MA5204 Homework 5

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Problem 1 (prime avoidance lemma)
Let A be a ring and let $I, J_1, J_2, ..., J_n \subseteq A$ be ideals such that at most two of $J_1, J_2, ..., J_n$ are not prime. If $I \nsubseteq J_i$ for all i, prove that $I \nsubseteq \bigcup_{1 \leq i \leq n} J_i$

Proof. We will prove by induction on n

base case: $n = 1$, clear

induction case: $n \geq 2$

Suppose that $I \subseteq \bigcup_{1 \leq i \leq n} J_i$. By induction hypothesis, for every i, we can pick an element x_i so that

$$x_i \in I \text{ and } x_i \notin \bigcup_{k \neq i} J_k$$

Assumption $I \subseteq \bigcup_{1 \leq i \leq n} J_i$ implies $x_i \in J_i$. Let

$$y = x_1 ... x_{n-1} + x_n \in I \subseteq \bigcup_{1 \leq i \leq n} J_i$$

subcase 1: $n = 2$, then $y = x_1 + x_2 \in J_1 \cup J_2$. Without loss of generality, let $y \in J_1$, then $x_2 = y - x_1 \in J_1$, this is a contradiction by construction of x_2.

subcase 2: $n \geq 3$. Without loss of generality, let J_n be prime. If $y \in J_n$, then $x_1 ... x_{n-1} = y - x_n \in J_n$, then at least one of $x_1, ..., x_{n-1}$ is an element of J_n which is a contradiction. If $y \in J_i$ for some $1 \leq i \leq n-1$, then $x_n = y - x_1 ... x_{n-1} \in J_i$ which is also a contradiction □

Problem 2
Problem consists of two parts

1. Let $A \hookrightarrow B$ be an integral ring extension of Noetherian rings. Prove that $\dim B = \dim A$

2. Let k be a field and let A be a finitely generated k-algebra. Recalled that we prove *Noetherian normalization* which say we can find an injective map $\phi : k[X_1, ..., X_n] \hookrightarrow A$ such that A is finitely generated as a $k[x_1, ..., x_n]$-module where $x_i = \phi(X_i)$. Prove that $\dim A = n$, so n is uniquely determined by A

Proof of part 1.
Since both A and B are Noetherian, any ascending chain of primes of strict inclusions must be finite

- $(\dim A \leq \dim B)$

  For any finite chain of primes $\mathfrak{p}_0 \subsetneq ... \subsetneq \mathfrak{p}_n$ in A, by lying over theorem for integral ring extension, pick $\mathfrak{q}_0$ in B so that $\mathfrak{q}_0 \cap A = \mathfrak{p}_0$. Going-up theorem induces a chain of primes $\mathfrak{q}_0 \subseteq ... \subseteq \mathfrak{q}_n$ in B. Moreover, $\mathfrak{q}_i = \mathfrak{q}_{i+1}$ implies $\mathfrak{p}_i = \mathfrak{p}_{i+1}$. Hence, the chain of primes in B is also of strict inclusions. Hence, $\dim A \leq \dim B$

- $(\dim B \leq \dim A)$

  For any finite chain of primes $\mathfrak{q}_0 \subsetneq ... \subsetneq \mathfrak{q}_n$ in B, let $\mathfrak{p}_i = \mathfrak{q}_i \cap A$, then $\mathfrak{p}_0 \subseteq ... \subseteq \mathfrak{p}_n$ is a chain of primes in A. By AM Corollary 5.9, $\mathfrak{p}_i = \mathfrak{p}_{i+1}$ implies $\mathfrak{q}_i = \mathfrak{q}_{i+1}$. Hence the chain of primes in A is also of strict inclusions, so $\dim B \leq \dim A$

□

Lemma 3

Given a ring extension $A \hookrightarrow B$, let $x \in B$, the following are equivalent:

1. x is integral over A
2. the ring $A[x] \subseteq B$ is finitely generated A-module
3. $A[x]$ is contained in a subring C of B such that C is also a finitely generated A-module
4. there exists a faithful $A[x]$-module M which is finitely generated as an A-module An R-module M is faithful if and only if $\operatorname{ann}_R(M) = \{r \in R : rM = 0\} = 0$ if and only if $R \to \operatorname{Hom}_R(M, M)$ is injective.

Proof of part 2. Since A is finitely generated as an $k[x_1, ..., x_n]$-module, for any $x \in A$, $k[x_1, ..., x_n][x] \subseteq A$ is also finitely generated as an $k[x_1, ..., x_n]$-module. By 2, x is integral over $k[x_1, ..., x_n]$. Therefore, $\phi : k[X_1, ..., X_n] \hookrightarrow A$ is an integral ring extension. By part 1, $\dim A = \dim k[X_1, ..., X_n] = n$ □

Problem 4

Let $\mathfrak{p}$ be a prime ideal in a Noetherian ring A. Prove that $\operatorname{ht}\mathfrak{p} + \dim A/\mathfrak{p} \leq \dim A$

Proof. Since A is Noetherian, then $A/\mathfrak{p}$ is also Noetherian, let $\mathfrak{p}_0 \subsetneq ... \subsetneq \mathfrak{p}_n = \mathfrak{p}$ be a finite chain of primes in A and $\mathfrak{q}_0 \subsetneq ... \subsetneq \mathfrak{q}_m$ be a finite chain of primes in $A/\mathfrak{p}$. Let $\tilde{\mathfrak{q}}_i \subseteq A$ be a lift of $\mathfrak{q}_i$, then we have the chain of primes in A of length at least $n + m$

$$\mathfrak{p}_0 \subsetneq ... \subsetneq \mathfrak{p}_n = \mathfrak{p} \subseteq \tilde{\mathfrak{q}}_0 \subsetneq ... \subsetneq \tilde{\mathfrak{q}}_m$$

Then, $\operatorname{ht}\mathfrak{p} + \dim A/\mathfrak{p} \leq \dim A$ □

Problem 5

Notation as in 3, find an example with $\dim A = 2$, $\operatorname{ht}\mathfrak{p} = 1$ and $\dim A/\mathfrak{p} = 0$. In particular, deduce that the inequality in 3 can be strict.

Proof. Consider the ideal $\mathfrak{a} = (zx, zy) = (z)(x, y)$ in $\mathbb{C}[x, y, z]$, let $A = \mathbb{C}[x, y, z]/\mathfrak{a}$. Note that (z) and (x, y) are the only minimal primes over $\mathfrak{a}$ in $\mathbb{C}[x, y, z]$, hence

$$\dim A = \max\left\{\dim \frac{\mathbb{C}[x, y, z]}{(z)}, \dim \frac{C[x, y, z]}{(x, y)}\right\} = 2$$

Let $\mathfrak{m} = (z - 1, x, y) \supseteq \mathfrak{a}$ be a maximal prime in $\mathbb{C}[x, y, z]$ which is also maximal in A, then

$$A/\mathfrak{m} = \frac{\mathbb{C}[x, y, z]}{(xz, zy, z - 1, x, y)} = \frac{\mathbb{C}[x, y, z]}{(z - 1, x, y)} \cong \mathbb{C}$$

Hence, $\dim A/\mathfrak{m} = 0$. It remains to show that $\operatorname{ht}_A \mathfrak{m} = 1$. Since (z) and (x, y) are the only minimal primes over $\mathfrak{a}$ in $\mathbb{C}[x, y, z]$, every prime $\mathfrak{p}$ in A lifted to $\mathbb{C}[x, y, z]$ must either contain (z) or (x, y). If $\mathfrak{p} \subseteq \mathfrak{m}$ in A, because $(z) \nsubseteq \mathfrak{m}$, then $\mathfrak{p}$ must contain (x, y). Hence,

$$\operatorname{ht}_A \mathfrak{m} = \operatorname{ht}_{\mathbb{C}[x,y,z]/(x,y)} \mathfrak{m} = \operatorname{ht}_{\mathbb{C}[z]}(z - 1) = 1$$

□

Problem 6

Problem consists of three parts

1. Let A be a Noetherian ring and let $x \in A$ be a non zero-divisor. Prove that $\dim A/(x) \leq \dim A - 1$
2. Prove that if moreover A is a local ring and x is not a unit, then equality holds in 1
3. Deduce that if A is any local Noetherian ring, then $\operatorname{depth}_A A \leq \dim A$

Lemma 7

If A is Noetherian and $\mathfrak{p} \subseteq A$ be a minimum prime, then any element of $\mathfrak{p}$ is a zero divisor.

Proof of part 1. (Assuming $0 < \dim A < +\infty$) Given any maximal chain of primes in $A/(x)$ of strict inclusions

$$\mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_n$$

the chain is lifted into a chain of primes in A of strict inclusions

$$\tilde{\mathfrak{p}}_0 \subsetneq \ldots \subsetneq \tilde{\mathfrak{p}}_n$$

Since $x \in \tilde{\mathfrak{p}}_0$ is a non zero-divisor, hence $\tilde{\mathfrak{p}}_0$ is not a minimal prime, we can extend the lifted chain in A by at least one. Hence

$$\dim A/(x) \leq \dim A - 1$$

$\square$

Lemma 8

Let $(A, \mathfrak{m})$ be a Noetherian local ring and $x_1, \ldots, x_m$ be some elements in $\mathfrak{m}$ with $m \leq \dim A$, then the following are equivalent

1. $x_1, \ldots, x_m$ can be extended into a system of parameters
2. $\dim A/(x_1, \ldots, x_m) \leq \dim A - m$
3. $\dim A/(x_1, \ldots, x_m) = \dim A - m$

Proof of part 2. (Assuming $0 < \dim A < +\infty$) Since x is not a unit, $x \in \mathfrak{m}$. By part 1 and Lemma 8 ($2 \implies 3$), $\dim A/(x) = \dim A - 1$ $\square$

Proof of part 3. We will prove by induction

base case: If $\dim A = 0$, since A is Noetherian local of dimension 0, A is Artinian local, so $\mathfrak{m}$ is nilpotent, that is, element every element of A must be either a unit or a zero-divisor. We need to show that $\operatorname{depth}_A A = 0$, that is there is no A-regular sequence in A. Suppose there is an A-regular sequence $x_1, \ldots, x_r$, then x_1 cannot be unit since $A/(x_1, \ldots, x_r)A \neq 0$. Moreover, x_1 cannot be zero divisor since $x_1 : A \to A$ is injective. So, no A-regular sequence exists in A, hence $\operatorname{depth}_A A = 0$

induction case: when $\dim A = n > 0$, if $\operatorname{depth}_A A = 0$, we are done. If $\operatorname{depth}_A A = r > 0$, pick a maximal A-regular sequence $a, a_2, \ldots, a_r \in A$ of length r. Note that,

$$\frac{A/(a)}{(a_2, \ldots, a_i)A/(a)} = \frac{A}{(a, a_2, \ldots, a_i)} \neq 0$$

and each $a_i : A/(a, ..., a_i) \to A/(a, ..., a_i)$ injective imply $\bar{a}_2, ..., \bar{a}_r \in A/(a)$ is a $A/(a)$-regular sequence of length $r-1$, then

$$\operatorname{depth}_A A - 1 \leq \operatorname{depth}_{A/(a)} A/(a)$$

Since a is not a zero divisor, $\dim A/(a) \leq \dim A - 1 = n - 1$, by induction hypothesis for $A/(a)$

$$\operatorname{depth}_{A/(a)} A/(a) \leq \dim A/(a) \leq \dim A - 1$$

Hence,

$$\operatorname{depth}_A A \leq \dim A$$

□

Problem 9

Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring. In class we define $\operatorname{embdim} A = \dim_k \mathfrak{m}/\mathfrak{m}^2$ and proved that $\dim A \leq \operatorname{embdim} A$. Prove that *every* pair of nonnegative integers (r, s) with $r \leq s$ occurs at $(\dim A, \operatorname{embdim} A)$ for some A

Proof. We construct ring $A = \mathbb{C}[x_1, ..., x_s]_{(x_1,...,x_s)}$ with $(\dim A, \operatorname{embdim} A) = (s, s)$ and ring $B = \frac{A}{(x_1^2,...,x_{s-r}^2)}$ with $(\dim B, \operatorname{embdim} B) = (r, s)$ for every pair $r \leq s$

1. (s, s)

   Consider $\mathbb{C}[x_1, ..., x_s]$ of dimension s, $\mathfrak{m} = (x_1, ..., x_s)$ is a maximal ideal of maximal height. Localizing at $\mathfrak{m}$ gives $A = \mathbb{C}[x_1, ..., x_s]_\mathfrak{m}$ and A is also of dimension s. Since $\mathfrak{m}$ is generated by s elements, then $\dim A \leq \operatorname{embdim} A \leq s = \dim A$. So, $\operatorname{embdim} A = \dim A = s$

2. (r, s) for $r \leq s$

   Consider

   $$I_i = (x_1^2, ..., x_i^2)$$

   Note that, x_i^2 is not a zero divisor in $A/I_{i-1} = (\mathbb{C}[x_1, ..., x_s]/I_{i-1})_\mathfrak{m}$ because if a nonzero element $\bar{f}(x)/\bar{g}(x) \in (\mathbb{C}[x_1, ..., x_s]/I_{i-1})_\mathfrak{m}$ for some $f(x) \in \mathbb{C}[x_1, ..., x_s], g(x) \in \mathbb{C}[x_1, ..., x_s] - \mathfrak{m}$ satisfies

   $$x_i^2 \frac{f(x)}{g(x)} = 0 \in A/I_{i-1}$$

   Then, $x_i^2 f(x)h(x) \in I_{i-1}$ for some $h(x) \in \mathbb{C}[x_1, ..., x_s] - \mathfrak{m}$. But $\bar{f}(x)/\bar{g}(x)$ being nonzero implies $f(x)h(x) \notin I_{i-1}$, this is a contradiction.

   Moreover, x_i^2 is not a unit in $A/I_{i-1} = (\mathbb{C}[x_1, ..., x_s]/I_{i-1})_\mathfrak{m}$ because if a nonzero element $\bar{f}(x)/\bar{g}(x) \in (\mathbb{C}[x_1, ..., x_s]/I_{i-1})_\mathfrak{m}$ for some $f(x) \in \mathbb{C}[x_1, ..., x_s], g(x) \in \mathbb{C}[x_1, ..., x_s] - \mathfrak{m}$ satisfies

   $$x_i^2 \frac{f(x)}{g(x)} = 1 \in A/I_{i-1}$$

   Then, $x_i^2 f(x)h(x) - g(x)h(x) \in I_{i-1}$ some $h(x) \in \mathbb{C}[x_1, ..., x_s] - \mathfrak{m} = \mathbb{C}$, this is a contradiction.

   Now, for each i, from Problem 6, x_i^2 is not a zero divisor and not a unit in $\dim A/I_{i-1}$, then

   $$\dim A/I_i = \dim A/I_{i-1} - 1$$

Let $B = A/I_{s-r}$, then

$$\dim B = \dim A/I_{s-r} = \dim A - s + r = r$$

We want to show that $\operatorname{embdim} B = \operatorname{embdim} A = s$, let $\mathfrak{m}_B$ denote the projection of $\mathfrak{m}$ under the map $A \twoheadrightarrow B$ and $k_B = B/\mathfrak{m}_B$. Note that, $I_{s-r} \subseteq \mathfrak{m}^2 \subseteq \mathfrak{m}$, then $\mathfrak{m}_B/\mathfrak{m}_B^2 = \mathfrak{m}/\mathfrak{m}^2$ and $k_B = B/\mathfrak{m}_B = A/\mathfrak{m} = k$ (third isomorphism theorem). So

$$\operatorname{embdim} B = \dim_{k_B} \mathfrak{m}_B/\mathfrak{m}_B^2 = \dim_k \mathfrak{m}/\mathfrak{m}^2 = \operatorname{embdim} A = s$$

□

Problem 10

Let A be a Noetherian ring. Prove that $\dim A[[x]] = \dim A + 1$

Proof. $x \in A[[x]]$ is not a zero-divisor and not a unit in the Noetherian ring $A[[x]]$, then $A \cong \frac{A[[x]]}{(x)}$ implies

$$\dim A = \dim \frac{A[[x]]}{(x)} = \dim A[[x]] - 1$$

□

Problem 11

Let k be a field and A, B be finitely generated k-algebras, so then $A \otimes_k B$ is also a finitely generated k-algebra. Prove that $\dim A \otimes_k B = \dim A + \dim B$

Lemma 12 (Noether normalization theorem)

Let k be a field and A be a finite-type k-algebra, then there exists $x_1, ..., x_n \in A$ so that

$$\begin{aligned} \phi : k[X_1, ..., X_n] &\hookrightarrow A \\ X_i &\mapsto x_i \end{aligned}$$

and A is finitely generated as a module over the image or equivalently A is integral over $k[x_1, ..., x_n]$. Moreover, $x_1, ..., x_n$ are algebraically independent over k, that is, x_i does not satisfy any nontrivial polynomial equation with coefficients in k

Lemma 13

Given an A-module M, M is flat if and only if $I \otimes M \to A \otimes M \xrightarrow{\sim} M$ induced from $I \hookrightarrow A$ is injective. In particular, if $A = k$ is a field, then every k-module is flat.

Proof. By Noether normalization theorem, and Problem 2, we have $\dim A = n, \dim B = m$ and injective maps

$$\phi_A : k[X_1, ..., X_n] \hookrightarrow A$$
$$\phi_B : k[Y_1, ..., Y_m] \hookrightarrow B$$

Every k-module is flat, $k[Y_1, ..., Y_m]$ is flat,

$$k[X_1, ..., X_n, Y_1, ..., Y_m] = k[X_1, ..., X_n] \otimes k[Y_1, ..., Y_m] \hookrightarrow A \otimes k[Y_1, ..., Y_m]$$

is injective. Every k-module is flat, A is flat

$$A \otimes k[Y_1, ..., Y_m] \hookrightarrow A \otimes B$$

is injective. Hence, the composition is injective

$$\phi : k[X_1, ..., X_n, Y_1, ..., Y_m] \hookrightarrow A \otimes B$$

Since each A and B is finitely generated as a module over $\phi(k[X_1, ..., X_n])$ and $\phi(k[Y_1, ..., Y_m])$ respectively. Hence, $A \otimes B$ is finitely generated over $\phi(k[X_1, ..., X_n, Y_1, ..., Y_m])$. By Problem 2, $\dim A \otimes B = n + m$ □

Problem 14

Let $(A, \mathfrak{m})$ be a Noetherian local ring, let $\hat{A}$ be the $\mathfrak{m}$-adic completion of A. Prove that $\dim A = \dim \hat{A}$

Lemma 15 (Matsumura CRT p63 - some results for local Noetherian ring)

Let $(A, \mathfrak{m})$ be a local Noetherian ring, then

1. $\bigcap_{n \geq 1} \mathfrak{m}^n = \ker(\psi : A \to \hat{A}) = 0$
2. For M a finitely generated A-module and $N \subseteq M$ a submodule

$$\bigcap_{n \geq 1} (N + \mathfrak{m}^n M) = N$$

3. The completion $\hat{A}$ of A is faithfully flat over A; hence $A \subseteq \hat{A}$ and $I\hat{A} \cap A = I$ for any ideal I of A
4. $\hat{A}$ is again a Noetherian local ring, with maximal ideal $\mathfrak{m}A$ and it has the same residue class field as A; moreover, $\hat{A}/\mathfrak{m}^n\hat{A} = A/\mathfrak{m}^n$ for all $n \geq 1$
5. If A is a complete local ring, the for any ideal $I \neq A$, A/I is afgain a complete local ring.

Lemma 16 (result from lecture April 4)

A local homomorphism of Notherian rings $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ is a ring map $\phi : A \to B$ so that $\mathfrak{m} = \mathfrak{n} \cap A$, then

$$\dim B \leq \dim A + \dim B/\mathfrak{p}B$$

if the map is flat, then the equality holds.

Proof. By Lemma 15, $A \to \hat{A}$ is a flat ring extension of Noetherian local rings. By Lemma 16 and $\mathfrak{m}\hat{A}$ is maximal in $\hat{A}$

$$\dim \hat{A} = \dim A + \dim \hat{A}/\mathfrak{m}\hat{A} = \dim A$$

□