complex_analysis_ahlfors

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Chapter 1

COMPLEX NUMBERS

Chapter 2

COMPLEX FUNCTIONS

Chapter 3

ANALYTIC FUNCTION AS MAPPING

3.1 ELEMENTARY POINT SET TOPOLOGY

3.2 CONFORMALITY

3.2.1 ARCS AND CLOSED CURVES

3.2.2 ANALYTIC FUNCTIONS IN REGIONS

Definition 1 (Analytic Function on an Open Set) *A complex-valued function $f(z)$ defined on an open set Ω is said to be analytic in Ω if it has a derivative at each point of Ω. An analytic function is also called holomorphic*

Definition 2 (Analytic Function on an Arbitrary Set) *A function $f(z)$ is analytic on an arbitrary point set A if it is the restriction to A of a function which is analytic in some open set containing A*

Theorem 1 *An analytic function in a region [1] Ω whose derivative vanishes [2] identically must reduce to a constant. The same is true if either the real part, the imaginary part, the modulus, or the argument is constant.*

3.2.3 CONFORMAL MAPPING

3.2.4 LENGTH AND AREA

3.3 LINEAR TRANSFORMATION

3.3.1 THE LINEAR GROUP

3.3.2 THE CROSS RATIO

Definition 3 (Cross Ratio) *The cross ratio (z_1, z_2, z_3, z_4) is the image of z_1 under the linear transformation which carries z_2, z_3, z_4 into $1, 0, \infty$*

Theorem 2 *If z_1, z_2, z_3, z_4 are distinct points in the extended plane and T any linear transformation, then $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$*

Theorem 3 *The cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a circle (or a straight line)*

Theorem 4 *A linear transformation carries circles to circles (straight line is a special circle)*

3.3.3 SYMMETRY

Definition 4 *The points z, z^* are said to be symmetric with respect to the circle C through z_1, z_2, z_3 if $(z^*, z_1, z_2, z_3) = (z, z_1, z_2, z_3)$*

Theorem 5 (The Symmetry Principle) *If a linear transformation carries a circle C_1 into a circle C_2, then it transforms any pair of symmetric points with respect to C_1 into a pair of symmetric points with respect to C_2*

[1] a connected open set does not include ∞

[2] at one point

3.3.4 ORIENTED CIRCLES

3.3.5 FAMILY OF CIRCLES

3.4 ELEMENTARY CONFORMAL MAPPING

3.4.1 THE USE OF LEVEL CURVES

3.4.2 A SURVEY OF ELEMENTARY MAPPINGS

3.4.3 ELEMENTARY RIEMANN SURFACES

Chapter 4

COMPLEX INTEGRATION

4.1 FUNDAMENTAL THEOREM

4.1.1 LINE INTEGRALS

4.1.2 RECTIFIABLE ARCS

4.1.3 LINE INTEGRALS AS FUNCTIONS OF ARCS

Theorem 6 (Fundamental Theorem of Line Integral) *The line integral $\int_\gamma (pdx + qdy)$ defined in Ω depends only on the end point of γ if and only if there exists a function $U(x, y)$ in Ω with partial derivatives $\frac{\partial U}{\partial x} = p$ and $\frac{\partial U}{\partial y} = q$*

Theorem 7 *The integral $\int_\gamma f(z)dz$ with continuous f depends only on the end points of γ if and only if f is the derivative of an analytic function in Ω*

4.1.4 CAUCHY THEOREM FOR A RECTANGLE

Theorem 8 (Cauchy Theorem) *Let ∂R denote the boundary curve of a rectangle R. If the function $f(z)$ is analytic on R then*

$$\int_{\partial R} f(z)dz$$

Theorem 9 (Cauchy Theorem) *Let $f(z)$ be analytic on the set R' obtained from a rectangle R by omitting a finite number of interior point ζ_j. If it is true that*

$$\lim_{z \to \zeta_j} (z - \zeta_j) f(z) = 0$$

for all j, then

$$\int_{\partial R} f(z)dz = 0$$

4.1.5 CAUCHY THEOREM IN A DISK

Theorem 10 (Cauchy Theorem) *If $f(z)$ is analytic in an open disk Δ, then*

$$\int_\gamma f(z)dz = 0$$

for every closed curve γ in Δ

Theorem 11 (Cauchy Theorem) *Let $f(z)$ be analytic in the region Δ' obtained by omitting a finite number of point ζ_j from an open disk Δ. If $f(z)$ satisfies the condition $\lim_{z \to \zeta_j} (z - \zeta_j) f(z) = 0$ for all j, then*

$$\int_\gamma f(z)dz = 0$$

for every closed curve γ in Δ'

4.2 CAUCHY INTEGRAL FORMULA

4.2.1 THE INDEX OF A POINT WITH RESPECT TO A CLOSED CURVE

Lemma 1 *If the piecewise differentiable closed curve γ does not pass through the point a, then the value of the integral*

$$\int_\gamma \frac{dz}{z-a}$$

is a multiple of $2\pi i$

Definition 5 (Index of a Point with respect to a Curve - Winding Number) *The index of point a with respect to curve γ is defined as*

$$n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$$

Lemma 2 *Let z_1, z_2 be two points on a closed curve γ which does not pass through the origin. Denote the subarc from z_1 to z_2 in the direction of the curve by γ_1 and the subarc from z_2 to z_1 by γ_2. Suppose that z_1 lies in the lower half plane and z_2 in the upper half plane. If γ_1 does not meet the negative real axis and γ_2 does not meet the positive real axis, then $n(\gamma, 0) = 1$*

Theorem 12 (Jordan Curve Theorem) *Every Jordan curve* [1] *in the plane determines exactly two regions*

4.2.2 THE INTEGRAL FORMULA

Theorem 13 (Cauchy Integral Formula) *Suppose that $f(z)$ is analytic in an open disk Δ and let γ be a closed curve in Δ. For any point a not in γ*

$$n(\gamma, a)f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-a}$$

4.2.3 HIGHER DERIVATIVES

Lemma 3 *Suppose that $\phi(\zeta)$ is continuous on the arc γ. Then the function*

$$F_n(z) = \int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta - z)^n}$$

is analytic and each of the regions determined by γ and its derivatives is $F_n'(z) = nF_{n+1}(z)$

Theorem 14 (Morena Theorem) *If $f(z)$ is defined and continuous in a region Ω and if $\int_\gamma f(z)dz = 0$ for all closed curves γ in Ω, then $f(z)$ is analytic in Ω*

Theorem 15 (Liouville Theorem) *A function which is analytic and bounded in the whole plane must reduce to a constant*

4.3 LOCAL PROPERTIES OF ANALYTIC FUNCTION

4.3.1 REMOVABLE SINGULARITIES - TAYLOR THEOREM

Theorem 16 (Removable Singularities) *Suppose that $f(z)$ is analytic in the region Ω' obtained by omitting a point from a region Ω. A necessary and sufficient condition that there exist an analytic function in Ω which coincides with $f(z)$ in Ω' is that $\lim_{z\to a}(z-a)f(z) = 0$. The extended function is uniquely determined. a is said to be a removable singularity.*

Theorem 17 (Taylor Theorem) *If $f(z)$ is analytic in a region Ω containing a, it is possible to write*

$$f(z) = \left[f(a) + \frac{f'(z)}{1!}(z-a) + \frac{f''(z)}{2!}(z-a)^2 + ... + \frac{f^{(n-1)}(z)}{(n-1))!}(z-a)^{n-1}\right] + f_n(z)(z-a)^n$$

[1] simple curve - a curve doesn't cut itself

4.3.2 ZEROS AND POLES

Proposition 1 *If $f(z)$ is analytic in Ω and if $f(z) = 0$ on a set A which has an accumulation point in Ω* [2] *then $f(z)$ is identically equal to 0 on Ω*

Proposition 2 (Zeros) *Suppose $f(z)$ is not identically zero. Then if $f(a) = 0$, there exists a first derivative $f^{(h)}(a)$ which is different from zero. Then, a is said to be a zero of order h. Furthermore, by Taylor Theorem,*

$$f(z) = (z-a)^h f_h(z)$$

where $f_h(z)$ is analytic and $f_h(a) \neq 0$

Definition 6 (Isolated Singularity) *Let $f(z)$ be an analytic in a neighbourhood of a except a, then a is said to be an isolated singularity*

Proposition 3 (Poles) *If $\lim_{z\to a} f(z) = \infty$, then a is said to be a pole of $f(z)$. If a is an isolated singularity, there exists $\delta > 0$ such that $f(z) \neq 0$ on $0 < |z-a| < \delta$. Then $g(z) = \frac{1}{f(z)}$ is defined and analytic $0 < |z-a| < \delta$ and a is a removable singularity. Let h be the order of the zero at a of $g(z)$. a is also said to be the pole of order h of $f(z)$. Similarly, by Taylor Theorem,*

$$f(z) = (z-a)^{-h} f_h(z)$$

where $f_h(z)$ is analytic and different from zero in $|z-a| < \delta$

Definition 7 (Meromorphic) *An analytic function $f(z)$ in a region Ω except for poles is said to be meromorphic in Ω*

Proposition 4 (Algebraic Order) *Consider the conditions*

1. $\lim_{z\to a} |z-a|^\alpha |f(z)| = 0$
2. $\lim_{z\to a} |z-a|^\alpha |f(z)| = \infty$

If there exists an integer h such that 1 holds for all $\alpha > h$ and 2 holds for all $\alpha < h$, then h is said to be the algebraic order of $f(z)$ at a. It is positive in the case of pole, negative in the case of zero, and zero if $f(a) \neq 0$ and analytic at a

Definition 8 (Essential Isolated Singularity) *An isolated singularity which is neither removable or pole*

Theorem 18 (Casorati-Weierstrass Theorem) *An analytic function comes arbitrary close to any complex value in every neighbourhood of an essential singularity*

4.3.3 THE LOCAL MAPPING

Theorem 19 *Let z_j be the zeros of an analytic function $f(z)$ on a disk Δ and does not vanish identically, each zero be counted as many times as its order indicates. For every closed curve γ in Δ which does not pass through a zero*

$$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz$$

where the sum has only a finite number of nonzero terms

Theorem 20 *Suppose that $f(z)$ is analytic at z_0, $f(z_0) = w_0$, and $f(z) - w_0$ has a zero of order n at z_0. If $\epsilon > 0$ is sufficiently small, there exists $\delta > 0$ such that for all a in $|a - w_0| < \delta$, the equation $f(z) = a$ has exactly n roots in the disk $|z - z_0| < \epsilon$*

Corollary 1 *A nonconstant analytic function maps an open set to an open set*

Corollary 2 *If $f(z)$ is analytic at z_0 with $f'(z_0) \neq 0$, it maps a neighbourhood of z_0 conformally and topologically onto a region.*

4.3.4 THE MAXIMUM PRINCIPLE

Theorem 21 (The Maximum Principle) *If $f(z)$ is analytic and nonconstant in a region Ω, then its absolute value $|f(z)|$ has no maximum in Ω*

Theorem 22 (The Maximum Principle) *If $f(z)$ is defined an continuous on a closed bounded set E and analytic on the interior of E, then the maximum of $|f(z)|$ on E is assumed on the boundary of E*

Theorem 23 *If $f(z)$ is analytic for $|z| < 1$ and satisfies the conditions $|f(z)| \leq 1$, $f(0) = 0$, then $|f(z)| \leq |z|$ and $|f'(0) \leq 1|$. If $|f(z)| = |z|$ for some $z \neq 0$ or if $|f'(0)| = 1$ then $f(z) = cz$ with the constant c of absolute value 1*

[2] $\overline{A} \cap \Omega \neq \varnothing$

4.4 THE GENERAL FORM OF CAUCHY THEOREM

4.4.1 CHAINS AND CYCLES

Definition 9 (Chains) *Let $\gamma_1, \gamma_2, ..., \gamma_n$ form a subdivision of the arc γ. Then the sum $\gamma_1 + \gamma_2 + ... + \gamma_n$ is said to be a chain*

4.4.2 SIMPLE CONNECTIVITY

Definition 10 (Simply Connected) *A region is said to be simply connected if its complement with respect to the extended plane is connected.*

Theorem 24 *A region Ω is simply connected if and only if $n(\gamma, a) = 0$ for all cycles γ in Ω and all points a belong to the complement of Ω*

4.4.3 HOMOLOGY

Definition 11 (Homologous) *A cycle γ in an open set Ω is said to be homologous to zero with respect to Ω if $n(\gamma, a) = 0$ for all points a in the complement of Ω. Denoted by $\gamma \sim 0 \pmod{\Omega}$* [3]

4.4.4 THE GENERAL STATEMENT OF CAUCHY THEOREM

Theorem 25 (Cauchy Theorem) *If $f(z)$ is analytic in Ω, then*

$$\int_\gamma f(z)dz = 0$$

for every cycle γ which is homologous to zero in Ω

Corollary 3 *If $f(z)$ is analytic in a simply connected region Ω, then $\int_\gamma f(z)dz = 0$ for all cycles in γ in Ω*

Corollary 4 *If $f(z)$ is analytic and nonzero in a simply connected region Ω then it is possible to define a single-value analytic branches of $\log f(z)$ and $\sqrt[n]{f(z)}$ in Ω*

4.4.5 PROOF OF CAUCHY THEOREM

4.4.6 LOCALLY EXACT DIFFERENTIALS

Theorem 26 *A differential $pdx + qdy$ is said to be locally exact in Ω if it is exact in some neighbourhood of each point in Ω If $pdx + qdy$ is locally exact in Ω, then*

$$\int_\gamma (pdx + qdy) = 0$$

for every cycle $y \sim 0$ in Ω

4.4.7 MULTIPLY CONNECTION REGIONS

Definition 12 (Multiply Connected) *A region which is not simply connected is said to be multiply connected. A region is said to have finite connectivity n if its complement has exactly n components.*

Proposition 5 (Homology Basis - Modules of Periodicity) *Let $A_1, A_2, ..., A_n$ be the components of the complement of Ω and $\infty \in A_n$. If γ is a cycle in Ω, then $n(\gamma, a)$ only depends on which components a belongs to, namely $n(\gamma, a) = c_i$ if $a \in A_i$, and $c_n = 0$. For each $A_i, i = 1, 2, ..., n-1$, we can find γ_i such that $n(\gamma_i, a) = 1$ for $a \in A_i$ and $n(\gamma_i, a) = 0$ $a \in A_j, j \neq i$. Then*

$$\gamma - (c_1\gamma_1 + c_2\gamma_2 + ... + c_{n-1}\gamma_{n-1}) \sim 0 \pmod{\Omega}$$

or

$$\gamma \sim c_1\gamma_1 + c_2\gamma_2 + ... + c_{n-1}\gamma_{n-1} \pmod{\Omega}$$

That is, every cycle is homologous to a linear combination of $\gamma_1, \gamma_2, ..., \gamma_n$. The set $\{\gamma_1, \gamma_2, ..., \gamma_n\}$ is said to be the homology basis for Ω For any analytic function $f(z)$ on Ω, then

$$\int_\gamma f(z)dz = \sum_{i=1}^{n} \left(c_i \int_{\gamma_i} f(z)dz \right)$$

the values $\int_{\gamma_i} f(z)dz, i = 1, 2, ..., n-1$ are said to be the modules of periodicity of the differential $f(z)dz$

[3] if Ω is simply connected, all cycle in Ω is homologous to zero

4.5 THE CALCULUS OF RESIDUES