differentiable_manifolds

this is my notes for ma5210 differential manifolds
taught by National University of Singapore Professor Loke Hung Yean in AY2024-2025

Khanh Nguyen

August 2024

Contents

Chapter 0

Preliminaries

0.1 Natural Number

Definition 1. *Denote* $[n] = \{1, 2, ..., n\}$

0.2 Analysis

Theorem 1 (Taylor theorem)**.** *A function* $f : \mathbb{R}^n \to \mathbb{R}$ *be differentiable at* $a \in \mathbb{R}^n$ *if and only if there is a linear functional* $L : \mathbb{R}^n \to \mathbb{R}$ *and a function* $h : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$f(x) = f(a) + L(x-a) + h(x)\|x-a\|$$

where $h(x) \to 0$ *as* $x \to 0$*. If this is the case, then* $L : \mathbb{R}^n \to \mathbb{R}$ *is unique and its Riesz representation* $L^* \in \mathbb{R}^n$ *is*

$$L^* = \left(\frac{\partial f}{\partial x_1}\bigg|_x, \frac{\partial f}{\partial x_2}\bigg|_x, ..., \frac{\partial f}{\partial x_n}\bigg|_x \right)$$

Definition 2 (smooth function)**.** *Let* D *be an open set in* $\mathbb{R}^n$*, a function* $f : D \to \mathbb{R}$ *is called smooth if it has partial derivatives of all orders and they are continuous at every point in* D*. The collection of all smooth functions is denoted by* $\mathcal{E}$

Definition 3 (real analytic)**.** *Let* D *be an open set in* $\mathbb{R}^n$*, a function* $f : D \to \mathbb{R}$ *is called real analytic if for all* $x_0 \in D$*, there exists a neighbourhood* $U \subseteq D$ *such that* f *can be written by a convergence power series*

$$f(x) = \sum_{m=0}^{\infty} P_m(x - x_0)$$

for all $x \in U$ *where* P_m *is a homogeneous polynomial* [1] *of degree* m *on* n *variables over* $\mathbb{R}$*. The collection of all real analytic functions is denoted by* $\mathcal{A}$*.*

Remark 1. *real analytic implies smoothness*

Definition 4 (complex holomorphic)**.** *Let* D *be an open set in* $\mathbb{C}^n$*, a function* $f : D \to \mathbb{C}$ *is called complex holomorphic if for all* $x_0 \in D$*, there exists a neighbourhood* $U \subseteq D$ *such that* f *can be written by a convergence power series*

$$f(x) = \sum_{m=0}^{\infty} P_m(x - x_0)$$

for all $x \in U$ *where* P_m *is a homogeneous polynomial of degree* m *on* n *variables over* $\mathbb{C}$*. The collection of all complex holomorphic functions is denoted by* $\mathcal{O}$*.*

Theorem 2 (inverse function theorem for $\mathcal{S}$-function)**.** *Let* D *be an open set in* K^n *for* K *being either* $\mathbb{R}$ *or* $\mathbb{C}$ *and* $F : D \to K^n$ *be an* S*-function where* $\mathcal{S}$ *is either* $\mathcal{E}$*,* $\mathcal{A}$*, or* $\mathcal{O}$*. Suppose* $p \in D$ *is regular point, that is the Jacobian* dF *at* p *is invertible*

$$dF = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & ... & \frac{\partial F_1}{\partial x_n} \\ ... & ... & ... \\ \frac{\partial F_n}{\partial x_1} & ... & \frac{\partial F_n}{\partial x_n} \end{bmatrix}$$

Then, there exists open neighbourhoods $U \subseteq D$ *of* p *and* $W \subseteq \mathbb{R}^n$ *of* $F(p)$ *such that* $F : U \to W$ *is a bijection and* $F^{-1} : W \to U$ *is an* $\mathcal{S}$*-function.*

[1] $a_{20}x^2 + a_{11}xy + a_{02}y^2$ is an example of homogeneous polynomial of degree 2 over 2 variables

Theorem 3 (maximum modulus principle)**.** *Let $f : D \to \mathbb{C}$ be a holomorphic function on some connected open set $D \subseteq \mathbb{C}$. If z_0 has an open neighbourhood $N \subseteq D$ such that*

$$|f(z_0)| \geq |f(z)|$$

for all $z \in N$ then f is a constant on D

Proposition 1 (chain rule)**.** *$f : \mathbb{R}^2 \to \mathbb{R}$, $x : \mathbb{R} \to \mathbb{R}, y : \mathbb{R} \to \mathbb{R}$, define $g : \mathbb{R} \to \mathbb{R}$ by*

$$g(t) = f(x(t), y(t))$$

Then,

$$\left.\frac{dg}{dt}\right|_{t_1} = \left(\left.\frac{\partial f}{\partial x}\right|_{x(t_1)} \left.\frac{dx}{dt}\right|_{t_1}\right) + \left(\left.\frac{\partial f}{\partial y}\right|_{y(t_1)} \left.\frac{dy}{dt}\right|_{t_1}\right)$$

Remark 2 (Cauchy-Riemann equation and complex derivative)**.** *This remark motivate the definition of complex holomorphic function. Let $f : \mathbb{C} \to \mathbb{C}$ be a function, write $f(z) = u(x, y) + iv(x, y)$ for $z \in \mathbb{C}$, $z = x + iy$, $x, y \in \mathbb{R}$, $u, v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we want to make the derivative $\frac{df}{dz}$ well-defined, that is,*

$$\left.\frac{df}{dz}\right|_{z_0} := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{(u(x, y) - u(x_0, y_0)) + i(v(x, y) - v(x_0, y_0))}{(x - x_0) + i(y - y_0)}$$

exists. Set $y = y_0$, then

$$\begin{aligned}
\left.\frac{df}{dz}\right|_{z_0} &= \lim_{x \to x_0} \frac{(u(x, y_0) - u(x_0, y_0)) + i(v(x, y_0) - v(x_0, y_0))}{x - x_0} \\
&= \left.\frac{\partial u}{\partial x}\right|_{z_0} + i\left.\frac{\partial v}{\partial x}\right|_{z_0}
\end{aligned}$$

Set $x = x_0$, then

$$\begin{aligned}
\left.\frac{df}{dz}\right|_{z_0} &= \lim_{y \to y_0} \frac{(u(x_0, y) - u(x_0, y_0)) + i(v(x_0, y) - v(x_0, y_0))}{i(y - y_0)} \\
&= -i\left.\frac{\partial u}{\partial y}\right|_{z_0} + \left.\frac{\partial v}{\partial y}\right|_{z_0}
\end{aligned}$$

We recover the Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

where all the partial derivatives exist. Use Taylor theorem we have

$$\begin{aligned}
u(x, y) &= u(x_0, y_0) + \left.\frac{\partial u}{\partial x}\right|_{x_0, y_0} (x - x_0) + \left.\frac{\partial u}{\partial y}\right|_{x_0, y_0} (y - y_0) + o(z - z_0) \\
v(x, y) &= v(x_0, y_0) + \left.\frac{\partial v}{\partial x}\right|_{x_0, y_0} (x - x_0) + \left.\frac{\partial v}{\partial y}\right|_{x_0, y_0} (y - y_0) + o(z - z_0)
\end{aligned}$$

where $\frac{o(z-z_0)}{\|z-z_0\|} \to 0$ as $z \to z_0$. Plug into the defintion of $\frac{df}{dz}$, we conclude that Cauchy-Riemann equation is both necessary and sufficient for $\frac{df}{dz}$ to be well-defined. And $\frac{df}{dz}$ is also written as

$$\left.\frac{df}{dz}\right|_{z_0} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

0.3 Linear Algebra

0.3.1 Tensor Product, Wedge Product

Definition 5 (bilinear map)**.** *Let V, W, Z be vector spaces over field K. A function $f : V \times W \to Z$ is K-bilinear if for every $w \in W$, the induced function $f_w : V \to Z$ defined by $f_w(v) = f(v, w)$ is linear and for every $v \in V$, the induced function $f_v : W \to Z$ defined by $f_v(w) = f(v, w)$ is linear. The collection of bilinear maps $V \to W$ is denoted by* $\mathrm{Bil}(V, W)$.

Proposition 2 (tensor product)**.** *Let V, W be vector spaces over field K, then there exists a vector space $V \otimes W$ over K and a bilinear map $g : V \times W \to V \otimes W$ such that for every vector space Z over K, there is a bijection*

$$F : \mathrm{Bil}(V \times W, Z) \to \mathrm{Hom}(V \otimes W, Z)$$

that makes the diagram below commutes

$$\begin{tikzcd} V \times W \arrow[r, "f"] \arrow[d, "g"'] & Z \\ V \otimes W \arrow[ur, "f_* = F(f)"'] & \end{tikzcd}$$

$V \otimes W$ is called tensor product and it is unique up to isomorphism.

Remark 3 (basis of tensor product)**.** *Let $\{v_1, ..., v_n\}$ be a basis of V and $\{w_1, ..., w_m\}$ be a basis of W, then*

$$\{v_i \otimes w_j : (i, j) \in [n] \times [m]\}$$

is a basis of $V \otimes W$ and

$$\dim(V \otimes W) = \dim V \times \dim W$$

Definition 6 (skew-symmetric)**.** *Let V, Z be vector spaces over field K. A function $f : V \times V \to Z$ is skew-symmetric if for every $v_1, v_2 \in V$,*

$$f(v_1, v_2) = -f(v_2, v_1)$$

The collection of bilinear skew-symmetric maps $V \times V \to Z$ is denoted by $\mathrm{Skew}(V, Z)$

Proposition 3 (wedge product)**.** *Let V be a vector space over field K, then there exists a vector space $V \wedge V$ over K and a bilinear skew-symmetric map $g : V \times V \to V \wedge V$ such that for every vector space Z over K, there is a bijection*

$$F : \mathrm{Skew}(V, Z) \to \mathrm{Hom}(V \wedge V, Z)$$

that makes the diagram below commutes

$$\begin{tikzcd} V \times V \arrow[r, "f"] \arrow[d, "g"'] & Z \\ V \wedge V \arrow[ur, "f_* = F(f)"'] & \end{tikzcd}$$

$V \times V$ is called wedge project and it is unique up to isomorphism.

Remark 4 (basis of wedge product)**.** *Let $\{v_1, ..., v_n\}$ be a basis of V, then*

$$\{v_i \wedge v_j : (i, j) \in [n] \times [n]\}$$

is a basis of $V \wedge V$. Note that $v_i \wedge v_j = -v_j \wedge v_i$, then

$$\dim(V \wedge V) = \binom{\dim V}{2}$$

Remark 5 (n-th skew-symmetric and n-th wedge product)**.** *$f : V^n \to Z$ is called n-th skew-symmetric if for every permutation $\sigma \in S^n$*

$$f(v_1, ..., v_n) = \mathrm{sign}(\sigma) f(\sigma(v_1, ..., v_n))$$

The n-th wedge product exists and denoted by $\wedge^n V$

Remark 6 (basis of wedge product)**.** *Let $\dim V = n$, then each basis vector of $\wedge^k V$ corresponding to a sorted subset of $[n]$, then*

$$\dim(\wedge^k V) = \binom{\dim V}{k}$$

Proposition 4. *Some observations*

- *If $\{v_1, ..., v_n\}$ is linear dependent then $\wedge^n v_i = 0$*
- $\mathrm{span}\{v_1, ..., v_n\} = \mathrm{span}\{w_1, ..., w_n\}$ *if and only if $\wedge^n v_i = c \wedge^n w_i$ for some constant c*

Definition 7 (determinant)**.** *Let the h be the canonical map $K^{n\times n} \to (K^n)^n$ which maps an $n \times n$ matrix into its n column vectors. Let $p : (K^n)^n \to \wedge^n K^n \cong K$ be the canonical skew-symmetric bilinear map. Let $k : \wedge^n K^n \cong K \to K$ be the unique linear map such that $kph(I) = 1$.*

$$\begin{tikzcd} K^{n\times n} \arrow[r, "h"] \arrow[rrr, bend right, "\det"'] & (K^n)^n \arrow[r, "p"] & \wedge^n K^n \cong K \arrow[r, "k"] & K \end{tikzcd}$$

Remark 7 (graded-structure of wedge product)**.** *(wedge product can be defined for R-module) Let R be a commutative ring and M be a free R-module, for $n \in \mathbb{N}$, let $\wedge^n M$ be the n-th wedge product of M. Let*

$$\wedge M = \bigoplus_{n=1}^{\infty} \wedge^n M$$

be the direct product of R-modules. Define multiplication on $\wedge M$ by

$$\wedge : \wedge^p M \times \wedge^q M \to \wedge^{p+q} M$$
$$(x_1 \wedge x_2 ... \wedge x_p, x_{p+1} \wedge x_{p+2} \wedge ... \wedge x_{p+q}) \mapsto x_1 \wedge x_2 \wedge ... \wedge x_{p+q}$$

That makes $\wedge M$ be a grade-commutative ring with multiplication $\wedge : \wedge M \times \wedge M \to \wedge M$

0.3.2 Complex Structure

Definition 8 (minimal polynomial)**.** *Let $T : V \to V$ be a linear map. The minimal polynomial of T is the unique monic polynomial p (leading coefficient is 1) of smallest degree such that $p(T) = 0$. Moreover, the zeros of p are exactly the collection of eigenvalues of T and if the roots of p are distinct and T is diagonalizable*

Definition 9 (complex structure)**.** *Let V be a real vector space, suppose $J : V \to V$ is $\mathbb{R}$-linear invertible such that $J^2 = -I$. Then J is called a complex structure on V*

Proposition 5 (complex structure decomposes V into $V^{1,0} \oplus V^{0,1}$)**.** *Let V be a real vector space with a complex structure $J : V \to V$. Since $J^2 = -I$, the minimal polynomial of J is $p(x) = x^2 + 1$. Since p has two distinct roots $\{+i, -i\}$ which is the eigenvalues of J, then J is diagonalizable. We can write V into a direct sum of two eigenspaces $V^{1,0}, V^{0,1}$ of two eigenvalues $\{+i, -i\}$ correspondingly*

$$V = V_{1,0} \oplus V_{0,1}$$

Remark 8 (complex structure makes V a complex vector space)**.** *If J is a complex structure on a real vector space V, define the complex scalar multiplication by*

$$\cdot : \mathbb{C} \times V \to V$$
$$((a + bi), v) \mapsto av + bJv$$

That makes V a complex vector space, denoted by (V, J)

Definition 10 (complexification)**.** *Let V be a real vector space with basis $\{v_\alpha\}_{\alpha\in A}$. The space $\mathbb{C}$ of complex numbers is a real vector space with a basis $\{1, i\}$. The complexification of V is $V \otimes_{\mathbb{R}} \mathbb{C}$ which has a $\mathbb{R}$-basis $\{v_\alpha \otimes_R 1 : \alpha \in A\} \cup \{v_\alpha \otimes_R i : \alpha \in A\}$. $V \otimes_{\mathbb{R}} \mathbb{C}$ is a complex vector space denoted by $(V \otimes_{\mathbb{R}} \mathbb{C}, i)$ by complex structure*

$$i : V \otimes_{\mathbb{R}} \mathbb{C} \to V \otimes_{\mathbb{R}} \mathbb{C}$$
$$v \otimes_{\mathbb{R}} z \mapsto v \otimes_{\mathbb{R}} iz$$

$(V \otimes_{\mathbb{R}} \mathbb{C}, i)$ has a $\mathbb{C}$-basis $\{v_\alpha \otimes_R 1 : \alpha \in A\}$. There is an $\mathbb{R}$-linear inclusion defined by

$$V \hookrightarrow V \otimes_{\mathbb{R}} \mathbb{C}$$
$$v_\alpha \mapsto v_\alpha \otimes_R 1$$

Remark 9 (decomposition of the complexification of a real vector space with a complex structure)**.** *Let V be a real vector space of dimension $2n$ with a complex structure $J : V \to V$, then J induces a complex structure on $V \otimes_{\mathbb{R}} \mathbb{C}$ by*

$$J_{\mathbb{C}} : V \otimes_{\mathbb{R}} \mathbb{C} \to V \otimes_{\mathbb{R}} \mathbb{C}$$
$$v \otimes_{\mathbb{R}} z \mapsto J(v) \otimes_{\mathbb{R}} z$$

The complex structure $J_{\mathbb{C}}$ decomposes $V \otimes_{\mathbb{R}} \mathbb{C}$ into the direct sum of two eigenspaces

$$V \otimes_{\mathbb{R}} \mathbb{C} = V_{1,0} \oplus V_{0,1}$$

For any $v \in V, z \in \mathbb{C}$, define the conjugation on $V \otimes_{\mathbb{R}} \mathbb{C}$ by

$$\overline{v \otimes_{\mathbb{R}} z} = v \otimes_{\mathbb{R}} \overline{z}$$

Then the conjugation gives a vector space isomorphism $c : V_{1,0} \to V_{0,1}$, that is, if $v \otimes z \in V_{1,0}$, then $\overline{v \otimes z} \in V_{0,1}$ and vice versa.

$$\begin{aligned}
J_{\mathbb{C}}(\overline{v \otimes z}) &= J_{\mathbb{C}}(v \otimes \overline{z}) \\
&= Jv \otimes \overline{z} \\
&= \overline{Jv \otimes z} \\
&= \overline{J_{\mathbb{C}}(v \otimes z)} \\
&= \overline{i(v \otimes z)} && (v \otimes z \in V_{1,0}) \\
&= \overline{v \otimes iz} \\
&= v \otimes \overline{iz} \\
&= v \otimes (-i)\overline{z} \\
&= (-i)(v \otimes \overline{z}) \\
&= (-i)\overline{v \otimes z}
\end{aligned}$$

Therefore, $\mathbb{C} - \dim V_{1,0} = \mathbb{C} - \dim V_{0,1} = n$. As a real vector space, $V_{1,0}$ is of dimension $2n$, we can define a $\mathbb{R}$-linear isomorphism $t : V \to V_{1,0}$

$$\begin{tikzcd}
V \arrow[r, hook, "inc"] \arrow[dr, dashed, "t"'] & V \otimes_{\mathbb{R}} \mathbb{C} \arrow[d, "proj"] \\
& V_{1,0}
\end{tikzcd}$$

This is indeed an isomorphism, for any $v \in V$, let $v \otimes 1 = v_{1,0} + v_{0,1}$, then

$$v_{1,0} + v_{0,1} = v \otimes 1 = v \otimes \overline{1} = \overline{v \otimes 1} = \overline{v_{1,0} + v_{0,1}} = \overline{v_{1,0}} + \overline{v_{0,1}}$$

As $v_{1,0}, \overline{v_{0,1}} \in V_{1,0}$ and $v_{0,1}, \overline{v_{1,0}} \in V_{0,1}$, then $v_{1,0} = \overline{v_{0,1}}$ and $v_{0,1} = \overline{v_{1,0}}$, then we can define the inverse of $t : V \to V_{1,0}$ by

$$v_{1,0} \mapsto v_{1,0} + \overline{v_{1,0}} \mapsto v$$

Hence, $t : V \to V_{1,0}$ is a bijection. Moreover, for any $v \in V \otimes_{\mathbb{R}} \mathbb{C}$, let $v = v_{1,0} + v_{0,1}$, then

$$tJv = t(iv_{1,0} - iv_{0,1}) = itv_{1,0} = tiv$$

Recall that J makes V into a complex vector space, then when view V as a complex vector space, $t : V \to V_{1,0}$ is a $\mathbb{C}$-linear isomorphism $t : (V, J) \to (V \otimes_{\mathbb{R}} \mathbb{C}, i)$. Note that, V and $V_{1,0}$ are not the same subspace in $V \otimes_{\mathbb{R}} \mathbb{C}$

0.3.3 Discussion on the natural isomorphism between $((-)_{\mathbb{C}})^*$ and $((-)^*)_{\mathbb{C}}$

Remark 10 (Dual of the complexification is complexification of the dual). *https://math.stackexchange.com/a/4718945/700,122*

Let V be a real vector space, let $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ be the dual space of V, let $(V^*)_{\mathbb{C}} = V^* \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of dual of V. For simplification, let $\dim V = 2$ and let $\{f_1, f_2\} \subseteq V^*$ be a basis for V^*. Then

$$(V^*)_{\mathbb{C}} = \mathbb{C} - \mathrm{span}\{f_1, f_2\}$$

Let $f \in (V^*)_{\mathbb{C}}$, then

$$\begin{aligned}
f &= (a_1 + ib_1)f_1 + (a_2 + ib_2)f_2 \\
&= (a_1 f_1 + a_2 f_2) + i(b_1 f_1 + b_2 f_2)
\end{aligned}$$

Therefore,

$$(V^*)_{\mathbb{C}} = \mathbb{C} - \mathrm{span}\{f_1, f_2\} = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$$

Let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of V, let $(V_{\mathbb{C}})^* = \operatorname{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$ be the dual of complexification of V. Let $f \in (V^*)_{\mathbb{C}} = \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C})$, then let $\tilde{f} \in (V_{\mathbb{C}})^* = \operatorname{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$ be defined by

$$f(v \otimes z) = zf(v)$$

Let $\tilde{g} \in (V_{\mathbb{C}})^* = \operatorname{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$, then let $g \in (V^*)_{\mathbb{C}} = \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C})$ be the restriction of $\tilde{g}$ on V. The two maps define an isomorphism between $(V^*)_{\mathbb{C}}$ and $(V_{\mathbb{C}})^*$. TODO - dual and complexification are functors, and this isomorphism is natural

Let K be a field, let Vect_K be the category of vector spaces over K.

Definition 11 (dual space contravariant functor from Vect_K into Vect_K)**.** *The dual space functor $D = (-)^*$ is a contravariant functor from Vect_K into Vect_K defined as follows: Let $V \in \operatorname{ob}\operatorname{Vect}_K$ then D induces $D(V) = V^* = \operatorname{Hom}(V, K)$ is the usual dual space of V. Let $f : V \to W$ be a linear map where $V, W \in \operatorname{ob}\operatorname{Vect}_K$, then D induces $D(f) = f^* : W^* \to V^*$ defined by $g \mapsto gf$ for every $g \in W^* = \operatorname{Hom}(W, K)$*

$$\begin{tikzcd}
V \arrow[r, "f"] & W \\
V^* & W^* \arrow[l, "D(f)=f^*"]
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "f"] \arrow[dr, dashed, "D(f)(g)=gf"'] & W \arrow[d, "g"] \\
& K
\end{tikzcd}$$

Definition 12 (complexification covariant functor from $\operatorname{Vect}_{\mathbb{R}}$ into $\operatorname{Vect}_{\mathbb{C}}$)**.** *The complexification functor $C = (-)_{\mathbb{C}}$ is a contravariant functor from $\operatorname{Vect}_{\mathbb{R}}$ to $\operatorname{Vect}_{\mathbb{C}}$ defined as follows: Let $V \in \operatorname{ob}\operatorname{Vect}_{\mathbb{R}}$, then C induces $C(V) = V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} \in \operatorname{ob}\operatorname{Vect}_{\mathbb{C}}$ where complex multiplication is defined by*

$$\mathbb{C} \times (V \otimes_{\mathbb{R}} \mathbb{C}) \to V \otimes_{\mathbb{R}} \mathbb{C}$$
$$\left(y, \sum_{i \in I} v_i \otimes_{\mathbb{R}} z_i\right) \mapsto \sum_{i \in I} v_i \otimes_{\mathbb{R}} yz_i$$

Let $f : V \to W$ be a linear map where $V, W \in \operatorname{ob}\operatorname{Vect}_{\mathbb{R}}$, then C induces $C(f) = f \otimes_{\mathbb{R}} 1 : V_{\mathbb{C}} \to W_{\mathbb{C}}$ defined by the linear extension of $v \otimes_{\mathbb{R}} z \mapsto f(v) \otimes_{\mathbb{R}} z$

Proposition 6. *Let $D_{\mathbb{R}}$ and $D_{\mathbb{C}}$ denote the dual space functors on $\operatorname{Vect}_{\mathbb{R}}$ and $\operatorname{Vect}_{\mathbb{C}}$ respectively, then there exists a natural isomorphism*

$$\mu : D_{\mathbb{C}} C \to C D_{\mathbb{R}}$$

Proof. For each $V, W \in \operatorname{Vect}_{\mathbb{R}}$, we will construct the maps such that the diagram below commutes

$$\begin{tikzcd}
V \arrow[d, "f"'] & (V_{\mathbb{C}})^* \arrow[r, "\mu_V"'] \arrow[rr, bend left, "1"] & (V^*)_{\mathbb{C}} \arrow[r, "\nu_V"'] \arrow[rr, bend left, "1"] & (V_{\mathbb{C}})^* \arrow[r, "\mu_V"'] & (V^*)_{\mathbb{C}} \\
W & (W_{\mathbb{C}})^* \arrow[r, "\mu_W"] \arrow[u, "(f\otimes 1)^*"] \arrow[rr, bend right, "1"'] & (W^*)_{\mathbb{C}} \arrow[r, "\nu_W"] \arrow[u, "f^*\otimes 1"] \arrow[rr, bend right, "1"'] & (W_{\mathbb{C}})^* \arrow[r, "\mu_W"] \arrow[u, "(f\otimes 1)^*"] & (W^*)_{\mathbb{C}} \arrow[u, "f^*\otimes 1"]
\end{tikzcd}$$

Let $g : V \otimes \mathbb{C} \to \mathbb{C}$ be an object of $(V_{\mathbb{C}})^*$, then $\mu_V(g)$ is an object of $(V^*)_{\mathbb{C}}$ defined by

$$\mu_V(g) = [v \mapsto \operatorname{Re} g(v \otimes 1)] \otimes_R 1 + [v \mapsto \operatorname{Im} g(v \otimes 1)] \otimes_R i$$

Let $h \otimes y$ where $h : V \to \mathbb{R}$ be an object of $(V^*)_{\mathbb{C}}$, then $\nu_V(h)$ is an object of $(V_{\mathbb{C}})^*$ defined by

$$v \otimes z \mapsto yzh(v)$$

ν_V is then extended linearly. We can verify that $\nu_V \mu_V = 1$ and $\mu_V \nu_V = 1$, that is, μ_V is an isomorphism.
Suppose in the diagram commutes except $y_1 b = a y_2$,

$$\begin{tikzcd}
{} \arrow[r, "x_1"'] \arrow[rr, bend left, "1"] & {} \arrow[r, "y_1"'] \arrow[rr, bend left, "1"] & {} \arrow[r, "x_1"'] & {} \\
{} \arrow[r, "x_2"] \arrow[u, "a"] \arrow[rr, bend right, "1"'] & {} \arrow[r, "y_2"] \arrow[u, "b"] \arrow[rr, bend right, "1"'] & {} \arrow[r, "x_2"] \arrow[u, "a"] & {} \arrow[u, "b"]
\end{tikzcd}$$

Then $y_1 b = a y_2$ can be induced from the commutativity of the rest

$$y_1 b = y_1 b x_2 y_2 = y_1 x_1 a y_2 = a y_2$$

Therefore, for naturality, it suffices to prove that

$$\mu_V (f^* \otimes 1) = (f \otimes 1)^* \mu_W$$

This is indeed true from the construction above. Hence, μ is a natural isomorphism. Moreover, from julio_es_sui_glace, this is a tensor-hom adjunction. □

0.4 Algebra

Definition 13 (vector space)**.** *An abelian group V is a vector space over field K if there is a scalar multiplication $\cdot : K \times V \to V$ satisfying*

1. $1v = v$ *for every* $1 \in K, v \in V$
2. $a(bv) = (ab)v$ *for every* $a, b \in K, v \in V$
3. $a(v + u) = av + au$ *for every* $a \in K, v, u \in V$

Definition 14 (module over a commutative ring)**.** *Let R be a commutative ring, an abelian group $(M, +)$ is a module over R, namely, R-module if there is an operation $\cdot : R \times M \to M$ such that*

$$r(x + y) = rx + ry$$
$$(r + s)x = rx + sx$$
$$(rs)x = rsx$$

for all $r, s \in R$, $x, y \in M$. If R is unital, then $1x = x$. When K is a field, then an K-module is a K-vector space. A basis of an R-module M is a collection of elements E such that every element in M can be written as finite linear combination of elements in E and moreover, every finite collection $\{e_1, e_2, ..., e_n\} \subseteq E$ is linearly independent, that is,

$$r_1 e_1 + r_2 e_2 + ... + r_n e_n = 0$$

for $r_1, r_2, ..., r_n \in R$, then $r_1 = r_2 = ... = r_n = 0$

Definition 15 (algebra over an commutative ring)**.** *Let R be a commutative ring, an ring $(A, +, \times)$ is an algebra over R, namely R-algebra is an R-module and*

$$r(xy) = rxy = x(ry)$$

for all $r \in R$ and $x, y \in A$.

Definition 16 (graded-ring)**.** *A graded ring is a ring that is decomposed into a direct sum*

$$R = \bigoplus_{n=0}^{\infty} R_n = R_0 \oplus R_1 \oplus ...$$

of additive groups such that $R_n R_m \subseteq R_{n+m}$ for all nonnegative integers n, m. A non-zero element of R_n is called homogeneous of degree n. An example for graded-ring is the polynomial ring.

Definition 17 (graded-commutative ring)**.** *In a graded ring R, the multiplication is called graded-commutative if given two homogeneous elements $x \in R_p$ and $y \in R_q$, then*

$$xy = (-1)^{pq} yx$$

A graded-ring with graded-commutative multiplication is called graded-commutative ring.

Definition 18 (exact sequence)**.** *In an abelian category, given a diagram $A \xrightarrow{f} B \xrightarrow{g} C$, it is called exact at B if f factors through $\ker g$ by an epimorphism $A \to \ker g$*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, dashed, "epi"'] & B \arrow[r, "g"] & C \\ \ker g \arrow[ur, "i"'] & & \end{tikzcd}$$

A sequence is called exact if if it is exact everywhere. In the context where f, g are maps, then this is equivalent to

$$\operatorname{im} f = \ker g$$

0.5 Topology

Proposition 7. *closed subset of a compact space is compact*

Proposition 8. *compact subset of a Hausdorff space is closed*

Definition 19 (quotient topology)**.** *A surjection $\pi : X \twoheadrightarrow Y$ where X is a topological space induces a topology on Y called quotient topology as follows: $U \subseteq Y$ is open if and only $\pi^{-1}U$ is open. In other words, quotient topology is the densest topology such that π is continuous.*

0.5.1 Fiber Bundle

Definition 20 (group action)**.** *Let K be a group and F be a set. K is called acting on F if there exists a map $\cdot : K \times F \to F$ such that*

$$1f = f \text{ and } (ab)f = a(b(f))$$

where 1 is the identity of K and for every $a, b \in K$, $f \in F$. If K is a topological group and F is a topological space, then $\cdot : K \times F \to F$ is required to be continuous

Definition 21 (Bredon - bundle projection)**.** *Let X, B, F be Hausdorff spaces. A continuous surjection $\pi : X \to B$ is called a bundle projection if there exists an open cover $\{U_i\}_{i \in I}$ of X such that for each $U \in \{U_i\}_{i \in I}$, there is a homeomorphism $\phi : U \times F \to \pi^{-1}U$ such that the composition $\pi\phi : U \times F \to U$ is the canonical projection $(u, f) \mapsto u$*

$$\begin{tikzcd}
\pi^{-1}U \arrow[d, two heads, "\pi"'] & U \times F \arrow[l, "\phi"'] \arrow[dl, dashed, two heads, "\pi\phi"] \\
U &
\end{tikzcd}$$

X is called total space, B is called base space, F is called fiber, ϕ is called trivialization (local trivialization)

Definition 22 (Bredon - fiber bundle)**.** *Let $\pi : X \to B$ be a bundle projection with fiber F. Let K be a topological group acting on F. $\pi : X \to B$ is called a fiber bundle if $U = U_i \cap U_j \neq \varnothing$ is the intersection of two local trivializations*

$$\begin{tikzcd}
U \times F \arrow[r, "\phi_i"] \arrow[rr, dashed, bend right, "\phi_j^{-1}\phi_i"'] & \pi^{-1}U & U \times F \arrow[l, "\phi_j"']
\end{tikzcd}$$

then there exists a map $\theta : U \to K$ such that restricted to U, we have $\phi_i(u, f) = \phi_j(u, \theta(u)f)$

$$\begin{aligned}
\phi_j^{-1}\phi_i : U \times F &\to U \times F \\
(u, f) &\mapsto (u, \theta(u)f)
\end{aligned}$$

K is called structure group.

Definition 23 (Bredon - vector bundle)**.** *A vector bundle is a fiber bundle where fiber $F = \mathbb{R}^r$ or $\mathbb{C}^r$ and structure group $K = GL(F)$*

0.6 Category Theory

Definition 24 (limit, colimit)**.** *TODO*

Definition 25 (adjunction)**.** *TODO*

Chapter 1

Manifolds

1.1 Manifolds

Definition 26 (Hausdorff space and second-countable space, locally homeomorphic)**.** *Given a topological space* $(M, \mathcal{T})$

- $(M, \mathcal{T})$ *is said to be Hausdorff if for every pair* $x, y \in M$*, there exists open neighbourhood* U *of* x *and* V *of* y *such that* $U \cap V = \varnothing$
- $(M, \mathcal{T})$ *is said to be second-countable if* $\mathcal{T}$ *has a countable basis, i.e. there exists a countable collection of open sets* $\mathcal{B}$ *such that every open set in* $\mathcal{T}$ *is a union of open sets in* $\mathcal{B}$*.* [1]
- $(M, \mathcal{T})$ *is said to be locally homeomorphic to* K^n *where* K *is either* $\mathbb{R}$ *or* $\mathbb{C}$ *if for every point* $x \in M$*, there exists an open neighbourhood* U_x *and a homeomorphism* $h_x : U_x \to U'_x$ *where* U'_x *is an open subset of* K^n

Remark 11. *The Hausdorff condition means the topology has a lot of open sets. On the other hand, the second-countable condition limits how many open sets there are.*

Definition 27 (topological manifold [2], chart, atlas, transition function)**.** *A topological space* $(M, \mathcal{T})$ *is called a topological manifold of dimension n over field* K *(either* $\mathbb{R}$ *or* $\mathbb{C}$*) if it is Hausdorff, second-countable, and locally homeomorphic to* K^n*. The homeomorphism* $h_x : U_x \to U'_x$ *is called chart, the collection* $\{h_x : U_x \to U'_x\}_{x \in M}$ *is called atlas. The atlas is also defined by* $\{h_i : U_i \to U'_i\}_{i \in I}$ *where* $\{U_i\}$ *is an open cover of* M*. Let* $h_i : U_i \to U'_i$ *and* $h_j : U_j \to U'_j$ *be two charts such that* $U = U_i \cap U_j \neq \varnothing$*, then the homeomorphism*

$$t_{ij} = h_j h_i^{-1} : h(U_i) \to h(U_j)$$

is called transition function.

Definition 28 ($\mathcal{S}$-manifold)**.** *A topological manifold is called a* $\mathcal{S}$*-manifold where* S *is either* $\mathcal{E}$*,* $\mathcal{A}$*, or* $\mathcal{O}$ *if the transition function is* $\mathcal{S}$*-function.*

Remark 12. *We can assume that there is an atlas* C *being a maximal set of charts* [3]*.*

Definition 29 (algebraic manifold)**.** *A complex holomorphic manifold with transition function being rational function (fraction of polynomials) is called algebraic manifold.*

1.2 Functions on manifolds (Part 1)

Definition 30 ($\mathcal{S}$-structure $\mathcal{S}_M$)**.** *Let* $(M, \mathcal{T})$ *together with atlas* $\{(U_i, h_i)\}_{i \in I}$ *be an* $\mathcal{S}$*-manifold. An* $\mathcal{S}$*-structure* S_M *on an* $\mathcal{S}$*-manifold* M *is a family of* K*-valued continuous functions defined on the open sets of* M *satisfying the following: Let* V *be an open set of* M *and a function* $f : V \to K$ *is in* $\mathcal{S}_M$ *if for every* $i \in I$*, let* f *be restricted to* $V \cap U_i$*, then the composition* fh^{-1} *is of* $\mathcal{S}$*-function*

$$\begin{tikzcd} V \cap U_i \subseteq M \arrow[r, "f"] \arrow[d, "h_i"'] & K \\ h_i(V \cap U_i) \subseteq K^n \arrow[ur, dashed, "fh_i^{-1}"'] & \end{tikzcd}$$

[1]remark: given a countable collection of sets $\mathcal{B}$, the collection of all finite intersections of $\mathcal{B}$ is countable.

[2]analogous to n-dimensional vector space over K

[3]by Zorn's lemma

Note that, later on, we will also denote the collection of $\mathcal{S}$-functions on M by $\mathcal{S}(M) \subseteq \mathcal{S}_M$

Proposition 9. *A manifold structure on a topological space M is characterized by its $\mathcal{S}$-structure $\mathcal{S}_M$, that is, if A_1 and A_2 are two atlases on M, that makes two manifolds $M_1 = (M, A_1)$ and $M_2 = (M, A_2)$ then $\mathcal{S}_{M_1} = \mathcal{S}_{M_2}$ implies A_1 and A_2 are compatible*

Proof. Let $h_1 : U_1 \to U_1' \subseteq \mathbb{R}^n$ and $h_2 : U_2 \to U_2' \subseteq \mathbb{R}^n$ be two charts from A_1 and A_2 respectively such that $U = U_1 \cap U_2 \neq \varnothing$. Let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be the canonical projection into the i-th coordinate. Then, the restriction of $\pi_i h_2$ on U, namely $\pi_i h_2|_U : U \to \mathbb{R}$ is an $\mathcal{S}$-function with respective to A_2, by the premise, it is also an $\mathcal{S}$-function with respective to A_1. Therefore, $\pi_i h_2 h_1^{-1}|_{h_1(U)} : h_1(U) \to \mathbb{R}$ is an $\mathcal{S}$-function. As this is true for all $i \in [n]$, the transition function $h_2 h_1^{-1}|_{h_1(U)} : h_1(U) \to h_2(U)$ is an $\mathcal{S}$-function. Therefore, A_1 is compatible with A_2 □

Definition 31 ($\mathcal{S}$-morphism)**.** *An $\mathcal{S}$-morphism $F : (M, \mathcal{S}_M) \to (N, \mathcal{S}_N)$ is a continuous map $F : M \to N$ such that $f \in \mathcal{S}_N$ implies $fF \in \mathcal{S}_M$*

$$\begin{tikzcd}
(M, \mathcal{S}_M) \arrow[r, "F"] \arrow[dr, dashed, "fF"'] & (N, \mathcal{S}_N) \arrow[d, "f"] \\
 & K
\end{tikzcd}$$

That is, an $\mathcal{S}$-morphism is a continuous map $F : M \to N$ and it induces a map

$$\begin{aligned} F_* : \mathcal{S}_N &\to \mathcal{S}_M \\ f &\mapsto fF \end{aligned}$$

Definition 32 ($\mathcal{S}$-isomorphism)**.** *An $\mathcal{S}$-morphism $F : (M, \mathcal{S}_M) \to (N, \mathcal{S}_N)$ is an $\mathcal{S}$-isomorphism if $F : M \to N$ is a homeomorphism and $F^{-1} : (N, \mathcal{S}_N) \to (M, \mathcal{S}_M)$ is also an $\mathcal{S}$-morphism. Hence, $\mathcal{S}$-isomorphism induces a bijection between $\mathcal{S}_M$ and $\mathcal{S}_N$ which in turn induces an equivalence of manifold structure between $(M, \mathcal{S}_M)$ and $(N, \mathcal{S}_N)$*

Remark 13 ($\mathcal{S}$-morphism from neighbourhoods)**.** *Let $f : X \to Y$, for each $y \in Y$, there is an open neighbourhood $U_y \subseteq Y$ of y, such that $U_x = f^{-1}U_y$, $f : U_x \to U_y$ is an $\mathcal{S}$-morphism, then $f : X \to Y$ is an $\mathcal{S}$-morphism.*

1.3 Functions on manifolds (Part 2): Partition of Unity

Definition 33 (refinement, locally finite, paracompact)**.** *Let M be a topological space.*

1. *Let $\mathcal{U}_A = \{U_\alpha\}_{\alpha \in A}$ and $\mathcal{U}_B = \{V_\beta\}_{\beta \in B}$ be two open covers of M. $\mathcal{U}_B$ is said to be a refinement of $\mathcal{U}_A$ if for each $V_\beta \in \mathcal{U}_B$, there exists a $U_\alpha \in \mathcal{U}_A$ such that $V_\beta \subseteq U_\alpha$*

2. *Let $\mathcal{U}_A = \{U_\alpha\}_{\alpha \in A}$ be a cover of M (not necessarily open). $\mathcal{U}_A$ is called locally finite if for every $x \in M$, there exists an open neighbourhood W of x such that $W \cap U_\alpha \neq \varnothing$ for finitely many $\alpha \in A$*

3. *A topological space is paracompact if every open cover has an open locally finite refinement.*

Proposition 10. *Let M be a smooth manifold with atlas $\{(U_i, h_i)\}_{i \in I}$, then*

1. *There is a countable subset $J \subseteq I$ such that $\{(U_j, h_j)\}_{j \in J}$ is also an atlas of M*

2. *For each $j \in J$, U_j is paracompact*

3. *M is paracompact*

Proof.

1. M is second-countable, hence let $\{U_j\}_{j \in J}$ a countable basis of M. Construct charts on $\{U_j\}_{j \in J}$ by restriction from $\{(U_i, h_i)\}_{i \in I}$ as $\{U_j\}_{j \in J}$ is a refinement of $\{(U_i, h_i)\}_{i \in I}$

2. each U_j is homeomorphic to open set in $\mathbb{R}^n$, hence it is paracompact (*google it, metrizable implies paracompact*).

3. combine (1) and (2)

□

Lemma 1. *For $r > 0$, denote the open cube*

$$C(r) = \{(x_1, ..., x_d) \in \mathbb{R}^d : \forall i \in [d], -r < x_i < +r\}$$

There exists a non-negative smooth function on $\mathbb{R}^d$ which equals 1 on $C(1)$ and 0 on $\mathbb{R}^d \setminus C(2)$

Proof. *result in analysis, skip* □

Definition 34 (partition of unity)**.** *A partition of unity on a smooth manifold M is a collection $\{\phi_i\}_{i\in I}$ of non-negative smooth functions on M such that the collection of supports $\{\operatorname{supp}\phi_i\}_{i\in I}$ is locally finite and for every $x \in M$*

$$\sum_{i\in I}\phi_i(x) = 1$$

Theorem 4 (countable partition of unity)**.** *Let M be a smooth manifold and $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. Then there exists a countable partition of unity $\{\phi_i\}_{i\in\mathbb{N}}$ such that for every $i \in \mathbb{N}$, $\operatorname{supp}\phi_i$ is compact and $\{\operatorname{supp}\phi_i\}_{i\in\mathbb{N}}$ is a refinement of $\{U_\alpha\}_{\alpha\in A}$*

Proof. *a little tedious - defer to the end of semester* □

Theorem 5 (arbitrary partition of unity)**.** *Let M be a smooth manifold and $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. Then there exists a partition of unity $\{\phi_\alpha\}_{\alpha\in A}$ such that for every $\alpha \in A$, $\operatorname{supp}\phi_\alpha$ is compact and $\{\phi_\alpha\}_{\alpha\in A}$ is a refinement of $\{U_\alpha\}_{\alpha\in A}$*

Proof. Let $\{\phi_i\}_{i\in\mathbb{N}}$ be a countable partition of unity. Let $f : \mathbb{N} \to A$ be a function such that $\operatorname{supp}\phi_i \subseteq U_{f(i)}$. Then for each $\alpha \in \operatorname{im} f \subseteq A$, define $\phi_\alpha = \sum_{i\in f^{-1}\alpha}\phi_i$ and for each $\beta \notin \operatorname{im} f \subseteq A$, $\phi_\beta = 0$ □

Corollary 1. *Let M be a smooth manifold and $A \subset G \subseteq M$ such that A is closed and G is open, then there exists a smooth non-negative function $\phi : M \to \mathbb{R}$ such that $\operatorname{supp}\phi \subseteq G$ and it attains its maximum value 1 in A*

Proof. Note that $\{G, M \setminus A\}$ is an open cover of M. There is a partition of unity $\{\phi, \psi\}$ such that the $\operatorname{supp}\phi \subseteq G$ and $\operatorname{supp}\psi \subseteq M \setminus A$ □

1.4 Submanifolds

Definition 35 ($\mathcal{S}$-submanifold [4])**.** *Let M be an $\mathcal{S}$-manifold of dimension m over field K, let N be a closed subset of M, $0 \le n \le m$ and K^n is the n-dimensional canonical subspace of K^m. N is called an $\mathcal{S}$-submanifold of M of dimension n if for every point $x_0 \in N$, there is a chart $h : U \to U' \subseteq K^m$ of M where $x_0 \in U$ then*

$$h(U \cap N) = U' \cap K^n$$

The restriction of $h : U \to U'$ into $U \cap N$ is evidently the chart of N so that the $\mathcal{S}$-submanifold N is a $\mathcal{S}$-manifold. The difference $m - n$ is called the K-codimension of N

Definition 36 ($\mathcal{S}$-embedding)**.** *An $\mathcal{S}$-morphism $f : (M, \mathcal{S}_M) \to (N, \mathcal{S}_N)$ is an $\mathcal{S}$-embedding if f is an $\mathcal{S}$-isomorphism onto an $\mathcal{S}$-submanifold of N*

Theorem 6 (Whitney)**.** *Let M be a smooth n-manifold, then there exists a smooth embedding $f : M \to \mathbb{R}^{2n+1}$*

1.4.1 Projective Space

Definition 37 (projective space)**.** *Let V be a n-dimensional vector space over field K. For any $x \in V$, let $[x] = \{kx : k \in K\}$ denote the 1 dimensional subspace of V spanned by x. Let*

$$P(V) = \{[x] : x \in V \setminus \{0\}\}$$

$P(V)$ is called the projective space of V. Let $\pi : V \setminus \{0\} \to P(V)$ be defined by $\pi(x) = [x]$, π being surjective gives $P(V)$ the quotient topology. We denote $P_n(\mathbb{R}) = P(\mathbb{R}^n), P_n(\mathbb{C}) = P(\mathbb{C}^n)$

Proposition 11. *:*

- *$P_n(\mathbb{R})$ is a real analytic manifold of dimension n*
- *$P_n(\mathbb{C})$ is a complex holomorphic manifold of dimension n*

Proof.

$\pi : K^n \setminus \{0\} \to P_n(K)$ is an open map. Let $U \subseteq K^n \setminus \{0\}$ be an open set, then $\pi^{-1}\pi U = \{kx : k \in K, x \in U\}$ is open. By definition of quotient topology, $\pi^{-1}\pi U$ open implies πU open. Therefore, $P_n(K)$ has a countable basis induced from countable basis of $K^n \setminus \{0\}$. As $\pi : K^n \setminus \{0\} \to P_n(K)$ factors through $S^n(K)$, $P_n(K)$ is compact. As $S^n(K)$ is Hausdorff and preimage of 2 points on $P_n(K)$ are 4 points on $S^n(K)$, $P_n(K)$ is also Hausdorff

[4] analogous to subspace of a vector space

- $P_n(\mathbb{R})$ is a real analytic manifold of dimension n

  For each $i = 0, 1, ..., n$, let

$$U_i = \{x = (x_0, x_1, ..., x_n) \in \mathbb{R}^{n+1} : x_i \neq 0\}$$

  Each πU_i is open, so $\{\pi U_i\}_{i=0}^n$ is an open cover of $P_n(\mathbb{R})$. Define a chart $h_i : \pi U_i \to U_i' \subseteq \mathbb{R}^n$ on $P_n(\mathbb{R})$ by

$$h_i([x]) = \left(\frac{x_0}{x_i}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_n}{x_i}\right)$$

  Note that, we can identify πU_i by the hyperplane $V_i = \left\{\left(\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}\right) : x \in U_i\right\}$, then h_i is a homeomorphism.

$$\begin{tikzcd}
\mathbb{R}^{n+1} & P_n(\mathbb{R}) & \mathbb{R}^n \\
U_i \arrow[u, hook] \arrow[r, two heads, "\pi"] \arrow[rr, bend right, dashed] & \pi U_i \arrow[u, hook] \arrow[r, "h_i"] & U_i' \arrow[u, hook]
\end{tikzcd}$$

  Now we will check if the transition function is smooth. Let $h_1 : U_1 \to U_1'$, $h_2 : U_2 \to U_2'$ be two charts such that $U = U_1 \cap U_2 \neq \varnothing$, let $z = (z_0, ..., z_{j-1}, z_{j+1}, ..., z_n) \in U_2' \subseteq \mathbb{R}^n$, then

$$h_2^{-1}(z) = [(z_0, ..., z_{j-1}, 1, z_{j+1}, ..., z_n)]$$

  As $h_2^{-1}(z) \in U$, so $z_i \neq 0$, then

$$h_1 h_2^{-1}(z) = \left(\frac{z_0}{z_i}, ..., \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, ..., \frac{z_{j-1}}{z_i}, \frac{1}{z_i}, \frac{z_{j+1}}{z_i}, \frac{z_n}{z_i}\right)$$

  As $\mathbb{R}^{n+1}$ is second-countable, $P_n(\mathbb{R}) = \pi\mathbb{R}^{n+1}$ is also second-countable. Now, we will prove that $P_n(\mathbb{R})$ is Hausdorff. Let $[x], [y] \in P_n(\mathbb{R})$, if $[x], [y]$ are in the same U_i, as U_i is homeomorphic to a Hausdorff space, $[x], [y]$ can be separated by open sets, if $[x], [y]$ are in U_i, U_j and $U_i \cap U_i = \varnothing$, U_i, U_j are evidently the separating sets. if $U = U_i \cap U_i \neq \varnothing$, write

$$[x] = [(x_0, ..., x_{i-1}, x_i, x_{i+1}, ..., x_{j-1}, 0, x_{j+1}, ..., x_n)]$$
$$[y] = [(y_0, ..., y_{i-1}, 0, y_{i+1}, ..., y_{j-1}, y_j, y_{j+1}, ..., y_n)]$$

  we construct two open sets in $\mathbb{R}^{n+1}$

$$U_x = \{z \in \mathbb{R}^{n+1} : |z_i| > |z_j|\}$$
$$U_y = \{z \in \mathbb{R}^{n+1} : |z_i| < |z_j|\}$$

  Then, $[x] \in \pi U_x$, $[y] \in \pi U_y$ and U_x, U_y do not intersect.

- $P_n(\mathbb{C})$ is a complex holomorphic manifold of dimension n

  same proof as above

□

1.4.2 Matrices of Constant Rank

Proposition 12 (matrices of rank $\geq m$)**.** *Let $m \leq k \leq n$, then*

$$M_{k,n}^{\geq m} = \{A \in K^{k \times n} : \text{rank}(A) \geq m\}$$

is an open subset of $K^{k\times n}$. In particular, $GL(K^r) = M_{r,r}^{\geq r}$ is an open subset of $K^{r\times r}$

Proof.

A matrix in $K^{k\times n}$ of rank m must have at least one $m\times m$ minor that is non-singular, denote $\{\sigma_i : K^{k\times n}\to K^{m\times m}\}$ be the collection of $m\times m$ minors, then

$$M_{k,n}^{\geq m} = \bigcup_{\sigma_i}\{A\in K^{k\times n} : \det \sigma_i A \neq 0\}$$

$\det\sigma_i : K^{k\times n}\to K$ is continuous, and $K\backslash\{0\}$ is open (in both case of $\mathbb{R}$ and $\mathbb{C}$), therefore, each $\{A\in K^{k\times n} : \det\sigma_i A\neq 0\}$ is open, hence $M_{k,n}^{\geq m}$ is an open subset of $K^{k\times n}$ ☐

Proposition 13 (Wells example 1.7 - matrices of rank m)**.** *Let $m\leq k\leq n$, then*

$$M_{k,n}^{m}(K) = \{A\in K^{k\times n} : \operatorname{rank}(A) = m\}$$

is a real analytic manifold or complex holomorphic manifold when $K=\mathbb{R}$ or $K=\mathbb{C}$ of dimension $(k-m)(n-m)$.

Proof.

TODO - long argument ☐

1.4.3 Grassmannian

Definition 38 (Grassmannian)**.** *Let V if a n-dimensional vector space over field K. Let k be an integer such that $0\leq k\leq n$, and let*

$$G_k(V) := \{W \textit{ is a subspace of } V : \dim W = k\}$$

$G_k(V)$ is called Grassmannian manifold. We denote $G_{k,n}(\mathbb{R}) = G_k(\mathbb{R}^n), G_{k,n}(\mathbb{C}) = G_k(\mathbb{C}^n)$

Proposition 14. *: $G_{k,n}(\mathbb{R})$ is a compact smooth manifold of dimension $k(n-k)$. $G_{k,n}(\mathbb{C})$ is an algebraic manifold of dimension $k(n-k)$.*

Proof. Define $\pi : M_{k,n}^k(K)\to G_{k,n}(K)$ be the map from matrices $K^{k\times n}$ of rank k to the Grassmannian by

$$\pi(W) = \text{ row space of } W$$

Let the topology on $G_{k,n}(K)$ be the quotient topology induced by π. Note that, $W_1, W_2\in M_{k,n}^k(K)$ have the same row space if and only if

$$W_1 = UW_2$$

where $U\in GL_k(K)$ is a $K^{k\times k}$ invertible linear transformation, that defines an equivalence relation on $M_{k,n}^k(K)$ and the equivalence classes is $G_{k,n}(K)$.

prove that $G_{k,n}(K)$ is second-countable, compact and Hausdorff

Given $W\in M_{k,n}^k(K)$, W must have k independent columns, let $\sigma\subseteq[n]$ be the k independent columns of W, let $\sigma : K^{k\times n}\to K^{k\times k}$ be the selection of k columns according to σ, that is, $\sigma\in\{0,1\}^{n\times k}$ and $W\sigma\in GL_k(K)$ consists of k columns of W from σ

Now, for each $\sigma\subseteq[n]$ such that $|\sigma| = k$, let

$$U_\sigma = \{W\in M_{n,k}^k(K) : \det(W\sigma)\neq 0\}$$

As $W\mapsto\det(W\sigma)$ is continuous, U_σ is an open set of $M_{n,k}^k(K)$. As every $W\in M_{n,k}^k(K)$ belongs to at least one of U_σ, $\bigcup_{\sigma\subseteq[n]} U_\sigma$ is an open cover of $M_{n,k}^k(K)$. We define an equivalent relation on $M_{n,k}^k(K)$ as follows: $W_1\sim W_2$ if and only if there exists $\sigma\subseteq[n]$ such that $W_1, W_2\in U_\sigma$ and $(W_1\sigma)^{-1}W_1 = (W_2\sigma)^{-1}W_2$. We will show that this equivalence relation coincides with Grassmannian by showing that if W_1, W_2 have the same row space then $W_1\sim W_2$.

Since $W_1\in M_{n,k}^k(K)$, let $\sigma\subseteq[n]$ such that $(W\sigma)\in GL_k(K)$, as $W_1 = UW_2$ with U invertible, we have

$$(W_1\sigma)^{-1}W_1 = (W_1\sigma)^{-1}UW_2 = (U^{-1}W_1\sigma)^{-1}W_2 = (W_2\sigma)^{-1}W_2$$

Hence, $W_2\in U_\sigma$ and $(W_1\sigma)^{-1}W_1 = (W_2\sigma)^{-1}W_2$. Now, we prove that under $\sim$, $M_{n,k}^k(K)$ induces a manifold of dimension $k(n-k)$.

We have shown that every fiber of Grassmannian belong to the same $\mathcal{U}_\alpha$, therefore, $\mathcal{U}_\alpha/\sim$ is open in $G_{k,n}(K)$. For each $\sigma\subseteq[n]$, let

$$V_\sigma = \{W\in U_\alpha : W\sigma = I_k\}\cong K^{(n-k)\times k}$$

Every $W\in U_\sigma$ is equivalent to one of $V\in V_\sigma$ by the map $h_\sigma : U_\sigma\to V_\sigma$ defined by

$$h_\sigma(W) = (W\sigma)^{-1}W$$

And no two $V_1, V_2 \in V_\sigma$ are equivalent, hence, $U_\sigma / \sim \cong V_\sigma \cong K^{(n-k)\times k}$. That defines a chart $h_\sigma : U_\sigma / \sim \to K^{(n-k)\times k}$. Therefore, $G_{k,n}(K)$ is a manifold of dimension $k(n-k)$. It remains to show that transition function is smooth.

Let $U = U_\alpha \cap U_\beta \neq \varnothing$. Let $W \in h(U_\alpha)$, that is both $W\alpha, W\beta$ are invertible and $W_\alpha = I_k$, the transition function $t_{\alpha\beta} : V_\alpha \to V_\beta$ is

$$t_{\alpha\beta}(W) = (W\beta)^{-1}W$$

Note that $W \mapsto (W\beta)^{-1}W$ is a sequence of elementary row operations that put W into its row echelon form, therefore $t_{\alpha\beta}(W)$ is a rational function, hence in the case $K = \mathbb{R}$, it is smooth. □

1.4.4 Projective Hyperplane

Definition 39 (hyperplane, projective hyperplane)**.** *Consider the surjective map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to P^n(\mathbb{C})$. For any $a \in \mathbb{C}^{n+1} \setminus \{0\}$, define*

$$\tilde{H} = \{x \in \mathbb{C}^{n+1} : x \cdot a = 0\}$$

and

$$H = \pi(\tilde{H})$$

$\tilde{H}$ is called hyperplane and H is called projective hyperplane.

Proposition 15. *H is an algebraic submanifold of $P^n(\mathbb{C})$ of dimension $n-1$*

Proof. *todo* □

1.5 Complex compact manifolds

Theorem 7. *Let X be a connected compact complex holomorphic manifold and let $f \in \mathcal{O}(X)$. Then f is constant i.e. global holomorphic function on a manifold is constant.*

Proof. As X is compact, the function $F : X \to \mathbb{R}$ defined by $F(x) = |f(x)|$ attains its maximum r. Let $S = F^{-1}r$. S is necessary closed as the singleton r is closed. We will prove that S is also open, i.e. for every $x \in X$, x is an interior point.

Let $x \in S$, and $h : U \to U' \subseteq \mathbb{C}^n$ be a chart containing x. Suppose $h(x) = 0$. Let $B \subseteq U' \subseteq \mathbb{C}^n$ be an open ball centered at 0, for every $z \in B$, the function $g(\lambda) = fh^{-1}(\lambda z)$ is a holomorphic function on the open set $\{\lambda \in \mathbb{C} : \lambda z \in B\} \subseteq \mathbb{C}$ of one variable (λ) having its maximum absolute value at $\lambda = 0$. By maximum modulus principle, g is constant, so $f(x) = fh^{-1}(0) = g(0) = g(1) = fh^{-1}(z)$. That is, f is constant on the open neighbourhood $h^{-1}(B)$ of x, then x is an interior point. □

Corollary 2. *There is no compact holomorphic submanifold of $\mathbb{C}^n$ of positive dimension.*

Proof. Suppose $X \subseteq \mathbb{C}^n$ be a compact holomorphic submanifold. Let X_0 be a connected component of X, then X_0 is both open and closed in X. As X_0 is closed in a compact space X, X_0 is compact. As X_0 is open in X, it is a manifold. Hence, X_0 is a connected compact complex holomorphic manifold. Since X is a submanifold of $\mathbb{C}^n$, it inherits the charts of $\mathbb{C}^n$. The open set X_0 of X inherits the charts of X. Hence, let $x = (x_1, x_2, ..., x_n) \in X_0$, then function $x \to x_i$ is complex holomorphic. As X_0 is connected, compact, complex holomorphic, it is a constant function. This shows that X_0 is a single point. $\dim X = \dim X_0 = 0$ □

Theorem 8 (Serre [5], Chow [6])**.** *A complex holomorphic submanifold of $P^n(\mathbb{C})$ is a projective algebraic manifold*

Definition 40 (projective algebraic manifold)**.** *A holomorphic manifold X has an embedding into $P^n(\mathbb{C})$ is called projective algebraic manifold. Moreover, as X is homeomorphic to a closed set in compact space $P^n(\mathbb{C})$, X is compact by defintion.*

Proposition 16. *The Grassmannian manifold $G_{k,n}(\mathbb{C})$ are projective algebraic manifold.*

Proof. We have shown that $G_{k,n}(\mathbb{C})$ is an algebraic manifold. We will show that $G_{k,n}(\mathbb{C})$ is projective, i.e. $G_{k,n}(\mathbb{C})$ is a closed subset of $P^N(\mathbb{C})$ for some $N \in \mathbb{N}$.

Let $V \in G_{k,n}(\mathbb{C})$ be a k-dimensional subspace of $\mathbb{C}^n$. Let $B = \{v_1, ..., v_k\}$ be a basis of V, then $\wedge^k v_i$ is a non-zero vector in $\wedge^k \mathbb{C}^n$. Define a function $t : G_{k,n}(\mathbb{C}) \to P(\wedge^k \mathbb{C}^n)$ by

$$t(V) = \left[\wedge^k v_i\right]$$

t is well-defined and it is an injection that maps k-dimensional subspace of $\mathbb{C}^n$ to a line in $\wedge^k \mathbb{C}^n$. The map is called Plucker embedding.

finish the proof by showing $\operatorname{im} t$ is a submanifold of $P(\wedge^k \mathbb{C}^n)$: (1) t is injective (2) $\operatorname{im} t$ is closed (3) $\operatorname{im} t$ is an algebraic subset (4) $\operatorname{im} t$ is an algebraic submanifold □

[5] Fields medal in 1954, said he loved rock climbing at his 80 years of age
[6] A Chinese mathematician

1.6 Grassmannian is a complex algebraic manifold

this is merely the proof of previous part

1.7 The Category of manifolds

Definition 41 (Category)**.** *A category $\mathcal{C}$ consists of a collection of objects denoted by* $\mathrm{ob}\,\mathcal{C}$ *and for each ordered pair $A, B \in \mathrm{ob}\,\mathcal{C}$, there is a collection of morphisms denoted by $\mathrm{Hom}(A, B)$. A morphism $f \in \mathrm{Hom}(A, B)$ is also denoted by $f : A \to B$. Morphisms satisfy the following*

- *(composition) if $f : A \to B, g : B \to C$, then there exists a morphism $gf : A \to C$. That is, there exists a map*

$$\begin{aligned} \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) &\to \mathrm{Hom}(A, C) \\ (f, g) &\mapsto gf \end{aligned}$$

  for every $A, B, C \in \mathrm{ob}\,\mathcal{C}$

- *(associativity) if $f : A \to B$, $g : B \to C$, $h : C \to D$, then*

$$h(gf) = (hg)f$$

- *(identity) for every object $A \in \mathrm{ob}\,\mathcal{C}$, there exists a morphism $1_A : A \to A$ called identity such that for every morphism $f : A \to B$, $f = f 1_A$ and for every morphism $g : C \to A$, $g = 1_A g$*

Proposition 17 (the category of $\mathcal{S}$-manifold)**.** *The collection of $\mathcal{S}$-manifolds and $\mathcal{S}$-morphisms form a category.*

Chapter 2

Vector Bundles

2.1 Vector Bundles

Definition 42 (Wells - vector bundle)**.** *Let E, X be Hausdorff spaces and K be either $\mathbb{R}$ or $\mathbb{C}$. A continuous surjective map $\pi : E \to X$ is called a K-vector bundle of rank r if the following satisfy*

1. *for every $p \in X$, the fiber $E_p = \pi^{-1}p$ is a vector space of dimension r over K*
2. *there is an open cover $\mathcal{U}$ of X such that for every $U \in U$, there is a homeomorphism $h : \pi^{-1}U \to U \times K^r$ such that for every $p \in U$, $h(E_p) = \{p\} \times K^r$ and the composition h^p is a K-vector space isomorphism where $\{p\} \times K^r \to K^r$ is the canonical isomorphism.*

$$\begin{tikzcd}
\pi^{-1}U \arrow[r, "h"] \arrow[d, two heads, "\pi"] & U \times K^r \arrow[dl, dashed] \\
U &
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi^{-1}U \arrow[r, "h"] & U \times K^r & \\
E_q \arrow[u, hook] \arrow[r, "h"] \arrow[rr, dashed, bend right, "h^q"'] & \{q\} \times K^r \arrow[u, hook] \arrow[r, "\cong"] & K^r
\end{tikzcd}$$

The pair (U, h) is called local trivialization. The space E is called total space and the space X is called base space.

Definition 43 (transition function of vector bundle)**.** *Let $\pi : E \to X$ be a K-vector bundle of rank r. Let (U_α, h_α) and (U_β, h_β) be two local trivializations such that $U_\alpha \cap U_\beta \neq \varnothing$, then the restrictions below are also homeomorphism*

$$h_\alpha : \pi^{-1}(U_\alpha \cap U_\beta) \to (U_\alpha \cap U_\beta) \times K^r$$
$$h_\beta : \pi^{-1}(U_\alpha \cap U_\beta) \to (U_\alpha \cap U_\beta) \times K^r$$

then, we have an homeomorphism $h_\beta h_\alpha^{-1} : (U_\alpha \cap U_\beta) \times K^r \to (U_\alpha \cap U_\beta) \times K^r$. For each $q \in U_\alpha \cap U_\beta$, the composition $g_{\beta\alpha}(q) = h^q_\beta (h^q_\alpha)^{-1}$ is a K-vector space isomorphism

$$\begin{tikzcd}
 & (U_\alpha \cap U_\beta) \times K^r \arrow[rr, dashed, bend left, "h_\beta h_\alpha^{-1}"] & \pi^{-1}(U_\alpha \cap U_\beta) \arrow[l, "h_\alpha"'] \arrow[r, "h_\beta"] & (U_\alpha \cap U_\beta) \times K^r & \\
K^r \arrow[rrrr, dashed, bend right, "g_{\beta\alpha}(q)"'] & \{q\} \times K^r \arrow[l, "\cong"'] \arrow[u, hook] & E_q \arrow[l, "h_\alpha"'] \arrow[r, "h_\beta"] \arrow[u, hook] & \{q\} \times K^r \arrow[r, "\cong"] \arrow[u, hook] & K^r
\end{tikzcd}$$

We can also write $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(K^r)$. The function $g_{\alpha\beta}$ is called transition function of the K-vector bundle $\pi : E \to X$

Proposition 18. *Given K-vector bundle, on $U_\alpha \cap U_\beta \cap U_\gamma$, $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = I$*

Proof. For any $q \in U_\alpha \cap U_\beta \cap U_\gamma$,

$$g_{\alpha\beta}(q) g_{\beta\gamma}(q) g_{\gamma\alpha}(q) = h^q_\alpha (h^q_\beta)^{-1} h^q_\beta (h^q_\gamma)^{-1} h^q_\gamma (h^q_\alpha)^{-1} = I$$

□

Proposition 19. *If X is a manifold, then $\pi : E \to X$ makes E being a manifold.*

Proof.

1. Constructing new atlas on X

   For each $x \in X$, let $\phi_x : V_x \to V'_x \subseteq K^n$ be a chart containing x, let $h_x : \pi^{-1} W_x \to W_x \times K^r$ be a local trivialization containing x. Let $U_x = V_x \cap W_x$ be open in V_x, then the restriction $\phi_x : U_x \to f_x(U_x) \subseteq K^n$ is a chart compatible with the atlas of X, the restriction $h_x : \pi^{-1} U_x \to U_x \times K^r$ is a local trivialization of $\pi : E \to X$.

2. Atlas on X induces an atlas on E.

   U_x is open and $\pi : E \to X$ is continuous, so $\{\pi^{-1} U_x\}_{x \in X}$ is an open cover of E. U_x is open and $\phi_x : U_x \to U'_x$ is an open map, so U'_x is open. Then, the composition

   $$(\phi_x \times 1) h_x : \pi^{-1} U_x \to U_x \times K^r \to U'_x \times K^r \subseteq \mathbb{R}^{n+r}$$

   is a homeomorphism. That makes an atlas on E

3. Construct a countable basis on X so that each basic open set is contained in one chart.

   Lemma 2. *If X is a second-countable space with an open cover $\{U_i\}_{i \in I}$, then there exists a countable basis $\{V_N\}_{N \in \mathbb{N}}$ so that each V_N is a subset of one of U_i.*

   Let $O \subseteq X$ be any open set, then $O = \bigcup_{i \in I} O \cap U_i$. Each $O \cap U_i \subseteq U_i$ is open, so it can be written as a union of sets in $\{V_N\}_{N \in \mathbb{N}}$ where each set is contained in U_i. Therefore, O can be written as a union of sets in $\{V_N\}_{N \in \mathbb{N}}$ where each set is contained in one of U_i

4. E is Hausdorff and second-countable

   E is Hausdorff by the premise. Let $\{W_M\}_{M \in \mathbb{N}}$ be a countable basis of K^r, $\{V_N\}_{N \in \mathbb{N}}$ be a countable basis of X so that each V_N is contained in one of U_x. We will show that $\{h^{-1}_{x(N)}(V_N \times W_M)\}_{(N,M) \in \mathbb{N} \times \mathbb{N}}$ where $x : \mathbb{N} \to X$ is a map that assigns each V_N to one U_x

   Let $O \subseteq E$ be an open set, write

   $$O = \bigcup_{x \in X} O \cap \pi^{-1} U_x$$

   Each $O \cap \pi^{-1} U_x$ is homeomorphic to an open set in $U_x \times K^r$ and that open set can be written as a union of sets in $\{V_N \times W_M\}_{(N,M) \in \mathbb{N} \times \mathbb{N}}$. Therefore, $O \cap \pi^{-1} U_x$ can be written as a union of sets in $\{h^{-1}_{x(N)}(V_N \times W_M)\}_{(N,M) \in \mathbb{N} \times \mathbb{N}}$.

5. If X is a manifold, then $\pi : E \to X$ makes E being a manifold

   We have proved that E is Hausdorff, second-countable and constructed an atlas on E

6. Some words on transition function

   Let $U_x \cap U_y \neq \varnothing$, then the manifold transition function $t_{xy} : (U'_x \cap U'_y) \times K^r \to (U'_x \cap U'_y) \times K^r$ is defined by

   $$t_{xy} = (\phi_y \times 1) h_y h_x^{-1} (\phi_x \times 1)^{-1}$$

$$\begin{tikzcd}
U_x \times K^r & \pi^{-1} U_x \arrow[l, "h_x"'] & & \pi^{-1} U_y \arrow[r, "h_y"] & U_y \times K^r \\
(U'_x \cap U'_y) \times K^r \arrow[rrrr, bend right, "t_{xy}"'] & (U_x \cap U_y) \times K^r \arrow[l, "\phi_x \times 1"'] \arrow[ul, hook] & \pi^{-1}(U_x \cap U_y) \arrow[l, "h_x"'] \arrow[r, "h_y"] \arrow[ul, hook] \arrow[ur, hook] & (U_x \cap U_y) \times K^r \arrow[r, "\phi_y \times 1"] \arrow[ur, hook] & (U'_x \cap U'_y) \times K^r
\end{tikzcd}$$

$\square$

Definition 44 ($\mathcal{S}$-bundle)**.** *A K-vector bundle $\pi : E \to X$ of rank r is called $\mathcal{S}$-bundle if E and X are $\mathcal{S}$-manifold, π is an $\mathcal{S}$-morphism, and local trivializations are $\mathcal{S}$-isomorphisms. Note that, local trivializations are $\mathcal{S}$-isomorphisms if and only if transition functions $g_{ji} : U_i \cap U_j \to GL(K^r)$ are $\mathcal{S}$-morphisms.*

Remark 14 (constructing vector bundle from transition function $g_{\alpha\beta}$)**.** *Given a collection of open sets $\{U_i\}_{i\in I}$ on an $\mathcal{S}$-manifold X and for each $U_i \cap U_j \neq \varnothing$, there is an $\mathcal{S}$-function*

$$g_{ij} : U_i \cap U_j \to GL(K^r)$$

so that $g_{ii} = I_r$ for all $i \in I$ and if $U_i \cap U_j \cap U_k \neq \varnothing$, then $g_{ij}g_{jk}g_{ki} = I_r$. Then, there exists a $\mathcal{S}$-bundle $\pi : E \to X$ having transition function g_{ij} on $U_i \cap U_j$. Moreover, if two vector bundle $\pi_E : E \to X$ and $\pi_F : F \to X$ have the same transition functions, then they are $\mathcal{S}$-bundle isomorphic.

Proof.

1. Make $\{U_i\}_{i\in I}$ an atlas

   For each $x \in X$, let $\{V_x, \phi_x\}$ be a chart and $U_x \subseteq \{U_i\}_{i\in I}$ containing x. Then, let $W_x = V_x \cap U_x$. We have an atlas $\{W_x, \phi_x\}_{x\in X}$ such that there is a transition function $g_{yx} : U_x \cap U_y \to GL(K^r)$. From now, we assume that there is a chart $\phi_i : U_i \to U_i' \subseteq K^n$ for each $i \in I$

2. Construction of E

   Let
   $$\tilde{E} = \coprod_{i\in I}(U_i \times K^r)$$
   be the disjoint union of product of U_i and K^r. Define an equivalence on $\tilde{E}$ as follows: for $(x, v) \in U_\beta \times K^r$ and $(y, w) \in U_\alpha \times K^r$, $(x, v) \sim (y, w)$ if and only $y = x$ and $w = g_{\beta\alpha}v$. Let $E = \tilde{E}/\sim$ be the set of equivalence classes. Define the quotient topology on E from the surjection $p : \tilde{E} \twoheadrightarrow E$ that sends $(x, v) \mapsto [x, v]$. Define the surjection $\pi : E \to X$ by $\pi([x, v]) = x$.

   $$\begin{tikzcd} \tilde{E} \arrow[r, two heads, "p"'] \arrow[rr, bend left, two heads, "\pi p"] & E \arrow[r, two heads, "\pi"'] & X \end{tikzcd}$$

   We will show that, $\pi : E \to X$ is indeed a $\mathcal{S}$-bundle.

3. Construct an atlas and local trivializations on E

   As $\pi p : \tilde{E} \to X$ is continuous, for any $i \in I$, U_i is open in X implies $p^{-1}\pi^{-1}U_i$ is open in $\tilde{E}$, so $\pi^{-1}U_i$ is open in E. Therefore, $\{\pi^{-1}U_i\}_{i\in I}$ is a open cover of E. Furthermore,
   $$p^{-1}\pi^{-1}U_i = (U_i \times K^r) \amalg \left(\coprod_{j\in I: j\neq i, U_j\cap U_i\neq\varnothing}((U_j \cap U_i) \times K^r)\right)$$
   Since each $g_{ii} = I$, every distinct pair $(x^{(1)}, v_i^{(1)}), (x^{(2)}, v_i^{(2)}) \in U_i \times K^r$ are not equivalent. Since g_{ij} is invertible for each $j \neq i$, every point $(x, v_j) \in (U_j \cap U_i) \times K^r$ is equivalent to one point in $U_i \times K^r$, namely $(x, g_{ij}(x)v_j)$. Then, there exists a bijection $h_i : \pi^{-1}U_i \to U_i \times K^r$.

   $$\begin{tikzcd} p^{-1}\pi^{-1}U_i \arrow[r, two heads, "p"'] \arrow[rr, bend left, "h_i p"] & \pi^{-1}U_i \arrow[r, "h_i"'] \arrow[d, "\pi"] & U_i \times K^r \\ & U_i & \end{tikzcd}$$

   and the composition $h_i p : p^{-1}\pi^{-1}U_i \to U_i \times K^r$ is the projection into $U_i \times K^r$. Moreover, the bijection is homeomorphism. h_i is continuous because for each open set in $U \times O \in U_i \times K^r$ where O is open in K^r and U is open in U_i, then
   $$p^{-1}h_i^{-1}(U \times O) = \coprod_{j\in I: U_j\cap U_i\neq\varnothing}(U \times g_{ij}O)$$
   is open. Therefore, $h_i^{-1}(U \times O)$ is open. h_i is open map because for each open set $O \subseteq \pi^{-1}U_i$, $p^{-1}O$ is open so $h_iO = (h_ip)(p^{-1}O)$ is also open due to projection from coproduct topology.

   h_i is a local trivializations and $(\phi_i \times 1)h_i : \pi^{-1}U_i \to U_i' \times K^r \subseteq K^n \times K^r$ is a chart.

4. $\pi^{-1}x$ is a vector space for all $x \in X$

   Using the same argument, we have the homeomorphism $\pi^{-1}x \cong \{x\} \times K^r$ for $x \in X$ and the map $h^x : \pi^{-1}x \to K^r$ is the canonical projection map. Take the vector space structure of $\pi^{-1}x$ to be induced from K^r, i.e. $h^x = I$

5. E is Hausdorff

   Let $[x_1, v_1], [x_2, v_2] \in E$

   If $[x_1, v_1], [x_2, v_2]$ belong to the same $\pi^{-1}U_i$, $\pi^{-1}U_i \cong U_i \times K^r$ is a Hausdorff space, there exists open sets separating $[x_1, v_1], [x_2, v_2]$.

   If $x_1 \neq x_2$, let $x_1 \in A_1 \subseteq U_1, x_2 \in A_2 \subseteq U_2$, so that $A_1 \cap A_2 = \varnothing$. This is possible since X is Hausdorff. Then $p(A_1 \times K^r)$ and $p(A_2 \times K^r)$ are disjoint in E since every $a \in A_1, b \in A_2$ are different.

6. $\pi : X \to E$ is a vector bundle

   From the above, $\pi : X \to E$ is a vector bundle.

7. E is a manifold

   Because $\pi : X \to E$ is a vector bundle and X is a manifold.

8. E is $\mathcal{S}$-manifold

   From the above, the manifold transition function on E is $t_{ij} : (U'_i \cap U'_j) \times K^r \to (U'_i \cap U'_j) \times K^r$

   $$t_{ij} = (\phi_j \times 1) h_j h_i^{-1} (\phi_i \times 1)^{-1}$$

   So that

   $$\begin{aligned} t_{ij} : (U'_i \cap U'_j) \times K^r &\to (U'_i \cap U'_j) \times K^r \\ (u, v) &\mapsto (\phi_j \phi_i^{-1} u, (g_{ji} \phi_i^{-1} u)(v)) \end{aligned}$$

   We will prove that $(u, v) \mapsto (g_{ji}\phi_i^{-1}u)(v)$ is an $\mathcal{S}$-function $(U'_i \cap U'_j) \times K^r \to K^r$. Note that $g_{ji}\phi_i^{-1} : U'_i \cap U'_j \to GL(K^r)$ is an $\mathcal{S}$-function as it is a composition of $\mathcal{S}$-functions.

   Lemma 3. *$g_{ji} : U_i \cap U_j \to GL(K^r)$ is an $\mathcal{S}$-function if and only if h_i, h_i are $\mathcal{S}$-isomorphisms.*

   Proof. Let $f_{mn} : K^{r \times r} \to K$ be defined by taking the (m, n) entry of a matrix in $K^{r \times r}$. Consider the function $f_{mn} g_{ji} \phi_i^{-1} : U'_i \cap U'_j \to K$. Since $GL(K^r)$ is an open set in $K^{r \times r}$, then f_{mn} is an $\mathcal{S}$-function.

   (a) $g_{ji} : U_i \cap U_j \to GL(K^r)$ is an $\mathcal{S}$-function implies h_i, h_i are $\mathcal{S}$-isomorphisms.
   If $g_{ji} : U_i \cap U_j \to GL(K^r)$ is an $\mathcal{S}$-function, then $f_{mn} g_{ji} \phi_i^{-1} : U'_i \cap U'_j \to K$ is an $\mathcal{S}$-function for each $(m, n) \in [r] \times [r]$. Let $u \in U'_i \cap U'_j \subseteq K^n$, then view $GL(K^r)$ as a collection of matrices, each entry of the $r \times r$ matrix $g_{ji}\phi_i^{-1}u$ is a $\mathcal{S}$-function of u. Let $v \in K^r$, then view K^r as a collection of vectors, each entry of the r-dimensional vector $(g_{ji}\phi_i^{-1}u)(v)$ is an $\mathcal{S}$-function of (u, v). Then, $t_{ij} : (U'_i \cap U'_j) \times K^r \to (U'_i \cap U'_j) \times K^r$ is an $\mathcal{S}$-function. Hence, $h_j h_i^{-1}$ is an $\mathcal{S}$-morphism. From the other direction, $h_i h_j^{-1}$ is also an $\mathcal{S}$-morphism. As the map is homeomorphism, $h_j h_i^{-1}$ and $h_i h_j^{-1}$ are $\mathcal{S}$-isomorphism. One can show that each h_i and h_j are $\mathcal{S}$-isomorphism.

   (b) $g_{ji} : U_i \cap U_j \to GL(K^r)$ is an $\mathcal{S}$-function is implied by h_i, h_i are $\mathcal{S}$-isomorphisms.
   TODO

   □

9. $\pi : E \to X$ is an $\mathcal{S}$-morphism

   Let $f : U \to K$ be an $\mathcal{S}$-function on an open set $U \subseteq X$. Suppose $U \subseteq U_i$, then $f\pi : \pi^{-1}U \to K$ is defined by

   $$f\pi(x, v) = f(x)$$

   This is an $\mathcal{S}$-function. Extend to the case where U intersects with multiple $U_i \subseteq X$.

10. $(\phi_i \times 1)h_i : \pi^{-1}U_i \to U'_i \times K^r \subseteq K^n \times K^r$ is $\mathcal{S}$-isomorphism.

    Because it is a composition of two $\mathcal{S}$-isomorphisms.

11. $\pi_E : E \to X$, $\pi_F : F \to X$ are $\mathcal{S}$-bundle isomorphic.

    Let $U \subseteq X$ be an open set. Suppose $U \subseteq U_i \cap U_j$, then we have the following commutative diagram

$$\begin{tikzcd}
& \pi_E^{-1}U \arrow[dl, "h_i^E"'] \arrow[dr, "h_j^E"] \arrow[dd, dashed] & \\
U \times K^r \arrow[rr, dashed] & & U \times K^r \\
& \pi_F^{-1}U \arrow[ul, "h_i^F"] \arrow[ur, "h_j^F"'] &
\end{tikzcd}$$

where the diagonal map $U \times K^r \to U \times K^r$ is defined by

$$(u, v) \mapsto (u, g_{ji}(u)(v))$$

Let $f : \pi_E^{-1}U \to \pi_F^{-1}U$ be defined by $f = (h_i^F)^{-1}h_i^E$. From the commutativity, we can extend $f : \pi_E^{-1}U \to \pi_F^{-1}U$ into $f : E \to F$. *TODO - check*

$\square$

Definition 45 (trivial bundle)**.** *Let M be an $\mathcal{S}$-manifold then $\pi : M \times K^r \to M$ defined by $\pi(m, x) = m$ is an $\mathcal{S}$-bundle called trivial bundle.*

2.2 Tangent Spaces

Definition 46 (direct limit or injective limit (restricted context))**.** *Let M be a topological space. For every open set $U \subseteq M$, let $\mathcal{F}(U)$ be some collection of K-valued functions on U. Let $p \in M$, define the equivalence as follows: two functions $f : U \to K$, $g : V \to K$ where U, V are two open neighbourhoods p are equivalent if there exists an open set $W \subseteq U \cap V$ being an open neighbourhood of p such that the restrictions on W agree. We write $f \sim g$. Form the disjoint union*

$$P = \coprod_{U \subseteq M : U \text{ open}, p \in U} \mathcal{F}(U)$$

The direct limit of $\mathcal{F}$ at p is the equivalence class

$$P/\sim = \coprod_{U \subseteq M : U \text{ open}, p \in U} \mathcal{F}(U)/\sim$$

and denoted by $\varinjlim_{U \subseteq M : U \text{ open}, p \in U} \mathcal{F}(U)$

direct limit under the partial order of open set containment

Remark 15 (filter and direct limit)**.** *In a partial ordered set $(P, \leq)$, a filter $\mathcal{F}$ is a subset with the following properties*

1. *$\mathcal{F}$ is non-empty*
2. *if $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then there exists $C \in \mathcal{F}$ such that $C \leq A$ and $C \leq B$*
3. *if $A \in \mathcal{F}$ and $D \in P$ such that $A \leq D$, then $D \in \mathcal{F}$*

Define $\leq$ in P as follows: $f \leq g$ if and only if domain of f is a subset of domain of g and f and g agree on the domain of f. Then, given any f, the collection of functions that is equivalent to f forms a filter. Moreover, it is a proper filter and an ultrafilter. Therefore, we can identify the collection of direct limits of $\mathcal{F}$ by the collection of corresponding ultrafilters.

Proposition 20 (algebra structure of direct limit)**.** *If $\mathcal{F}(U)$ forms a K-algebra for every open set $U \subseteq M$ with scalar multiplication, vector addition, vector multiplication defined by*

$$(af)(x) = af(x)$$
$$(f + g)(x) = f(x) + g(x)$$
$$(fg)(x) = f(x)g(x)$$

Then, $\varinjlim_{U \subseteq M : U \text{ open}, p \in U} \mathcal{F}(U)$ forms a K-algebra with scalar multiplication, vector addition, vector multiplication defined by

$$a[f] = [af]$$
$$[f] + [g] = [f + g]$$
$$[f][g] = [fg]$$

For any open set $U \subseteq M$, the map $phi : \mathcal{F}(U) \to \varinjlim_{U \subseteq M : U \text{ open}, p \in U} \mathcal{F}(U)$ defined by

$$f \mapsto [f]$$

is a K-algebra homomorphism

Definition 47 (direct limit of smooth functions, germs of differentiable functions). *Let M be a smooth manifold, $p \in M$, and U be an open neighbourhood of p. Let $\mathcal{E}_M(U)$ be the space of smooth functions on U. Define*

$$\mathcal{E}_{M,p} = \varinjlim_{U \subseteq M : U \, open, p \in U} \mathcal{E}(U)$$

The set $\mathcal{E}_{M,p}$ is an $\mathbb{R}$-algebra and is called the algebra of germs of differentiable functions. In other words, each germ in $\mathcal{E}_{M,p}$ is a collection of functions where each function defined on an open neighbourhood of p and any two functions agree on some open neighbourhood of p

what is the dimension of $\mathcal{E}_{M,p}$

Proposition 21. *Some propositions*

1. *Given an open neighbourhood U_i of p, $\mathcal{E}_{M,p}$ gives an algebra homomorphism*

$$\phi_i : \mathcal{E}_M(U_i) \to \mathcal{E}_{M,p}$$

2. *Let $f \in \mathcal{E}_{M,p}$ be a germ, then there exists an open neighbourhood U_1 of p and a function $f_1 \in \mathcal{E}_M(U_1)$ such that $f = \phi(f_1)$*

3. *Let $U_1, U_2 \subseteq U$ be two open neighbourhoods of p, we have two maps ϕ_1, ϕ_2. Let $f_1 \in \mathcal{E}_M(U_1)$ and $f_2 \in \mathcal{E}_M(U_2)$, suppose $f = \phi_1(f_1) = \phi_2(f_2)$, show that there exists $U_3 \subseteq U_1 \cap U_2$ being an open neighbourhood of p such that $f_1|_{U_3} = f_2|_{U_3}$*

4. *Let $f \in \mathcal{E}_{M,p}$. Define $f(p) = f_1(p) \in \mathbb{R}$ for some $f_1 \in \mathcal{E}_M(U_1)$ such that $\phi_1(f_1) = f$. Show that $f(p)$ is independent of the choice of f_1*

Proof. $f(p)$ is independent of the choice of f_1: Let $f_2 \in \mathcal{E}_M(U_2)$ such that $\phi_2(f_2) = f$. As $\phi_1(f_1) = \phi_2(f_2)$, then f_1 and f_2 belong to the same equivalence class, then there exists $U \subseteq U_1 \cap U_2$ such that $p \in U$ and $f_1|_U = f_2|_U$, then $f_1(p) = f_2(p)$ $\square$

Definition 48 (derivation, tangent space). *A derivation of the $\mathbb{R}$-algebra $\mathcal{E}_{M,p}$ is a linear map $D : \mathcal{E}_{M,p} \to \mathbb{R}$ satisfying Leibniz's law:*

$$D(fg) = D(f)g(p) + f(p)D(g)$$

for all $f, g \in \mathcal{E}_{M,p}$. The tangent space at $p \in M$ is the vector space of all derivations of $\mathcal{E}_{M,p}$, denoted by $T_p(M)$ which is a subspace of the vector space of all linear maps $\mathcal{E}_{M,p} \to \mathbb{R}$

Proposition 22 (tangent space on $\mathbb{R}^n$). *Let $\mathbb{R}^n$ be a manifold equipped with the usual chart. Let $p \in \mathbb{R}^n$, the tangent space $T_p(\mathbb{R}^n)$ is a $\mathbb{R}$-vector space of dimension n spanned by*

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_p, \left.\frac{\partial}{\partial x_2}\right|_p, ..., \left.\frac{\partial}{\partial x_n}\right|_p \right\}$$

Proof. $\left.\frac{\partial}{\partial x_i}\right|_p : \mathcal{E}_{\mathbb{R}^n,p} \to \mathbb{R}$ defined by

$$\left.\frac{\partial}{\partial x_i}[f]\right|_p = \left.\frac{\partial f}{\partial x_i}\right|_p$$

for all $[f] \in \mathcal{E}_{\mathbb{R}^n,p}$ is a derivation of $\mathcal{E}_{\mathbb{R}^n,p}$. We will show that

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_p, \left.\frac{\partial}{\partial x_2}\right|_p, ..., \left.\frac{\partial}{\partial x_n}\right|_p \right\}$$

is exactly the basis of $T_p(\mathbb{R}^n)$. Let $D \in T_p(\mathbb{R}^n)$, let $1 : \mathbb{R}^n \to \mathbb{R}$ be the constant function 1, then

$$D(1) = D(1 \cdot 1) = D(1)1 + 1D(1) = 2D(1)$$

Then, $D(1) = 0$. Let $f \in \mathcal{E}_{M,p}$, by Taylor theorem,

$$f(x) = f(p) + \sum_{i=1}^n \left(\left.\frac{\partial f}{\partial x_i}\right|_p \right)(x_i - p_i) + \sum_{i,j \in [n]^2} g_{ij}(x_i - p_i)(x_j - p_j)$$

where $x_i - p_i$ is the germ of a function that maps $a = (a_1, a_2, ..., a_n) \in \mathbb{R}^n$ to $a_i - p_i \in \mathbb{R}$ and g_{ij} is a germ of smooth function $\mathbb{R}^n \to \mathbb{R}$. Apply D on both side

$$Df = f(p)D(1) + \sum_{i=1}^n \left(\left.\frac{\partial f}{\partial x_i}\right|_p \right) D(x_i - p_i) + \sum_{i,j \in [n]^2} D(g_{ij}(x_i - p_i)(x_j - p_j))$$

We have

$$D(g_{ij}(x_i - p_i)(x_j - p_j)) = D(g_{ij}(x_i - p_i))(x_j - p_j)(p) + (g_{ij}(x_i - p_i))(p)D((x_j - p_j))$$

As $(x_j - p_j)(p) = 0$ and $(g_{ij}(x_i - p_i))(p) = 0$, then $D(g_{ij}(x_i - p_i)(x_j - p_j)) = 0$. Therefore,

$$Df = \sum_{i=1}^{n} \left(\left. \frac{\partial f}{\partial x_i} \right|_p \right) D(x_i - p_i)$$

As $D(x_i - p_i)$ is a constant, D is a linear combination of $\left\{ \left. \frac{\partial}{\partial x_1} \right|_p, \left. \frac{\partial}{\partial x_2} \right|_p, ..., \left. \frac{\partial}{\partial x_n} \right|_p \right\}$

□

Remark 16 (tangent space on $\mathbb{C}^n$). *TODO*

Definition 49 (Jacobian). *Suppose $f : M \to N$ is a smooth map between smooth manifolds. Let $p \in M$ and $q = f(p) \in N$. Let $U \subseteq M$ be an open neighbourhood of p and $V \subseteq N$ be an open neighbourhood of q such that $f(U) \subseteq V$. Given any $\phi \in \mathcal{E}_N(V)$, by definition of manifold homomorphism, f induces a smooth map $\phi f i \in \mathcal{E}_M(U)$*

$$\begin{tikzcd} U \arrow[r, hook, "i"] \arrow[drr, dashed, "\phi f i \in \mathcal{E}_M(U)"'] & f^{-1}(V) \arrow[r, two heads, "f"] & V \arrow[d, "\phi \in \mathcal{E}_N(V)"] \\ & & \mathbb{R} \end{tikzcd}$$

That is, f induces a linear map $f^* : \mathcal{E}_N(V) \to \mathcal{E}_M(U)$ between spaces of smooth functions, that induces a linear map $f^*_p : \mathcal{E}_{N,q} \to \mathcal{E}_{M,p}$ between spaces of germs of smooth functions

$$\begin{tikzcd} \mathcal{E}_N(V) \arrow[r] \arrow[d, two heads] & \mathcal{E}_M(U) \arrow[d, two heads] \\ \mathcal{E}_{N,q} \arrow[r, dashed, "f^*_p"] & \mathcal{E}_{M,p} \end{tikzcd}$$

We define the linear map $df_p : T_p(M) \to T_q(N)$ by $D \mapsto Df^*_p$ for all $D \in T_p(M)$

$$\begin{tikzcd} \mathcal{E}_{N,q} \arrow[r, "f^*_p"] \arrow[dr, dashed, "df_p D = Df^*_p"'] & \mathcal{E}_{M,p} \arrow[d, "D"] \\ & \mathbb{R} \end{tikzcd}$$

The linear map df_p is called Jacobian of f at p

Proposition 23. *$df_p : T_p(M) \to T_p(N)$ is well-defined and linear.*

Proposition 24 (reconcile with Jacobian in calculus). *If $M = \mathbb{R}^m, N = \mathbb{R}^n$ equipped with the usual manifold structure, let $f : M \to N$ be a smooth map, let $p \in M$ and $q = f(p) \in N$. Then the Jacobian can be written as a matrix $J \in \mathbb{R}^{n \times m}$*

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & ... & \frac{\partial f_1}{\partial x_m} \\ ... & ... & ... \\ \frac{\partial f_n}{\partial x_1} & ... & \frac{\partial f_n}{\partial x_m} \end{bmatrix}$$

Proof. We will show that $df_p \left(\left. \frac{\partial}{\partial x_i} \right|_p \right)$ can be written as a linear combination of basis of $T_q(N) = \left\{ \left. \frac{\partial}{\partial y_1} \right|_q, ..., \left. \frac{\partial}{\partial y_n} \right|_q \right\}$ with coefficients $\left\{ \left. \frac{\partial f_1}{\partial x_i} \right|_p, ..., \left. \frac{\partial f_n}{\partial x_i} \right|_p \right\}$, that is, for every $\phi \in \mathcal{E}_N(V)$, $[\phi] \in \mathcal{E}_{N,p}$,

$$df_p \left(\left. \frac{\partial}{\partial x_i} \right|_p \right) [\phi] = \sum_{j=1}^{n} \left. \frac{\partial \phi}{\partial y_j} \right|_q \left. \frac{\partial f_j}{\partial x_i} \right|_p$$

We have

$$\begin{aligned}
df_p\left(\left.\frac{\partial}{\partial x_i}\right|_p\right)[\phi] &= \left.\frac{\partial}{\partial x_i}\right|_p f_p^*[\phi] && \text{(definition of Jacobian)}\\
&= \left.\frac{\partial}{\partial x_i}\right|_p [f^*\phi] && \text{(definition of } f_p^* : \mathcal{E}_{N,q} \to \mathcal{E}_{M,p})\\
&= \left.\frac{\partial}{\partial x_i}\right|_p [\phi f i] && \text{(definition of } f^* : \mathcal{E}_N(V) \to \mathcal{E}_M(U))\\
&= \left.\frac{\partial(\phi f i)}{\partial x_i}\right|_p && \text{(definition of } \left.\frac{\partial}{\partial x_i}\right|_p)
\end{aligned}$$

Note that $fi = f|_U : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ and $\phi : V \subseteq \mathbb{R}^n \to \mathbb{R}$, by chain rule

$$\begin{aligned}
df_p\left(\left.\frac{\partial}{\partial x_i}\right|_p\right)[\phi] &= \left.\frac{\partial(\phi f i)}{\partial x_i}\right|_p\\
&= \sum_{j=1}^n \left.\frac{\partial \phi}{\partial y_j}\right|_{y=fi(p)=q} \left.\frac{\partial fi}{\partial x_i}\right|_{x=p}\\
&= \sum_{j=1}^n \left.\frac{\partial \phi}{\partial y_j}\right|_q \left.\frac{\partial f}{\partial x_i}\right|_p
\end{aligned}$$

□

Remark 17 (basis of a tangent space on real smooth M)**.** *Given a real smooth manifold M, let $\phi^{-1} : U' \to U$ where $U' \subseteq \mathbb{R}^n$ be a chart, the ϕ^{-1} is a smooth isomorphism between two manifolds. Let $\left\{\left.\frac{\partial}{\partial x_1}\right|_{\phi(p)}, ..., \left.\frac{\partial}{\partial x_n}\right|_{\phi(p)}\right\}$ be a basis of $T_{\phi(p)}(\mathbb{R}^n)$ and $d\phi^{-1}{}_{\phi(p)} : T_{\phi(p)}(\mathbb{R}^n) \to T_p(M)$, then*

$$\{D_1, D_2, ..., D_n\} = \left\{d\phi^{-1}{}_{\phi(p)} \left.\frac{\partial}{\partial x_1}\right|_{\phi(p)}, ..., d\phi^{-1}{}_{\phi(p)} \left.\frac{\partial}{\partial x_n}\right|_{\phi(p)}\right\}$$

is a basis of $T_p(M)$.

Proposition 25 (chain rule)**.** *Let $f : M \to N$, $g : N \to P$ be smooth maps between manifolds. Then*

$$d(gf)_p = dg_{f(p)} df_p$$

Proof. *TODO* □

2.3 Tangent Bundles

Proposition 26 (dimension of tangent space)**.** *Let M be a smooth manifold of dimension n. Let $h : U \to U' \subseteq \mathbb{R}^n$ be a chart with $p \in U$. Then, h is a smooth map between smooth manifolds where $\mathbb{R}^n$ is equipped with usual atlas, so there exists a linear map*

$$dh_p : T_p(M) \to T_{h(p)}(\mathbb{R}^n)$$

Furthermore, as h is a homeomorphism, all maps in the construction of dh_p are isomorphisms, so dh_p is an isomorphism, therefore

$$\dim T_p(M) = \dim T_{h(p)}(\mathbb{R}^n) = n$$

Definition 50 (tangent bundle of an open set in $\mathbb{R}^n$ as a smooth manifold)**.** *Let $M \subseteq \mathbb{R}^n$ be an open set equipped with the usual atlas. Let*

$$T(M) = \coprod_{p \in M} T_p(M)$$

Note that, $T_p(M) = \mathrm{span}\{\frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_n}|_p\}$. We equip a topology on $T(M)$ as follows: define the set function as follows:

$$\begin{aligned}
\psi : T(M) &\to M \times \mathbb{R}^n\\
\left(p, \left(a_1 \left.\frac{\partial}{\partial x_1}\right|_p, ..., a_1 \left.\frac{\partial}{\partial x_n}\right|_p\right)\right) &\mapsto (p, (a_1, ..., a_n))
\end{aligned}$$

where $M \times \mathbb{R}^n$ is equipped with the product topology. Then, equip $T(M)$ with a topology such that $U \subseteq T(M)$ is open if and only if $\psi(U) \subseteq M \times \mathbb{R}^n$ is open. The canonical projection $\pi : T(M) \to M$ is a vector bundle and called tangent bundle of M

note, in this case, ψ is a set isomorphism so it is a homeomorphism, i.e. $T(M) \cong M \times \mathbb{R}^n$, the tangent bundle is a trivial bundle

Definition 51 (tangent bundle of a smooth manifold)**.** *Let M be a smooth manifold. Let*

$$T(M) = \coprod_{p \in M} T_p(M)$$

and $\pi : T(M) \to M$ be the canonical projection. Let $\{U_\alpha, \phi_\alpha\}$ be an atlas of M then let

$$T(U_\alpha) = \pi^{-1} U_\alpha = \coprod_{p \in U_\alpha} T_p(M) \subseteq T(M)$$

The smooth homeomorphism $\phi_\alpha : U_\alpha \to U'_\alpha \subseteq K^n$ is a smooth map between manifolds induces an invertible linear map for any $p \in U_\alpha$, namely Jacobian

$$d\phi_{\alpha,p} : T_p(M) \to T_{\phi_\alpha(p)}(\mathbb{R}^n)$$

That induces an isomorphism in sets

$$h_\alpha : \left(T(U_\alpha) = \coprod_{p \in U_\alpha} T_p(M) \right) \to \left(T(U'_\alpha) = \coprod_{\phi_\alpha(p) \in U'_\alpha} T_{\phi_\alpha(p)}(\mathbb{R}^n) \right)$$

As $U'_\alpha \subseteq \mathbb{R}^n$, equip $T(U'_\alpha)$ with the topology of the tangent bundle $T(U'_\alpha) \to U'_\alpha$ so that $T(U'_\alpha) \cong U'_\alpha \times \mathbb{R}^n \cong U_\alpha \times \mathbb{R}^n$. Equip the a topology on $T(U_\alpha)$ such that $U \subseteq T(U_\alpha)$ is open if and only $h_\alpha(U) \subseteq T(U'_\alpha)$ is open. We have the homeomorphism

$$h_\alpha : T(U_\alpha) \to U_\alpha \times \mathbb{R}^n$$

Equip the topology on $T(M)$ induced from topologies on $T(U_\alpha)$ for all α. Then, $T(M)$ is smooth manifold of dimension $2n$ with atlas $\{(U_\alpha, h_\alpha)\}$. The canonical projection $\pi : T(M) \to M$ is a vector bundle and is called tangent bundle.

Proposition 27. *$\pi : T(M) \to M$ is a smooth vector bundle.*

Proof.

1. $\pi : T(M) \to M$ is a bundle projection

   For each U_α, there is a homeomorphism $h_\alpha : T(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ and the composition $\pi h_\alpha^{-1} : U_\alpha \times \mathbb{R}^n \to U_\alpha$ is

   $$\pi h_\alpha^{-1}(x, v) = x$$

   which is the canonical projection. Therefore, $\pi : T(M) \to M$ is a bundle projection

2. $\pi : T(M) \to M$ is a vector bundle

   For each $x \in M$, $\pi^{-1} x = T_x(M)$ which is a real vector space of dimension n. By definition, the restriction of $h_\alpha : T(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ on $\{x\}$ is the Jacobian $h_\alpha|_{T_x(M)} = d_\phi : T_x(M) \to T_{\phi_\alpha(x)}(\mathbb{R}^n)$ which is a invertible linear map. Therefore, any transition is a composition of two invertible linear map, hence a linear map, i.e. $g_{ji} : U_i \cap U_j \to GL(K^r)$

3. vector bundle transition function is smooth

   The transition map $g_{ji} : U_i \cap U_j \to GL(K^r)$ is defined by

   $$p \mapsto (d\phi_{\beta,p})^{-1} d\phi_{\alpha,p}$$

   By chain rule, $(d\phi_{\beta,p})^{-1} d\phi_{\alpha,p} = d(\phi_\alpha \phi_\beta^{-1})_{\phi_\beta(p)}$ which is a map $T_{\phi_\beta(p)}(\mathbb{R}^n) \to T_{\phi_\alpha(p)}(\mathbb{R}^n)$, when viewed as a $n \times n$ matrix is a smooth function of p. $GL(K^r)$ is an open set in $\mathbb{R}^{n \times n}$, then $g_{ji} : U_i \cap U_j \to GL(K^r)$ is a smooth map.

4. $T(M)$ is a smooth manifold.

   As proved above, g_{ji} being smooth implies the manifold transition function on $T(M)$ being smooth.

5. $\pi : T(M) \to M$ is a smooth map

   As proved above.

6. $T(M)$ is a smooth vector bundle.

   All conditions have been checked

□

Definition 52 (tangent bundle of a holomorphic manifold)**.** *TODO*

2.4 Universal Bundles

Definition 53 (canonical vector bundle on Grassmannian)**.** *Let $G_{r,n}(K)$ be a Grassmannian*

$$G_{r,n}(K) = \{V \textit{ subspace of } K^n : \dim_V = r\}$$

Let

$$U_{r,n}(K) = \coprod_{V \in G_{r,n}(K)} V$$

be the disjoint union of all r-dimensional K-linear subspaces V in K^n. There is a canonical projection

$$\pi : U_{r,n}(K) \to G_{r,n}(K)$$

given by $pi(v) = V$ if $v \in V \subseteq K^n$. For $p \in G_{r,n}$, $\pi^{-1}p$ is a subspace of dimension r in K^n. Furthermore, to equip a topology on $U_{r,n}$, note that each V is a subset of K^n, consider the inclusion map

$$i : U_{r,n}(K) \hookrightarrow G_{r,n}(K) \times K^n$$

where $G \times K^n$ is equipped with the product topology. Let $U_{r,n}(K)$ inherit the subspace topology. That makes $\pi : U_{r,n}(K) \to G_{r,n}(K)$ a vector bundle which is called canonical vector bundle (or tautological bundle) over $G_{r,n}(K)$. Moreover it is an algebraic vector bundle (both $\mathbb{R}$ and $\mathbb{C}$)

two youtube videos - every vector bundle on compact Hausdorff space is a pullback of a canonical bundle over $G_{r,n}$ `https://www.youtube.com/watch?v=_nhVEKRi640` `https://www.youtube.com/watch?v=Z9RQhf220c4`

Remark 18 (case $r = 1$, $K = \mathbb{R}$)**.** *The vector bundle is*

$$\pi : U_{1,n}(\mathbb{R}) \to G_{1,n}(\mathbb{R}) = P^{n-1}(\mathbb{R})$$

For each $\alpha \in \{0, 1, ..., n-1\}$, let

$$U_\alpha = \{V = [x_0, x_1, ..., x_{n-1}] \subseteq P^{n-1}(\mathbb{R}) : x_\alpha \neq 0\}$$

The preimage under $\pi : U_{1,n}(\mathbb{R}) \to G_{1,n}(\mathbb{R})$ is

$$\pi^{-1}U_\alpha = \{(v, V) : V \in U_\alpha, v \in V\}$$

Define the local trivializations $h_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times \mathbb{R}$ as follows: each $v \in V \subseteq \mathbb{R}^n$ can be written uniquely as

$$v = t\frac{1}{x_\alpha}(x_0, x_1, ..., x_\alpha, ..., x_{n-1}) \in \mathbb{R}^n$$

for $t \in \mathbb{R}$ where $V = [x_0, x_1, ..., x_\alpha, ..., x_{n-1}]$, then the map $v \mapsto t$ is well-defined. Define

$$h_\alpha(v, V) = (V, t)$$

1. *$\pi : U_{1,n}(\mathbb{R}) \to G_{1,n}(\mathbb{R})$ is continuous*

   Any open set $O \subseteq G_{1,n}(\mathbb{R})$ have preimage under π being

$$\pi^{-1}O = \coprod_{V \in O} V$$

   On the other hand, $O \times \mathbb{R}$ is open in $G_{1,n}(\mathbb{R})$, then

$$\coprod_{V \in O} V = i^{-1}(O \times \mathbb{R})$$

   is open. Hence, π is continuous

2. *$\pi : U_{1,n}(\mathbb{R}) \to G_{1,n}(\mathbb{R})$ is surjective*

   obvious

3. *$U_{1,n}(\mathbb{R})$ is Hausdorff*

   obvious

4. $h_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times \mathbb{R}$ *is a homeomorphism.*

   TODO - somehow this was not discussed in Wells book - well, prof was right, don't take Wells too seriously

5. $\pi : U_{1,n}(\mathbb{R}) \to G_{1,n}(\mathbb{R})$ *is a bundle projection*

   From above

6. *Transition function of vector bundle* $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(\mathbb{R})$

$$g_{\beta\alpha}([x_0, ..., x_\alpha, ..., x_\beta, ..., x_{n-1}]) = \frac{x_\beta}{x_\alpha}$$

for every $[x_0, ..., x_\alpha, ..., x_\beta, ..., x_{n-1}] \in U_\alpha \cap U_\beta$*, that is,* $x_\alpha \neq 0, x_\beta \neq 0$*. The transition is algebraic, that is smooth, then the vector bundle is smooth. TODO - verify this by uniqueness of VB construction - skip this for now*

2.5 Homomorphisms and Direct Sums

Definition 54 (restriction of vector bundle)**.** *Let* $\pi : E \to X$ *be an* $\mathcal{S}$*-bundle and* U *is an open set of* X*, then the restriction of* E *to* U*, denoted by* $E|_U = \pi^{-1}(U)$ *induces an* $\mathcal{S}$*-bundle* $\pi|_{E|_U} : E|_U \to U$*.*

$$\begin{tikzcd} E|_U = \pi^{-1}U \arrow[r, hook] \arrow[d, "\pi"'] & E \arrow[d, two heads, "\pi"] \\ U \arrow[r, hook] & X \end{tikzcd}$$

Definition 55 (homomorphism of $\mathcal{S}$-bundles)**.** *Let* $\pi_E : E \to X$ *and* $\pi_F : F \to X$ *be* $\mathcal{S}$*-bundles over* X*. A homomorphism of* $\mathcal{S}$*-bundles* $f : E \to F$ *is an* $\mathcal{S}$*-morphism between total spaces which preserves fibers, i.e.* f *commutes with* π_E, π_F *and* K*-linear on each fiber.*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, two heads, "\pi_E"'] & F \arrow[dl, two heads, "\pi_F"] \\ X & \end{tikzcd} \qquad \begin{tikzcd} K^r \arrow[r, "linear"] & K^r \\ E_p \arrow[u, "\cong"] \arrow[r, "f"] \arrow[d, two heads, "\pi_E"'] & F_p \arrow[u, "\cong"'] \arrow[dl, two heads, "\pi_F"] \\ \{p\} & \end{tikzcd}$$

Definition 56 (isomorphism of $\mathcal{S}$-bundles)**.** *A* $\mathcal{S}$*-bundle homomorphism* $f : E \to F$ *is an* $\mathcal{S}$*-bundle isomorphism if it is a* $\mathcal{S}$*-isomorphism on total spaces and invertible* K*-linear map on each fiber. Two* $\mathcal{S}$*-bundles are equivalent if there exists an* $\mathcal{S}$*-bundle isomorphism between them.*

Proposition 28. *Let* X *be an* $\mathcal{S}$*-manifold,* $\mathcal{S}$*-bundle isomorphism defines an equivalence relation on the* $\mathcal{S}$*-bundles over* X

Proof. trivial □

Definition 57 (direct sum of vector bundles)**.** *Let* $\pi_E : E \to X$ *and* $\pi_F : F \to X$ *be two vector bundles, define*

$$E \oplus F = \coprod_{p \in X} (E_p \oplus F_p)$$

and the projection $\pi : E \oplus F \to X$ *is given by* $\pi(v) = p$ *if* $v \in E_p \oplus F_p$*. For each* $p \in X$*, we can find a neighbourhood* U *of* p *and local trivializations*

$$h_E : E|_U \to U \times K^n \text{ and } h_F : F|_U \to U \times K^m$$

Define $h_{E\oplus F} : (E \oplus F)|_U \to U \times (K^n \oplus K^m)$ *by*

$$h_{E\oplus F}(u \oplus v) = (p, h_E(u) \oplus h_F(v))$$

for $u \in E_p, v \in F_p$*. For any* $p \in U_\alpha \cap U_\beta$ *where* U_α, U_β *have local trivializations, the direct product transition function is*

$$\begin{aligned} g^{E\oplus F}_{\beta\alpha} : U_\alpha \cap U_\beta &\to GL(K^n \times K^m) \\ p &\mapsto g^E_{\beta\alpha}(p) \oplus g^F_{\beta\alpha}(p) \\ p &\mapsto \left(u \oplus v \mapsto g^E_{\beta\alpha}(p)(u) \oplus g^F_{\beta\alpha}(p)(v)\right) \end{aligned}$$

where $g^E_{\beta\alpha}$ *and* $g^F_{\beta\alpha}$ *are transition functions of* π_E *and* π_F

Proposition 29. *Suppose that X is an $\mathcal{S}$-manifold and E, F are $\mathcal{S}$-bundles over X, then $E \oplus F$ is an $\mathcal{S}$-bundle over X.*

Proof. The existence of transition functions implies the construction is a vector bundle. We just need to show that the transition functions are $\mathcal{S}$-function. We have

$$g_{\beta\alpha}^{E\oplus F}(p) = \begin{bmatrix} g_{\beta\alpha}^{E}(p) & 0 \\ 0 & g_{\beta\alpha}^{F}(p) \end{bmatrix}$$

is a matrix of $\mathcal{S}$-functions. Hence, $g_{\beta\alpha}$ is an $\mathcal{S}$-function. □

Proposition 30. *Let $E \to X$, $F \to X$ be vector bundles*

1. *$E \otimes F$, is also a vector bundle, and it is also $\mathcal{S}$-bundle if E, F are.*

$$E \otimes F = \coprod_{p\in X} (E_p \otimes F_p)$$

$$\begin{aligned} h_{E\otimes F} : (E \otimes F)|_U &\to U \times (K^n \otimes K^m) \\ u \otimes v &\mapsto (p, h_E(u) \otimes h_F(v)) && \textit{(for } u \in E_p, v \in F_p\textit{)} \\ g_{\beta\alpha}^{E\otimes F} : U_\alpha \cap U_\beta &\to GL(K^n \otimes K^m) \\ p &\mapsto \big(u \otimes v \mapsto g_{\beta\alpha}^{E}(p)(u) \otimes g_{\beta\alpha}^{F}(p)(v)\big) && \textit{(for } u \in E_p, v \in F_p\textit{)} \end{aligned}$$

2. *$\wedge^n E$, is also a vector bundle, and it is also $\mathcal{S}$-bundle if E, F are.*

$$\wedge^n E = \coprod_{p\in X} \wedge^n E_p$$

$$\begin{aligned} h_{\wedge^n E} : (\wedge^n E)\Big|_U &\to U \times \wedge^n K^n \\ \wedge_{i=1}^n u_i &\mapsto (p, \wedge_{i=1}^n h_E(u_i)) && \textit{(for } \wedge_{i=1}^n u_i \in \wedge^n E_p\textit{)} \\ g_{\beta\alpha}^{\wedge^n E} : U_\alpha \cap U_\beta &\to GL(\wedge^n K^r) \\ p &\mapsto \big(\wedge_{i=1}^n u_i \mapsto \wedge_{i=1}^n g_{\beta\alpha}^{E}(p)(u_i)\big) && \textit{(for } \wedge_{i=1}^n u_i \in \wedge^n E_p\textit{)} \end{aligned}$$

3. $\operatorname{Hom}(E, F)$ *is also a vector bundle, and it is also $\mathcal{S}$-bundle if E, F are.*

$$\operatorname{Hom}(E, F) = \coprod_{p\in X} \operatorname{Hom}(E_p, F_p)$$

$$\begin{aligned} h_{\operatorname{Hom}(E,F)} : \operatorname{Hom}(E, F)|_U &\to U \times \operatorname{Hom}(K^n, K^m) \\ l &\mapsto (p, h_F l h_E^{-1}) && \textit{(for } l \in \operatorname{Hom}(E_p, F_p)\textit{)} \\ g_{\beta\alpha}^{E\otimes F} : U_\alpha \cap U_\beta &\to GL \operatorname{Hom}(K^n, K^m) \\ p &\mapsto \big(l \mapsto g_{\beta\alpha}^{F}(p) l g_{\beta\alpha}^{E}(p)\big) && \textit{(for } l \in \operatorname{Hom}(E_p, F_p)\textit{)} \end{aligned}$$

Proof. TODO □

2.6 Exact Sequences

Definition 58 (subbundle)**.** *Let $\pi : E \to X$ be an $\mathcal{S}$-bundle of rank r. An $\mathcal{S}$-submanifold $F \subseteq E$ is called an $\mathcal{S}$-subbundle of E if*

1. *$F \cap E_x$ is a subspace of E_x for all $x \in X$*

2. *$\pi|_F : F \to X$ has the structure of an $\mathcal{S}$-bundle induced by the $\mathcal{S}$-bundle structure of E. That is, given $U \subseteq X$, and $h : E|_U \to U \times K^r$ is a local trivialization, then the diagram below commutes*

$$\begin{tikzcd} E|_U \arrow[r, "h"] & U \times K^r \\ F|_U \arrow[u] \arrow[r, "h"'] & U \times K^s \arrow[u, "\mathrm{id} \times i"'] \end{tikzcd}$$

where $i : K^s \to K^r$ is the canonical inclusion

Definition 59 (kernel and image of vector bundle homomorphism)**.** *Let $f : E \to F$ is a vector bundle homomorphism of K-vector bundles over a space X. Define*

$$\ker f = \coprod_{x \in X} \ker f_x \text{ and } \operatorname{im} f = \coprod_{x \in X} \operatorname{im} f_x$$

where $f_x = f|_{E_x} : E_x \to F_x$. Note that, $\ker f$ and $\operatorname{im} f$ might not be vector bundle.

Remark 19. *$\ker f$ and $\operatorname{im} f$ might not be vector bundle. e.g. given trivial bundle $\pi_E : E \to X$, $\pi_F : F \to X$ where $E = F = X \times \mathbb{R}^2, X = \mathbb{R}$. Let $f : E \to F$ be defined by $(x, v) \mapsto (x, xv)$. Then, $E_x \to F_x$ is $v \mapsto xv$, then $\operatorname{im} f|_x$ is $\mathbb{R}$ if $x \neq 0$ and 0 if $x = 0$*

Proposition 31. *Let $f : E \to F$ be an $\mathcal{S}$-homomorphism of $\mathcal{S}$-bundles over X, if f has constant rank on X, then $\ker f$, $\operatorname{im} f$ are $\mathcal{S}$-subbundles of E and F respectively.*

Proof. *TODO - need read* □

Proposition 32. *$f : E \to F$ is injective or surjective then f has constant rank.*

2.7 Sections and Frames of Vector Bundles

2.7.1 Sections

Definition 60 (section)**.** *A $\mathcal{S}$-section of and $\mathcal{S}$-bundle $\pi : E \to X$ is an $\mathcal{S}$-morphism $s : X \to E$ such that $\pi s = 1$. Let $\mathcal{S}(X, E) \subseteq \mathcal{S}(X, E)$ denote the set of $\mathcal{S}$-sections. More generally, for a open set $U \subseteq X$, $\mathcal{S}(U, E)$ denotes the $\mathcal{S}$-sections of $E|_U$*

Remark 20 (section of trivial bundle)**.** *For a trivial bundle $\pi : M \times K^r \to M$, $\mathcal{S}(M, M \times K^r)$ can be identified as the collection of $\mathcal{S}$-functions $M \to K^r$ on M*

Remark 21 (zero section)**.** *Let $\pi : E \to X$ be an $\mathcal{S}$-bundle. Let $0 : X \to E$ defined by $x \mapsto (x, 0)$. $0 : X \to E$ is called the zero bundle and $0 : X \to E$ is an $\mathcal{S}$-isomorphism.*

Proposition 33. *Let $pi_E : E \to X$ and $\pi_F : F \to X$ be two $\mathcal{S}$-bundles, then $\pi_{\operatorname{Hom}(E,F)} : \operatorname{Hom}(E, F) \to X$ is also an $\mathcal{S}$-bundle. Let $s \in \mathcal{S}(X, \operatorname{Hom}(E, F))$, for any $x \in X$,*

$$s(x) : E_x \to F_x$$

is a K-linear invertible map. Define $f : E \to F$ such that $f(v) = s(x)v$ for $v \in E_x$. Then f is a $\mathcal{S}$-bundle homomorphism.

Proof. *TODO* □

Proposition 34 (constructing sections)**.** *Let $E \to X$ be an $\mathcal{S}$-bundle of rank r. Let $f_\alpha : U_\alpha \to K^r$ be $\mathcal{S}$-morphism satisfying $f_\beta(u) = g_{\beta\alpha}(u) f_\alpha(u)$ for $u \in U_\alpha \cap U_\beta \neq 0$. We can construct an $\mathcal{S}$-section ϕ as follows: Each f_α gives a section on $U_\alpha \times K^r \cong E|_{U_\alpha}$. These sections agree on overlapping regions.*

Remark 22 (sections on universal bundle)**.** *TODO - wtf*

Proposition 35 (space of sections is a module)**.** *Sections $\mathcal{S}(X, E)$ is a K-vector space, let $s, t \in \mathcal{S}(X, E)$, $k \in K$,*

$$(s + t)(x) = s(x) + t(x)$$
$$(ks)(x) = ks(x)$$

Moreover, $\mathcal{S}(X, E)$ is also a module over $\mathcal{S}(X)$ which is the space of K-valued $\mathcal{S}$-functions on X

2.7.2 Frames

Definition 61 (frame)**.** *Let $E \to X$ be an $\mathcal{S}$-bundle of rank r, let U be an open subset on X. A frame for E over U is a set of $\mathcal{S}$-sections*

$$s = (s_1, s_2, ..., s_r)$$

where $s_j \in \mathcal{S}(U, E)$ such that $(s_1(x), s_2(x), ..., s_r(x))$ is a basis for E_x for every $x \in U$. Note that, section $\mathcal{S}(U, E)$ is a $\mathcal{S}(U)$-module and a frame is the basis, that is, $\mathcal{S}(U, E)$ is a free $\mathcal{S}(U)$-module

Remark 23 (matrix form of an $\mathcal{S}$-section on frame)**.** *Let $\eta \in \mathcal{E}(U, E)$ be an $\mathcal{S}$-section on $U \subseteq X$ of vector bundle $E \to X$ and $f = \{f_1, f_2, ..., f_r\}$ be a frame on U. Then, we can write η by*

$$\eta = \eta(f)_1 f_1 + \eta(f)_2 f_2 + ... + \eta(f)_r f_r$$

where each $\eta(f)_j \in \mathcal{S}(U)$ is an $\mathcal{S}$-function on U. Let

$$\eta(f) = (\eta(f)_1, \eta(f)_2, ..., \eta(f)_r) \in M_r[\mathcal{S}(U)]$$

be a vector of dimension r where each entry is an $\mathcal{S}$-function on U.

Proposition 36. *A local trivialization $h : E|_U \to U \times K^r$ induces a frame for E over U*

Proof. The local trivialization induces an isomorphism of sets

$$\tilde{h} : \mathcal{S}(U, E|_U) \to \mathcal{S}(U, U \times K^r)$$

Let $(e_1, e_2, ..., e_r)$ be a basis for K^r, then $s_i : X \to E$ is defined by

$$s_i(x) = h^{-1}(x, e_i)$$

Then $(s_1, s_2, ..., s_r)$ is a frame for E over U □

Proposition 37. *TODO - frame induces local trivialization*

Definition 62 (change of frames)**.** *Let $E \to X$ be an $\mathcal{S}$-bundle over field K and $U \subseteq X$ be an open set. Let f, h be frames on U, that is,*

$$f = (f_1, f_2, ..., f_r)$$
$$h = (h_1, h_2, ..., h_r)$$

where each f_i, h_i is an $\mathcal{S}$-section on U and for each $x \in U$, $(f_1(x), f_2(x), ..., f_r(x))$, $(h_1(x), h_2(x), ..., h_r(x))$ are bases of the fiber E_x. Then, there exists a $\mathcal{S}$-morphism

$$g : U \to GL(K^r)$$

so that, for each $x \in U$, $g(x)$ is the change of basis from $f(x)$ to $h(x)$, g is called the change of frames.

Remark 24 (matrix form of change of frames)**.** *Let $g : U \to GL(K^r)$ be a change of frames from frame f to frame h, if write frames f, h as row vectors of $\mathcal{S}$-sections, that is, $f = (f_1, f_2, ..., f_r) \in M_r[\mathcal{S}(U, E)]$, $h = (h_1, h_2, ..., h_r) \in M_r[\mathcal{S}(U, E)]$ where $f_\sigma, h_\sigma \in \mathcal{S}(U, E)$ and g as a matrix of $\mathcal{S}$-functions, that is, $g = [g_{\rho\sigma}] \in M_{r \times r}[\mathcal{S}(U)]$, then*

$$h = fg$$

2.8 Real-Valued Differential Forms $\mathcal{E}^p(U) = \mathcal{E}(U, \wedge^p T^*(M))$

2.8.1 Real-Valued Differential Forms

Definition 63 (cotangent bundle, exterior algebra bundles)**.** *Consider the tangent bundle $T(M) \to M$ of a smooth manifold M*

1. *The cotangent bundle $T^*(M)$ is the vector bundle whose fiber at $x \in M$ is $T^*_x(M) = \mathrm{Hom}_{\mathbb{R}}(T_x(M), \mathbb{R})$ which is a linear functional on $T_x(M)$ (basis are dx, dy, dz while in tangent bundle are $\partial/\partial x, \partial/\partial y, \partial/\partial z$)*

2. *The exterior algebra bundles $\wedge^n T(M)$ and $\wedge^n T^*(M)$ are the vector bundles whose fibers at $x \in M$ are $\wedge^n T_x(M)$ and $\wedge^n T^*_x(M)$ respectively. We also define*

$$\wedge T(M) = \bigoplus_{p=0}^{n} \wedge^p T(M)$$
$$\wedge T^*(M) = \bigoplus_{p=0}^{n} \wedge^p T^*(M)$$

3. *The exterior algebra bundles $Sym^n T(M)$ and $Sym^n T^*(M)$ are the vector bundles whose fibers at $x \in M$ are $Sym^n T_x(M)$ and $Sym^n T^*_x(M)$ respectively. Define*

$$SymT(M) = \bigoplus_{p=0}^{n} Sym^p T(M)$$
$$SymT^*(M) = \bigoplus_{p=0}^{n} Sym^p T^*(M)$$

Definition 64 (differential form)**.** *Let $U \subseteq M$ be an open set of smooth manifold M, let*

$$\mathcal{E}^p(U) = \mathcal{E}(U, \wedge^p T^*(M))$$

A section $f \in \mathcal{E}^p(U)$ is called a smooth differential p-form on U

Definition 65 (a basis for $\mathcal{E}^p(U)$ for $U \subseteq \mathbb{R}^n$)**.** *Let $x = (x_1, ..., x_n) \in U$, then a basis of $T_x(\mathbb{R}^n)$ is*

$$\left\{ \frac{\partial}{\partial x_1}\bigg|_x, ..., \frac{\partial}{\partial x_n}\bigg|_x \right\}$$

Let $dx_j|_x : T_x(\mathbb{R}^n) \to \mathbb{R}$ be the linear functional such that $\frac{\partial}{\partial x_j}|_x \mapsto 1$ and $\frac{\partial}{\partial x_k}|_x \mapsto 0$ for $k \neq j$. Then, a basis of $T^*_x(\mathbb{R}^n)$ is

$$\{dx_1|_x, ..., dx_n|_x\}$$

Let $h : T^*(\mathbb{R}^n)|_U \to U \times \mathbb{R}^n$ be a local trivialization so that h maps the basis $\{dx_1|_x, ..., dx_n|_x\}$ into the canonical basis of $\mathbb{R}^n$. Moreover, let $s \in \mathcal{E}(U, T^*(\mathbb{R}^n))$ be a section

$$\begin{tikzcd}
T^*(\mathbb{R}^n)|_U \arrow[r, "h"] & U \times \mathbb{R}^n \\
U \arrow[u, "s"] \arrow[ur, dashed, "hs"'] &
\end{tikzcd}$$

Then, $hs : U \to U \times \mathbb{R}^n$ applied on $x \in U$ is $hs(x) = (x, (f_1(x), ..., f_n(x)))$ where $f_i \in \mathcal{E}(U)$ are a real-valued smooth functions on U. Define $dx_j : U \to T^*(\mathbb{R}^n)|_U$ such that $x \mapsto dx_j|_x$, then $dx_j(x) = (0, ..., 0, 1, 0, ..., 0)$ (1 at the j-th coordinate). Therefore, $\mathcal{E}^1(U) = \mathcal{E}(U, T^*(\mathbb{R}^n))$ is an $\mathcal{E}(U)$-module with basis $\{dx_1, ..., dx_n\}$ with

$$s = f_1 dx_1 + ... + f_n dx_n$$

the scalar multiplication defined by

$$\begin{aligned} \mathcal{E}(U) \times \{dx_1, ..., dx_n\} &\to \mathcal{E}(U, T^*(\mathbb{R}^n)) \\ (f \cdot dx_j)(x) &\mapsto f(x) dx_j(x) \end{aligned}$$

Similar, a basis of $\mathcal{E}(U)$-module $\mathcal{E}^p(U) = \mathcal{E}(U, \wedge^p T^*(\mathbb{R}^n))$ is [1]

$$\{dx_\sigma : \sigma \subseteq [n], |\sigma| = p\}$$

2.8.2 Real-Valued Exterior Derivative

Definition 66 (exterior derivative $d : \mathcal{E}^p(U) \to \mathcal{E}^{p+1}(U)$ for $U \subseteq \mathbb{R}^n$)**.** *Define $d : \mathcal{E}^p(U) \to \mathcal{E}^{p+1}(U)$ for $U \subseteq \mathbb{R}^n$ as follows:*

1. *$p = 0$, $f \in \mathcal{E}^0(U) = \mathcal{E}(U)$*

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

2. *$p \geq 1$, $f \in \mathcal{E}^p(U) = \mathcal{E}(U, \wedge^p T^*(\mathbb{R}^n))$, then we can write f as a $\mathcal{E}(U)$-linear combination $f = \sum_{\sigma \subseteq [n] : |\sigma| = p} f_\sigma dx_\sigma$ where each $f_\sigma \in \mathcal{E}(U)$, define*

$$df = \sum_{\sigma \subseteq [n] : |\sigma| = p} df_\sigma \wedge dx_\sigma = \sum_{\sigma \subseteq [n] : |\sigma| = p} \left(\sum_{i=1}^{n} \frac{\partial f_\sigma}{\partial x_i} dx_i \right) \wedge dx_\sigma$$

[1] note that, σ is an ordered subset of $[n]$

Definition 67 (exterior derivative $d : \mathcal{E}^p(M) \to \mathcal{E}^{p+1}(M)$)**.** *: Let $h : U \to U' \subseteq \mathbb{R}^n$ be a chart, there exists a map $d : \mathcal{E}^p(U') \to \mathcal{E}^{p+1}(U')$. We have isomorphisms $\mathcal{E}^p(U) \cong \mathcal{E}^p(U')$ and $\mathcal{E}^{p+1}(U) \cong \mathcal{E}^{p+1}(U')$, define $d : \mathcal{E}^p(U) \to \mathcal{E}^{p+1}(U)$ so that the diagram below commutes*

$$\begin{tikzcd}
\mathcal{E}^p(U) \arrow[r, dotted, "d"] \arrow[d, "\cong"'] & \mathcal{E}^{p+1}(U) \arrow[d, "\cong"'] \\
\mathcal{E}^p(U') \arrow[r, "d"] & \mathcal{E}^{p+1}(U')
\end{tikzcd}$$

Let M be a manifold with atlas $\{h_i : U_i \to U_i' \subseteq \mathbb{R}^n\}$. Then, $d : \mathcal{E}^p(M) \to \mathcal{E}^{p+1}(M)$ is the unique operator such that the diagram below commutes for every chart

$$\begin{tikzcd}
\mathcal{E}^p(M) \arrow[r, dashed, "d"] \arrow[d, two heads, "r_U^M"'] & \mathcal{E}^{p+1}(M) \arrow[d, two heads, "r_U^M"] \\
\mathcal{E}^p(U) \arrow[r, "d"] & \mathcal{E}^{p+1}(U)
\end{tikzcd}$$

TODO verify that, on the intersection $U_i \cap U_j$, the exterior derivatives agree, that is, let $d_i : \mathcal{E}^p(U_i) \to \mathcal{E}^{p+1}(U_i)$, $d_j : \mathcal{E}^p(U_j) \to \mathcal{E}^{p+1}(U_j)$, then $d_i|_{\mathcal{E}^p(U_i \cap U_j)} = d_j|_{\mathcal{E}^p(U_i \cap U_j)}$

Definition 68 (*axiomatic definition* of exterior derivative $d : \mathcal{E}^p(M) \to \mathcal{E}^{p+1}(M)$)**.** *When $U \subseteq M = \mathbb{R}^n$, then $d : \mathcal{E}^p(U) \to \mathcal{E}^{p+1}(U)$ is defined to be the unique $\mathbb{R}$-linear map satisfies the following:*

1. *when $p = 0$, $f \in \mathcal{E}^0(U) = \mathcal{E}(U)$, then*
$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

2. *if $a \in \mathcal{E}^p(U)$ and $b \in \mathcal{E}^q(U)$, then*
$$d(a \wedge b) = da \wedge b + (-1)^p a \wedge db$$

in 2, let $a = 1 \in \mathcal{E}^0(U) = \mathcal{E}(U)$, then $d^2 = 0$ (Poincaré lemma)

2.9 Pullback of vector bundles

Definition 69 ($\mathcal{S}$-bundle morphism)**.** *An $\mathcal{S}$-bundle morphism between two $\mathcal{S}$-bundles $\pi_E : E \to X$ and $\pi_F : F \to Y$ is an $\mathcal{S}$-morphism $f : E \to F$ that takes fibers of E into fibers of F and if restricted to one fiber, the map is linearly invertible.*

Remark 25. *As we can identify the base space with its zero section, therefore, an $\mathcal{S}$-bundle morphism defines a map between base spaces factoring through their zero sections.*

Proposition 38 (pullback)**.** *Given an $\mathcal{S}$-morphism $f : X \to Y$ and an $\mathcal{S}$-bundle $\pi_F : F \to Y$, then there exists a bundle $\pi_E : E \to X$ so that the following diagram commutes*

$$\begin{tikzcd}
E \arrow[r, dashed, "g"] \arrow[d, dashed, two heads, "\pi_E"'] & F \arrow[d, two heads, "\pi_F"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Moreover, E is unique upto $\mathcal{S}$-bundle isomorphism. E is called pullback of F by $f : X \to Y$ denoted by $E = f^*F$, $\pi_E = f^*\pi_F$, $g = f^*f$

Proof. TODO □

Proposition 39. *Let $\pi_E : E \to X$ and $\pi_F : F \to Y$ be $\mathcal{S}$-bundles. If $f : E \to F$ is an $\mathcal{S}$-morphism of total spaces that preserves fibers and a linear on each fiber, then $f : E \to F$ can decomposed into a $\mathcal{S}$-bundle homomorphism and an $\mathcal{S}$-bundle morphism*

$$\begin{tikzcd}
E \arrow[r, dashed, "h"'] \arrow[rr, bend left, "f"] \arrow[d, two heads, "\pi_E"'] & G \arrow[r, dashed, "k"'] \arrow[d, dashed, "\pi_G"] & F \arrow[d, two heads, "\pi_F"] \\
X \arrow[r, dashed, "\cong"] & X \arrow[r, dashed] & Y
\end{tikzcd}$$

Proof. TODO □

Chapter 3

Almost Complex Manifolds

3.1 Almost Complex Manifold

Proposition 40 (complex structure on manifold)**.** *Let X be a holomorphic manifold of dimension n, then X is also a real smooth manifold of dimension $2n$, let X_0 denoted the corresponding real manifold. Let $T_x(X)$ be the complex tangent space at $x \in X$ and $T_x(X_0)$ be the real tangent space at $x \in X_0$. Then,*

1. *The real tangent space $T_x(X_0)$ is canonically isomorphic to the underlying real vector space of complex tangent space $T_x(X)$*
2. *Complex multiplication by i on $T_x(X)$ induces a complex structure J_x on the real tangent space $T_x(X_0)$*

Proof.

Let $x \in U$ and $\phi : U \to U' \subseteq \mathbb{C}^n$ be a chart containing x. Let the canonical map $h : U' \to U'' \subseteq \mathbb{R}^{2n}$ defined by

$$(x_1 + iy_1, ..., x_n + iy_n) \mapsto (x_1, y_1, ..., x_n, y_n)$$

Then, $h\phi : U \to U'' \subseteq \mathbb{R}^{2n}$ is the chart that makes X_0 to be a real smooth manifold. The bases on $T_x(X)$ and $T_x(X_0)$ are induced from the Jacobian of the $\mathcal{S}$-isomorphism from $\phi : U \to U'$ and $h\phi : U \to U''$ respectively.

$$T_x(X) \xrightarrow{d\phi_x} T_{\phi(x)}(U') \qquad T_{h\phi(x)}(U'') \xleftarrow{d(h\phi)_x} T_x(X_0)$$

where

$$T_{\phi(x)}(U') = \mathbb{C} - \text{span}\left\{ \frac{\partial}{\partial z_1}\bigg|_{\phi(x)}, ..., \frac{\partial}{\partial z_n}\bigg|_{\phi(x)} \right\}$$
$$T_{h\phi(x)}(U'') = \mathbb{R} - \text{span}\left\{ \frac{\partial}{\partial x_1}\bigg|_{h\phi(x)}, \frac{\partial}{\partial y_1}\bigg|_{h\phi(x)}, ..., \frac{\partial}{\partial x_n}\bigg|_{h\phi(x)}, \frac{\partial}{\partial y_n}\bigg|_{h\phi(x)} \right\}$$

Define $\mathbb{R}$-linear invertible map $T_{\phi(x)}(U') \to T_{h\phi(x)}(U'')$ as follows:

$$(a+ib)\frac{\partial}{\partial z_j}\bigg|_{\phi(x)} \mapsto \frac{1}{2}\left(a\frac{\partial}{\partial x_j}\bigg|_{h\phi(x)} + b\frac{\partial}{\partial y_j}\bigg|_{h\phi(x)} \right)$$

That induces a $\mathbb{R}$-linear invertible map $T_x(X) \to T_x(X_0)$, the complex structure J_x on $T_x(X_0)$ is induced naturally

$$\begin{tikzcd} T_x(X) \arrow[r, "v \mapsto iv"] \arrow[d, "\cong"'] & T_x(X) \arrow[d, "\cong"] \\ T_x(X_0) \arrow[r, dashed, "J_x"] & T_x(X_0) \end{tikzcd}$$

Note that, $J_x : T_x(X_0) \to T_x(X_0)$ does not depend on the choice of chart $\phi : U \to U'$

□

Definition 70 (almost complex manifold)**.** *Let X be a real smooth manifold of dimension $2n$. Suppose that J is a smooth vector bundle isomorphism (automorphism on tangent bundle $T(X) \to T(X)$)*

$$J : T(X) \to T(X)$$

such that $J_x : T_x(X) \to T_x(X)$ is a complex structure for $T_x(X)$, that is, $J_x^2 = -1$. J is called an almost complex structure for smooth manifold X. If X is equipped with an almost complex structure, then (X, J) is called an almost complex manifold.

Proposition 41. *A holomorphic manifold X induces an almost complex structure on its underlying smooth manifold X_0*

Proof. *TODO - the map from each point x to its complex structure induced by multiplication by i on complex tangent space is an almost complex structure* □

3.2 Complex-Valued Differential Forms of Type (p, q) $\mathcal{E}^r(X)_{\mathbb{C}} = \mathcal{E}(X, \wedge^r T^*(X)_{\mathbb{C}})$

3.2.1 Complex-Valued Differential Forms and Complex-Valued Exterior Derivative

Definition 71 (complex-valued differential forms and exterior derivative $d : \mathcal{E}^r(X)_{\mathbb{C}} \to \mathcal{E}^{r+1}(X)_{\mathbb{C}}$)**.** *Let X be a smooth manifold of dimension m, the complexification of cotangent bundle is*

$$\mathcal{E}^r(X)_{\mathbb{C}} = \mathcal{E}(X, \wedge^r T^*(X)_{\mathbb{C}}) = \mathcal{E}\left(X, \coprod_{x \in X} \wedge^r T_x^*(X)_{\mathbb{C}}\right) = \mathcal{E}\left(X, \coprod_{x \in X} \wedge^r T_x^*(X) \otimes_{\mathbb{R}} \mathbb{C}\right)$$

Sections in $\mathcal{E}^r(X)_{\mathbb{C}}$ are called the complex-valued smooth differential r-forms. The complex-valued exterior derivative $d : \mathcal{E}^r(X)_{\mathbb{C}} \to \mathcal{E}^{r+1}(X)_{\mathbb{C}}$ is defined as follows: let $f \in \mathcal{E}^r(X)_{\mathbb{C}}$, then for $x \in X$

$$f(x) \in \wedge^r T_x^*(X) \otimes_{\mathbb{R}} \mathbb{C} \cong \wedge^r T_x^*(X) \oplus \wedge^r T_x^*(X)$$

then we can write $f = f_{\mathrm{Re}} + i f_{\mathrm{Im}}$ where $f_{\mathrm{Re}}, f_{\mathrm{Im}} \in \mathcal{E}^r(X) = \mathcal{E}(X, \wedge^r T^*(X))$. We define $d : \mathcal{E}^r(X)_{\mathbb{C}} \to \mathcal{E}^{r+1}(X)_{\mathbb{C}}$ by

$$df = df_{\mathrm{Re}} + i df_{\mathrm{Im}}$$

3.2.2 Complex-Valued Differential Forms of Type (p, q)

Definition 72 (bundle $T(X)^{1,0}$, $T(X)^{0,1}$)**.** *Let (X, J) be an almost complex manifold. Then, J extends into a $\mathbb{C}$-linear bundle isomorphism on $T(X)_{\mathbb{C}} = \coprod_{x \in X} T_x(X)_{\mathbb{C}}$*

$$J_x : T_x(X)_{\mathbb{C}} \to T_x(X)_{\mathbb{C}}$$

satisfying $J_x^2 = -1$. Let $T_x(X)^{1,0}$ and $T_x(X)^{0,1}$ be the $+i$ and $-i$ eigenspaces of J_x, then

$$T_x(X)_{\mathbb{C}} = T_x(X)^{1,0} \oplus T_x(X)^{0,1}$$

We set

$$T(X)^{1,0} = \coprod_{x \in X} T_x(X)^{1,0}$$
$$T(X)^{0,1} = \coprod_{x \in X} T_x(X)^{0,1}$$

That are smooth bundles over X. Moreover, there is an $\mathbb{C}$-linear isomorphism $Q_x : T_x(X)^{1,0} \to T_x(X)^{0,1}$ which extends into an isomorphism of smooth vector bundles over X

$$Q : T(X)_{\mathbb{C}} \to T(X)_{\mathbb{C}}$$
$$Q : T(X)^{1,0} \to T(X)^{0,1}$$

Similarly, as J makes $T_x(X)$ into a complex vector space denoted by $T_x(X)_J$, there is an $\mathbb{C}$-linear isomorphism $T_x(X)_J \to T_x(X)^{1,0}$ which extends into an isomorphism of smooth vector bundles over X

$$T(X)_J \to T(X)^{1,0}$$

Proof. *TODO* □

Definition 73 (complex-valued differential forms of type (p, q))**.** *When (X, J) is an almost complex manifold, we also have the decomposition for complexification of cotangent space*

$$T_x^*(X)_{\mathbb{C}} = T_x^*(X)^{1,0} \oplus T_x^*(X)^{0,1}$$

Define

$$\wedge^{p,q} T_x^*(X)_{\mathbb{C}} = \left(\wedge^p T_x^*(X)_{\mathbb{C}}^{1,0}\right) \wedge \left(\wedge^q T_x^*(X)_{\mathbb{C}}^{0,1}\right)$$

which extend into a smooth bundles over X

$$\wedge^{p,q}T^*(X)_{\mathbb{C}} = \coprod_{x \in X} \wedge^{p,q}T_x^*(X)_{\mathbb{C}}$$

We denote the section

$$\mathcal{E}^{p,q}(X)_{\mathbb{C}} = \mathcal{E}(X, \wedge^{p,q}T^*(X)_{\mathbb{C}})$$

A section $f \in \mathcal{E}^{p,q}(X)_{\mathbb{C}}$ is called complex-valued smooth differential (p,q)-form

Proposition 42.

$$\mathcal{E}^r(X)_{\mathbb{C}} = \mathcal{E}(X, \wedge^r T^*(X)_{\mathbb{C}}) = \bigoplus_{p+q=r} \mathcal{E}^{p,q}(X)_{\mathbb{C}}$$

Proof. proof in the next section. □

3.2.3 Complex-Valued Exterior Derivative of Type (p,q)

Definition 74 (local representation of exterior derivative of type (p,q))**.** *Let (X,J) be an almost complex manifold of dimension n, let $\{w_1, w_2, ..., w_n\}$ be a frame over an open subset $U \subseteq X$ for $T^*(X)^{1,0} \subseteq T^*(X)_{\mathbb{C}}$. Then, $\{\overline{w}_1, \overline{w}_2, ..., \overline{w}_n\}$ is a frame over U for $T^*(X)^{0,1} \subseteq T^*(X)_{\mathbb{C}}$. A frame for $\wedge^{p,q}T^*(X)$ is given by* [1]

$$\{w^I \wedge \overline{w}^J : I, J \subseteq [n], |I| = p, |J| = q\}$$

So that, a section $s \in \mathcal{E}^{p,q}(U)_{\mathbb{C}} \subseteq \mathcal{E}^r(U)_{\mathbb{C}}$ can be written as

$$s = \sum_{I,J} a_{IJ} w^I \wedge \overline{w}^J$$

where each $a_{IJ} \in \mathcal{E}(U)_{\mathbb{C}} = \mathcal{E}(U, \mathbb{C}) = \mathcal{E}^0(U, \mathbb{C})$, then

$$ds = \sum_{I,J} da_{IJ} \wedge w^I \wedge \overline{w}^J + a_{IJ} d(w^I \wedge \overline{w}^J)$$

TODO - note that $d(w^I \wedge \overline{w}^J)$ might not be zero

Definition 75 $(\partial, \overline{\partial})$**.** *Note that, a frame for $\wedge^r T^*(X)$ is given by $(r = p+q)$*

$$\{w^I \wedge \overline{w}^J : I, J \subseteq [n], |I| + |J| = r\}$$

Then, $\mathcal{E}^r(X)_{\mathbb{C}} = \bigoplus_{p,q:p+q=r} \mathcal{E}^{p,q}(X)_{\mathbb{C}}$. Let

$$\pi_{p,q} : \mathcal{E}^r(X)_{\mathbb{C}} \to \mathcal{E}^{p,q}(X)_{\mathbb{C}}$$

be the canonical projection, then there is a restriction from complex-valued exterior derivative

$$d : \mathcal{E}^{p,q}(X)_{\mathbb{C}} \to \mathcal{E}^{p+q+1}(X)_{\mathbb{C}} = \bigoplus_{r+s=p+q+1} \mathcal{E}^{r,s}(X)_{\mathbb{C}}$$

Define $\partial : \mathcal{E}^{p,q}(X)_{\mathbb{C}} \to \mathcal{E}^{p+1,q}(X)_{\mathbb{C}}$ and $\overline{\partial} : \mathcal{E}^{p,q}(X)_{\mathbb{C}} \to \mathcal{E}^{p,q+1}(X)_{\mathbb{C}}$ as follows

$$\begin{tikzcd} \mathcal{E}^{p,q}(X)_{\mathbb{C}} \arrow[r, "d"'] \arrow[rr, bend left, "\partial"] & \mathcal{E}^{p+q+1}(X)_{\mathbb{C}} \arrow[r, "\pi_{p+1,q}"'] & \mathcal{E}^{p+1,q}(X)_{\mathbb{C}} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{E}^{p,q}(X)_{\mathbb{C}} \arrow[r, "d"] \arrow[rr, bend right, "\overline{\partial}"'] & \mathcal{E}^{p+q+1}(X)_{\mathbb{C}} \arrow[r, "\pi_{p,q+1}"] & \mathcal{E}^{p,q+1}(X)_{\mathbb{C}} \end{tikzcd}$$

Let $\mathcal{E}^*(X)_{\mathbb{C}} = \bigoplus_{r=0}^m \mathcal{E}^r(X)_{\mathbb{C}}$, we extend the above operators to

$$\partial : \mathcal{E}^*(X)_{\mathbb{C}} \to \mathcal{E}^*(X)_{\mathbb{C}}$$
$$\overline{\partial} : \mathcal{E}^*(X)_{\mathbb{C}} \to \mathcal{E}^*(X)_{\mathbb{C}}$$

[1] recall that $w^{\{1,3,4\}} = w_1 \wedge w_3 \wedge w_4$

Definition 76 (integrable almost complex manifold)**.** *If $d = \partial + \overline{\partial}$, then we call (X, J) integrable*

Proposition 43. *When (X, J) is integrable, then $\partial^2 = \overline{\partial}^2 = 0$*

Proof.

$$0 = d^2 = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2$$

Each term on the right is at different summand of $\mathcal{E}^{p+q+2}(X)$. Then each of them is zero. □

Theorem 9. *The induced almost complex structure on a complex manifold is integrable.*

Proof. (sketch proof) reduce the problem into the case where $U \subseteq X = \mathbb{C}^n$, $X_0 = \mathbb{R}^{2n}$, then construct a basis for $T^*(X_0)_{\mathbb{C}}$, $T^*(X_0)^{1,0}$, and $T^*(X_0)^{0,1}$. That is,

$$T_x(X_0)_{\mathbb{C}} = \mathbb{C} - \text{span}\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} : i = 1, 2, ..., n\right\}$$
$$T_x(X_0)^{1,0} = \mathbb{C} - \text{span}\left\{\frac{\partial}{\partial z_i} : i = 1, 2, ..., n\right\}$$
$$T_x(X_0)^{0,1} = \mathbb{C} - \text{span}\left\{\frac{\partial}{\partial \overline{z}_i} : i = 1, 2, ..., n\right\}$$

where $\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right)$, $\frac{\partial}{\partial \overline{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right)$, then one can write

$$\partial = \sum_{i=1}^{n} \frac{\partial}{\partial z_i} dz_i$$
$$\overline{\partial} = \sum_{i=1}^{n} \frac{\partial}{\partial \overline{z}_i} d\overline{z}_i$$

Then, $d = \partial + \overline{\partial}$. *Moreover, from to Cauchy Riemann equation, $\overline{\partial}$ applied on $\mathcal{O}^0(X) \subseteq \mathcal{E}^0(X_0)$ (holomorphic function) yields 0* □

Theorem 10 (Newlander, Nirenberg)**.** *Let (X, J) be an integrable almost complex manifold. Then there exists a unique complex structure O_X on X (holomorphic manifold) which induces the almost complex structure J.*

3.2.4 Complex-Valued Differential Forms with Coefficients

Definition 77 (complex-valued differential forms with coefficients)**.** *Let $E \to X$ be a complex smooth vector bundle over a real manifold, let*

$$\mathcal{E}^p(X, E) = \mathcal{E}(X, \wedge^p T^*(X)_{\mathbb{C}} \otimes E)$$

$\mathcal{E}^p(X, E)$ is called the complex-valued differential p-forms with coefficients in E on X. When $E = \mathbb{C}$, we recover the complex-valued differential form of degree p

$$\mathcal{E}^p(X) = \mathcal{E}^p(X, \mathbb{C}) = \mathcal{E}(X, \wedge^p T^*(X)_{\mathbb{C}} \otimes \mathbb{C}) = \mathcal{E}(X, \wedge^p T^*(X)_{\mathbb{C}})$$

Proposition 44 (isomorphism of sheaves)**.** *(refer sheaf in chapter 4) The sheaf of vector bundle $\wedge^p T^*(X) \otimes E \to X$ is denoted by $\mathcal{E}^p(E)$, the sheaf of vector bundle $\wedge^p T^*(X) \to X$ is denoted by $\mathcal{E}^p$, then there is an isomorphism of sheaves*

$$\tau : \mathcal{E}^p \otimes_{\mathcal{E}} \mathcal{E}(E) \to \mathcal{E}^p(E)$$

That induces a map of sections on $U \subseteq X$

$$\mathcal{E}^p(U) \otimes_{\mathcal{E}(U)} \mathcal{E}(U, E) \to \mathcal{E}^p(U, E)$$
$$\phi \otimes \xi \mapsto \phi \cdot \xi$$

where $\phi \in \mathcal{E}^p(U)$, $\xi \in \mathcal{E}(U, E)$, and $\phi \cdot \xi \in \mathcal{E}^p(U, E)$.

Proof. Proved in chapter 4 □

Remark 26 (local representation of $\mathcal{E}^p(U,E)$)**.** *Let $f = (e_1, e_2, ..., e_r) \in M_r[\mathcal{E}(U,E)]$ be a frame for E over U, then for each $\xi \in \mathcal{E}^p(U,E) \cong \mathcal{E}^p(U) \otimes_{\mathcal{E}(U)} \mathcal{E}(U,E)$, we can write*

$$\xi = \xi^1(f) \cdot e_1 + \xi^2(f) \cdot e_2 + ... + \xi^r(f) \cdot e_r$$

where each $\xi^\rho(f) \in \mathcal{E}^p(U)$. In this way we have local representation

$$\begin{aligned} \mathcal{E}^p(U,E) &\to M_r[\mathcal{E}^p(U)] \\ \xi &\mapsto \xi(f) = (\xi^1(f), \xi^2(f), ..., \xi^r(f)) \end{aligned}$$

Proposition 45 (local representation of $\mathcal{E}^p(U,E)$ on change of frame)**.** *Let g be a change of frame on $E \to U$, then*

$$\xi(fg) = g^{-1} \cdot \xi(f)$$

where $\xi(f), \xi(fg) \in M_r[\mathcal{E}^p(U)]$, $g \in M_{r\times r}[\mathcal{E}(U)]$

Proof. *TODO - does not look easy* □

Chapter 4

Sheaf Theory

4.1 Presheaves

Definition 78 (category of open sets, presheaf)**.** *Let X be a topological space, the category of open sets denoted by $\mathrm{Op}(X)$ is the category where objects are open sets of X and morphisms and open set inclusions. If we drop the empty set from $\mathrm{Op}(X)$, the remaining object is also a category, denoted by $\mathrm{Op}(X)_*$.*

A presheaf $\mathcal{F}$ over X is a functor on $\mathrm{Op}(X)_*^{op}$, that is

1. *An assignment to each nonempty open set $U \subseteq X$ of an object $\mathcal{F}(U)$*
2. *For each pair of open sets $U, V \subseteq X$ such that $U \supseteq V$, there exists a map*

$$r_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$$

   And these restrictions satisfy the following:

   (a) $r_U^U : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map on $\mathcal{F}(U)$
   (b) if $U \supseteq V \supseteq W$, then $r_W^U = r_V^U r_W^V$

An element of $\mathcal{F}(U)$ is called section, the map r_V^U is called restriction map. If the codomain category is an algebraic structure such as group, ring, etc, then the presheaf is called presheaf of groups, presheaf of rings, etc.

Definition 79 (category of presheaves, morphisms of presheaves)**.** *The collection of presheaves over X form a category where objects are presheaves and morphisms are natural transformations between them. The morphisms are called morphisms of presheaves. That is, if $\mathcal{F}, \mathcal{G}$ are presheaves over X, a morphism of presheaves $h : \mathcal{F} \to \mathcal{G}$ is a collection of maps*

$$h_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

for every open set U in X and the diagram below commutes

$$\begin{tikzcd} U & \mathcal{F}(U) \arrow[r, "h_U"] \arrow[d, "r_V^U"] & \mathcal{G}(U) \arrow[d, "r_V^U"] \\ V \arrow[u, hook] & \mathcal{F}(V) \arrow[r, "h_V"] & \mathcal{G}(V) \end{tikzcd}$$

The presheaf $\mathcal{F}$ is called subpresheaf of $\mathcal{G}$ if h_U are inclusions

Remark 27 (constant presheaf)**.** *Let S be a set, the constant presheaf $\mathcal{F}$ is defined by $\mathcal{F}(U) = S$ for all U and $\mathcal{F}(r_V^U) : S \to S$ is the identity map.*

Definition 80 (restriction of presheaf)**.** *Let $\mathcal{F}$ be a presheaf on topological space X, let $U \subseteq X$ be an open subset of X, then the restriction of presheaf $\mathcal{F}|_U$ is a presheaf on subspace U defined by*

$$\mathcal{F}|_U(V) = \mathcal{F}(V)$$

for every open subset $V \subseteq U$.

4.2 Sheaves

Definition 81 (sheaf). *A presheaf $\mathcal{F}$ over a topological space X is called sheaf if the following holds: Let $U = \bigcup_{i\in I} U_i$ be a union of open sets in X then*

1. *If $s, t \in \mathcal{F}(U)$ such that $r^U_{U_i}(s) = r^U_{U_i}(t)$ for all $i \in I$, then $s = t$*
2. *Given $s_i \in \mathcal{F}(U_i)$ for all $i \in I$ such that*
$$r^U_{U_i\cap U_j}(s_i) = r^U_{U_i\cap U_j}(s_j)$$
for all $U_i \cap U_j \neq \varnothing$, then there exists an $s \in \mathcal{F}(U)$ so that $r^U_{U_i}(s) = s_i$ for all $i \in I$

Definition 82 (category of sheaves, morphisms of sheaves). *The collection of sheaves over X form a category where objects are sheaves and morphisms are natural transformations between them. The morphisms are called morphisms of sheaves.*

Remark 28 (constant sheaf). *Constant presheaf is not a sheaf. Let $X = U_1 \coprod U_2$ be disjoint union of two open sets and $A = \{s_1, s_2\}$ be the set of two elements. Let $\mathcal{F}(U_1) = \mathcal{F}(U_2) = \mathcal{F}(X) = A$ be the constant presheaf. As $s_1 \in \mathcal{F}(U_1)$ and $s_2 \in \mathcal{F}(U_2)$. Suppose $\mathcal{F}$, by condition 2, there exists $s \in \mathcal{F}(X)$ such that $r^X_{U_1}s_1 = r^X_{U_2}s_2$. Since the restriction in constant presheaf is the identity map, then $s_1 = s_2$. Contradiction.*

TODO: construct constant sheaf

4.3 Sheaves and Vector Bundles

Definition 83 (sheaf of $\mathcal{R}$-modules). *Let X be a topological space*

1. *Let $\mathcal{R}$ be a presheaf of commutative rings over X and ρ^U_V be the $\mathcal{R}$-restriction.*
2. *Let $\mathcal{M}$ be a presheaf of abelian groups over X and r^U_V be the $\mathcal{M}$-restriction.*

Suppose for any open set $U \subseteq X$, $\mathcal{M}(U)$ is an $\mathcal{R}(U)$-module and if $\alpha \in \mathcal{R}(U)$ and $f \in \mathcal{M}(U)$, then

$$r^U_V(\alpha f) = \rho^U_V(\alpha) r^U_V(f)$$

for all $U \supseteq V$. That is, the diagram below commutes

$$\begin{tikzcd}
\mathcal{R}(U) \arrow[d, "\rho^U_V"'] & \mathcal{M}(U) \arrow[d, "r^U_V"'] & \mathcal{R}(U)\times\mathcal{M}(U) \arrow[r] \arrow[d, "\rho^U_V\times r^U_V"'] & \mathcal{M}(U) \arrow[d, "r^U_V"] \\
\mathcal{R}(V) & \mathcal{M}(V) & \mathcal{R}(V)\times\mathcal{M}(V) \arrow[r] & \mathcal{M}(V)
\end{tikzcd}$$

Then, $\mathcal{M}$ is called a presheaf of $\mathcal{R}$-modules. If $\mathcal{M}$ is a sheaf, then $\mathcal{M}$ is called a sheaf of $\mathcal{R}$-modules

Remark 29 (sheaf of $\mathcal{S}$-modules from $\mathcal{S}$-bundle). *Let X be an $\mathcal{S}$-manifold over field K, then we define the presheaf $\mathcal{S}$ as follows: for any open set U in X, let $\mathcal{S}(U)$ be the collection of $\mathcal{S}$-functions on U. If $U \supseteq V$, then $\mathcal{S}$-restriction $\rho^U_V : \mathcal{S}(U) \to \mathcal{S}(V)$ is the usual restriction of functions. $\mathcal{S}$ is a presheaf of commutative K-algebras, that is, commutative rings. Let $E \to X$ be an $\mathcal{S}$-bundle, then define a presheaf $\mathcal{S}(E) = \mathcal{S}_X(E)$ as follows: for any open set U in X, let*

$$\mathcal{S}(E)(U) = \mathcal{S}(U, E)$$

be the collection of $\mathcal{S}$-sections of E on U. For any open subset $V \subseteq U$, we define

$$r^U_V : \mathcal{S}(E)(U) \to \mathcal{S}(E)(V)$$

to be the usual restriction of sections. Then, $\mathcal{S}(E)$ is a sheaf of $\mathcal{S}$-sections of vector bundle E. Each $\mathcal{S}(E)(U)$ is an $\mathcal{S}$-section on U which is a $\mathcal{S}(U)$-module. It can be verified that $\mathcal{S}(E)$ is a sheaf of $\mathcal{S}$-modules.

Definition 84 (ideal sheaf). *Let X be a topological space and $\mathcal{R}$ be a sheaf of commutative rings on X. Let $\mathcal{J}$ be a subsheaf of $\mathcal{R}$. In particular, there is an inclusion of ideal for every open set U*

$$\mathcal{J}(U) \subseteq \mathcal{R}(U)$$

Then, $\mathcal{J}$ is called an ideal sheaf in $\mathcal{R}$

Definition 85 (direct sum, free sheaf). *Let $\mathcal{R}$ be a sheaf of commutative rings over a topological space X*

1. *For a positive integer p, we define the sheaf $\mathcal{R}^p$ by*

$$\mathcal{R}^p(U) = \bigoplus_{i=1}^{p} \mathcal{R}(U)$$

This is sheaf of $\mathcal{R}$-modules and called direct sum of sheaf $\mathcal{R}$

2. *If $\mathcal{M}$ is a sheaf of $\mathcal{R}$-modules such that $\mathcal{M}$ is isomorphic to $\mathcal{R}^p$ for some $p \geq 0$, then $\mathcal{M}$ is called a free sheaf of modules*

3. *If $\mathcal{M}$ is a sheaf of $\mathcal{R}$-modules such that each $x \in X$ has a neighbourhood U such that $\mathcal{M}|_U$ is free, then $\mathcal{M}$ is called a locally free sheaf.*

Remark 30 (vector bundles are locally free sheaves)**.** *Let $E \to M$ be a smooth bundle of rank r. Let $x \in M$, there exists an open neighbourhood U of x and a local trivialization $h_U : E|_U \to U \times \mathbb{R}^r$ that induces an isomorphism of smooth bundles then an isomorphism of sheaves*

$$\mathcal{E}(E|_U) \cong \mathcal{E}(U \times \mathbb{R}^r)$$

On the other hand,

$$\mathcal{E}(U \times \mathbb{R}^r) \cong \bigoplus_{i=1}^{r} \mathcal{E}|_U$$

where each $\mathcal{E}|_U$ is the restriction of sheaf $\mathcal{E}$ (the sheaf of smooth functions) on U. Hence, $\mathcal{E}(E)$ is a locally free $\mathcal{E}$-module

Proof. We will show that $\mathcal{E}(U \times \mathbb{R}^r) \cong \bigoplus_{i=1}^{r} \mathcal{E}|_U$. Let $V \subseteq U$, then $f \in \mathcal{E}(U \times \mathbb{R}^r)(V) = \mathcal{E}(V, (U \times \mathbb{R}^r)|_V)$ is a $\mathcal{E}$-section if and only if

$$f(x) = (x, g(x)) = (x, (g_1(x), g_2(x), ..., g_r(x)))$$

where each g_i is a smooth function $V \to \mathbb{R}$. We define the bijection

$$\mathcal{E}(U \times \mathbb{R}^r)(V) \to \bigoplus_{i=1}^{r} \mathcal{E}|_U(V)$$
$$f \mapsto (g_1, g_2, ..., g_r)$$

The bijection is an isomorphism of $\mathcal{E}|_U(V)$-modules. ∎

Theorem 11. *Let M be a connected $\mathcal{S}$-manifold. Then there exists a one-to-one correspondence between (isomorphism classes of) $\mathcal{S}$-bundles over M and (isomorphism classes of) locally free sheaves of $\mathcal{S}$-modules over M*

Proof. We will show that locally free sheaf gives a vector bundle. Suppose that $\mathcal{F}$ is a locally free sheaf of $\mathcal{S}$-modules over M. Then $M = \bigcup_{i \in I} U_i$ is a union of open sets and for each $i \in I$, there exists an isomorphism of sheaves

$$g_i : \mathcal{F}|_{U_i} \to \mathcal{S}^r|_{U_i}$$

for some non-negative integer r. Note that, r does not depend on U_i since M is connected. Let

$$g_{ij} : \mathcal{S}^r|_{U_i \cap U_j} \to \mathcal{S}^r|_{U_i \cap U_j}$$

be defined by $g_{ij} = g_i g_j^{-1}$. The map g_{ij} is a morphism of sheaves, it determine a $\mathcal{S}$-map

$$g_{ij} \in \text{Hom}(\mathcal{S}(U_i \cap U_j)^r, \mathcal{S}(U_i \cap U_j)^r) = \text{Hom}(U_i \cap U_j, GL(\mathbb{R}^r))$$

The collections g_{ij} satisfies

1. $g_{ii} = 1$ for all $i \in I$
2. $g_{ij} g_{jk} = g_{ik}$ for all $i, j, k \in I$

which determines a $\mathcal{S}$-bundle.
TODO - finish the proof by showing the constructed smooth bundle induces the same sheaf ∎

4.4 Sheafification, Stalk

Definition 86 (sheafification)**.** *If $\mathcal{F}$ is a presheaf on X, then a morphism of presheaves*

$$sh : \mathcal{F} \to \mathcal{F}^{sh}$$

called sheafification of $\mathcal{F}$ if $\mathcal{F}^{sh}$ is a sheaf and for every sheaf $\mathcal{G}$, every morphism of presheaves $g : \mathcal{F} \to \mathcal{G}$ factors through $sh : \mathcal{F} \to \mathcal{F}^{sh}$ by a unique morphism of sheaves, that is, the following diagram commutes

$$\begin{tikzcd} \mathcal{F} \arrow[r, "sh"] \arrow[dr, "g"'] & \mathcal{F}^{sh} \arrow[d, dashed, "f"] \\ & \mathcal{G} \end{tikzcd}$$

The sheaf $\mathcal{F}^{sh}$ is called the sheaf generated by $\mathcal{F}$

Definition 87 (stalk of a sheaf)**.** *Let $\mathcal{F}$ be a presheaf over X, let $x \in X$, consider*

$$\tilde{\mathcal{F}}_x = \coprod_{U \ni x} \mathcal{F}(U)$$

the disjoint union of all $\mathcal{F}(U)$ such that U contains x. Define an equivalence relation on $\tilde{\mathcal{F}}_x$ as follows:
Let $f \in \mathcal{F}(U_1)$, $g \in \mathcal{F}(U_2)$, then $f \sim g$ if and only if there is an open subset V such that $x \in V \subseteq U_1 \cap U_2$ and

$$r_V^{U_1} f = r_V^{U_2} g$$

The stalk of $\mathcal{F}$ at x is

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U) = \tilde{\mathcal{F}}_x / \sim$$

the set of equivalence classes. Note 1: stalk is the colimit of the diagram generated by all open set containing x, namely direct limit Note 2: a cocomplete category is a category having all of its colimits. That is, stalk is well-defined for a presheaf to a cocomplete category, for example abelian category

Remark 31. *Some properties of stalk*

1. *Let U be an open set containing x, then there is a map (if the codomain category is cocomplete, then this is a morphism)*

   $$r_x^U : \mathcal{F}(U) \to \mathcal{F}_x$$

   we will denote $r_x^U(f) = f_x$ for $f \in \mathcal{F}(U)$

2. *If $f_x \in \mathcal{F}_x$, then there exists an open set U containing x and $f \in \mathcal{F}(U)$ such that $r_x^U(f) = f_x$*

3. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves over X, then it induces a map $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$*

4. *Suppose that $\mathcal{F}$ is a sheaf, let $f, g \in \mathcal{F}(U)$, if $f_x = h_x$ for all $x \in U$, then $f = h$*

5. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves, suppose $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is a bijection for all $x \in U$. Then ϕ is an isomorphism of sheaves.*

Proof.

3. Let $f_x \in \mathcal{F}_x$, there exists an open set U containing x and a map $f \in \mathcal{F}(U)$ such that $r_x^U f = f_x$, define $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ by

   $$\phi_x f_x = r_x^U \phi_U f$$

   We will show that ϕ_x does not depend on the choice of representative f. Let $\overline{f}$ be another representative of f_x on U, as f and $\overline{f}$ belong the the same stalk, there exists an open set $V \subseteq U$ such that $r_V^U f = r_V^U \overline{f}$, by naturality of ϕ and direct limit of $\mathcal{G}_x$, we have $r_V^U \phi_U = \phi_V r_V^U$ and $r_x^U = r_x^V r_V^U$

   $$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\phi_U"] \arrow[d, "r_V^U"'] & \mathcal{G}(U) \arrow[d, "r_V^U"] \\ \mathcal{F}(V) \arrow[r, "\phi_V"] & \mathcal{G}(V) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{G}(U) \arrow[r, "r_V^U"] \arrow[dr, dashed, "r_x^U"'] & \mathcal{G}(V) \arrow[d, dashed, "r_x^V"] \\ & \mathcal{G}_x \end{tikzcd}$$

Hence,

$$r_x^U \phi_U f = r_x^V r_V^U \phi_U f = r_x^V \phi_V r_V^U f = r_x^V \phi_V r_V^U \overline{f} = r_x^U \phi_U \overline{f}$$

4. For every $x \in U$, as $f_x = h_x$, there exists an open set U_x such that $f|_{U_x} = h|_{U_x}$. As $\{U_x : x \in U\}$ is an open cover for U, hence by the property of sheaf, $f = h$

5. (a) ϕ_U is injective
   Let $f, h \in \mathcal{F}(U)$ such that $\phi_U(f) = \phi_U(h) \in \mathcal{G}(U)$, then for any $x \in U$, $r_x^U \phi_U f = r_x^U \phi_U h$, then $r_x^U f_x = r_x^U h_x$. Since ϕ_x is injective, $f_x = h_x$. From the previous part, $f = h$

   (b) ϕ_U is surjective
   Let $g \in \mathcal{G}(U)$ and $g_x = r_x^U g \in \mathcal{G}_x$, since ϕ_x is surjective, there exists $f_x \in \mathcal{F}_x$ such that $\phi_x f_x = g_x$, hence by defintion of ϕ_x, there exists open set $V \ni x$ and $f \in \mathcal{F}(V)$ such that $r_x^V \phi_V f = \phi_x f_x = g_x = r_x^U g$. Therefore, there exists $W \ni x$ such that $\phi_W r_W^V f = r_W^V \phi_V f = r_W^U g$. That is, for each $x \in U$, there is an open neighbourhood W such that there exists an $f \in \mathcal{F}(W)$ and $\phi_W f = r_W^U g$. Let f_1, f_2 are the two maps on W_1, W_2, then

$$r_{W_1 \cap W_2}^{W_1} f_1 = r_{W_1 \cap W_2}^{W_2} f_2$$

   since they have the same stalks on $W_1 \cap W_2$. By definition of sheaf, there exists $f \in U$ such that $\phi_U f = g$, that is, ϕ_U is surjective.

   (c) bijective morphism of sheaves is an isomorphism of sheaves *TODO*

□

Remark 32 (construction of sheafification). *TODO*

4.5 Tensor Product of Sheaves

Definition 88 (tensor product of sheaves). *Let $\mathcal{F}, \mathcal{G}$ be sheaves on X, let*

$$\mathcal{T}(U) = \mathcal{F}(U) \otimes_{\mathcal{E}(U)} \mathcal{G}(U)$$

Then, $\mathcal{T}$ is a presheaf over X. We define the tensor product of sheaves by

$$\mathcal{F} \otimes_{\mathcal{E}} \mathcal{G} = \mathcal{T}^{sh}$$

Lemma 4 (local morphisms of sheaves induce global morphism of sheaves). *If $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on X and there exists an open cover $\{U_i\}_{i \in I}$ for X such that $\tau_{U_i} : \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$ is a morphism of sheaves for all $i \in I$, then there exists a unique morphism of sheaves $\tau : \mathcal{F} \to \mathcal{G}$. Moreover, if each τ_{U_i} is an isomorphsim of sheaves, then τ is also an isomorphsim of sheaves.*

TODO - we can extend this lemma to (morphisms of stalks induce global morphism of sheaves)

Proof. We will construct a map $\tau_V : \mathcal{F}(V) \to \mathcal{G}(V)$ for any open set $V \subseteq X$ that is compatible with any τ_{U_i}, it will make $\tau : \mathcal{F} \to \mathcal{G}$ a sheaf morphism. For any $U \in \{U_i\}_{i \in I}$ with $\tau_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is the corresponding isomorphism induced from the isomorphism of sheaves on U, define $\tau_V : \mathcal{F}(V) \to \mathcal{G}(V)$ as follows: For any $f \in \mathcal{F}(V)$, let

$$g_U = \tau_U r_U^V f$$

For any $U_i, U_j \in \{U_i\}_{i \in I}$ with $U_i \cap U_j \neq \varnothing$, the diagram below commutes

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[rr, "r^V_{U_i}"] \arrow[dr, "r^V_{U_j}"] \arrow[dd, dashed, "\tau_V"'] & & \mathcal{F}(U_i) \arrow[dr, "r^{U_i}_{U_i \cap U_j}"] \arrow[dd, "\tau_{U_i}" near start] & \\
& \mathcal{F}(U_j) \arrow[rr, "r^{U_j}_{U_i \cap U_j}" near start] \arrow[dd, "\tau_{U_j}" near end] & & \mathcal{F}(U_i \cap U_j) \arrow[dd, "\tau_{U_i \cap U_j}"] \\
\mathcal{G}(V) \arrow[rr] \arrow[dr] & & \mathcal{G}(U_i) \arrow[dr, "r^{U_i}_{U_i \cap U_j}"] & \\
& \mathcal{G}(U_j) \arrow[rr, "r^{U_j}_{U_i \cap U_j}"] & & \mathcal{G}(U_i \cap U_j)
\end{tikzcd}$$

We have

$$r^{U_i}_{U_i\cap U_j} g_{U_i} = r^{U_i}_{U_i\cap U_j} \tau_{U_i} r^V_{U_i} f = r^{U_j}_{U_i\cap U_j} \tau_{U_j} r^V_{U_j} f = r^{U_j}_{U_i\cap U_j} g_{U_j}$$

Therefore, by definition of sheaf, there exists a unique $g_V \in \mathcal{G}(V)$ such that for any $U \in \{U_i\}_{i\in I}$, the diagram below commutes

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "r^V_U"] \arrow[d, dashed, "\tau_V"'] & \mathcal{F}(U) \arrow[d, "\tau_U"] \\ \mathcal{G}(V) \arrow[r, "r^V_U"] & \mathcal{G}(U) \end{tikzcd}$$

That defines a unique map $\tau_V : \mathcal{F}(V) \to \mathcal{G}(V)$. Then, the collection $\{\tau_V : V \subseteq X\}$ defines a unique sheaf morphism $\tau : \mathcal{F} \to \mathcal{G}$. When τ_{U_i} are sheaf isomorphisms, we can construct another unique sheaf morphism $\tau^{-1} : \mathcal{G} \to \mathcal{F}$. It can be verified that τ is a natural isomorphism between two functors $\mathcal{F}$ and $\mathcal{G}$ □

Proposition 46 (tensor product of sheaves of vector bundles). *Let $E \to X$ and $F \to X$ be vector bundles over X. Then there is a sheaf isomorphism*

$$\tau : \mathcal{E}(E) \otimes_{\mathcal{E}} \mathcal{E}(F) \to \mathcal{E}(E \otimes F)$$

where $\mathcal{E}(E)$ and $\mathcal{E}(F)$ are sheaves induced from sections of vector bundles.

Proof. Let

$$\mathcal{T}(-) = \mathcal{E}(E)(-) \otimes_{\mathcal{E}(-)} \mathcal{E}(F)(-)$$

be the tensor product of presheaves and $\mathcal{T}^{sh}$ be the tensor product of sheaves. For every $x \in X$, pick $U \subseteq X$ containing x small enough such that $E|_U \to U$ and $F|_U \to U$ are trivial bundles, then $\mathcal{T}|_U = \mathcal{T}^{sh}|_U$. We will construct a sheaf isomorphism $\mathcal{T}|_U \to \mathcal{E}(E \otimes F)|_U$

$$\begin{tikzcd} \mathcal{T}(U) \arrow[r, "t"] \arrow[d, "r^U_V"] & \mathcal{E}(E\otimes F)(U) \arrow[d, "r^U_V"] \\ \mathcal{T}(V) \arrow[r, "t"] & \mathcal{E}(E\otimes F)(V) \end{tikzcd}$$

Let $e = (e_1, e_2, ..., e_m)$ and $f = (f_1, f_2, ..., f_n)$ be frames of E and F on U. Then, every element $\xi \in \mathcal{T}(U)$ can be written as

$$\xi = \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} (e_i \otimes_{\mathcal{E}(U)} f_j)$$

where $\xi_{ij} \in \mathcal{E}(U)$. And every element $\eta \in \mathcal{E}(E \otimes F)(U)$ can be written as

$$\eta(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} \eta_{ij}(x)(e_i(x) \otimes f_i(x))$$

where $\eta_{ij} \in \mathcal{E}(U)$. Hence, there exists a natural isomorphism of sheaves from $\mathcal{T}|_U = \mathcal{T}^{sh}|_U$ to $\mathcal{E}(E \otimes F)|_U$ defined on $U \subseteq X$. Local sheaf isomorphisms induce a global sheaf isomorphism, which can be verified to be also natural

$$\tau : \mathcal{E}(E) \otimes_{\mathcal{E}} \mathcal{E}(F) \to \mathcal{E}(E \otimes F)$$

□

Chapter 5

Differential Geometry

5.1 Connection, Connection Matrix, Curvature Matrix

Definition 89 (connection). *Let $E \to X$ be a complex vector bundle over a real smooth manifold X, a connection D is a $\mathbb{C}$-linear map*

$$D : \mathcal{E}(X, E) \to \mathcal{E}^1(X, E)$$

such that for any $U \subseteq X$, $\phi \in \mathcal{E}(U)$ and $\xi \in \mathcal{E}(U, E)$, then D satisfy

$$D(\phi\xi) = d\phi \cdot \xi + \phi D\xi$$

Moreover, we also define connection for arbitrary forms, that is $D : \mathcal{E}^p(X, E) \to \mathcal{E}^{p+1}(X, E)$ such that for $\phi \cdot \xi \in \mathcal{E}^p(U) \otimes_{\mathcal{E}(U)} \mathcal{E}(U, E) \cong \mathcal{E}^p(U, E)$

$$D(\phi \cdot \xi) = d\phi \cdot \xi + (-1)^p \phi \wedge D\xi$$

Remark 33 (local representation of connection, connection matrix). *Let $f = e = (e_1, e_2, ..., e_r) \in M_r[\mathcal{E}(U, E)]$ be a frame over $U \subseteq X$. Let $\xi \in \mathcal{E}(U, E)$ with $\xi(f) \in M_r[\mathcal{E}(U)]$ so that $\xi = e \cdot \xi(f) = \sum_{\sigma=1}^r \xi(f)^\sigma e_\sigma$ and $\phi \in \mathcal{E}^p(U)$. Then*

$$\begin{aligned} D\xi &= \sum_{\sigma=1}^r D(\xi^\sigma(f) e_\sigma) \\ &= \sum_{\sigma=1}^r d\xi^\sigma(f) \cdot e_\sigma + \xi^\sigma(f) De_\sigma \end{aligned}$$

As $De_\sigma \in \mathcal{E}^1(U, E) \cong \mathcal{E}^1(U) \otimes_{E(U)} \mathcal{E}(U, E)$, we can write

$$De_\sigma = \sum_{\rho=1}^r \theta(f)_{\rho\sigma} \cdot e_\rho$$

Or in matrix form

$$De = e \cdot \theta(f)$$

The matrix of 1-forms $\theta(f) \in M_{r\times r}[\mathcal{E}^1(U)]$ is called connection matrix. Then, D can be described locally by its connection matrix

$$D\xi = e \cdot (d + \theta(f))\xi(f)$$

Definition 90 (curvature matrix). *Let $E \to X$ be a vector bundle of rank r with connection D, let f be a frame over $U \subseteq X$, define the curvature matrix $\Theta(f) \in M_{r\times r}[\mathcal{E}^2(U)]$ (matrix of 2-forms) by*

$$\Theta(f) = d\theta(f) + \theta(f) \wedge \theta(f)$$

Lemma 5. *Let $g \in M_{r\times r}[\mathcal{E}(U)]$ be a change of frame $h = fg$ where $f = (f_1, f_2, ..., f_r) \in M_r[\mathcal{E}(U, E)]$ and $h = (h_1, h_2, ..., h_r) \in M_r[\mathcal{E}(U, E)]$, then*

1. *$dg + \theta(f)g = g\theta(h) \in M_{r\times r}[\mathcal{E}^1(U)]$*
2. *$\Theta(h) = g^{-1}\Theta(f)g \in M_{r\times r}[\mathcal{E}^2(U)]$*

Proof.

1. $dg + \theta(f)g = g\theta(h)$
   We will show that
   $$f \cdot [(dg) + \theta(f)g] = f \cdot (dg) + (Df)g = Dh = h \cdot \theta(h) = f \cdot [g\theta(h)] \in M_r[\mathcal{E}^1(U,E)]$$
   Indeed
   $$\begin{aligned} Dh_\sigma &= D\left(\sum_{\rho=1}^{r} f_\rho g_{\rho\sigma}\right) \\ &= \sum_{\rho=1}^{r} D(g_{\rho\sigma} f_\rho) \\ &= \sum_{\rho=1}^{r} dg_{\rho\sigma} \cdot f_\rho + \sum_{\rho=1}^{r} g_{\rho\sigma} D f_\rho \end{aligned}$$
   $f \cdot [(dg) + \theta(f)g] = f \cdot [g\theta(h)]$ implies $dg + \theta(f)g = g\theta(h)$ (*TODO check - may be it is true if we interpret these are operators*)
2. $\Theta(h) = g^{-1}\Theta(f)g$
   Taking exterior derivative on both sides of $dg + \theta(f)g = g\theta(h)$
   $$d\theta(f)g - \theta(f) \wedge dg = dg \wedge \theta(h) + gd\theta(h)$$
   Then,
   $$\begin{aligned} g\Theta(h) &= gd\theta(h) + g\theta(h) \wedge \theta(h) \\ &= [d\theta(f)g - \theta(f) \wedge dg - dg \wedge \theta(h)] + (dg + \theta(f)g) \wedge \theta(h) \\ &= d\theta(f)g - \theta(f) \wedge dg - dg \wedge \theta(h) + dg \wedge \theta(h) + \theta(f)g \wedge \theta(h) \\ &= d\theta(f)g - \theta(f) \wedge dg + \theta(f)g \wedge \theta(h) \\ &= d\theta(f)g - \theta(f) \wedge dg + \theta(f) \wedge g\theta(h) \\ &= d\theta(f)g - \theta(f) \wedge dg + \theta(f) \wedge (dg + \theta(f)g) \\ &= d\theta(f)g - \theta(f) \wedge dg + \theta(f) \wedge dg + \theta(f) \wedge \theta(f)g \\ &= d\theta(f)g + \theta(f) \wedge \theta(f)g \\ &= (d\theta(f) + \theta(f) \wedge \theta(f))g \\ &= \Theta(f)g \end{aligned}$$

□

Lemma 6. *(note, notation overuse here - equality of operators)*
$$[d + \theta(f)]^2 = \Theta(f)$$

Proof. Let $\xi \in \mathcal{E}(U,E)$, then $\xi = \sum_{\rho=1}^{r} \xi^\rho(f) f_\rho$ where each $\xi^\rho(f) \in \mathcal{E}(U)$. Let $\xi(f) = (\xi^1(f), \xi^2(f), ..., \xi^r(f)) \in M_r[\mathcal{E}(U)]$

$$\begin{aligned} (d+\theta)^2\xi(f) &= (d+\theta)(d+\theta)\xi(f) \\ &= (d+\theta)(d\xi(f) + \theta\xi(f)) \\ &= d(d\xi(f)) + d(\theta\xi(f)) + \theta \wedge d\xi(f) + \theta \wedge \theta\xi(f) \\ &= d(\theta\xi(f)) + \theta \wedge d\xi(f) + \theta \wedge \theta\xi(f) \\ &= d\theta\xi(f) - \theta \wedge d\xi(f) + \theta \wedge d\xi(f) + \theta \wedge \theta\xi(f) \\ &= d\theta\xi(f) + \theta \wedge \theta\xi(f) \\ &= \Theta\xi(f) \end{aligned}$$

□

5.2 Curvature

Remark 34 (build up to curvature)**.** *Let $E \to X$ be a complex vector bundle over a real smooth manifold X. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X, let $f_\alpha = (e_1^\alpha, e_2^\alpha, ..., e_r^\alpha)$ be a frame of E on U_α. Suppose on every U_α, we have an $r \times r$ matrix of p-forms*

$$\chi(f_\alpha) \in M_{r\times r}[\mathcal{E}^p(U_\alpha)]$$

Let $U = U_\alpha$ with frame $f = f_\alpha$, with trivialization $E|_U \cong U \times \mathbb{C}^r$, we have an isomorphism

$$M_r[\mathcal{E}(U)] \cong \mathcal{E}(U, E|_U) = \mathcal{E}(U, E)$$

Given a frame f, each element $\phi(f) \in M_{r\times r}[\mathcal{E}(U)]$ defines a map

$$\begin{aligned} \phi(f) : M_r[\mathcal{E}(U)] &\to M_r[\mathcal{E}(U)] \\ \xi(f) &\mapsto \phi(f)\xi(f) \end{aligned}$$

that is a map in $\mathrm{Hom}(\mathcal{E}(U, E), \mathcal{E}(U, E)) \cong \mathcal{E}(U, \mathrm{Hom}(E, E))$. Hence, we have an inclusion

$$M_{r\times r}[\mathcal{E}(U)] \hookrightarrow \mathcal{E}(U, \mathrm{Hom}(E, E))$$

(this is actually an isomorphism) Therefore,

$$M_{r\times r}[\mathcal{E}^p(U)] \cong \mathcal{E}^p(U) \otimes_{\mathcal{E}(U)} M_{r\times r}[\mathcal{E}(U)] \hookrightarrow \mathcal{E}^p(U) \otimes_{\mathcal{E}(U)} \mathcal{E}(U, \mathrm{Hom}(E, E)) \cong \mathcal{E}^p(U, \mathrm{Hom}(E, E))$$

Hence, each matrix $\chi(f_\alpha) \in M_{r\times r}[\mathcal{E}^p(U_\alpha)]$ corresponds to an element

$$\chi_\alpha \in \mathcal{E}^p(U_\alpha, \mathrm{Hom}(E, E))$$

Lemma 7 (condition to extend $\chi_\alpha(f_\alpha)$ globally)**.** *Suppose $V = U_\alpha \cap U_\beta \neq \varnothing$, we have two frames of E over V, namely $f_\alpha, f_\beta \in M_r[\mathcal{E}(V, E)]$, so that, $f_\beta = f_\alpha g$ for some change of frame $g \in M_{r\times r}[\mathcal{E}(V)]$. Then, the restrictions*

$$r_V^{U_\alpha}\chi_\alpha = r_V^{U_\beta}\chi_\beta \in \mathcal{E}^p(V, \mathrm{Hom}(E, E))$$

if and only if $\chi_\beta(f_\beta) = g^{-1}\chi_\alpha(f_\alpha)g$

Proof. Let $\xi \in \mathcal{E}(V, E)$ be a section with $\xi(f_\alpha) \in M_r[\mathcal{E}(V)]$, then we have

$$\begin{aligned} (r_V^{U_\alpha}\chi_\alpha)(\xi) &= f_\alpha \cdot \chi_\alpha(f_\alpha)\xi(f_\alpha) \\ (r_V^{U_\beta}\chi_\beta)(\xi) &= f_\beta \cdot \chi_\beta(f_\beta)\xi(f_\beta) \end{aligned}$$

Then, $(r_V^{U_\alpha}\chi_\alpha)(\xi) = (r_V^{U_\beta}\chi_\beta)(\xi)$ if and only if

$$\begin{aligned} f_\beta \cdot \chi_\beta(f_\beta)\xi(f_\beta) &= f_\alpha \cdot \chi_\alpha(f_\alpha)\xi(f_\alpha) \\ &= f_\beta g^{-1} \cdot \chi_\alpha(f_\alpha) g \xi(f_\beta) \\ &= f_\beta \cdot (g^{-1}\chi_\alpha(f_\alpha)g)\xi(f_\beta) \end{aligned}$$

That is, $\chi_\beta(f_\beta) = g^{-1}\chi_\alpha(f_\alpha)g$. The converse direction is the same. □

Definition 91 (curvature)**.** *Let $E \to X$ be a complex smooth vector bundle on a real smooth manifold X, let $D : \mathcal{E}(X, E) \to \mathcal{E}^1(X, E)$ be a connection on E, then the curvature matrix of D induces a global 2-form, namely the curvature of connection D on E*

$$\Theta \in \mathcal{E}^2(X, \mathrm{Hom}(E, E))$$

Let $\xi \in \mathcal{E}(U, E)$ be a local section with $\xi(f) \in M_r[\mathcal{E}(U)]$ and frame $f = (e_1, e_2, ..., e_r) \in M_r[\mathcal{E}(U, E)]$, then

$$\Theta\xi = e \cdot \Theta(f)\xi(f) = e \cdot (d\theta(f) + \theta(f) \wedge \theta(f))\xi(f) = e \cdot (d + \theta(f))^2\xi(f)$$

From the definition, we have

$$D^2 = \Theta : \mathcal{E}(X, E) \to \mathcal{E}^2(X, E)$$

5.3 The Bianchi identity

Definition 92 (Lie algebra, Lie bracket). *Let A be a complex vector space and $[\cdot,\cdot] : A \times A \to A$ be a $\mathbb{C}$-bilinear form on A. A is called Lie algebra if the following holds:*

$$
\begin{aligned}
&[x, y] + [y, x] = 0 \\
&[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0
\end{aligned}
$$

The bilinear form is called Lie bracket

Definition 93 (Lie bracket on $\mathcal{E}^*(X, \mathrm{Hom}(E, E))$). *Define*

$$\mathcal{E}^*(X, \mathrm{Hom}(E, E)) = \oplus_{p=0}^{\infty} \mathcal{E}^p(X, \mathrm{Hom}(E, E))$$

Let $\chi \in \mathcal{E}^p(X, \mathrm{Hom}(E, E))$, let f be a frame over $U \subseteq X$, then χ as a local representation

$$\chi(f) \in M_{r\times r}[\mathcal{E}^p(U)]$$

Let $\psi \in \mathcal{E}^p(X, \mathrm{Hom}(E, E))$ with its local representation $\psi(f) \in M_{r\times r}[\mathcal{E}^p(U)]$, define

$$[\chi(f), \psi(f)] = \chi(f) \wedge \psi(f) - (-1)^{pq}\psi(f) \wedge \chi(f) \in M_{r\times r}[\mathcal{E}^{p+q}(U)]$$

Then, the corresponding element in $\mathcal{E}^{p+q}(U, \mathrm{Hom}(E, E))$ is denoted by

$$[\chi, \psi]_f \in \mathcal{E}^{p+q}(U, \mathrm{Hom}(E, E))$$

In the case of $\mathcal{E}^*(X) = \bigoplus_{p=0}^{\infty} \mathcal{E}^p(X)$, the wedge product makes this structure to be a graded-commutative $\mathcal{E}(X)$-algebra. Here, the Lie bracket makes $\mathcal{E}^*(X, \mathrm{Hom}(E, E))$ to be a grade-commutative $\mathcal{E}(X)$-algebra

Proposition 47. *$[\chi, \psi]_f$ is independent of the choice of frame*

Proof. Let $f = (f_1, f_2, ..., f_r) \in M_r[\mathcal{E}(U, E)]$ and $h = fg = (h_1, h_2, ..., h_r) \in M_r[\mathcal{E}(U, E)]$ where $g \in M_{r\times r}[\mathcal{E}(U)]$ is the change of frame. Then

$$
\begin{aligned}
\chi(h) = g^{-1}\chi(f)g = g^{-1} \wedge \chi(f) \wedge g \\
\psi(h) = g^{-1}\psi(f)g = g^{-1} \wedge \psi(f) \wedge g
\end{aligned}
$$

Then,

$$
\begin{aligned}
[\chi(h), \psi(h)] &= [g^{-1}\chi(f)g, g^{-1}\psi(f)g] \\
&= g^{-1}\chi(f) \wedge \psi(f)g - (-1)^{pq}g^{-1}\psi(f) \wedge \chi(f)g \\
&= g^{-1}[\chi(f), \psi(f)]g
\end{aligned}
$$

Hence, $[\chi, \psi]_h = [\chi, \psi]_f$ □

Proposition 48 (Bianchi identity). *Let $E \to X$ be a complex smooth bundle over a real smooth manifold X with a connection D. Let $\theta(f)$ and $\Theta(f)$ be the connection matrix and curvature matrix with respective to a frame f over U, then*

$$d\Theta(f) = [\Theta(f), \theta(f)]$$

Proof.

$$
\begin{aligned}
[\Theta, \theta] &= \Theta \wedge \theta - (-1)^{2\times 1}\theta \wedge \Theta \\
&= \Theta \wedge \theta - \theta \wedge \Theta \\
&= (d\theta + \theta \wedge \theta) \wedge \theta - \theta \wedge (d\theta + \theta \wedge \theta) \\
&= d\theta \wedge \theta - \theta \wedge d\theta \\
&= d\theta \wedge \theta + (-1)^1\theta \wedge d\theta \\
&= d^2\theta + d(\theta \wedge \theta) \\
&= d(d\theta + \theta \wedge \theta) \\
&= d\Theta
\end{aligned}
$$

□

5.4 Hermitian Metrics on Vector Bundles

Definition 94 (symmetric Hermitian form, Hermitian symmetric matrix)**.** *Let V be a complex vector space (or more generally, a commutative ring with a complex structure like $\mathcal{E}^p(U) = \mathcal{E}(U, \wedge^p T^*(X)_{\mathbb{C}})$). A symmetric Hermitian form $\langle \cdot, \cdot \rangle$ on V is a map*

$$\langle \cdot, \cdot \rangle : V \times V \to V$$

that satisfies the following

$$\langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle$$
$$\langle u, v \rangle = \overline{\langle v, u \rangle}$$

for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{C}$. Given a basis, the matrix of a symmetric Hermitian form is called Hermitian symmetric matrix.

Definition 95 (Hermitian inner product, positive definite Hermitian symmetric matrix)**.** *A symmetric Hermitian form is called Hermitian inner product if $\langle u, u \rangle \geq 0$ for all $u \in V$ and $\langle u, u \rangle = 0$ if and only if $u = 0$. Given a basis, the matrix of a Hermitian inner product is called positive definite Hermitian symmetric matrix.*

Definition 96 (Hermitian metric, Hermitian vector bundle)**.** *Let $E \to X$ be an complex smooth bundle over a real smooth manifold X, if we have a Hermitian inner product $h_x : \langle \cdot, \cdot \rangle_x$ on each fiber E_x, h is called Hermitian metric on E if for any open set $U \subseteq X$ and $\xi, \eta \in \mathcal{E}(U, E)$, the function*

$$\begin{aligned} \langle \xi, \eta \rangle : U &\to \mathbb{C} \\ x &\mapsto \langle \xi(x), \eta(x) \rangle_x \end{aligned}$$

is a smooth function on U. An smooth bundle equipped with a Hermitian metric is called Hermitian vector bundle.

One can define higher order Hermitian form by the following: Let $f \cdot \xi \in \mathcal{E}^p(U) \otimes \mathcal{E}(U, E) \cong \mathcal{E}^p(U, E)$, $g \cdot \eta \in \mathcal{E}^q(U) \otimes \mathcal{E}(U, E) \cong \mathcal{E}^p(U, E)$, then

$$\langle f \cdot \xi, g \cdot \eta \rangle = (f \wedge \overline{g}) \langle \xi, \eta \rangle$$

Definition 97 (local represention of Hermitian metric)**.** *Given a frame $f = (e_1, e_2, ..., e_r) \in M_r[\mathcal{E}(U, E)]$ on $U \subseteq X$, given two sections ξ, η with $\xi = \sum_{i=1}^r \xi^i(f) e_i$ and $\eta = \sum_{i=1}^r \eta^i(f) e_i$ where $\xi^i(f), \eta^i(f) \in \mathcal{E}(U)$. Then,*

$$\begin{aligned} \langle \xi, \eta \rangle &= \left\langle \sum_{i=1}^r \xi^i(f) e_i, \sum_{i=1}^r \eta^i(f) e_i \right\rangle \\ &= \sum_{i=1}^r \sum_{j=1}^r \overline{\eta^j(f)} \langle e_j, e_i \rangle \xi^i(f) \\ &= \overline{\eta(f)}^T h(f) \xi(f) && \text{(matrix form)} \end{aligned}$$

where $\xi(f), \eta(f) \in M_r[\mathcal{E}(U)]$, $h(f) \in M_{r \times r}[\mathcal{E}(U)]$ and each entry $h(f)_{ij} = \langle e_j, e_i \rangle$. Note that, $h(f)(x)$ is a positive definite Hermitian symmetric matrix for every $x \in U$. On the other hand, a complex smooth map from U to the set of positive definite Hermitian symmetric matrices together with a frame f defines a Hermitian metric on U

Theorem 12. *Every vector bundle admits a Hermitian metric*

Proof. Sketch Proof: (1) Pick a locally finite covering of X where each open set has a local trivialization (2) in each local trivialization, define the canonical frame (3) take the positive definite Hermitian symmetric matrix to be identity on each local trivialization, (4) use partition of unity to construct a global Hermitian metric □

Proposition 49. *If $f \in M_r[E(U, E)]$ is a frame and $g \in M_{r \times r}[E(U)]$ is a change of frames, then the Hermitian matrix corresponding to the Hermitian metric with respect to the frame fg is*

$$h(fg) = \overline{g}^T h(f) g$$

Proof.

$$\begin{aligned}
h(fg)_{ij} &= \langle (fg)_j, (fg)_i \rangle \\
&= \left\langle \sum_{k=1}^{r} f_k g_{kj}, \sum_{l=1}^{r} f_l g_{li} \right\rangle \\
&= \sum_{k=1}^{r} \sum_{l=1}^{r} \langle f_k g_{kj}, f_l g_{li} \rangle \\
&= \sum_{k=1}^{r} \sum_{l=1}^{r} (\overline{g}^T)_{il} \langle f_k, f_l \rangle g_{kj} \\
&= \sum_{k=1}^{r} \sum_{l=1}^{r} (\overline{g}^T)_{il} h(f)_{lk} g_{kj}
\end{aligned}$$

□

5.5 Connections on Hermitian Vector Bundles

Definition 98 (connection compatible with Hermitian metric)**.** *A connection D on $E \to X$ is called compatible with a Hermitian metric h on E if*

$$d\langle \xi, \eta \rangle = \langle D\xi, \eta \rangle + \langle \xi, D\eta \rangle$$

for all $\xi, \eta \in \mathcal{E}(X, E)$

Remark 35 (local representation of connection compatible with Hermitian metric)**.** *Let $E \to X$ be a complex smooth vector bundle over a real manifold X. Let (D, h) be a connection D that is compatible with Hermitian metric h. Let $f = e = (e_1, e_2, ..., e_r) \in M_r[\mathcal{E}(U, E)]$ be a frame on $U \subseteq X$. That give a Hermitian matrix $h(f) \in M_{r\times r}[\mathcal{E}(U)]$ and a connection matrix $\theta(f) \in M_{r\times r}[\mathcal{E}^1(U)]$, we have*

$$\begin{aligned}
dh_{\rho\sigma} &= d\langle e_\sigma, e_\rho \rangle \\
&= \langle De_\sigma, e_\rho \rangle + \langle e_\sigma, De_\rho \rangle \\
&= \left\langle \sum_{\tau=1}^{r} \theta_{\tau\sigma} e_\tau, e_\rho \right\rangle + \left\langle e_\sigma, \sum_{\mu=1}^{r} \theta_{\mu\rho} e_\mu \right\rangle \\
&= \sum_{\tau=1}^{r} \theta_{\tau\sigma} h_{\rho\tau} + \sum_{\mu=1}^{r} \overline{\theta_{\mu\rho}} h_{\mu\sigma} \\
&= (h\theta)_{\rho\sigma} + ({}^t\overline{\theta} h)_{\rho\sigma}
\end{aligned}$$

That is, (note that, conjugate transpose also denoted by ${}^t\overline{A} = A^\dagger$)

$$d\theta = h\theta +^t \overline{\theta} h$$

TODO check sufficient

Proposition 50. *Let $E \to X$ be a Hermitian vector bundle with Hermitian metric h, then there exists a connection D compatible with h*

Proof. Given a Hermitian metric h, by Gram-Schmidt process, we have a frame $f = e = (e_1, e_2, ..., e_r) \in M_r[\mathcal{E}(U, E)]$ so that

$$\{e_1(x), e_2(x), ..., e_r(x)\} \subseteq E_x$$

is a orthonormal set for all $x \in U$. That is, $h(f)$ is the identity matrix. Hence, D is compatible with h locally on U is equivalent to

$$0 = \theta +^t \overline{\theta}$$

Setting $\theta = 0$ yields a compatible connection. In that case, for any section $\xi \in \mathcal{E}(U, E)$, we have

$$D\xi = e \cdot d\xi(f)$$

with $\xi = \sum_{\rho=1}^{r} \xi(f)^\rho e_\rho = e \cdot \xi(f)$. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover for X. Using the construction above, we have a connection D_α that is compatible with h on U_α. Let $\{\phi_\alpha : U_\alpha \to \mathbb{R}\}$ be a partition of unity for $\{U_\alpha\}$, let

$$D = \sum_{\alpha \in A} \phi_\alpha D_\alpha$$

D is compatible with h globally. Let $\xi, \eta \in \mathcal{E}(X, E)$,

$$\begin{aligned}
\langle D\xi, \eta \rangle + \langle \xi, D\eta \rangle &= \left\langle \left(\sum_{\alpha \in A} \phi_\alpha D_\alpha \right) \xi, \eta \right\rangle + \left\langle \xi, \left(\sum_{\alpha \in A} \phi_\alpha D_\alpha \right) \eta \right\rangle \\
&= \sum_{\alpha \in A} \phi_\alpha (\langle D_\alpha \xi|_{U_\alpha}, \eta|_{U_\alpha} \rangle + \langle \xi|_{U_\alpha}, D_\alpha \eta|_{U_\alpha} \rangle) \\
&= \sum_{\alpha \in A} \phi_\alpha d \langle \xi|_{U_\alpha}, \eta|_{U_\alpha} \rangle \\
&= d\langle \xi, \eta \rangle
\end{aligned}$$

□

5.6 Canonical Connections on Hermitian Holomorphic Vector Bundles

Definition 99 (Hermitian holomorphic vector bundle). *Let $E \to X$ be a holomorphic vector bundle. X is a holomorphic manifold that is also a real smooth manifold, then $E \to X$ admits a Hermitian metric h. $E \to X$ is called a Hermitian holomorphic vector bundle*

Remark 36 (decomposition of connection on vector bundle). *Let $E \to X$ be a comlex smooth vector bundle over an almost complex manifold X. Note that, since there is an almost complex structure on the cotangent space $T^*(X)_\mathbb{C}$, then $T^*(X)_\mathbb{C} = T^{1,0}(X)_\mathbb{C} \oplus T^{0,1}(X)_\mathbb{C}$, we have the decomposition*

$$\mathcal{E}^1(X, E) = \mathcal{E}^{1,0}(X, E) \oplus \mathcal{E}^{0,1}(X, E)$$

where $\mathcal{E}^{1,0}(X, E) = \mathcal{E}(X, T^{1,0}(X)_\mathbb{C} \otimes E)$ and $\mathcal{E}^{0,1}(X, E) = \mathcal{E}(X, T^{0,1}(X)_\mathbb{C} \otimes E)$. Let $D : \mathcal{E}(X, E) \to \mathcal{E}^1(X, E) = \mathcal{E}(X, T^*(X)_\mathbb{C} \otimes E)$ be a connection, composing D with the projections, we have

$$\begin{aligned}
D' &: \mathcal{E}(X, E) \to \mathcal{E}^{1,0}(X, E) \\
D'' &: \mathcal{E}(X, E) \to \mathcal{E}^{0,1}(X, E) \\
D &= D' + D''
\end{aligned}$$

Theorem 13 (canonical connection). *If h is a Hermitian metric on a holomorphic vector bundle $E \to X$, then h induces a connection $D(h)$ on E that is compatible with h and moreover for any open set $W \subseteq X$, if $\xi \in \mathcal{O}(W, E)$ is a homomorphic section of E, then $D''\xi = 0$. D is called the canonical connection.*

Proof. Let D be a connection on E, let $f = e = (e_1, e_2, ..., e_r) \in M_r[\mathcal{O}(U, E)]$ be a holomorphic frame on $U \subseteq X$. Let $\theta(f) = \theta(f)^{1,0} + \theta(f)^{0,1}$ where $\theta(f)^{1,0} \in M_{r\times r}[\mathcal{E}^{1,0}(U)]$, $\theta(f)^{0,1} \in M_{r\times r}[\mathcal{E}^{0,1}(U)]$. Since, X is holomorphic, hence integrable, then $d = \partial + \overline{\partial}$ where $\partial : \mathcal{E}^{p,q}(U) \to \mathcal{E}^{p+1,q}(U)$ and $\overline{\partial} : \mathcal{E}^{p,q}(U) \to \mathcal{E}^{p,q+1}(U)$

For $\xi \in \mathcal{O}(U, E) \subseteq \mathcal{E}(U, E)$, $\xi = \sum_{\rho=1}^{r} \xi(f)^\rho e_\rho$ and $\xi(f) = (\xi(f)^1, \xi(f)^2, ..., \xi(f)^r) \in M_r[\mathcal{O}(U)] \subseteq M_r[\mathcal{E}(U)]$, we have

$$\begin{aligned}
D\xi(f) &= (d + \theta(f))\xi(f) \\
&= (\partial + \overline{\partial} + \theta(f)^{1,0} + \theta(f)^{0,1})\xi(f) \\
&= (\partial + \theta(f)^{1,0})\xi(f) + (\overline{\partial} + \theta(f)^{0,1})\xi(f)
\end{aligned}$$

Then,

$$\begin{aligned}
D'\xi(f) &= (\partial + \theta(f)^{1,0})\xi(f) \\
D''\xi(f) &= (\overline{\partial} + \theta(f)^{0,1})\xi(f) = \theta(f)^{0,1}\xi(f)
\end{aligned}$$

Note that, $\overline{\partial}\xi(f) = 0$ since entries of $\xi(f)$ are holomorphic functions. (*recall an exercise on defining $\partial, \overline{\partial}$ in $\mathbb{C}^n$*)

Since D is compatible, then

$$\partial h + \overline{\partial} h = dh = h\theta +^t \overline{\theta} h$$

To make $D'' = 0$, $\theta = \theta^{1,0} \in M_{r\times r}[\mathcal{E}^{1,0}(U)] \subseteq M_{r\times r}[\mathcal{E}^1(U)]$ is a matrix of $(1,0)$-forms, then ${}^t\overline{\theta} \in M_{r\times r}[\mathcal{E}^{0,1}(U)] \subseteq M_{r\times r}[\mathcal{E}^1(U)]$ is a matrix of $(0,1)$-forms Hence,

$$\partial h = h\theta \text{ and } \overline{\partial h} =^t \overline{\theta} h$$

This gives $\theta = h^{-1}\partial h$. *TODO - complete the proof with the following lemma* □

Lemma 8. *Let D_α, D_β be connections on U_α, U_β, let $V = U_\alpha \cap U_\beta \neq \varnothing$, then*

$$r_V^{U_\alpha} D_\alpha = r_V^{U_\beta} D_\beta$$

Then, there exists a connection D on $U_\alpha \cup U_\beta$

Proposition 51. *Let D be a canonical connection of a holomorphic bundle $E \to X$ Hermitian metric h, let $\theta(f)$ and $\Theta(f)$ be the connection matrix and curvature matrix for frame f. Then*

1. *$\theta(f)$ is of type $(1,0)$ and $\partial\theta(f) = -\theta(f) \wedge \theta(f)$*
2. *$\Theta(f) = \overline{\partial}\theta(f)$ is of type $(1,1)$*
3. *$\overline{\partial}\Theta(f) = 0$ and $\partial\Theta(f) = [\Theta(f), \theta(f)]$*

Proof.

1. $\theta(f)$ is of type $(1,0)$ and $\partial\theta(f) = -\theta(f) \wedge \theta(f)$
   Note that, $0 = \partial(hh^{-1}) = (\partial h)h^{-1} + h\partial(h^{-1})$, then $\partial(h^{-1}) = -h^{-1}(\partial h)h^{-1}$

$$\begin{aligned}
\partial\theta(f) &= \partial(h^{-1}\partial h) \\
&= \partial(h^{-1}) \wedge \partial h + h^{-1}\partial^2 h \\
&= -h^{-1}(\partial h)h^{-1} \wedge \partial h \\
&= -h^{-1}(\partial h) \wedge h^{-1}\partial h \\
&= -\theta \wedge \theta
\end{aligned}$$

2. $\Theta(f) = \overline{\partial}\theta(f)$ is of type $(1,1)$

$$\Theta = d\theta + \theta \wedge \theta = d\theta - \partial\theta = \overline{\partial}\theta$$

3. $\overline{\partial}\Theta(f) = 0$ and $\partial\Theta(f) = [\Theta(f), \theta(f)]$

$$\overline{\partial}\Theta = \overline{\partial}^2\theta = 0$$

   Then, $\partial\Theta(f) = [\Theta(f), \theta(f)]$ is followed from Bianchi identity.

□

5.7 de Rham Cohomology, Bolbeault Cohomology

We have the exact sequence of sheaves

$$0 \xrightarrow{d} \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \xrightarrow{d} ...$$

$$0 \xrightarrow{d} \mathcal{E}^0(E) \xrightarrow{d} \mathcal{E}^1(E) \xrightarrow{d} \mathcal{E}^2(E) \xrightarrow{d} ...$$

Definition 100 (de Rham cohomology)**.**

$$H^j(X) = \frac{\ker(d : \mathcal{E}^j(X) \to \mathcal{E}^{j+1}(X))}{\operatorname{im}(d : \mathcal{E}^{j-1}(X) \to \mathcal{E}^j(X))}$$
$$H^j(X,E) = \frac{\ker(d : \mathcal{E}^j(X,E) \to \mathcal{E}^{j+1}(X,E))}{\operatorname{im}(d : \mathcal{E}^{j-1}(X,E) \to \mathcal{E}^j(X,E))}$$

We have other exact sequences of sheaves

$$0 \xrightarrow{\overline{\partial}} \mathcal{E}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{E}^{p,1} \xrightarrow{\overline{\partial}} \mathcal{E}^{p,2} \xrightarrow{\overline{\partial}} ...$$

$$0 \xrightarrow{\overline{\partial}} \mathcal{E}^{p,0}(E) \xrightarrow{\overline{\partial}} \mathcal{E}^{p,1}(E) \xrightarrow{\overline{\partial}} \mathcal{E}^{p,2}(E) \xrightarrow{\overline{\partial}} ...$$

$$0 \xrightarrow{\partial} \mathcal{E}^{0,p} \xrightarrow{\partial} \mathcal{E}^{1,p} \xrightarrow{\partial} \mathcal{E}^{2,p} \xrightarrow{\partial} ...$$

$$0 \xrightarrow{\partial} \mathcal{E}^{0,p}(E) \xrightarrow{\partial} \mathcal{E}^{1,p}(E) \xrightarrow{\partial} \mathcal{E}^{2,p}(E) \xrightarrow{\partial} ...$$

Definition 101 (Dolbeault cohomology)**.**

$$H^{p,q}(X) = \frac{\ker(\overline{\partial} : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p,q+1}(X))}{\operatorname{im}(\overline{\partial} : \mathcal{E}^{p,q-1}(X) \to \mathcal{E}^{p,q}(X))}$$
$$H^{p,q}(X, E) = \frac{\ker(\overline{\partial} : \mathcal{E}^{p,q}(X, E) \to \mathcal{E}^{p,q+1}(X, E))}{\operatorname{im}(\overline{\partial} : \mathcal{E}^{p,q-1}(X, E) \to \mathcal{E}^{p,q}(X, E))}$$