Some theorems we might missed in Real Analysis

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Definition 1 (left limit and right limit) *Let $f : [a, b] \to \mathbb{R}$ and $x_0 \in [a, b]$. Let $\epsilon > 0$, the left/right limit is defined as*

$$f(x_0^-) = \lim_{\epsilon \to 0} \sup f((x_0 - \epsilon, x_0)) = \lim_{\epsilon \to 0} \inf f((x_0 - \epsilon, x_0))$$

$$f(x_0^+) = \lim_{\epsilon \to 0} \sup f((x_0, x_0 + \epsilon)) = \lim_{\epsilon \to 0} \inf f((x_0, x_0 + \epsilon))$$

If f is monotonically increasing

$$f(x_0^-) = \sup f([a, x_0))$$

$$f(x_0^+) = \inf f((x_0, b])$$

Definition 2 (removable discontinuity) $f(x^-) = f(x^+) \neq f(x)$

Definition 3 (jump discontinuity) $f(x^-) \neq f(x^+)$

Theorem 1 *Let $f : [a, b] \to \mathbb{R}$ be a monotonically increasing function. Then, f has at most countable number of discontinuities. Furthermore, they are all jump discontinuities.*

Theorem 2 (Heine–Cantor theorem) *A continuous function $f : A \to B$ on a compact metric space into a metric space is uniformly continuous, i.e. given any $\epsilon > 0$, there exists a $\delta > 0$ such that for every open ball $B = \mathcal{B}(a, \delta)$, $f(B)$ has diameter less than ϵ*