ma5209 assignment 1

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1 Problem 1

Define a category Ho(Top) in the following way. For objects, take the class of topological spaces. A morphism from X to Y is a homotopy class of continuous maps from X to Y. Show that there are unique notions of composition and identity for which the evident "function" from objects and morphisms in Top to those of Ho(Top) constitute a functor. What is an isomorphism in Ho(Top)? If A is a set and for each $\alpha \in A$ we are given a space X_α, construct the product of X_α's and coproduct of X_α's in Ho(Top).

Similarly, let Ch be the category of chain complexes and chain maps. Define Ho(Ch) and the functor Ch $\to$ Ho(Ch). Explain why the singular chain complex functor and the n-th homology functor define functors on Ho(Top) $\to$ Ho(Ch) $\to$ Ab where Ab is the category of abelian groups and homomorphisms.

1.1 Definition of Ho(Top)

As being homotopic is an equivalence relation, define the identity and composition in Ho(Top) as follows:

- identity: the identity map of an object X in Ho(Top) is defined as the homotopy class of $1 : X \to X$, namely $[1]$
- composition: let $[f] : X \to Y, [g] : Y \to Z$ be two morphisms in Ho(Top) with representatives $f : X \to Y, g : Y \to Z$ that are two morphisms in Top. Then the composition is defined by

  $$[g][f] = [gf]$$

  where $[gf]$ denotes the homotopy class of the composition gf in Top

We will prove that Ho(Top), identity, and composition form a category by verifying the following:

1. composition is well-defined
2. $[1]$ is the identity of X in Ho(Top)
3. composition satisfies associativity

Proof.

1. composition is well-defined:

   Let $f_1 : X \to Y, g_1 : Y \to Z$ be two other representatives of $[f], [g]$, we will show that $g_1 f_1$ are homotopic to gf. Let $F : X \times I \to Y$ be the homotopy from f to f_1, $G : Y \times I \to Z$ be the homotopy from g to g_1, define $H : X \times I \to Z$ by

   $$H(x,t) = GF_1(x,t) = G(F(x,t),t)$$

   where $F_1 : X \times I \to Y \times I$ is defined by $F_1(x,t) = (F(x,t),t)$. Now, H is continuous because both G and F_1 are continuous. The continuity of F_1 is as follows: as any open set in $Y \times I$ is generated by the pair $O_Y \times O_I$ where $O_Y \subseteq Y$ and $O_I \subseteq I$ are two open subsets. We have $F_1^{-1}(O_Y \times O_I) = F_1^{-1}(O_Y \times I) \cap F_1^{-1}(Y \times O_I) = F^{-1}(O_Y) \cap X \times O_I$

2. $[1]$ is the identity of X in Ho(Top):

   Given $[f] : X \to Y$, then $[f][1_Y] = [f 1_Y] = [f]$ and $[1_X][f] = [1_X f] = [f]$. The equality is due to $1_X, 1_Y$ being the identity in Top

3. composition satisfies associativity:

   This is due to associativity of composition in Top

   $$[h]([g][f]) = [h][gf] = [h(gf)] = [(hg)f] = [hg][f] = ([h][g])[f]$$

□

Define the evident "functor" $F : \mathrm{Top} \to \mathrm{Ho(Top)}$ as follows

- on objects: $X \mapsto X$
- on morphisms: $f \mapsto [f]$ where $f : X \to Y$ is a continuous map from X to Y and $[f]$ is the homotopy class of f

We will prove that F is indeed a functor by verifying the following

1. $F(1_X) = 1_{F(X)}$ where $1_X : X \to X$ is the identity map of X in Top and $1_{F(X)}$ is the identity map of $F(X)$ in Ho(Top)
2. $F(gf) = F(g)F(f)$ where $f : X \to Y, g : Y \to Z$ are morphisms in Top

Proof.

1. $F(1_X) = 1_{F(X)}$:
   This is true by definition of identity in Ho(Top)
2. $F(gf) = F(g)F(f)$:
   This is true by definition of composition in Ho(Top)

$$F(gf) = [gf] = [g][f] = F(g)F(f)$$

□

1.2 Isomorphism in Ho(Top)

An isomorphism $[f] : X \to Y$ in Ho(Top) is a morphism such that there exists $[g] : Y \to X$ such that $[f][g] = 1$ and $[g][f] = 1$. That is, f is a homotopy equivalence.

1.3 Product in Ho(Top)

The product space $\prod_\alpha X_\alpha$ (Cartesian product of sets with product topology) is the product in Top. Define the following objects and morphisms

$$\begin{tikzcd}
& X_\alpha \\
W \arrow[ur, "f_\alpha"] \arrow[r, dashed, "h"'] \arrow[r, dashed, bend right, "h_1"'] & \prod_\alpha X_\alpha \arrow[u, "p_\alpha"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& F(X_\alpha) \\
F(W) \arrow[ur, "F(f_\alpha)"] \arrow[r, dashed, "F(h)"'] \arrow[r, dashed, bend right, "F(h_1)"'] & F(\prod_\alpha X_\alpha) \arrow[u, "F(p_\alpha)"']
\end{tikzcd}$$

Given any $F(W) \in \mathrm{Ho(Top)}$, there is a $F(h)$ such that the diagram commutes. We will prove the uniqueness of $F(h)$. Suppose there is another map $F(h_1)$ that makes the diagram commutes, we will prove that $F(h) = F(h_1)$. Indeed, for every $\alpha \in A$.

$$F(p_\alpha h) = F(p_\alpha)F(h) = F(f_\alpha) = F(p_\alpha)F(h_1) = F(p_\alpha h_1)$$

That is, $p_\alpha h$ is homotopic to $p_\alpha h_1$ for every $\alpha \in A$ then there exists a continuous map $H_\alpha : W \times I \to X_\alpha$ for every $\alpha \in A$ such that $H_\alpha(w, 0) = p_\alpha h(w)$, $H_\alpha(w, 1) = p_\alpha h_1(w)$. Construct $H : W \times I \to \prod_\alpha X_\alpha$ as follows: (evaluation map)

$$H(w, t) = \prod_\alpha H_\alpha(w, t) = (H_\alpha(w, t))_\alpha$$

This is a homotopy from h to h_1, hence $F(h) = F(h_1)$, that is, $F(h)$ is unique

$$\prod_{\alpha \in A} F(X_\alpha) = F\left(\prod_{\alpha \in A} X_\alpha\right)$$

1.4 Coproduct in Ho(Top)

The disjoint union space $\coprod_\alpha X_\alpha$ (disjoint union of sets with disjoint union topology) is the coproduct in Top. Define the following objects and morphisms

$$\begin{tikzcd} X_\alpha \arrow[d, "i_\alpha"] \arrow[dr, "f_\alpha"] & \\ \coprod_\alpha X_\alpha \arrow[r, dashed, "h"] \arrow[r, dashed, bend right=60, "h_1"'] & W \end{tikzcd} \qquad \begin{tikzcd} F(X_\alpha) \arrow[d, "F(i_\alpha)"'] \arrow[dr, "F(f_\alpha)"] & \\ F(\coprod_\alpha X_\alpha) \arrow[r, dashed, "F(h)"] \arrow[r, dashed, bend right=60, "F(h_1)"'] & F(W) \end{tikzcd}$$

Given any $F(W) \in \mathrm{Ho(Top)}$, there is a $F(h)$ such that the diagram commutes. We will prove the uniqueness of $F(h)$. Suppose there is another map $F(h_1)$ that makes the diagram commutes, we will prove that $F(h) = F(h_1)$. Indeed, for every $\alpha \in A$

$$F(hi_\alpha) = F(h)F(i_\alpha) = F(f_\alpha) = F(h_1)F(i_\alpha) = F(h_1 i_\alpha)$$

That is hi_α is homotopic to $h_1 i_\alpha$ for every $\alpha \in A$ then there exists a continuous map $H_\alpha : X_\alpha \times I \to W$ for every $\alpha \in A$ such that $H_\alpha(x_\alpha, 0) = hi_\alpha(x_\alpha)$, $H_\alpha(x_\alpha, 1) = h_1 i_\alpha(x_\alpha)$. Construct $H : \coprod_\alpha X_\alpha \times I \to W$ as follows:

$$H(x,t) = H_\alpha(x,t) \text{ if } x \in X_\alpha$$

This is a homotopy from h to h_1, hence $F(h) = F(h_1)$. That is, $F(h)$ is unique

$$\coprod_{\alpha \in A} F(X_\alpha) = F\left(\coprod_{\alpha\in A} X_\alpha\right)$$

1.5 Definition of Ho(Ch)

Define the objects and morphisms in Ho(Ch)

- objects: chain complexes
- morphisms: chain maps

As being chain homotopic is an equivalence relation, define the identity and composition in Ho(Ch)

- identity: the identity map of a chain complex X in Ho(Top) is defined as the chain homotopy class of the identity chain map $1 : X \to X$, namely [1]
- composition: let $[f] : X \to Y, [g] : Y \to Z$ be two morphisms in Ho(Ch) with representatives $f : X \to Y, g : Y \to Z$ that are two morphisms in Ch. Then the composition is defined as

  $$[g][f] = [gf]$$

  where $[gf]$ denotes the chain homotopy class of gf in Ch

We will prove that Ho(Ch), identity, and composition form a category by verifying the following:

1. composition is well-defined
2. [1] is the identity of X in Ho(Ch)
3. composition satisfies associativity

Proof.

1. composition is well-defined:

   Let $f_1 : X \to Y, g_1 : Y \to Z$ be two other representatives of $[f], [g]$, we will show that $g_1 f_1$ is chain homotopic to gf.

$$\begin{tikzcd} X_{n-1} \arrow[dr, "h_f"'] & X_n \arrow[l, "\partial"'] \arrow[d, "f-f_1"'] \arrow[dr, "h_f"] & X_{n+1} \arrow[l, "\partial"'] \\ Y_{n-1} \arrow[dr, "h_g"'] & Y_n \arrow[l, "\partial"'] \arrow[d, "g-g_1"'] \arrow[dr, "h_g"] & Y_{n+1} \arrow[l, "\partial"'] \\ Z_{n-1} & Z_n \arrow[l, "\partial"] & Z_{n+1} \arrow[l, "\partial"] \end{tikzcd}$$

$$
\begin{aligned}
gf &= (g_1 + \partial h_g + h_g\partial)(f_1 + \partial h_f + h_f\partial) && (f, f_1), (g, g_1) \text{ are chain homotopic} \\
&= g_1(f_1 + \partial h_f + h_f\partial) + (\partial h_g + h_g\partial)(f_1 + \partial h_f + h_f\partial) \\
&= g_1 f_1 + g_1\partial h_f + g_1 h_f\partial + (\partial h_g + h_g\partial)(f_1 + \partial h_f + h_f\partial) \\
&= g_1 f_1 + \partial g_1 h_f + g_1 h_f\partial + (\partial h_g + h_g\partial)(f_1 + \partial h_f + h_f\partial) && (\partial g_1 = g_1\partial)
\end{aligned}
$$

$$
\begin{aligned}
(\partial h_g + h_g\partial)(f_1 + \partial h_f + h_f\partial) &= \partial h_g f_1 + \partial h_g\partial h_f + \partial h_g h_f\partial + h_g\partial f_1 + h_g\partial\partial h_f + h_g\partial h_f\partial \\
&= \partial h_g f_1 + \partial h_g\partial h_f + \partial h_g h_f\partial + h_g\partial f_1 + h_g\partial h_f\partial && (\partial\partial = 0) \\
&= (\partial h_g f_1 + \partial h_g h_f\partial + h_g\partial f_1) + (\partial h_g\partial h_f + h_g\partial h_f\partial) && (+ \text{ is commutative, associative}) \\
&= [\partial h_g(f_1 + h_f\partial) + h_g(f_1 + h_f\partial)\partial] + (\partial h_g\partial h_f + h_g\partial h_f\partial) && (\partial f_1 = f_1\partial)
\end{aligned}
$$

Therefore, let $h_{gf} = g_1 h_f + h_g(f_1 + h_f\partial) + h_g\partial h_f$, then

$$gf = g_1 f_1 + \partial h_{gf} + h_{gf}\partial$$

2. [1] is the identity of X in Ho(Ch):

   Given $[f] : X \to Y$, then $[f][1_Y] = [f1_Y] = [f]$ and $[1_X][f] = [1_X f] = [f]$. The equality is due to $1_X, 1_Y$ being the identity in Ch

3. composition satisfies associativity:

   This is due to associativity of composition in Ch

$$[h]([g][f]) = [h][gf] = [h(gf)] = [(hg)f] = [hg][f] = ([h][g])[f]$$

□

Define the "functor" $F : \text{Ch} \to \text{Ho(Ch)}$ as follows

- on objects: $X \mapsto X$
- on morphisms: $f \mapsto [f]$ where $f : X \to Y$ is a chain map from X to Y and $[f]$ is the chain homotopy class of f

We will prove that F is indeed a functor by verifying the following

1. $F(1_X) = 1_{F(X)}$ where $1_X : X \to X$ is the identity map of X in Ch and $1_{F(X)}$ is the identity map of $F(X)$ in Ho(Ch)
2. $F(gf) = F(g)F(f)$ where $f : X \to Y, g : Y \to Z$ are morphisms in Ch

Proof.

1. $F(1_X) = 1_{F(X)}$:

   This is true by definition of identity in Ch

2. $F(gf) = F(g)F(f)$:

$$F(gf) = [gf] = [g][f] = F(g)F(f)$$

□

1.6 Singular chain complex functor and singular homology functor on Ho(Top), Ho(Ch)

$$
\begin{tikzcd}
\text{Top} \arrow[r, "C_\bullet"] \arrow[d, two heads] & \text{Ch} \arrow[r, "H_\bullet"] \arrow[d, two heads] & \text{Ab} \\
\text{Ho(Top)} \arrow[r, dashed] & \text{Ho(Ch)} \arrow[ur, dashed] &
\end{tikzcd}
$$

Define the "functor" $C_\bullet : \text{Ho(Top)} \to \text{Ho(Ch)}$ as follows

- on objects: same with $C_\bullet : \mathrm{Top} \to \mathrm{Ch}$
- on morphisms: $[f] \mapsto f \mapsto C_\bullet(f) \mapsto [C_\bullet(f)]$ where $[f]$ is the homotopy class of a morphism f in Top, $[C_\bullet(f)]$ is the chain homotopy class of a morphism $C_\bullet(f)$ in Ch.

We will prove that $C_\bullet : \mathrm{Ho(Top)} \to \mathrm{Ho(Ch)}$ is indeed a functor by verifying the following

1. $C_\bullet([1]) = [1]$
2. $C_\bullet([g][f]) = C_\bullet([g])C_\bullet([f])$ where $f : X \to Y, g : Y \to Z$ are morphisms in Top

Proof.

1. $C_\bullet([1]) = [1]$:
   This is true by the definition of $C_\bullet$
2. $C_\bullet([g][f]) = C_\bullet([g])C_\bullet([f])$:

$$\begin{aligned}
C_\bullet([g][f]) &= C_\bullet([gf]) && \text{(composition in Ho(Top))} \\
&= [C_\bullet(gf)] && (C_\bullet : \mathrm{Ho(Top)} \to \mathrm{Ho(Ch)} \text{ on morphisms } [gf]) \\
&= [C_\bullet(g)C_\bullet(f)] && (\text{functor } C_\bullet : \mathrm{Top} \to \mathrm{Ch}) \\
&= [C_\bullet(g)][C_\bullet(f)] && \text{(composition in Ho(Ch))} \\
&= C_\bullet([g])C_\bullet([f]) && (C_\bullet : \mathrm{Ho(Top)} \to \mathrm{Ho(Ch)} \text{ on morphisms } [f] \text{ and } [g])
\end{aligned}$$

□

Define the "functor" $H_n : \mathrm{Ho(Ch)} \to \mathrm{Ab}$ as follows

- on objects: same with $H_n : \mathrm{Ch} \to \mathrm{Ab}$
- on morphisms: $[C_\bullet(f)] \mapsto C_\bullet(f) \mapsto H_n(f)$

We will prove that $H_n : \mathrm{Ho(Ch)} \to \mathrm{Ab}$ is indeed a functor by verifying the following

1. $H_n([1]) = 1$
2. $H_n([C_\bullet(g)][C_\bullet(f)]) = H_n([C_\bullet(g)])H_n([C_\bullet(f)])$ where where $C_\bullet(f) : X \to Y, C_\bullet(g) : Y \to Z$ are morphisms in Ch

Proof.

1. $H_n([1]) = 1$:
   This is true by the definition of H_n
2. $H_n([C_\bullet(g)][C_\bullet(f)]) = H_n([C_\bullet(g)])H_n([C_\bullet(f)])$:

$$\begin{aligned}
H_n([C_\bullet(g)][C_\bullet(f)]) &= H_n([C_\bullet(g)C_\bullet(f)]) && \text{(composition in Ho(Ch))} \\
&= H_n([C_\bullet(gf)]) && (\text{functor } C_\bullet : \mathrm{Top} \to \mathrm{Ch}) \\
&= H_n(gf) && (H_n : \mathrm{Ho(Ch)} \to \mathrm{Ab} \text{ on morphism } [C_\bullet(gf)]) \\
&= H_n(g)H_n(f) && (\text{functor } H_n : \mathrm{Top} \to \mathrm{Ab}) \\
&= H_n([C_\bullet(g)])H_n([C_\bullet(f)]) && (H_n : \mathrm{Ho(Ch)} \to \mathrm{Ab} \text{ on morphism } [C_\bullet(f)] \text{ and } [C_\bullet(g)])
\end{aligned}$$

□

2 Problem 2

- Put a CW structure on the product two finite CW complexes.
- Show that the composite of two cofibrations is a cofibration.

2.1 CW structure on the product two finite CW complexes

Given two cell complexes X, Y. We define a CW structure on Z as follows

$$\begin{tikzcd}
\coprod_{\alpha\in A_n} \partial D^n \arrow[r, hook] \arrow[d, "a_n"'] & \coprod_{\alpha\in A_n} D^n \arrow[d, "\overline{a_n}"] \\
X_{n-1} \arrow[r, hook] & X_n
\end{tikzcd}
\qquad
\begin{tikzcd}
\coprod_{\beta\in B_n} \partial D^n \arrow[r, hook] \arrow[d, "b_n"'] & \coprod_{\beta\in B_n} D^n \arrow[d, "\overline{b_n}"] \\
Y_{n-1} \arrow[r, hook] & Y_n
\end{tikzcd}$$

$$\begin{tikzcd}
\coprod_{\gamma\in C_n} \partial D^n \arrow[r, hook] \arrow[d, "c_n"'] & \coprod_{\gamma\in C_n} D^n \arrow[d, "\overline{c_n}"] \\
Z_{n-1} \arrow[r, hook] & Z_n
\end{tikzcd}$$

where $C_n = \coprod_{i+j=n} A_i \times B_j$ be the disjoint union of $A_i \times B_j$.

Some notes on CW structure:

a_n is the attaching map, $\overline{a_n}$ is the characteristic map. Note that, attaching map is a restriction of characteristic map on the boundary of $\coprod D^n$. In the interior of $\coprod D^n$, characteristic map is a homeomorphism. X_n is the quotient of $\coprod_{\alpha\in A_n} D^n$ under the equivalence class defined by a_n (or $\overline{a_n}$). Let $a_n^\alpha : \partial D^n \to X_{n-1}, \overline{a_n^\alpha} : D^n \to X_n$ are attaching map and characteristic map corresponding to $\alpha \in A_n$. Similar notations for Y and Z.

If D^n is a n-dimensional cube, we can show that

$$D^{i+j} = D^i \times D^j$$
$$\partial D^{i+j} = \partial D^i \times D^j \cup D^i \times \partial D^j$$

For $\gamma = (\alpha, \beta) \in A_i \times B_j$, define attaching map

$$c_n^\gamma : \partial D^{i+j} \to X_{i-1} \times Y_j \cup X_i \times Y_{j-1} \subseteq Z_{n-1}$$

that maps $\partial D^i \times D^j$ to $X_{i-1} \times Y_j$, maps $D^i \times \partial D^j$ to $X_i \times Y_{j-1}$ as follows:

$$c_n^\gamma(x, y) = \left(\overline{a_i^\alpha}(x), \overline{b_j^\beta}(y)\right)$$

where $(x, y) \in \partial D^i \times D^j \cup D^i \times \partial D^j$. We are left to prove that $Z = X \times Y$ by verifying $Z_n = \bigcup_{i+j=n} X_i \times Y_j$, that is, $\bigcup_{i+j=n} X_i \times Y_j$ is the pushout of Z_n's diagram and finish the proof by the argument on finiteness of X, Y, Z

Proof.

Consider one of the pair (i, j), we show that $W_{ij} = X_i \times Y_j$ is the pushout of the diagram below:

$$\begin{tikzcd}
\coprod_{\gamma\in A_i\times B_j} \partial D^{i+j} \arrow[r, hook] \arrow[d, "c_n"'] & \coprod_{\gamma\in A_i\times B_j} D^{i+j} \arrow[d, "\overline{c}_n"] \\
X_{i-1} \times Y_j \cup X_i \times Y_{j-1} \arrow[r, hook] & W_{ij}
\end{tikzcd}$$

Let $(x, y), (x_1, y_1) \in \coprod_{\gamma\in A_i\times B_j} D^i \times D^j$.

$(x, y), (x_1, y_1)$ are in the same equivalence class (under $\overline{c_n}$) if and only if $\overline{a_i^\alpha}(x) = \overline{a_i^\alpha}(x_1)$ and $\overline{b_j^\beta}(y) = \overline{b_j^\beta}(y_1)$ if and only if x, x_1 be in the same equivalence class of $\overline{a_n}$ and y, y_1 be in the same equivalence class of $\overline{b_n}$, that is, x, x_1 identify the same point on X_i and y, y_1 identify the same point on Y_i. Therefore, the pushout of the diagram is exactly $W_{ij} = X_i \times Y_j$

Hence, $Z_n = \bigcup_{i+j=n} X_i \times Y_j$ is the pushout of the diagram consists of disjoint union over the finite collection of pairs (i, j)

As X, Y are finite, let $X = X_m, Y = Y_n$, as $X_0 \subseteq X_1 \subseteq ... \subseteq X$, $Y_0 \subseteq Y_1 \subseteq ... \subseteq Y$, we have

$$Z = Z_{m+n} = X_m \times Y_n = X \times Y$$

$\square$

2.2 Composite of cofibrations

$$
\begin{tikzcd}
 & & W \\
Z \arrow[r] \arrow[urr, "h_Z"] & Z \times I \arrow[ur, dashed, "H_Z"] & \\
Y \arrow[u, "g"] \arrow[r, "i_1"'] & Y \times I \arrow[u, "g\times 1"'] \arrow[uur, dashed, "H_Y"] & \\
X \arrow[u, "f"] \arrow[r, "i_1"'] & X \times I \arrow[u, "f\times 1"'] \arrow[uuur, "H_X"'] &
\end{tikzcd}
$$

Suppose $f : X \to Y, g : Y \to Z$ are cofibrations, there is a homotopy $H_X : X \times I \to W$ and a map $h_Z : Z \to W$. If $gf : X \to Z$ is a cofibration, homotopy extension property states that there exists $H_Z : Z \times I \to W$ such that that diagram commutes.

Indeed, let $h_Y : Y \to W$ be defined by $h_Y = h_Z g$. Since $f : X \to Y$ is a cofibration, given $H_X : X \times I \to W$ and $h_Y : Y \to W$, there exists $H_Y : Y \times I \to W$ such that the diagram commutes. Since $g : Y \to Z$ is a cofibration, given $H_Y : Y \times I \to W$ and $h_Z : Z \to W$, there exists $H_Z : Z \times I \to W$ such that the diagram commutes

3 Problem 3

For an invertible linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$. show that the induced map on $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \approx \tilde{H}_{n-1}(\mathbb{R}^n - \{0\}) \approx \mathbb{Z}$ is $\mathbb{1}$ or $-\mathbb{1}$ according to whether the determinant of f is positive or negative.

3.1 Preliminaries

We adopt the definition of reduced homology in Hatcher.

Definition 1 (reduced homology). *Let X be a non-empty topological spaces and $C_\bullet : \mathrm{Top} \to \mathrm{Ab}$ be singular chain functor. Reduced homology is the homology of the chain complex*

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\epsilon} C_0(X) \xleftarrow{\partial} C_1(X) \xleftarrow{\partial} \ldots$$

where $\epsilon : C_0(X) \to \mathbb{Z}$ is the augmentation map. The augmented chain complex is denoted by $\tilde{C}_\bullet(X)$ and the reduced homology is denoted by $\tilde{H}_\bullet(X)$

Remark 1 (relationship with singular homology).

$$
\begin{aligned}
H_0(X) &= \tilde{H}_0(X) \oplus \mathbb{Z} \\
H_n(X) &= \tilde{H}_n(X) \text{ for } n \geq 1
\end{aligned}
$$

Remark 2 (reduced homology of common spaces). *Reduced homology of common spaces*

- $\tilde{H}_n(*) = 0$*: homology of a contractible space is the trivial group*
- $\tilde{H}_0(X) = \bigoplus_{i=1}^{n-1} \mathbb{Z}$*: if X has n path-components*

Remark 3 (relative homology on reduced homology). *Short exact sequence of chains*

$$
\begin{tikzcd}
0 \arrow[r] & \tilde{C}_n(A) \arrow[r, hook] \arrow[d, "\partial"] & \tilde{C}_n(X) \arrow[r, two heads] \arrow[d, "\partial"] & \tilde{C}_n(X)/\tilde{C}_n(A) \arrow[r] \arrow[d, dashed, "\partial"] & 0 \\
0 \arrow[r] & \tilde{C}_{n-1}(A) \arrow[r, hook] & \tilde{C}_{n-1}(X) \arrow[r, two heads] & \tilde{C}_{n-1}(X)/\tilde{C}_{n-1}(A) \arrow[r] & 0
\end{tikzcd}
$$

Note that, this is identical to the short exact sequence of chains for singular homology except $\tilde{C}_{-1}(A)$ and $\tilde{C}_{-1}(X)$. The induced long exact sequence

$$\begin{tikzcd}
 & & \dots \arrow[dll, "\partial"'] \\
\tilde{H}_n(A) \arrow[r, "i"] & \tilde{H}_n(X) \arrow[r, two heads, "p"] & H_n(X,A) \arrow[dll, "\partial"'] \\
\tilde{H}_{n-1}(A) \arrow[r, "i"] & \tilde{H}_{n-1}(X) \arrow[r, two heads, "p"] & H_{n-1}(X,A) \arrow[dll, "\partial"'] \\
\dots
\end{tikzcd}$$

Definition 2 ($\partial : H_{n+1}(C) \to H_n(A)$)**.** *Definition of the connecting homomorphism* $\partial : H_{n+1}(C) \to H_n(A)$

$$\begin{tikzcd}
n+1 & & b \arrow[r, mapsto, "p"] \arrow[d, mapsto, "\partial"] & c \arrow[d, mapsto, "\partial"] \\
n: & a \arrow[r, mapsto, "i"] \arrow[d, mapsto, "\partial"] & \partial b \arrow[r, mapsto, "p"] \arrow[d, mapsto, "\partial"] & 0 \\
n-1: & \partial a \arrow[r, mapsto, "i"] & \partial^2 b = 0 &
\end{tikzcd}$$

Given $[c] \in H_{n+1}(C)$*, (1) take any representative* $c \in Z_{n+1}(C)$*. As* $p : B_{n+1} \to C_{n+1}$ *is surjective, (2) take any* $b \in B_{n+1}$ *such that* $pb = c$*. As* $p\partial b = \partial pb = \partial c = 0$ *and* $\ker(p : B_n \to C_n) = \operatorname{im}(i : A_n \to B_n)$*, take* $a \in A_n$ *such that* $ia = \partial b$*, this choice is unique as* i *is injective.* $i\partial a = \partial ia = \partial^2 b = 0$*, as* i *is an injective homomorphism,* $\partial a = 0$*, then* $a \in Z_n(A)$*. The construction is done by* $[c] \mapsto [a]$

Another result from Hatcher:

Lemma 1. *Given two short exact sequences of chain complexes with chain maps* $\alpha : A_n \to A'_n, \beta : B_n \to B'_n, \gamma : C_n \to C'_n$*, such that the diagram below commutes*

$$\begin{tikzcd}
0 \arrow[r] & A_\bullet \arrow[r, "i"] \arrow[d, "\alpha"] & B_\bullet \arrow[r, "p"] \arrow[d, "\beta"] & C_\bullet \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A'_\bullet \arrow[r, "i'"] & B'_\bullet \arrow[r, "p'"] & C'_\bullet \arrow[r] & 0
\end{tikzcd}$$

Then the induced long exact sequence diagram commutes

$$\begin{tikzcd}
\dots \arrow[r] & H_{n+1}(A_\bullet) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & H_{n+1}(B_\bullet) \arrow[r, "p_*"] \arrow[d, "\beta_*"] & H_{n+1}(C_\bullet) \arrow[r, "\partial"] \arrow[d, "\gamma_*"] & H_n(A_\bullet) \arrow[r] \arrow[d, "\alpha*"] & \dots \\
\dots \arrow[r] & H_{n+1}(A'_\bullet) \arrow[r, "i'_*"] & H_{n+1}(B'_\bullet) \arrow[r, "p'_*"] & H_{n+1}(C'_\bullet) \arrow[r, "\partial'"] & H_n(A'_\bullet) \arrow[r] & \dots
\end{tikzcd}$$

where $H_n : \mathrm{Ch} \to \mathrm{Ab}$ *is a functor*

Proof.

The first two squares commute since H_n is a functor. For the third square, recall the definition of $\partial : H_n(C) \to H_{n-1}(A)$

$$\partial[c] = [a]$$

where $c = pb$ and $ia = \partial b$. We have

$$\gamma c = \gamma pb = p'\beta b$$
$$i'\alpha a = \beta ia = \beta\partial b = \partial\beta b$$

then by the definition of connecting homomorphism $\partial : H_n(C') \to H_{n-1}(A')$, we have

$$\partial[\gamma c] = [\alpha a]$$

Again, H_n is a functor,

$$[\gamma c] = H_n(\gamma)[c] = \gamma_*[c]$$
$$[\alpha a] = H_n(\alpha)[a] = \alpha_*[a] = \alpha_*\partial[c]$$

That is, the last third square commutes ☐

Definition 3 (degree)**.** *For* $n > 0$*, let* $f : S^n \to S^n$*, then* $f_* : \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)$ *is a multiplication* $\mathbb{Z} \to \mathbb{Z}$ *of* m*.* m *is called the degree of* f

Lemma 2. *Degree of a refection is* -1

3.2 Main Proof

Let's denote $X = \mathbb{R}^n, A = \mathbb{R}^n - \{0\}$. Any linear map f in $GL(\mathbb{R}^n)$ can by transformed into either the identity 1 or a reflection r by Gaussian elimination, each row operation is either row-swap, row-scale, row-sum which can be written as a smooth map of time t, that is, any linear map is homotopic to either 1 (if $\det f > 0$) or r (if $\det f < 0$). Moreover, the homotopy applies for the case of pair of spaces (X, A)

Long exact sequence of (X, A) implies the connecting homomorphism $\partial : H_n(X, A) \to \tilde{H}_{n-1}(A)$ is an isomorphism.

$$\begin{tikzcd} & \tilde{H}_n(X) = 0 \arrow[r, two heads, "p_*"] & H_n(X,A) \arrow[dll, "\partial"'] \\ \tilde{H}_{n-1}(A) \arrow[r, "i_*"] & \tilde{H}_{n-1}(X) = 0 & \end{tikzcd}$$

The diagram below commutes

$$\begin{tikzcd} 0 \arrow[r] & \tilde{C}_\bullet(A) \arrow[r, hook] \arrow[d, "f_\#"] & \tilde{C}_\bullet(X) \arrow[r, two heads] \arrow[d, "f_\#"] & \tilde{C}_\bullet(X,A) \arrow[r] \arrow[d, "f_\#"] & 0 \\ 0 \arrow[r] & \tilde{C}_\bullet(A) \arrow[r, hook] & \tilde{C}_\bullet(X) \arrow[r, two heads] & \tilde{C}_\bullet(X,A) \arrow[r] & 0 \end{tikzcd}$$

where $f_\#$ is induced from f in the level of chain. By Lemma 1 the diagram below commutes, , f_* is induced from f_* in the level of homology

$$\begin{tikzcd} C_n(X,A) \arrow[d, "f_\#"'] & H_n(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"'] & \tilde{H}_{n-1}(A) \arrow[d, "f_*"] & \tilde{C}_{n-1}(A) \arrow[d, "f_\#"] \\ C_n(X,A) & H_n(X,A) \arrow[r, "\partial"] & \tilde{H}_{n-1}(A) & \tilde{C}_{n-1}(A) \end{tikzcd}$$

As ∂ is an isomorphism between $\mathbb{Z}$ and $\mathbb{Z}$ (isomorphism sends 1 to either 1 or -1), it suffices to show for the case of reduced homology $\tilde{H}_{n-1}(A)$

Proof.

If f is homotopic to the identity 1, the induced map in $\tilde{H}_{n-1}(A)$ is the identity map 1

If f is homotopic to a reflection r, let $g : S^{n-1} \to S^{n-1}$ be the restriction of r (g is a reflection on S^{n-1}, $\tilde{H}_{n-1}(g) = -1$), $i : S^{n-1} \to A$ be the inclusion map, $p : A \to S^{n-1}$ be the deformation retraction of A into S^{n-1}.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & A \\ S^{n-1} \arrow[r, "g"'] & S^{n-1} \arrow[u, "i"'] \end{tikzcd} \qquad \begin{tikzcd} \tilde{H}_{n-1}(A) \arrow[r, "\tilde{H}_{n-1}(f)"] \arrow[d, "\tilde{H}_{n-1}(p)"'] & \tilde{H}_{n-1}(A) \\ \tilde{H}_{n-1}(S^{n-1}) \arrow[r, "\tilde{H}_{n-1}(g)"'] & \tilde{H}_{n-1}(S^{n-1}) \arrow[u, "\tilde{H}_{n-1}(i)"'] \end{tikzcd}$$

Since the left diagram (diagram in Top) commutes, $\tilde{H}_{n-1} : \text{Top} \to \text{Ab}$ is a functor, the right diagram (diagram in Ab) commutes. As i and p are homotopy equivalence ($pi \simeq 1, ip \simeq 1$), $\tilde{H}_{n-1}(i) = \tilde{H}_{n-1}(p) = 1$, then

$$\tilde{H}_{n-1}(f) = \tilde{H}_{n-1}(i)\tilde{H}_{n-1}(g)\tilde{H}_{n-1}(p) = 1(-1)1 = -1$$

the induced map in $\tilde{H}_{n-1}(A)$ is -1. $\square$

4 Problem 4

A polynomial $f(z)$ with complex coefficients, viewed as a map $\mathbb{C} \to \mathbb{C}$ can always be extended to a continuous map of one-point compactifications $\hat{f} : S^2 \to S^2$. Show that the degree of $\hat{f}$ equals the degree of f as a polynomial. Show also that the local degree of $\hat{f}$ at a root of f is the multiplicity of the root.

4.1 Preliminaries

Lemma 3. *On S^1 (unit circle in $\mathbb{C}$),* $\deg z^n = n$

Lemma 4 (Hatcher proposition 2.33)**.** $\deg \Sigma f = \deg f$ *where $\Sigma f : \Sigma S^n \to \Sigma S^n$ is the suspension of $f : S^n \to S^n$ and $\Sigma S^n \cong S^{n+1}$ is the suspension of S^n*

4.2 Degree of $\hat{f}$

By lemma 3 and 4, in S^2, $\deg \Sigma z^n = \deg z^n = n$. Moreover, there exists a homotopy from Σz^n to z^n (write z^n in polar coordinate). Hence, in S^2, $\deg z^n = n$

Let $f(z) = a_n z^n + ... + a_1 z + a_0$ defined on S^2, there exist two maps $H_1 : S^2 \times I \to S^2$ and $H_2 : S^2 \times I \to S^2$ as follows

$$H_1(z,t) = a_n^t z^n$$
$$H_2(z,t) = t a_n z^n + (1-t) f(z)$$

Both maps are continuous on $\mathbb{C} \times I$ and $\{\infty\} \times I$, hence they are homotopies $z^n \to a_n z^n$, $a_n z^n \to \hat{f}(z)$. Therefore, in S^2, $\deg \hat{f} = \deg z^n = n$

4.3 Local degree of $\hat{f}$

Let $\{x_1, ..., x_n\}$ be the roots of $f(z)$. Let disjoint path-connected open sets $\{U_1, ..., U_n\}$ such that $x_i \in U_i$ and $V = \hat{f}(\bigcup_{i=1}^n U_i)$, hence, V is also path-connected. By definition, local degree of $\hat{f}$ at x_i is the induced function $\hat{f}_*$

$$H_2(U_i, U_i - \{x_i\}) = \mathbb{Z} \xrightarrow{\hat{f}_*} H_2(V, V - \{0\}) = \mathbb{Z}$$

Given the commutative diagram below

$$\begin{tikzcd}
0 \arrow[r] & C_\bullet(U_i - \{x_i\}) \arrow[r, "i"] \arrow[d, "f_\#"] & C_\bullet(U_i) \arrow[r, "p"] \arrow[d, "f_\#"] & C_\bullet(U_i, U_i - \{x_i\}) \arrow[r] \arrow[d, "f_\#"] & 0 \\
0 \arrow[r] & C_\bullet(V - \{0\}) \arrow[r, "i'"] & C_\bullet(V) \arrow[r, "p'"] & C_\bullet(V, V - \{0\}) \arrow[r] & 0
\end{tikzcd}$$

By Lemma 1 and exactness, ∂, ∂' are isomorphisms, the square is commutative, the induced maps are the same.

$$\begin{tikzcd}
H_2(U_i) = 0 \arrow[r, "p_*"] & H_2(U_i, U_i - \{x_i\}) = \mathbb{Z} \arrow[r, "\partial"] \arrow[d, "\hat{f}_*"] & H_1(U_i - \{x_i\}) = \mathbb{Z} \arrow[r, "i_*"] \arrow[d, "\hat{f}_*"] & H_1(U_i) = 0 \\
H_2(V) = 0 \arrow[r, "p'_*"] & H_2(V, V - \{0\}) = \mathbb{Z} \arrow[r, "\partial'"] & H_1(V - \{0\}) = \mathbb{Z} \arrow[r, "i'_*"] & H_1(V) = 0
\end{tikzcd}$$

Hence, local degree of $\hat{f}$ at x_i is the degree of $\hat{f}$ restricted to $U_i - \{x_i\} \to V - \{0\}$. Now write $f(z) = (z - x_i)^{m_i} g(z)$ where $g(z) \neq 0$ on U_i and m_i is the multiplicity of root x_i. There exists a map $H_3 : (U_i - \{x_i\}) \times I \to V - \{0\}$ as follows

$$H_3(z,t) = t(z - x_i)^{m_i} + (1-t) f(z)$$

As H_3 is continuous on its domain, hence it is a homotopy from $(z - x_i)^{m_i}$ to $f(z)$. Therefore, $\deg \hat{f}$ restricted to $U_i - \{x_i\} \to V - \{0\}$ is m_i

5 Problem 5

Let X be the quotient space of S^2 under identifications $x \sim -x$ for x in the equator S^1. Compute the homology groups $H_i(X)$. Do the same for S^3 with antipodal points of equatorial $S^2 \subset S^3$ identified.

Proposition 1 (cellular boundary formula). *$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$ where $d_{\alpha\beta}$ is the degree of the map $S_\alpha^{n-1} \to X^{n-1} \to S_\beta^{n-1}$ that is the composition of the attaching map of e_α^n with the quotient map collapsing $X^{n-1} - e_\beta^{n-1}$ to a point.*

5.1 S^2 with antipodal points of equatorial identified

Define the CW structure $X_0 \subseteq X_1 \subseteq X_2 = X_3 = ... = X$ as follows

- X_0 is a single point
- $X_1 \cong S^1$, $a_1^{(1)} : S_1^0 \to X_0$ maps two points of S_1^0 to X_0
- $X_2 = X$, $a_1^{(2)}$ and $a_2^{(2)}$ wind around X_1 twice in opposite directions and D_1^2 and D_2^2 are the northern hemisphere and southern hemisphere

The pushout diagrams are as follows (subscripts are used to distinguish between multiple copies)

$$\begin{tikzcd} S^0_1 \arrow[r] \arrow[d, "a^{(1)}_\bullet"] & D^1_1 \arrow[d, "c^{(1)}_\bullet"] \\ X_0 \arrow[r] & X_1 \cong S^1 \end{tikzcd} \qquad \begin{tikzcd} S^1_1 \amalg S^1_1 \arrow[r] \arrow[d, "a^{(2)}_\bullet"] & D^2_2 \amalg D^2_2 \arrow[d, "c^{(2)}_\bullet"] \\ X_1 \cong S^1 \arrow[r] & X_2 \end{tikzcd}$$

We have the cellular chain complex

$$0 \xleftarrow{d_0} C^{CW}_0(X) = \mathbb{Z} \xleftarrow{d_1} C^{CW}_1(X) = \mathbb{Z} \xleftarrow{d_2} C^{CW}_2(X) = \mathbb{Z}^2 \xleftarrow{d_3} 0 \longleftarrow ...$$

We have

$$\mathbb{Z} = H_0(X) = H^{CW}_0(X) = \frac{\ker d_0}{\operatorname{im} d_1} = \frac{\mathbb{Z}}{\operatorname{im} d_1}$$

then, the map $d_1 = 0$ and $H_0(X) = \mathbb{Z}$ since $\operatorname{im} d_1 = k\mathbb{Z}$ for $k \in \mathbb{Z}$ implies $k = 0$. As $a^{(2)}_1, a^{(2)}_2$ composed with the quotient map collapsing $X_1 - e^1_1$ are maps $S^1 \to S^1$ that wind around S^1 twice in opposite directions, then $d_{11} = +2, d_{12} = -2$, and

$$d_2(e^2_1) = d_{11}e^1_1 = +2e^1_1$$
$$d_2(e^2_2) = d_{12}e^1_1 = -2e^1_1$$

That is, $\operatorname{im} d_2 = 2\mathbb{Z}$ and $\ker d_2 = \operatorname{span}(1,1)$. Hence,

$$H_1(X) = \frac{\ker d_1}{\operatorname{im} d_2} = \frac{\mathbb{Z}}{2\mathbb{Z}}$$

$C^{CW}_3(X) = C^{CW}_4(X) = ... = 0$ as $\mathcal{A}_3 = \mathcal{A}_4 = ... = \varnothing$, then $d_3 = d_4 = ... = 0$. Hence,

$$H_2(X) = \frac{\ker d_2}{\operatorname{im} d_3} \cong \frac{\mathbb{Z}}{0} = \mathbb{Z}$$
$$H_3(X) = H_4(X) = ... = 0$$

5.2 S^3 with antipodal points of equatorial identified

Define the CW structure $X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 = X_4 = ... = X$ as follows

- X_0 is a single point
- $X_1 \cong S^1$, $a^{(1)}_1$ maps two points of S^0_1 to X_0
- $X_2 \cong \mathbb{R}P^2$, $a^{(2)}_1$ winds around X_1 twice.
- $X_3 = X$, $a^{(3)}_1, a^{(3)}_2$ are maps from S^2 to $X_2 \cong \mathbb{R}P^2$ identifying antipodal points on S^2 to the same point on $\mathbb{R}P^2$ and D^3_1, D^3_2 are northern hemisphere and southern hemisphere.

The pushout diagrams are as follows (subscripts are used to distinguish between multiple copies)

$$\begin{tikzcd} S^0_1 \arrow[r] \arrow[d, "a^{(1)}_\bullet"] & D^1_1 \arrow[d, "c^{(1)}_\bullet"] \\ X_0 \arrow[r] & X_1 \cong S^1 \end{tikzcd} \qquad \begin{tikzcd} S^1_1 \arrow[r] \arrow[d, "a^{(2)}_\bullet"] & D^2_1 \arrow[d, "c^{(2)}_\bullet"] \\ X_1 \cong S^1 \arrow[r] & X_2 \cong \mathbb{R}P^2 \end{tikzcd} \qquad \begin{tikzcd} S^2_1 \amalg S^2_2 \arrow[r] \arrow[d, "a^{(3)}_\bullet"] & D^3_1 \amalg D^3_2 \arrow[d, "c^{(3)}_\bullet"] \\ X_2 \cong \mathbb{R}P^2 \arrow[r] & X_3 \end{tikzcd}$$

We have the cellular chain complex

$$0 \xleftarrow{d_0} C^{CW}_0(X) = \mathbb{Z} \xleftarrow{d_1} C^{CW}_1(X) = \mathbb{Z} \xleftarrow{d_2} C^{CW}_2(X) = \mathbb{Z} \xleftarrow{d_3} C^{CW}_3(X) = \mathbb{Z}^2 \xleftarrow{d_4} 0 \longleftarrow ...$$

Similar to previous part, $d_1 = 0$ and $H_0(X) = \mathbb{Z}$. As $a_1^{(2)}$ composed with the quotient map collapsing $X_1 - e_1^1$ is map $S^1 \to S^1$ that wind around S^1 twice, then $d_{11} = +2$, and

$$d_2(e_1^2) = d_{11}e_1^1 = +2e_1^1$$

That is, $\operatorname{im} d_2 = 2\mathbb{Z}$ and $\ker d_2 = 0$. Hence

$$H_1(X) = \frac{\ker d_1}{\operatorname{im} d_2} = \frac{\mathbb{Z}}{2\mathbb{Z}}$$

As $\ker d_2 = 0$, then $d_3 = 0$ because $\operatorname{im} d_3 \subseteq \ker d_2$, then $\ker d_3 = \mathbb{Z}^2$. Hence,

$$H_2(X) = \frac{\ker d_2}{\operatorname{im} d_3} = \frac{0}{0} = 0$$

$C_4^{CW}(X) = C_5^{CW}(X) = ... = 0$ as $\mathcal{A}_4 = \mathcal{A}_5 = ... = \varnothing$, then $d_4 = d_5 = ... = 0$. Hence,

$$H_3(X) = \frac{\ker d_3}{\operatorname{im} d_4} \cong \frac{\mathbb{Z}^2}{0} = \mathbb{Z}^2$$
$$H_4(X) = H_5(X) = ... = 0$$