Some words on commutative algebra from the perspective of smooth manifold and vector bundle

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Let X be a smooth manifold. Let $\mathcal{E}$ be the canonical sheaf of rings of smooth functions. In particular, for each open set U on X, $\mathcal{E}(U)$ is a ring of smooth functions on U, that is, each smooth function on U is an element of the ring $\mathcal{E}(U)$
Let x be a point on X, then the set of smooth functions $\mathfrak{m}_x$ that vanish at x is a maximal ideal of $\mathcal{E}(X)$

$$\mathfrak{m}_x = \{f \in \mathcal{E}(X) : f(x) = 0\}$$

Each element of the residue field $k_x = \mathcal{E}(X)/\mathfrak{m}_x$ is a equivalence class of smooth functions in which $f \sim g$ if and only if $(f-g)(x) = 0$. That is, the evaluation function $\mathrm{Ev}_x : \mathcal{E}(X) \to \mathbb{R}$ at x defines an $\mathbb{R}$-linear isomorphism

$$\begin{aligned} k_x &\xrightarrow{\sim} \mathbb{R} \\ \bar{f} &\mapsto \mathrm{Ev}_x(f) = f(x) \end{aligned}$$

Quotient at the maximal ideal at a point x gives the field of X. On the other hand, $\mathfrak{m}_x$ is prime, localizing $\mathcal{E}(X)$ at $\mathfrak{m}_x$ gives the ring

$$\mathcal{E}(X)_{\mathfrak{m}_x} = \mathcal{E}(X) \times (\mathcal{E}(X) - \mathfrak{m}_x)/\sim$$

where $(f_1, g_1) \sim (f_2, g_2)$ if and only if there exists $t \in \mathcal{E}(X) - \mathfrak{m}_x$ such that $t(g_1 f_2 - g_2 f_1) = 0$. Since t is nonzero at x, then it is necessary that $g_1(x) f_2(x) - g_2(x) f_1(x) = 0$. By smoothness$g_1 f_2 - g_2 f_1$ must be zero on a neighbourhood of x. On the other hand, given any neighbourhood of x, we can choose t so that it is zero outside of that neighbourhood. Hence, the localization of $\mathcal{E}(X)$ on a maximal ideal $\mathfrak{m}_x$ gives the germs of smooth functions at x.
Let $\pi : E \to X$ be a smooth bundle. Let $\mathcal{E}(-, E)$ be the canonical sheaf of $\mathcal{E}$-module of smooth sections. In particular, for each open set U on X, $\mathcal{E}(U, E)$ is an $\mathcal{E}(U)$-module of smooth sections on U, that is, each smooth section on U is an element of the $\mathcal{E}(U)$-module $\mathcal{E}(U, E)$
Let x be a point on X, let $E_x = \pi^{-1}(x)$ be the fiber at x, the base change of $\mathcal{E}(X, E)$ by the map $\phi_x : \mathcal{E}(X) \to k_x$ is

$$k_x \otimes_{\mathcal{E}(X)} \mathcal{E}(X, E) \cong \frac{\mathcal{E}(X, E)}{\mathfrak{m}_x \mathcal{E}(X, E)}$$

where $\mathfrak{m}_x \mathcal{E}(X, E) = \{f \in \mathcal{E}(X, E) : f(x) = 0\}$ is the submodule of smooth sections that vanishes at x. Again the evaluation function $\mathrm{Ev}_x : \mathcal{E}(X, E) \to E_x$ at x defines an $\mathbb{R}$-linear isomorphism

$$\begin{aligned} \frac{\mathcal{E}(X, E)}{\mathfrak{m}_x \mathcal{E}(X, E)} &\xrightarrow{\sim} E_x \\ \bar{f} &\mapsto \mathrm{Ev}_x(f) = f(x) \end{aligned}$$

Base change by the residue field at x gives the fiber at x.