MA5216 Homework 2

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1 QUESTION 1

1.1 Exercise 3.4.23

Problem 1.1 (Exercise 3.4.23)

For two symmetric $(0,2)$-tensors h, k define the *Kulkarni-Nomizu product* as the $(0,4)$-tensor

$$\begin{aligned}&(h \circ k)(v_1, v_2, v_3, v_4)\\ &= \frac{1}{2}h(v_1, v_4) \cdot k(v_2, v_3) + \frac{1}{2}h(v_2, v_3) \cdot k(v_1, v_4) - \frac{1}{2}h(v_1, v_3) \cdot k(v_2, v_4) - \frac{1}{2}h(v_2, v_4)k(v_1, v_3)\end{aligned}$$

(5) Show that $\nabla_X(h \circ k) = (\nabla_X h) \circ k + h \circ (\nabla_X k)$

(6) Show that (M, g) has constant curvature c if and only if the $(0,4)$-curvature tensor satisfies $R = c \cdot (g \circ g)$

1.1.1 (5)

Let a, b, c, d be vector fields. Let the $(0,4)$-tensors $h * k$ and $h \star k$ be defined by

$$\begin{aligned}(h * k)(a, b, c, d) &= h(a, d) \cdot k(b, c) + h(b, c) \cdot k(a, d)\\ (h \star k)(a, b, c, d) &= h(a, c) \cdot k(b, d) + h(b, d) \cdot k(a, c) = (h * k)(a, b, d, c)\end{aligned}$$

Then, we can write the *Kulkarni-Nomizu product* as

$$h \circ k = \frac{1}{2}(h * k - h \star k)$$

If we assume product rules for $h * k$ and $h \star k$, then

$$\nabla_X(h \circ k) = \frac{1}{2}((\nabla_X h) * k + h * (\nabla_X k) - (\nabla_X h) \star k - h \star (\nabla_X k)) = (\nabla_X h) \circ k + h \circ (\nabla_X k)$$

Hence, it suffices to prove the product rule for $h * k$ tensor. Hence, we can assume that

$$(h \circ k)(a, b, c, d) = h(a, d) \cdot k(b, c) + h(b, c) \cdot k(a, d)$$

Note that, the order of operations is as follows: $\nabla_X h(a, b) := \nabla_X(h(a, b)) = D_X(h(a, b))$. We have

$$
\begin{aligned}
&(\nabla_X(h \circ k))(a,b,c,d) \\
&= \nabla_X((h \circ k)(a,b,c,d)) \\
&\quad - (h \circ k)(\nabla_X a,b,c,d) - (h \circ k)(a,\nabla_X b,c,d) - (h \circ k)(a,b,\nabla_X c,d) - (h \circ k)(a,b,c,\nabla_X d) \\
&= \nabla_X(h(a,d) \cdot k(b,c)) + \nabla_X(h(b,c) \cdot k(a,d)) \\
&\quad - (h \circ k)(\nabla_X a,b,c,d) - (h \circ k)(a,\nabla_X b,c,d) - (h \circ k)(a,b,\nabla_X c,d) - (h \circ k)(a,b,c,\nabla_X d) \\
&= \nabla_X h(a,d) \cdot k(b,c) + h(a,d) \cdot \nabla_X k(b,c) + \nabla_X h(b,c) \cdot k(a,d) + h(b,c) \cdot \nabla_X k(a,d) \\
&\quad - h(\nabla_X a,d) \cdot k(b,c) - h(b,c) \cdot k(\nabla_X a,d) - h(a,d) \cdot k(\nabla_X b,c) - h(\nabla_X b,c) \cdot k(a,d) \\
&\quad - h(a,d) \cdot k(b,\nabla_X c) - h(b,\nabla_X c) \cdot k(a,d) - h(a,\nabla_X d) \cdot k(b,c) - h(b,c) \cdot k(a,\nabla_X d) \\
&= (\nabla_X h)(a,d) \cdot k(b,c) + (\nabla_X h)(b,c) \cdot k(a,d) + h(a,d) \cdot (\nabla_X k)(b,c) + h(b,c) \cdot (\nabla_X k)(a,d) \\
&= ((\nabla_X h) \circ k)(a,b,c,d) + (h \circ (\nabla_X k))(a,b,c,d)
\end{aligned}
$$

1.1.2 (6)

Note that, the sectional curvature of the subspace spanned by $\{u,v\}$ is

$$\sec(v,w) = \frac{R(w,v,v,w)}{g(v,v)g(w,w) - g(v,w)^2}$$

We have

$$(g \circ g)(w,v,v,w) = g(w,w) \cdot g(v,v) - g(w,v) \cdot g(v,w) = g(v,v)g(w,w) - g(v,w)^2$$

Then, $R = c \cdot (g \circ g) \iff \sec(v,w) = c$. If c is a constant, then $R = c \cdot (g \circ g) \iff (M,g)$ has constant curvature

1.2 Exercise 3.4.24

Problem 1.2 (Exercise 3.4.24)
Define the $(0,2)$ *Schouten tensor*

$$P = \frac{2}{n-2}\operatorname{Ric} - \frac{\operatorname{scal}}{(n-1)(n-2)} \cdot g$$

for Riemannian manifolds of dimension $n > 2$
(1) Show that if P vanishes on M then $\operatorname{Ric} = 0$
(2) Show that the decomposition

$$P = \frac{\operatorname{scal}}{n(n-1)} g + \frac{2}{n-2}\left(\operatorname{Ric} - \frac{\operatorname{scal}}{n} g\right)$$

of the *Schouten tensor* is orthogonal
(5) Show that (M, g) has constant curvature when $n > 2$ if and only if

$$R = P \circ g \text{ and } \operatorname{Ric} = \frac{\operatorname{scal}}{n} g$$

(6) Show that

$$\operatorname{Ric}(X, Y) = \sum_{i=1}^{n} (P \circ g)(X, E_i, E_i, Y)$$

for any orthonormal frame E_i

1.2.1 (1)

We have

$$\begin{aligned}
\operatorname{tr} P &= \frac{2}{n-2}\operatorname{tr}\operatorname{Ric} - \frac{\operatorname{scal}}{(n-1)(n-2)} \cdot \operatorname{tr} g \\
&= \frac{2}{n-2}\operatorname{scal} - \frac{\operatorname{scal}}{(n-1)(n-2)} n \\
&= \frac{\operatorname{scal}}{n-1}
\end{aligned}$$

Then *Schouten tensor* vanishes, that is $P = 0$, implies $\operatorname{scal} = 0$, so $\operatorname{Ric} = \frac{n-2}{2} P = 0$

1.2.2 (2)

Given two $(0,2)$-tensors $T = T_{ij}\sigma^i \otimes \sigma^j$ and $S = S_{ij}\sigma^i \otimes \sigma^j$, then the pointwise inner product induced by metric g is defined by the smooth function

$$\langle T, S\rangle = g^{ik} g^{jl} T_{ij} S_{kl}$$

We want to show that

$$0 = \left\langle \frac{\operatorname{scal}}{n(n-1)} g, \frac{2}{n-2}\left(\operatorname{Ric} - \frac{\operatorname{scal}}{n} g\right)\right\rangle = \frac{2\operatorname{scal}}{n(n-1)(n-2)}\left\langle g, \operatorname{Ric} - \frac{\operatorname{scal}}{n} g\right\rangle$$

That is equivalent to showing $\langle g, \operatorname{Ric}\rangle = \frac{\operatorname{scal}}{n}\langle g, g\rangle$, we have

$$\langle g, \operatorname{Ric}\rangle = g^{ik} g^{jl} g_{ij} \operatorname{Ric}_{kl}$$

At every point, we can choose an orthonormal frame so that $g_{ij} = g^{ij} = \delta_{ij}$, hence

$$\langle g, \mathrm{Ric}\rangle = \mathrm{Ric}_{ii} = \mathrm{tr}\,\mathrm{Ric} = \mathrm{scal}$$

On the other hand,

$$\langle g, g\rangle = g^{ik} g^{jl} g_{ij} g_{kl} = g_{ii} = \mathrm{tr}\, g = n$$

Hence, the decomposition is orthogonal.

1.2.3 (5)

$$P \circ g = \frac{2}{n-2}(\mathrm{Ric} \circ g) - \frac{\mathrm{scal}}{(n-1)(n-2)}(g \circ g)$$

($\implies$) when (M, g) has constant curvature, $R = c(g \circ g)$, then

$$\begin{aligned}
\mathrm{Ric}(v, w) &= \sum_i R(E_i, w, v, E_i) \\
&= c \sum_i (g \circ g)(E_i, w, v, E_i) \\
&= c \sum_i g(E_i, E_i) g(w, v) - g(E_i, v) g(w, E_i) \\
&= c \left[\left(\sum_i g(E_i, E_i) g(w, v) \right) - \left(\sum_i g(E_i, v) g(w, E_i) \right) \right] \\
&= c(ng(w, v) - g(w, v)) \\
&= c(n-1) g(w, v)
\end{aligned}$$

So, $\mathrm{Ric} = c(n-1)g$ and $\mathrm{scal} = cn(n-1)$, then

$$\mathrm{Ric} = \frac{\mathrm{scal}}{n} g$$

Moreover,

$$\begin{aligned}
P \circ g &= \frac{2}{n-2}(\mathrm{Ric} \circ g) - \frac{\mathrm{scal}}{(n-1)(n-2)}(g \circ g) \\
&= \frac{2}{n-2} c(n-1)(g \circ g) - \frac{cn(n-1)}{(n-1)(n-2)}(g \circ g) \\
&= cg \circ g = R
\end{aligned}$$

($\impliedby$) when $\mathrm{Ric} = \frac{\mathrm{scal}}{n} g$, we have

$$\begin{aligned}
R &= P \circ g \\
&= \frac{2}{n-2}(\mathrm{Ric} \circ g) - \frac{\mathrm{scal}}{(n-1)(n-2)}(g \circ g) \\
&= \frac{2}{n-2} \frac{\mathrm{scal}}{n}(g \circ g) - \frac{\mathrm{scal}}{(n-1)(n-2)}(g \circ g) \\
&= \frac{\mathrm{scal}}{n(n-1)}(g \circ g)
\end{aligned}$$

We will show that scal is a constant function on M. The exterior dervative of a tensor ω satisfies

$$(d\omega)(X_0, ..., X_k) = \sum_i (-1)^i \nabla_{X_i} \omega(X_0, ..., \hat{X}_i, ..., X_k)$$

So

$$(d\,\mathrm{scal})(v) = \nabla_v \,\mathrm{scal} = (\nabla\,\mathrm{scal})(v)$$

On the other hand, by proposition 3.1.5 (the contracted Bianchi identity), we have

$$d\,\mathrm{scal} = -2\nabla^* \,\mathrm{Ric} = -\frac{2}{n}\nabla^*(\mathrm{scal}\, g)$$

where

$$(\nabla^* S)(X_2, ..., X_r) = -\sum_i (\nabla_{E_i} S)(E_i, X_2, ..., X_r)$$

for any (s, r)-tensor S. Hence, for any vector field $v = v^i E_i$

$$(d\,\mathrm{scal})(v) = -\frac{2}{n}(\nabla^*(\mathrm{scal}\, g))(v) = \frac{2}{n}(\nabla_{E_i}(\mathrm{scal}\, g))(E_i, v)$$

Note that, Riemannian connection is compatible with metric, proposition 2.2.5 implies $\nabla g = 0$, we have

$$\begin{aligned}
(d\,\mathrm{scal})(v) &= \frac{2}{n}(\nabla_{E_i}\,\mathrm{scal}) g(E_i, v) + \frac{2}{n}\,\mathrm{scal}(\nabla_{E_i} g)(E_i, v) \\
&= \frac{2}{n}(\nabla_{E_i}\,\mathrm{scal}) g(E_i, v) \\
&= \frac{2}{n} v^i (\nabla\,\mathrm{scal})(E_i) \\
&= \frac{2}{n}(\nabla\,\mathrm{scal})(v)
\end{aligned}$$

When $n \geq 3$, we have $\big(1 - \frac{2}{n}\big)\nabla\,\mathrm{scal} = d\,\mathrm{scal} - d\,\mathrm{scal} = 0$, so scal is a constant function

1.2.4 (6)

Let $X = x^j E_j$, $Y = y^j E_j$, we have (where summands with index i is a sum over $i = 1, ..., n$, tensor $P(X, -)$ is $Y \mapsto P(X, Y)$)

$$\begin{aligned}
&2(P \circ g)(X, E_i, E_i, Y) \\
&= P(X, Y) g(E_i, E_i) + P(E_i, E_i) g(X, Y) - P(X, E_i) g(E_i, Y) - P(E_i, Y) g(X, E_i) \\
&= P(X, Y)(\mathrm{tr}\, g) + (\mathrm{tr}\, P) g(X, Y) - y^i P(X, E_i) - x^i P(E_i, Y) \\
&= nP(X, Y) + (\mathrm{tr}\, P) g(X, Y) - y^i P(X, -)(E_i) - x^i P(-, Y)(E_i) \\
&= nP(X, Y) + (\mathrm{tr}\, P) g(X, Y) - P(X, Y) - P(X, Y) \\
&= (n - 2) P(X, Y) + (\mathrm{tr}\, P) g(X, Y)
\end{aligned}$$

We have, $\operatorname{tr} P = \frac{\mathrm{scal}}{n-1}$, so

$$\begin{aligned}
&2(P \circ g)(X, E_i, E_i, Y) \\
&= (n-2)P(X,Y) + (\operatorname{tr} P)g(X,Y) \\
&= (n-2)\left(\frac{2}{n-2}\operatorname{Ric}(X,Y) - \frac{\mathrm{scal}}{(n-1)(n-2)}g(X,Y)\right) + \left(\frac{\mathrm{scal}}{n-1}\right)g(X,Y) \\
&= 2\operatorname{Ric}(X,Y)
\end{aligned}$$

1.3 Exercise 3.4.25

Problem 1.3 (Exercise 3.4.25)
The *Weyl tensor* W is defined implicitly through

$$\begin{aligned}
R &= \frac{\mathrm{scal}}{n(n-1)}(g \circ g) + \frac{2}{n-2}\left(\operatorname{Ric} - \frac{\mathrm{scal}}{n} \cdot g\right) \circ g + W \\
&= P \circ g + W
\end{aligned}$$

(2) Show that

$$\sum_{i=1}^{n} W(X, E_i, E_i, Y) = 0$$

for any orthonormal frame E_i

1.3.1 (2)

From previous part, we have

$$(P \circ g)(X, E_i, E_i, Y) = \operatorname{Ric}(X,Y) = R(E_i, Y, X, E_i) = R(X, E_i, E_i, Y)$$

Hence,

$$W(X, E_i, E_i, Y) = R(X, E_i, E_i, Y) - (P \circ g)(X, E_i, E_i, Y) = 0$$

2 QUESTION 2

2.1 Exercise 4.7.4

Problem 2.1 (Exercise 4.7.4)
Assume that a Riemmanian manifold (M, g) has a function f such that

$$\operatorname{Hess} f = \lambda(x)g + \mu(f)df^2 = \lambda g + (\mu \circ f)df^2$$

where $\lambda : M \to \mathbb{R}$ and $\mu : \mathbb{R} \to \mathbb{R}$. Show that the metric is locally a warped product

For any smooth function $\phi : \mathbb{R} \to \mathbb{R}$, we have ($\circ$ denotes composition, $\cdot$ denotes pointwise multiplication)

$$\begin{aligned}
&\operatorname{Hess}(\phi \circ f) \\
&= (\phi'' \circ f) \cdot df^2 + (\phi' \circ f) \operatorname{Hess} f \\
&= (\phi'' \circ f) \cdot df^2 + (\phi' \circ f)(\lambda g + (\mu \circ f) df^2) \\
&= [(\phi'' \circ f) + (\phi' \circ f) \cdot (\mu \circ f)] df^2 + (\phi' \circ f) \cdot \lambda g
\end{aligned}$$

For any $p \in M$, we choose function ϕ satisfying the ODE

$$\phi''(x) + \phi'(x) \cdot \mu(x) = 0$$

locally around $x = f(p)$, then $\operatorname{Hess}(\phi \circ f) = (\phi' \circ f) \cdot \lambda g$ locally around p, by Brinkman, the Riemannian structure is locally a warped product.

2.2 Exercise 4.7.5

Problem 2.2 (Exercise 4.7.5)
Show that if $\operatorname{Hess} f = \lambda g$ then $\lambda = \frac{\Delta f}{\dim M}$

Taking contraction both sides gives

$$\Delta f = \operatorname{tr}(\operatorname{Hess} f) = \lambda \operatorname{tr} g = \lambda(\dim M)$$

Hence, $\lambda = \frac{\Delta f}{\dim M}$

3 QUESTION 3

3.1 Exercise 4.7.12

Problem 3.1 (Exercise 4.7.12)
Let (N^{n-1}, g_N) have constant curvature c with $n > 2$. Consider the warped product metric $(M, g) = (I \times N, dr^2 + \rho^2(r) g_N)$
(1) Show that the curvature of g is given by

$$\begin{aligned}
R &= \frac{c - \dot{\rho}^2}{\rho^2} g_r \circ g_r - 2\frac{\ddot{\rho}}{\rho} dr^2 \circ g_r \\
&= \frac{c - \dot{\rho}^2}{\rho^2} g \circ g - 2\left(\frac{\ddot{\rho}}{\rho} + \frac{c - \dot{\rho}^2}{\rho^2}\right) dr^2 \circ g
\end{aligned}$$

(2) Show that the *Weyl tensor* vanishes

3.1.1 (1)

Since r is a distance function, follow 4.2.3, we have $\operatorname{Hess} r = \frac{\dot{\rho}}{\rho} g_r$ and

$$R(\cdot, \partial_r, \partial_r, \cdot) = -\frac{\ddot{\rho}}{\rho} g_r$$

By Tangential Curvature Equation and Mixed Curvature Equation, if X, Y, Z, W are tangent vector fields on N, then

$$\begin{aligned} R(X, Y, Z, W) &= R^r(X, Y, Z, W) - \mathrm{II}(X, W)\,\mathrm{II}(Y, Z) + \mathrm{II}(X, Z)\,\mathrm{II}(Y, W) \\ R(X, Y, Z, \partial_r) &= -(\nabla_X \mathrm{II})(Y, Z) + (\nabla_Y \mathrm{II})(X, Z) \end{aligned}$$

Follow 4.2.3, $R(X, Y, Z, \partial_r) = 0$. On the other hand, g_r is a metric of constant curvature $\frac{c}{\rho^2}$, then

$$R^r(X, Y, Z, W) = \frac{c}{\rho^2} g_r(X \wedge Y, W \wedge Z)$$

By proposition 3.2.1, r is a distance function, $\mathrm{II} = \operatorname{Hess} r = \frac{\dot{\rho}}{\rho} g_r$, so

$$R(X, Y, Z, W) = \frac{c - \dot{\rho}^2}{\rho^2} g_r(X \wedge Y, W \wedge Z) = \frac{c - \dot{\rho}^2}{\rho^2} (g_r \circ g_r)(X, Y, Z, W)$$

The equality is proved by the following reductions:

1. R and $\frac{c-\dot{\rho}^2}{\rho^2} g_r \circ g_r - 2\frac{\ddot{\rho}}{\rho} dr^2 \circ g_r$ are multilinear, it suffices to prove the equality for vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}$ where each is either tangent to N or orthogonal to N.
2. R and $\frac{c-\dot{\rho}^2}{\rho^2} g_r \circ g_r - 2\frac{\ddot{\rho}}{\rho} dr^2 \circ g_r$ have the same symmetries (permutation of parameters $S^4 \to \{-1, +1\}$), hence it suffices to prove the equality in any order of parameters.
3. $dr^2 \circ g_r$ is nonzero if and only if there are precisely two terms orthogonal to N. We already proved the equality when there is zero or one term orthogonal to N. If there are three or more terms orthogonal to N, both sides are zeros. The only case we need to consider is when there are exactly two terms orthogonal to N, that is $R(X, \partial_r, \partial_r, W)$

We have

$$\begin{aligned} R(X, \partial_r, \partial_r, W) &= -\frac{\ddot{\rho}}{\rho} g_r(X, W) \\ &= 2\frac{\ddot{\rho}}{\rho} (dr^2 \circ g_r)(X, \partial_r, \partial_r, W) \\ &= \left(\frac{c - \dot{\rho}^2}{\rho^2} g_r \circ g_r - 2\frac{\ddot{\rho}}{\rho} dr^2 \circ g_r \right)(X, \partial_r, \partial_r, W) \end{aligned}$$

Therefore,

$$R = \frac{c - \dot{\rho}^2}{\rho^2} g_r \circ g_r - 2\frac{\ddot{\rho}}{\rho} dr^2 \circ g_r$$

Now, we rewrite R in terms of g and dr. Note that, *Kulkarni-Nomizu product* is bilinear and $dr^2 \circ dr^2 = 0$, so

$$g \circ g = (dr^2 + g_r) \circ (dr^2 + g_r) = g_r \circ g_r + 2dr^2 \circ g_r$$

Hence,

$$R = \frac{c-\dot{\rho}^2}{\rho^2} g \circ g - 2\left(\frac{\ddot{\rho}}{\rho} + \frac{c-\dot{\rho}^2}{\rho^2}\right) dr^2 \circ g$$

3.1.2 (2)

We will show that $P \circ g = R$, equivalently

$$P = \frac{2}{n-2}\operatorname{Ric} - \frac{\operatorname{scal}}{(n-1)(n-2)} \cdot g = \frac{c-\dot{\rho}^2}{\rho^2} g - 2\left(\frac{c-\dot{\rho}^2}{\rho^2} + \frac{\ddot{\rho}}{\rho}\right) dr^2$$

First, we write Ric and scal in terms of dr and g. Let $\{E_i\}_{i=1}^n$ be an orthonormal frame so that E_1 is parallel to ∂_r, then $E_2, \ldots, E_n$ are in N. Note that, $g_r(E_i, E_i) \neq 0$ if and only if $i \neq 1$, so $\operatorname{tr} g_r = n-1$, we have

$$\begin{aligned}
\operatorname{tr}(g_r \circ g_r)(Y, Z) &= (g_r \circ g_r)(E_i, Y, Z, E_i) \\
&= g_r(E_i, E_i) g_r(Y, Z) - g_r(E_i, Z) g_r(Y, E_i) \\
&= (\operatorname{tr} g_r) g_r(Y, Z) - g_r(Y, Z) \\
&= (n-2) g_r(Y, Z)
\end{aligned}$$

Note that, $dr(E_i) \neq 0$ if and only if $i = 1$, so $dr(E_i)dr(Z)g_r(Y, E_i) = dr(Y)dr(E_i)g_r(E_i, Z) = 0$ and $dr(E_i)dr(E_i) = \operatorname{tr} dr^2 = 1$ we have

$$\begin{aligned}
\operatorname{tr}(dr^2 \circ g_r) &= (dr^2 \circ g_r)(E_i, Y, Z, E_i) \\
&= \frac{1}{2} dr(E_i)dr(E_i) g_r(Y, Z) + \frac{1}{2} dr(Y)dr(Z) g_r(E_i, E_i) \\
&\quad - \frac{1}{2} dr(E_i)dr(Z) g_r(Y, E_i) - \frac{1}{2} dr(Y)dr(E_i) g_r(E_i, Z) \\
&= \frac{1}{2} g_r(Y, Z) + \frac{1}{2} dr(Y)dr(Z)(\operatorname{tr} g_r) \\
&= \frac{1}{2} g_r(Y, Z) + \frac{1}{2}(n-1) dr^2(Y, Z)
\end{aligned}$$

Hence, $\operatorname{tr}(g_r \circ g_r) = (n-2) g_r$ and $\operatorname{tr}(dr^2 \circ g_r) = \frac{1}{2} g_r + \frac{1}{2}(n-1) dr^2$, we have

$$\begin{aligned}
\operatorname{Ric} &= \frac{c-\dot{\rho}^2}{\rho^2} \operatorname{tr}(g_r \circ g_r) - 2\frac{\ddot{\rho}}{\rho} \operatorname{tr}(dr^2 \circ g_r) \\
&= \frac{(n-2)(c-\dot{\rho}^2)}{\rho^2} g_r - \frac{\ddot{\rho}}{\rho}(g_r + (n-1) dr^2) \\
&= \left(\frac{(n-2)(c-\dot{\rho}^2)}{\rho^2} - \frac{\ddot{\rho}}{\rho}\right) g_r - (n-1)\frac{\ddot{\rho}}{\rho} dr^2 \\
&= \left(\frac{(n-2)(c-\dot{\rho}^2)}{\rho^2} - \frac{\ddot{\rho}}{\rho}\right)(g - dr^2) - (n-1)\frac{\ddot{\rho}}{\rho} dr^2 \\
&= \left(\frac{(n-2)(c-\dot{\rho}^2)}{\rho^2} - \frac{\ddot{\rho}}{\rho}\right) g - (n-2)\left(\frac{c-\dot{\rho}^2}{\rho^2} + \frac{\ddot{\rho}}{\rho}\right) dr^2
\end{aligned}$$

$$
\begin{aligned}
\mathrm{scal} &= \operatorname{tr}\mathrm{Ric} \\
&= \left(\frac{(n-2)(c-\dot\rho^2)}{\rho^2} - \frac{\ddot\rho}{\rho}\right)\operatorname{tr} g - (n-2)\left(\frac{c-\dot\rho^2}{\rho^2} + \frac{\ddot\rho}{\rho}\right)\operatorname{tr} dr^2 \\
&= n\left(\frac{(n-2)(c-\dot\rho^2)}{\rho^2} - \frac{\ddot\rho}{\rho}\right) - (n-2)\left(\frac{c-\dot\rho^2}{\rho^2} + \frac{\ddot\rho}{\rho}\right) \\
&= \frac{(n-1)(n-2)(c-\dot\rho^2)}{\rho^2} - \frac{2(n-1)\ddot\rho}{\rho}
\end{aligned}
$$

Now, we write P in terms of dr and g

$$
\begin{aligned}
P &= \frac{2}{n-2}\mathrm{Ric} - \frac{\mathrm{scal}}{(n-1)(n-2)} \cdot g \\
&= \frac{2}{n-2}\left(\left(\frac{(n-2)(c-\dot\rho^2)}{\rho^2} - \frac{\ddot\rho}{\rho}\right)g - (n-2)\left(\frac{c-\dot\rho^2}{\rho^2} + \frac{\ddot\rho}{\rho}\right)dr^2\right) \\
&\quad - \frac{1}{(n-1)(n-2)}\left(\frac{(n-1)(n-2)(c-\dot\rho^2)}{\rho^2} - \frac{2(n-1)\ddot\rho}{\rho}\right)g \\
&= \frac{2(c-\dot\rho^2)}{\rho^2}g - \frac{2\ddot\rho}{(n-2)\rho}g - 2\left(\frac{c-\dot\rho^2}{\rho^2} + \frac{\ddot\rho}{\rho}\right)dr^2 \\
&\quad - \frac{c-\dot\rho^2}{\rho^2}g + \frac{2\ddot\rho}{(n-2)\rho}g \\
&= \frac{c-\dot\rho^2}{\rho^2}g - 2\left(\frac{c-\dot\rho^2}{\rho^2} + \frac{\ddot\rho}{\rho}\right)dr^2
\end{aligned}
$$

Hence

$$
R - P \circ g = W = 0
$$