homological_algebra

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1 CATEGORICAL CONSTRUCTION OF HOMOLOGICAL ALGEBRA

Definition 1 (initial object, terminal object, pointed category, zero map, kernel). *Given a category C, an object 0 is initial if for all $X \in \operatorname{ob} C$, there is only one map in $\operatorname{Hom}(0, X)$, an object $*$ is terminal if for all $X \in \operatorname{ob} C$, there is only one map in $\operatorname{Hom}(X, *)$. Category C is called pointed if it has initial and terminal objects and the unique map $0 \to *$ is an isomorphism.*

If C is a pointed category, we use the same symbol 0 for both initial object and terminal object. There exists a zero map between any two objects $M, N \in \operatorname{ob} C$, defined by

$$\begin{tikzcd} M \arrow[r] \arrow[dr, "0"'] & 0 \arrow[d] \\ & N \end{tikzcd}$$

the composition of $M \to 0$ and $0 \to N$. Let $f : M \to N$ be a morphism in C, a kernel of f is a map $i : K \to M$ such that $fi = 0$ and such map is universal, that is, if $j : L \to M$ with $fj = 0$, then it factors through K

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[rr, bend left, "0"] & M \arrow[r, "f"] & N \\ L \arrow[u, dashed] \arrow[ur, "j"'] \arrow[urr, bend right, "0"'] & & \end{tikzcd}$$

Category C has kernels if every morphism has a kernel.

Definition 2 (preadditive category, Ab-enriched category). *A category C is called preadditive category (or* Ab*-enriched category) if for any two objects $M, N \in \operatorname{ob} C$, $\operatorname{Hom}(M, N)$ is an abelian group and composition is bilinear, that is, if f, g, h are morphisms in C*

$$f(g + h) = fg + fh$$
$$(f + g)h = fh + gh$$

Definition 3 (additive category). *A category C is additive if*

1. *C is pointed*
2. *C is preadditive*
3. *C admits finite biproduct*

(biproduct) is when product and coproduct coincide

Definition 4 (abelian category). *A category C is abelian if*

1. *C is additive*
2. *every map in C has kernel and cokernel*
3. *every monomorphism in C is the kernel of its cokernel*
4. *every epimorphism in C is the cokernel of its kernel*

Definition 5 (projective class). *Let C be a* ***pointed category with kernels****. A projective class in C is a pair $(\mathcal{P}, \mathcal{E})$ where $\mathcal{P}$ is a collection of objects (called* ***projectives****) and $\mathcal{E}$ is a collection of morphisms (called* ***epimorphisms****) such that*

1. *An object P is* ***projective*** *if and only if P has the universal lifting property against every* ***epimorphism*** *$M \to N$, that is, given any* ***epimorphism*** *$M \to N$, if there is a map $P \to N$, then it factors through M*

$$\begin{tikzcd} M \arrow[r, two heads, "epi"] & N \\ & P \arrow[u] \arrow[ul, dashed] \end{tikzcd}$$

2. *A morphism $f : M \to N$ is an* ***epimorphism*** *if and only if every* ***projective*** *has the universal lifting property against f, that is, given any* ***projective*** *P, if there is a map $P \to N$, then it factors through M*

$$\begin{tikzcd} M \arrow[r, two heads, "f"] & N \\ & P \arrow[u] \arrow[ul, dashed] \end{tikzcd}$$

3. *C has enough* ***projectives****, that is, given any object $M \in \operatorname{ob} C$, for every* ***projective*** *P, there exists an* ***epimorphism*** *$P \to M$.*

Definition 6 (chain complex, acyclic chain complex, exact sequence)**.** *In a* ***pointed category with kernels****, a chain complex is a sequence such that given any subsequence $A \to B \to C$, $A \to B$ factors through $\ker(B \to C)$, that is, there exists a map $A \to \ker(B \to C)$ such that the diagram below commutes*

$$\begin{tikzcd} \dots \arrow[r] & A \arrow[r] \arrow[d, dashed] & B \arrow[r] & C \arrow[r] & \dots \\ & \ker(B \to C) \arrow[ur] & & & \end{tikzcd}$$

Equivalently, the composition $A \to B \to C$ is the zero map.

If there is a notion of epimorphism and the map $A \to \ker(B \to C)$ is an epimorphism, then the sequence is called exact at B. A sequence is called exact sequence or an acyclic chain complex if it is exact everywhere, possibly except the two ends.

Definition 7 (chain map, chain homotopy)**.** *Given two chain complexes $C_\bullet, D_\bullet$ in a* ***pointed category with kernels****, for each $n \in \mathbb{Z}$, there is a map $f_n : C_n \to D_n$ such that the diagram below commutes, then $f_\bullet$ is called a chain map*

$$\begin{tikzcd} \dots & C_{n-1} \arrow[l] \arrow[d, "f_{n-1}"] & C_n \arrow[l] \arrow[d, "f_n"] & C_{n+1} \arrow[l] \arrow[d, "f_{n+1}"] & \dots \arrow[l] \\ \dots & D_{n-1} \arrow[l] & D_n \arrow[l] & D_{n+1} \arrow[l] & \dots \arrow[l] \end{tikzcd}$$

Chain complexes and chain maps form a category and it is called the category of chain complexes.

Given two chain complexes $C_\bullet, D_\bullet$ in a ***pointed preadditive category with kernels****. Let $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ be two chain maps. A chain homotopy from $f_\bullet$ to $g_\bullet$ is a collection of maps $h_n : C_{n-1} \to D_n$ such that $\partial h_{n+1} + h_n \partial = f_n - g_n$*

$$\begin{tikzcd} \dots & C_{n-1} \arrow[l, "\partial"'] \arrow[dr, "h_n"] & C_n \arrow[l, "\partial"'] \arrow[dr, "h_{n+1}"] & C_{n+1} \arrow[l, "\partial"'] & \dots \arrow[l, "\partial"'] \\ \dots & D_{n-1} \arrow[l, "\partial"] & D_n \arrow[l, "\partial"] & D_{n+1} \arrow[l, "\partial"] & \dots \arrow[l, "\partial"] \end{tikzcd}$$

Theorem 1 (fundamental theorem of homological algebra - FTHA)**.** *Let C be a* ***pointed category with kernels*** and $(\mathcal{P}, \mathcal{C})$ *be a projective class in C. Given $f : M \to M'$ in C and the diagram below*

$$\begin{tikzcd} 0 & M \arrow[l] \arrow[d, "f"] & P_0 \arrow[l, "\epsilon"'] \arrow[d, dashed, "f_0"] & P_1 \arrow[l, "d"'] \arrow[d, dashed, "f_1"] & \dots \arrow[l, "d"'] \\ 0 & M' \arrow[l] & P'_0 \arrow[l, "\epsilon'"'] & P'_1 \arrow[l, "d'"'] & \dots \arrow[l, "d'"'] \end{tikzcd}$$

where both chains are chain complexes, the top chain consists of projectives P_n and the bottom chain is acyclic. Then,

- *There exists a chain map defined by $f_n : P_n \to P'_n$*
- *If C is preadditive, the lift is unique upto chain homotopy.*

Proof.

1. The first statement is proved by induction

$$\begin{tikzcd}
P_{n-2} \arrow[dd, "f_{n-2}"'] & P_{n-1} \arrow[l] \arrow[dd, "f_{n-1}"'] & & P_n \arrow[ll] \arrow[dl, dashed] \arrow[dd, dashed, "f_n"] \\
& & K'_{n-1} \arrow[dl] & \\
P'_{n-2} & P'_{n-1} \arrow[l] & & P'_n \arrow[ll] \arrow[ul, two heads]
\end{tikzcd}$$

Suppose there exist maps $f_{n-1} : P_{n-1} \to P'_{n-1}$ and $f_{n-2} : P_{n-2} \to P'_{n-2}$. Let $K'_{n-1} = \ker(P'_{n-1} \to P'_{n-2})$.
Since the bottom chain is acyclic, the map $P'_n \to P'_{n-1}$ factors through K'_{n-1} by an epimorphism.
Since the top chain is a chain complex, the composition $P_n \to P_{n-1} \to P'_{n-1} \to P'_{n-2}$ equals $P_n \to P_{n-1} \to P_{n-2} \to P'_{n-2}$ and equals 0 zero, so $P_n \to P_{n-1} \to P'_{n-1}$ factors through K'_{n-1}
Since P_n is projective and $P'_n \to K'_{n-1}$ is an epimorphism, $P_n \to K'_{n-1}$ factors through P'_n by a map $f_n : P_n \to P'_n$
Base case: $n = 0$, let $P_{n-1} = M, P'_{n-1} = M'$, $P_{n-2} = 0, P'_{n-2} = 0$ and $f_{n-1} = f, f_{n-2} = 0$

2. Let $f_\bullet^{(1)}, f_\bullet^{(2)} : P_\bullet \to P'_\bullet$ be any two lifts from $f : M \to M'$

$$\begin{tikzcd}
M \arrow[d, "f"'] & P_\bullet \arrow[l, "\epsilon"'] \arrow[d, dashed, bend right, "f_\bullet^{(1)}"'] \arrow[d, dashed, bend left, "f_\bullet^{(2)}"] \\
M' & P'_\bullet \arrow[l, "\epsilon'"]
\end{tikzcd}$$

We will prove that $g_\bullet = f_\bullet^{(1)} - f_\bullet^{(2)}$ is chain homotopic to zero, that is to find maps $h_{n+1} : P_n \to P'_{n+1}$ such that $d'h + hd = g$

$$\begin{tikzcd}
0 \arrow[d, "0"'] & P_0 \arrow[l, "d"'] \arrow[d, "g_0"'] & P_1 \arrow[l, "d"'] \arrow[d, "g_1"'] & \dots \arrow[l, "d"'] \\
0 & P'_0 \arrow[l, "d'"] & P'_1 \arrow[l, "d'"] & \dots \arrow[l, "d'"]
\end{tikzcd}$$

Suppose there exists map $h_{n-1} : P_{n-2} \to P'_{n-1}$ and $h_{n-2} : P_{n-3} \to P'_{n-2}$ such that

$$g_{n-2} - h_{n-2}d = d'h_{n-1}$$

$$\begin{tikzcd}
P_{n-3} \arrow[dr, "h_{n-2}"] & P_{n-2} \arrow[l, "d"'] \arrow[dr, "h_{n-1}"] & P_{n-1} \arrow[l, "d"'] \\
P'_{n-3} & P'_{n-2} \arrow[l, "d'"] & P'_{n-1} \arrow[l, "d'"]
\end{tikzcd}$$

Consider the map $g_{n-1} - h_{n-1}d : P_{n-1} \to P'_{n-1}$,

$$\begin{aligned}
d'(g_{n-1} - h_{n-1}d) &= d'g_{n-1} - d'h_{n-1}d && \text{(preadditive)} \\
&= d'g_{n-1} - (g_{n-2} - h_{n-2}d)d && \text{(induction)} \\
&= d'g_{n-1} - g_{n-2}d && \text{(preadditive, } dd = 0\text{)} \\
&= 0 && (g_\bullet \text{ is a chain map})
\end{aligned}$$

Let $K'_{n-1} = \ker(d' : P'_{n-1} \to P'_{n-2})$.
Since the bottom chain is acyclic, the map $d' : P'_n \to P'_{n-1}$ factors through K'_{n-1} by an epimorphism.

$$\begin{tikzcd}
P_{n-2} \arrow[ddr, "h_{n-1}"] & P_{n-1} \arrow[l, "d"'] \arrow[d, dashed] \arrow[ddr, dashed, "h_n"] & \\
& K'_{n-1} \arrow[d] & \\
P'_{n-2} & P'_{n-1} \arrow[l, "d'"] & P'_n \arrow[l, "d'"] \arrow[ul, dashed, two heads]
\end{tikzcd}$$

As $d'(g_{n-1} - h_{n-1}d) = 0$, $g_{n-1} - h_{n-1}d$ factors through K'_{n-1}, that is, $g_{n-1} - h_{n-1}d$ equals the composition $P_{n-1} \to K'_{n-1} \to P'_{n-1}$

Since P_{n-1} is projective and $P'_n \to K'_{n-1}$ is an epimorphism, $P_{n-1} \to K'_{n-1}$ factors through P'_n by a map $h_n : P_{n-1} \to P'_n$, that is, the $d'h_n$ equals the composition $P_{n-1} \to P'_n \to K'_{n-1} \to P'_{n-1}$ and equals the composition $P_{n-1} \to K'_{n-1} \to P'_{n-1}$, hence

$$d'h_n = g_{n-1} - h_{n-1}d$$

Base case: $n = 0$, let $P_{n-2} = 0, P'_{n-2} = 0$, $P_{n-1} = M, P'_{n-1} = M'$, $h_{n-1} = 0$, then

$$d'(g_{n-1} - h_{n-1}d) = 0 \qquad (d' : P'_{n-1} \to P'_{n-2} \text{ is the zero map } M' \to 0)$$

□

Definition 8 (resolution, projective resolution)**.** *Let M be an object in a **pointed category with kernels**. A resolution of M is an exact sequence*

$$0 \longleftarrow M \xleftarrow{\epsilon} P_0 \xleftarrow{d} P_1 \xleftarrow{d} \dots$$

If P_n are projectives in a projective class $(\mathcal{P}, \mathcal{E})$, then the sequence is called $\mathcal{P}$-projective resolution.

Corollary 1. *Let M be an object in a **pointed preadditive category with kernels**. Any two projective resolutions of M are of the same chain homotopy type.*

Definition 9 (additive functor)**.** *Let C, D be **preadditive categories**, a functor $F : C \to D$ is additive if for every $M, N \in \operatorname{ob} C$,*

$$\operatorname{Hom}(M, N) \to \operatorname{Hom}(F(M), F(N))$$

is a homomorphism of abelian groups

Remark 1. *Additive functor preserves chain complex. That is, if $C_\bullet$ is a chain complex, then*

$$\dots \xleftarrow{d} C_{n-1} \xleftarrow{d} C_n \xleftarrow{d} C_{n+1} \xleftarrow{d} \dots$$

$$\dots \xleftarrow{Fd} FC_{n-1} \xleftarrow{Fd} FC_n \xleftarrow{Fd} FC_{n+1} \xleftarrow{Fd} \dots$$

the bottom sequence is also a chain complex.

$$(Fd)(Fd) = F(dd) = F(0) = 0$$

Remark 2. *Additive functor preserves chain homotopy. That is, if $f, g : C_\bullet \to D_\bullet$ are chain homotopic by a chain homotopy h, then, $F(h)$ is a chain homotopy from Ff to Fg*

$$\begin{aligned}
(Fd)(Fh) + (Fh)(Fd) &= F(dh) + F(hd) && \textit{(F is a functor)} \\
&= F(dh + hd) && \textit{(F is additive)} \\
&= F(f - g) && \textit{(}f \simeq g \textit{ by } h\textit{)} \\
&= F(f) - F(g) && \textit{(F is a functor)}
\end{aligned}$$

Remark 3. *a functor between additive categories is additive if and only if it preserves finite coproducts - Mac Lane's "Categories for the working mathematician"*

TODO - https://math.stackexchange.com/questions/793029/do-covariant-functors-preserve-direct-sums

Definition 10 (left derived functor)**.** *Let C be a **pointed preadditive category with kernels**, a projective class $(\mathcal{P}, \mathcal{E})$. Let $F : C \to \mathrm{Ab}$ be an additive functor, then the left derived functor F with respect to $(\mathcal{P}, \mathcal{E})$ are $L_nF : C \to \mathrm{Ab}$ defined by*

$$(L_nF)(X) = H_n(FP_\bullet)$$

where $X \leftarrow P_0 \leftarrow P_1 \leftarrow \dots$ is a $\mathcal{P}$-projective resolution.

Remark 4. *As F is additive and any two projective resolutions of an object are chain homotopic, the left derived functor is well-defined.*

Moreover, left derived functor is functorial TODO

Remark 5 (motivation of left derived functor). *something related to extending $FA \to FB \to FC \to 0$ to $... \to L_2 FA \to L_2 FB \to L_2 FC \to L_1 FA \to L_1 FB \to L_1 FC \to FA \to FB \to FC \to 0$ so that if F is an exact functor (both left and right exact) then $L_1 F = 0$, that is, L_1 measures how far F from being exact. In case of* Tor, Tor_1 *measures how far $\otimes M$ from being exact, or how far M from being projective/free*

Proof.
can be proved, take a look at my notes □

2 UNIVERSAL COEFFICIENT THEOREM FOR CHAIN COMPLEXES OF R-MODULES

Remark 6 (R-Mod is abelian). *If R is a commutative ring, the category of R-modules is abelian*

Remark 7 (projective module, projective class in R-Mod). *In the category* R-Mod, *there is a projective class $(\mathcal{P}, \mathcal{E})$ defined by epimorphism being surjective homomorphism. Then, the following are equivalent*

1. $P \in \mathrm{ob}\, \text{R-Mod}$ *is projective*
2. *Every short exact sequence $0 \to M \to N \to P \to 0$ splits*
3. *P is a direct summand of a free R-module, that is, there exists $Q \in \mathrm{ob}\, \text{R-Mod}$ such that $P \oplus Q$ is a free R-module.*

Proof. *TODO* □

Definition 11 (Tor functor). *In the category* R-Mod, *define* $\mathrm{Tor}_n : \text{R-Mod} \times \text{R-Mod} \to \mathrm{Ab}$ *such that* $\mathrm{Tor}(-, M)$ *is the n-th left derived functor of $(- \times M)$*

$$\mathrm{Tor}_n(N, M) = (L_n(- \otimes M))(N) = H_n(P_\bullet \otimes M)$$

where $N \leftarrow P_0 \leftarrow P_1 \leftarrow ...$ is a projective resolution of N

Remark 8 (some properties of Tor).

$$\begin{aligned} \mathrm{Tor}_n(A, B) &= \mathrm{Tor}_n(B, A) \\ \mathrm{Tor}_n(\varinjlim_\alpha A_\alpha, B) &= \varinjlim_\alpha \mathrm{Tor}_n(A_\alpha, B) \\ \mathrm{Tor}_n(P, B) &= 0 && \text{(if } P \text{ is projective)} \end{aligned}$$

TODO

Definition 12 (direct sum of chain complexes of R-module - product, coproduct, biproduct). *In the category* Ch(R-Mod), *the direct sum* $\oplus : \text{Ch(R-Mod)} \times \text{Ch(R-Mod)} \to \text{Ch(R-Mod)}$ *is defined as follows:*

$$(C_\bullet \oplus D_\bullet)_n = C_n \oplus D_n$$

where $C_\bullet, D_\bullet \in \mathrm{ob}\,\text{Ch(R-Mod)}$ *and the boundary map* $\partial : (C_\bullet \oplus D_\bullet)_n \to (C_\bullet \oplus D_\bullet)_{n-1}$ *is defined by*

$$\begin{aligned} \partial : (C \oplus D)_n &\to (C \oplus D)_{n-1} \\ c \oplus d &\mapsto \partial c \oplus \partial d \end{aligned}$$

Definition 13 (tensor product of chain complexes of R-module). *In the category* Ch(R-Mod), *the tensor product* $\otimes : \text{Ch(R-Mod)} \times \text{Ch(R-Mod)} \to \text{Ch(R-Mod)}$ *is defined as follows:*

$$(C_\bullet \otimes D_\bullet)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

where $C_\bullet, D_\bullet \in \mathrm{ob}\,\text{Ch(R-Mod)}$ *and the boundary map* $\partial : (C_\bullet \otimes D_\bullet)_n \to (C_\bullet \otimes D_\bullet)_{n-1}$ *is the linear extension of* $\partial : C_p \otimes D_q \to (C_\bullet \otimes D_\bullet)_{n-1}$ *where*

$$\partial(c \otimes d) = \partial c \otimes d + (-1)^p c \otimes \partial d$$

Proof. *TODO - bilinear chain map factors through tensor product* □

Lemma 1 (on the spot categorical lemma). *Let $F, G : C \to D$ be functors that are natural to the identity functor. Let $f : A \to B$ be a morphism in C, if $FA = GA$ and $FB = GB$, then $Ff = Gf$ if $C = D$, then it produces a bunch of commutative squares, easy to prove*

$$\begin{tikzcd} A \arrow[d, "f"] & FA \arrow[r, "="] \arrow[d, "Ff"] & GA \arrow[d, "Gf"] \\ B & FB \arrow[r, "="] & GB \end{tikzcd}$$

Theorem 2 (universal coefficient theorem - UCT)**.** *In the category* R-Mod *where R is a PID. Let $C_\bullet$ be a chain complex of free R-modules and N be an R-module. There is a natural short exact sequence (the map $H_n(C_\bullet) \otimes N \to H_n(C_\bullet \times N)$ is natural)*

$$0 \longrightarrow H_n(C_\bullet) \otimes N \longrightarrow H_n(C_\bullet \otimes N) \longrightarrow \mathrm{Tor}_1(H_{n-1}(C_\bullet), N) \longrightarrow 0$$

and this sequence splits (but not naturally)

Proof.
We have the short exact sequence of chain complexes of free R-modules

$$0 \longrightarrow Z_\bullet \hookrightarrow C_\bullet \longrightarrow B_{\bullet-1} \twoheadrightarrow 0$$

where $Z_n = \ker(\partial : C_n \to C_{n-1})$ and $B_n = \mathrm{im}(\partial : C_{n+1} \to C_n)$ and the boundary maps on $Z_\bullet$ and $B_{\bullet-1}$ are zeros. Note that, B_{n-1} is free since it is a submodule of free R-module C_n, then the sequence splits. Then, the sequence below is exact and splits

$$0 \longrightarrow Z_\bullet \otimes N \hookrightarrow C_\bullet \otimes N \twoheadrightarrow B_{\bullet-1} \otimes N \longrightarrow 0$$

where the boundary maps on $Z_\bullet \otimes N$ and $B_{\bullet-1} \otimes N$ are zeros and boundary map on $C_\bullet \otimes N$ is $\partial \otimes 1 : c \otimes n \mapsto \partial c \otimes n$
That induces a long exact sequence

$$\begin{tikzcd} & \ldots \arrow[r] & H_{n+1}(B_{\bullet-1} \otimes N) \arrow[dll, "(i_n)_*"'] \\ H_n(Z_\bullet \otimes N) \arrow[r] & H_n(C_\bullet \otimes N) \arrow[r] & H_n(B_{\bullet-1} \otimes N) \arrow[dll, "(i_{n-1})_*"'] \\ H_{n-1}(Z_\bullet \otimes N) \arrow[r] & \ldots & \end{tikzcd}$$

where the connecting homomorphism $(i_n)_*$ is induced by inclusion map $i_n : B_{n-1} \otimes N \to Z_n \otimes N$

$$\begin{tikzcd} & & B_{n-1} \otimes N \arrow[dll, hook', "i_n"'] \arrow[dl, hook'] \\ Z_n \otimes N & C_n \otimes N \arrow[l] & \end{tikzcd}$$

That induces a short exact sequence

$$0 \longrightarrow \mathrm{coker}(i_n)_* \longrightarrow H_n(C_\bullet \otimes N) \longrightarrow \ker(i_{n-1})_* \longrightarrow 0$$

We have another short exact sequence

$$0 \longrightarrow B_n \xhookrightarrow{j} Z_n \twoheadrightarrow H_n(C_\bullet) \longrightarrow 0$$

Then, the sequence below is exact (*see the motivation of left derived functor*)

$$\begin{tikzcd} \ldots \arrow[r] & \mathrm{Tor}_1(Z_n, N) \arrow[r] & \mathrm{Tor}_1(H_n(C_\bullet), N) \arrow[dll] & \\ B_n \otimes N \arrow[r, "j \otimes 1 = i_n"'] & Z_n \otimes N \arrow[r, two heads] & H_n(C_\bullet) \otimes N \arrow[r] & 0 \end{tikzcd}$$

Note that, both i_n and $(i_n)_*$ are induced from the inclusion $B_n \to Z_n$ and we have a natural transformation of functors from the category of pairs of a chain complex and an R-module to Ab,

$$(B_\bullet, N) \mapsto H_n(B_\bullet) \otimes N$$
$$(B_\bullet, N) \mapsto H_n(B_\bullet \otimes N)$$

so $(i_n)_* = i_n$

$$B_n \hookrightarrow Z_n$$

$$\begin{tikzcd}
B_n \otimes N = H_n(B_\bullet) \otimes N \arrow[r, hook, "j\otimes 1 = i_n"] \arrow[d, "\cong"'] & Z_n \otimes N = H_n(Z_\bullet) \otimes N \arrow[d, "\cong"] \\
H_n(B_\bullet \otimes N) \arrow[r, "(i_n)_*"'] & H_n(Z_\bullet \otimes N)
\end{tikzcd}$$

As Z_n is free, $\mathrm{Tor}_1(Z_n, N) = 0$, then

$$\begin{aligned}
\mathrm{coker}(i_n)_* &= H_n(C_\bullet) \otimes N \\
\ker(i_n)_* &= \mathrm{Tor}_1(H_n(C_\bullet), N)
\end{aligned}$$

We have the short exact sequence

$$0 \longrightarrow H_p(C_\bullet) \otimes N \longrightarrow H_n(C_\bullet \otimes N) \longrightarrow \mathrm{Tor}_1(H_{n-1}(C_\bullet), N) \longrightarrow 0$$

The split of this sequence is from the map $\beta : H_n(C_\bullet \otimes N) \to \mathrm{coker}(i_n)_*$ induced from projection map $C_n \to Z_n$ □

3 KÜNNETH THEOREM FOR CHAIN COMPLEXES OF R-MODULES

Theorem 3 (Künneth theorem)**.** *In the category* R-Mod *where R is a PID, let $C_\bullet, D_\bullet$ be chain complexes of R-modules, and $C_\bullet$ is degree-wise free (each C_n is a free R-module). Then, there is a natural short exact sequence (homology cross product is natural)*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(D_\bullet) \xrightarrow{\times} H_n(C_\bullet \otimes D_\bullet) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(C_\bullet), H_q(D_\bullet)) \longrightarrow 0$$

and this sequence splits (but not naturally)

Proof.

1. **Case 1**: boundary map of $C_\bullet$ is zero

   The boundary map $(C_\bullet \otimes D_\bullet)_n \to (C_\bullet \otimes D_\bullet)_{n-1}$ is the linear extension of

$$\begin{aligned}
\partial : C_p \otimes D_q &\to C_p \otimes D_{q-1} \\
c \otimes d &\mapsto (-1)^{|c|} c \otimes \partial d
\end{aligned}$$

   Hence, we can write the tensor product $C_\bullet \otimes D_\bullet$ as a direct sum of chain complexes

$$C_\bullet \otimes D_\bullet = \bigoplus_p C_p \otimes D_{\bullet - p}$$

   We have

$$\begin{aligned}
H_n(C_\bullet \otimes D_\bullet) &= H_n\left(\bigoplus_p C_p \otimes D_{\bullet-p}\right) \\
&= \bigoplus_p H_n(C_p \otimes D_{\bullet-p}) \\
&= \bigoplus_p C_p \otimes H_n(D_{\bullet-p}) && (C_p \text{ is free, cons of UCT}) \\
&= \bigoplus_{p+q=n} C_p \otimes H_q(D_\bullet) && (\text{shifted chain complex})
\end{aligned}$$

2. **Case 2**: $C_\bullet$ is an arbitrary chain complex

   We have the short exact sequence of chain complexes of free R-modules

$$0 \longrightarrow Z_\bullet \hookrightarrow C_\bullet \longrightarrow B_{\bullet-1} \twoheadrightarrow 0$$

where $Z_n = \ker(\partial : C_n \to C_{n-1})$ and $B_n = \operatorname{im}(\partial : C_{n+1} \to C_n)$ and the boundary maps on $Z_\bullet$ and $B_{\bullet-1}$ are zeros. Note that, B_{n-1} is free since it is a submodule of free R-module C_n, then the sequence splits. Then, the sequence below is exact and splits (*different from the proof of UCT, this uses split, direct sum, tensor product of sequence of chain complexes*)

$$0 \longrightarrow Z_\bullet \otimes D_\bullet \hookrightarrow C_\bullet \otimes D_\bullet \twoheadrightarrow B_{\bullet-1} \otimes D_\bullet \longrightarrow 0$$

That induces a long exact sequence in homology

$$\begin{tikzcd} & \dots \arrow[r] & H_{n+1}(B_{\bullet-1} \otimes D_\bullet) \arrow[dll, "(i_n)_*"'] \\ H_n(Z_\bullet \otimes D_\bullet) \arrow[r] & H_n(C_\bullet \otimes D_\bullet) \arrow[r] & H_n(B_{\bullet-1} \otimes D_\bullet) \arrow[dll, "(i_{n-1})_*"'] \\ H_{n-1}(Z_\bullet \otimes D_\bullet) \arrow[r] & \dots & \end{tikzcd}$$

where the connecting homomorphism $(i_n)_*$ are induced by inclusion map $i_n : (B_{\bullet-1} \otimes D_\bullet)_{n+1} \to (Z_\bullet \otimes D_\bullet)_n$

$$\begin{tikzcd} & & (B_{\bullet-1} \otimes D_\bullet)_{n+1} \arrow[dll, "i_n"'] \arrow[dl, hook] \\ (Z_\bullet \otimes D_\bullet)_n & (C_\bullet \otimes D_\bullet)_n \arrow[l, hook'] & \end{tikzcd}$$

That induces a short exact sequence

$$0 \longrightarrow \operatorname{coker}(i_n)_* \longrightarrow H_n(C_\bullet \otimes D_\bullet) \longrightarrow \ker(i_{n-1})_* \longrightarrow 0$$

We have another short exact sequence

$$0 \longrightarrow B_p \xhookrightarrow{j} Z_p \twoheadrightarrow H_p(C_\bullet) \longrightarrow 0$$

Then, the sequence below is exact (*see the motivation of left derived functor*)

$$\begin{tikzcd} & \dots \arrow[r] & \operatorname{Tor}_1(Z_p, H_q(D_\bullet)) \arrow[r] & \operatorname{Tor}_1(H_p(C_\bullet), H_q(D_\bullet)) \arrow[dlll] & \\ B_p \otimes H_q(D_\bullet) \arrow[rr, hook, "j\otimes 1 = i_n"'] & & Z_p \otimes H_q(D_\bullet) \arrow[r, two heads] & H_p(C_\bullet) \otimes H_q(D_\bullet) \arrow[r] & 0 \end{tikzcd}$$

Take the direct sum over all pairs $p+q=n$ and note that both i_n and $(i_n)_*$ are induced from the the inclusion $B_p \to Z_p$ we have a natural transformation of functors from the category of pairs of chain complexes to Ab

$$(A_\bullet, B_\bullet) \mapsto H_n(A_\bullet \otimes B_\bullet)$$
$$(A_\bullet, B_\bullet) \mapsto \bigoplus_{p+q=n} H_p(A_\bullet) \otimes H_q(B_\bullet)$$

so $(i_n)_* = i_n$

$$B_p \hookrightarrow Z_p$$

$$\begin{tikzcd} \bigoplus_{p+q=n} B_p \otimes H_q(D_\bullet) \arrow[r, hook, "j\otimes 1 = i_n"] \arrow[d, "="'] & \bigoplus_{p+q=n} Z_p \otimes H_q(D_\bullet) \arrow[d, "="] \\ H_{n+1}(B_{\bullet-1} \otimes D_\bullet) \arrow[r, "(i_n)_*"'] & H_n(Z_\bullet \otimes D_\bullet) \end{tikzcd}$$

As Z_p is free, $\mathrm{Tor}_1(Z_p, H_q(D_\bullet)) = 0$, then

$$\mathrm{coker}(i_n)_* = \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(D_\bullet)$$
$$\ker(i_n)_* = \bigoplus_{p+q=n} \mathrm{Tor}_1(H_p(C_\bullet), H_q(D_\bullet))$$

We have the short exact sequence

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(D_\bullet) \xrightarrow{\times} H_n(C_\bullet \otimes D_\bullet) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(C_\bullet), H_q(D_\bullet)) \longrightarrow 0$$

The split of this sequence is from the map $\beta : H_n(C_\bullet \otimes D_\bullet) \to \mathrm{coker}(i_n)_*$ induced from projection map $(C_\bullet \otimes D_\bullet)_n \to (Z_\bullet \otimes D_\bullet)_n$

□

4 REAL-WORLD APPLICATIONS IN ALGEBRAIC TOPOLOGY

4.1 UNIVERSAL COEFFICIENT THEOREM FOR TOPOLOGICAL SPACES

Theorem 4 (universal coefficient theorem - UCT)**.** *Let X be a topological space and $C_\bullet(X)$ be the singular chain complex of X. The singular chain complex with coefficients N be defined by*

$$C_\bullet(X, N) = C_\bullet(X) \otimes N$$

The homology group with coefficients N is defined by

$$H_n(X, N) = H_n(C_\bullet(X, N))$$

Then, there is a short exact sequence

$$0 \longrightarrow H_n(X) \otimes N \longrightarrow H_n(X, N) \longrightarrow \mathrm{Tor}_1(H_{n-1}(X), N) \longrightarrow 0$$

and this sequence splits (but not naturally)

4.2 EILENBERG-ZILBER THEOREM

Remark 9 ($\mathrm{Fun}(C, \text{R-Mod})$)**.** *Given a category C, $\mathrm{Fun}(C, \text{R-Mod})$ is a **pointed preadditive category with kernels** (more precisely, abelian category - will define in the future).*

Given a morphism $F \to G$ in $\mathrm{Fun}(C, \text{R-Mod})$ (a natural transformation from F to G), then the kernel of $F \to G$ is a morphism $K \to F$ such that $K(X)$ is the kernel of $F(X) \to G(X)$ for all $X \in \mathrm{ob}\, C$

Remark 10 (models define projective class in $\mathrm{Fun}(C, \text{R-Mod})$)**.** *Let $\mathcal{M}$ be any set of objects in C (called models), then $\mathcal{M}$ defines a projective class $(\mathcal{P}, \mathcal{E})$ in $\mathrm{Fun}(C, \text{R-Mod})$ where a morphism $G \to F$ is an epimorphism (relative to $\mathcal{M}$) if for all $M \in \mathcal{M}$, $G(M) \twoheadrightarrow F(M)$ is surjective. Then, the following are equivalent*

1. *$P \in \mathrm{ob}\,\mathrm{Fun}(C, \text{R-Mod})$ is projective*
2. *P is a **retract of coproduct** of $R\,\mathrm{Hom}(M, -)$ for some $M \in \mathcal{M}$ where $\mathrm{Hom}(M, -)$ is a functor $\text{R-Mod} \to \text{Set}$, R is the free R-module functor $\text{Set} \to \text{R-Mod}$. In the case of R-module, **retract of coproduct** is the **direct summand** of a R-module*

Proof. *TODO - prove using Yoneda lemma* □

Remark 11. *Let $C = \text{Top}$ and model $\mathcal{M} = \{\Delta^n : n = 0, 1, ...\}$, then for each n, $R\,\mathrm{Hom}(\Delta^n, -)$ is projective. Note that, $R\,\mathrm{Hom}(\Delta^n, -)$ is the n-singular chain complexes with coefficients in R denoted by C_n. Moreover, the sequence below is a projective resolution of the zero-th homology functor H_0*

$$0 \longleftarrow H_0 \xleftarrow{\epsilon} C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} \dots$$

Let Ev_X be the evaluation functor of topological space X, then the left derived functor Ev_X on H_0 is the n-th singular homology of X

$$(L_n \mathrm{Ev}_X)(H_0) = H_n(C_\bullet(X)) = H_n(X)$$

Theorem 5 (Eilenberg-Zilber theorem)**.** *Let X, Y be topological spaces and $C_\bullet$ be the singular chain complex functor from* Top *to* R-Mod*, then there are two chain maps*

$$F : C_\bullet(X \times Y) \to C_\bullet(X) \otimes C_\bullet(Y)$$
$$G : C_\bullet(X) \otimes C_\bullet(Y) \to C_\bullet(X \times Y)$$

such that FG and GF are chain homotopic to identity. That is, $C_\bullet(X \times Y)$ and $C_\bullet(X) \times C_\bullet(Y)$ are of the same chain homotopy type.

Proof.
idea of proof
In the category of Top $\times$ Top, let model $\mathcal{M} = \{(\Delta^p, \Delta^q) : p, q \geq 0\}$. These functors Top $\times$ Top $\to$ R-Mod are projective

$$C_n(X \times Y) = R[\mathrm{Hom}(\Delta^n, X) \times \mathrm{Hom}(\Delta^n, Y)]$$
$$C_p(X) \otimes C_q(Y) = R[\mathrm{Hom}(\Delta^p, X) \times \mathrm{Hom}(\Delta^q, Y)]$$

As $H_0(X \times Y) \to H_0(X) \to H_0(Y)$ is an isomorphism, by FTHA, that induces a chain homotopy equivalence.

$$\begin{tikzcd}
0 & H_0(X \times Y) \arrow[l] \arrow[d, "="'] & C_0(X \times Y) \arrow[l] \arrow[d, dashed] & C_1(X \times Y) \arrow[l] \arrow[d, dashed] & \ldots \arrow[l] \\
0 & H_0(X) \otimes H_0(Y) \arrow[l] & (C_\bullet(X) \otimes C_\bullet(Y))_0 \arrow[l] & (C_\bullet(X) \otimes C_\bullet(Y))_1 \arrow[l] & \ldots \arrow[l]
\end{tikzcd}$$

□

Corollary 2. *Same chain homotopy type induces isomorphism in homology, that is*

$$H_n(X \times Y) \cong H_n(C_\bullet(X) \otimes C_\bullet(Y))$$

4.3 KÜNNETH THEOREM FOR TOPOLOGICAL SPACES

Theorem 6 (Künneth theorem)**.** *In the category* R-Mod *where R is a PID, let $C_\bullet, D_\bullet$ be chain complexes of R-modules, and $C_\bullet$ is degree-wise free (each C_n is a free R-module). Then, there is a short exact sequence*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(D_\bullet) \xrightarrow{\times} H_n(C_\bullet \otimes D_\bullet) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(C_\bullet), H_q(D_\bullet)) \longrightarrow 0$$

and this sequence splits (but not naturally)

Theorem 7 (Künneth theorem)**.** *Let X, Y be topological spaces and R be a PID, there is a natural short exact sequence*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(X; R) \otimes H_q(Y; R) \xrightarrow{\times} H_n(X \times Y) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(X; R), H_q(Y; R)) \longrightarrow 0$$

and this sequence splits (but not naturally).

Corollary 3. *If $H_\bullet(X; R)$ is torsion free over R (of the form $R \oplus R \oplus \ldots$ without any R/nR term), then*

$$H_n(X \times Y) = \bigoplus_{p+q=n} H_p(X; R) \otimes H_q(Y; R)$$