

complex_analysis_ahlfors

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ANALYTIC FUNCTION AS MAPPING

3.1 ELEMENTARY POINT SET TOPOLOGY

3.2 CONFORMALITY

3.2.1 ARCS AND CLOSED CURVES

3.2.2 ANALYTIC FUNCTIONS IN REGIONS

Definition 1 (Analytic Function on an Open Set) A complex-valued function $f(z)$ defined on an open set Ω is said to be analytic in Ω if it has a derivative at each point of Ω . An analytic function is also called holomorphic

Definition 2 (Analytic Function on an Arbitrary Set) A function $f(z)$ is analytic on an arbitrary point set A if it is the restriction to A of a function which is analytic in some open set containing A

Theorem 1 An analytic function in a region ¹ Ω whose derivative vanishes ² identically must reduce to a constant. The same is true if either the real part, the imaginary part, the modulus, or the argument is constant.

3.2.3 CONFORMAL MAPPING

3.2.4 LENGTH AND AREA

3.3 LINEAR TRANSFORMATION

3.3.1 THE LINEAR GROUP

3.3.2 THE CROSS RATIO

Definition 3 (Cross Ratio) The cross ratio (z_1, z_2, z_3, z_4) is the image of z_1 under the linear transformation which carries z_2, z_3, z_4 into $1, 0, \infty$

Theorem 2 If z_1, z_2, z_3, z_4 are distinct points in the extended plane and T any linear transformation, then $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$

Theorem 3 The cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a circle (or a straight line)

Theorem 4 A linear transformation carries circles to circles (straight line is a special circle)

3.3.3 SYMMETRY

Definition 4 The points z, z^* are said to be symmetric with respect to the circle C through z_1, z_2, z_3 if $(z^*, z_1, z_2, z_3) = (z, z_1, z_2, z_3)$

Theorem 5 (The Symmetry Principle) If a linear transformation carries a circle C_1 into a circle C_2 , then it transforms any pair of symmetric points with respect to C_1 into a pair of symmetric points with respect to C_2

¹a connected open set does not include ∞

²at one point

3.3.4 ORIENTED CIRCLES

3.3.5 FAMILY OF CIRCLES

3.4 ELEMENTARY CONFORMAL MAPPING

3.4.1 THE USE OF LEVEL CURVES

3.4.2 A SURVEY OF ELEMENTARY MAPPINGS

3.4.3 ELEMENTARY RIEMANN SURFACES

Chapter 4

COMPLEX INTEGRATION

4.1 FUNDAMENTAL THEOREM

4.1.1 LINE INTEGRALS

4.1.2 RECTIFIABLE ARCS

4.1.3 LINE INTEGRALS AS FUNCTIONS OF ARCS

Theorem 6 (Fundamental Theorem of Line Integral) *The line integral $\int_{\gamma} (pdx + qdy)$ defined in Ω depends only on the end point of γ if and only if there exists a function $U(x, y)$ in Ω with partial derivatives $\frac{\partial U}{\partial x} = p$ and $\frac{\partial U}{\partial y} = q$*

Theorem 7 *The integral $\int_{\gamma} f(z)dz$ with continuous f depends only on the end points of γ if and only if f is the derivative of an analytic function in Ω*

4.1.4 CAUCHY THEOREM FOR A RECTANGLE

Theorem 8 (Cauchy Theorem) *Let ∂R denote the boundary curve of a rectangle R . If the function $f(z)$ is analytic on R then*

$$\int_{\partial R} f(z)dz = 0$$

Theorem 9 (Cauchy Theorem) *Let $f(z)$ be analytic on the set R' obtained from a rectangle R by omitting a finite number of interior point ζ_j . If it is true that*

$$\lim_{z \rightarrow \zeta_j} (z - \zeta_j)f(z) = 0$$

for all j , then

$$\int_{\partial R} f(z)dz = 0$$

4.1.5 CAUCHY THEOREM IN A DISK

Theorem 10 (Cauchy Theorem) *If $f(z)$ is analytic in an open disk Δ , then*

$$\int_{\gamma} f(z)dz = 0$$

for every closed curve γ in Δ

Theorem 11 (Cauchy Theorem) *Let $f(z)$ be analytic in the region Δ' obtained by omitting a finite number of point ζ_j from an open disk Δ . If $f(z)$ satisfies the condition $\lim_{z \rightarrow \zeta_j} (z - \zeta_j)f(z) = 0$ for all j , then*

$$\int_{\gamma} f(z)dz = 0$$

for every closed curve γ in Δ'

4.2 CAUCHY INTEGRAL FORMULA

4.2.1 THE INDEX OF A POINT WITH RESPECT TO A CLOSED CURVE

Lemma 1 *If the piecewise differentiable closed curve γ does not pass through the point a , then the value of the integral*

$$\int_{\gamma} \frac{dz}{z-a}$$

is a multiple of $2\pi i$

Definition 5 (Index of a Point with respect to a Curve - Winding Number) *The index of point a with respect to curve γ is defined as*

$$n(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a}$$

Lemma 2 *Let z_1, z_2 be two points on a closed curve γ which does not pass through the origin. Denote the subarc from z_1 to z_2 in the direction of the curve by γ_1 and the subarc from z_2 to z_1 by γ_2 . Suppose that z_1 lies in the lower half plane and z_2 in the upper half plane. If γ_1 does not meet the negative real axis and γ_2 does not meet the positive real axis, then $n(\gamma, 0) = 1$*

Theorem 12 (Jordan Curve Theorem) *Every Jordan curve¹ in the plane determines exactly two regions*

4.2.2 THE INTEGRAL FORMULA

Theorem 13 (Cauchy Integral Formula) *Suppose that $f(z)$ is analytic in an open disk Δ and let γ be a closed curve in Δ . For any point a not in γ*

$$n(\gamma, a)f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)dz}{z-a}$$

4.2.3 HIGHER DERIVATIVES

Lemma 3 *Suppose that $\phi(\zeta)$ is continuous on the arc γ . Then the function*

$$F_n(z) = \int_{\gamma} \frac{\phi(\zeta)d\zeta}{(\zeta-z)^n}$$

is analytic and each of the regions determined by γ and its derivatives is $F'_n(z) = nF_{n+1}(z)$

Theorem 14 (Morena Theorem) *If $f(z)$ is defined and continuous in a region Ω and if $\int_{\gamma} f(z)dz = 0$ for all closed curves γ in Ω , then $f(z)$ is analytic in Ω*

Theorem 15 (Liouville Theorem) *A function which is analytic and bounded in the whole plane must reduce to a constant*

4.3 LOCAL PROPERTIES OF ANALYTIC FUNCTION

4.3.1 REMOVABLE SINGULARITIES - TAYLOR THEOREM

Theorem 16 (Removable Singularities) *Suppose that $f(z)$ is analytic in the region Ω' obtained by omitting a point from a region Ω . A necessary and sufficient condition that there exist an analytic function in Ω which coincides with $f(z)$ in Ω' is that $\lim_{z \rightarrow a} (z-a)f(z) = 0$. The extended function is uniquely determined. a is said to be a removable singularity.*

Theorem 17 (Taylor Theorem) *If $f(z)$ is analytic in a region Ω containing a , it is possible to write*

$$f(z) = \left[f(a) + \frac{f'(z)}{1!}(z-a) + \frac{f''(z)}{2!}(z-a)^2 + \dots + \frac{f^{(n-1)}(z)}{(n-1)!}(z-a)^{n-1} \right] + f_n(z)(z-a)^n$$

¹simple curve - a curve doesn't cut itself

4.3.2 ZEROS AND POLES

Proposition 1 If $f(z)$ is analytic in Ω and if $f(z) = 0$ on a set A which has an accumulation point in Ω then $f(z)$ is identically equal to 0 on Ω

Proposition 2 (Zeros) Suppose $f(z)$ is not identically zero. Then if $f(a) = 0$, there exists a first derivative $f^{(h)}(a)$ which is different from zero. Then, a is said to be a zero of order h . Furthermore, by Taylor Theorem,

$$f(z) = (z - a)^h f_h(z)$$

where $f_h(z)$ is analytic and $f_h(a) \neq 0$

Definition 6 (Isolated Singularity) Let $f(z)$ be analytic in a neighbourhood of a except a , then a is said to be an isolated singularity

Proposition 3 (Poles) If $\lim_{z \rightarrow a} f(z) = \infty$, then a is said to be a pole of $f(z)$. If a is an isolated singularity, there exists $\delta > 0$ such that $f(z) \neq 0$ on $0 < |z - a| < \delta$. Then $g(z) = \frac{1}{f(z)}$ is defined and analytic $0 < |z - a| < \delta$ and a is a removable singularity. Let h be the order of the zero at a of $g(z)$. a is also said to be the pole of order h of $f(z)$. Similarly, by Taylor Theorem,

$$f(z) = (z - a)^{-h} f_h(z)$$

where $f_h(z)$ is analytic and different from zero in $|z - a| < \delta$

Definition 7 (Meromorphic) An analytic function $f(z)$ in a region Ω except for poles is said to be meromorphic in Ω

Proposition 4 (Algebraic Order) Consider the conditions

1. $\lim_{z \rightarrow a} |z - a|^\alpha |f(z)| = 0$
2. $\lim_{z \rightarrow a} |z - a|^\alpha |f(z)| = \infty$

If there exists an integer h such that ?? holds for all $\alpha > h$ and ?? holds for all $\alpha < h$, then h is said to be the algebraic order of $f(z)$ at a . It is positive in the case of pole, negative in the case of zero, and zero if $f(a) \neq 0$ and analytic at a

Definition 8 (Essential Isolated Singularity) An isolated singularity which is neither removable or pole

Theorem 18 (Casorati-Weierstrass Theorem) An analytic function comes arbitrary close to any complex value in every neighbourhood of an essential singularity

4.3.3 THE LOCAL MAPPING

Theorem 19 Let z_j be the zeros of an analytic function $f(z)$ on a disk Δ and does not vanish identically, each zero be counted as many times as its order indicates. For every closed curve γ in Δ which does not pass through a zero

$$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz$$

where the sum has only a finite number of nonzero terms

Theorem 20 Suppose that $f(z)$ is analytic at z_0 , $f(z_0) = w_0$, and $f(z) - w_0$ has a zero of order n at z_0 . If $\epsilon > 0$ is sufficiently small, there exists $\delta > 0$ such that for all a in $|a - w_0| < \delta$, the equation $f(z) = a$ has exactly n roots in the disk $|z - z_0| < \epsilon$

Corollary 1 A nonconstant analytic function maps an open set to an open set

Corollary 2 If $f(z)$ is analytic at z_0 with $f'(z_0) \neq 0$, it maps a neighbourhood of z_0 conformally and topologically onto a region.

4.3.4 THE MAXIMUM PRINCIPLE

Theorem 21 (The Maximum Principle) If $f(z)$ is analytic and nonconstant in a region Ω , then its absolute value $|f(z)|$ has no maximum in Ω

Theorem 22 (The Maximum Principle) If $f(z)$ is defined and continuous on a closed bounded set E and analytic on the interior of E , then the maximum of $|f(z)|$ on E is assumed on the boundary of E

Theorem 23 If $f(z)$ is analytic for $|z| < 1$ and satisfies the conditions $|f(z)| \leq 1$, $f(0) = 0$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. If $|f(z)| = |z|$ for some $z \neq 0$ or if $|f'(0)| = 1$ then $f(z) = cz$ with the constant c of absolute value 1

$${}^2\bar{A} \cap \Omega \neq \emptyset$$

4.4 THE GENERAL FORM OF CAUCHY THEOREM

4.4.1 CHAINS AND CYCLES

Definition 9 (Chains) Let $\gamma_1, \gamma_2, \dots, \gamma_n$ form a subdivision of the arc γ . Then the sum $\gamma_1 + \gamma_2 + \dots + \gamma_n$ is said to be a chain

4.4.2 SIMPLE CONNECTIVITY

Definition 10 (Simply Connected) A region is said to be simply connected if its complement with respect to the extended plane is connected.

Theorem 24 A region Ω is simply connected if and only if $n(\gamma, a) = 0$ for all cycles γ in Ω and all points a belong to the complement of Ω

4.4.3 HOMOLOGY

Definition 11 (Homologous) A cycle γ in an open set Ω is said to be homologous to zero with respect to Ω if $n(\gamma, a) = 0$ for all points a in the complement of Ω . Denoted by $\gamma \sim 0 \pmod{\Omega}$ ³

4.4.4 THE GENERAL STATEMENT OF CAUCHY THEOREM

Theorem 25 (Cauchy Theorem) If $f(z)$ is analytic in Ω , then

$$\int_{\gamma} f(z) dz = 0$$

for every cycle γ which is homologous to zero in Ω

Corollary 3 If $f(z)$ is analytic in a simply connected region Ω , then $\int_{\gamma} f(z) dz = 0$ for all cycles in γ in Ω

Corollary 4 If $f(z)$ is analytic and nonzero in a simply connected region Ω then it is possible to define a single-valued analytic branches of $\log f(z)$ and $\sqrt[n]{f(z)}$ in Ω

4.4.5 PROOF OF CAUCHY THEOREM

4.4.6 LOCALLY EXACT DIFFERENTIALS

Theorem 26 A differential $pdx + qdy$ is said to be locally exact in Ω if it is exact in some neighbourhood of each point in Ω . If $pdx + qdy$ is locally exact in Ω , then

$$\int_{\gamma} (pdx + qdy) = 0$$

for every cycle $\gamma \sim 0$ in Ω

4.4.7 MULTIPLY CONNECTION REGIONS

Definition 12 (Multiply Connected) A region which is not simply connected is said to be multiply connected. A region is said to have finite connectivity n if its complement has exactly n components.

Proposition 5 (Homology Basis - Modules of Periodicity) Let A_1, A_2, \dots, A_n be the components of the complement of Ω and $\infty \in A_n$. If γ is a cycle in Ω , then $n(\gamma, a)$ only depends on which components a belongs to, namely $n(\gamma, a) = c_i$ if $a \in A_i$, and $c_n = 0$. For each $A_i, i = 1, 2, \dots, n-1$, we can find γ_i such that $n(\gamma_i, a) = 1$ for $a \in A_i$ and $n(\gamma_i, a) = 0$ for $a \in A_j, j \neq i$. Then

$$\gamma - (c_1\gamma_1 + c_2\gamma_2 + \dots + c_{n-1}\gamma_{n-1}) \sim 0 \pmod{\Omega}$$

or

$$\gamma \sim c_1\gamma_1 + c_2\gamma_2 + \dots + c_{n-1}\gamma_{n-1} \pmod{\Omega}$$

That is, every cycle is homologous to a linear combination of $\gamma_1, \gamma_2, \dots, \gamma_n$. The set $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ is said to be the homology basis for Ω . For any analytic function $f(z)$ on Ω , then

$$\int_{\gamma} f(z) dz = \sum_{i=1}^n \left(c_i \int_{\gamma_i} f(z) dz \right)$$

the values $\int_{\gamma_i} f(z) dz, i = 1, 2, \dots, n-1$ are said to be the modules of periodicity of the differential $f(z) dz$

³if Ω is simply connected, all cycle in Ω is homologous to zero

4.5 THE CALCULUS OF RESIDUES