compactness

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this text presents the idea of local-to-global principle as a definition of compactness

In mathematics, one often encounters the notion of compactness as the existence in a finite subcover in a cover of a compact set. The notion of compactness seems intuitive at first as it often described as finiteness since it shares many properties with finite sets. In fact, one can consider a finite set is the compact subset of the set of natural numbers as finite sets resemble two notions: discrete and finiteness.

1 Finite Union Property

A property is a characteristic that identifies subsets of a given set. *Finite Union Property* is a property that is generalized to all finite union of sets. Formally,

Definition 1 (Finite Union Property) *Let X be any set (one might call it the universal set) and property p identify a family P of subsets of X. That is, $P = \{x \in \mathcal{P}(X) : p(x)\}$. Property p (or family P) is said to have* Finite Union Property (FUP) *if*

$$U_1, U_2, ..., U_N \in P \implies \bigcup_{n=1}^{N} U_n \in P$$

where $N \in \mathbb{N}$

So, what is so special about *FUP*? Before answer that question, let's see some example of properties that exhibit *FUP*.

1. Let X be any non-empty set, property p_1 identifies all finite subsets of X
2. Let $\mathbb{N}$ be the set of natural numbers, property p_2 identifies all the subsets of $\mathbb{N}$ with its maximum element being even, e.g. $\{8\}, \{1, 4\}, \{n \in \mathbb{N} : n \leq 1000\}, ...$
3. Let $f : S \to \mathbb{R}$ be a real-valued continuous function on $S \subseteq \mathbb{R}$, property p_3 identifies all open subset of $O \subseteq S$ if the image of O is bounded

4. Let Ω be a sample space, property p_4 identifies all events $E \subseteq \Omega$ with probability greater than 0.1

Example 1 and example 2 are on finite sets while example 3 and 4 are on arbitrary sets. So *FUP* is a universal property that works on certain structures.

2 Compactness

In order to give a relation to compactness, let's recall the definition of compactness.

Definition 2 (Compactness) *A topological space $(X, \mathcal{T})$ is said to be compact if for every family of open sets $\mathcal{O} = \{O_i : i \in I\}$ where I is an index set such that $\bigcup_{i \in I} O_i = X$, then there exists a finite subset $J \subseteq I$ such that $\bigcup_{j \in J} O_j = X$*

In many textbooks, compactness is often considered as a generalization of finiteness. Indeed, compactness and finiteness share many common properties.

The statements [**?**] below are true for any finite set X

- (**bounded**) If $f : X \to \mathbb{R}$ is a real-valued function on X, then f is bounded
- (**maximum**) If $f : X \to \mathbb{R}$ is a real-valued function on X, then f attains a maximum value
- (**constant subsequence**) if $x_1, x_2, ..., x_n, ... \in X$ is a sequence of points in X, then there exists a constant subsequence $x_{n_1}, x_{n_2}, ...$

While for a compact set $X \subseteq \mathbb{R}$, a similar set of statements are true:

- (**bounded**) If $f : X \to \mathbb{R}$ is a *continuous* real-valued function on X, then f is bounded
- (**maximum**) If $f : X \to \mathbb{R}$ is a *continuous* real-valued function on X, then f attains a maximum value
- (**convergence subsequence**) if $x_1, x_2, ..., x_n, ... \in X$ is a sequence of points in X, then there exists a convergence subsequence $x_{n_1}, x_{n_2}, ...$

The three statements above exhibit different equivalent notions of compactness in metric space; namely, *local-to-global principle*, *limit point compactness*, *sequential compactness*. With the machinery of *FUP*, we are ready to give a equivalent notion of compactness from *FUP*

Proposition 1 (Finite Union Compactness) *A topological space $(X, \mathcal{T})$ is compact if and only if for every property p on X with* FUP*, for each point $x \in X$, there is an open neighbourhood O_x (not necessarily distinct) that satisfies p implies X satisfies p.*

In the perspective of *local-to-global principle*, *Finite Union Compactness* gives sufficient conditions to generalize p into the whole set if every point has a neighbourhood satisfying p. In fact, *Finite Union Compactness* is equivalent to *Compactness.*

Back to the four examples at the beginning. p_1 identifies the whole set X if and only if X is finite. If p_2 identifies a set U, then U must be finite. If p_3 identifies S, then S must be compact.

Now, we have a better notion of compactness. We will define discreteness as follows

Definition 3 (Discreteness) *A set X is said to be discrete if it has a bijective mapping onto a subset of* $\mathbb{N}$

Compactness of discrete sets is then defined as follows

Proposition 2 (Compactness of discrete sets) *A discrete set with the discrete topology is compact if and only if it is finite*

Equivalently, we can say that any finite set resembles discreteness and compactness. One property enables countability, the other enables *local-to-global principles.*

3 Proof

3.1 Proposition 1

Compactness $\implies$ Finite Union Compactness

Let $(X, \mathcal{T})$ be a compact topological space with a property p defined on X that has *FUP*. For each point $x \in X$, there is an open neighbourhood O_x satisfies p. Clearly, $\bigcup_{x \in X} O_x = X$. By *Compactness*, there is a subcover $O_{x_1}, O_{x_2}, ..., O_{x_N}$ for $N \in \mathbb{N}$, i.e. $\bigcup_{n=1}^{N} O_{x_n} = X$. By *FUP*, $p(X) = p(\bigcup_{n=1}^{N} O_{x_n})$ is true since it is a finite union of sets satisfying p

Finite Union Compactness $\implies$ Compactness

Let $(X, \mathcal{T})$ be a compact topological space with an open cover $\mathcal{O} = \{O_i : i \in I\}$ where I is an index set. Let p be the property on X that identifies all open sets that can be covered by a finite number of open sets in $\mathcal{O}$. Clearly, p is a *FUP*. Invoke Axiom of Choice, for each $x \in X$, choose an open set $O_x \in \mathcal{O}$ containing x (not necessary distinct). Each O_x satisfies p. By *Finite Union Compactness*, p is true for the whole set X, that is, X can be covered by a finite number of open sets in $\mathcal{O}$, namely a finite subcover of $\mathcal{O}$

3.2 Proposition 2

Finiteness $\implies$ Compactness

Trivial

Compactness $\implies$ Finiteness

Let U be discrete and compact with discrete topology and let p be a property defined on U that identifies all finite sets. Clearly, each singleton in U satisfies p. By *Finite Union Compactness*, p is true for U, that is, U is finite.