{i=1}^{8} \sum_{j=1}^{8} \Delta C_q^2(i,j) \cdot \frac{\partial}{\partial b_q^2} \sigma \left(\sum_{p=1}^{6} \sum_{u=-2}^{2} \sum_{v=-2}^{2} S_p^1(i-u,j-v) \cdot k_{p,q}^2(u,v) + b_q^2 \right) \tag{43}$$

$$= \sum_{i=1}^{8} \sum_{j=1}^{8} \Delta C_q^2(i,j) \cdot C_q^2(i,j) \left(1 - C_q^2(i,j) \right) \tag{44}$$

$$= \sum_{i=1}^{8} \sum_{j=1}^{8} \Delta C_{q,\sigma}^2(i,j) \tag{45}$$

2.5 $\Delta k_{1,p}^1$ (size 5×5)

Similar to the derivation of $\Delta k_{p,q}^2$, we should first obtain ΔS_p^1, the error on S1 layer. Then, upsampling will be performed to get ΔC_p^1, the error on C1 layer. Finally, following the same

way, we can calculate $\Delta k_{1,p}^1$.

$$\Delta S_p^1(i,j) = \frac{\partial L}{\partial S_p^1(i,j)} \tag{46}$$

$$= \sum_{q=1}^{12} \sum_{u=-2}^{2} \sum_{v=-2}^{2} \frac{\partial L}{\partial C_{q,\sigma}^2(i+u,j+v)} \cdot \frac{\partial C_{q,\sigma}^2(i+u,j+v)}{\partial S_p^1(i,j)} \tag{47}$$

$$= \sum_{q=1}^{12} \sum_{u=-2}^{2} \sum_{v=-2}^{2} \Delta C_{q,\sigma}^2(i+u,j+v) \cdot \frac{\partial}{\partial S_p^1(i,j)} \left(\sum_{p=1}^{6} \sum_{u=-2}^{2} \sum_{v=-2}^{2} S_p^1(i,j) \cdot k_{p,q}^2(u,v) + b_q^2 \right) \tag{48}$$

$$= \sum_{q=1}^{12} \sum_{u=-2}^{2} \sum_{v=-2}^{2} \Delta C_{q,\sigma}^2(i+u,j+v) \cdot k_{p,q}^2(u,v) \tag{49}$$

Rotating $k_{p,q}^2$ 180 degrees, we get $k_{p,q,rot180}^2(-u,-v) = k_{p,q}^2(u,v)$. Therefore,

$$\Delta S_p^1(i,j) = \sum_{q=1}^{12} \sum_{u=-2}^{2} \sum_{v=-2}^{2} \Delta C_{q,\sigma}^2(i-(-u),j-(-v)) \cdot k_{p,q,rot180}^2(-u,-v) \tag{50}$$

$$\Longrightarrow \Delta S_p^1 = \sum_{q=1}^{12} \Delta C_{q,\sigma}^2 * k_{p,q,rot180}^2 \tag{51}$$

By upsampling, we get the error on C1 layer,

$$\Delta C_p^1(i,j) = \frac{1}{4} \Delta S_p^1 \left(\lceil i/2 \rceil, \lceil j/2 \rceil \right), \ i,j = 1,2,\cdots,24 \tag{52}$$

Now, we are ready to calculate $\Delta k_{1,p}^1$,

$$\Delta k_{1,p}^1(u,v) = \frac{\partial L}{\partial k_{1,p}^1(u,v)} \tag{53}$$

$$= \sum_{i=1}^{24} \sum_{j=1}^{24} \frac{\partial L}{\partial C_p^1(i,j)} \cdot \frac{\partial C_p^1(i,j)}{\partial k_{1,p}^1(u,v)} \tag{54}$$

$$= \sum_{i=1}^{24} \sum_{j=1}^{24} \Delta C_p^1(i,j) \cdot \frac{\partial}{\partial k_{1,p}^1(u,v)} \sigma \left(\sum_{u=-2}^{2} \sum_{v=-2}^{2} I(i-u,j-v) \cdot k_{1,p}^1(u,v) + b_p^1 \right) \tag{55}$$

$$= \sum_{i=1}^{24} \sum_{j=1}^{24} \Delta C_p^1(i,j) \cdot C_p^1(i,j) \left(1 - C_p^1(i,j) \right) \cdot I(i-u,j-v) \tag{56}$$

By the same token, rotate I 180 degrees, and let

$$\Delta C_{p,\sigma}^1(i,j) = \Delta C_p^1(i,j) \cdot C_p^1(i,j) \left(1 - C_p^1(i,j) \right). \tag{57}$$

Finally,

$$\Delta k_{1,p}^1(u,v) = \sum_{i=1}^{24} \sum_{j=1}^{24} I_{rot180}(u-i,v-j) \cdot \Delta C_{p,\sigma}^1(i,j) \tag{58}$$

$$\Longrightarrow \Delta k_{1,p}^1 = I_{rot180} * \Delta C_{p,\sigma}^1 \tag{59}$$

2.6 Δb_p^1 (size 1×1)

$$\Delta b_p^1 = \frac{\partial L}{\partial b_p^1} \tag{60}$$

$$= \sum_{i=1}^{24} \sum_{j=1}^{24} \frac{\partial L}{\partial C_p^1(i,j)} \cdot \frac{\partial C_p^1(i,j)}{\partial b_p^1} \tag{61}$$

$$= \sum_{i=1}^{24} \sum_{j=1}^{24} \Delta C_p^1(i,j) \cdot \frac{\partial}{\partial b_p^1} \sigma \left(\sum_{u=-2}^{2} \sum_{v=-2}^{2} I(i-u,j-v) \cdot k_{1,p}^1(u,v) + b_p^1 \right) \tag{62}$$

$$= \sum_{i=1}^{24} \sum_{j=1}^{24} \Delta C_p^1(i,j) \cdot C_p^1(i,j) \left(1 - C_p^1(i,j) \right) \tag{63}$$

$$= \sum_{i=1}^{24} \sum_{j=1}^{24} \Delta C_{p,\sigma}^1(i,j) \tag{64}$$

3 Parameter Update

We need to set a learning rate $\alpha \in (0,1]$.

$$k_{1,p}^1 \leftarrow k_{1,p}^1 - \alpha \cdot \Delta k_{1,p}^1 \tag{65}$$

$$b_p^1 \leftarrow b_p^1 - \alpha \cdot \Delta b_p^1 \tag{66}$$

$$k_{p,q}^2 \leftarrow k_{p,q}^2 - \alpha \cdot \Delta k_{p,q}^2 \tag{67}$$

$$b_q^2 \leftarrow b_q^2 - \alpha \cdot \Delta b_q^2 \tag{68}$$

$$W \leftarrow W - \alpha \cdot \Delta W \tag{69}$$

$$b \leftarrow b - \alpha \cdot \Delta b \tag{70}$$

References

[1] Palm, Rasmus Berg. "Prediction as a candidate for learning deep hierarchical models of data." *Technical University of Denmark* 5 (2012).

[2] LeCun, Yann A., Leon Bottou, Genevieve B. Orr, and Klaus-Robert MÃijller. "Efficient backprop." In *Neural networks: Tricks of the trade*, pp. 9-48. Springer Berlin Heidelberg, 2012.