

MA4271 Homework 3

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October 2023

1 Problem

Problem 1 Prove that the absolute value of the torsion τ at a point of an asymptotic curve whose curvature is nowhere zero is given by

$$|\tau| = \sqrt{-K}$$

where K is the Gaussian curvature of the surface at the given point.

Let $\alpha(s)$ be an asymptotic curve on surface S whose curvature is nowhere zero. The normal curvature of $\alpha(s)$ is

$$\langle N, \kappa n \rangle = 0$$

where N denotes the unit normal vector of surface, n denotes the unit normal vector of $\alpha(s)$, κ denotes the curvature of α . As $\kappa \neq 0$, n exists everywhere. Torsion of α is defined as $b'(s) = \tau(s)n(s)$. Restrict N on α , as $N(s)$ is orthogonal to $n(s)$ and $\alpha'(s) \in T_{\alpha(s)}S$, we have

$$N(s) = \lambda(n(s) \times \alpha'(s)) = \lambda b(s)$$

where λ is either $+1$ or -1 . Then

$$\tau(s) = ||b'(s)|| = ||N'(s)||$$

We can write $N'(s)$ in term of Gauss map (differential of N along the direction of tangent vector)

$$\begin{aligned} N'(s) &= dN_{\alpha(s)}(\alpha'(s)) \\ &= dN_{\alpha(s)}(e_1 \cos \theta + e_2 \sin \theta) && (\text{let } \alpha'(s) = e_1 \cos \theta + e_2 \sin \theta) \\ &= e_1 k_1 \cos \theta + e_2 k_2 \sin \theta && (\text{eigen direction}) \end{aligned}$$

Then $||N'(s)||^2 = k_1^2(\cos \theta)^2 + k_2^2(\sin \theta)^2$. On the other hand, by Euler formula, $0 = k_1(\cos \theta)^2 + k_2(\sin \theta)^2$, Then

$$\begin{aligned} ||N'(s)||^2 &= k_1 k_1 (\cos \theta)^2 + k_2 k_2 (\sin \theta)^2 \\ &= k_1 (-k_2 (\sin \theta)^2) + k_2 (-k_1 (\cos \theta)^2) \\ &= -k_1 k_2 = -K \end{aligned}$$

Then

$$\tau(s) = ||N'(s)|| = \sqrt{-K}$$

Problem 2 Suppose that S_1 and S_2 intersect along a regular curve C and make an angle $\theta(p), p \in C$. Assume that C is a line of curvature of S_1 . Prove that $\theta(p)$ is constant if and only if C is a line of curvature of S_2

(\Leftarrow) $C = \alpha(s)$ is the line of curvature of both S_1 and S_2 , by Oline Rodrigues,

$$\begin{aligned} N'_1(s) &= \lambda_1(s)\alpha'(s) \\ N'_2(s) &= \lambda_2(s)\alpha'(s) \end{aligned}$$

where $N_1(s), N_2(s)$ is the restriction of unit normal vectors of S_1, S_2 on C , $-\lambda_1(s), -\lambda_2(s)$ are the corresponding principle curvatures

As $\alpha'(s)$ is orthogonal to $N_1(s)$ and $N_2(s)$

$$\begin{aligned}\langle N_1'(s), N_2(s) \rangle &= \lambda_1(s) \langle \alpha'(s), N_2(s) \rangle = 0 \\ \langle N_2'(s), N_1(s) \rangle &= \lambda_2(s) \langle \alpha'(s), N_1(s) \rangle = 0\end{aligned}$$

Then

$$\frac{d}{ds} \langle N_1(s), N_2(s) \rangle = \langle N_1'(s), N_2(s) \rangle + \langle N_2'(s), N_1(s) \rangle = 0$$

Then $\theta = \langle N_1, N_2 \rangle$ is constant along C

(\implies) $C = \alpha(s)$ is the line of curvature of S_1 , and $\theta = \langle N_1, N_2 \rangle$ is constant along C , similarly by Oline Rodrigues

$$N_1'(s) = \lambda_1(s) \alpha'(s)$$

Then

$$\langle N_1'(s), N_2(s) \rangle = \lambda_1(s) \langle \alpha'(s), N_2(s) \rangle = 0$$

Then

$$\langle N_2'(s), N_1(s) \rangle = \frac{d}{ds} \langle N_1(s), N_2(s) \rangle - \langle N_1'(s), N_2(s) \rangle = 0$$

As $N_2(s)$ is a unit vector, then $N_2'(s)$ is orthogonal to $N_2(s)$. Since both $\alpha'(s)$ and $N_2'(s)$ are orthogonal two both $N_1(s)$ and $N_2(s)$, then they are parallel, i.e. $N_2'(s) = \lambda(s) \alpha'(s)$. By Oline Rodrigues, C is the line of curvature of S_2

Problem 3 Let $\lambda_1, \dots, \lambda_m$ be the normal curvatures at $p \in S$ along directions making angles $0, 2\pi/m, \dots, (m-1)2\pi/m$ with a principle direction. Prove that

$$\lambda_1 + \dots + \lambda_m = mH$$

where H is the mean curvature at p

Denote $\theta_i = 2\pi \frac{i-1}{m}$, by Euler formula,

$$\sum_{i=1}^m \lambda_i = \sum_{i=1}^m [k_1 (\cos \theta_i)^2 + k_2 (\sin \theta_i)^2]$$

where k_1, k_2 are the principle curvatures ¹. If we rotate the n directions by $\frac{\pi}{2}$, we have

$$\begin{aligned}\sum_{i=1}^m \lambda'_i &= \sum_{i=1}^m [k_1 (\cos(\theta_i + \pi/2))^2 + k_2 (\sin(\theta_i + \pi/2))^2] \\ &= \sum_{i=1}^m [k_1 (\sin \theta_i)^2 + k_2 (-\cos \theta_i)^2] \quad (\cos(\theta + \pi/2) = \sin \theta, \sin(\theta + \pi/2) = -\cos \theta,)\end{aligned}$$

where $\lambda'_1, \dots, \lambda'_m$ are the corresponding normal curvatures after rotation. Then

$$\left(\sum_{i=1}^m \lambda_i + \sum_{i=1}^m \lambda'_i \right) = m(k_1 + k_2) = 2mH$$

and

$$\left(\sum_{i=1}^m \lambda_i - \sum_{i=1}^m \lambda'_i \right) = \sum_{i=1}^m (k_1 - k_2) [(\cos \theta_i)^2 - (\sin \theta_i)^2] = (k_1 - k_2) \sum_{i=1}^m \cos(2\theta_i)$$

Note that

$$\sum_{i=1}^m \cos(2\theta_i) = \sum_{i=1}^m \operatorname{Re}[e^{i2\theta_i}] = \operatorname{Re} \left[\sum_{i=1}^m e^{i2\theta_i} \right] = \operatorname{Re}[0] = 0$$

where \mathbf{i} denotes the imaginary unit. Hence,

$$\lambda_i = \frac{1}{2} \left(\sum_{i=1}^m \lambda_i + \sum_{i=1}^m \lambda'_i \right) = mH$$

¹ k_1 is either maximum or minimum principle curvature

Problem 4 Show that a surface which is compact has an elliptic point.

Let C be a compact regular surface. Pick $c \in \mathbb{R}^3$. For any $x \in C$, the map $\phi(x) = \|x - c\|$ is continuous, then it admits a maximum value, namely $r = \max_{x \in C} \|x - c\|$. Let $p \in C \cap \mathcal{S}_r(c)$ on the intersection of C and the sphere of radius r centered at c .

Let N be any plane containing p that intersects C at more than 1 point², then N intersects $\mathcal{S}_r(c)$ at more than 1 point³, then $N \cap \mathcal{S}_r(c)$ is a sphere of radius $0 < r_N \leq r$, then the curvature of $N \cap C$ is bounded below by a positive number, i.e. $\kappa_N \geq \frac{1}{r_N} \geq \frac{1}{r}$. Therefore, any principle curvature of C at p is bounded below by a positive number, then it must be an elliptic point or hyperbolic point.

Furthermore, C is contained in one of the half space separated by $T_p \mathcal{S}_r(c)$, then C cannot be a hyperbolic point.

Problem 5 Obtain the asymptotic curves of the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$

In the parameterization

$$x(u, v) = \left(\frac{\cos(u+v)}{\cos(u-v)}, \frac{\sin(u+v)}{\cos(u-v)}, \tan(u-v) \right)$$

Claim: asymptotic curves are the coordinate curves

Let $a = \cos(u+v)$, $b = \sin(u+v)$, $c = \cos(u-v)$, $d = \sin(u-v)$, then we have

- $a^2 + b^2 = 1, c^2 + d^2 = 1$
- $ac - bd = \cos(2u), ac + bd = \cos(2v)$
- $ad + bc = \sin(2u), ad - bc = -\sin(2v)$
- $a_u = -b, a_v = -b$
- $b_u = a, b_v = a$
- $c_u = -d, c_v = d$
- $d_u = c, d_v = -c$

Calculation

$$\begin{aligned} x &= \left(\frac{a}{c}, \frac{b}{c}, \frac{d}{c} \right) \\ x_u &= \left(\frac{ad - bc}{c^2}, \frac{ac + bd}{c^2}, \frac{1}{c^2} \right) \\ x_v &= \left(\frac{-ad - bc}{c^2}, \frac{ac - bd}{c^2}, \frac{-1}{c^2} \right) \\ N &= x_u \times x_v = \left(\frac{-2a}{c^3}, \frac{2b}{c^3}, \frac{2d}{c^3} \right) \\ N_u &= \left(\frac{2bc - 6ad}{c^4}, \frac{2ac + 6bd}{c^4}, \frac{2 + 4d^2}{c^4} \right) \\ N_v &= \left(\frac{2bc + 6ad}{c^4}, \frac{2ac - 6bd}{c^4}, \frac{-2 - 4d^2}{c^4} \right) \end{aligned}$$

Second fundamental form

$$\begin{aligned} e &= -\langle N_u, x_u \rangle = 0 \\ f &= -\langle N_u, x_v \rangle > 0 \\ g &= -\langle N_v, x_v \rangle = 0 \end{aligned}$$

Then the asymptotic curves are the coordinate curves w.r.t the parameterization

2 Appendix

²normal section guarantees the existence of N

³because N intersects $C \subseteq \mathcal{B}_r(c)$ at more than 1 point