Some words on affine schemes

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1 AFFINE SCHEME

Definition 1.1 (affine scheme)
An affine scheme $(X, \mathcal{O})$ is a locally ringed space that is isomorphic to the spectrum of a ring

Given any ring A, the spectrum of A as a set is the collection of all primes of A and denoted by $\operatorname{Spec} A$. Endow $X = \operatorname{Spec} A$ with the a topology where every closed set is defined by the collection of primes containing some ideal I

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{p} \supseteq I\}$$

The topology is called Zariski topology. For every $f \in A$, the set

$$X_f = \operatorname{Spec} A - V(f) = \{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{p} \not\ni f\}$$

of all primes not containing f is an open set. Turn out, $\{X_f : f \in A\}$ generates the Zariski topology. Moreover, there is a sheaf $\mathcal{O}$ on X that makes it a locally ringed space (a sheaf of rings that every stalk is a local ring) defined on each basic open set X_f by localization of A at f

$$\mathcal{O}(X_f) = A_f$$

and $\mathcal{O}(X) = A$. Given any $f, g \in A$ so that $X_f \supseteq X_g$, that is $g \in \sqrt{(f)}$ ($g^n = fh$ for some $h \in A$ and $n \geq 1$), f is a unit in A_g, universal property gives a unique restriction map $\mathcal{O}(X_f) \to \mathcal{O}(X_g)$

$$\begin{aligned} A_f &\to A_g \\ \frac{x}{f^m} &\mapsto x\left(\frac{h}{g^n}\right)^m \end{aligned}$$

These make $(\operatorname{Spec} A, \mathcal{O})$ a locally ringed space.

Examples:

Let k be a field

- $\operatorname{Spec} k = \{(0)\}$ is a singleton set
- $\operatorname{Spec} \mathbb{Z} = \{(0), (2), (3), (5), ...\}$
- $\operatorname{Spec} \mathbb{Z}/6\mathbb{Z} = \{(2), (3)\}$
- $k[x, y]$ is the ring of polynomials defined on xy plane. By Nullstellensatz, the maximal ideals in $k[x, y]$ are precisely

$$(x - a, y - b)$$

for all $a, b \in k$. Those maximal ideals are called **closed points** and correspond to points on xy plane. The other prime ideals are called **generic points** and correspond to other geometric objects. Let $\mathfrak{p} = (f(x,y))$ be a prime ideal in $k[x,y]$, then the closure of $\mathfrak{p}$ consists of $\mathfrak{p}$ itself and all closed points that are the zeros of $f(x,y) = 0$. In particular, (0) corresponds to the "classical plane" xy, $(x^2+y^2-1), (y^2-x^3)$ correspond to some "classical lines" on the xy plane. Moreover, the height of prime ideals $(0), (x^2+y^2-1), (y^2-x^3)$ is the codimension of the corresponding geometric shapes: $\operatorname{ht}(0) = 0, \operatorname{ht}(y^2-x^3) = 1, \operatorname{ht}(x-1, y+2) = 2$. The dimension of the whole space is defined as its Krull dimension which is 2 in this case.

- $k[x,y]/(xy)$ is the ring of polynomials defining on $xy = 0$. The geometric picture consists of two lines cross each other. The spectrum is more or less the same with the case of $k[x,y]$ and its dimension is still 2.
- $k[x,y]/(x^2+y^2-1)$ is the ring of polynomials defining on x^2+y^2-1 which is the unit circle. Since (x^2+y^2-1) is of height 1, the Krull dimension $k[x,y]/(x^2+y^2-1)$ is 1 that coincides with the geometric dimension.
- $k[\epsilon]/(\epsilon^2) = \{a + b\epsilon : a, b \in k\}$ is the ring of dual numbers. ϵ is nilpotent, hence belongs to all prime ideals. (ϵ) is the unique maximal ideal of $k[\epsilon]/(\epsilon^2)$. *TODO - apparently, this have something to do with* $\operatorname{Hom}(\operatorname{Spec} k[\epsilon]/(\epsilon^2), X) \cong T_x X$ *or function that is everywhere zero but not zero*
- $k[x] \times k[y]$ is the product of two polynomial rings. *TODO*

Remark 1.2
In the affine scheme $(X, \mathcal{O}) \cong \operatorname{Spec} A$ defined as above, elements of A are called **functions**, primes of A are called **points**, sending function $f \in A$ into $f(\mathfrak{p}) \in \mathcal{O}(X_\mathfrak{p}) = A_\mathfrak{p}$ by the canonical map $A \to A_\mathfrak{p}$ is function evaluation.

2 SCHEME

Definition 2.1 (scheme)
A scheme $(X, \mathcal{O})$ is a locally ringed space such that each point $x \in X$ admits a local neighbourhood $(U, \mathcal{O}|_U)$ being an affine scheme.