Scheme Theory

this is my note for scheme theory. this is loosely based on (1) Borcherds online lecture (2) Vakil FOAG (3) Görtz - Wedhorn AG 1

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Contents

Chapter 1

AFFINE SCHEME

1.1 DEFINITION OF AFFINE SCHEME

Definition 1.1.1 (ringed space, locally ringed space)

A ringed space $(X, \mathscr{O})$ is a topological space X together with a sheaf of rings $\mathscr{O}$. A ringed space is a locally ringed space if for every point $x \in X$, the stalk $\mathscr{O}_x$ is a local ring.

Definition 1.1.2 (affine scheme)

An affine scheme is a locally ringed space that is isomorphic to the spectrum of some ring A

1.2 CONSTRUCT AFFINE SCHEME FROM RING

Let A be a ring, let $X = \operatorname{Spec} A$ be the set of prime ideals in A

$$\operatorname{Spec} A = \{ \text{prime } \mathfrak{p} \subseteq A \}$$

We put a topology on $\operatorname{Spec} A$ generated by the basis of open sets

$$D(f) = \{ \mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p} \}$$

for every $f \in A$. The constructed topology is called Zariski's topology. We put a sheaf of rings $\mathscr{O}$ on $\operatorname{Spec} A$ generated by

$$\mathscr{O}(D(f)) = A_f$$

If $D(g) \subseteq D(f)$, that is $g \in \sqrt{(f)}$, $g^n = fh$ for some $h \in A$ and $n \geq 1$. Since f is a unit in A_g, the restriction map is well-defined and unique

$$A_f \to A_g$$
$$\frac{x}{f^m} \mapsto x \left(\frac{h}{g^n} \right)^m$$

Under this construction, $(X, \mathscr{O})$ is an affine scheme. An element $f \in A$ is called **function**, a element $x \in X$ is called **point**, when refering x as a prime ideal in A, we write $\mathfrak{p}_x \subseteq A$, and function evaluation is equivalent to sending f to

the residue field of stalk $\mathscr{O}_X = A_{\mathfrak{p}_x}$.

When A is a polynomial ring of n variables over ring R, we write $\mathbb{A}_R^n = \operatorname{Spec} A$

1.3   SOME EXAMPLES OF AFFINE SCHEME

example: (integers) Let $A = \mathbb{Z}$, then

$$X = \operatorname{Spec} \mathbb{Z} = \{(2), (3), (5), ..., (0)\}$$

for each prime number $p \in \mathbb{Z}$, (p) is a closed point. (0) is a generic point, and closure of (0) is the whole space X.

example: (field) Let k be a field. Let $A = k$, then

$$X = \operatorname{Spec} k = \{(0)\}$$

is a singleton set.

example: (polynomial ring of two variables over an algebraically closed field) Let A be an arbitrary ring, maximal ideals are closed points and other non-maximal primes are generic points. Moreover, the closure of a point $\mathfrak{p}$ is the set of prime ideals containing $\mathfrak{p}$

$$V(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} A : \mathfrak{p} \subseteq \mathfrak{q}\}$$

In particular, let k be an algebraically closed field, let $A = k[x, y]$, by Nullstellensatz, the set of closed points are

$$\{(x - a, y - b) : a \in k, b \in k\}$$

X admits other generic points (0) and (f) for every irreducible $f \in k[x, y]$. By dimensionality argument, the prime (f) is of height 1 and the closure of (f) consists of (f) and $(x - a, y - b)$ for $(a, b) \in k^2$ in the vanishing set of f. Similarly, closure of (0) is the whole space.

1.4   MORPHISM OF AFFINE SCHEMES

Remark 1.4.1 (inverse image, direct image)

Let $\psi : X \to Y$ be a continuous map. The inverse image functor ψ^{-1} and direct image functor ψ_* is an adjoint pair between the category of sheaves on X and the category of sheaves on Y. Let $\mathscr{F}$ and $\mathscr{G}$ be a sheaf on X and a sheaf on Y respectively, then

$$\hom_{\operatorname{Sh}(X)}(\psi^{-1}\mathscr{G}, \mathscr{F}) \cong \hom_{\operatorname{Sh}(Y)}(\mathscr{G}, \psi_*\mathscr{F})$$

The direct image functor ψ_* is defined as follows: for every open subset $V \subseteq Y$, then

$$(\psi_*\mathscr{F})(V) = \mathscr{F}(\psi^{-1}(U))$$

The inverse image functor ψ^{-1} is defined as follows: for every open subset $U \subseteq X$, then

$$(\psi^{-1}\mathscr{G})(U) = \operatorname{colim}_{V \subseteq Y : \psi(U) \subseteq V} \mathscr{G}(V)$$

> **Definition 1.4.2** (morphism of ringed spaces)
>
> A morphism of ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is defined by a continuous map $\psi : X \to Y$ and a morphism of sheaves of rings $\psi^\flat : \mathscr{O}_Y \to \psi_* \mathscr{O}_X$. By adjunction between ψ^{-1} and ψ_*, this is equivalent to a morphism of sheaves of rings $\psi^\# : \psi^{-1} \mathscr{O}_Y \to \mathscr{O}_X$

> **Definition 1.4.3** (morphism of locally ringed spaces, morphism of affine schemes)
>
> A morphism of ringed spaces $(\psi, \psi^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of locally ringed spaces if for every $x \in X$, the induced map on stalks
> $$\psi_x^\# : (\psi^{-1} O_Y)_x = \mathscr{O}_{Y, \psi(x)} \to \mathscr{O}_{X,x}$$
> is a local homomorphism. A morphism of affine schemes is a morphism of locally ringed spaces.

The local homomorphism condition of the map between stalks ensures that any zero function $g \in \mathscr{O}_{Y, \psi(x)}$ at $\psi(x) \in Y$ will be sent to a zero function $\psi_x^\#(g) \in \mathscr{O}_{X,x}$ at $x \in X$

> **Proposition 1.4.4** (equivalence between commutative rings and affine schemes)
>
> The functor Spec from the opposite category of commutative rings into the category of affine schemes is fully faithful and essentially surjective.
> $$\mathrm{Spec} : \mathrm{CRing}^{\mathrm{op}} \xrightarrow{\sim} \mathrm{AffSch}$$

1.5 CONSTRUCT MORPHISM OF AFFINE SCHEMES FROM MAP OF RINGS

Let $\phi : A \to B$ be a map of rings, let $X = \mathrm{Spec}\, B$ and $Y = \mathrm{Spec}\, A$ be the corresponding affine schemes. Then, the corresponding morphism of locally ringed space $(\psi, \psi^\flat) : X \to Y$ is defined as follows: for every $x \in X$

$$\psi : X \to Y$$
$$x \mapsto \phi^{-1}(x)$$

The local homomorphism $\psi_x^\# : \mathscr{O}_{Y, \psi(x)} = A_{\psi(x)} \to B_x$ is induced from $\phi : A \to B$

$$
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\downarrow & & \downarrow \\
A_{\psi(x)} & \xrightarrow{\psi_x^\#} & B_x
\end{array}
$$

1.6 SOME EXAMPLES OF MORPHISM OF AFFINE SCHEMES

example: (quotient) Let A be a ring and $\mathfrak{a} \subseteq A$ be an ideal. The map $A \twoheadrightarrow A/\mathfrak{a}$ induces an injective map of sets

$$\mathrm{Spec}\, A/\mathfrak{a} \hookrightarrow \mathrm{Spec}\, A$$

As sets, we have $\mathrm{Spec}\, A/\sqrt{\mathfrak{a}} = \mathrm{Spec}\, A/\mathfrak{a} = \mathrm{Spec}\, A/\mathfrak{a}^n$ for every $n \geq 1$. However, as affine schemes, in general, they are different, $A/\sqrt{\mathfrak{a}}$ is reduced, i.e. has no nilpotent while $\mathrm{Spec}\, A/\mathfrak{a}$ and $\mathrm{Spec}\, A/\mathfrak{a}^n$ might have nilpotents.

Furthermore, when $\mathfrak{a} = \mathfrak{p}_x = x$ is prime, quotient by $\mathfrak{p}_x$ is the action of taking closed subscheme [1] of X

keep all primes containing $\mathfrak{p}_x$. equivalently, keep all points contained in x

example: (localization) Let A be a ring and $S \subseteq A$ be a multiplicatively closed subset. Localization at S induces an injective map of sets

$$\operatorname{Spec} S^{-1}A \hookrightarrow \operatorname{Spec} A$$

Furthermore, when $A - S = \mathfrak{p}_x = x$ is prime, localization at $\mathfrak{p}_x$ is the action of taking open subscheme [2] of X

keep all primes contained in $\mathfrak{p}_x$. equivalently, keep all points containing x

example: (disjoint union) Let A, B be rings, then the projection $A \times B \to A$ corresponds to the monomorphism of affine schemes

$$\operatorname{Spec} A \hookrightarrow \operatorname{Spec} A \amalg \operatorname{Spec} B \cong \operatorname{Spec} A \times B$$

Categorically, $A \times B$ is product in rings and $\operatorname{Spec} A \amalg \operatorname{Spec} B$ is coproduct in affine schemes

example: (non-trivial fiber) Consider the ring map $\phi : k[u] \to k[x, y]$ defined by $u \mapsto y^2 - x$. It induces a morphism of affine schemes

$$\psi : \operatorname{Spec} k[x, y] \to \operatorname{Spec} k[u]$$
$$(0) \mapsto (0)$$
$$(x - a, y - b) \mapsto (u - (b^2 - a))$$
$$(y^2 - x) \mapsto (u)$$
$$(f) \mapsto (0)$$

for every $a, b \in k$ and f is irreducible other than multiple of $y^2 - x$. *(check this carefully)*.

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[1] meaning will be revealed later
[2] meaning will be revealed later