

# Scheme Theory

*this is my note for scheme theory. this is loosely based on (1) Görtz - Wedhorn AG 1 (2) Borcherds online lecture (3) Vakil FOAG*

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# Chapter 1

## AFFINE SCHEME

### 1.1 DEFINITION OF AFFINE SCHEME

**Definition 1.1.1** (ringed space, locally ringed space)

A ringed space  $(X, \mathcal{O})$  is a topological space  $X$  together with a sheaf of rings  $\mathcal{O}$ . A ringed space is a locally ringed space if for every point  $x \in X$ , the stalk  $\mathcal{O}_x$  is a local ring.

**Definition 1.1.2** (affine scheme)

An affine scheme is a locally ringed space that is isomorphic to the spectrum of some ring  $A$

### 1.2 CONSTRUCT AFFINE SCHEME FROM RING

Let  $A$  be a ring, let  $X = \text{Spec } A$  be the set of prime ideals in  $A$

$$\text{Spec } A = \{\text{prime } \mathfrak{p} \subseteq A\}$$

We put a topology on  $\text{Spec } A$  generated by the basis of open sets

$$D(f) = \{\mathfrak{p} \in \text{Spec } A : f \notin \mathfrak{p}\}$$

for every  $f \in A$ . The constructed topology is called Zariski's topology and the open subsets of the form  $D(f)$  for some  $f \in A$  are called **principal open subsets**. We put a sheaf of rings  $\mathcal{O}$  on  $\text{Spec } A$  generated by

$$\mathcal{O}(D(f)) = A_f$$

If  $D(g) \subseteq D(f)$ , that is  $g \in \sqrt{(f)}$ ,  $g^n = fh$  for some  $h \in A$  and  $n \geq 1$ . Since  $f$  is a unit in  $A_g$ , the restriction map is well-defined and unique

$$\begin{aligned} A_f &\rightarrow A_g \\ \frac{x}{f^m} &\mapsto x \left( \frac{h}{g^n} \right)^m \end{aligned}$$

Under this construction,  $(X, \mathcal{O})$  is an affine scheme. An element  $f \in A$  is called **function**, a element  $x \in X$  is called **point**, when refering  $x$  as a prime ideal in  $A$ , we write  $\mathfrak{p}_x \subseteq A$ , and function evaluation is equivalent to sending  $f$  to the residue field of stalk  $\mathcal{O}_X = A_{\mathfrak{p}_x}$ .

When  $A$  is a polynomial ring of  $n$  variables over ring  $R$ , we write  $\mathbb{A}_R^n = \text{Spec } A$

*(I see many proofs, people assumed that open set is just principle open set. maybe the argument generalizes easily to general open set. anyway, if needed, below is the defintion of sheaf on open set)*

On  $\text{Spec } A$ , any open set is of the form

$$D(S) = \{\mathfrak{p} \in \text{Spec } A : \mathfrak{p} \cap S = \emptyset\}$$

for some multiplicatively closed subset  $S \subseteq A$ . An the ring over  $D(S)$  is

$$\mathcal{O}(D(S)) = S^{-1}A$$

If  $D(T) \subseteq D(S)$ , then  $S \subseteq T$ , the restriction map is the canonical map

$$\begin{aligned} \mathcal{O}(D(S)) &\rightarrow \mathcal{O}(D(T)) \\ S^{-1}A &\rightarrow T^{-1}A \\ \frac{a}{s} &\mapsto \frac{a}{s} \end{aligned}$$

Note that, this map corresponds to the inclusion of affine schemes

$$D(T) \hookrightarrow D(S)$$

## 1.3 SOME EXAMPLES OF AFFINE SCHEME

### Remark 1.3.1 (irreducible)

A topological space  $X$  is reducible if it can be written as a union  $X = X_1 \cup X_2$  of two non-empty proper closed subsets.

1. A topological space is irreducible if it is not reducible. Equivalently,  $X$  is irreducible if all non-empty open sets of  $X$  are dense or any two non-empty open sets have non-empty intersection.
2. A subset  $F$  of topological space  $X$  is reducible or irreducible if  $F$  as a subspace is reducible or irreducible, respectively.
3. An irreducible component is a maximal irreducible subset.

### Remark 1.3.2 (closed, generic, specialization, generization, maximal)

Let  $X$  be a topological space

1. A point  $x \in X$  is called closed if  $\{x\} \subseteq X$  is closed
2. A point  $\eta \in X$  is called generic if  $\overline{\{\eta\}} = X$
3. Let  $x, y \in X$ ,  $y$  is called a specialization of  $x$  if  $y \in \overline{\{x\}}$ . In that case,  $x$  is called generization of  $y$
4. A point  $x \in X$  is called maximal if  $\overline{\{x\}}$  is an irreducible component of  $X$

**example:** (integers) Let  $A = \mathbb{Z}$ , then

$$X = \text{Spec } \mathbb{Z} = \{(2), (3), (5), \dots, (0)\}$$

for each prime number  $p \in \mathbb{Z}$ ,  $(p)$  is a closed point.  $(0)$  is a generic point, and closure of  $(0)$  is the whole space  $X$ .

**example:** (field) Let  $k$  be a field. Let  $A = k$ , then

$$X = \text{Spec } k = \{(0)\}$$

is a singleton set.

**example:** (polynomial ring of two variables over an algebraically closed field) Let  $A$  be an arbitrary ring, maximal ideals are closed points. Moreover, the closure of a point  $\mathfrak{p}$  is the set of prime ideals containing  $\mathfrak{p}$

$$V(\mathfrak{p}) = \{\mathfrak{q} \in \text{Spec } A : \mathfrak{p} \subseteq \mathfrak{q}\}$$

In particular, let  $k$  be an algebraically closed field, let  $A = k[x, y]$ , by Nullstellensatz, the set of closed points are

$$\{(x - a, y - b) : a \in k, b \in k\}$$

$X$  admits other generic points  $(0)$  and  $(f)$  for every irreducible  $f \in k[x, y]$ . By dimensionality argument, the prime  $(f)$  is of height 1 and the closure of  $(f)$  consists of  $(f)$  and  $(x - a, y - b)$  for  $(a, b) \in k^2$  in the vanishing set of  $f$ . Similarly, closure of  $(0)$  is the whole space.

## 1.4 MORPHISM OF AFFINE SCHEMES

**Remark 1.4.1** (inverse image, direct image)

Let  $\psi : X \rightarrow Y$  be a continuous map. The inverse image functor  $\psi^{-1}$  and direct image functor  $\psi_*$  is an adjoint pair between the category of sheaves on  $X$  and the category of sheaves on  $Y$ . Let  $\mathcal{F}$  and  $\mathcal{G}$  be a sheaf on  $X$  and a sheaf on  $Y$  respectively, then

$$\text{Hom}_{\text{Sh}(X)}(\psi^{-1}\mathcal{G}, \mathcal{F}) \cong \text{Hom}_{\text{Sh}(Y)}(\mathcal{G}, \psi_*\mathcal{F})$$

The direct image functor  $\psi_*$  is defined as follows: for every open subset  $V \subseteq Y$ , then

$$(\psi_*\mathcal{F})(V) = \mathcal{F}(\psi^{-1}(U))$$

The inverse image functor  $\psi^{-1}$  is defined as follows: for every open subset  $U \subseteq X$ , then

$$(\psi^{-1}\mathcal{G})(U) = \text{colim}_{V \subseteq Y : \psi(U) \subseteq V} \mathcal{G}(V)$$

**Definition 1.4.2** (morphism of ringed spaces)

A morphism of ringed spaces  $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  is defined by a continuous map  $\psi : X \rightarrow Y$  and a morphism of sheaves of rings  $\psi^\flat : \mathcal{O}_Y \rightarrow \psi_*\mathcal{O}_X$ . By adjunction between  $\psi^{-1}$  and  $\psi_*$ , this is equivalent to a morphism of sheaves of rings  $\psi^\# : \psi^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ . Moreover,  $\psi^\#$  induces a map on stalks

$$\psi_x^\# : (\psi^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y, \psi(x)} \rightarrow \mathcal{O}_{X, x}$$

**Remark 1.4.3** (compatible germs)

Given a sheaf of sets  $\mathcal{F}$  on  $X$ , for every open subset  $U \subseteq X$ , the natural map from sections into product of stalks

$$\mathcal{F}(U) \hookrightarrow \prod_{x \in U} \mathcal{F}_x$$

is injective. An element  $(s_x)_{x \in U} \subseteq \prod_{x \in U} \mathcal{F}_x$  in the image if this map is called compatible germs.

From the equivalence between set of sections and set of compatible germs, one can identify a morphism of ringed spaces by its topological space map  $\psi : X \rightarrow Y$  and its map on stalks  $\psi_x^\# : (\psi^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y, \psi(x)} \rightarrow \mathcal{O}_{X, x}$ .

**Definition 1.4.4** (morphism of locally ringed spaces, morphism of affine schemes)

A morphism of ringed spaces  $(\psi, \psi^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  is a morphism of locally ringed spaces if for every  $x \in X$ , the induced map on stalks  $\psi_x^\# : (\psi^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y, \psi(x)} \rightarrow \mathcal{O}_{X, x}$  is a local homomorphism. A morphism of affine schemes is a morphism of locally ringed spaces.

We obtain the category of affine schemes, denoted by  $\text{AffSch}$ . The local homomorphism condition of the map between stalks ensures that any zero function  $g \in \mathcal{O}_{Y, \psi(x)}$  at  $\psi(x) \in Y$  will be sent to a zero function  $\psi_x^\#(g) \in \mathcal{O}_{X, x}$  at  $x \in X$

**Proposition 1.4.5** (equivalence between commutative rings and affine schemes)

The functor  $\text{Spec}$  from the opposite category of commutative rings into the category of affine schemes is fully faithful and essentially surjective.

$$\text{Spec} : \text{CRing}^{\text{op}} \xrightarrow{\sim} \text{AffSch}$$

## 1.5 CONSTRUCT MORPHISM OF AFFINE SCHEMES FROM MAP OF RINGS

Let  $\phi : A \rightarrow B$  be a map of rings, let  $X = \text{Spec } B$  and  $Y = \text{Spec } A$  be the corresponding affine schemes. Then, the corresponding morphism of locally ringed space  $(\psi, \psi^\#) : X \rightarrow Y$  is defined as follows: for every  $x \in X$

$$\begin{aligned} \psi : X &\rightarrow Y \\ x &\mapsto \phi^{-1}(x) \end{aligned}$$

The local homomorphism  $\psi_x^\# : \mathcal{O}_{Y, \psi(x)} = A_{\psi(x)} \rightarrow B_x$  is induced from  $\phi : A \rightarrow B$

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ A_{\psi(x)} & \xrightarrow{\psi_x^\#} & B_x \end{array}$$

## 1.6 SOME EXAMPLES OF MORPHISM OF AFFINE SCHEMES

**example:** (quotient) Let  $A$  be a ring and  $\mathfrak{a} \subseteq A$  be an ideal. The map  $A \rightarrow A/\mathfrak{a}$  induces an injective map of sets

$$\text{Spec } A/\mathfrak{a} \hookrightarrow \text{Spec } A$$

As sets, we have  $\text{Spec } A/\sqrt{\mathfrak{a}} = \text{Spec } A/\mathfrak{a} = \text{Spec } A/\mathfrak{a}^n$  for every  $n \geq 1$ . However, as affine schemes, in general, they are different,  $A/\sqrt{\mathfrak{a}}$  is reduced, i.e. has no nilpotent while  $\text{Spec } A/\mathfrak{a}$  and  $\text{Spec } A/\mathfrak{a}^n$  might have nilpotents.

Furthermore, when  $\mathfrak{a} = \mathfrak{p}_x = x$  is prime, quotient by  $\mathfrak{p}_x$  is the action of taking closed subscheme <sup>1</sup> of  $X$

*keep all primes containing  $\mathfrak{p}_x$ . equivalently, keep all points contained in  $x$*

**example:** (localization) Let  $A$  be a ring and  $S \subseteq A$  be a multiplicatively closed subset. Localization at  $S$  induces an injective map of sets

$$\text{Spec } S^{-1}A \hookrightarrow \text{Spec } A$$

Furthermore, when  $A - S = \mathfrak{p}_x = x$  is prime, localization at  $\mathfrak{p}_x$  is the action of taking open subscheme <sup>2</sup> of  $X$

*keep all primes contained in  $\mathfrak{p}_x$ . equivalently, keep all points containing  $x$*

**example:** (disjoint union) Let  $A, B$  be rings, then the projection  $A \times B \rightarrow A$  corresponds to the monomorphism of affine schemes

$$\text{Spec } A \hookrightarrow \text{Spec } A \amalg \text{Spec } B \cong \text{Spec } A \times B$$

Categorically,  $A \times B$  is product in rings and  $\text{Spec } A \amalg \text{Spec } B$  is coproduct in affine schemes

**example:** (non-trivial fiber) Consider the ring map  $\phi : k[u] \rightarrow k[x, y]$  defined by  $u \mapsto y^2 - x$ . It induces a morphism of affine schemes

$$\begin{aligned}\psi : \text{Spec } k[x, y] &\rightarrow \text{Spec } k[u] \\ (0) &\mapsto (0) \\ (x - a, y - b) &\mapsto (u - (b^2 - a)) \\ (y^2 - x) &\mapsto (u) \\ (f) &\mapsto (0)\end{aligned}$$

for every  $a, b \in k$  and  $f$  is irreducible other than multiple of  $y^2 - x$ . (*check this carefully*).

**example:** (inclusion of open subschemes) Let  $A$  be a ring, and  $U \subseteq V \subseteq \text{Spec } A$  be open sets. Then, the inclusion map of open affine subschemes corresponds to the canonical ring map between global sections.

$$\begin{aligned}U &\hookrightarrow V \\ \mathcal{O}(U) &\hookleftarrow \mathcal{O}(V)\end{aligned}$$

If  $U = D(T)$  and  $V = D(S)$ , then  $S \subseteq T$  and the canonical ring map is

$$\begin{aligned}S^{-1}A = \mathcal{O}(V) &\rightarrow T^{-1}A = \mathcal{O}(U) \\ \frac{a}{s} &\mapsto \frac{a}{s}\end{aligned}$$

In particular, the affine scheme map induced from  $\text{Spec } A \rightarrow S^{-1}A = \text{Spec } \mathcal{O}(V)$  is precisely  $V \hookrightarrow \text{Spec } A$

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<sup>1</sup>meaning will be revealed later

<sup>2</sup>meaning will be revealed later

# Chapter 2

## SCHEME

### 2.1 DEFINITION OF SCHEME

**Definition 2.1.1** (scheme, morphism of schemes)

A scheme is a locally ringed space  $(X, \mathcal{O})$  such that for every point  $x \in X$ , there is an open neighbourhood  $U_x$  containing  $x$  so that  $(U_x, \mathcal{O}|_{U_x})$  is an affine scheme. Morphism of schemes is morphism of locally ringed spaces.

We obtain the category of schemes, denoted by  $\text{Sch}$

**Definition 2.1.2** (scheme over  $S$ )

Let  $S$  be a scheme, the category  $(\text{Sch}/S)$  of schemes over  $S$  is the category where an object is a morphism of schemes  $X \rightarrow S$  and a morphism is a morphism of schemes  $X \rightarrow Y$  so that  $X \rightarrow S$  factors through  $Y$  by the map  $X \rightarrow Y$

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ & \searrow & \swarrow \\ & S & \end{array}$$

The collection of morphisms from  $X \rightarrow S$  into  $Y \rightarrow S$  is denoted by  $\text{Hom}_S(X, Y)$ . If  $S = \text{Spec } A$ , we also call scheme over  $S$  as scheme over  $A$

Since  $\text{Spec } \mathbb{Z}$  is a terminal object in the category of schemes, the category of schemes is canonically equivalent the category of schemes over  $\mathbb{Z}$  (*proof for  $\text{Spec } \mathbb{Z}$  is terminal in next section*)

### 2.2 MORPHISM INTO AFFINE SCHEMES, GLUING OF MAPPINGS

**Definition 2.2.1** (open subscheme)

Let  $(X, \mathcal{O}_X)$  be a scheme and  $U \subseteq X$  be an open subset. Then, the locally ringed space  $(U, \mathcal{O}_X|_U)$  is a scheme and we call it an open subscheme of  $X$ . If the

**Lemma 2.2.2**

Let  $X$  be a scheme and  $U, V$  be affine open subschemes of  $X$ . Then for every  $x \in U \cap V$ , there exists an open subscheme  $W \subseteq U \cap V$  containing  $x$  such that  $W$  is principal open in  $U$  as well as in  $V$ .

**Proposition 2.2.3 (gluing of morphisms)**

Let  $X, Y$  be locally ringed spaces. Then  $U \mapsto \text{Hom}(U, Y)$  sending open subset  $U \subseteq X$  into the set of morphisms  $(U, \mathcal{O}_X|_U) \rightarrow (Y, \mathcal{O}_Y)$  is a sheaf of sets on  $X$ .

In other words, if  $X = \bigcup_i U_i$  is an open covering, then a family of morphisms  $U_i \rightarrow Y$  glues into a morphism  $X \rightarrow Y$  if and only if the morphisms coincide on intersections  $U_i \cap U_j$ . Moreover, in that case, the resulting morphism  $X \rightarrow Y$  is uniquely determined.

*TODO - proof idea - one can identify the morphism by map between stalks, it feels local hence gluable*

**Proposition 2.2.4**

Let  $X$  be a locally ringed space and  $Y = \text{Spec } A$  an affine scheme. Then, the natural map

$$\begin{aligned} \text{Hom}(X, Y) &\rightarrow \text{Hom}(A, \Gamma(X)) \\ (f, f^\flat) &\mapsto f_Y^\flat \end{aligned}$$

is a bijection.

*Proof. (TODO - use lemma above to prove for the case  $X$  is a scheme, full proof in EGAInew 1.6.3)*  $\square$

$\text{Spec } \mathbb{Z}$  being terminal object in the category of schemes follows this proposition. Moreover, when  $A = \Gamma(X)$ , there corresponds to  $\text{id}_{\Gamma(X)}$  a morphism

$$c_X : X \rightarrow \text{Spec } \Gamma(X)$$

which we call canonical.

## 2.3 MORPHISM FROM AFFINE POINT

Let  $X$  be a scheme. Let  $x \in X$  and  $U \subseteq X$  be an affine open subscheme of  $x$ , let  $U = \text{Spec } A$ . Let  $\mathfrak{p} \in \text{Spec } A$  corresponds to  $x$ . Then, the natural map  $A \rightarrow A_{\mathfrak{p}}$  induces a morphism

$$j_x : \text{Spec } \mathcal{O}_{X,x} = \text{Spec } A_{\mathfrak{p}} \rightarrow \text{Spec } A = U \subseteq X$$

*(here, stalk at  $x \in X$  is the same as stalk at  $x \in X$  because stalk is defined as colimit, so it doesn't matter if we restrict to  $U$  or not - Görtz - Wedhorn didn't explain it clearly, so below is my attempt <sup>1</sup>)*

*proof for  $j_x : \text{Spec } \mathcal{O}_{X,x} \rightarrow X$  being independent of choice of  $U$ .* Let  $U$  and  $V$  be two open affine subschemes containing  $x$ , let  $U = \text{Spec } A$  and  $V = \text{Spec } B$ , let  $\mathfrak{p} \in \text{Spec } A$  and  $\mathfrak{q} \in \text{Spec } B$  correspond to  $x$ . Then  $A_{\mathfrak{p}} = B_{\mathfrak{q}}$  and we have two maps  $j_x^U : \text{Spec } \mathcal{O}_{X,x} \rightarrow U$  and  $j_x^V : \text{Spec } \mathcal{O}_{X,x} \rightarrow V$ . By definition,  $\mathcal{O}_{X,x}$  is the colimit of functor  $\mathcal{O}_X$  from the category of open sets containing  $x$ . Since the contravariant functor  $\text{Spec}$  is fully faithful, it preserves limits,  $\text{Spec } \mathcal{O}_{X,x}$  is the limit of the functor  $\text{Spec } \mathcal{O}_X$ . By universal property of limit, let  $W = U \cap V$ , then  $j_x^U$  and  $j_x^V$  factor through a unique map  $\text{Spec } \mathcal{O}_{X,x} \rightarrow W$  into their intersection and inclusions of schemes  $W \hookrightarrow U$  and  $W \hookrightarrow V$

<sup>1</sup>math.SE <https://math.stackexchange.com/q/5116709/700122>

$$\begin{array}{ccc}
& \mathcal{O}_X(U) & \\
\swarrow & & \downarrow \\
\mathcal{O}_{X,x} & \longleftarrow \dashrightarrow & \mathcal{O}_X(W) \\
\uparrow & & \uparrow \\
& \mathcal{O}_X(V) &
\end{array}
\quad
\begin{array}{ccc}
U & & \\
\swarrow & \uparrow & \downarrow \\
\text{Spec } \mathcal{O}_{X,x} & \dashrightarrow & W \\
\searrow & \downarrow & \\
& V &
\end{array}$$

□

Image of  $j_x$  in  $X$  is the intersection of all open subsets containing  $x$  (consider the map  $A \rightarrow A_{\mathfrak{p}}$ , its kernel is  $\mathfrak{a} = \{a \in A : as = 0 \text{ for some } s \notin \mathfrak{p}\}$ . Hence, the image of  $j_x$  are those primes in  $A$  containing  $\mathfrak{a}$ , ...)

Let  $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$  be the residue field at  $x$ , we obtain the map of schemes

$$i_x : \text{Spec } \kappa(x) \rightarrow \text{Spec } \mathcal{O}_{X,x} \rightarrow X$$

which is called canonical. The image point of  $i_x$  in  $X$  is  $x$ .

Now let  $k$  be a field and  $f : \text{Spec } k \rightarrow X$  be a map, let  $x$  be the image point on  $X$  of the singleton  $p \in \text{Spec } k$ . The map  $f$  induces a map on stalks  $f_p^{\#} : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{\text{Spec } k, p} = k$ . Since  $\mathcal{O}_{X,x}$  is local,  $\mathcal{O}_{X,x} \rightarrow k$  factors through  $\mathcal{O}_{X,x} \rightarrow \kappa(x)$ . Hence,  $\text{Spec } k \rightarrow \text{Spec } \mathcal{O}_{X,x}$  factors through  $\text{Spec } \kappa(x) \rightarrow \text{Spec } \mathcal{O}_{X,x}$ .

$$\begin{array}{ccc}
\kappa(x) & \xleftarrow{\quad} & \mathcal{O}_{X,x} \\
\downarrow & \nearrow f_p^{\#} & \\
k & &
\end{array}
\quad
\begin{array}{ccc}
\text{Spec } \kappa(x) & \xrightarrow{\quad} & \text{Spec } \mathcal{O}_{X,x} \xrightarrow{\quad} X \\
\uparrow & \nearrow f & \\
\text{Spec } k & \xrightarrow{\quad} & X
\end{array}$$

Moreover,  $f : \text{Spec } k \rightarrow X$  factors through  $i_x : \kappa(x) \rightarrow X$  (this is very nice but the book didn't explain clearly why  $i_x$  has such a nice property - I guess what's outside a proof is informal)

### Proposition 2.3.1

There is a bijection of sets

$$\text{Hom}(\text{Spec } k, X) \rightarrow X \times \text{Hom}(\kappa(x), k)$$

## 2.4 MORPHISM INTO AFFINE POINT

(from Borcherds lecture on gluing schemes)

Let  $k$  be a field and  $R$  be a ring, the map  $\text{Spec } R \rightarrow \text{Spec } k$  that factors through  $\text{Spec } k[x]$  corresponds to an element of  $R$ , that is, a regular function on the global section of  $\text{Spec } R$

$$\begin{array}{ccc}
R & \xleftarrow{\quad} & k[x] \\
\swarrow & & \searrow \\
& k &
\end{array}$$

$$\begin{array}{ccc}
\text{Spec } R & \xrightarrow{\quad} & \text{Spec } k[x] \\
\searrow & & \swarrow \\
& \text{Spec } k &
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{\quad} & \text{Spec } k[x] \\
\searrow & & \swarrow \\
& \text{Spec } k &
\end{array}$$

Similar, any map from scheme  $X$  into  $\text{Spec } k$  that factors through  $\text{Spec } k[x]$  is a regular function on global section of  $X$  since  $X$  can be covered by affine open subschemes.

## 2.5 GLUING SCHEMES, DISJOINT UNION OF SCHEMES

### Definition 2.5.1 (gluing of schemes)

A gluing of schemes consists of the following data: given an index set  $I$

- for all  $i \in I$ , a scheme  $U_i$
- for all  $i, j \in I$ , an open subscheme  $U_{ij} \subseteq U_i$  such that  $U_{ii} = U_i$  for all  $i \in I$
- for all  $i, j \in I$ , an isomorphism  $\phi_{ji} : U_{ij} \rightarrow U_{ji}$  of schemes such that the cocycle condition holds

$$\phi_{kj}\phi_{ji} = \phi_{ki}$$

on  $U_{ij} \cap U_{ik}$  for all  $i, j, k \in I$  if  $\phi_{ji}(U_{ij} \cap U_{ik}) \subseteq U_{jk}$

For  $i = j = k$ , the cocycle condition implies that  $\phi_{ii} = \text{id}_{U_i}$  and for  $i = k$ ,  $\phi_{ij}^{-1} = \phi_{ji}$  and that  $\phi_{ji}$  is an isomorphism  $U_{ij} \cap U_{ik} \rightarrow U_{ji} \cap U_{jk}$

(it looks like we can make the definition a bit stronger by imposing cocycle condition on those overlapped region where it makes sense, for example  $\phi_{ji}(U_{ij} \cap U_{ik}) \cap U_{jk}$ , but somehow the authors didn't do so, I don't understand now, will do later)

### Proposition 2.5.2

Given a gluing of schemes, there exists a scheme  $X$  together with morphism  $\psi : U_i \rightarrow X$  such that

- for all  $i \in I$ , the map  $\psi_i$  is an isomorphism from  $U_i$  into an open subscheme of  $X$
- $\psi_j \phi_{ji} = \psi_i$  on  $U_{ij}$  for all  $i, j \in I$
- $X = \bigcup_{i \in I} \psi_i(U_i)$
- $\psi_i(U_i) \cap \psi_j(U_j) = \psi_i(U_{ij}) = \psi_j(U_{ji})$  for all  $i, j \in I$

Furthermore,  $(X, \psi_i)$  is unique up to unique isomorphism.

(there's a similar result in differential manifold using sheaf. it seems like sheaf is the categorical notion tool for gluing, just like morphism is the categorial notion of mapping, (co)limit is the categorical notion of sequential limit)

Gluing of morphisms implies the universal property of gluing schemes

### Remark 2.5.3 (universal property of gluing schemes)

If  $T$  is a scheme and for every  $i \in I$ , there is a map  $\xi_i : U_i \rightarrow T$  such that  $\xi_j \phi_{ji} = \xi_i$  on  $U_{ij}$  for all  $i, j \in I$ , then there exists a unique morphism  $\xi : X \rightarrow T$  with  $\xi \psi_i = \xi_i$  for every  $i \in I$

(the book requires each  $\xi_i$  is an isomorphism into its image but it's clearly not required)

## 2.6 SOME EXAMPLES OF GLUING SCHEMES

**example:** (gluing two schemes) When the index set  $I = \{1, 2\}$ , any two open subsets  $U_{12} \subseteq U_1$  and  $U_{21} \subseteq U_2$  and an isomorphism  $\phi : U_{12} \xrightarrow{\sim} U_{21}$  is a gluing of schemes. Let  $X$  be the glued scheme, for any open subset  $V \subseteq X$ , we have

$$\Gamma(V, \mathcal{O}_X) = \{(s1, s2) \in \Gamma(V \cap U_1, \mathcal{O}_{U_1}) \times \Gamma(V \cap U_2, \mathcal{O}_{U_2}) : \phi^*(s_2|_{V \cap U_{21}}) = s_1|_{V \cap U_{12}}\}$$

(this is just a pushout/colimit in schemes and pullback/limit in rings)

**example:** (affine line with double origin) Let  $k$  be a field, let  $U_1 = U_2 = \text{Spec } k[x]$ , let  $U_{12} = U_{21} = \text{Spec } k[x] - \{(x)\}$  and the gluing morphism  $U_{12} \rightarrow U_{21}$  the identity map. The global section is  $\Gamma(\mathcal{O}_X) = k[x]$  ([exercise 3.26 -> showing  \$X\$  is not affine](#))

**example:** (projective space) Let  $R$  be a ring. Define the projective space  $\mathbb{P}_R^n$  over  $R$  by gluing  $n+1$  copies of affine space  $\mathbb{A}_R^n$ , that is,  $U_i = \mathbb{A}_R^n$  for  $i = 0, \dots, n$ . For convenience, let

$$A^i = R[Y_0, \dots, \widehat{Y}_i, \dots, Y_n]$$

( $\widehat{Y}_i$  means  $Y_i$  is omitted) and we can view these rings as subring of  $R[X_0, \dots, X_N, X_0^{-1}, \dots, X_N^{-1}]$  by the inclusion  $Y_j \mapsto X_j X_i^{-1}$ , then  $U_i = \text{Spec } A^i$ . For each pair  $i \neq j$ , set  $U_{ij}$  to be the principal open set  $D(Y_j) = A^i[Y_j^{-1}]$ , let the gluing map  $\phi_{ji} : U_{ij} \rightarrow U_{ji}$  be induced by the equality  $A_{Y_j}^i \rightarrow A_{Y_i}^j$  in  $R[X_0, \dots, X_N, X_0^{-1}, \dots, X_N^{-1}]$  as follows:

$$\begin{array}{ccc} A^i[Y_j^{-1}] = R[Y_0, \dots, \widehat{Y}_I, \dots, Y_N, Y_j^{-1}] & & A^j[Y_i^{-1}] = R[Y_0, \dots, \widehat{Y}_I, \dots, Y_N, Y_i^{-1}] \\ \downarrow Y_j^{-1} \quad \searrow & & \swarrow \quad \downarrow Y_i^{-1} \\ X_j^{-1} X_i & R[X_0, \dots, X_N, X_0^{-1}, \dots, X_N^{-1}] & X_i^{-1} X_j \end{array}$$

This scheme is called projective space of relative dimension  $n$  over  $R$ . We consider the open subscheme  $U_i$  of  $\mathbb{P}_R^n$  and denoted them by  $D_+(X_i)$ . The canonical ring morphism  $R \rightarrow \Gamma(\mathbb{P}_R^n)$  is an ismorphism. Hence, projective space is not affine.

## 2.7 BASIC PROPERTIES - TOPOLOGICAL PROPERTIES

**Definition 2.7.1** (connected, quasi-compact, irreducible)

A scheme is called connected/quasi-compact/irreducible if the underlying topological space is connected/quasi-compact/irreducible.

**Definition 2.7.2** (injective, surjective, bijective)

A morphism of schemes  $f : X \rightarrow Y$  is called injective/surjective/bijective if the underlying set function is injective/surjective/bijective.

**Definition 2.7.3** (open, closed, homeomorphism)

A morphism of schemes  $f : X \rightarrow Y$  is called open/closed/homeomorphism if the underlying continuous map is open/closed/homeomorphism.

## 2.8 BASIC PROPERTIES - NOETHERIAN SCHEMES