MA4271 Homework 2

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1 Problem

Problem 1 *Is the set $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, z = 0\}$ a regular surface? Is the set $B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1, z = 0\}$ a regular surface?*

(A) Put $p = (1, 0, 0)$. For any open neighbourhood $V \subseteq \mathbb{R}^3$ containing p, p is a boundary point of A then p is also a boundary point of $V \cap A$. Therefore, $V \cap A$ containing a boundary point hence, it is not homeomorphic to any open set in $\mathbb{R}^2$. Therefore, A is not a regular surface.

(B) Let $D = \{(u, v) : u^2 + v^2 < 1\} \subseteq \mathbb{R}^2$ be the unit disk in $\mathbb{R}^2$. There is a single parameterization $f : D \to B$ of B as defined by

$$f(u, v) = (u, v, 0)$$

f is a homeomorphism, smooth, and its differential is one-to-one.

$$df = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore, B is a regular surface.

Problem 2 *Let two points $p(t)$ and $q(t)$ move with the same speed, p starting from $(0, 0, 0)$ and moving along z axis and q starting at $(a, 0, 0), a \neq 0$ and moving parallel to the y axis. Show that the line joining $p(t)$ and $q(t)$ describes a set in $\mathbb{R}^3$ given by*

$$y(x - a) + zx = 0$$

Moreover, is this a regular surface?

Write $p(t)$ and $q(t)$ as follows

$$p(t) = (0, 0, t)$$
$$q(t) = (a, t, 0)$$

Any point $r = (x, y, z)$ on the line containing $p(t)$ and $q(t)$ has the form

$$
\begin{aligned}
r(\alpha, t) = (x, y, z) &= p(t) + \alpha(q(t) - p(t)) \\
&= (0, 0, t) + \alpha((a, t, 0) - (0, 0, t)) \\
&= (\alpha a, \alpha t, (1 - \alpha)t)
\end{aligned}
$$

For each $(\alpha, t) \in \mathbb{R}^2$, $r(\alpha, t) = (x, y, z)$ satisfies $y(x - a) + zx = 0$. On the other hand, for each (x, y, z) satisfies $y(x - a) + zx = 0$, there is a unique (α, t) such that $r(\alpha, t) = (x, y, z)$. Therefore, the set $S = \{(x, y, z) \in \mathbb{R}^3 : y(x - a) + zx\}$ describes the line joining $p(t)$ and $q(t)$.

The parameterization $(\alpha, t) \mapsto (x, y, z)$ is a homeomorphism, smooth and the differential is

$$
df = \begin{bmatrix} a & 0 \\ t & \alpha \\ -t & 1 - \alpha \end{bmatrix}
$$

This differential is one-to-one everywhere since α and $1 - \alpha$ cannot be zero at the same time (span of row space is 2). Therefore, $y(x - a) + zx = 0$ is a regular surface. [1]

Problem 3 *Let S^2 and H be defined as following*

$$
\begin{aligned}
S^2 &= \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \\
H &= \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}
\end{aligned}
$$

Denote by $N = (0, 0, 1)$ and $S = (0, 0, -1)$ the north and south poles of S^2, respectively. Let $F : S^2 \setminus \{N, S\} \to H$ be defined as following: for each $p \in S^2 \setminus \{N, S\}$, let the perpendicular from p to z axis meet Oz at q. Consider the half line l starting at q and containing p. Then $F(p) = l \cap H$. Prove that F is smooth

Since both S^2 and H are surfaces of revolution (invariant under rotation) on the z axis. With an appropriate change of parameters, let $p = (0, y, z), y > 0$, then $F(p) = (0, \sqrt{1 + z^2}, z)$. We will construct a parameterization for each p and $F(p)$ [2]

Let $f_1 : \mathcal{B}_\delta(0, z) \subseteq \mathbb{R}^2 \to S^2$ be a parameterization on a neighbourhood of p with δ small enough such that $1 - v^2 > 0$

$$
\begin{aligned}
f_1(u, v) &= (ua(u, v), a(u, v), v) \\
df_1 &= \begin{bmatrix} a + u\frac{\partial a}{\partial u} & u\frac{\partial a}{\partial v} \\ \frac{\partial a}{\partial u} & \frac{\partial a}{\partial v} \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

[1] there is another argument for regular surface using regular value

[2] Informally, we will take the intersection between l and the plane $y = 1$

where $a(u,v) = \sqrt{\frac{1-v^2}{u^2+1}} > 0$, f_1 is a homeomorphism, smooth, and its differential is one-to-one ($a > 0$ so the first row and the last row of df_1 is linearly independent)

Let $f_2 : \mathcal{B}_\delta(0,z) \subseteq \mathbb{R}^2 \to H$ be a parameterization on a neighbourhood of $F(p)$.

$$f_2(u,v) = (ub(u,v), b(u,v), v)$$
$$df_2 = \begin{bmatrix} b + u\frac{\partial b}{\partial u} & u\frac{\partial b}{\partial v} \\ \frac{\partial b}{\partial u} & \frac{\partial b}{\partial v} \\ 0 & 1 \end{bmatrix}$$

where $b(u,v) = \sqrt{\frac{1+v^2}{u^2+1}} > 0$, f_2 is a homeomorphism, smooth, and its differential is one-to-one ($b > 0$ so the first row and the last row of df_2 is linearly independent)

Under these two parameterizations, $f_2^{-1} \circ F \circ f_1$ is the identity map which is smooth. Hence, F is smooth.

Problem 4 *Let $a \neq 0$, $b \neq 0$, and $c \neq 0$. Show that each of the equations*

$$x^2 + y^2 + z^2 = ax$$
$$x^2 + y^2 + z^2 = by$$
$$x^2 + y^2 + z^2 = cz$$

define a regular surface and that they all intersect orthogonally.

Let $f(x,y,z) = x^2 + y^2 + z^2 - ax$, 0 is a regular value since $f_x(0) = a \neq 0$. Therefore, $x^2 + y^2 + z^2 = ax$ defines a regular surface. Similar proofs for the other two cases.

Lemma 1 *A regular surface defined by $f(x,y,z) = 0$ has its unit normal vector being the normalized gradient.*

Consider $f(x,y,z) = x^2 + y^2 + z^2 - ax$ and $g(x,y,z) = x^2 + y^2 + z^2 - by$. The gradient of each function is

$$Df = (2x - a, 2y, 2z)$$
$$Dg = (2x, 2y - b, 2z)$$

Then

$$\begin{aligned} Df \cdot Dg &= (4x^2 - 2ax) + (4y^2 - 2yb) + 4z^2 \\ &= 2[(x^2 + y^2 + z^2 - ax) + (x^2 + y^2 + z^2 - by)] \end{aligned}$$

Let $p = (x, y, z)$ on the intersection of two regular surfaces f and g (the intersection is non-empty since it contains $(0, 0, 0)$). Then

$$(Df \cdot Dg)|_p = 0$$

Hence, the two normal vectors are orthogonal. Similar proof for the other two cases.

Problem 5 *Show that the area A of a bounded region of the surface $z = f(x, y)$ is*

$$A = \int\int_Q \sqrt{1 + f_x^2 + f_y^2} dxdy$$

where Q is the normal projection of P onto the xy plane

P can be parameterized by

$$x(u, v) = (u, v, f(u, v))$$

where $(u, v) \in Q$. The area element is

$$||x_u \times x_v|| = ||(1, 0, f_u) \times (0, 1, f_v)|| = ||(-f_u, -f_v, 1)|| = \sqrt{1 + f_u^2 + f_v^2}$$

Hence, area is

$$A = \int_Q ||x_u \times x_v|| = \int_Q \sqrt{1 + f_u^2 + f_v^2}$$

2 Appendix

Proof of Lemma 1

At $p \in S$ where S is a regular surface defined by $f(x, y, z) = 0$. There exists a neighbourhood V of p where V is a graph of a smooth function. Let V be defined by $h : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ as $h(x, y) = (x, y, g(x, y))$ on an open set $U \subseteq \mathbb{R}^2$. Consider the function $f \circ h : U \to \mathbb{R}$ where $(f \circ h)(x, y) = 0$ for all $(x, y) \in U$

$$\begin{aligned} 0 &= D(f \circ h) \\ &= (Df) \circ (Dh) \\ &= (f_x, f_y, f_z) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ g_x & g_y \end{bmatrix} \end{aligned}$$

Therefore, $g_x = -f_x/f_z$ and $g_y = -f_y/f_z$. On the other hand, we can construct the (scaled) normal vector at p by

$$
\begin{aligned}
\alpha n &= h_x \times h_y \\
&= (1, 0, g_x) \times (0, 1, g_y) \\
&= (-g_x, -g_y, 1) \\
&= (f_x, f_y, f_z)/f_z \\
&= Df/f_z
\end{aligned}
$$