The dynamics of a trajectory in $\mathbb{R}^n$

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$x : [0, \infty) \to \mathbb{R}^n$

Given by some ODE

$$\dot{x} = f(x)$$

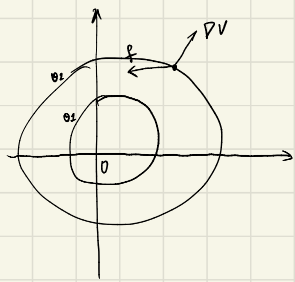
Lyapunov function is a certificate on the stability of this system.

Let $V : E \to [0, \infty)$ be a candidate for Lyapunov function defined in a subset $E \subseteq \mathbb{R}^n$

V is a Lyapunov function if it satisfies two condition:

(1) $V(0) = 0$

(2) $\nabla V \cdot f < 0$



The proof of stability relies on two lemmas

(1) $0 \leq \vartheta_1 \leq \vartheta_2 \Rightarrow E_{\vartheta_1} \subseteq E_{\vartheta_2}$

where $E_\vartheta = \{x : V(x) \leq \vartheta\}$

(2) Every bounded monotone sequence converges.

If we define a function in time $\vartheta : [0, \infty) \to [0, \infty)$

$$\vartheta(t) = V(x(t)) = (V \circ x)(t)$$

The derivative of ϑ

$$\dot{\vartheta} = \nabla V \cdot \frac{dx}{dt} = \nabla V \cdot f < 0$$

This is a monotonically decreasing continuous function bounded at 0.

ϑ decreases implies the stationary position of x stays within $E_{V(x_0)} = \{x : V(x) \leq V(x_0)\}$