ma5209 - algebraic topology

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Chapter 1

CATEGORY AND HOMOLOGICAL ALGEBRA

1.1 BASIC HOMOLOGICAL ALGEBRA

Definition 1 (chain complex). A chain complex A_{\bullet} is a sequence of abelian groups and homomorphisms, namely boundary operator

$$\ldots \xleftarrow{\partial} A_{-2} \xleftarrow{\partial} A_{-1} \xleftarrow{\partial} A_0 \xleftarrow{\partial} A_1 \xleftarrow{\partial} A_2 \xleftarrow{\partial} \ldots$$

such that $\partial^2 = 0$, that is, $\operatorname{im}(\partial : A_{n+1} \to A_n) \subseteq \ker(\partial : A_n \to A_{n-1})$

Definition 2 (cycle, boundary, homology group). Given a chain complex A_{\bullet} with boundary operator ∂ .

• An n-cycle is an element of $Z_n(A)$ defined by

$$Z_n(A_{\bullet}) = \ker(\partial : A_n \to A_{n-1})$$

• An n-boundary is an element of $B_n(A)$ defined by

$$B_n(A_{\bullet}) = \operatorname{im}(\partial : A_{n+1} \to A_n)$$

• The nth homology group of A is the quotient

$$H_n(A_{\bullet}) = \frac{Z_n(A_{\bullet})}{B_n(A_{\bullet})}$$

Definition 3 (chain map). Let A_{\bullet} , B_{\bullet} be chain complexes. A chain map $g: A_{\bullet} \to B_{\bullet}$ is a sequence of homomorphisms such that $\partial g = g\partial$. The following square commutes

$$\begin{array}{ccc} A_n & \stackrel{\partial}{\longrightarrow} & A_{n-1} \\ \downarrow^g & & \downarrow^g \\ B_n & \stackrel{\partial}{\longrightarrow} & B_{n-1} \end{array}$$

Proposition 1. Let $g: A \to B$ be a chain map. Then, g determines a homomorphism $H_n(g) = g_*: H_n(A) \to H_n(B)$ for all n

$$A_{n} \xrightarrow{g} B_{n}$$

$$\downarrow \subseteq \qquad \qquad \downarrow \subseteq$$

$$Z_{n}(A) \xrightarrow{g} Z_{n}(B)$$

$$\downarrow \qquad \qquad \downarrow$$

$$H_{n}(A) \xrightarrow{g_{*}} H_{n}(B)$$

Proof.

 $(g: Z_n(A) \to Z_n(B))$ well-defined - g maps cycles to cycles) Let $\sigma \in Z_n(A)$, then $\partial \sigma = 0$ and $\partial g \sigma = g \partial \sigma = g \partial \sigma = 0$ $(g: B_n(A) \to B_n(B))$ well-defined - g maps boundaries to boundaries) Let $\sigma \in B_n(A)$, then $\sigma = \partial b$ where $b \in C_{n+1}(A)$, and $g\sigma = g \partial b = \partial g b \in B_n(B)$

 $(g_*: H_n(A) \to H_n(B))$ well-defined) Any element of $Z_n(A)$ in equivalence class $[a] \in H_n(A)$ can be written as $a + \partial c$, then $g(a + \partial c) = ga + g\partial c = ga$. That is, g maps elements from the same equivalence class to the same value.

 $(g_*: H_n(A) \to H_n(B))$ is a homomorphism) Let $a + \partial c, b + \partial d \in Z_n(A)$ be in the equivalence classes $[a], [b] \in H_n$. Then $g(a + \partial c + b + \partial d) = g(a + \partial c) + g(b + \partial d)$. Therefore, $g_*([a] + [b]) = g_*[a] + g_*[b]$

Proposition 2. Let $1: A \to A, f: B \to C, g: A \to B$ be chain maps.

1.
$$H_n(1) = 1$$

2.
$$H_n(fg) = H_n(f)H_n(g)$$

Proof.

(??) obvious

$$(??) H_n(fg)[a] = [fga] = H_n(f)[ga] = H_n(f)H_n(g)[a]$$

Definition 4 (chain homotopy). Let $g_0, g_1 : A_{\bullet} \to B_{\bullet}$ be chain maps. A chain homotopy from g_0 to g_1 is a sequence of homomorphisms $h : A_n \to B_{n+1}$ such that $\partial h + h\partial = g_0 - g_1$

$$\dots \stackrel{\partial}{\longleftarrow} A_0 \stackrel{\partial}{\longleftarrow} A_1 \stackrel{\partial}{\longleftarrow} A_2 \stackrel{\partial}{\longleftarrow} \dots$$

$$\dots \stackrel{\partial}{\longleftarrow} B_0 \stackrel{\partial}{\longleftarrow} B_1 \stackrel{\partial}{\longleftarrow} B_2 \stackrel{\partial}{\longleftarrow} \dots$$

If there is a chain homotopy from g_0 to g_1 , we said g_0 and g_1 are chain homotopic or of the same chain homotopy type.

Proposition 3. Two chain homotopic maps induce the same map in homology. That is, if g_0, g_1 are chain homotopic, then $g_{0*} = g_{1*}$

Proof. Let
$$a \in Z_n(A)$$
, then $\partial a = 0$ and $g_0(a) = g_1(a) + \partial ha + h\partial a = g_1(a) + \partial ha$. So, $[g_0(a)] = [g_1(a)]$

Definition 5 (chain homotopy equivalence). A map $f: X \to Y$ is a chain homotopy equivalence if there exists $g: Y \to X$ such that $fg: Y \to Y$ is chain homotopic to 1_Y and $gf: X \to X$ is chain homotopic to 1_X

Definition 6 (short exact sequence). A sequence of groups and homomorphisms

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

is a short exact sequence if one of the following equivalent conditions satisfies

- (change needed) p is surjective and i = inclusion of ker p
- (change needed) i is injective and p = projection onto coker i
- i is injective, p is surjective, and im $i = \ker p$

Definition 7 (exact). Given sequences of groups

- $A \xrightarrow{i} B \xrightarrow{p} C$ is exact at B if im $i = \ker p$
- ullet ... $\longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow \dots$ is exact if it is exact at every group

Theorem 1. Short exact sequence of chain complexes induces long exact sequence in homology. That is, Let $0 \longrightarrow A \stackrel{i}{\longrightarrow} B \stackrel{p}{\longrightarrow} C \longrightarrow 0$ be a short exact sequence of chain complexes and i, p be chain maps.

$$0 \longrightarrow A_{n+1} \xrightarrow{i} B_{n+1} \xrightarrow{p} C_{n+1} \longrightarrow 0$$

$$\downarrow \partial \qquad \qquad \downarrow \partial \qquad \qquad \downarrow \partial \qquad \qquad \downarrow \partial$$

$$0 \longrightarrow A_{n} \xrightarrow{i} B_{n} \xrightarrow{p} C_{n} \longrightarrow 0$$

$$\downarrow \partial \qquad \qquad \downarrow \partial \qquad \qquad \downarrow \partial$$

$$0 \longrightarrow A_{n-1} \xrightarrow{i} B_{n-1} \xrightarrow{p} C_{n-1} \longrightarrow 0$$

$$\downarrow \partial \qquad \qquad \downarrow \partial \qquad \qquad \downarrow \partial$$

$$\vdots \qquad \qquad \downarrow \partial \qquad \qquad \downarrow \partial$$

$$\vdots \qquad \qquad \downarrow \partial \qquad \qquad \downarrow \partial$$

$$\vdots \qquad \qquad \downarrow \partial \qquad \qquad \downarrow \partial$$

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$$\vdots \qquad \qquad \vdots \qquad \qquad \vdots \qquad \qquad \vdots \qquad \qquad \vdots$$

Then there exists connecting homomorphisms ∂ such that the sequence

$$H_{n+1}(A) \xrightarrow{i_*} H_{n+1}(B) \xrightarrow{p_*} H_{n+1}(C)$$

$$H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{p_*} H_n(C)$$

$$H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{p_*} H_{n-1}(C)$$

is exact.

Proof.

(1 defining of $\partial: H_{n+1}(C) \to H_n(A)$)

Given $[c] \in H_{n+1}(C)$, (1) take any representative $c \in Z_{n+1}(C)$. As $p: B_{n+1} \to C_{n+1}$ is surjective, (2) take any $b \in B_{n+1}$ such that pb = c. As $p\partial b = \partial pb = \partial c = 0$ and $\ker(p: B_n \to C_n) = \operatorname{im}(i: A_n \to B_n)$, take $a \in A_n$ such that $ia = \partial b$, this choice is unique as i is injective. $i\partial a = \partial ia = \partial^2 b = 0$, as i is an injective homomorphism, $\partial a = 0$, then $a \in Z_n(A)$. The construction is done by $[c] \mapsto [a]$

We will verify that the homology class of a does not depend on (1) the representative c and (2) the value of b.

(1.1 [a]does not depend on the choice of b)

$$n+1: \qquad \qquad d \stackrel{i}{\longmapsto} b-b_1 \stackrel{p}{\longmapsto} 0$$

$$\downarrow \partial \qquad \qquad \downarrow \partial \qquad \qquad \downarrow \partial$$

$$n: \qquad \partial d=a-a_1 \stackrel{i}{\longrightarrow} \partial b-\partial b_1$$

Let $b_1 \in B_{n+1}$ such that $pb_1 = c$ and $a_1 \in A_n$ such that $ia_1 = \partial b_1$. We want to prove that a and a_1 belong to the same homology class, that is $a - a_1$ is a boundary. As $p(b - b_1) = c - c = 0$, there exists a unique $d \in A_{n+1}$ such that $id = b - b_1$. Then, $i\partial d = \partial id = \partial (b - b_1) = \partial b - \partial b_1 = ia - ia_1 = i(a - a_1)$. As i is injective, $a - a_1 = \partial d$, that is, $a - a_1$ is a boundary. (1.1 [a] does not depend on the choice of c)

$$n+2: \qquad \qquad f \xrightarrow{p} e \\ \downarrow \partial \qquad \downarrow \partial \\ n+1: \qquad d \xrightarrow{i} b-b_1-\partial f \xrightarrow{p} 0 \qquad \partial f \xrightarrow{p} c-c_1 = \partial e \\ \downarrow \partial \qquad \qquad a-a_1 = \partial d$$

Let $c_1 = c - \partial e$ for $e \in C_{n+2}$ be in the same homology class with $c, b_1 \in B_{n+1}$ such that $pb_1 = c_1$, and $a_1 \in A_n$ such that $ia_1 = \partial b_1$. We want to prove that a and a_1 belong to the same homology class, that is $a - a_1$ is a boundary. Let $f \in B_{n+2}$ such that pf = e. Then, $p\partial f = \partial pf = \partial e$. Hence, $p(b - b_1 - \partial f) = c - c_1 - \partial e = 0$. There exists unique $d \in A_{n+1}$ such that $id = b - b_1 - \partial f$, then $i\partial d = \partial id = \partial (b - b_1 - \partial f) = \partial b - \partial b_1 - \partial^2 f = ia - ia_1 = i(a - a_1)$. As i is injective, $a - a_1 = \partial d$, that is, $a - a_1$ is a boundary.

 $(1.3 \partial \text{ is a homomorphism})$

Let $c_1 \in Z_{n+1}(C)$, $b_1 \in B_{n+1}$ such that $pb_1 = c_1$, and $a_1 \in A_n$ such that $ia_1 = \partial b_1$. We want to prove that $\partial(c + c_1) = a + a_1$. Indeed, this is done verifying $p(b + b_1) = c + c_1$ and $i(a + a_1) = \partial(b + b_1)$.

(2 exactness at $H_n(A), H_n(B), H_n(C)$) It is equivalent to $\operatorname{im} \partial = \ker i_*, \operatorname{im} i_* = \ker p_*, \operatorname{im} p_* = \ker \partial$

 $(2.1 \text{ im } \partial \subseteq \ker i_* \iff \text{ for all } [c] \in H_{n+1}(C), \ i_* \partial [c] = 0) \ i_* \partial [c] = i_* [a] = [ia] = [\partial b] = 0$

 $(2.2 \text{ im } i_* \subseteq \ker p_* \iff \text{ for all } [a] \in H_n(A), \ p_*i_*[a] = 0) \ p_*i_*[a] = p_*[ia] = [pia] = 0$

(2.3 im $p_* \subseteq \ker \partial \iff$ for all $[b] \in H_n(B)$, $\partial p_*[b] = 0$) $\partial p_*[b] = \partial [pb] = i^{-1}(\partial^*b)$ where $\partial^* : B_n \to B_{n-1}$ denotes the boundary operator. As $b \in Z_n(B)$, $\partial^*b = 0$ and i is an injective homomorphism, then $\partial p_*[b] = i^{-1}(\partial^*b) = i^{-1}(0) = 0$

 $(2.4 \text{ im } \partial \supseteq \ker i_* \iff \text{for all } [\overline{a}] \in H_n(A), \text{ if } i_*[\overline{a}] = 0, \text{ then there exists } [c] \in H_{n+1}(C) \text{ such that } \partial[c] = [\overline{a}])$ $i_*[\overline{a}] = 0 \implies [i\overline{a}] = 0, \text{ that is } i\overline{a} \text{ is a boundary, then there exists } b \in B_n \text{ such that } \partial^*b = i\overline{a}. \text{ Take } c = pb.$ $(2.5 \text{ im } i_* \supseteq \ker p_*) \text{ TODO} - \text{look at the notes from prof}$ $(2.6 \text{ im } p_* \supseteq \ker \partial \iff \text{for all } [c] \in H_n(C), \text{ if } \partial[c] = 0, \text{ then there exists } [\overline{b}] \in H_n(B) \text{ such that } p_*[\overline{b}] = [c])$

As p is surjective, let $b \in B_n$ such that pb = c, $a \in A_{n-1}$ such that $ia = \partial^* b$, then $[a] = \partial[c] = 0$, that is a is a boundary. Then, there exists $d \in A_n$ such that $a = \partial^* d$. Take $\bar{b} = b - id \in B_n$, then $p_*[\bar{b}] = [p(b - id)] = [pb - pid] = [pb] = [c]$, and $\bar{b} \in Z_n(B)$ because $\partial^* \bar{b} = \partial^* (b - id) = \partial^* b - \partial^* id = \partial^* b - i\partial^* d = \partial^* b - ia = 0$

Definition 8 (short exact sequence splits). A short exact sequence splits if TODO

1.2 BASIC CATEGORY THEORY

1.2.1 CATEGORY, FUNCTOR, NATURAL TRANSFORMATION

Definition 9 (category). A category C is a class ob C of objects such that

- for all $X, Y \in \text{ob } C$, a class C(X, Y) (or $\text{hom}_C(X, Y)$, hom(X, Y)) of morphisms
- for all $X \in ob C$, $1_X \in C(X,X)$: the identity morphism
- for all $X, Y, Z \in ob C$, there exists a function

$$\circ: C(Y,Z) \times C(X,Y) \to C(X,Z)$$
$$(q,f) \mapsto qf$$

 $f \in C(X,Y)$ is also written as $f: X \to Y$

and

1. Let
$$f: X \to Y$$
, $1_Y f = f$, $f1_X = f$

$$X \xrightarrow{f} Y \qquad X$$

$$\downarrow_{1_Y f = f} \downarrow_{1_Y} \qquad \downarrow_{1_X} \downarrow_{f \downarrow_{X} = f}$$

$$Y \qquad X \xrightarrow{f} Y$$

2. Let
$$f: X \to Y, g: Y \to Z, h: Z \to W, h(gf) = (hg)f$$

Remark 1. :

Some remarks on categories

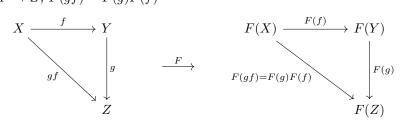
- Set: the category of sets and functions
- Top: the category of topological spaces and continuous maps
- Grp: the category of groups and homomorphisms
- Ab: the category of abelian groups and homomorphisms

Some remarks on morphisms: given a morphism $f: A \to B$

- f is said to be a monomorphism if for all $g_1, g_2: X \to A$, $fg_1 = fg_2 \implies g_1 = g_2$
- f is said to be an epimorphism if for all $g_1, g_2 : B \to X$, $g_1 f = g_2 f \implies g_1 = g_2$
- f is said to be an isomorphism if there exists $g: B \to A$ such that $fg = 1_B$ and $gf = 1_A$

Definition 10 (functor). Let C, D be categories. A functor from C to $D, F: C \to D$ is

- $F : \operatorname{ob} C \to \operatorname{ob} D$
- for all $X, Y \in \text{ob } C$, there is a function $F: C(X,Y) \to C(F(X),F(Y))$ such that
 - for all $X \in ob C$, $F(1_X) = 1_{F(X)}$
 - for all $f: X \to Y, g: Y \to Z, F(gf) = F(g)F(f)$



Remark 2. Some remarks on functors

- identity functor: $1: C \to C$
- forgetful functors: $U: \text{Grp} \to \text{Set}, \ U: \text{Top} \to Set$
- $singular\ homology:\ H_n: Top \to Ab$
- free abelian group functor: $\mathbb{Z} : \operatorname{Set} \to \operatorname{Ab}$

Definition 11 (natural transformation). Let C, D be categories and $F, G: C \to D$ be functors. A natural transformation from F to G is an assignment of each $X \in \text{ob } C$ to a morphism $\theta_X: F(X) \to G(X)$ in D such that for all $f: X \to Y$ in C. The diagram below commutes

$$\begin{array}{ccc}
X & F(X) & \xrightarrow{\theta_X} G(X) \\
\downarrow^f & F(f) \downarrow & \downarrow^{G(f)} \\
Y & F(Y) & \xrightarrow{\theta_Y} G(Y)
\end{array}$$

Notation:

- ob Fun(C, D): functors $C \to D$
- Fun(C, D)(F, G): natural transformations $F \to G$

Remark 3. Some remarks on natural transformations

• The boundary operator ∂ is a natural transformation from C_n to C_{n-1} where C_n is the functor from the Top to the category of free abelian group of n-chains

$$\begin{array}{ccc}
X & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) \\
\downarrow^f & C_n(f) = f_{\#} & & \downarrow^{C_{n-1}(f) = f_{\#}} \\
Y & C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y)
\end{array}$$

- The connecting homomorphism ∂ is a natural transformation from $H_n(C)$ to $H_{n-1}(A)$ where $H_n(C)$ and $H_{n-1}(A)$ are two functors from the category of short exact sequence of chain complexes to Ab and defined by
 - $-H_n(C)(upper\ chain) = H_n(C_{\bullet})$
 - $-H_n(C)(lower\ chain) = H_n(C'_{\bullet})$
 - $H_n(C)(f) = H_n(f) = f_*$
 - $-H_{n-1}(A)(upper\ chain) = H_{n-1}(A_{\bullet})$
 - $-H_{n-1}(A)(lower\ chain) = H_{n-1}(A'_{\bullet})$
 - $H_{n-1}(A)(f) = H_n(f) = f_*$

$$0 \longrightarrow A_{\bullet} \longrightarrow B_{\bullet} \longrightarrow C_{\bullet} \longrightarrow 0 \qquad H_{n}(C_{\bullet}) \stackrel{\partial}{\longrightarrow} H_{n-1}(A_{\bullet})$$

$$\downarrow f \qquad \downarrow f \qquad \downarrow f \qquad \downarrow H_{n}(C)(f) = f_{*} \downarrow \qquad \downarrow H_{n-1}(A)(f) = f_{*}$$

$$0 \longrightarrow A'_{\bullet} \longrightarrow B'_{\bullet} \longrightarrow C'_{\bullet} \longrightarrow 0 \qquad H_{n}(C'_{\bullet}) \stackrel{\partial}{\longrightarrow} H_{n-1}(A'_{\bullet})$$

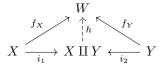
1.2.2 UNIVERSAL MAPPING PROPERTY

Definition 12 (product). Let $X, Y \in \text{ob } C$, the product an object Z together with $p_1: Z \to X$, $p_2: Z \to Y$. Moreover, the triplet (Z, p_1, p_2) is universal, that is, for each diagram $X \xleftarrow{f_X} W \xrightarrow{f_Y} Y$, there exists a unique $h: W \to Z$ such that the diagram commutes

$$X \stackrel{f_X}{\longleftarrow} X \times Y \stackrel{f_Y}{\longrightarrow} Y$$

The coproduct is written by $Z = X \times Y$

Definition 13 (coproduct). Let $X,Y \in \text{ob } C$, the coproduct an object Z together with $i_1: X \to Z$, $i_2: Y \to Z$. Moreover, the triplet (Z,i_1,i_2) is universal, that is, for each diagram $X \xrightarrow{f_X} W \xleftarrow{f_Y} Y$, there exists a unique $h: Z \to W$ such that the diagram commutes



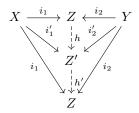
The coproduct is written by $Z = X \coprod Y$

Remark 4. Some remarks on coproduct

- direct sum on abelian groups is a coproduct
- disjoint union on sets is a coproduct
- free product on groups is a coproduct
- coproduct is unique upto isomorphism

Proof.

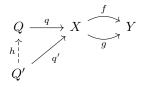
Let $(Z, i_1, i_2), (Z', i'_1, i'_2)$ be both coproduct of X, Y. Using universal property, we can construct unique $h: Z \to Z'$ and $h': Z' \to Z$ such that the diagram below commutes



By uniqueness of h, h', we must have $h'h = 1_Z$. Similarly, $hh' = 1_{Z'}$

• Construction of coproduct from pushout: coproduct of X, Y is the pushout of $X \longleftarrow 0 \longrightarrow Y$ where 0 is the initial object.

Definition 14 (equalizer). Let $f, g: X \to Y$, the equalizer is an object Q and a map $Q \to X$ such that fq = gq. Moreover, the pair (Q, q) is universal, that is, if (Q', q') satisfies the same condition, there exists a unique map $h: Q' \to Q$ such that the diagram below commutes



Definition 15 (coequalizer). Let $f, g: X \to Y$, the coequalizer is an object Q and a map $q: Y \to Q$ such that qf = qg. Moreover, the pair (Q, q) is universal, that is, if (Q', q') satisfies the same conditions, there exists a unique map $h: Q \to Q'$ such that the diagram below commutes

$$X \xrightarrow{g} Y \xrightarrow{q} Q$$

$$\downarrow h$$

$$Q'$$

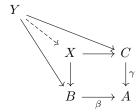
Definition 16 (pullback). The pullback of $C \xrightarrow{\gamma} A \xleftarrow{\beta} B$ is X and $X \to B, X \to C$ such that the diagram below commutes

$$\begin{array}{ccc} X & \longrightarrow & C \\ \downarrow & & & \downarrow^{\gamma} \\ B & \stackrel{\beta}{\longrightarrow} & A \end{array}$$

Moreover, X is universal, that is, if there exists Y and the commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & C \\ \downarrow & & \downarrow^{\gamma} \\ B & \stackrel{\beta}{\longrightarrow} & A \end{array}$$

then there exists a unique map $Y \to X$ such that the diagram below commutes



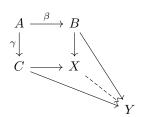
Definition 17 (pushout). The pushout of $C \xleftarrow{\gamma} A \xrightarrow{\beta} B$ is X and $A \to X$, $B \to X$ such that the diagram below commutes

$$\begin{array}{ccc}
A & \xrightarrow{\beta} & B \\
\uparrow \downarrow & & \downarrow \\
C & \longrightarrow & X
\end{array}$$

Moreover, X is universal, that is, if there exists Y and the commutative diagram

$$\begin{array}{ccc} A & \stackrel{\beta}{\longrightarrow} & B \\ \uparrow \downarrow & & \downarrow \\ C & \longrightarrow & Y \end{array}$$

then there exists a unique map $X \to Y$ such that the diagram below commutes



Remark 5. Some remarks on pushout

ullet Construction of pushout from coproduct and coequalizer: the pushout of $C \xleftarrow{\gamma} A \xrightarrow{\beta} B$, is the coequalizer of

$$A \underbrace{\bigcap_{i_2 \gamma}^{i_1 \beta}}_{i_2 \gamma} B \coprod C$$

- Consider the category of sets, then the pushout $X = B \coprod C / \sim$ where the equivalence relation is defined by $\beta(a) \sim \gamma(a)$ for all $a \in A$
- $\mathbb{R}P^{n-1}$: real projective space space of lines through the origin in \mathbb{R}^n , $\mathbb{R}P^n$ is the pushout of $\mathbb{R}P^{n-1} \longleftarrow S^{n-1} \longleftarrow e^n$, the diagram below commutes

$$S^{n-1} \longleftrightarrow D^n$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathbb{R}P^{n-1} \longrightarrow \mathbb{R}P^n$$

where S^{n-1} denotes the equator, D^n denotes the closed upper hemisphere.

• $\mathbb{C}P^{n-1}$: complex projective space - space of lines through the origin in \mathbb{C}^n , similarly, $\mathbb{C}P^n$ is the pushout of the diagram below

$$S^{2n-1} \longleftrightarrow D^{2n}$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathbb{C}P^{n-1} \longleftrightarrow \mathbb{C}P^n$$

Definition 18 (base change, cobase change). :

- 1. if X is the pullback of $C \xrightarrow{\gamma} A \xleftarrow{\beta} B$ then the transform from $B \to A$ to $X \to C$ is called base change.
- 2. if X is the pushout of $C \xleftarrow{\gamma} A \xrightarrow{\beta} B$, then the transform from $A \to B$ to $C \to X$ is called cobase change.

Definition 19 (direct limit). Given a category C with

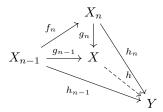
$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \dots$$

the direct limit of this sequence is an object X with maps $g_n: X_n \to X$ for all n such that the diagram below commutes for all n

$$X_n \xrightarrow{f_n} X_n$$

$$X_{n-1} \xrightarrow{g_{n-1}} X$$

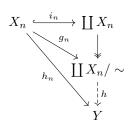
Moreover, (X, g_{\bullet}) is universal, that is, if (Y, h_{\bullet}) is also a direct limit, then there exists a unique map $h: X \to Y$ such that the diagram below commutes for all n



The direct limit is written by $X = \lim_{n \to \infty} X_n$

Remark 6. Some remarks on direct limit

- Construction of direct limit from pushout: the direct limit of $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \dots$ is X such that X is the pushout of $X_{n-1} \xleftarrow{1} X_{n-1} \xrightarrow{f_n} X_n$ for all n
- Consider the category of sets, then the direct limit of (X_n) is $\coprod X_n / \sim$ where the equivalence relation is defined by $(x \in X_{n-1}) \sim (f_n(x) \in X_n)$



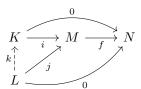
- if each f_n is mono, $\lim X_n = \bigcup X_n$
- if each f_n is inclusion of a subspace, then $\varinjlim X_n$ describe a topology on $\bigcup X_n$, called weak topology, that is, $U \subseteq \bigcup X_n$ is said to be open if $U \cap X_n$ open for all n

1.2.3 PREADDITIVE CATEGORY, ADDITIVE FUNCTOR, PROJECTIVE CLASS, DERIVE FUNCTOR

Definition 20 (initial object, terminal object, zero, pointed category, zero map). Let C be a category. 0 is initial if #C(0,X) = 1 for all X, * is terminal if #C(X,*) = 1 for all X. C is called pointed if it has initial and terminal objects and the unique map $0 \to *$ is an isomorphism. We write 0 for both initial and terminal objects. Consequently, given any two objects M, N there exists a zero map

$$M \longrightarrow 0 \longrightarrow N$$

Definition 21 (kernel). Given a pointed category C, let $f: M \to N$, a kernel of f is a map $i: K \to M$ such that fi = 0. Moreover, i is universal, that is, if $j: L \to M$ such that fj = 0, then there exists a unique $k: L \to K$ such that the diagram below commutes

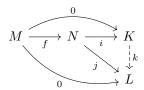


The category C has kernels if every morphism has a kernel.

Remark 7. Some remarks on kernels

• Construction of kernel from equalizer: the kernel of $f: X \to Y$ is the equalizer of $f: X \to Y$ and $0: X \to Y$

Definition 22 (cokernel). Given a pointed category C, let $f: M \to N$, a cokernel of f is a map $i: N \to K$ such that if = 0. Moreover, i is universal, that is, if $j: N \to L$ such that jf = 0, then there exists a unique $k: K \to L$ such that the diagram below commutes



Definition 23 (preadditive category, Ab-enriched category). A preadditive category is a category C together with an abelian group structure on Hom(M, N) for all $M, N \in \text{ob } C$ such that composition map are bilinear. That is,

$$fh + gh = (f + g)h$$
 and $hf + hg = h(f + g)$

Preadditive category is also called Ab-enriched category, or enriched over a monoidal category.

Definition 24 (additive functor). A functor F between preadditive categories is additive if

$$\operatorname{Hom}(M,N) \to \operatorname{Hom}(F(M),F(N))$$

is a homomorphism between two abelian groups.

Definition 25 (P-element, P-exact, P-epic - https://arxiv.org/pdf/math/9912157.pdf). For any two objects P, A, a map $P \to A$ is called P-element of A. If P is a collection of objects, then P-element is a P-element for some $P \in P$. Let C be a (locally small) pointed category. Let $P \in ob C$, a sequence

$$A \longrightarrow B \longrightarrow C$$

is said to be P-exact if the composite $A \to B \to C$ is the zero map and

$$C(P,A) \longrightarrow C(P,B) \longrightarrow C(P,C)$$

is an exact sequence of pointed sets (based point is the zero map). If \mathcal{P} is a class of objects in C, a \mathcal{P} -exact sequence in one which is P-exact for all $P \in \mathcal{P}$. A map $A \to B$ is called P-epic if it induces a surjection of P-elements.

Definition 26 (projective class - https://arxiv.org/pdf/math/9912157.pdf). A projective class on a pointed category C is $(\mathcal{P}, \mathcal{E})$ where \mathcal{P} is a class of objects (projectives) and \mathcal{E} is a class of morphisms (epimorphisms) such that

- 1. \mathcal{E} is precisely the collection of all \mathcal{P} -exact sequences
- 2. P is precisely the collection of all object P such that each sequence in \mathcal{E} is P-exact.
- 3. any map $A \to B$ can be extended to a sequence $P \to A \to A$ in \mathcal{E} with $P \in \mathcal{P}$

Definition 27 (projective class). Let C be a pointed category with kernels. A projective class in C is $(\mathcal{P}, \mathcal{E})$ where \mathcal{P} is a class of objects (projectives) and \mathcal{E} is a class of morphisms (epimorphisms) such that

1. $P \in \text{ob } C \text{ lies in } \mathcal{P} \text{ if and only if for all } f: M \to N \text{ in } \mathcal{E}, \text{ the map } C(P,M) \to C(P,N) \text{ defined by } g \mapsto fg \text{ is onto.}$

$$P \xrightarrow{g} M$$

$$fg \xrightarrow{f} N$$

2. A morphism $f: M \to N$ lies in \mathcal{E} if and only if for all $P \in \mathcal{P}$, the map $C(P, M) \to C(P, N)$ defined by $g \mapsto fg$ is onto.

$$P \xrightarrow{g} M$$

$$\downarrow f$$

$$\downarrow f$$

$$N$$

3. For any object $M \in \text{ob } C$, there exists $P \to M$ in \mathcal{E} such that $P \in P$. This condition is also called C has enough projectives

Definition 28 (left derived functor). Given a pointed preadditive category C with kernels and projective class (P, E) in C. Let $F: C \to Ab$ be an additive functor (it maps chain complexes to chain complexes). The left derived functors of F with respect to P are $L_nF: C \to Ab$ for $n \ge 0$ defined by

$$L_n F(X) = H_n(F(P_{\bullet}))$$

where $X \leftarrow P_{\bullet}$ is a \mathcal{P} -projective resolution. (analogous to Tor functor)

Remark 8. Some remarks on left derived functor

• Let C = R-Mod, \mathcal{P} be the categorical projectives (= direct summands of free R-modules), $F : R\text{-Mod} \to Ab$ defined by $A \mapsto A \oplus_R B$ where $B \in R\text{-Mod}$, then

$$L_n F(A) = \operatorname{Tor}_n^R(A, B)$$

• $F(X) \leftarrow L_0F(X)$ might not be an isomorphism. It is an isomorphism if and only if F is right exact

Definition 29 (retract). P is a retract of F is there exists a commutative diagram



The map $P \to F$ is called section and the map $F \to P$

Proposition 4. Class of projectives \mathcal{P} is closed under coproduct and retract

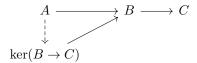
1.2.4 AXIOMATISE FUNDAMENTAL THEOREM OF HOMOLOGICAL ALGEBRA

Theorem 2 (fundamental theorem of homological algebra). Let C be a pointed category with kernels. Let (P, \mathcal{E}) be a projective class in C.

1. For any $M \in \text{ob } \mathcal{C}$, there is an exact sequence w.r.t $(\mathcal{P}, \mathcal{E})$ (\mathcal{P} -projective resolution)

$$0 \longleftarrow M \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \dots$$

where $P_n \in \mathcal{P}$ and exactness $A \to B \to C$ is defined by the composition being the zero map and $A \to B$ is factored through $\ker(B \to C)$ by an epimorphism, that is, there exists an epimorphism $A \to \ker(B \to C)$ such that the diagram below commutes



2. Any map $f: M \to M'$ lifts to maps of projectives

$$0 \longleftarrow M \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \dots$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longleftarrow M' \longleftarrow P'_0 \longleftarrow P'_1 \longleftarrow \dots$$

Furthermore, if C is preadditive, the lifts are unique upto chain homotopy.

Proof. TODO

1.2.5 YONEDA LEMMA

Definition 30 (functor category, evaluation functor). Let A, B be categories, the functor category $\operatorname{Fun}(A, B)$ is defined by functors $A \to B$ as objects and natural transformations as morphisms. Let $X \in \operatorname{ob} A$, define the evaluation functor $\operatorname{Ev}_X : \operatorname{Fun}(A, B) \to B$ by

$$\operatorname{Ev}_X(F) = F(X)$$

where $F \in \text{ob Fun}(A, B)$. define application of morphism

Definition 31 (hom functor). Let C be a locally small category, for any $A \in \text{ob } C$, the hom functor $h_A : C \to \text{Set is defined } by$

$$X \xrightarrow{h_A} C(A, X)$$

$$f \downarrow \qquad \qquad \downarrow h_A(f)$$

$$Y \xrightarrow{h_A} C(A, Y)$$

where $h_A(f)(g) = fg$ for any $g \in C(A, X)$. h_A is also written as C(A, -)

Remark 9. Some remarks on hom functor

• The singular chain complex functor $C_n : \text{Top} \to \text{Ab}$ can be defined by

$$C_n = \mathbb{Z}\operatorname{Top}(\Delta^n, -)$$

Theorem 3 (Yoneda lemma). Given a locally small category C and $A \in \text{ob } C$. Let $F : C \to \text{Set } be$ a functor, then both h_A and F are objects in Fun(C, Set) and the set of morphisms $h_A \to F$ in Fun(C, Set) is isomorphic to F(A),

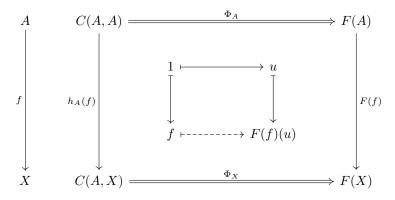
$$\operatorname{Nat}(h_A, F) \cong F(A)$$

where $Nat(h_A, F)$ denotes the set of natural transformation from h_A to F. Since, $F(A) = Ev_A(F)$, then

$$\Phi: \operatorname{Hom}(A,-) \to \operatorname{Ev}_A$$

is a natural isomorphism. We say that Ev_A is corepresentable. (Φ, A) is called corepresentation of Ev_A Proof.

Let $\Phi: h_A \to F$ be a natural transformation



By commutativity, $\Phi_X: C(A,X) \to F(X)$ is defined by $f \mapsto F(f)(u)$. That is,

1. for each $u \in F(A)$, there is a unique natural transformation Φ defined by

$$\Phi_X(f) = F(f)(u)$$

2. for each natural transformation Φ , there is unique $u \in F(A)$ defined by

$$u = \Phi_A(1)$$

Remark 10. Some remarks on Yoneda lemma

- Take C be a category such objects are sets, take F to be the forgetful functor and $A \in ob C$. Then, the set of natural transformations $h_A \to F$ is exactly A
- in C = Set, each functor $F : C \to \text{Set}$ corresponds to a natural transformation $1 \to F$. From Yoneda lemma, for each natural transformation $1 \to F$, the map $X \to F(X)$ is factored through $X \to C(X,X)$ defined by $x \mapsto 1$

Lemma 1. Let C be any category, then Fun(C, Ab) is preadditive. That is, given any two functors $G, F : C \to Ab$, the class of natural transformations of $G \to F$ is an abelian group.

Proposition 5 (Yoneda lemma for Ab-enriched functor). Given C is preadditive, for any $A \in \text{ob } C$ and functor $F: C \to \text{Ab}$, then we have an isomorphism between abelian groups

$$Nat(h_A, F) \cong F(A)$$

Corollary 1. Given $X, Y \in \text{ob } C$, $X \cong Y$ if and only if $C(X, -) \cong C(Y, -)$

1.2.6 ACYCLIC MODEL

Remark 11 (kernel of natural transformation between Ab-enriched functors). Given Fun(C, Ab) is pointed, has kernels, and preadditive. Let $F, G \in \text{Fun}(C, \text{Ab})$, and $v : G \to F$ be a natural transformation, then

$$\ker(v:G\to F)(X) = \ker(v_X:G(X)\to F(X))$$

where $X \in \text{ob } C$

Definition 32 (acyclic models). Let \mathcal{M} be any set of objects in C ("models"). It defines a projective class in $\operatorname{Fun}(C,\operatorname{Ab})$ as follows: an object is projective (relative to \mathcal{M}) if it is a retract of a coproduct of $\mathbb{Z}C(M,-)s$. A morphism $Y:G\to F$ is an epimorphism (relative to \mathcal{M}) if $G(M) \twoheadrightarrow F(M)$ is surjective for all $M\in \mathcal{M}$

Remark 12. Some remarks on acyclic model

- $\mathbb{Z}C(M,-)$ is analogous to free \mathbb{Z} -module (free abelian group), then the coproduct is direct product and retract is direct summand (A is a summand of B if $B = A \oplus C$)
- C = Top, $\mathcal{M} = \{\Delta^n : n \geq 0\}$, this models \mathcal{M} recovers projectives being singular n-chains and left derive functor of evaluation map at X of H_0 is H_n (lecture 16)

Proof. We will show the that the construction is indeed a projective class in Fun(C, Ab) by the following

1. If H is a projective and $G \to F$ epimorphism, then there is a lift from $H \to F$ to $H \to G$ (forward direction of (1) and (2))

It suffices to show for the case if H = C(M, -).

$$C(M,-) \longrightarrow F \qquad F(M)$$

By Yoneda lemma, let natural transformation $C(M,-) \to F$ determined by $v \in F(M)$

$$C(M,N) \to F(N)$$

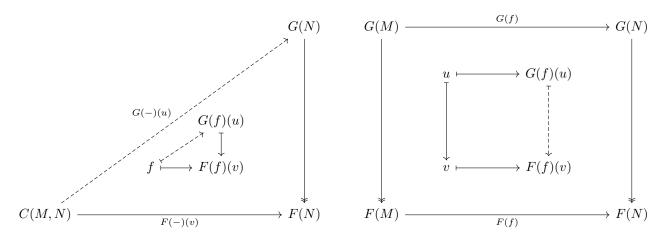
 $f \mapsto F(f)(v)$

Since $G(M) \to F(M)$ is surjective, let $u \in G(M)$ such that $u \mapsto v$ by $G(M) \to F(M)$. Construct the natural transformation $C(M, -) \to G$ by

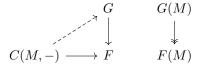
$$C(M,N) \to G(N)$$

 $f \mapsto G(f)(u)$

The lift make diagram commutes is due to naturality of $G \to F$ as in the diagram below



2. If for every projective H there is a lift from $H \to F$ to $H \to G$, then $G \to F$ epimorphism (backward direction of (2)) Let H = C(M, -),



FOR LATER - TOO LONG

1.3 EILENBERG-STEENROD-MILNOR AXIOMS

Definition 33 (category of pairs of topological spaces). The category of pairs of topological spaces Top₂ is defined as follows

- objects: pair (X, A) where $A \subseteq X$
- morphisms: continuous map $f:(X,A)\to (Y,B)$ that is a continuous map $f:X\to Y$ such that $f(A)\subseteq B$
- composition law for morphisms: compose a set maps

Axiom 1 (Eilenberg-Steenrod-Milnor axioms). Let C be a category of pairs of topological spaces (X, A) such that if $(X, A) \in O$ ob C, then $(A, \varnothing) \to (X, \varnothing) \to (X, A)$ are also in C. A homology theory on C is a functor $H_n : OD C \to AD$, $n \in \mathbb{Z}$ and a natural transformation (connecting homomorphism) $\partial : H_n(X, A) \to H_{n-1}(A, \varnothing)$ such that

- 1. **homotopy axiom**: if $f \simeq g$, then $H_n(f) = H_n(g)$ where $f, g : (X, A) \to (Y, B)$ are homotopic if they are homotopic as both a function $X \to Y$ and $A \to B$.
- 2. exactness axiom: for all $(X, A) \in ob C$, the sequence below is exact

$$H_n(A) \xrightarrow{\dots} H_{n+1}(X, A)$$

$$H_n(X) \xrightarrow{\dots} H_n(X, A)$$

$$H_{n-1}(A) \xrightarrow{\dots} \dots$$

where $H_n(X)$ denotes $H_n(X,\emptyset)$

- 3. excision axiom: if $(X, A) \in \text{ob } C$ and $U \subseteq \overline{U} \subseteq \text{int } A$ and $(X U, A U) \hookrightarrow (X, A)$ in C, then $H_n(X U, A U) \cong H_n(X, A)$
- 4. additivity axiom (Milnor): homology commutes with coproduct. That is, let $\{X_{\alpha} : \alpha \in \mathcal{A}\}$ be a set of topological spaces. If $(X_{\alpha}, \varnothing) \in \text{ob } C$ for all $\alpha \in \mathcal{A}$ and $(\coprod X_{\alpha}, \varnothing) \in \text{ob } C$, then

$$\bigoplus_{i_{\alpha}} H_n(X_{\alpha}, \varnothing) \xrightarrow{\cong} H_n(\coprod X_{\alpha}, \varnothing)$$

$$H_n(X_{\alpha}, \varnothing)$$

5. dimension axiom: $H_n(*,\varnothing) = \begin{cases} \mathbb{Z}, & n=0 \\ 0, & n \neq 0 \end{cases}$

Remark 13. Some remarks on Eilenberg-Steenrod axioms

- we omit dimension axiom: extraordinary cohomology theory.
- varying of dimension axiom: homology with coefficients. Let A be any abelian group, $H_n(*,\varnothing;A) = \begin{cases} A, & n=0\\ 0, & n\neq 0 \end{cases}$

Chapter 2

ALGEBRAIC TOPOLOGY

2.1 SINGULAR HOMOLOGY

2.1.1 DEFINITION OF SINGULAR HOMOLOGY

Definition 34 (n-simplex, barycentric coordinate, barycentre). The standard n-simplex is the convex hull of $e_0, e_1, ..., e_n$ in \mathbb{R}^{n+1} where $e_0, e_1, ..., e_n$ are the standard basis vectors, denoted by Δ^n ,

$$\Delta^n = \{ x \in \mathbb{R}^{n+1} : \sum_{i=0}^n x_i = 1, x_i \ge 0 \}$$

We also write $\Delta^n = [e_0, ..., e_n] = [0, ..., n]$. $\{x_0, ..., x_n\}$ is called the barycentric coordinate. The point $b = \left(\frac{1}{n+1}, ..., \frac{1}{n+1}\right)$ is called the barycentre.

Definition 35 (singular n-simplex). A singular n-simplex in X is a continuous function $\sigma: \Delta^n \to X$. The set of singular n-simplicies in X is denoted by $S_n(X)$

Definition 36 (singular n-chain). A singular n-chain is an element of the free abelian group $C_n(X) = \mathbb{Z}[S_n(X)]$ generated by S_n . Alternatively, n-chain is a finite formal linear combination of singular n-simplicies with integer coefficients. If $n \leq 0$, $C_n(X) = 0$ is defined to be the trivial group.

Definition 37 (face map). The face map $d^i: \Delta^{n-1} \to \Delta^n$ is the affine map sending vertices to vertices, in order, missing i. That is,

$$d^{i}(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \ge i \end{cases}$$

Proposition 6. If i < j, then $d^j \circ d^i = d^i \circ d^{j-1}$. Equivalently, $d_i d_j = d_{j-1} d_i$

Definition 38 (boundary operator). Define the boundary operator $\partial: C_n(X) \to C_{n-1}(X)$ as a linear extension of $\partial: S_n(X) \to C_{n-1}(X)$

$$\partial \sigma = \sum_{i=0}^{n} (-1)^{i} \sigma \circ d^{i} = \sum_{i=0}^{n} (-1)^{i} d_{i} \sigma \in C_{n-1}(X)$$

where $\sigma \in S_n(X)$.

Proposition 7. $\partial^2 = 0$

$$C_{n-1}(X) \stackrel{\partial}{\longleftarrow} C_n(X) \stackrel{\partial}{\longleftarrow} C_{n+1}(X)$$

Proposition 8 (singular homology). The singular n-chains $C_n(X)$ together with the boundary operator ∂ is a chain complex, written as $C_{\bullet}(X)$. The corresponding n-homology group is denoted by $H_n(X)$.

2.1.2 PATH-COMPONENT DECOMPOSITION

Define $x \sim y$ if and only if there exists a path in X from x to y, then \sim is an equivalent relation. Each equivalence class of \sim is called a path component of X. The set of path components of X is denoted by $\pi_0(X)$. Since each singular n-simplex $\sigma: \Delta^n \to X$ land entirely in a single path component, we have

$$S_n(X) = \coprod_{\alpha \in \pi_0(X)} S_n(X_\alpha)$$

where X_{α} denotes the subspace induced by α , and

$$C_n(X) = \bigoplus_{\alpha \in \pi_0(X)} C_n(X_\alpha)$$

Proposition 9. $H_n(X) \cong \bigoplus_{\alpha \in \pi_0(X)} H_n(X_\alpha)$

2.1.3 SOME SPECIAL CASES: $H_0, H_n(*)$

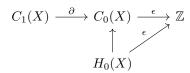
Assume X is path connected, then $Z_0(X) = C_0(X)$, and $H_0(X) = \frac{Z_0(X)}{B_0(X)} = \frac{C_0(X)}{\operatorname{im}(\partial:C_1(X) \to C_0(X))} = \operatorname{coker} \partial C_0(X)$

Definition 39 (augmentation map). The augmentation map $\epsilon: C_n(X) \to \mathbb{Z}$ is defined by

$$\epsilon \left(\sum_{x \in S_n(X)} a_x x \right) = \sum_{x \in S_n(X)} a_x$$

Lemma 2. Let $\epsilon: C_0(X) \to \mathbb{Z}$, then $\epsilon \partial = 0$

Proof. For each $\sigma \in S_1(X)$, then $\epsilon \partial \sigma = \epsilon (d_0 \sigma - d_1 \sigma) = \epsilon d_0 \sigma - \epsilon d_1 \sigma = 1 - 1 = 0$.



Lemma 3. If X is path connected, then the augmentation map induces an isomorphism $\epsilon: H_0(X) \to \mathbb{Z}$

Proof.

(well-defined) Let a and $a + \partial b$ be two elements from the same cosets of $B_0(X)$, then

$$\epsilon[a + \partial b] = \epsilon(a + \partial b) = \epsilon a = \epsilon[a]$$

(homomorphism)

$$\epsilon([a] + [b]) = \epsilon[a + b] = \epsilon(a + b) = \epsilon a + \epsilon b = \epsilon[a] + \epsilon[b]$$

(surjective) Let any $x \in S_0(X), n \in \mathbb{Z}$, then

$$\epsilon[nx] = \epsilon(nx) = n\epsilon x = n$$

(injective) As ϵ is a homomorphism, we just need to prove that the kernel of ϵ is 0, that is, for all $c \in C_0(X)$, if $\epsilon c = 0$, then [c] = 0, that is, $c \in B_0(X)$. Let $c = \sum_{x \in S_0(X)} a_x x$ and $\epsilon c = 0$, then $\sum_{x \in S_0(X)} a_x = 0$. We can rewrite

$$c = \sum_{x \in S_0(X)} a_x x = (x_1 + \dots + x_n) - (y_1 + \dots + y_n)$$

where $x_1, ..., x_n, y_1, ..., y_n \in S_0(X)$. For each i = 1, ..., n, pick $\sigma_i \in S_1(X)$ such that $d_0\sigma_i = x_i$ and $d_1\sigma_i = y_i$ (pick $\sigma_i : \Delta^1 \to X$ mapping the end points of Δ^1 to x_i and y_i), then $\sum_{i=1}^n \sigma_i \in C_1(X)$, and

$$\partial \left(\sum_{i=1}^{n} \sigma_{i}\right) = \sum_{i=1}^{n} (d_{0}\sigma_{i} - d_{1}\sigma_{i}) = \sum_{i=1}^{n} (x_{i} - y_{i}) = c$$

Corollary 2. $H_0(X) = \bigoplus_{a \in \pi_0(X)} \mathbb{Z}$ for any space X

Proposition 10. $H_n(*) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}$ where * denotes the space of one point.

Proof. For any $n = 0, 1, ..., S_n(*) = *$ then $C_n(*) = \mathbb{Z}$. For n > 0, and $* \in S_n(X)$,

$$\partial * = \sum_{i=0}^{n} (-1)^i d_i * = \sum_{i=0}^{n} (-1)^i * = \begin{cases} 0, & n \text{ odd} \\ *, & n \text{ even} \end{cases}$$

We have the diagram

$$0 \stackrel{\partial=0}{\longleftarrow} C_0(*) \stackrel{\partial=0}{\longleftarrow} C_1(*) \stackrel{\partial=1}{\longleftarrow} C_2(*) \stackrel{\partial=0}{\longleftarrow} C_3(*) \stackrel{\partial=1}{\longleftarrow} \dots$$

 $H_0(*) = \mathbb{Z}$ by the previous explanation. If n > 0 (n odd or even), $Z_n(*) = B_n(*) = C_n(*) = \mathbb{Z}$, then $H_n(*) = 0$.

2.1.4 INDUCED FUNCTION BETWEEN TWO TOPOLOGICAL SPACES

Proposition 11 (naturality of induced map). A function $f: X \to Y$ induces a function $f_\#: C_n(X) \to C_n(Y)$ defined by a linear extension of $f_\#: S_n(X) \to S_n(Y)$ with $f_\#\sigma = f\sigma$

$$\begin{array}{ccc}
\Delta^n & \xrightarrow{\sigma} & X \\
\downarrow^f \\
f_\# \sigma = f \sigma & \downarrow^f \\
Y
\end{array}$$

Then, $f_{\#}\partial = \partial f_{\#}$, that is, the following square commutes

$$C_n(X) \xrightarrow{\partial} C_{n-1}(X)$$

$$\downarrow f_{\#} \qquad \qquad \downarrow f_{\#}$$

$$C_n(Y) \xrightarrow{\partial} C_{n-1}(Y)$$

Proof. Let $\sigma \in S_n(X)$, then

$$f_{\#}\partial\sigma = f_{\#} \sum_{i=0}^{n} (-1)^{i} \sigma d^{i} = f \sum_{i=0}^{n} (-1)^{i} \sigma d^{i} = \sum_{i=0}^{n} (-1)^{i} f \sigma d^{i} = \partial f_{\#}\sigma$$

Remark 14 (chain map). The function $f_{\#}: C_n(X) \to C_n(Y)$ induced from $f: X \to Y$ is a chain map of from chain complex $C_{\bullet}(X)$ to chain complex $C_{\bullet}(Y)$. The induced map in homology is written as $H_n(f) = f_*: H_n(A) \to H_n(B)$

2.1.5 HOMOTOPY

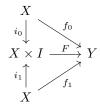
Definition 40 (homotopy). Let $f_0, f_1 : X \to Y$ be continuous maps. A homotopy from f_0 to f_1 is a continuous map $F : X \times I \to Y$ where I = [0,1] such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. Being homotopic is an equivalence relation on maps: $(X,Y) \to [X,Y]$ where (X,Y) denotes the set of all maps from X to Y and [X,Y] denotes the equivalence classes of homotopy of maps from X to Y. f_0 and f_1 being homotopic is denoted by $f_0 \simeq f_1$.

Theorem 4 (homotopy invariance). Two homotopic maps induce the same map in homology. That is, if $f_0 \simeq f_1$, then $H_n(f_0) = H_n(f_1) : H_n(X) \to H_n(Y)$ for all n

Proof. Consequence of Proposition ?? and Proposition ??

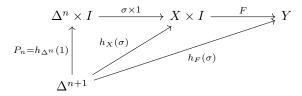
prism operator

Let $f_0, f_1: X \to Y$ be homotopic, the diagram below commutes



where $i_0: x \mapsto (x,0)$, and $i_1: x \mapsto (x,1)$

Definition 41 (prism operator). Given homotopy $F: X \times I \to Y$, the prism operator $h_F: C_n(X) \to C_{n+1}(Y)$ is defined by a linear extension of $h_F: S_n(X) \to C_{n+1}(Y)$ as follows. Let $\sigma \in S_n(X)$,

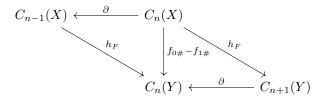


Let $[v_0,...,v_n],[w_1,...,w_n]$ be the bases of the prism $\Delta^n \times I$. Define $\sigma_i:\Delta^{n+1} \to \Delta^n \times I$ be the affine map from Δ^{n+1} to the simplex $[v_0,v_1,...,v_i,w_i,w_{i+1},...,w_n]$. Define $P_n=\sum_{i=0}^n (-1)^i\sigma_i$, and $h_F(\sigma)=F(\sigma\times 1)P_n$ is then defined accordingly.

Proposition 12. $\partial P_n = -P_{n-1}\partial + i_1 - i_0$ and $\partial h_F = -h_F\partial + f_1 - f_0$ where $i_1 : \Delta^n \mapsto (\Delta^n, 1), i_0 : \Delta^n \mapsto (\Delta^n, 0)$

Proof. TODO

Proposition 13. Two homotopic maps induce chain homotopic maps. That is, if $f_0 \simeq f_1 : X \to Y$, then $f_{0\#}, f_{1\#} : C_n(X) \to C_n(Y)$ are chain homotopic



Proof. Proposition ??

Definition 42 (homotopy equivalence, homotopy type). A map $f: X \to Y$ is a homotopy equivalence if there exists $g: Y \to X$ such that $1_Y \simeq fg: Y \to Y$ and $1_X \simeq gf: X \to X$. If there exists a homotopy equivalence between X and Y, X and Y are said to be of the same homotopy type.

Definition 43 (contractible). X is contractible if $X \to *$ is a homotopy equivalence

Corollary 3. Any homotopy equivalence induces an isomorphism in homology

Proof. As $fg \simeq 1_Y$, then $H_n(f)H_n(g) = 1_{H_n(Y)}$. Similarly, $H_n(g)H_n(f) = 1_{H_n(X)}$. Therefore, $H_n(f)$ is an isomorphism \square

2.1.6 LOCALITY OF $H_*(-)$

Let \mathcal{U} be an open cover of a topological space X.

Definition 44 (*U*-small). A singular simplex $\sigma: \Delta^n \to X$ is *U*-small if its image lies entirely in a single open set $U \in \mathcal{U}$. Let $C_n^{\mathcal{U}}(X)$ denote the subgroup of $C_n(X)$ generated by *U*-small singular simplicies, namely *U*-small n-chains

$$C_n^{\mathcal{U}}(X) \longrightarrow C_n(X)$$

$$\downarrow_{\partial} \qquad \qquad \downarrow_{\partial}$$

$$C_{n-1}^{\mathcal{U}}(X) \longrightarrow C_{n-1}(X)$$

The boundary map, homology on U-small chains is defined accordingly.

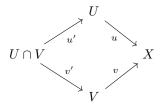
Theorem 5 (locality principle). The inclusion map $C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ is a chain homotopy equivalence

Corollary 4. $H_n(C_n^{\mathcal{U}}(X)) = H_n(C_n(X)) = H_n(X)$

Proof. chain homotopy equivalence induces isomorphism in homology

2.1.7 LONG EXACT SEQUENCE IN HOMOLOGY

Special case: $\mathcal{U} = \{U, V\}, X = U \cup V, U, V \text{ open. Define } u, v, u', v' \text{ being the inclusion maps.}$



Proposition 14. The diagram below is a short exact sequence

$$C_n(U \cap V) \xrightarrow{i = \begin{bmatrix} u'_{\#} \\ -v'_{\#} \end{bmatrix}} C_n(U) \oplus C_n(V) \xrightarrow{p = \begin{bmatrix} u_{\#}, v_{\#} \end{bmatrix}} C_n^{\mathcal{U}}(X)$$

$$\downarrow \subseteq$$

$$C_n(X)$$

where $i = c \mapsto (c, -c)$ and $p = (c_1, c_2) \mapsto c_1 + c_2$

Theorem 6 (Mayer-Vietoris sequence). Special case: $U = \{U, V\}$, $X = U \cup V$, U, V open. There exists a long exact sequence

$$H_{n}(U \cap V) \stackrel{i_{*}}{\longleftarrow} H_{n}(U) \oplus H_{n}(V) \stackrel{p_{*}}{\longrightarrow} H_{n}(C_{n}^{\mathcal{U}}(X))$$

$$H_{n-1}(U \cap V) \stackrel{i_{*}}{\longleftarrow} H_{n-1}(U) \oplus H_{n-1}(V) \stackrel{p_{*}}{\longrightarrow} H_{n-1}(C_{n-1}^{\mathcal{U}}(X))$$

$$\downarrow 0$$

2.1.8 PROOF OF LOCALITY PRINCIPLE

barycentric subdivision

Definition 45 (cone operator). Given a star-shaped subset D of \mathbb{R}^n (there exists a point p such that the line segment from p to any point $x \in D$ belongs to D). The cone operator $c_p : C_n(D) \to C_{n+1}(D)$ is defined by a linear extension of $c_p : S_n(D) \to C_{n+1}(D)$. Let $\sigma \in S_n(D)$, then $c_p(\sigma)$ is defined by

$$c_p(\sigma): (x_0, x_1, ..., x_{n+1}) \mapsto x_0 p + (1 - x_0) \sigma(\overline{x_1}, ..., \overline{x_{n+1}})$$

where $\overline{x_i} = \frac{x_i}{x_1 + \dots + x_{n+1}}$

Definition 46 (subdivision operator). The subdivision operator $\operatorname{Sd}_n: C_n(\Delta^n) \to C_n(\Delta^n)$ is defined by

$$Sd_0 = 1$$

$$Sd_n = c_{b(n)} Sd_{n-1} \partial$$

where $c_{b(n)}$ is the cone operator with respective to the barycentre of Δ^n

Lemma 4. Given a chain $s \in C_n(\Delta^p)$, we can define a unique operator, a linear transformation

$$S^{\bullet}: C_n(\bullet) \to C_n(\bullet)$$

such that

- 1. $S^{\Delta^p}(1) = s$ where $1: \Delta^p \mapsto \Delta^p$ is the identity simplex
- 2. for any map $f: X \to Y$, the diagram below commutes

$$\begin{array}{ccc} C_p(X) & \xrightarrow{S^X} & C_n(X) \\ f_\# \downarrow & & f_\# \downarrow \\ C_p(Y) & \xrightarrow{S^Y} & C_n(Y) \end{array}$$

Proof. (Uniqueness of S^X)

$$C_{p}(\Delta^{p}) \xrightarrow{S^{\Delta^{p}}} C_{n}(\Delta^{p}) \qquad 1 \longmapsto S^{\Delta^{p}} \qquad s$$

$$\sigma_{\#} \downarrow \qquad \sigma_{\#} \downarrow \qquad \qquad \downarrow \sigma_{\#} \qquad \downarrow f_{\#}$$

$$C_{p}(X) \xrightarrow{S^{X}} C_{n}(X) \qquad \sigma \longmapsto S^{X} S^{X} \sigma_{\#} 1 = \sigma_{\#} S^{\Delta^{p}} 1$$

Let $\sigma: \Delta^n \to X$, the commutativity implies the unique definition of S^X by $S^X \sigma = S^X \sigma_\# 1 = \sigma_\# S^{\Delta^p} 1 = \sigma_\# s$

Definition 47 (subdivision operator). Given $\operatorname{Sd}_n: C_n(\Delta^n) \to C_n(\Delta^n)$, by Lemma ??, naturality of S^X induces a unique map $\operatorname{Sd}_n: C_n(X) \to C_n(X)$ such that for all $\sigma: \Delta^n \to X$, $\operatorname{Sd}_n \sigma = \operatorname{Sd}_n \sigma_\# 1 = \sigma_\# \operatorname{Sd}^n 1$ where $1: \Delta^n \to \Delta^n$ is the identity simplex.

Proposition 15 (boundary of cone operator). Given the cone operator $c_p: C_n(D) \to C_{n+1}(D)$ then its boundary is

$$\partial c_p = \begin{cases} 1 - c_p \partial, & n > 0 \\ 1 - p\epsilon, & n = 0 \end{cases}$$

where $\epsilon: C_0(X) \to \mathbb{Z}$ is the augmentation map

Proposition 16. $Sd_n: C_n(X) \to C_n(X)$ is a chain map, that is $Sd_n \partial = \partial Sd_{n-1}$ apply formula

Proposition 17. Sd is chain homotopic to 1

Proof

Let defined $T_n: C_n(X) \to C_{n+1}(X)$ as follows:

$$T_0 = 0$$

$$T_n = c_{b(n)} (1 - \operatorname{Sd}_n - T_{n-1} \partial)$$

We will prove that $\partial T_n = 1 - \operatorname{Sd}_n - T_{n-1}\partial$ by induction. Base case: n = 1, then

$$\partial T_{1} = \partial c_{b(1)} (1 - \operatorname{Sd}_{1} - T_{0} \partial)$$

$$= (1 - c_{b(1)} \partial) (1 - \operatorname{Sd}_{1})$$

$$= 1 - \operatorname{Sd}_{1} - c_{b(1)} \partial + c_{b(1)} \partial \operatorname{Sd}_{1}$$

$$= 1 - \operatorname{Sd}_{1} - c_{b(1)} \partial + c_{b(1)} \operatorname{Sd}_{0} \partial$$

$$= 1 - \operatorname{Sd}_{1} - c_{b(1)} \partial + c_{b(1)} \partial$$

$$= 1 - \operatorname{Sd}_{1}$$

$$= 1 - \operatorname{Sd}_{1}$$

Induction case, suppose $\partial T_{n-1} = 1 - \operatorname{Sd}_{n-1} - T_{n-2}\partial$, then

$$\begin{split} \partial T_n &= (1 - c_p \partial) (1 - \operatorname{Sd}_n - T_{n-1} \partial) \\ &= (1 - \operatorname{Sd}_n - T_{n-1} \partial) - c_p (\partial - \partial \operatorname{Sd}_n - \partial T_{n-1} \partial) \\ &= (1 - \operatorname{Sd}_n - T_{n-1} \partial) - c_p (\partial - \partial \operatorname{Sd}_n - (1 - \operatorname{Sd}_{n-1} - T_{n-2} \partial) \partial) \\ &= 1 - \operatorname{Sd}_n - T_{n-1} \partial \end{split}$$

Proposition 18. $\operatorname{Sd}^k: C_{\bullet}(X) \to C_{\bullet}(X)$ is a chain map

Proof. Let g be a chain map, then $\partial gg = g\partial g = gg\partial$

Corollary 5. 1 and Sd^k are chain homotopic

Proof. Let $G = 1 + \text{Sd} + \text{Sd}^2 + ... + \text{Sd}^{k-1}$, then

$$1 - \mathrm{Sd}^k = G(1 - \mathrm{Sd}) = G(\partial T + T\partial) = \partial(GT) + (GT)\partial$$

Theorem 7 (locality principle).

Proof.

TODO - assignment 1

2.1.9 RELATIVE HOMOLOGY

Definition 48 (pair, quotient chain complex, relative homology). A pair (X, A) is a space X together with a subspace $A \subseteq X$

$$0 \longrightarrow C_n(A) \hookrightarrow C_n(X) \longrightarrow C_n(X)/C_n(A) \longrightarrow 0$$

$$\downarrow \partial \qquad \qquad \downarrow \partial \qquad \qquad \downarrow \partial$$

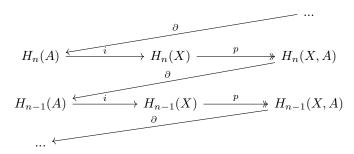
$$0 \longrightarrow C_{n-1}(A) \hookrightarrow C_{n-1}(X) \longrightarrow C_{n-1}(X)/C_{n-1}(A) \longrightarrow 0$$

$$\dots \hookrightarrow c + a \longrightarrow [c] \longrightarrow 0$$

$$\downarrow \partial \qquad \qquad \downarrow \partial$$

$$\dots \hookrightarrow \partial c + \partial a \longrightarrow [\partial c] \longrightarrow 0$$

The boundary map on quotient chain complex $C_n(X)/C_n(A)$ is induced from the boundary map on chain complex $C_n(X)$ such that the diagram remains commutative. The quotient chain complex is also denoted by $C_n(X,A) = C_n(X)/C_n(A)$. The relative homology is defined by the homology on $C_n(X,A)$ and denoted by $H_n(X,A) = H_n(C_n(X,A))$. The short exact sequence $C_n(A) \stackrel{\iota}{\smile} C_n(X) \stackrel{p}{\longrightarrow} C_n(X)/C_n(A)$ induces a long exact sequence



Remark 15. $C_n(X,\varnothing) = C_n(X)$, then $H_n(X,\varnothing) = H_n(X)$

Definition 49 (map of pairs). $f:(X,A)\to (Y,B)$ is a map of pairs of $f:X\to Y$ and $f(A)\subseteq B$

Definition 50 (naturality of induced map in pair). Let $f:(X,A)\to (Y,B)$, the induced map $f_\#:C_n(X,A)\to C_n(Y,B)$ is well-defined since the map $f_\#:C_n(X)\to C_n(Y)$ takes $C_n(A)$ to $C_n(B)$. The naturality is followed by naturality in absolute chains

$$C_n(X,A) \xrightarrow{\partial} C_{n-1}(X,A)$$

$$\downarrow f_{\#} \qquad \qquad \downarrow f_{\#}$$

$$C_n(Y,B) \xrightarrow{\partial} C_{n-1}(Y,B)$$

Theorem 8 (excision). Let (X, A) be a pair and $U \subseteq X$ such that $\overline{U} \subseteq \text{int } A$. Then,

$$H_n(X-U,A-U) \cong H_n(X,A)$$

Proof.

TODO

Proof.

TODO

Lemma 5 (five lemma). Given $A \to B \to C \to D \to E$ and $A' \to B' \to C' \to D' \to E'$ exact. Then

 $C \to C'$ is a monomorphism if

- ullet $D \to D'$ is a monomorphism
- ullet $B \to B'$ is a monomorphism
- $A \rightarrow A'$ is a epimorphism

 $C \to C'$ is a epimorphism if

- $B \to B'$ is a epimorphism
- $D \rightarrow D'$ is a epimorphism
- $E \to E'$ is a monomorphism

Proof.

TODO

Definition 51 (pointed space, reduced homology). A pointed space is a pair (X, *) such that * is a single point. The reduced homology is a pointed space is denoted by $\overline{H}_n(X) = H_n(X, *)$

Proposition 19. Some propositions on reduced homology

- $H_0(X) = \overline{H}_0(X) \oplus \mathbb{Z}$
- $H_n(X) = \overline{H}_n(X)$ for $n \ge 1$

2.1.10 EXPRESSING RELATIVE HOMOLOGY IN TERMS OF REDUCE HOMOLOGY

Definition 52 (cone). Let A be a topological space, the cone on A is defined by

$$CA = A \times [0,1]/A \times \{0\}$$

Note: $A/\emptyset = A \coprod *$, then $C\emptyset = *$

Lemma 6. The cone on any space A is contractible, that is $CA \simeq *$. In other words, any space can be embedded into a contractible space.

Proof.

TODO

Definition 53 (cone of the pair). The cone of the pair (X, A) is

$$X \cup_A CA = X \coprod CA/a \sim (a,1)$$

where $a \sim (a, 1)$ denotes the equivalence relation between point $a \in A$ and point $(a, 1) \in CA$

Proposition 20. $H_n(X,A) = \overline{H}_n(X \cup_A CA)$

Proof.

TODO

Definition 54 (suspension). The cone of pair (CA, A), denoted by ΣA

2.2 CELLULAR HOMOLOGY

closure-finite weak topology

Definition 55 (cell complex). A cell complex is a space X with a filtration $X_0 \subseteq X_1 \subseteq ... \subseteq X$ by subspaces such that

- 1. X_0 is discrete
- 2. for each n, X_n is the pushout of the diagram below

$$\coprod_{\alpha \in \mathcal{A}_n} \partial D^n \longleftrightarrow \coprod_{\alpha \in \mathcal{A}_n} D^n$$

$$\downarrow \qquad \qquad \downarrow$$

$$X_{n-1} \longleftrightarrow X_n$$

where D^n is a space homeomorphic to an n-dimensional disk, ∂D^n is the boundary of D^n (S^{n-1}) . X_n is called the n-skeleton of X, $\coprod \partial D^n \to X_{n-1}$ is called the attaching map, $\coprod D^n$ is called the characteristic map.

3.
$$X = \lim_{n \to \infty} X_n$$

Remark 16. Some remarks on cell complex

- $\mathbb{R}P^{n-1}$ is a cell complex with $(\mathbb{R}P^{n-1})_k = \mathbb{R}P^k \subseteq \mathbb{R}P^n$. $\mathbb{R}P^\infty = \bigcup \mathbb{R}P^n = \lim \mathbb{R}P^n$
- S^n is a cell complex
- Grassmannians, $Gr_k(\mathbb{R}^n)$, the set of k-dimensional subspace of \mathbb{R}^n , is a cell complex

Definition 56 (finite type, finite dimensional). A cell complex $X_0 \subseteq X_1 \subseteq ... \subseteq X$ is of finite type if X_0 is finite and each A_n is finite $(|X_0|)$ is the number of 0-cells, $|A_n|$ is the number of n-cells). X is of finite dimensional if $A_n = \emptyset$ for n >> 0. X is of finite if it is both finite type and finite dimensional.

2.2.1 THE CONSTRUCTION OF CELLULAR HOMOLOGY

Let X_0 be the pushout of the diagram below

$$\coprod_{\alpha \in \mathcal{A}_0} \partial D^0 = \varnothing \hookrightarrow \longrightarrow \coprod_{\alpha \in \mathcal{A}_0} D^0$$

$$\downarrow \qquad \qquad \downarrow$$

$$X_{-1} = \varnothing \hookrightarrow X_0$$

Then, from the definition of cell complex,

$$X_n/X_{n-1} = \prod_{\alpha \in \mathcal{A}_n} D^n / \prod_{\alpha \in \mathcal{A}_n} \partial D^n = \bigvee_{\alpha \in \mathcal{A}_n} S^n$$

where $S^n = D^n/\partial D^n$ (why do the two quotient topologies coincide? think about it. X_n/X_{n-1} makes X_{n-1} to be one point). The wedge sum \vee is defined by follows: Let $(A, *_A), (B, *_B)$ be pointed spaces, then

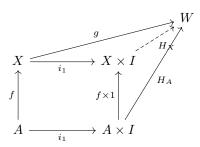
$$A \vee B = A \coprod B/*_A \sim *_B$$

Remark 17. Wedge sum is the coproduct in the category of pointed spaces

Let H be a homology functor satisfying Eilenberg-Steenrod axioms. Given $A \subseteq X$, as CA contractible, $X \cup_A CA \simeq X/A$, then $\overline{H}_q(X \cup_A CA) \cong \overline{H}_q(X/A)$. Therefore, $H_q(X,A) = \overline{H}_q(X/A)$

$$H_a(X,A) \xrightarrow{\cong} \overline{H}_a(X \cup_A CA) \xrightarrow{\cong} \overline{H}_a(X/A)$$

Definition 57 (cofibration). A map $f: A \to X$ is a cofibration if the homotopy $H_A: A \times I \to W$ can be extended to $H_X: X \times I \to W$, namely, homotopy extension property. That is, given the diagram below, there exists a map $H_X: X \times I \to W$ such that the diagram below commutes



where i_1 denotes the map $x \mapsto (x,1)$

Proof. Let $Z = X \cup_A (A \times I)$ be the pushout of $X \leftarrow_f A \xrightarrow{i_1} A \times I$ where $X \cup_A (A \times I) = X \coprod (A \times I) / \sim$ with $f(a) \sim (a,1)$ for all $a \in A$. There exists a retraction $r: X \times I \to Z$. Define H = hr where $h: Z \to W$ is the unique map from pushout Z to W

Lemma 7. Some lemmas on cofibration

- 1. the composite of cofibrations is a cofibration
- 2. the coproduct of (closed) cofibrations is a (closed) cofibration
- 3. (closed) cofibration is closed under cobase change, that is, if $f:A\to X$ is a (closed) cofibration and $Y=X\cup_A B$ is the pushout of $B\xleftarrow{g}A\xrightarrow{f}X$, then $\overline{f}:B\to Y$ is a (closed) cofibration

$$\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow g & & \downarrow \\
B & \xrightarrow{\overline{f}} & Y
\end{array}$$

where $X \cup_A B = X \coprod B / \sim$ with $f(a) \sim g(a)$ for all $a \in A$

Corollary 6. $X_q \hookrightarrow X_n$ for $q \leq n$ is a closed cofibration

Proof. the composite of cofibrations is a cofibration (inclusion map is a cofibration)

Corollary 7. $H_q(X_n, X_{n-1}) = \overline{H}_q(X_n/X_{n-1}) = \overline{H}_q(\bigvee_{\alpha \in \mathcal{A}_n} S^n) = \bigoplus_{\alpha \in \mathcal{A}_n} \overline{H}_q(S^n) = \begin{cases} 0, & q \neq n \\ \mathbb{Z}\mathcal{A}_n, & q = n \end{cases}$ where \mathbb{Z} denotes the free abelian group functor, that is, $\mathbb{Z}\mathcal{A}_n$ is the free abelian group generated by \mathcal{A}_n

Proof. In the category of pointed space, morphisms are the continuous maps that fix the basepoint and wedge sum is the coproduct. As H_q is a functor on that category. Let X,Y be pointed spaces, we must have the coproduct of two abelian groups $H_q(X)$ and $H_q(Y)$ being $H_q(X \vee Y)$, that is, $H_q(X) \oplus H_q(Y) = H_q(X \vee Y)$

The long exact sequence for (X_n, X_{n-1})

$$H_{q+1}(X_n, X_{n-1})$$

$$H_q(X_{n-1}) \xrightarrow{i_*} H_q(X_n) \longrightarrow H_q(X_n, X_{n-1})$$

the map $i_*: H_q(X_{n-1}) \to H_q(X_n)$ is an isomorphism unless q=n,n-1. This is due to $H_q(X_n,X_{n-1})=0$ and $H_{q+1}(X_n,X_{n-1})$ otherwise. Therefore,

Corollary 8. $H_q(X_n) \cong H_q(X_{n-1}) \cong ... \cong H_q(X_0) = 0$ for $q > n \geq 0$. That is, homology groups do not see the skeleton of dimension less than its.

Corollary 9. $H_q(X_n) \cong H_q(X_{n+1}) \cong ... \cong H_q(X)$ for q < n. That is, homology groups see the same thing on skeleton of dimension more than its.

When q = n, the long exact sequence

$$0 = H_n(X_{n-1}) \xrightarrow[i_*]{} H_q(X_n) \xrightarrow[\partial]{} H_q(X_n, X_{n-1}) = \mathbb{Z}A_n$$

Then

Corollary 10. $H_n(X_n)$ is free abelian (a subgroup of $H_n(X_n, X_{n-1}) = \mathbb{Z}A_n$). Moreover, the number of generators of $H_n(X_n)$ is bounded above by number of n-cells.

Now using the lemmas, we put the long exact sequence of the pairs $(X_{n+1}, X_n), (X_n, X_{n-1})$ together,

Definition 58 (cellular chain complex, cellular homology). The abelian groups $H_n(X_n, X_{n-1})$ and homomorphisms $d_n: H_{n+1}(X_{n+1}, X_n) \to H_n(X_n, X_{n-1})$ form a chain complex and called cellular chain complex of the cell complex $X_0 \subseteq X_1 \subseteq ... \subseteq X$ with respect to homology H. Denote $H_n(X_n, X_{n-1})$ by $C_n^{CW}(X)$. We have

$$C_n^{CW}(X) = \mathbb{Z}\mathcal{A}_n$$

That is, the number of generators of $C_n^{CW}(X)$ is exactly the number of n-cells. Cellular homology is defined from the cellular chain complex and denoted by $H_n(C_n^{CW}(X)) = H_n^{CW}(X)$.

Theorem 9. $H_n(X) = H_n^{CW}(X)$ and the isomorphism is also natural in the cell complex. *Proof.*

$$H_n(X) = H_n(X_{n+1})$$

$$= \operatorname{im}(H_n(X_n) \to H_n(X_{n+1}))$$

$$= \frac{H_n(X_n)}{\ker(H_n(X_n) \to H_n(X_{n+1}))}$$

$$= \frac{H_n(X_n)}{\operatorname{im} \partial_{n+1}}$$

$$= \frac{\operatorname{im} i_n}{\operatorname{im} d_n}$$

$$= \frac{\ker d_{n-1}}{\operatorname{im} d_n}$$

$$= \frac{\operatorname{Her} d_{n-1}}{\operatorname{im} d_n}$$

$$= H_n^{CW}(X)$$

$$(first isomorphism theorem)$$

$$(exactness)$$

$$(i_n is injective)$$

$$(exactness)$$

Theorem 10 (cellular boundary formula). TODO

2.2.2 COEFFICIENTS IN AN ABELIAN GROUP

Definition 59. Let A be an abelian group, cellular chain complex of X is

$$C_n^{CW}(X;A) = H_n(X_n, X_{n-1}; A) = H_n(X_n, X_{n-1}) \otimes A = \bigoplus_{\alpha \in \mathcal{A}_n} A = \left(\bigoplus_{\alpha \in \mathcal{A}_n} \mathbb{Z}\right) \otimes A$$
$$H_n(X;A) = H_n^{CW}(X;A)$$

2.3 HOMOLOGY WITH COEFFICIENTS

2.3.1 TENSOR PRODUCT

Definition 60 (tensor product). Let A, B, C be abelian groups. A bilinear map $A \times B \to C$ is a function $f: A \times B \to C$ such that

$$f(a + a', b) = f(a, b) + f(a', b)$$

$$f(a, b + b') = f(a, b) + f(a, b')$$

The set of bilinear maps $A \times B \to C$ is denoted by Bil(A, B; C). There exists an abelian group $A \otimes B$ and bilinear map $A \times B \to A \otimes B$ such that for any C and any bilinear map $g: A \times B \to C$, there exists a unique homomorphism $A \otimes B \to C$ such that the following diagram commutes

Moreover, $A \otimes B$ is universal. universal mapping property

Remark 18. :

- 1. If $A \times B \to D$ is bilinear and $D \to C$ is a homomorphism, then the composite $A \times B \to D \to C$ is bilinear
- 2. Let B be an abelian group, then $\otimes B$, the right tensoring of B, is a functor on the category of abelian groups Ab
- 3. $\mathbb{Z} \otimes B = B$

Proof.

1. trivial

- 2. consequence of $A \times B \to A \otimes B$ being a functor
- 3. define the map $f: Z \times B \to B$ by f(n, b) = nb and prove that B satisfies the tensor product factoring property. Hence, $\mathbb{Z} \otimes B = B$

Proposition 21. $\operatorname{Hom}(A \otimes B, C) \cong \operatorname{Bil}(A, B; C) \cong \operatorname{Hom}(A, \operatorname{Hom}(B, C))$

Proof. Hom $(A \otimes B, C) \cong \text{Bil}(A, B; C)$ is done by the definition of tensor product. Bil $(A, B; C) \cong \text{Hom}(A, \text{Hom}(B, C))$ is done by natural evaluation map.

Corollary 11. $\operatorname{Hom}((\bigoplus_{\alpha} A_{\alpha}) \otimes B, C) = \operatorname{Hom}(\bigoplus_{\alpha} A_{\alpha} \otimes B, C)$

Proof.

$$\operatorname{Hom}\left(\left(\bigoplus_{\alpha} A_{\alpha}\right) \otimes B, C\right) = \operatorname{Hom}\left(\bigoplus_{\alpha} A_{\alpha}, \operatorname{Hom}(B, C)\right)$$

$$= \prod_{\alpha} \operatorname{Hom}(A_{\alpha}, \operatorname{Hom}(B, C))$$

$$= \prod_{\alpha} \operatorname{Hom}(A_{\alpha} \otimes B, C)$$

$$= \operatorname{Hom}\left(\bigoplus_{\alpha} A_{\alpha} \otimes B, C\right)$$

Lemma 8. Let C be a category, then $f \in C(X,Y)$ is an isomorphism in C if and only if for all $W \in ob C$, $C(Y,W) \to C(X,W)$ defined by $g \mapsto gf$ is bijective

Proof. By surjectivity, $1 \in C(X, X)$ has a preimage $g \in C(Y, X)$ such that $g \mapsto gf = 1$. Now, $fg \in C(Y, Y)$ is mapped to $g = gfg \in C(X, Y)$. Moreover, $1 \in C(Y, Y)$ is also mapped to $g \in C(X, Y)$. By injectivity, fg = 1.

Proposition 22. $\beta: \bigoplus (A_{\alpha} \otimes B) \to (\bigoplus A_{\alpha}) \otimes B$ is an isomorphism

Proof. From the corollary above

Lemma 9. Suppose $A' \to A \to A'' \to 0$ in Ab. Then $A' \to A \to A'' \to 0$ is exact if and only if for all C, $\operatorname{Hom}(A',C) \leftarrow \operatorname{Hom}(A,C) \leftarrow \operatorname{Hom}(A'',C) \leftarrow 0$ is exact.

Proof. no proof provided

Corollary 12 (right exactness of tensoring abelian group). If $A' \to A \to A'' \to 0$ is exact, then $A' \otimes B \to A \otimes B \to A'' \otimes B \to 0$ is also exact

Proof.

A sequence of exact sequences

- 1. $A' \rightarrow A \rightarrow A'' \rightarrow 0$
- 2. $\operatorname{Hom}(A', \operatorname{Hom}(B, C)) \to \operatorname{Hom}(A, \operatorname{Hom}(B, C)) \to \operatorname{Hom}(A'', \operatorname{Hom}(B, C)) \to 0$
- 3. $\operatorname{Hom}(A' \otimes B, C) \to \operatorname{Hom}(A \otimes B, C) \to \operatorname{Hom}(A'' \otimes B, C) \to 0$
- 4. $A' \otimes B \to A \otimes B \to A'' \otimes B \to 0$

Proposition 23 (an example of free resolution of abelian group). Let A be an arbitrary abelian group, if $S \subseteq A$ generates A, then the natural projection $\eta: \mathbb{Z}S \to A$ is a surjection where \mathbb{Z} is the free abelian group functor. Let $F_0 = \mathbb{Z}S$, $F_1 = \ker \eta$ (F_1 is free abelian as it is a subgroup of F_0) and $M: F_1 \to F_0$ be the inclusion map, then the sequence below is short exact

$$0 \longrightarrow F_1 \stackrel{M}{\longrightarrow} F_0 \stackrel{\eta}{\longrightarrow} A \longrightarrow 0$$

Let $F = \bigoplus_{\alpha} \mathbb{Z}$ be a free abelian group, then $F \otimes B = (\bigoplus_{\alpha} \mathbb{Z}) \otimes B = \bigoplus_{\alpha} (\mathbb{Z} \otimes B) = \bigoplus_{\alpha} B$. Therefore, for any abelian group B, the sequence below is also exact

$$\bigoplus_{S_1} B \xrightarrow{M} \bigoplus_{S_0} B \xrightarrow{\eta} A \otimes B \longrightarrow 0$$

Remark 19. Let $A = \mathbb{Z}/m\mathbb{Z}$, $S = \{1\}$, then $m\mathbb{Z} \cong \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow A \longrightarrow 0$ is exact. And

is exact. Therefore, $\mathbb{Z}/m\mathbb{Z}\otimes B=B/mB$

Lemma 10. There is a natural isomorphism $A \otimes B \cong B \otimes A$

Proof. short \Box

Lemma 11 (right exactness of tensoring free abelian group). If F is a free abelian group and $A' \longrightarrow A \longrightarrow A''$ is exact at A, then

$$A' \otimes F \longrightarrow A \otimes F \longrightarrow A'' \otimes F$$

is exact at $A \otimes F$

Proof. As F is free, write
$$F = \bigoplus_{\alpha} \mathbb{Z}$$
, then $A \otimes F = A \otimes \bigoplus_{\alpha} \mathbb{Z} = \bigoplus_{\alpha} A$

Proposition 24. :

1. In the category $Ch \times Ab$ where Ch is the category of chain complexes and Ab is the category of abelian groups, then there is a natural transformation α

$$H_n(C_{\bullet}) \otimes A \xrightarrow{\alpha_{C_{\bullet},A}} H_n(C_{\bullet} \otimes A)$$

$$\downarrow \qquad \qquad \downarrow$$

$$H_n(C'_{\bullet}) \otimes A' \xrightarrow{\alpha_{C'_{\bullet},A'}} H_n(C'_{\bullet} \otimes A')$$

where $\alpha: H_n(C_{\bullet}) \times A \to H_n(C_{\bullet} \otimes A)$ is defined by a bilinear map as follows

$$[z], a \mapsto [z \otimes a]$$

for $z \in C_{\bullet}$, $a \in A$.

2. α is compatible with direct sum

3. if A is free abelian then α is an isomorphism. An example when α is not an isomorphism:

$$n-1$$
 n $n+1$ $n+1$
 C_{ullet} $0 \longleftarrow \mathbb{Z} \stackrel{2}{\longleftarrow} \mathbb{Z} \longleftarrow 0$
 $H(C_{ullet}) \otimes \mathbb{Z}/2$ $\mathbb{Z}/2$ 0
 $C_{ullet} \otimes \mathbb{Z}/2$ $0 \longleftarrow \mathbb{Z}/2 \stackrel{0}{\longleftarrow} \mathbb{Z}/2 \longleftarrow 0$
 $H(C_{ullet}) \otimes \mathbb{Z}/2$ $0 \longleftarrow \mathbb{Z}/2 \stackrel{0}{\longleftarrow} \mathbb{Z}/2 \longleftarrow 0$

Proof. $(\alpha: H_n(C_{\bullet}) \otimes A \to H_n(C_{\bullet} \otimes A)$ is mono)

Let C_{\bullet} be a chain complex and A be an abelian group, the natural projection $F_0 = \mathbb{Z}S \to A$ where $S \subseteq A$ generates A, $F_1 = \ker(F_0 \to A)$. Suppose C_{\bullet} is a free abelian group, we have two short exact sequences

$$0 \longrightarrow F_1 \stackrel{d}{\longrightarrow} F_0 \longrightarrow A \longrightarrow 0$$

$$0 \longrightarrow C_{\bullet} \otimes F_1 \xrightarrow{1 \otimes d} C_{\bullet} \otimes F_0 \longrightarrow C_{\bullet} \otimes A \longrightarrow 0$$

Define the natural boundary maps to make $C_{\bullet} \otimes F_1, C_{\bullet} \otimes F_0, C_{\bullet} \otimes A$ chain complexes. The short exact sequence of chain complexes induces a long exact sequence in homology

$$H_n(C_{\bullet} \otimes F_1) \xrightarrow[H_n(1 \otimes d)]{} H_n(C_{\bullet} \otimes F_0) \xrightarrow{} H_n(C_{\bullet} \otimes A)$$

$$H_{n-1}(C_{\bullet} \otimes F_1) \xrightarrow[H_{n-1}(1 \otimes d)]{} H_{n-1}(C_{\bullet} \otimes F_0) \xrightarrow{} H_{n-1}(C_{\bullet} \otimes A)$$

Hence, coker $H_n(1 \otimes d) \to H_n(C_{\bullet} \otimes A)$ is a monomorphism and $H_n(C_{\bullet} \otimes A) \to \ker H_{n-1}(1 \otimes d)$ is a epimorphism. Hence, the sequence below is exact

$$0 \longrightarrow \operatorname{coker} H_n(1 \otimes d) \longrightarrow H_n(C_{\bullet} \otimes A) \longrightarrow \ker H_{n-1}(1 \otimes d) \longrightarrow 0$$

Since $H_n(C_{\bullet})$ is abelian, we have another exact sequence

$$H_n(C_{\bullet}) \otimes F_1 \xrightarrow{1 \otimes d} H_n(C_{\bullet}) \otimes F_0 \longrightarrow H_n(C_{\bullet}) \otimes A \longrightarrow 0$$

The diagram below commutes

$$H_{n}(C_{\bullet}) \otimes F_{1} \xrightarrow{1 \otimes d} H_{n}(C_{\bullet}) \otimes F_{0} \longrightarrow H_{n}(C_{\bullet}) \otimes A \longrightarrow 0$$

$$\downarrow^{\alpha(\cong)} \qquad \downarrow^{\alpha(\cong)} \qquad \downarrow^{\alpha} \qquad \downarrow^$$

We have $H_n(C_{\bullet}) \otimes A = \operatorname{coker}(1 \otimes d) = \operatorname{coker} H_n(1 \otimes d)$. Since the sequence $0 \to \operatorname{coker} H_n(1 \otimes d) \to H_n(C_{\bullet} \otimes A) \to \ker H_{n-1}(1 \otimes d) \to 0$ is exact, then $\operatorname{coker} H_n(1 \otimes d) \to H_n(C_{\bullet} \otimes A)$ is mono. Hence $\alpha : H_n(C_{\bullet}) \otimes A \to H_n(C_{\bullet} \otimes A)$ is mono.

2.3.2 *R*-MODULE

Definition 61 (R-module). Given a commutative ring R, an R-module is an abelian group M with bilinear map $R \times M \to M$ such that

- 1m = m
- r(sm) = (rs)m

for all $r, s \in R, m \in M$

Remark 20. Some remarks on R-module

- R-module is similar to group acting on a set
- R-module is a generalization of a vector space where the field of scalars (\mathbb{R} or \mathbb{C}) is taken as a commutative ring R
- R-module is a generalization of abelian group that is every abelian group is a Z-module.

Definition 62 (free R-module). An R-module M is free if there is a subset B, namely basis, such that

- B generates M: every element of M can be written as a linear combination of elements in B
- elements of B are linearly independent: for every $\{e_1, e_2, ..., e_n\} \subseteq B$, if $r_1e_1 + r_2e_2 + ... + r_ne_n = 0$, then $r_1 = r_2 = ... = r_n = 0$

Definition 63 (tensor product of R-modules). Let M, N, L be R-modules and a bilinear map $f: M \times N \to L$ such that

$$f(rm, n) = rf(m, n) = f(m, rn)$$

Define the tensor product $M \otimes_R N$ with universal property as follows

$$Bil_R(M, N; L) = Hom_R(M \otimes_R N, L)$$

The story above works for chain complexes over R-module, that is, the sequence below is exact

$$0 \longrightarrow F_1 \longrightarrow F_0 \stackrel{\epsilon}{\longrightarrow} A \longrightarrow 0$$

if F_1 is free. In particular, if R is a principal ideal domain (PID) .

2.3.3 FUNDAMENTAL THEOREM OF HOMOLOGICAL ALGEBRA

Definition 64 (resolution, free resolution). Let \mathcal{M} be the category of R-modules (more generally, abelian category). A resolution of $M \in \text{ob } \mathcal{M}$ is a chain complex of R-modules (F_{\bullet}, d) together with a map $\epsilon : F_0 \to M$ such that the sequence below is exact

$$0 \longleftarrow M \stackrel{\epsilon}{\longleftarrow} F_0 \stackrel{d}{\longleftarrow} F_1 \stackrel{d}{\longleftarrow} F_2 \stackrel{d}{\longleftarrow} \dots$$

d is called boundary map, ϵ is called augmentation map. If F_{\bullet} are free, the sequence is called free resolution of M

Remark 21. Some remarks on resolution

• If we write free resolution as a chain map

$$0 \longleftarrow F_0 \xleftarrow{d} F_1 \xleftarrow{d} F_2 \xleftarrow{d} \dots$$

$$\downarrow \qquad \qquad \downarrow^{\epsilon} \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longleftarrow M \longleftarrow 0 \longleftarrow 0 \longleftarrow \dots$$

then ϵ is an isomorphism in homology.

• If A is a \mathbb{Z} -module (abelian group), then $0 \longleftarrow A \xleftarrow{\epsilon} \mathbb{Z}S \xleftarrow{d} \ker \epsilon \longleftarrow 0$ is the free resolution of A where $S \subseteq A$ generates A

Proposition 25. Free resolution always exists

Proof. Denote M by F_{-1} , 0 by F_{-2} . Let $S_n = \ker(F_{n-1} \to F_{n-2})$, construct $F_n = RS_n$ free R-module generated by S_n

$$F_{n-2} \longleftarrow F_{n-1} \longleftarrow F_n = RS_n$$

$$S_n = \ker(F_{n-1} \to F_{n-2})$$

ask prof why in her proof, M has a generator

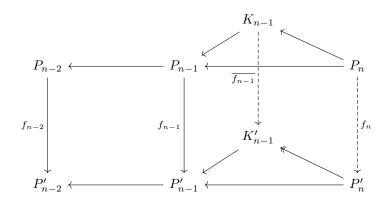
Theorem 11 (fundamental theorem of homological algebra, FTHA). Let $f: M \to M'$ be a map of R-modules and let

be such that P_n is free, the top chain is a chain complex, the bottom chain is exact. Then

- 1. for each n, there exists a chain map (lift) $f_n: P_n \to P'_n$ lifting f
- 2. any two lifts are chain homotopic

Proof.

1. We will prove the first statement by induction.



Suppose $f_{n-1}: P_{n-1} \to P'_{n-1}$ and $f_{n-2}: P_{n-2} \to P'_{n-2}$ were constructed. Let $K_{n-1} = \ker(P_{n-1} \to P_{n-2})$ and $K'_{n-1} = \ker(P'_{n-1} \to P'_{n-2})$. Since the both chains are chain complexes, there are maps $P_n \to K_{n-1}$ and $P'_n \to K'_{n-1}$. Moreover, since the bottom chain is exact, the map $P'_n \to K'_{n-1}$ is surjective. We also have the restriction of f_{n-1} on K_{n-1} by $\overline{f_{n-1}}: K_{n-1} \to K'_{n-1}$. This map is well-defined because if $x \in K_{n-1} \subseteq P_{n-1}$, x is mapped to 0 through $P_{n-1} \to P'_{n-2} \to P'_{n-2}$, by commutativity, x is also mapped to 0 through $P_{n-1} \to P'_{n-1} \to P'_{n-2}$, that is, x is mapped to K'_{n-1} through f_{n-1} .

Now, since $P'_n \to K'_{n-1}$ is surjective, there exists a map $f_n: P_n \to P'_n$ such that the diagram remains commutative. For the base case when n=0, let $P_{n-1}=M, P'_{n-1}=M', P_{n-2}=P'_{n-2}=0, f_{n-1}=f, f_{n-2}=0$

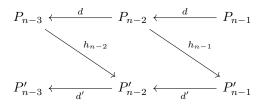
2. Let $f_{\bullet}^{(1)}, f_{\bullet}^{(2)}: P_{\bullet} \to P_{\bullet}'$ are two lifts.

$$M \leftarrow_{\epsilon} P_{\bullet}$$

$$f \downarrow \qquad f^{(1)} (\) \downarrow f^{(2)} \downarrow f^{(2)} \downarrow \qquad M' \leftarrow_{\epsilon'} P'_{\bullet}$$

We will prove that $g_{\bullet} = f_{\bullet}^{(1)} - f_{\bullet}^{(2)}$ is chain homotopic to zero, that is, find map $h_{n+1}: P_n \to P'_{n+1}$ such that d'h + hd = g

Suppose $h_{n-1}: P_{n-2} \to P'_{n-1}$ and $h_{n-2}: P_{n-3} \to P'_{n-2}$ were constructed



Consider the term below

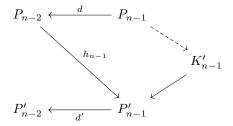
$$d'(g_{n-1} - h_{n-1}d) = d'g_{n-1} - d'h_{n-1}d$$

$$= d'g_{n-1} - (g_{n-2} - h_{n-2}d)d$$

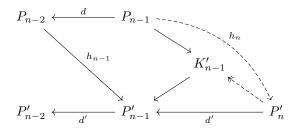
$$= d'g_{n-1} - g_{n-2}d$$

$$= 0$$
(top chain is a chain complex)
$$= 0$$

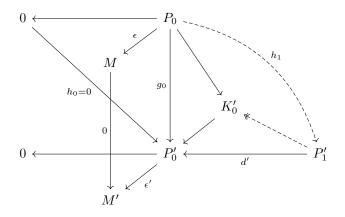
Hence, there exists a map $P_{n-1} \to K'_{n-1}$ where $K'_{n-1} = \ker(P'_{n-1} \to P'_{n-2})$ such that the diagram below commutes



Using the same argument as above, there exists a surjective map $P'_n \to K'_{n-1}$ such that the diagram below commutes. Then, $h_n: P_{n-1} \to P'_n$ is constructed from $P_{n-1} \to K'_{n-1}$ and surjectivity of $P'_n \to K'_{n-1}$.

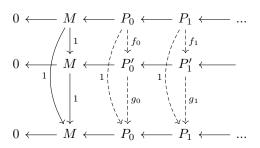


For the base case, let $K_0' = \ker \epsilon'$, the map $M \to M'$ is zero, then $g_0 : P_0 \to P_0'$ is factored through K_0' . The map $P_1 \to K_0'$ is surjective since $M' \leftarrow P_0' \leftarrow P_1'$ is exact. Hence, $h_1 : P_0 \to P_1'$ can be constructed.



Corollary 13. A free resolution of M is well-defined as a chain homotopy type.

Proof. Suppose we have two free resolutions $0 \leftarrow M \leftarrow P_{\bullet}$ and $0 \leftarrow M \leftarrow P'_{\bullet}$



By FTHA, there are three lifts $1_{\bullet}: P_{\bullet} \to P_{\bullet}, f_{\bullet}: P_{\bullet} \to P'_{\bullet}, g_{\bullet}: P'_{\bullet} \to P_{\bullet}$. As both 1_{\bullet} and $g_{\bullet}f_{\bullet}$ are both lifts $P_{\bullet} \to P_{\bullet}$. By FTHA, $g_{\bullet}f_{\bullet} \simeq 1$. Similarly, $f_{\bullet}g_{\bullet} \simeq 1$. Hence, the two free resolutions are of the same chain homotopy type.

TODO - whether free resolution of two R-modules can be chain homotopic?

Corollary 14. There exists a functor R-Mod \rightarrow Ho(Ch(R-Mod)) where Ho denotes the category of homology

Proof. Given M being an R-module, the functor is defined by the homology of the chain complex from free resolution of M. The functor is well-defined because a free resolution defines a chain homotopy type and chain homotopic chain complexes induces the same homology. Given any map between R-modules M and M', the functor is defined by the chain map between free resolutions of M and M'. The functor is well-defined because two chain homotopic maps induces the same map in homology.

2.3.4 ADDITIVE FUNCTOR, TOR FUNCTOR R-Mod \rightarrow Ab

Remark 22. Let $F: R\text{-Mod} \to Ab$ be a functor from R-module to abelian group. Let P_{\bullet} be a chain complex of R-module

$$0 \longleftarrow P_0 \longleftarrow^d P_1 \longleftarrow^d \dots$$

$$0 \longleftarrow FP_0 \xleftarrow{Fd} FP_1 \xleftarrow{Fd} \dots$$

then,

$$(Fd)(Fd) = F(dd) = F(0)$$

If F(0) = 0, then F sends a chain complex to a chain complex.

Definition 65 (additive functor R-Mod \rightarrow Ab). $F: \text{R-Mod} \rightarrow \text{Ab}$ is a additive functor if for any $M, N \in \text{ob R-Mod}$,

$$\operatorname{Hom}_R(M,N) \to \operatorname{Hom}(F(M),F(N))$$

is a homomorphism of abelian groups.

Remark 23. Some remarks on additive functor of R-Mod \rightarrow Ab

- in the categories R-Mod or Ab, Hom is an abelian groups. More generally, they are preadditive categories or Abenriched categories where Hom is equipped with abelian group structure and composition is bilinear.
- if $F : R\text{-Mod} \to Ab$ is additive, then F sends a chain complex to a chain complex, that is, there is an extension $Ch(R\text{-Mod}) \to Ch(Ab)$
- tensor product is an additive functor.

Lemma 12. If F is additive and h is a chain homotopy $f_0 \simeq f_1 : C_{\bullet} \to C'_{\bullet}$ in R-Mod, then F(h) is a chain homotopy $F(f_0) \simeq F(f_1) : F(C_{\bullet}) \to F(C'_{\bullet})$

Proof. Suppose $d'h + hd = f_0 - f_1$ is a chain homotopy. Then

$$F(d')F(h) + F(h)F(d) = F(d'h) + F(hd)$$
 (F is a functor)
 $= F(d'h + hd)$ (F is an additive functor)
 $= F(f_0 - f_1)$
 $= F(f_0) - F(f_1)$ (F is an additive functor)

Corollary 15. Chain homotopy type of $F(P_{\bullet})$ is well-defined (where $M \leftarrow P_{\bullet}$ is a free resolution)

Remark 24. TODO

Definition 66 (Tor functor). Let R be a ring and M be R-modules, define the Tor functor as the homology of chain complex $P_{\bullet} \otimes_R N$

$$\operatorname{Tor}_n^R(M,N) = H_n(P_{\bullet} \otimes_R N)$$

for $M \leftarrow P_{\bullet}$ is any free resolution.

Remark 25. Some remarks on Tor

- $\operatorname{Tor}_0^R(M,N) = M \otimes_R N$
- Some example of Tor TODO

Proposition 26. Tor is well-defined and functorial in both variables, that is, $\operatorname{Tor}_n^R(M,-)$ and $\operatorname{Tor}_n^R(-,N)$ are functors $R\operatorname{-Mod} \to \operatorname{Ab}$.

2.3.5 UNIVERSAL COEFFICIENT THEOREM

Theorem 12 (universal coefficient theorem). Let R be a PID and N be an R-module, then for any chain complex C_{\bullet} , then there is a short exact sequence

$$0 \longrightarrow H_n(C_{\bullet}) \otimes_R N \stackrel{\alpha}{\longrightarrow} H_n(C_{\bullet} \otimes_R N) \longrightarrow \operatorname{Tor}_1^R(H_{n-1}(C_{\bullet}), N) \longrightarrow 0$$

Proof. Since R is a PID, N admits a free resolution

$$0 \longleftarrow N \longleftarrow P_0 \xleftarrow{d} P_1 \longleftarrow 0$$

By exactness of tensor product, the sequence below is exact

$$0 \longleftarrow C_{\bullet} \otimes_{R} N \longleftarrow C_{\bullet} \otimes_{R} P_{0} \stackrel{1 \otimes d}{\longleftarrow} C_{\bullet} \otimes_{R} P_{1} \longleftarrow 0$$

The short exact sequence induces a long exact sequence

$$H_n(C_{\bullet} \otimes_R P_1) \xrightarrow{H_n(1 \otimes_R d)} H_n(C_{\bullet} \otimes_R P_0) \longrightarrow H_n(C_{\bullet} \otimes_R N) \longrightarrow H_{n-1}(C_{\bullet} \otimes_R P_1) \xrightarrow{H_{n-1}(1 \otimes_R d)} H_{n-1}(C_{\bullet} \otimes_R P_0)$$

That induces a short exact sequence

$$0 \longrightarrow \operatorname{coker} H_n(1 \otimes_R d) \longrightarrow H_n(C_{\bullet} \otimes_R N) \longrightarrow \ker H_{n-1}(1 \otimes_R d) \longrightarrow 0$$

As discussed earlier, coker $H_n(1 \otimes_R d) = H_n(C_{\bullet}) \otimes_R N$. Furthermore, we can identify $\ker H_n(1 \otimes_R d)$ by $\ker(1 \otimes_R d)$

$$H_{n-1}(C_{\bullet}) \otimes_{R} P_{1} \xrightarrow{1 \otimes_{R} d} H_{n-1}(C_{\bullet}) \otimes_{R} P_{0}$$

$$\downarrow^{\alpha(\cong)} \qquad \qquad \downarrow^{\alpha(\cong)}$$

$$H_{n-1}(C_{\bullet} \otimes_{R} P_{1}) \xrightarrow{H_{n-1}(1 \otimes_{R} d)} H_{n-1}(C_{\bullet} \otimes_{R} P_{0})$$

And $\ker(1 \otimes_R d) = \operatorname{Tor}_1^R(H_{n-1}(C_{\bullet}), N)$ by exactness of tensor product, the sequence below is exact

$$0 \longleftarrow H_{n-1}(C_{\bullet}) \otimes_R N \longleftarrow H_{n-1}(C_{\bullet}) \otimes_R P_0 \stackrel{1 \otimes d}{\longleftarrow} H_{n-1}(C_{\bullet}) \otimes_R P_1 \longleftarrow 0$$

Remark 26. Let $C_{\bullet} = C_{\bullet}(X; R) = C_{\bullet}(X) \otimes_{\mathbb{Z}} R$, then

$$0 \longrightarrow H_n(X;R) \otimes_R N \stackrel{\alpha}{\longrightarrow} H_n(X;N) \longrightarrow \operatorname{Tor}_1^R(H_{n-1}(X;R),N) \longrightarrow 0$$

2.3.6 KÜNNETH THEOREM

Definition 67 (tensor product of chain complexes). Let C_{\bullet} , D_{\bullet} be chain complexes of R-module. Define the tensor product $C_{\bullet} \otimes D_{\bullet}$ by

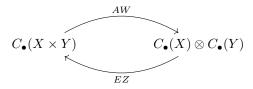
$$(C_{\bullet} \otimes D_{\bullet})_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and define the boundary map $\partial: (C_{\bullet} \otimes D_{\bullet})_n \to (C_{\bullet} \otimes D_{\bullet})_{n-1}$ by a extension of $\partial: C_p \otimes D_q \to (C_{\bullet} \otimes D_{\bullet})_{n-1}$

$$\partial(x \otimes y) = \partial x \otimes y + (-1)^p x \otimes \partial y$$

where $x \in C_p, y \in D_q$

Theorem 13 (Eilenberg-Zilber theorem). The Alexander-Whitney (AW) map and Eilenberg-Zilber (EZ) map are natural transformation of functors $\text{Top}^2 \to \text{Ch}(\text{Ab})$ which are naturally chain homotopy inverses



TODO: method of proof

Remark 27. Two chain complexes of functors $\mathrm{Top}^2 \to \mathrm{Ab}$

$$(X,Y) \mapsto C_n(X \times Y)$$

 $(X,Y) \mapsto (C_{\bullet}(X) \otimes C_{\bullet}(Y))_n$

Theorem 14. There is a natural short exact sequence

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C_{\bullet}) \otimes_R H_q(D_{\bullet}) \longrightarrow H_n(C_{\bullet} \otimes D_{\bullet}) \longrightarrow \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R(H_p(C_{\bullet}), H_q(D_{\bullet})) \longrightarrow 0$$

if R is a PID, C_{ullet} degree-wise free chain complex and this sequence splits

Corollary 16. Let R be a PID and $C'_{\bullet}, C_{\bullet}$ degree-wise R-free. If $C'_{\bullet} \to C_{\bullet}$ and $D'_{\bullet} \to D_{\bullet}$ are homology isomorphisms, so is $C'_{\bullet} \otimes D'_{\bullet} \to C_{\bullet} \otimes D_{\bullet}$

Chapter 3

OUT OF PLACE DISCUSSION

3.1 (BROUWER) DEGREE OF $f: S^n \to S^n$ ()