Scheme Theory

this is my note for scheme theory. this is mainly based on (1) Görtz - Wedhorn AG 1 and loosely based on (2) Vakil FOAG and (3) Borcherds online lectures on youtube

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Contents

Chapter 1

AFFINE SCHEME

1.1 DEFINITION OF AFFINE SCHEME

Definition 1.1.1 (ringed space, locally ringed space)
A ringed space $(X, \mathscr{O})$ is a topological space X together with a sheaf of rings $\mathscr{O}$. A ringed space is a locally ringed space if for every point $x \in X$, the stalk $\mathscr{O}_x$ is a local ring.

Definition 1.1.2 (affine scheme)
An affine scheme is a locally ringed space that is isomorphic to the spectrum of some ring A

1.1.1 CONSTRUCT AFFINE SCHEME FROM RING

Let A be a ring, let $X = \operatorname{Spec} A$ be the set of prime ideals in A

$$\operatorname{Spec} A = \{\text{prime } \mathfrak{p} \subseteq A\}$$

We put a topology on $\operatorname{Spec} A$ generated by the basis of open sets

$$D(f) = \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}\}$$

for every $f \in A$. The constructed topology is called Zariski's topology and the open subsets of the form $D(f)$ for some $f \in A$ are called **principal open subsets**. We put a sheaf of rings $\mathscr{O}$ on $\operatorname{Spec} A$ generated by

$$\mathscr{O}(D(f)) = A_f$$

If $D(g) \subseteq D(f)$, that is $g \in \sqrt{(f)}$, $g^n = fh$ for some $h \in A$ and $n \geq 1$. Since f is a unit in A_g, the restriction map is well-defined and unique

$$\begin{aligned} A_f &\to A_g \\ \frac{x}{f^m} &\mapsto x\left(\frac{h}{g^n}\right)^m \end{aligned}$$

Under this construction, $(X, \mathscr{O})$ is an affine scheme. An element $f \in A$ is called **function**, a element $x \in X$ is called

point, when refering x as a prime ideal in A, we write $\mathfrak{p}_x \subseteq A$, and function evaluation is equivalent to sending f to the residue field of stalk $\mathscr{O}_X = A_{\mathfrak{p}_x}$.
When A is a polynomial ring of n variables over ring R, we write $\mathbb{A}_R^n = \operatorname{Spec} A$
(I see many proofs, people assumed that open set is just principle open set. maybe the argument generalizes easily to general open set. anyway, if needed, below is the defintion of sheaf on open set)
On $\operatorname{Spec} A$, any open set is of the form

$$D(S) = \{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{p} \cap S = \varnothing\}$$

for some multiplicatively closed subset $S \subseteq A$. An the ring over $D(S)$ is

$$\mathscr{O}(D(S)) = S^{-1}A$$

If $D(T) \subseteq D(S)$, then $S \subseteq T$, the restriction map is the canonical map

$$\begin{gathered} \mathscr{O}(D(S)) \to \mathscr{O}(D(T)) \\ S^{-1}A \to T^{-1}A \\ \frac{a}{s} \mapsto \frac{a}{s} \end{gathered}$$

Note that, this map corresponds to the inclusion of affine schemes

$$D(T) \hookrightarrow D(S)$$

1.1.2 SOME EXAMPLES OF AFFINE SCHEME

Remark 1.1.3 (irreducible)
A topological space X is reducible if it can be written as a union $X = X_1 \cup X_2$ of two non-empty proper closed subsets.

1. Let X be a topological space, X is irreducible if one of the following condiotions
   (a) X is not reducible
   (b) every non-empty open subset of X is dense
   (c) any two non-empty open subsets of X have non-empty intersection
2. A subset F of topological space X is reducible/irreducible if F as a subspace is reducible/irreducible
3. An irreducible component is a maximal irreducible subset.

Remark 1.1.4 (closed, generic, specialization, generization, maximal)
Let X be a topological space

1. A point $x \in X$ is called closed if $\{x\} \subseteq X$ is closed
2. A point $\eta \in X$ is called generic if $\overline{\{\eta\}} = X$
3. Let $x, y \in X$, y is called a specialization of x if $y \in \overline{\{x\}}$. In that case, x is called generization of y
4. A point $x \in X$ is called maximal if $\overline{\{x\}}$ is an irreducible component of X

example: (integers) Let $A = \mathbb{Z}$, then

$$X = \operatorname{Spec} \mathbb{Z} = \{(2), (3), (5), ..., (0)\}$$

for each prime number $p \in \mathbb{Z}$, (p) is a closed point. (0) is a generic point, and closure of (0) is the whole space X.
example: (field) Let k be a field. Let $A = k$, then

$$X = \operatorname{Spec} k = \{(0)\}$$

is a singleton set.
example: (polynomial ring of two variables over an algebraically closed field) Let A be an arbitrary ring, maximal ideals are closed points. Moreover, the closure of a point $\mathfrak{p}$ is the set of prime ideals containing $\mathfrak{p}$

$$V(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} A : \mathfrak{p} \subseteq \mathfrak{q}\}$$

In particular, let k be an algebraically closed field, let $A = k[x, y]$, by Nullstellensatz, the set of closed points are

$$\{(x - a, y - b) : a \in k, b \in k\}$$

X admits other generic points (0) and (f) for every irreducible $f \in k[x, y]$. By dimensionality argument, the prime (f) is of height 1 and the closure of (f) consists of (f) and $(x - a, y - b)$ for $(a, b) \in k^2$ in the vanishing set of f. Similarly, closure of (0) is the whole space.

1.2 MORPHISM OF AFFINE SCHEMES

Remark 1.2.1 (inverse image, direct image)
Let $\psi : X \to Y$ be a continuous map. The inverse image functor ψ^{-1} and direct image functor ψ_* is an adjoint pair between the category of sheaves on X and the category of sheaves on Y. Let $\mathscr{F}$ and $\mathscr{G}$ be a sheaf on X and a sheaf on Y respectively, then

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(\psi^{-1}\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathscr{G}, \psi_*\mathscr{F})$$

The direct image functor ψ_* is defined as follows: for every open subset $V \subseteq Y$, then

$$(\psi_*\mathscr{F})(V) = \mathscr{F}(\psi^{-1}(U))$$

The inverse image functor ψ^{-1} is defined as follows: for every open subset $U \subseteq X$, then

$$(\psi^{-1}\mathscr{G})(U) = \mathrm{colim}_{V \subseteq Y : \psi(U) \subseteq V}\, \mathscr{G}(V)$$

Definition 1.2.2 (morphism of ringed spaces)
A morphism of ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is defined by a continuous map $\psi : X \to Y$ and a morphism of sheaves of rings $\psi^\flat : \mathscr{O}_Y \to \psi_*\mathscr{O}_X$. By adjunction between ψ^{-1} and ψ_*, this is equivalent to a morphism of sheaves of rings $\psi^\# : \psi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$. Moreover, $\psi^\#$ induces a map on stalks

$$\psi^\#_x : (\psi^{-1}O_Y)_x = \mathscr{O}_{Y,\psi(x)} \to \mathscr{O}_{X,x}$$

Remark 1.2.3 (compatible germs)
Given a sheaf of sets $\mathscr{F}$ on X, for every open subset $U \subseteq X$, the natural map from sections into product of stalks

$$\mathscr{F}(U) \hookrightarrow \prod_{x \in U} \mathscr{F}_x$$

is injective. An element $(s_x)_{x \in U} \subseteq \prod_{x \in U} \mathscr{F}_x$ in the image if this map is called compatible germs.

From the equivalence between set of sections and set of compatible germs, one can identify a morphism of ringed spaces by its topological space map $\psi : X \to Y$ and its map on stalks $\psi^\#_x : (\psi^{-1}O_Y)_x = \mathscr{O}_{Y,\psi(x)} \to \mathscr{O}_{X,x}$.

Definition 1.2.4 (morphism of locally ringed spaces, morphism of affine schemes)
A morphism of ringed spaces $(\psi, \psi^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of locally ringed spaces if for every $x \in X$, the induced map on stalks $\psi^\#_x : (\psi^{-1}O_Y)_x = \mathscr{O}_{Y,\psi(x)} \to \mathscr{O}_{X,x}$ is a local homomorphism. A morphism of affine schemes is a morphism of locally ringed spaces.

We obtain the category of affine schemes, denoted by AffSch. The local homomorphism condition of the map between stalks ensures that any zero function $g \in \mathscr{O}_{Y,\psi(x)}$ at $\psi(x) \in Y$ will be sent to a zero function $\psi^\#_x(g) \in \mathscr{O}_{X,x}$ at $x \in X$

Proposition 1.2.5 (equivalence between commutative rings and affine schemes)
The functor Spec from the opposite category of commutative rings into the category of affine schemes is fully faithful and essentially surjective.

$$\mathrm{Spec} : \mathrm{CRing}^{\mathrm{op}} \xrightarrow{\sim} \mathrm{AffSch}$$

1.2.1 CONSTRUCT MORPHISM OF AFFINE SCHEMES FROM MAP OF RINGS

Let $\phi : A \to B$ be a map of rings, let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ be the corresponding affine schemes. Then, the corresponding morphism of locally ringed space $(\psi, \psi^\flat) : X \to Y$ is defined as follows: for every $x \in X$

$$\begin{aligned} \psi : X &\to Y \\ x &\mapsto \phi^{-1}(x) \end{aligned}$$

The local homomorphism $\psi_x^\# : \mathscr{O}_{Y,\psi(x)} = A_{\psi(x)} \to B_x$ is induced from $\phi : A \to B$

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \\ A_{\psi(x)} \arrow[r, "\psi_x^\#"] & B_x \end{tikzcd}$$

1.2.2 SOME EXAMPLES OF MORPHISM OF AFFINE SCHEMES

example: (quotient) Let A be a ring and $\mathfrak{a} \subseteq A$ be an ideal. The map $A \twoheadrightarrow A/\mathfrak{a}$ induces an injective map of sets

$$\operatorname{Spec} A/\mathfrak{a} \hookrightarrow \operatorname{Spec} A$$

As sets, we have $\operatorname{Spec} A/\sqrt{\mathfrak{a}} = \operatorname{Spec} A/\mathfrak{a} = \operatorname{Spec} A/\mathfrak{a}^n$ for every $n \geq 1$. However, as affine schemes, in general, they are different, $A/\sqrt{\mathfrak{a}}$ is reduced, i.e. has no nilpotent while $\operatorname{Spec} A/\mathfrak{a}$ and $\operatorname{Spec} A/\mathfrak{a}^n$ might have nilpotents.
Furthermore, when $\mathfrak{a} = \mathfrak{p}_x = x$ is prime, quotient by $\mathfrak{p}_x$ is the action of taking closed subscheme [1] of X

> *keep all primes containing $\mathfrak{p}_x$. equivalently, keep all points contained in x*

example: (localization) Let A be a ring and $S \subseteq A$ be a multiplicatively closed subset. Localization at S induces an injective map of sets

$$\operatorname{Spec} S^{-1}A \hookrightarrow \operatorname{Spec} A$$

Furthermore, when $A - S = \mathfrak{p}_x = x$ is prime, localization at $\mathfrak{p}_x$ is the action of taking open subscheme [2] of X

> *keep all primes contained in $\mathfrak{p}_x$. equivalently, keep all points containing x*

example: (disjoint union) Let A, B be rings, then the projection $A \times B \to A$ corresponds to the monomorphism of affine schemes

$$\operatorname{Spec} A \hookrightarrow \operatorname{Spec} A \amalg \operatorname{Spec} B \cong \operatorname{Spec} A \times B$$

Categorically, $A \times B$ is product in rings and $\operatorname{Spec} A \amalg \operatorname{Spec} B$ is coproduct in affine schemes
example: (non-trivial fiber) Consider the ring map $\phi : k[u] \to k[x, y]$ defined by $u \mapsto y^2 - x$. It induces a morphism of affine schemes

[1] meaning will be revealed later
[2] meaning will be revealed later

$$
\begin{aligned}
\psi : \operatorname{Spec} k[x,y] &\to \operatorname{Spec} k[u] \\
(0) &\mapsto (0) \\
(x-a, y-b) &\mapsto (u-(b^2-a)) \\
(y^2-x) &\mapsto (u) \\
(f) &\mapsto (0)
\end{aligned}
$$

for every $a, b \in k$ and f is irreducible other than multiple of $y^2 - x$. *(check this carefully)*.

example: (inclusion of open subschemes) Let A be a ring, and $U \subseteq V \subseteq \operatorname{Spec} A$ be open sets. Then, the inclusion map of open affine subschemes corresponds to the canonical ring map between global sections.

$$
\begin{aligned}
U &\hookrightarrow V \\
\mathscr{O}(U) &\leftarrow \mathscr{O}(V)
\end{aligned}
$$

If $U = D(T)$ and $V = D(S)$, then $S \subseteq T$ and the canonical ring map is

$$
\begin{aligned}
S^{-1}A = \mathscr{O}(V) &\to T^{-1}A = \mathscr{O}(U) \\
\frac{a}{s} &\mapsto \frac{a}{s}
\end{aligned}
$$

In particular, the affine scheme map induced from $\operatorname{Spec} A \to S^{-1}A = \operatorname{Spec} \mathscr{O}(V)$ is precisely $V \hookrightarrow \operatorname{Spec} A$

Chapter 2

SCHEME

2.1 DEFINITION OF SCHEME AND MORPHISM OF SCHEMES

Definition 2.1.1 (scheme, morphism of schemes)
A scheme is a locally ringed space $(X, \mathscr{O})$ such that for every point $x \in X$, there is an open neighbourhood U_x containing x so that $(U_x, \mathscr{O}|_{U_x})$ is an affine scheme. Morphism of schemes is morphism of locally ringed spaces.

We obtain the category of schemes, denoted by Sch

Definition 2.1.2 (scheme over S)
Let S be a scheme, the category (Sch/S) of schemes over S is the category where an object is a morphism of schemes $X \to S$ and a morphism is a morphism of schemes $X \to Y$ so that $X \to S$ factors through Y by the map $X \to Y$

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

The collection of morphisms from $X \to S$ into $Y \to S$ is denoted by $\mathrm{Hom}_S(X, Y)$. If $S = \mathrm{Spec}\, A$, we also call scheme over S as scheme over A

Since $\mathrm{Spec}\, \mathbb{Z}$ is a terminal object in the category of schemes, the category of schemes is canonically equivalent the category of schemes over $\mathbb{Z}$ *(proof for $\mathrm{Spec}\, \mathbb{Z}$ is terminal in next section)*

2.2 GLUING OF MORPHISMS

Definition 2.2.1 (open subscheme)
Let $(X, \mathscr{O}_X)$ be a scheme and $U \subseteq X$ be an open subset. Then, the locally ringed space $(U, \mathscr{O}_X|_U)$ is a scheme and we call it an open subscheme of X. If the

Lemma 2.2.2

Let X be a scheme and U, V be open affine subschemes of X. Then for every $x \in U \cap V$, there exists an open subscheme $W \subseteq U \cap V$ containing x such that W is principal open in U as well as in V.

Proposition 2.2.3 (gluing of morphisms)

Let X, Y be locally ringed spaces. Then $U \mapsto \operatorname{Hom}(U, Y)$ sending open subset $U \subseteq X$ into the set of morphisms $(U, \mathscr{O}_X|_U) \to (Y, \mathscr{O}_Y)$ is a sheaf of sets on X.

In other words, if $X = \bigcup_i U_i$ is an open covering, then a family of morphisms $U_i \to Y$ glues into a morphism $X \to Y$ if and only if the morphisms coincide on intersections $U_i \cap U_j$. Moreover, in that case, the resulting morphism $X \to Y$ is uniquely determined.

TODO - proof idea - one can identify the morphism by map between stalks, it feels local hence gluable

Proposition 2.2.4

Let X be a locally ringed space and $Y = \operatorname{Spec} A$ an affine scheme. Then, the natural map

$$\begin{aligned} \operatorname{Hom}(X, Y) &\to \operatorname{Hom}(A, \Gamma(X)) \\ (f, f^\flat) &\mapsto f^\flat_Y \end{aligned}$$

is a bijection.

Proof. *(TODO - use lemma above to prove for the case X is a scheme, full proof in EGAInew 1.6.3)* □

$\operatorname{Spec} \mathbb{Z}$ being terminal object in the category of schemes follows this proposition. Moreover, when $A = \Gamma(X)$, there corresponds to $\mathrm{id}_{\Gamma(X)}$ a morphism

$$c_X : X \to \operatorname{Spec} \Gamma(X)$$

which we call canonical.

2.3 SOME BASIC MORPHISMS

2.3.1 MORPHISM FROM AFFINE POINT

Let X be a scheme. Let $x \in X$ and $U \subseteq X$ be an open affine subscheme of x, let $U = \operatorname{Spec} A$. Let $\mathfrak{p} \in \operatorname{Spec} A$ corresponds to x. Then, the natural map $A \to A_\mathfrak{p}$ induces a morphism

$$j_x : \operatorname{Spec} \mathscr{O}_{X,x} = \operatorname{Spec} A_\mathfrak{p} \to \operatorname{Spec} A = U \subseteq X$$

(*here, stalk at $x \in X$ is the same as stalk at $x \in X$ because stalk is defined as colimit, so it doesn't matter if we restrict to U or not - Görtz - Wedhorn didn't explain it clearly, so below is my attempt* [1])

proof for $j_x : \operatorname{Spec} \mathscr{O}_{X,x} \to X$ being independent of choice of U. Let U and V be two open affine subschemes containing x, let $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$, let $\mathfrak{p} \in \operatorname{Spec} A$ and $\mathfrak{q} \in \operatorname{Spec} B$ correspond to x. Then $A_\mathfrak{p} = B_\mathfrak{q}$ and we have two maps $j_x^U : \operatorname{Spec} \mathscr{O}_{X,x} \to U$ and $j_x^V : \operatorname{Spec} \mathscr{O}_{X,x} \to V$. By definition, $\mathscr{O}_{X,x}$ is the colimit of functor $\mathscr{O}_X$ from the

[1] math.SE https://math.stackexchange.com/q/5116709/700122

category of open sets containing x. Since the contravariant functor Spec is fully faithful, it preserves limits, $\operatorname{Spec} \mathscr{O}_{X,x}$ is the limit of the functor $\operatorname{Spec} \mathscr{O}_X$. By universal property of limit, let $W = U \cap V$, then j_x^U and j_x^V factor through a unique map $\operatorname{Spec} \mathscr{O}_{X,x} \to W$ into their intersection and inclusions of schemes $W \hookrightarrow U$ and $W \hookrightarrow V$

$$\begin{tikzcd}
 & \mathscr{O}_X(U) \arrow[dl] \arrow[d] \\
\mathscr{O}_{X,x} & \mathscr{O}_X(W) \arrow[l, dashed] \\
 & \mathscr{O}_X(V) \arrow[ul] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & U \\
\operatorname{Spec} \mathscr{O}_{X,x} \arrow[ur] \arrow[r, dashed] \arrow[dr] & W \arrow[u, hook] \arrow[d, hook] \\
 & V
\end{tikzcd}$$

□

Image of j_x in X is the intersection of all open subsets containing x *(consider the map $A \to A_{\mathfrak{p}}$, its kernel is $\mathfrak{a} = \{a \in A : as = 0$ for some $s \notin \mathfrak{p}\}$. Hence, the image of j_x are those primes in A containing $\mathfrak{a}$, ...)*
Let $\kappa(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$ be the residue field at x, we obtain the map of schemes

$$i_x : \operatorname{Spec} \kappa(x) \to \operatorname{Spec} \mathscr{O}_{X,x} \to X$$

which is called canonical. The image point of i_x in X is x.
Now let k be a field and $f : \operatorname{Spec} k \to X$ be a map, let x be the image point on X of the singleton $p \in \operatorname{Spec} k$. The map f induces a map on stalks $f_p^\# : \mathscr{O}_{X,x} \to \mathscr{O}_{\operatorname{Spec} k, p} = k$. Since $\mathscr{O}_{X,x}$ is local, $\mathscr{O}_{X,x} \to k$ factors through $\mathscr{O}_{X,x} \to \kappa(x)$. Hence, $\operatorname{Spec} k \to \operatorname{Spec} \mathscr{O}_{X,x}$ factors through $\operatorname{Spec} \kappa(x) \to \operatorname{Spec} \mathscr{O}_{X,x}$.

$$\begin{tikzcd}
\kappa(x) \arrow[d, dashed] & \mathscr{O}_{X,x} \arrow[l] \arrow[dl, "f_p^\#"] \\
k &
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{Spec} \kappa(x) \arrow[r] \arrow[rr, bend left, "i_x"] & \operatorname{Spec} \mathscr{O}_{X,x} \arrow[r] & X \\
\operatorname{Spec} k \arrow[u, dashed] \arrow[ur] \arrow[urr, "f"'] & &
\end{tikzcd}$$

Moreover, $f : \operatorname{Spec} k \to X$ factors through $i_x : \kappa(x) \to X$ *(this is very nice but GW didn't explain clearly why i_x has such a nice property - I guess what's outside a proof is informal)*

Proposition 2.3.1
There is a bijection of sets

$$\operatorname{Hom}(\operatorname{Spec} k, X) \to X \times \operatorname{Hom}(\kappa(x), k)$$

2.3.2 MORPHISM INTO AFFINE POINT

(from Borcherds lecture on gluing schemes)
Let k be a field and R be a ring, the map $\operatorname{Spec} R \to \operatorname{Spec} k$ that factors through $\operatorname{Spec} k[x]$ corresponds to an element of R, that is, a regular function on the global section of $\operatorname{Spec} R$

$$\begin{tikzcd} R & & k[x] \arrow[ll] \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} \operatorname{Spec} R \arrow[rr] \arrow[dr] & & \operatorname{Spec} k[x] \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr] \arrow[dr] & & \operatorname{Spec} k[x] \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}$$

Similar, any map from scheme X into $\operatorname{Spec} k$ that factors through $\operatorname{Spec} k[x]$ is a regular function on global section of X since X can be covered by open affine subschemes.

2.4 GLUING SCHEMES, DISJOINT UNION OF SCHEMES

Definition 2.4.1 (gluing of schemes)

A gluing of schemes consists of the following data: given an index set I

- for all $i \in I$, a scheme U_i
- for all $i, j \in I$, an open subscheme $U_{ij} \subseteq U_i$ such that $U_{ii} = U_i$ for all $i \in I$
- for all $i, j \in I$, an isomorphism $\phi_{ji} : U_{ij} \to U_{ji}$ of schemes such that the cocycle condition holds

$$\phi_{kj}\phi_{ji} = \phi_{ki}$$

  on $U_{ij} \cap U_{ik}$ for all $i, j, k \in I$ if $\phi_{ji}(U_{ij} \cap U_{ik}) \subseteq U_{jk}$

For $i = j = k$, the cocycle condition implies that $\phi_{ii} = \mathrm{id}_{U_i}$ and for $i = k$, $\phi_{ij}^{-1} = \phi_{ji}$ and that ϕ_{ji} is an isomorphism $U_{ij} \cap U_{ik} \to U_{ji} \cap U_{jk}$

(it looks like we can make the definition a bit stronger by imposing cocycle condition on those overlapped region where it makes sense, for example $\phi_{ji}(U_{ij} \cap U_{ik}) \cap U_{jk}$, but somehow the authors didn't do so, I don't understand now, will do later)

Proposition 2.4.2

Given a gluing of schemes, there exists a scheme X together with morphism $\psi : U_i \to X$ such that

- for all $i \in I$, the map ψ_i is an isomorphism from U_i into an open subscheme of X
- $\psi_j\phi_{ji} = \psi_i$ on U_{ij} for all $i, j \in I$
- $X = \bigcup_{i \in I} \psi_i(U_i)$
- $\psi_i(U_i) \cap \psi_j(U_j) = \psi_i(U_{ij}) = \psi_j(U_{ji})$ for all $i, j \in I$

Furthermore, (X, ψ_i) is unique up to unique isomorphism.

(there's a similar result in differential manifold using sheaf. it seems like sheaf is the categorical notion tool for gluing, just like morphism is the categorial notion of mapping, (co)limit is the categorical notion of sequential limit)

Gluing of morphisms implies the universal property of gluing schemes

> **Remark 2.4.3** (universal property of gluing schemes)
> If T is a scheme and for every $i \in I$, there is a map $\xi_i : U_i \to T$ such that $\xi_j \phi_{ji} = \xi_i$ on U_{ij} for all $i, j \in I$, then there exists a unique morphism $\xi : X \to T$ with $\xi \psi_i = \xi_i$ for every $i \in I$

(GW requires each ξ_i is an isomorphism into its image but it's clearly not required)

2.4.1 SOME EXAMPLES OF GLUING SCHEMES

example: (gluing two schemes) When the index set $I = \{1, 2\}$, any two open subsets $U_{12} \subseteq U_1$ and $U_{21} \subseteq U_2$ and an isomorphism $\phi : U_{12} \xrightarrow{\sim} U_{21}$ is a gluing of schemes. Let X be the glued scheme, for any open subset $V \subseteq X$, we have

$$\Gamma(V, \mathscr{O}_X) = \{(s1, s2) \in \Gamma(V \cap U_1, \mathscr{O}_{U_1}) \times \Gamma(V \cap U_2, \mathscr{O}_{U_2}) : \phi^\flat(s_2|_{V \cap U_{21}}) = s_1|_{V \cap U_{12}}\}$$

(this is just a pushout/colimit in schemes and pullback/limit in rings)

example: (affine line with double origin) Let k be a field, let $U_1 = U_2 = \operatorname{Spec} k[x]$, let $U_{12} = U_{21} = \operatorname{Spec} k[x] - \{(x)\}$ and the gluing morphism $U_{12} \to U_{21}$ the identity map. The global section is $\Gamma(\mathscr{O}_X) = k[x]$ (*exercise 3.26 -> showing X is not affine*)

example: (projective space) Let R be a ring. Define the projective space $\mathbb{P}^n_R$ over R by gluing $n + 1$ copies of affine space $\mathbb{A}^n_R$, that is, $U_i = \mathbb{A}^n_R$ for $i = 0, ..., n$. For convenience, let

$$A^i = R[Y_0, ..., \widehat{Y}_i, ..., Y_n]$$

($\widehat{Y}_i$ means Y_i is ommited) and we can view these rings as subring of $R[X_0, ..., X_N, X_0^{-1}, ..., X_N^{-1}]$ by the inclusion $Y_j \mapsto X_j X_i^{-1}$, then $U_i = \operatorname{Spec} A^i$. For each pair $i \neq j$, set U_{ij} to be the principal open set $D(Y_j) = A^i[Y_j^{-1}]$, let the gluing map $\phi_{ji} : U_{ij} \to U_{ji}$ be induced by the equality $A^i_{Y_j} \to A^j_{Y_i}$ in $R[X_0, ..., X_N, X_0^{-1}, ..., X_N^{-1}]$ as follows:

$$\begin{tikzcd}
A^i[Y_j^{-1}] = R[Y_0, ..., \widehat{Y}_I, ..., Y_N, Y_j^{-1}] \arrow[dr] & & A^j[Y_i^{-1}] = R[Y_0, ..., \widehat{Y}_I, ..., Y_N, Y_i^{-1}] \arrow[dl] \\
Y_j^{-1} \arrow[d, mapsto] & & Y_i^{-1} \arrow[d, mapsto] \\
X_j^{-1} X_i & R[X_0, ..., X_N, X_0^{-1}, ..., X_N^{-1}] & X_i^{-1} X_j
\end{tikzcd}$$

This scheme is called projective space of relative dimension n over R. We consider the open subscheme U_i of $\mathbb{P}^n_R$ and denoted them by $D_+(X_i)$. The canonical ring morphism $R \to \Gamma(\mathbb{P}^n_R)$ is an ismorphism. Hence, projective space is not affine.

2.5 BASIC PROPERTIES OF SCHEME AND MORPHISM OF SCHEMES

2.5.1 ON LOCAL PROPERTIES

In commutative algebra, one often observes many properties called local property in which a property on ring A propagate into its prime ideals and maximal ideals. Below is an example

Remark 2.5.1 (being zero is a local property)
Let A be a ring, let $x \in A$, then the following are equivalent

1. $x = 0$ in A
2. $x = 0$ in $A_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$
3. $x = 0$ in $A_{\mathfrak{m}}$ for every maximal $\mathfrak{m}$

There is an equivalent notion of local property. Since GW did not discuss it, Vakil only briefly mentioned it, below is my attempt to properly define local property
A property P on some collection of objects is a subset of the collection, if an object X belong to the subset, we said X has property P or X is P or $P(X)$

Definition 2.5.2 (stalk-local)
Let X be a scheme and let P be a property on schemes. P is called stalk-local if there exists a property Q on stalks of schemes such that $P(X) \iff Q(\mathscr{O}_{X,x})$ for every $x \in X$

stalk-local property on schemes are the equivalence of local property on rings.

Definition 2.5.3 (affine-local)
Let X be a scheme and let P be a property on schemes P is called affine-local if

1. (locality) if $P(X)$, then $P(U)$ for every open affine subscheme $U \subseteq X$
2. (gluability) if $\{U_i\}_{i \in I}$ is an open affine cover of X and $P(U_i)$ for every $i \in I$, then $P(X)$

Definition 2.5.4 (open-local)
Let X be a scheme and let P be a property on schemes P is called open-local if

1. (locality) if $P(X)$, then $P(U)$ for every open subscheme $U \subseteq X$
2. (gluability) if $\{U_i\}_{i \in I}$ is an open cover of X and $P(U_i)$ for every $i \in I$, then $P(X)$

Since the open affine sets generate the topology on X, it is easy to show that open-local and affine-local are equivalent. Moreover, stalk-local implies affine-local. Side note on sheaf: *(somehow I just randomly admire the beauty of sheaf)*

Proposition 2.5.5 (sections are determined by germs)
Let $\mathscr{F}$ be a sheaf on X and $U \subseteq X$ be an open set, then the canonical map from sections to stalks is injective

$$\mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}_x$$

This property has two important implications:

1. A sheaf $\mathscr{F}$ on X is completely determined by its étale space $E = \coprod_{x \in X} \mathscr{F}_x$ [2]
2. A section $s \in \mathscr{F}(U)$ is zero if and only if s is zero on all stalks on U

[2] topologized by the weakest topology generated by all compatible germs $(s_x)_{x \in U}$ for some $s \in \mathscr{F}(U)$ for some open $U \subseteq X$

The second point is actually the local property of being zero on rings. Back to local property, when $X = \operatorname{Spec} A$, then being zero is stalk-local. *(I will mumble something that I am not an expert in, so this is likely wrong)*
Another subtle issue is these local definitions use higher order logic, hence it will sometimes be more confusing. For example, below we characterize reduced scheme by two assertions

1. being a reduced scheme is an affine-local property
2. an affine scheme is reduced if and only if it is a reduced ring

The definition of reduced schemes is constructed recursively where the base case is affine scheme. This is not possible to define reduced scheme from open-local since its definition refer to itself.
Final thought, it seems like sheaf (or locality and gluability) is literally what is called "geometric" properties

2.5.2 TOPOLOGICAL PROPERTIES

Definition 2.5.6 (connected, quasi-compact, irreducible)
A scheme is called connected/quasi-compact/irreducible if the underlying topological space is connected/quasi-compact/irreducible.

All affine schemes are quasi-compact.

Definition 2.5.7 (injective, surjective, bijective)
A morphism of schemes $f : X \to Y$ is called injective/surjective/bijective if the underlying set function is injective/surjective/bijective.

Definition 2.5.8 (open, closed, homeomorphism)
A morphism of schemes $f : X \to Y$ is called open/closed/homeomorphism if the underlying continuous map is open/closed/homeomorphism.

2.5.3 NOETHERIAN SCHEMES

Definition 2.5.9 (locally Noetherian, Noetherian)
A scheme X is called locally Noetherian if every open affine subscheme $U \subseteq X$ is a Noetherian ring. A quasi-compact locally Noetherian scheme is called Noetherian.

Localization of Noetherian rings is Noetherian, hence a stalk of a locally Noetherian scheme is a Noetherian ring and every affine scheme of a Noetherian ring is Noetherian. However, being Noetherian scheme is not stalk-local, the converse statement is false.
Noetherian affine scheme is the equivalence to Noetherian ring in ring theory.

Proposition 2.5.10
An affine scheme X is Noetherian if and only if it is a Noetherian ring.

Lemma 2.5.11

Let A be a ring and $\operatorname{Spec} A = \bigcup_{i \in I} D(f_i)$ be a finite open cover of $\operatorname{Spec} A$ by principal open subsets [a]. Let M be an A-module, then M is finitely generated if and only if for all $i \in I$, the localization M_{f_i} is a finitely generated A_{f_i}-module.

[a]that is, the ideal generated by f_i is the whole ring A

Remark 2.5.12 (Noetherian topological space)

A topological space X is called Noetherian if one of the following conditions satisfies:

1. Every nonempty collection of open sets has a maximal element
2. Decending chain condition for closed sets.

A finite covering $X = \bigcup_{i \in I} X_i$ of Noetherian spaces is Notherian.

Hence, the topological space of an Noetherian scheme is Noetherian but the converse statement is false.

Remark 2.5.13 (subsets of Noetherian topological space)

Let X be a Noetherian topological space

1. Every subspace of X is Noetherian
2. Every open subset of X is quasi-compact
3. Every closed subset of X has only finitely many irreducible components.

Proposition 2.5.14

Let X be a (locally) Noetherian scheme, then every open subscheme $U \subseteq X$ is (locally) Noetherian.

2.5.4 GENERIC POINTS

Remark 2.5.15

Let X be a topological space, a subspace $Z \subseteq X$ is irreducible if and only if its closure is irreducbile

Remark 2.5.16

Let X be a topological space and $Z \subseteq X$, a point $z \in Z$ is called a generic point of Z if $\overline{\{z\}} = Z$.

As closure of an irreducible subset is irreducible, if a subset of X contains a generic point, it is irreducible. Moreover

Proposition 2.5.17

Let X be a scheme, there is a bijection map

$$\begin{aligned} X &\to \{\text{closed irreducible subsets of } X\} \\ x &\mapsto \overline{\{x\}} \end{aligned}$$

Proposition 2.5.18

Let $f : X \to Y$ be an open morphism of schemes and Y be irreducible with generic point η. Then X is irreducible if and only if the fiber $f^{-1}(\eta)$ is irreducible.

Proof. (the proof in GW was too short, skipped many point-set topology things, this is a better version)
It's clear that $\overline{f^{-1}(\eta)} \subseteq f^{-1}(\overline{\{\eta\}})$ since f is continuous. Moreover, since f is open, any open set $U \subseteq X$ is mapped into an open set $f(U) \subseteq Y$. $\{\eta\}$ is dense in Y, hence $\eta \in f(U)$, so $U \cap f^{-1}(\eta) \neq \varnothing$. $f^{-1}(\eta)$ cuts all open sets in X, so it is dense in X. □

Although schemes are never Hausdorff, they at least satisfy some weaker separation property.

Proposition 2.5.19

Let X be a scheme, then the underlying topological space X is Kolmogorov (T_0-space), that is, for any two distinct point $x, y \in X$, there exists an open subset of X containing exactly one of the points.

2.5.5 REDUCED SCHEME, INTEGRAL SCHEME

A reduced ring is a ring with no nilpotent

Definition 2.5.20 (reduced scheme)

A scheme is called reduced if every stalk is reduced.

(from here, we diverted from GW a bit so that it is easier to internalize)
By definition, being reduced is a stalk-local property, hence also an affine-local and open-local. Since being reduced ring is a local property, there for an affine scheme is reduced if and only if it is a reduced ring. One can characterized reduced scheme as follows:

Remark 2.5.21

A reduced scheme is characterized by the following two conditions:

1. being reduced scheme is an affine-local property
2. an affine scheme is reduced if and only if it is a reduced ring.

Moreover, a reduced scheme is also decided by its global sections of open subschemes

Proposition 2.5.22

A scheme is reduced if and only if the global sections of every open subscheme is a reduced ring.

Proof. The converse is easy since every open affine subscheme is an open subscheme. We will show the implication. Since sections are determined by germs, let $U \subseteq X$ be an open subscheme of a reduced scheme X, let $f \in \Gamma(U, \mathscr{O}_X)$, then $f = 0$ if and only if $f_x = 0$ in every stalk $\mathscr{O}_{X,x}$ with $x \in U$. Since every stalk is reduced, $f^n = 0$ implies $f_x^n = 0$ for every $x \in U$ implies $f_x = 0$ for every $x \in U$ implies $f = 0$. Hence, $\Gamma(U, \mathscr{O}_X)$ is reduced. □

Integral affine scheme is the equivalence to integral domain in ring theory.

Definition 2.5.23 (integral scheme)

A scheme is called integral if it is reduced and irreducible.

Proposition 2.5.24

An affine scheme is integral if and only if it is an integral domain.

Proof.

($\Longleftarrow$) When ring A is an integral domain, $\eta = (0)$ is a prime, hence $\overline{\{\eta\}} = V(\eta) = \operatorname{Spec} A$. Therefore, $\operatorname{Spec} A$ is irreducible. $\operatorname{Spec} A$ is reduced because every integral domain is reduced.

($\Longrightarrow$) When $X = \operatorname{Spec} A$ is integral, let $a, b \in A$ so that $ab = 0$, note that $D(a) \cap D(b) = D(ab) = 0$. Since X is irreducible, either $D(a)$ or $D(b)$ is empty. Suppose $D(a)$ is empty, then a belongs to every prime ideal. In other words, a belongs to the nilradical of A. On the other hand, X being reduced implies A being reduced, the nilradical of A is $\{0\}$. Hence $a = 0$. ☐

In an affine scheme, there is a one-to-one correspondence between open subschemes and localizations. Since open subscheme of an irreducible scheme is irreducible and being reduced scheme is an open-local property, every open subscheme of an integral affine scheme is also integral. We arrived at the first algebraic fact derived from geometry.

Remark 2.5.25

Every localization of an integral domain is an integral domain

Moreover, every stalk of an integral affine scheme is an integral domain. Scheme is covered by affine schemes, being reduced is open-local and open subscheme is irreducible, hence each affine scheme is integral. Hence, every stalk of an integral scheme is an integral domain. However, the converse is not true.

Proposition 2.5.26

Every stalk of an integral scheme is an integral domain.

Similar to reduced scheme, an integral scheme is also decided by its global sections of open subschemes

Proposition 2.5.27

A scheme is integral if and only if the global sections of every open subscheme is an integral domain.

Proof.

($\Longleftarrow$) It suffices to show that if $\Gamma(U, \mathscr{O}_X)$ is an integral domain for every open subscheme $U \subseteq X$, then X is irreducible. *(here, we just follow GW's book)* Suppose X is not irreducible, that is $X = U \cup V$ for some nonempty open subschemes U, V. Then

$$\Gamma(X, \mathscr{O}_X) = \Gamma(U \cup V, \mathscr{O}_X) = \Gamma(U, \mathscr{O}_X) \times \Gamma(V, \mathscr{O}_X)$$

The right hand side product contains zero divisors: $(1, 0) \cdot (0, 1) = (0, 0)$

($\Longrightarrow$) It suffices to show that $\Gamma(X, \mathscr{O}_X)$ is an integral domain since every localization of an integral domain is an integral domain.

(TODO - not sure why in GW's book, the proof is similar to my proof in affine case, they talked about $V(f) \subseteq \operatorname{Spec} A$ for some $f \in A = \Gamma(X, \mathscr{O}_X)$) then invoke the irreducibilty of X for $V(f)$, clearly X differs from $\operatorname{Spec} A$ ☐

FUNCTION FIELD OF AN INTEGRAL SCHEME

Around the generic point of an integral scheme, pick an open affine scheme. The affine scheme corresponds to an integral domain and the generic point corresponds to the zero ideal, hence its stalk is a field.

Definition 2.5.28 (function field)
Let X be an integral scheme and $\eta \in X$ be its generic point. Then the stalk $\mathscr{O}_{X,\eta}$ is a field which is called the function field of X and denoted by $K(X)$

For an integral scheme, every ring of functions is contained in the function field which is formalized as follows:

Proposition 2.5.29
Let X be an integral scheme with generic point η and let $K(X)$ be its function field

1. if $\operatorname{Spec} A \subseteq X$ is a non-empty open affine subscheme, then $\operatorname{Frac}(A) = K(X)$. if $x \in X$, then $K(X) = \operatorname{Frac}(\mathscr{O}_{X,x})$
2. let $U \subseteq V \subseteq X$ be non-empty open subschemes, the maps [a] below are injective
$$\Gamma(V, \mathscr{O}_X) \xrightarrow{\text{res}} \Gamma(U, \mathscr{O}_X) \to \mathscr{O}_{X,\eta} = K(X)$$
3. for every non-empty open subset $U \subseteq X$ and every open cover $\{U_i\}$ of U, then
$$\Gamma(U, \mathscr{O}_X) = \bigcap_i \Gamma(U_i, \mathscr{O}_X) = \bigcap_{x \in U} \mathscr{O}_{X,x}$$
where the intersections take place in $K(X)$ by the injective maps $\Gamma(U_i, \mathscr{O}_X) \to \mathscr{O}_{X,\eta} = K(X)$ and $\mathscr{O}_{X,x} \to \operatorname{Frac}(\mathscr{O}_{X,x}) = K(X)$

[a] since η is dense, η is contained in any open subset

2.6 PREVARIETIES AS SCHEMES

I skipped this part as prevariety seems like a uglier version of scheme, and this part feels like a historical note rather than something interesting

2.7 SUBMERSION AND IMMERSION

2.7.1 OPEN IMMERSION

Definition 2.7.1 (open immersion)
A morphism $j : Y \to X$ of schemes is called an open immersion if

1. the underlying continuous map is a homeomorphism from Y into an open subset $U \subseteq X$
2. the morphism of sheaves $\mathscr{O}_X \to j_*\mathscr{O}_Y$ induces an isomorphism $\mathscr{O}_X|_U \xrightarrow{\sim} j_*\mathscr{O}_Y$

2.7.2 CLOSED SUBSCHEME

Remark 2.7.2 (sheaf of ideals, quotient sheaf)
Given a ringed space $(X, \mathscr{O}_X)$, we said the subsheaf $\mathscr{J} \subseteq \mathscr{O}_X$ a sheaf of ideals if for every open subset $U \subseteq X$, the sections $\Gamma(U, \mathscr{J})$ is an ideal in $\Gamma(U, \mathscr{O}_X)$. The quotient sheaf which is a sheaf of rings, denoted by $\mathscr{O}_X/\mathscr{J}$, is defined as the sheaf associated to the presheaf

$$U \mapsto \mathscr{O}_X(U)/\mathscr{J}(U)$$

The canonical projection $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{J}$ is an epimorphism.

Definition 2.7.3
Let X be a scheme

1. A closed subscheme Z of X is a scheme such that Z is a closed subset of X, $i : Z \to X$ is the inclusion, and the sheaf $i_*\mathscr{O}_Z$ on X is isomorphic to a quotient sheaf $\mathscr{O}_X/\mathscr{J}$
2. A morphism of schemes $i : Z \to X$ is called a closed immersion if
   (a) the underlying continuous map is a hemeomorphism from Z into a closed subset of X
   (b) the morphism of sheaves $i^\flat : \mathscr{O}_X \to i_*\mathscr{O}_Z$ is an epimorphism.
3. If $Z \subseteq X$ is a closed subscheme, then there is a closed immersion $(i, i^\flat) : Z \to X$. On the other hand, if $i : Z \to X$ is a closed immersion, then Z is ismorphic to some closed subscheme of X.

(this is important) In part (1) GW specifically requires Z to be a scheme since this is not true for arbitrary $\mathscr{J}$. *(sheaf of ideals defines a closed subscheme if and only if it is a quasi-coherent sheaf)*
In an affine scheme, every closed subscheme is identified by an ideal. More formally,

Theorem 2.7.4
Let $X = \operatorname{Spec} A$ be an affine scheme, there is a bijection between the set of ideals of A and the set of closed subschemes of X given by

$$\mathfrak{a} \mapsto V(\mathfrak{a}) \cong \operatorname{Spec} A/\mathfrak{a}$$

In particular, every closed subscheme of an affine scheme is affine.

(GW didn't give any example, below is my take)
example: (affine scheme) Let A be a ring and $\mathfrak{a} \subseteq A$ be an ideal. Let $X = \operatorname{Spec} A$ and $\mathscr{J} \subseteq \mathscr{O}_X$ be the sheaf of ideals defined as follows: for every $f \in A$,

$$\mathscr{J}(D(f)) = \mathfrak{a}_f \subseteq A_f$$

where $\mathfrak{a}_f$ is the image of $\mathfrak{a}$ under the canonical map $A \to A_f$. Let $\mathscr{Q} = \mathscr{O}_X/\mathscr{J}$ be the quotient sheaf, that is

$$\mathscr{Q}(D(f)) = A_f/\mathfrak{a}_f = (A/\mathfrak{a})_f$$

This is a quasi-coherent sheaf. For every $x \in X$ with associated prime ideal $\mathfrak{p} \subseteq A$, the stalk of $\mathscr{Q}$ at x is

$$\mathscr{Q}_x = \operatorname{colim}_{f \notin \mathfrak{p}} A_f/\mathfrak{a}_f = \operatorname{colim}_{f \notin \mathfrak{p}} (A/\mathfrak{a})_f$$

If $\mathfrak{p}$ is a prime in $A/\mathfrak{a}$, in other words, $x \in V(\mathfrak{a})$, then $\mathscr{Q}_x = (A/\mathfrak{a})_\mathfrak{p}$. Otherwise, there exists $a \in \mathfrak{a}$ but $a \notin \mathfrak{p}$, then $(A/\mathfrak{a})_a = 0$ is in the colimit diagram of $\mathscr{Q}_x$, that is, there is a ring map $(A/\mathfrak{a})_a = 0 \to \mathscr{Q}_x$. Hence, $\mathscr{Q}_x = 0$.

$$\mathscr{Q}_x = \begin{cases} (A/\mathfrak{a})_{\mathfrak{p}} & \text{if } x \in V(\mathfrak{a}) \\ 0 & \text{otherwise} \end{cases}$$

The zero ring is not a local ring, hence $(X, \mathscr{Q})$ is not a scheme or any restriction to any closed set not inside $V(\mathfrak{a})$. From the theorem above, $(V(\mathfrak{a}), \mathscr{Q})$ is the canonical closed subscheme of the affine scheme X. *(talk about closed subset of $V(\mathfrak{a})$ being a scheme, not sure)*

2.7.3 SUBSCHEME AND IMMERSION

Open and closed subschemes are special cases of locally closed subscheme

Definition 2.7.5