Künneth Theorem

subtitle

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1 UNIVERSAL COEFFICIENT THEOREM

Theorem 1 (universal coefficient theorem)**.** *Let R be a PID, N be an R-module, $C_\bullet$ be chain complexes of R-module, and $C_\bullet$ is degree-wise free (each C_n is a free R-module). Then, there is a short exact sequence*

$$0 \longrightarrow H_n(C_\bullet) \otimes N \longrightarrow H_n(C_\bullet \otimes N) \longrightarrow \mathrm{Tor}_1(H_{n-1}(C_\bullet), N) \longrightarrow 0$$

and this sequence splits (but not naturally)

2 KÜNNETH THEOREM FOR CHAIN COMPLEXES OF R-MODULES

Definition 1 (direct sum of chain complexes of R-module)**.** *In the category of chain complexes of R-module ($\mathrm{Ch}(\text{R-Mod})$), let $C_\bullet, D_\bullet \in \mathrm{ob}\,\mathrm{Ch}(\text{R-Mod})$, define the direct sum $C_\bullet \oplus D_\bullet \in \mathrm{ob}\,\mathrm{Ch}(\text{R-Mod})$ as follows:*

$$(C_\bullet \oplus D_\bullet)_n = C_n \oplus D_n$$

and the boundary map $\partial : (C_\bullet \oplus D_\bullet)_n \to (C_\bullet \oplus D_\bullet)_{n-1}$ is defined by

$$\begin{aligned} \partial : (C \oplus D)_n &\to (C \oplus D)_{n-1} \\ c \oplus d &\mapsto \partial c \oplus \partial d \end{aligned}$$

Definition 2 (tensor product of chain complexes of R-module)**.** *In the category of chain complexes of R-module ($\mathrm{Ch}(\text{R-Mod})$), let $C_\bullet, D_\bullet \in \mathrm{ob}\,\mathrm{Ch}(\text{R-Mod})$, define the tensor product $C_\bullet \otimes D_\bullet \in \mathrm{ob}\,\mathrm{Ch}(\text{R-Mod})$ as follows:*

$$(C_\bullet \otimes D_\bullet)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and the boundary map $\partial : (C_\bullet \otimes D_\bullet)_n \to (C_\bullet \otimes D_\bullet)_{n-1}$ is the linear extension of $\partial : C_p \otimes D_q \to (C_\bullet \otimes D_\bullet)_{n-1}$ where

$$\partial(c \otimes d) = \partial c \otimes d + (-1)^{|c|} c \otimes \partial d$$

where $c \otimes d \in C_p \otimes D_q$ and $|c| = p$

Proof. *TODO - bilinear chain map factors through tensor product* □

Definition 3 (the Tor functor)**.** *TODO*

Theorem 2 (Künneth theorem for chain complexes)**.** *Let R be a PID, $C_\bullet, D_\bullet$ be chain complexes of R-module, and $C_\bullet$ is degree-wise free (each C_n is a free R-module). Then, there is a short exact sequence*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(D_\bullet) \xrightarrow{\times} H_n(C_\bullet \otimes D_\bullet) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(C_\bullet), H_q(D_\bullet)) \longrightarrow 0$$

and this sequence splits (but not naturally)

Proof. Consider the case where the boundary map in $C_\bullet$ is zero, that is, for all $c \in C_n$, $\partial c = 0$. Then,

$$\begin{aligned} \partial : C_p \otimes D_q &\to C_p \otimes D_{q-1} \\ c \otimes d &\mapsto (-1)^{|c|} c \otimes \partial d \end{aligned}$$

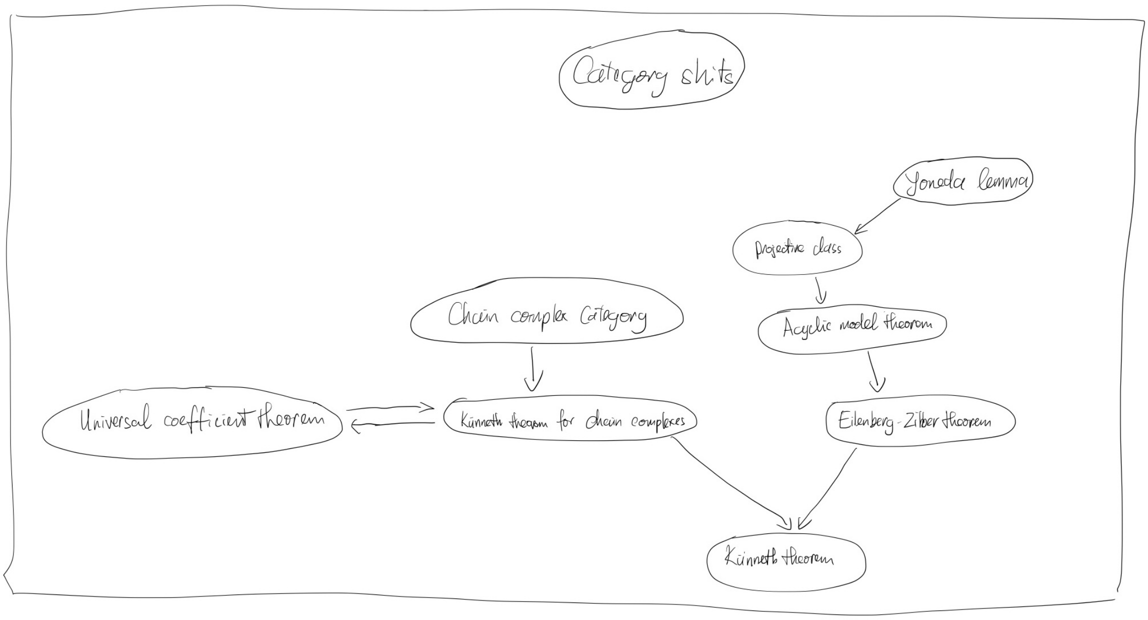

Figure 1: road map

Hence, $C_\bullet \otimes D_\bullet$ can be written as a direct sum of chain complexes $C_\bullet \otimes D_\bullet = \bigoplus_p C_p \otimes D_{\bullet - p}$. Therefore

$$\begin{aligned}
H_n(C_\bullet \otimes D_\bullet) &= H_n\left(\bigoplus_p C_p \otimes D_{\bullet-p}\right) \\
&= \bigoplus_p H_n(C_p \otimes D_{\bullet-p}) \\
&= \bigoplus_p C_p \otimes H_n(D_{\bullet-p}) && (C_p \text{ is free, cons of UCT}) \\
&= \bigoplus_{p+q=n} C_p \otimes H_q(D_\bullet) && (\text{shifted chain complex}) \\
&= \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(D_\bullet) && (C_p = H_p(C_\bullet))
\end{aligned}$$

Now let $C_\bullet$ be an arbitrary chain complex, we have the short exact sequence of chain complexes

$$0 \longrightarrow Z_\bullet \longrightarrow C_\bullet \longrightarrow B_{\bullet-1} \longrightarrow 0$$

where $Z_n = \ker(\partial : C_n \to C_{n-1})$ and $B_n = \operatorname{im}(\partial : C_{n+1} \to C_n)$ are n-cycle and n-boundary and consider $Z_\bullet, B_{\bullet-1}$ as chain complexes with zero boundary map. Each Z_n, B_n are free as they are submodules of free R-module C_n. As $B_{\bullet-1}$ is free, the sequence splits, hence, the sequence below is exact

$$0 \longrightarrow Z_\bullet \otimes D_\bullet \longrightarrow C_\bullet \otimes D_\bullet \longrightarrow B_{\bullet-1} \otimes D_\bullet \longrightarrow 0$$

That induces a long exact sequence in homology

$$\begin{tikzcd}
 & \dots \arrow[r] & H_{n+1}(B_{\bullet-1} \otimes D_\bullet) \arrow[dll, "(i_n)_*"'] \\
H_n(Z_\bullet \otimes D_\bullet) \arrow[r] & H_n(C_\bullet \otimes D_\bullet) \arrow[r] & H_n(B_{\bullet-1} \otimes D_\bullet) \arrow[dll, "(i_{n-1})_*"'] \\
H_{n-1}(Z_\bullet \otimes D_\bullet) \arrow[r] & \dots &
\end{tikzcd}$$

where the connecting homomorphisms $(i_n)_*, (i_{n-1})_*$ are induced by inclusion maps

$$\begin{tikzcd}
 & & (B_{\bullet-1} \otimes D_\bullet)_{n+1} \arrow[dll, "i_n"'] \arrow[dl, hook] \\
(Z_\bullet \otimes D_\bullet)_n & (C_\bullet \otimes D_\bullet)_n \arrow[l, hook'] &
\end{tikzcd}$$

From the long exact sequence, we have the short exact sequence

$$0 \longrightarrow \operatorname{coker}(i_n)_* \longrightarrow H_n(C_\bullet \otimes D_\bullet) \longrightarrow \ker(i_{n-1})_* \longrightarrow 0$$

Discussed in the previous argument, as $Z_\bullet$ and $B_{\bullet-1}$ are free,

$$H_n(Z_\bullet \otimes D_\bullet) = \bigoplus_{p+q=n} Z_p \otimes H_q(D_\bullet)$$
$$H_{n+1}(B_{\bullet-1} \otimes D_\bullet) = \bigoplus_{p+q=n} B_p \otimes H_q(D_\bullet)$$

Since tensor product is right-exact, exactness of the top sequence implies exactness of the bottom sequence

$$\begin{tikzcd}
0 \arrow[r] & B_p \arrow[r, hook, "j"] & Z_p \arrow[r, two heads] & H_p(C_\bullet) \arrow[r] & 0 \\
& B_p \otimes H_q(D_\bullet) \arrow[r, hook, "(i_*)_{p+q}=j\otimes 1"] & Z_p \otimes H_q(D_\bullet) \arrow[r, two heads] & H_p(C_\bullet) \otimes H_q(D_\bullet) \arrow[r] & 0
\end{tikzcd}$$

Hence, $\operatorname{coker}(i_*)_n = \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_p(D_\bullet)$. On the other hand, the top sequence is the free resolution of $H_p(C_\bullet)$. Then, $\operatorname{Tor}_1^R(H_p(C_\bullet), H_q(D_\bullet))$ is the first homology group of the bottom sequence

$$\begin{tikzcd}
2 & & 1 & & 0 & \\
0 \arrow[r] & B_p \arrow[r, hook, "j"] & & Z_p \arrow[r] & & 0 \\
0 \arrow[r] & B_p \otimes H_q(D_\bullet) \arrow[r, hook, "(i_*)_{p+q}=j\otimes 1"] & & Z_p \otimes H_q(D_\bullet) \arrow[r] & & 0
\end{tikzcd}$$

That is, $\ker(i_*)_{n-1} = \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R(H_p(C_\bullet), H_q(D_\bullet))$
TODO - here, there are to functors, one is $\otimes D$ composed with H and the other is $\otimes H(D)$ - need to prove
TODO - split □

3 KÜNNETH THEOREM FOR TOPOLOGICAL SPACES

3.1 FUNDAMENTAL THEOREM OF HOMOLOGICAL ALGEBRA

Definition 4 (initial object, terminal object, pointed category, zero map, kernel)**.** *Given a category C, an object 0 is initial if for all $X \in \operatorname{ob} C$, there is only one map in $\operatorname{Hom}(0, X)$, an object $*$ is terminal if for all $X \in \operatorname{ob} C$, there is only one map in $\operatorname{Hom}(X, *)$. Category C is called pointed if it has initial and terminal objects and the unique map $0 \to *$ is an isomorphism.*

If C is a pointed category, we use the same symbol 0 for both initial object and terminal object. There exists a zero map between any two objects $M, N \in \operatorname{ob} C$, defined by

$$\begin{tikzcd}
M \arrow[r] \arrow[dr, "0"'] & 0 \arrow[d] \\
& N
\end{tikzcd}$$

the composition of $M \to 0$ and $0 \to N$. Let $f : M \to N$ be a morphism in C, a kernel of f is a map $i : K \to M$ such that $fi = 0$ and such map is universal, that is, if $j : L \to M$ with $fj = 0$, then it factors through K

$$\begin{tikzcd}
K \arrow[r, "i"] \arrow[rr, bend left, "0"] & M \arrow[r, "f"] & N \\
L \arrow[u, dashed] \arrow[ur, "j"'] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

Category C has kernels if every morphism has a kernel.

Definition 5 (preadditive category, Ab-enriched category). *A category C is called preadditive category (or* Ab*-enriched category) if for any two objects $M, N \in \operatorname{ob} C$, $\operatorname{Hom}(M,N)$ is an abelian group and composition is bilinear, that is, if f, g, h are morphisms in C*

$$f(g+h) = fg + fh$$
$$(f+g)h = fh + gh$$

Definition 6 (chain complex, acyclic chain complex, exact sequence). *In a pointed category with kernels, a chain complex is a sequence such that given any subsequence $A \to B \to C$, $A \to B$ factors through $\ker(B \to C)$, that is, there exists a map $A \to \ker(B \to C)$ such that the diagram below commutes*

$$\begin{tikzcd}
... \arrow[r] & A \arrow[r] \arrow[d, dashed] & B \arrow[r] & C \arrow[r] & ... \\
& \ker(B \to C) \arrow[ur] & & &
\end{tikzcd}$$

If there is a notion of epimorphism and the map $A \to \ker(B \to C)$ is an epimorphism, then the sequence is called exact at B. A sequence is called exact sequence or an acyclic chain complex if it is exact everywhere, possibly except the two ends.

Definition 7 (chain map, chain homotopy). *Given two chain complexes $C_\bullet, D_\bullet$ in a pointed category with kernels, for each $n \in \mathbb{Z}$, there is a map $f_n : C_n \to D_n$ such that the diagram below commutes, then $f_\bullet$ is called a chain map*

$$\begin{tikzcd}
... & C_{n-1} \arrow[l] \arrow[d, "f_{n-1}"] & C_n \arrow[l] \arrow[d, "f_n"] & C_{n+1} \arrow[l] \arrow[d, "f_{n+1}"] & ... \arrow[l] \\
... & D_{n-1} \arrow[l] & D_n \arrow[l] & D_{n+1} \arrow[l] & ... \arrow[l]
\end{tikzcd}$$

Chain complexes and chain maps form a category and it is called the category of chain complexes.

Given two chain complexes $C_\bullet, D_\bullet$ in a pointed ***preadditive*** *category with kernels. Let $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ be two chain maps. A chain homotopy from $f_\bullet$ to $g_\bullet$ is a collection of maps $h_n : C_{n-1} \to D_n$ such that $\partial h_{n+1} + h_n \partial = f_n - g_n$*

$$\begin{tikzcd}
... & C_{n-1} \arrow[l, "\partial"'] \arrow[dr, "h_n"] & C_n \arrow[l, "\partial"'] \arrow[dr, "h_{n+1}"] & C_{n+1} \arrow[l, "\partial"'] & ... \arrow[l, "\partial"'] \\
... & D_{n-1} \arrow[l, "\partial"] & D_n \arrow[l, "\partial"] & D_{n+1} \arrow[l, "\partial"] & ... \arrow[l, "\partial"]
\end{tikzcd}$$

Definition 8 (projective class). *Let C be a pointed category with kernels. A projective class in C is a pair $(\mathcal{P}, \mathcal{E})$ where $\mathcal{P}$ is a collection of objects (called* ***projectives****) and $\mathcal{E}$ is a collection of morphisms (called* ***epimorphisms****) such that*

1. *An object P is* ***projective*** *if and only if P has the universal lifting property against every* ***epimorphism*** *$M \to N$, that is, given any* ***epimorphism*** *$M \to N$, if there is a map $P \to N$, then it factors through M*

$$\begin{tikzcd}
M \arrow[r, two heads, "epi"] & N \\
& P \arrow[u] \arrow[ul, dashed]
\end{tikzcd}$$

2. *A morphism $f : M \to N$ is an* ***epimorphism*** *if and only if every* ***projective*** *has the universal lifting property against f, that is, given any* ***projective*** *P, if there is a map $P \to N$, then it factors through M*

$$\begin{tikzcd}
M \arrow[r, two heads, "f"] & N \\
& P \arrow[u] \arrow[ul, dashed]
\end{tikzcd}$$

3. *C has enough* ***projectives****, that is, given any object $M \in \operatorname{ob} C$, for every* ***projective*** *P, there exists an* ***epimorphism*** *$P \to M$.*

Theorem 3 (fundamental theorem of homological algebra). *Let C be a pointed category with kernels and $(\mathcal{P}, \mathcal{C})$ be a projective class in C. Given $f : M \to M'$ in C and the diagram below*

$$\begin{tikzcd}
0 & M \arrow[l] \arrow[d, "f"] & P_0 \arrow[l, "\epsilon"'] \arrow[d, dashed, "f_0"] & P_1 \arrow[l, "d"'] \arrow[d, dashed, "f_1"] & ... \arrow[l, "d"'] \\
0 & M' \arrow[l] & P'_0 \arrow[l, "\epsilon'"'] & P'_1 \arrow[l, "d'"'] & ... \arrow[l, "d'"']
\end{tikzcd}$$

where both chains are chain complexes, the top chain consists of projectives P_n and the bottom chain is acyclic. Then,

- *There exists a chain map defined by $f_n : P_n \to P'_n$*
- *If C is preadditive, the lift is unique upto chain homotopy.*

Proof.

1. The first statement is proved by induction

$$\begin{tikzcd}
P_{n-2} \arrow[d, "f_{n-2}"'] & P_{n-1} \arrow[l] \arrow[d, "f_{n-1}"'] & & P_n \arrow[ll] \arrow[dl, dashed] \arrow[dd, dashed, "f_n"] \\
& & K'_{n-1} \arrow[dl] & \\
P'_{n-2} & P'_{n-1} \arrow[l] & & P'_n \arrow[ll] \arrow[ul, two heads]
\end{tikzcd}$$

Suppose there exist maps $f_{n-1} : P_{n-1} \to P'_{n-1}$ and $f_{n-2} : P_{n-2} \to P'_{n-2}$. Let $K'_{n-1} = \ker(P'_{n-1} \to P'_{n-2})$.
Since the bottom chain is acyclic, the map $P'_n \to P'_{n-1}$ factors through K'_{n-1} by an epimorphism.
Since the top chain is a chain complex, the composition $P_n \to P_{n-1} \to P'_{n-1} \to P'_{n-2}$ equals $P_n \to P_{n-1} \to P_{n-2} \to P'_{n-2}$ and equals 0 zero, so $P_n \to P_{n-1} \to P'_{n-1}$ factors through K'_{n-1}
Since P_n is projective and $P'_n \to K'_{n-1}$ is an epimorphism, $P_n \to K'_{n-1}$ factors through P'_n by a map $f_n : P_n \to P'_n$
Base case: $n = 0$, let $P_{n-1} = M, P'_{n-1} = M'$, $P_{n-2} = 0, P'_{n-2} = 0$ and $f_{n-1} = f, f_{n-2} = 0$

2. Let $f^{(1)}_\bullet, f^{(2)}_\bullet : P_\bullet \to P'_\bullet$ be any two lifts from $f : M \to M'$

$$\begin{tikzcd}
M \arrow[d, "f"'] & P_\bullet \arrow[l, "\epsilon"'] \arrow[d, dashed, bend right, "f^{(1)}_\bullet"'] \arrow[d, dashed, bend left, "f^{(2)}_\bullet"] \\
M' & P'_\bullet \arrow[l, "\epsilon'"]
\end{tikzcd}$$

We will prove that $g_\bullet = f^{(1)}_\bullet - f^{(2)}_\bullet$ is chain homotopic to zero, that is to find maps $h_{n+1} : P_n \to P'_{n+1}$ such that $d'h + hd = g$

$$\begin{tikzcd}
0 \arrow[d, "0"'] & P_0 \arrow[l, "d"'] \arrow[d, "g_0"'] & P_1 \arrow[l, "d"'] \arrow[d, "g_1"'] & ... \arrow[l, "d"'] \\
0 & P'_0 \arrow[l, "d'"] & P'_1 \arrow[l, "d'"] & ... \arrow[l, "d'"]
\end{tikzcd}$$

Suppose there exists map $h_{n-1} : P_{n-2} \to P'_{n-1}$ and $h_{n-2} : P_{n-3} \to P'_{n-2}$ such that

$$g_{n-2} - h_{n-2}d = d'h_{n-1}$$

$$\begin{tikzcd}
P_{n-3} \arrow[dr, "h_{n-2}"] & P_{n-2} \arrow[l, "d"'] \arrow[dr, "h_{n-1}"] & P_{n-1} \arrow[l, "d"'] \\
P'_{n-3} & P'_{n-2} \arrow[l, "d'"] & P'_{n-1} \arrow[l, "d'"]
\end{tikzcd}$$

Consider the map $g_{n-1} - h_{n-1}d : P_{n-1} \to P'_{n-1}$,

$$\begin{aligned}
d'(g_{n-1} - h_{n-1}d) &= d'g_{n-1} - d'h_{n-1}d && \text{(preadditive)} \\
&= d'g_{n-1} - (g_{n-2} - h_{n-2}d)d && \text{(induction)} \\
&= d'g_{n-1} - g_{n-2}d && \text{(preadditive, } dd = 0\text{)} \\
&= 0 && (g_\bullet \text{ is a chain map})
\end{aligned}$$

Let $K'_{n-1} = \ker(d' : P'_{n-1} \to P'_{n-2})$.
Since the bottom chain is acyclic, the map $d' : P'_n \to P'_{n-1}$ factors through K'_{n-1} by an epimorphism.

$$\begin{tikzcd}
P_{n-2} \arrow[ddr, "h_{n-1}"] & P_{n-1} \arrow[l, "d"'] \arrow[d, dashed] \arrow[ddr, dashed, "h_n"] & \\
& K'_{n-1} \arrow[d] & \\
P'_{n-2} & P'_{n-1} \arrow[l, "d'"] & P'_n \arrow[l, "d'"] \arrow[ul, dashed, two heads]
\end{tikzcd}$$

As $d'(g_{n-1} - h_{n-1}d) = 0$, $g_{n-1} - h_{n-1}d$ factors through K'_{n-1}, that is, $g_{n-1} - h_{n-1}d$ equals the composition $P_{n-1} \to K'_{n-1} \to P'_{n-1}$

Since P_{n-1} is projective and $P'_n \to K'_{n-1}$ is an epimorphism, $P_{n-1} \to K'_{n-1}$ factors through P'_n by a map $h_n : P_{n-1} \to P'_n$, that is, the $d'h_n$ equals the composition $P_{n-1} \to P'_n \to K'_{n-1} \to P'_{n-1}$ and equals the composition $P_{n-1} \to K'_{n-1} \to P'_{n-1}$, hence

$$d'h_n = g_{n-1} - h_{n-1}d$$

Base case: $n = 0$, let $P_{n-2} = 0, P'_{n-2} = 0$, $P_{n-1} = M, P'_{n-1} = M'$, $h_{n-1} = 0$, then

$$d'(g_{n-1} - h_{n-1}d) = 0 \qquad\qquad (d' : P'_{n-1} \to P'_{n-2} \text{ is the zero map } M' \to 0)$$

□

3.2 RESOLUTION AND TOR FUNCTOR

Definition 9 (resolution, projective resolution)**.** *Let M be an object in a pointed category with kernels. A resolution of M is an exact sequence*

$$0 \longleftarrow M \xleftarrow{\epsilon} P_0 \xleftarrow{d} P_1 \xleftarrow{d} \ldots$$

If P_n are projectives in a projective class $(\mathcal{P}, \mathcal{E})$, then the sequence is called $\mathcal{P}$-projective resolution.

Corollary 1. *Let M be an object in a pointed preadditive category with kernels. Any two projective resolutions of M are chain homotopy equivalent* [1] *or equivalently M defines a chain homotopy type.*

Definition 10 (Tor functor on R-Mod)**.** *TODO*

3.3 EILENBERG-ZILBER THEOREM

Proposition 1 (projective module, projective class in R-Mod)**.** *In the category of R-module (*R-Mod*), there is a projective class $(\mathcal{P}, \mathcal{E})$ defined by epimorphism being surjective homomorphism. Then, the following are equivalent*

1. *$P \in \operatorname{ob} \text{R-Mod}$ is projective*
2. *Every short exact sequence $0 \to M \to N \to P \to 0$ splits*
3. *P is a direct summand of a free R-module, that is, there exists $Q \in \operatorname{ob} \text{R-Mod}$ such that $P \oplus Q$ is a free R-module.*

Proof. *TODO* □

Proposition 2 (models define projective class in $\text{Fun}(C, \text{R-Mod})$)**.** *Given a category C, $\text{Fun}(C, \text{R-Mod})$ is a pointed preadditive category with kernels (more precisely, abelian category - will define in the future).*

Let $\mathcal{M}$ be any set of objects in C (called models), then $\mathcal{M}$ defines a projective class $(\mathcal{P}, \mathcal{E})$ in $\text{Fun}(C, \text{R-Mod})$ where a morphism $G \to F$ is an epimorphism (relative to $\mathcal{M}$) if for all $M \in \mathcal{M}$, $G(M) \twoheadrightarrow F(M)$ is surjective. Then, the following are equivalent

1. *$P \in \operatorname{ob} \text{Fun}(C, \text{R-Mod})$ is projective*
2. *P is a **retract of coproduct** of $R \operatorname{Hom}(M, -)$ for some $M \in \mathcal{M}$ where $\operatorname{Hom}(M, -)$ is a functor $\text{R-Mod} \to \text{Set}$, R is the free R-module functor $\text{Set} \to \text{R-Mod}$. In the case of R-module, **retract of coproduct** is the **direct summand** of a R-module*

Proof. *TODO - prove using Yoneda lemma* □

Theorem 4 (Eilenberg-Zilber theorem)**.** *TODO*

[1] two chain complexes $C_\bullet, D_\bullet$ are chain homotopy equivalent if there are two chain maps $f_\bullet : C_\bullet \to D_\bullet$, $g_\bullet : D_\bullet \to C_\bullet$ such that gf and fg are chain homotopic to identity