complex

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May 2023

this is the note for my learning on complex analysis

1 Definition of complex numbers

Definition 1 (Real Numbers)**.** *An ordered field R satisfies least upper bound property: Every non-empty subset of R with an upper bound has a least upper bound.*

Definition 2 (Complex Numbers)**.** *The set of complex numbers is defined as $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ with additional structure*

- *Complex Addition (+):* $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$
- *Complex Multiplication ($\cdot$):* $(a_1, b_1) \cdot (a_2, b_2) = (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1)$

$(\mathbb{C}, +, \cdot)$ *form the field of complex numbers.*

Theorem 1 (Real sub-field of Complex Numbers)**.** *The set $R = \{(a, 0) : a \in \mathbb{R}\} \subset \mathbb{C}$ together with Complex Addition and Complex Multiplication is Real Numbers*

In fact, Given any non-zero complex number (a, b) where $a, b \neq 0$, any set of the form $\{(\alpha a, \alpha b) : \alpha \in \mathbb{R}\}$ is also Real Numbers.
In the context of Complex Numbers, we only call the set $R = \{(a, 0) : a \in \mathbb{R}\}$ Real Numbers and write them as $a = (a, 0)$

1.1 Representation of Complex Numbers

1.1.1 Standard form

Theorem 2 (Standard form of Complex Numbers)**.** *Any complex number can be written as $a + ib$ where $a, b \in R$ and $i = (0, 1)$.*

Proof

$$\begin{aligned}
a + ib &= (a, 0) + (b, 0) \cdot (0, 1) && \text{(rewrite)} \\
&= (a, 0) + (b0 - 01, b1 + 00) && \text{(complex multiplication)} \\
&= (a, 0) + (0, b) && \text{(the field of real numbers)} \\
&= (a, b) && \text{(complex addition)}
\end{aligned}$$

From now on, instead of writing a complex number as (a, b), we will write them in standard form: $a + ib$

1.1.2 Polar form

Define the equivalent relation $\sim$ in the set of real numbers with element denoted as $\theta \in \mathbb{R}$

Definition 3 (2π periodic)**.**

$$\theta \sim \theta + 2\pi$$

Let the equivalent class Θ be defined by the equivalent relation $\sim$ on $\mathbb{R}$. Element of Θ is denoted as $[\theta]$

Definition 4 (Polar form of Complex Numbers)**.** *Any non-zero complex number can be written in polar form*

$$(r, [\theta]) \in (0, \infty) \times \Theta$$

where $r = \sqrt{a^2 + b^2}$ *and* $\theta = \arctan(b/a)$

Given $(r, [\theta])$ in polar form, we can rewrite it in standard form $a + ib$ where $a = r \cos\theta$, $b = r \sin\theta$
Multiplication in polar form
By the special structure of polar form, multiplication is simpler

$$(r_1, [\theta_1]) \cdot (r_2, [\theta_2]) = (r_1 r_2, [\theta_1 + \theta_2])$$

1.2 Complex function

several functions defined for the set of complex numbers

1.2.1 Modulus - Argument - Conjugate

Modulus $|\cdot|$

$$|a + ib| = \sqrt{a^2 + b^2}$$

Complex Numbers inherits the L2 norm in $\mathbb{R}^2$ making it a normed space.
Argument arg

$$\arg(a+ib) = \arctan(b/a)$$

where $\arctan(b/a)$ is defined to be in the range $[0, 2\pi)$ or $(-\pi, \pi]$
Conjugate $\bar{\cdot}$

$$\overline{a+ib} = a - ib$$

1.2.2 Inverses

Additive inverse

$$-(a+ib) = -a - ib$$

Multiplicative inverse

$$(a+ib)^{-1} = \frac{a-ib}{a^2+b^2}$$

1.2.3 Exponential

Definition 5 (Exponential function)**.**

$$e^{a+ib} = e^a(\cos b + i \sin b)$$

Theorem 3 (Polar form as Exponential)**.**

$$(r, [\theta]) = re^{i\theta}$$

Proof

$$\begin{aligned} re^{i\theta} &= r(\cos\theta + i\sin\theta) && \text{(Exponential)} \\ &= r\cos\theta + ir\sin\theta && \text{(the field of complex numbers)} \\ &= (r, [\theta]) && \text{(polar form definition)} \end{aligned}$$

From now on, instead of writing a complex number in polar form as $(r, [\theta])$, we will write them as complex exponential: $re^{i\theta}$
Inverse of Exponential
Exponential function is 2π periodic in imaginary axis, i.e

$$e^{a+ib} = e^{a+i(b+2\pi)}$$

Hence the inverse of exponential is defined as a multi-function

Definition 6 (Logarithm)**.** *Defining* log *as a multi-function being the inverse of exponential function*

$$\begin{aligned}\log : \mathbb{C} \setminus \{0\} &\to \mathcal{P}(\mathbb{C}) \\ \log(z) &= \{x : x \in \mathbb{C} \wedge e^x = z\} \\ \log(re^{i\theta}) &= \{\log r + i(\theta + 2\pi k) : k \in \mathbb{Z}\}\end{aligned}$$

Definition 7 (Power function)**.** [1]

$$z^n = e^{n \log z}$$

Write z in polar form

$$\begin{aligned}z^n &= (re^{i\theta})^n \\ &= e^{n \log(re^{i\theta})} \\ &= e^{n\{\log r + i(\theta+2\pi k):k\in\mathbb{Z}\}} \\ &= e^{n \log r} e^{\{in(\theta+2\pi k):k\in\mathbb{Z}\}}\end{aligned}$$

n is a positive integer
$z^n : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is a proper function [2]

$$\begin{aligned}z^n &= e^{n \log r} e^{\{in(\theta+2\pi k):k\in\mathbb{Z}\}} \\ &= r^n e^{in\theta}\end{aligned}$$

$m - th$ root, i.e $n = 1/m$ where $m \in \mathbb{N} \setminus \{0\}$
There are m $m - th$ roots of complex number $z \neq 0$

$$\begin{aligned}z^n = \sqrt[m]{z} &= e^{\frac{\log r}{m}} e^{\{i(\frac{\theta}{m} + \frac{2\pi}{m}k):k\in\mathbb{Z}\}} \\ &= \sqrt[m]{r}\{e^{i(\frac{\theta}{m} + \frac{2\pi}{m}k)} : k \in \mathbb{Z} \cup [0, m)\}\end{aligned}$$

n is irrational
There are infinitely many output values of power function
Properties of Exponential function and Power function
Exponential function

[1] in this note, given $f : A \to B$, we write $f(\overline{A}) = \{f(x) : x \in \overline{A}\}$ where $\overline{A} \subseteq A$
[2] one input, one output

- $e^{a+b} = e^a e^b$
- $e^{-a} = (e^a)^{-1}$

Power function

- $z^{a+b} = z^a z^b$
- $z^{-a} = (z^a)^{-1}$
- $z^{ab} = (z^a)^b$

2 Subspace of $\mathbb{R}^{2\times 2}$

The field of complex numbers is isomorphic to a subspace of the vector space of (2×2) matrices with real entries, i.e $\mathbb{R}^{2\times 2}$ with additional structure as follows Basis of the complex subspace is $\mathbf{B} = \{r, i\}$ where

$$r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

All complex numbers is represented as $z = ar + ib = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $a, b \in \mathbb{R}$. Furthermore, we define addition as matrix addition and multiplication as matrix multiplication.

2.1 Polar form

Let $r = \sqrt{a^2 + b^2}$ and $\theta = \arctan(b/a)$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Geometrically speaking, a complex number is a composition of a scaling operator and a rotation operator in $\mathbb{R}^2$