Scheme Theory

this is my note for scheme theory. this is loosely based on (1) Borcherds online lecture (2) Vakil FOAG (3) Görtz - Wedhorn AG 1

Nguyen Ngoc Khanh

December 22, 2025

Contents

Chapter 1

SCHEME

1.1 AFFINE SCHEME

1.1.1 DEFINITION OF AFFINE SCHEME

Definition 1.1.1 (ringed space, locally ringed space)
A ringed space $(X, \mathscr{O})$ is a topological space X together with a sheaf of rings $\mathscr{O}$. A ringed space is a locally ringed space if for every point $x \in X$, the stalk $\mathscr{O}_x$ is a local ring.

Definition 1.1.2 (affine scheme)
An affine scheme is a locally ringed space that is isomorphic to the spectrum of some ring A

1.1.2 CONSTRUCT AFFINE SCHEME FROM A RING

Let A be a ring. Define $X = \operatorname{Spec} A$ by the set of prime ideals in A

$$X = \operatorname{Spec} A = \{\text{prime } \mathfrak{p} \subseteq A\}$$

We put a topology on $\operatorname{Spec} A$ generated by the basis of open sets

$$D_f = \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}\}$$

for every $f \in A$. The constructed topology is called Zariski's topology. We put a sheaf of rings $\mathscr{O}$ on $\operatorname{Spec} A$ generated by

$$\mathscr{O}(D_f) = A_f$$

If $D_g \subseteq D_f$, that is $g \in \sqrt{(f)}$, $g^n = fh$ for some $h \in A$ and $n \geq 1$. Since f is a unit in A_g, the restriction map is well-defined and unique

$$\begin{aligned} A_f &\to A_g \\ \frac{x}{f^m} &\mapsto x\left(\frac{h}{g^n}\right)^m \end{aligned}$$

An element $f \in A$ is called **function**, a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ is called **point**, and function evaluation is equivalent to sending f to the residue field of stalk $\mathscr{O}_\mathfrak{p} = A_\mathfrak{p}$

1.1.3 SOME EXAMPLES OF AFFINE SCHEME

Let $A = \mathbb{Z}$, then

$$X = \operatorname{Spec} \mathbb{Z} = \{(2), (3), (5), ..., (0)\}$$

for each prime number $p \in \mathbb{Z}$, (p) is a closed point. (0) is a generic point, and closure of (0) is the whole space X.
Let k be an algebraically closed field. Let $A = k$, then

$$X = \operatorname{Spec} k = \{(0)\}$$

is a singleton set.
Let A be an arbitrary ring, maximal ideals are closed points and other non-maximal primes are generic points. Moreover, the closure of a point $\mathfrak{p}$ is the set of prime ideals containing $\mathfrak{p}$

$$V(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} A : \mathfrak{p} \subseteq \mathfrak{q}\}$$

In particular, let $A = k[x, y]$, by Nullstellensatz, the set of closed points are

$$\{(x - a, y - b) : a \in k, b \in k\}$$

X admits other generic points (0) and (f) for every irreducible $f \in k[x, y]$. By dimensionality argument, the prime (f) is of height 1 and the closure of (f) consists of (f) and $(x - a, y - b)$ for $(a, b) \in k^2$ in the vanishing set of f. Similarly, closure of (0) is the whole space.

1.1.4 QUOTIENT AND LOCALIZATION

Let A be a ring and $\mathfrak{p} \in \operatorname{Spec} A$
Recall the map $A \twoheadrightarrow A/\mathfrak{p}$, it induces an injective map

$$\operatorname{Spec} A/\mathfrak{p} \hookrightarrow \operatorname{Spec} A$$

Informally, quotient by $\mathfrak{p}$ is the action of taking closed subscheme

> *keep all (geometrically inside points = algebraically outside primes containing $\mathfrak{p}$)*

Recall the map $A \to A_\mathfrak{p}$, it induces an injective map

$$\operatorname{Spec} A_\mathfrak{p} \hookrightarrow \operatorname{Spec} A$$

Informally, localizing at $\mathfrak{p}$ is the action of taking quotient

> *keep all (geometrically outside points = algebraically inside primes contained in $\mathfrak{p}$)*

1.1.5 MORPHISM OF AFFINE SCHEMES

Definition 1.1.3 (inverse image, direct image)
Let $\pi : X \to Y$ be a continuous map. The inverse image functor π^{-1} and direct image functor π_* is an adjoint pair between the category of sheaves on X and the category of sheaves on Y. Let $\mathscr{F}$ and $\mathscr{G}$ be a sheaf on X and a sheaf on Y respectively, then

$$\hom_{\mathrm{Sh}(X)}(\pi^{-1}\mathscr{G}, \mathscr{F}) \cong \hom_{\mathrm{Sh}(Y)}(\mathscr{G}, \pi_*\mathscr{F})$$

The direct image functor π_* is defined as follows: for every open subset $V \subseteq Y$, then

$$(\pi_*\mathscr{F})(V) = \mathscr{F}(\pi^{-1}(U))$$

The inverse image functor π^{-1} is defined as follows: for every open subset $U \subseteq X$, then

$$(\pi^{-1}\mathscr{G})(U) = \operatorname{colim}_{V \subseteq Y : \pi(U) \subseteq V} \mathscr{G}(V)$$

Definition 1.1.4 (morphism of ringed spaces)
A morphism of ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is defined by a continuous map $\pi : X \to Y$ and a morphism of sheaves of rings $\pi^\flat : \mathscr{O}_Y \to \pi_*\mathscr{O}_X$. By adjunction between π^{-1} and π_*, this is equivalent to a morphism of sheaves of rings $\pi^\# : \pi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$

Definition 1.1.5 (morphism of locally ringed spaces, morphism of affine schemes)
A morphism of ringed spaces $(\pi, \pi^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of locally ringed spaces if for every $x \in X$, the induced map on stalks

$$\pi^\#_x : (\pi^{-1}O_Y)_x = \mathscr{O}_{Y,\pi(x)} \to \mathscr{O}_{X,x}$$

is a local homomorphism. A morphism of affine schemes is a morphism of locally ringed spaces.

The local homomorphism condition of the map between stalks ensures that any zero function $g \in \mathscr{O}_{Y,\pi(x)}$ at $\pi(x) \in Y$ will be sent to a zero function $\pi^\#_x(g) \in \mathscr{O}_{X,x}$ at $x \in X$

Proposition 1.1.6 (equivalence between commutative rings and affine schemes)
The functor Spec from the opposite category of commutative rings into the category of affine schemes is fully faithful and essentially surjective.

$$\mathrm{Spec} : \mathrm{CRing}^{\mathrm{op}} \xrightarrow{\sim} \mathrm{AffSch}$$

1.1.6 SOME EXAMPLES OF MORPHISM OF AFFINE SCHEMES

Consider the ring map $\phi : k[x] \to k[x, y]$ defined by $x \mapsto x$. It induces a morphism of affine schemes

$$\pi : \mathrm{Spec}\, k[x, y] \to \mathrm{Spec}\, k[x]$$

Let $t = (x - a, y - b) \in \mathrm{Spec}\, k[x, y]$ be a prime. Preimage of a prime under a ring map is a prime,

$$\pi(t) = \phi^{-1}(t) = (x - a)$$

The map between stalks at t is

$$\begin{gathered}\pi_t^{\#} : \mathscr{O}_{\operatorname{Spec} k[x], \pi(t)} \to \mathscr{O}_{\operatorname{Spec} k[x,y], t} \\ \pi_t^{\#} : k[x]_{(x-a)} \to k[x,y]_{(x-a, y-b)} \\ \frac{f}{g} \mapsto \frac{f}{g}\end{gathered}$$

Similarly, if $t = (f)$ for some irreducible polynomial f, then

$$\pi(t) = \phi^{-1}(t) = (0)$$

When

TODO

Consider the ring map $\phi : k[u] \to k[x, y]$ defined by $u \to x + y$