ma5205 - measure and integral

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Chapter 1

Measurable Space

1.1 Measurable Space

In $\mathbb{R}$, we define the volume of an interval to be the length of it. That is, if a closed [1] interval is of the form $I = [a, b]$, then its length is $b - a$. Additionally, we want several properties such as: *additivity*, i.e. if a set is a disjoint union of intervals then the volume of that set is the sum of volumes of the intervals, *complement*, i.e. if a set has volume then the complement also has. It leads to the definitions of σ-algebra and measure.

Definition 1 (σ-algebra) *Let X be an arbitrary set, M is said to be a σ-algebra on X if M consists of subsets of X with properties as follows:*

- $\varnothing, X \in M$
- $U \in M \implies X \setminus U \in M$ *(complement)*
- $U_1, U_2, ... \in M \implies \bigcup_{k \in \mathbb{N}} U_k \in M$ *(countable union[2])*

Definition 2 (Measurable Space) *Let X be an arbitrary set and a σ-algebra M defined on X. A function $\mu : M \to [0, +\infty]$ is called measure if $\mu(\varnothing) = 0$ and given a countable collection of disjoint sets $U_1, U_2, ... \in M$ then $\mu(\bigcup_{k \in \mathbb{N}} U_k) = \sum_{k \in \mathbb{N}} \mu(U_k)$. The triplet (X, M, μ) is called a measurable space. A set $U \in M$ is called a measurable set.*

1.2 Basis of a Measurable Space

Given an arbitrary collection of σ-algebras, the union and intersection are also σ-algebras. That enables us to talk about σ-algebra generated by a basis.

Definition 3 (σ-algebra generated by basis) *Let X be an arbitrary set and B be a collection of subsets of X. Define $\sigma(B)$ to be the smallest σ-algebra containing B, that is the intersection of all σ-algebra containing B. $\sigma(B)$ is also said to be the σ-algebra generated by B.*

Definition 4 (Basis of a σ-algebra) *Let X be an arbitrary set and M be a σ-algebra on X. A subset B of M is said to be a basis of M if $M = \sigma(B)$*

As discussed earlier, to define volume of a set in $\mathbb{R}$, we will construct a σ-algebra consisting all open intervals on $\mathbb{R}$ as follows

Definition 5 (Borel σ-algebra or Borel algebra) *Let $(X, \mathcal{T})$ be a topological space, the Borel σ-algebra on X, denoted by $\mathscr{B}(X)$ is the smallest σ-algebra defined on X containing all open sets.*

The Borel Measurable Space is then defined as

Definition 6 (Borel Measurable Space on $\mathbb{R}^n$) *$(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \mu)$ is the Borel Measurable Space on $\mathbb{R}^n$ if μ is defined by the volume of n-intervals.*

$$\mu\left(\prod_{k=1}^{n} [a_k, b_k]\right) = \prod_{k=1}^{n} (b_k - a_k)$$

[1] similarly for other types of interval
[2] or σ-union

1.3 Subspace of a Measurable Space

Definition 7 (Subspace of a Measurable Space) *Let (X, M, μ) be a measurable space and $E \in M$. Define the subspace (E, M_E, μ_E) by the restriction of M and μ as follows*

- $M_E = \{U \in M : U \subseteq E\}$
- $\mu_E(U) = \mu(U)$ *if* $U \in M_E$

1.4 Complete Measurable Space

I haven't understood enough to give a reason for complete measurable space

Definition 8 (Outer Measure) *Let (X, M, μ) be a measurable space. Define outer measure $\mu_e : \mathcal{P}(X) \to [0, +\infty]$ by the infimum of the measure of all measurable covers*

$$\mu_e(E) = \inf \{\mu(S) : E \subseteq S \in M\}$$

Outer Measure satisfies several properties

- $A \subseteq B \implies |A|_e \leq |B|_e$ (subset)
- $\left|\bigcup_{k\in\mathbb{N}} E_k\right| \leq \sum_{k\in\mathbb{N}} |E_k|$ (σ-subadditivity)

Definition 9 (Complete Measurable Space)

Definition 10 (Completion of a Measurable Space)

Definition 11 (Lebesgue Measurable Space) [3] *Lebesgue Measurable Space on $\mathbb{R}^n$ is defined as the completion of the Borel Measurable Space on $\mathbb{R}^n$*

1.5 Product Measure

Definition 12 (Product Measurable Space) *Let (X, M, μ) and (Y, N, ν) be measurable spaces. Define the product measurable space as $(X \times Y, \sigma(M \times N), \mu \times \nu)$ where the product measure $\mu \times \nu$ is defined as*

$$(\mu \times \nu)(E_X \times E_Y) = \mu(E_X)\nu(E_Y)$$

where $E_X \in M$ and $E_Y \in N$

1.6 Image Measure

[3] can also defined as the unique non-trivial measurable space on $\mathbb{R}^n$ that translation invariant

Chapter 2

Measurable Function

Definition 13 (Measurable Function) *Let (X, M, μ) and (Y, N, ν) be measurable spaces. A function $f : X \to Y$ is said to be measurable if*

$$E_Y \in N \implies f^{-1}(E_Y) \in M$$

In this text, we are particularly interested in the case where $Y = [0, +\infty]$ as it is the foundation to define Lebesgue integral. Let's first define a topology on the totally ordered set $[0, +\infty]$

Definition 14 (Order Topology) *Given a totally ordered set $(X, <)$, the order topology $\mathcal{T}$ on X is defined by the basis consists of rays as follows*

$$B = \{\{x :\in X : x < a\} : a \in X\} \cup \{\{x :\in X : b < x\} : b \in X\}$$

The Borel σ-algebra on $[0, +\infty]$ is now defined as follows

Definition 15 (Measurable Function on Half Extended Real Line) *Let $[0, +\infty]$ be equipped with the order topology. Define the Borel measurable space on $[0, +\infty]$ by $([0, +\infty], \mathscr{B}([0, +\infty]), \nu)$ where ν is defined by the length of intervals. Let (X, M, μ) be a measurable space. A function $f : X \to [0, +\infty]$ is said to be measurable if it is a measurable function to the Borel measurable space on $[0, +\infty]$.*

From now on, we will call *Measurable Function on Half Extended Real Line* by *Measurable Function* if there is no confusion. It is trivial to verify the proposition below.

Proposition 1 *Let (X, M, μ) be a measurable space. A function $f : X \to [0, +\infty]$ is measurable if and only if*

$$f^{-1}(a, +\infty] \in M$$

for all $a \in [0, +\infty]$

Chapter 3

Integration

3.1 Simple Function

Definition 16 (Indicator function) *Let $E \subseteq X$, the indicator function of E denoted by $\chi_E : X \to [0, +\infty]$ is defined by*

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

Definition 17 (Non-negative simple function) *A function is said to be simple if the set image is finite set. A non-negative simple function $\phi : X \to [0, +\infty]$ can be written as*

$$\phi = \sum_{n=1}^{N} a_n \chi_{E_n}$$

where $N \in \mathbb{N}$, $a_n \in [0, +\infty]$, $\{E_n\}$ are pairwise disjoint, and $X = \bigcup_{n=1}^{N} E_n$

Definition 18 (Integral of a non-negative simple function) *If $\phi = \sum_{n=1}^{N} a_n \chi_{E_n}$ is a non-negative simple function where E_n are measurable w.r.t some measurable space (X, M, μ). We define the integral of ϕ as follows*

$$\int_X \phi = \sum_{n=1}^{M} a_n \mu(E_n)$$

Theorem 1 (Monotone sequence of simple functions) *Given any $f : \mathbb{R}^n \to [0, +\infty]$, there exists a monotone increasing sequence of simple functions $\phi_k : \mathbb{R}^n \to [0, +\infty]$ that converges pointwise to f*

3.2 Integral of a non-negative function

Definition 19 (Integral of a non-negative function) *Let (X, M, μ) be a measurable space and $f : X \to [0, +\infty]$ be measurable. The integral of f is defined as*

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu : g \leq f \right\}$$

Definition 20 (Lebesgue integral of a non-negative function) *Let $(\mathbb{R}^n, M, \mu)$ be the Lebesgue measurable space on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to [0, +\infty]$ be measurable. The Lebesgue integral of f is defined as*

$$\int_{\mathbb{R}^n} f d\mu = \sup \left\{ \int_{\mathbb{R}^n} \phi d\mu : \phi \leq f, \phi \text{ is simple} \right\}$$

Proposition 2 (Area under the graph) *Let $(\mathbb{R}^n, M, \mu)$ be the Lebesgue measurable space on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to [0, +\infty]$ be measurable. Define the set under f as follows*

$$R(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq y < f(x)\}$$

Then,

$$\mu(R(f)) = \int_{\mathbb{R}^n} f d\mu$$

Definition 21 (Integral of arbitrary function) *Let $(\mathbb{R}^n, M, \mu)$ be the Lebesgue measurable space on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to [-\infty, +\infty]$ be measurable. The Lebesgue integral of f is defined as*

$$\int_{\mathbb{R}^n} f d\mu = \int_{\mathbb{R}^n} f^+ d\mu - \int_{\mathbb{R}^n} f^- d\mu$$

if either $\int_{\mathbb{R}^n} f^+ d\mu$ and $\int_{\mathbb{R}^n} f^- d\mu$ is finite.

Proposition 3 (Linearity of Lebesgue integral)

3.3 Convergence Theorems of a non-negative function

Proposition 4 (Monotone Convergence Theorem)

Proposition 5 (Fatou Lemma)

Theorem 2 (Lebesgue Dominated Convergence Theorem)

3.4 Product Measure Theorems

Proposition 6 (Sectional Property) *Let $(X, M_X, \mu_X), (Y, M_Y, \mu_Y)$ be measurable spaces. Given any $E \in \sigma(M_X \times M_Y)$. For almost every $x \in X$, the set $E_x = \{y \in Y : (x, y) \in E\} \in M_Y$*

Proposition 7 (Sectional Property for Lebesgue Measurable Space) *Let $X = \mathbb{R}^m, Y = \mathbb{R}^n$, E is a Lebesgue measurable set on $X \times Y$. For almost every $x \in X$, the set $E_x = \{y \in Y : (x, y) \in E\}$, namely section, is Lebesgue measurable on Y*

Theorem 3 (Fubini Theorem) *Let $X = \mathbb{R}^m, Y = \mathbb{R}^n$. If $f : X \times Y \to [-\infty, +\infty]$ is Lebesgue integrable, i.e. $f \in L(X \times Y)$, then*

- *For almost every $x \in X$, function $f_x : Y \to [-\infty, +\infty]$ defined by $f_x(y) = f(x, y)$ is Lebesgue integrable, i.e $f_x \in L(Y)$*
- *Function $g(x) : X \to (-\infty, +\infty)$ defined by $g(x) = \int_Y f_x(y) dy$ is Lebesgue integrable, i.e $g \in L(X)$*

And

$$\int_{X \times Y} f = \int_X \left[\int_Y f_x(y) dy \right] dx$$

3.5 Change of Variables

Chapter 4

Lebesgue Differentiation Theorem

4.1 Vitali Covering Lemma

Lemma 1 (Zorn Lemma) *Suppose a partially ordered set P has the property that every chain in P has an upper bound in P. Then the set P contains at least one maximal element.*

Lemma 2 (Axiom of Choice) *For any set X of nonempty sets, there exists a choice function f that is defined on X and maps each set of X to an element of that set.*

Lemma 3 (Simple Vitali Covering Lemma) *In a metric space, let F be a collection of balls such that $\sup_{B\in F} r(B) = R < +\infty$, then there is a (countable in a separable metric space) subcollection M of pairwise disjoint balls such that*

$$\bigcup_{B\in F} B \subseteq \bigcup_{B_k \in M} cB_k$$

where cB_k denotes the open ball constructed from B_k by simple enlarging it c times [1].

Let F be a collection of balls, define a partial order on the set of all disjoint collection of balls in F_n as follows: let A, B be any set of disjoint balls in F_n, define $A \leq B$ as $A \subseteq B$. Hence, any chain $\{A_i \in F : i \in I\}$ for any index set I, there is an upperbound $A = \bigcup_{i\in I} A_i$. By Zorn Lemma, F has a maximal subcollection M of disjoint balls.

Let F_1 be defined as follows

$$F_1 = \left\{ B \in F : \frac{R}{2} < r(B) \leq R \right\}$$

By Zorn Lemma, there is a maximal subcollection M_1 of disjoint balls in F_1. Similarly, let M_n be a maximal subcollection of disjoint balls of F_n

$$F_n = \left\{ B \in F : \frac{R}{2^{n+1}} < r(B) \leq \frac{R}{2^n} \text{ and } B \text{ does not intersect } M_1, M_2, ..., M_{n-1} \right\}$$

Given any ball $B \in F$, we will prove that every point $b \in B$ is not too far off from $M = \bigcup_{n\in\mathbb{N}} M_n$. Suppose $\frac{R}{2^{n+1}} < r(B) \leq \frac{R}{2^n}$, then B must intersect one of $M_1, M_2, ..., M_n$. If B does not intersect $M_1, M_2, ..., M_{n-1}$, by the construction of F_n, $B \in F_n$. On the other hand, by maximality of M_n, B intersects one of ball in M_n. Next, let $B_k \in M_k (k \leq n)$ intersects B, let c_k, c denote the centers of B_k and B. Then, for any $b \in B$, we have

$$\begin{aligned}
d(c_k, b) &\leq d(c_k, a) + d(a, c) + d(c, b) && (\text{let } a \in B_k \cap B) \\
&\leq r(B_k) + r(B) + r(B) && (a \in B_k, a \in B, b \in B) \\
&\leq r(B_k) + 4\frac{R}{2^{n+1}} && \left(r(B) \leq \frac{R}{2^n}\right) \\
&\leq 5r(B_k) && \left(\frac{R}{2^{n+1}} \leq \frac{R}{2^{k+1}} \leq r(B_k)\right)
\end{aligned}$$

[1] $\{x \in \mathbb{R}^n : ||x - x_0|| < r\} \mapsto \{x \in \mathbb{R}^n : ||x - x_0|| < cr\}$

Hence, any point in any ball in F is in one of the ball $5B_k$ where $B_k \in M_k$, $k \in \mathbb{N}$. Let $M = \bigcup_{k\in\mathbb{N}} M_k$ given any two balls $B_m, B_n \in M$ with $m < n$, by the construction of F_n, $B_n \in M_n \subseteq F_n$ then B_n does not intersect B_m. Hence, M is a pairwise disjoint subcollection of F and [2]

$$\bigcup_{B\in F} B \subseteq \bigcup_{B_k\in M} 5B_k$$

If the metric space is separable, let $M_Q = \{Q \cap B_k : B_k \in M\}$ where Q is a countable dense set. Each set in M_Q is non-empty and pairwise disjoint, hence is there a bijection from M to M_Q by Axiom of Choice. Hence, M is countable.

Lemma 4 (Simple Vitali Covering Lemma for Lebesgue measure) *Let $E \subseteq \mathbb{R}^n$ with $\mu_e(E) < +\infty$, let F be a collection of balls covering E, then for any $0 < \epsilon < \mu_e(E)$, there is a countable subcollection M of pairwise disjoint balls such that*

$$\sum_{B_k\in M} \mu(B_k) \geq \beta\mu_e(E)$$

Let $F_n = \{B \in F : r(B) < n\}$, then $(\overline{F_1} \cap E) \subseteq (\overline{F_2} \cap E) \subseteq ...$ and $\left(\bigcup_{n\in\mathbb{N}}(\overline{F_n} \cap E)\right) = E$ where $\overline{F_n} = \bigcup_{B\in F_n} B$, hence [3], $\mu_e(E) = \lim_{n\to+\infty} \mu_e(\overline{F_n} \cap E)$. Therefore, given any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\mu_e(E) - \epsilon < \mu_e(\overline{F_n} \cap E)$. By Simple Vitali Covering Lemma on F_n of bounded radius. there is a countable collection M of balls such that

$$\bigcup_{B\in F_n} B \subseteq \bigcup_{B_k\in M} cB_k$$

Hence,

$$\begin{aligned} c^n \sum_{B_k\in M} \mu(B_k) &\geq \mu_e\left(\bigcup_{B\in F_n} B\right) \\ &\geq \mu_e(\overline{F_n} \cap E) \\ &\geq \mu_e(E) - \epsilon \end{aligned}$$

Write $M = \bigcup_{k=1}^{+\infty} B_k$, then $\lim_{N\to+\infty} \sum_{k=1}^{N} \mu(B_k) = \mu(M) \geq \frac{1}{c^n}(\mu_e(E) - \epsilon)$. Choose appropriate ϵ, there exists $N \in \mathbb{N}$ such that

$$\sum_{k=1}^{N} \mu(B_k) \geq \beta\mu_e(E)$$

where $\beta > 0$

Given any $\epsilon > 0$, choose an open set $G \supseteq E$ such that $\mu_e(E) \leq \mu(G) < \mu_e(E) + \epsilon$. If the first condition satisfies, the second is immediate.

Let $G_1 = G, E_1 = E$. Let $\mathcal{V}_1$ be constructed by discarding all closed balls in $\mathcal{V}$ that are not contained in G_1. As $\mathcal{V}$ finely covers E_1 and G_1 is open, $\mathcal{V}_1$ still finely covers E_1 [4]. By Simple Vitali Covering Lemma for Lebesgue measure on E with cover $\mathcal{V}_1$, there is a countable subcollection $M_1 \subseteq \mathcal{V}_1$ such that

$$\sum_{B_i\in M_1} \mu(B_i) \geq \beta\mu_e(E)$$

Then,

$$\begin{aligned} \mu\left(E_1 \setminus \overline{M_1}\right) &\leq \mu\left(G_1 \setminus \overline{M_1}\right) \\ &= \mu(G_1) - \sum_{B_i\in M_1} \mu(B_i) && \text{(balls in } M_1 \text{ are are contained in } G_1\text{)} \\ &< (1-\beta)\mu_e(E_1) + \epsilon \\ &< (1-\beta/2)\mu_e(E_1) && \text{(choose } 0 < \epsilon < \beta\mu_e(E_1)/2\text{)} \end{aligned}$$

Let $M^{(k-1)} = \bigcup_{i=1}^{k-1} \overline{M_i}$ where $\overline{M_i} = \bigcup_{B\in M_i} B$. Similarly, Let $G_k = G \setminus M^{(k-1)}, E_k = E \setminus M^{(k-1)}$ be the remaining set after choosing $k-1$ finite subcollection M_i. Construct V_k by discarding all closed balls in $\mathcal{V}$ that are not contained in G_k. As $\mathcal{V}$ finely covers E_k and G_k is open, $\mathcal{V}_k$ still finely covers E_k. Simple Vitali Covering Lemma for Lebesgue measure on $E \setminus M^{(k-1)}$ with cover $\mathcal{V}_1$, there is a countable subcollection $M_k \subseteq \mathcal{V}_k$ such that

[2] we can make c smaller by choosing finer partition of R

[3] By Theorem 3.27, Richard L. Wheeden - Antoni Zygmund - Measure and Integral: An Introduction to Real Analysis - Second Edition

[4] A similar argument: for each $x \in G_1 \cap E$, choose one a sequence of decreasing closed balls containing x that is contained in $G_1 \cap E$

$$\sum_{B_i \in M_k} \mu(B_i) \geq \beta \mu_e(E_k)$$

Then, note that $\mu(G_k) = \mu(G) - \mu(M^{(k-1)}) < [\mu_e(E) - \mu(M^{(k-1)})] + \epsilon \leq \mu_e(E_k) + \epsilon$

$$\begin{aligned}\mu_e(E_{k+1}) = \mu_e\left(E_k \setminus \overline{M_k}\right) &\leq \mu\left(G_k \setminus \overline{M_k}\right) \\ &= \mu(G_k) - \sum_{B_i \in M_k} \mu(B_i) && \text{(balls in } M_k \text{ are are contained in } G_k) \\ &< (1 - \beta/2)\mu_e(E_1)\end{aligned}$$

Therefore,

$$\mu(E_k) < (1 - \beta/2)^{k-1}\mu(E)$$

Put $M = \bigcup_{k\in\mathbb{N}} M_k$, we have $(E \cap M^{(1)}) \subseteq (E \cap M^{(2)}) \subseteq ...$, and $\bigcup_k (E \cap M^{(k)}) = E \cap \overline{M}$ Then, [5]

$$\begin{aligned}\mu_e(E \cap \overline{M}) &= \lim_{k\to+\infty} \mu_e(E \cap M^{(k)}) \\ &= \lim_{k\to+\infty} [\mu_e(E) - \mu(E \setminus M^{(k)})] && \text{(Caratheodory criterion)} \\ &= \mu_e(E) - \lim_{k\to+\infty} \mu(E \setminus M^{(k)}) \\ &= \mu_e(E)\end{aligned}$$

Hence, $\mu_e(E \setminus \overline{M}) = \mu_e(E) - \mu_e(E \cap \overline{M}) = 0$

4.2 Lebesgue Differentiation Theorem

Lemma 5 (Hardy-Littlewood Maximal Function) *Given a Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{R}$, Hardy-Littlewood maximal function defined on ball is as follows*

$$f^*(x) = \sup_{r > 0} \frac{1}{\mu(\mathcal{B}_r(x))} \int_{\mathcal{B}_r(x)} |f(y)|dy$$

Some properties of Hardy-Littlewood maximal function:

1. *f^* is lower semicontinuous, i.e. $E_t = \{x \in \mathbb{R}^n : f^*(x) > t\}$ is open for all $t \in \mathbb{R}$*
2. *$\mu(E_t) \leq \frac{c}{t} \int_{\mathbb{R}^n} |f(y)|dy$ where $c \in \mathbb{R}$*

(Part 1) Given any $\alpha > 0$, let $x_1 \in E_t = \{x \in \mathbb{R}^n : f^*(x) > t\}$ that is, there exists $r > 0$ such that

$$\frac{1}{\mu(\mathcal{B}_r(x_1))} \int_{\mathcal{B}_r(x_1)} |f(y)|dy > \alpha t$$

where $\alpha > 1$. We will prove that there exists an open ball $\mathcal{B}_\epsilon(x_1)$ such that for every $x_2 \in \mathcal{B}_\epsilon(x_1)$, $f^*(x_2) > t$, that implies the set E_t is open.

Let $\epsilon > 0$, for any $x_2 \in \mathcal{B}_\epsilon(x_1)$, then $\mathcal{B}_r(x_1) \subseteq \mathcal{B}_{r+\epsilon}(x_2)$. Therefore

$$\begin{aligned}t &< \frac{1}{\alpha} \frac{1}{\mu(\mathcal{B}_r(x_1))} \int_{\mathcal{B}_r(x_1)} |f(y)|dy \\ &\leq \frac{1}{\alpha} \frac{\mu(\mathcal{B}_{r+\epsilon}(x_2))}{\mu(\mathcal{B}_r(x_1))} \frac{1}{\mu(\mathcal{B}_{r+\epsilon}(x_2))} \int_{\mathcal{B}_{r+\epsilon}(x_2)} |f(y)|dy\end{aligned}$$

Choose ϵ small enough such that $\frac{\mu(\mathcal{B}_{r+\epsilon}(x_2))}{\mu(\mathcal{B}_r(x_1))} < \alpha$, then

$$\frac{1}{\mu(\mathcal{B}_{r+\epsilon}(x_2))} \int_{\mathcal{B}_{r+\epsilon}(x_2)} |f(y)|dy > t$$

That is, $f^*(x_2) > t$. Hence, there exists $\epsilon > 0$ such that for all $x_2 \in \mathcal{B}_\epsilon(x_1)$, $f^*(x_2) > t$. E_t is open for any t.

[5] By Theorem 3.27, Richard L. Wheeden - Antoni Zygmund - Measure and Integral: An Introduction to Real Analysis - Second Edition

(Part 2) For each $x \in E_t$, let B_x be a ball centered at x such that $\frac{1}{\mu(B_x)} \int_{B_x} |f(y)|dy > t$. The collection $\{B_x : x \in E_t\}$ covered E_t, by Vitali Covering Lemma, given any $\epsilon > 0$, there is a finite subcollection M_ϵ of pairwise disjoint B_k such that

$$\sum_{B_k \in M_\epsilon} \mu(B_k) \geq \frac{1}{5}\mu(E_t) - \epsilon$$

Therefore,

$$\begin{aligned}
\mu(E_t) &\leq 5 \sum_{B_k \in M_\epsilon} \mu(B_k) + 5\epsilon \\
&< 5 \sum_{B_k \in M_\epsilon} \left(\frac{1}{t} \int_{B_k} |f(y)|dy \right) + 5\epsilon && \text{(by construction of } B_k\text{)} \\
&\leq \frac{5}{t} \int_{\mathbb{R}^n} |f(y)|dy + 5\epsilon && (B_k \text{ pairwise disjoint})
\end{aligned}$$

The statement is true for all $\epsilon > 0$, hence

$$\mu(E_t) \leq \frac{5}{t} \int_{\mathbb{R}^n} |f(y)|dy$$

Lemma 6 (Density of Continuous Function) *Given $f \in L(\mathbb{R}^n)$ and any $\epsilon > 0$, there exists a continous function g with compact support* [6] *such that*

$$\int_{\mathbb{R}^n} |f(x) - g(x)|dx < \epsilon$$

Theorem 4 (Lebesgue Differentiation Theorem) *Given $f \in L(\mathbb{R}^n)$, then for almost every $x \in \mathbb{R}^n$*

$$\limsup_{B \searrow x} \left| \frac{F(B)}{|B|} - f(x) \right| = \limsup_{B \searrow x} \frac{1}{\mu(B)} \left| \int_B [f(y) - f(x)]dy \right| = 0$$

where $B \searrow x$ denotes $B = \mathcal{B}_r(x)$ and $r \to 0^+$, $F(B) = \int_B f(y)dyt$

Let D_t be defined as

$$D_t = \left\{ x \in \mathbb{R}^n : \limsup_{B \searrow x} \frac{1}{\mu(B)} \left| \int_B [f(y) - f(x)]dy \right| > t \right\}$$

We will prove that $\mu_e(D_t) = 0$ for all $t > 0$, that will conclude the Lebesgue Differentiation Theorem.

Let g be a continous function, and any ball B centers at x, then

$$\begin{aligned}
\left| \int_B [f(y) - f(x)]dy \right| &\leq \int_B |f(y) - f(x)|dy \\
&\leq \int_B |f(y) - g(y)|dy + \int_B |g(y) - g(x)|dy + \mu(B)|g(x) - f(x)|
\end{aligned}$$

Hence,

$$\begin{aligned}
\limsup_{B \searrow x} \frac{1}{\mu(B)} \left| \int_B [f(y) - f(x)]dy \right| &\leq \limsup_{B \searrow x} \frac{1}{\mu(B)} \left(\int_B |f(y) - g(y)|dy + \int_B |g(y) - g(x)|dy + \mu(B)|g(x) - f(x)| \right) \\
&\leq \limsup_{B \searrow x} \frac{1}{\mu(B)} \int_B |f(y) - g(y)|dy + \limsup_{B \searrow x} \frac{1}{\mu(B)} \int_B |g(y) - g(x)|dy + \limsup_{B \searrow x} |g(x) - f(x)|
\end{aligned}$$

As g is continuous, $\limsup_{B \searrow x} \frac{1}{\mu(B)} \int_B |g(y) - g(x)|dy = 0$, then

$$\begin{aligned}
\limsup_{B \searrow x} \frac{1}{\mu(B)} \left| \int_B [f(y) - f(x)]dy \right| &\leq \limsup_{B \searrow x} \frac{1}{\mu(B)} \int_B |f(y) - g(y)|dy + |g(x) - f(x)| \\
&\leq \sup_B \frac{1}{\mu(B)} \int_B |f(y) - g(y)|dy + |g(x) - f(x)| \\
&= (f - g)^*(x) + |g(x) - f(x)|
\end{aligned}$$

[6] support of $f : X \to \mathbb{R}$ is $\{x \in X : f(x) \neq 0\}$

Therefore,

$$D_t \leq \left\{x \in \mathbb{R}^n : (f-g)^*(x) > \frac{t}{2}\right\} \cup \left\{x \in \mathbb{R}^n : |g(x) - f(x)| > \frac{t}{2}\right\}$$

As the two sets are measurable

$$\mu_e(D_t) \leq \mu\left(\left\{x \in \mathbb{R}^n : (f-g)^*(x) > \frac{t}{2}\right\}\right) + \mu\left(\left\{x \in \mathbb{R}^n : |g(x) - f(x)| > \frac{t}{2}\right\}\right)$$

By property of Hardy-Littlewood maximal function,

$$\mu\left(\left\{x \in \mathbb{R}^n : (f-g)^*(x) > \frac{t}{2}\right\}\right) \leq \frac{2c}{t} \int_{\mathbb{R}^n} |f(y) - g(y)| dy$$

By Chebyshev inequality,

$$\mu\left(\left\{x \in \mathbb{R}^n : |g(x) - f(x)| > \frac{t}{2}\right\}\right) \leq \frac{2}{t} \int_{\mathbb{R}^n} |f(y) - g(y)| dy$$

Therefore,

$$\mu_e(D_t) \leq \frac{2(c+1)}{t} \int_{\mathbb{R}^n} |f(y) - g(y)| dy$$

Given any $t > 0$ and $\epsilon > 0$, we can choose g close enough to f such that $\int_{\mathbb{R}^n} |f(y) - g(y)| dy < \frac{t\epsilon}{2(c+1)}$, then $\mu_e(D_t) < \epsilon$. Hence $\mu_e(D_t) = 0$ for all $t > 0$. Then

$$D = \left\{x \in \mathbb{R}^n : \limsup_{B \searrow x} \frac{1}{\mu(B)} \left| \int_B [f(y) - f(x)] dy \right| > 0\right\} = \bigcup_{k \in \mathbb{N}} D_{1/k}$$

Then $\mu_e(D) = 0$, that is $\limsup_{B \searrow x} \frac{1}{\mu(B)} \left|\int_B [f(y) - f(x)] dy\right| = 0$ for almost every $x \in \mathbb{R}^n$

4.3 Absolute Continuous Set Function

I wrote another pdf for this

Chapter 5

Convex function

Definition 22 (Convex set) *A set $C \in \mathbb{R}^n$ is said to be convex if for any $x, y \in C$, then $tx + (1-t)y \in C$ for all $t \in [0,1]$*

Definition 23 (Convex function) *A function $f : C \to \mathbb{R}$ defined on a convex set $C \in \mathbb{R}^n$ is said to be a convex function if for any $x, y \in C$, then $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for all $t \in [0,1]$*

Proposition 8 (The cone of convex functions) *The set of convex functions form a cone*

- *$\phi_1 + \phi_2$ convex if ϕ_1 and ϕ_2 are convex*
- *$c\phi$ convex if $c > 0$ and ϕ convex*

Proposition 9 (Completeness of convex functions) *If $\phi_k, k = 1, 2, ...$ convex and $\phi_k \to \phi$, then ϕ is convex*

Lemma 7 (Existence of supporting plane) *Given a convex function $f : C \to \mathbb{R}$, $C \subseteq \mathbb{R}^n$, for any $x_0 \in C$, there exists a n-plane $p : \mathbb{R}^n \to \mathbb{R}$ such that $p(x) = f(x)$ for all $x \in C$. Moreover, p can be chosen such that $p(x_0) = f(x_0)$*

Theorem 5 (Jensen Inequality) *Given a measurable space (X, M, μ) with $\mu(X) = 1$ and μ-measurable function $f : X \to \mathbb{R}$. Given a convex function $\phi : \mathbb{R} \to \mathbb{R}$, then $\phi(f)$ is μ-measurable and*

$$\phi\left(\int_X f d\mu\right) \leq \int_X \phi(f) d\mu$$

Let $\gamma = \int_X f d\mu$ and m be the slope of the supporting line at $(\gamma, \phi(\gamma))$, then for any $x \in X$

$$\phi(\gamma) + m(f(x) - \gamma) \leq \phi(f(x))$$

Then

$$\begin{aligned}\int_X \phi(\gamma) d\mu + m\left(\int_X f(x) d\mu - \int_X \gamma d\mu\right) &\leq \int_X \phi(f(x)) d\mu \\ \mu(X)\phi(\gamma) + m\left(\gamma - \mu(X)\gamma\right) &\leq \int_X \phi(f(x)) d\mu\end{aligned}$$

As $\mu(X) = 1$, yields the inequality.
If μ is the counting measure on a finite set, we have

$$\phi\left(\sum_{i=1}^n t_i f(x_i)\right) \leq \left(\sum_{i=1}^n t_i \phi(f(x_i))\right)$$

for $t_1, t_2, ... \geq 0$ and $\sum_{i=1}^n t_i = 1$

Chapter 6

L^p Space

6.1 L^p

Definition 24 (L^p) *Given a measurable space (X, M, μ) and a μ-measurable function $f : X \to [-\infty, +\infty]$. For any $0 < p \leq \infty$, define*

- $||f||_p = \left(\int_X |f|^p d\mu\right)^{1/p}$
- $||f||_\infty = \inf\{\alpha \in [-\infty, +\infty] : |\{x \in X : f(x) > \alpha\}| = 0\}$

The set $L^p(E)$ is then defined as

$$L^p(E) = \{f \text{ is } \mu\text{-measurable, defined on } X \ : ||f||_p < +\infty\}$$

Proposition 10 *If $\mu(X) < +\infty$ then $||f||_\infty = \lim_{p\to\infty} ||f||_p$*

Case 1:
not interesting
Case 2: $\mu(X) > 0, ||f||_\infty > 0$

$$\begin{aligned}
||f||_p &= \left(\int_X |f|^p\right)^{1/p} \\
&\leq \left(\int_X ||f||_\infty^p\right)^{1/p} && \text{(let } p \geq 1\text{)} \\
&= ||f||_\infty \mu(X)^{1/p}
\end{aligned}$$

Then, $\lim_{p\to\infty} ||f||_p \leq ||f||_\infty$. On the other hand,

By definition, given any $M < ||f||_\infty$, then the set $A = \{x \in X : |f(x)| > M\}$ has a positive μ-measure, then

$$\begin{aligned}
||f||_p &= \left(\int_X |f|^p\right)^{1/p} \\
&\geq \left(\int_A |f|^p\right)^{1/p} \\
&\geq \left(\int_A M^p\right)^{1/p} \\
&= M\mu(A)^{1/p}
\end{aligned}$$

Fix M, $\lim_{p\to\infty} \mu(A)^{1/p} = 0$. Then, $\lim_{p\to\infty} ||f||_p \geq M$ for all $M < ||f||_\infty$. Hence,

$$||f||_\infty = \lim_{p\to\infty} ||f||_p$$

Proposition 11 *Given $\mu(X) < +\infty$, if $0 < p_1 < p_2 \leq +\infty$, then $L^{p_1} \supset L^{p_2}$*

Case 1:
not interesting
Case 2: $\mu(X)$ finite and $p_2 < +\infty$

Jensen inequality

$$\phi\left(\frac{\int_X f^{p_1}}{\mu(X)}\right) \leq \left(\frac{\int_X \phi(f^{p_1})}{\mu(X)}\right)$$

Take $\phi(x) = x^{p_2/p_1}$, then

$$\left(\frac{\int_X f^{p_1}}{\mu(X)}\right)^{p_2/p_1} \leq \frac{\int_X f^{p_2}}{\mu(X)}$$

If $f \in L^{p_2}$, the LHS = RHS finite

Proposition 12 (Vector Space of L^p) *L^p for $0 < p \leq +\infty$ forms a vector space over the field of complex numbers*

- *$f + g \in L^p$ if $f, g \in L^p$*
- *$cf \in L^p$ if $c \in \mathbb{C}$ and $f \in L^p$*

Proposition 13 (Continuity of $L^p(\mathbb{R}^n)$) *If $f \in L^p(\mathbb{R}^n), 1 \leq p < +\infty$, then*

$$\lim_{|h| \to 0} ||f(x+h) - f(x)||_p = 0$$

to

6.2 Holder Inequality and Minkowski Inequality

Lemma 8 (Young Inequality) *Given $c > 0$, let $\phi : [0, c] \to \mathbb{R}$ be a continuous, strictly increasing function with $\phi(0) = 0$, then it is a bijection from $[0, c]$ into $[0, \phi(c)]$. Moreover, given any $a \in [0, c]$ and $b \in [0, \phi(c)]$, then*

$$ab \leq \int_0^a \phi + \int_0^b \phi^{-1}$$

The equality holds if (a, b) lies on the graph of ϕ, i.e. $b = \phi(a)$

First, we write $\int_0^b \phi^{-1}$ in term of integral of ϕ.

$$\begin{aligned}
\int_0^b \phi^{-1} &= |\{(y, x) \in [0, b] \times [0, \phi^{-1}(b)] : x < \phi^{-1}(y)\}| \\
&= |\{(x, y) \in [0, \phi^{-1}(b)] \times [0, b] : \phi(x) < y\}| \\
&= |[0, \phi^{-1}(b)] \times [0, b]| - |\{(x, y) \in [0, \phi^{-1}(b)] \times [0, b] : y < \phi(x)\}| \\
&= b\phi^{-1}(b) - \int_0^{\phi^{-1}(b)} \phi
\end{aligned}$$

Case 1: $\phi^{-1}(b) \leq a$

$$\begin{aligned}
\int_0^a \phi + \int_0^b \phi^{-1} &= \int_{\phi^{-1}(b)}^a \phi + b\phi^{-1}(b) \\
&\geq \int_{\phi^{-1}(b)}^a b + b\phi^{-1}(b) \\
&= ab
\end{aligned}$$

Case 2: $\phi^{-1}(b) \geq a$

$$\begin{aligned}
\int_0^a \phi + \int_0^b \phi^{-1} &= -\int_a^{\phi^{-1}(b)} \phi + b\phi^{-1}(b) \\
&\geq -\int_a^{\phi^{-1}(b)} b + b\phi^{-1}(b) \\
&= ab
\end{aligned}$$

If $1 \leq p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\phi(x) = x^{p-1}$, then $\phi^{-1}(y) = y^{q-1}$. Moreover,

$$ab \leq \int_0^a x^{p-1} + \int_0^b y^{q-1} = \frac{a^p}{p} + \frac{b^q}{q}$$

Theorem 6 (Holder Inequality) *Given a measurable space (X, M, μ) and μ-measurable functions $f, g : X \to [-\infty, +\infty]$. If $1 \le p \le +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$||fg||_1 \le ||f||_p ||g||_q$$

Case 1: $p = 1$ or $p = +\infty$

not interesting

Case 2: $1 < p < +\infty$

If $||f||_p = 0$ (or $||g||_q = 0$), then $f = 0$ (or $g = 0$) almost everywhere, then both sides equal 0. If $||f||_p = +\infty$ (or $||g||_q = +\infty$), then RHS equals $+\infty$. Hence, consider the case where both $||f||_p$, $||g||_q$ finite. We further assume that $||f||_p = 1$ and $||g||_q = 1$ (if not, divide both sides by $||f||_p||g||_q$),

Then

$$\begin{aligned} ||fg||_1 &= \int_X |f||g| d\mu \\ &\le \int_X \left(\frac{|f|^p}{p} + \frac{|g|^q}{q} \right) d\mu \\ &= \left(\frac{||f||_p^p}{p} + \frac{||g||_q^q}{q} \right) \\ &= 1 = ||f||_p ||g||_q \end{aligned}$$

Theorem 7 (Converse of Holder Inequality) *Given a measurable space (X, M, μ) and μ-measurable function $f : X \to [-\infty, +\infty]$, let $1 \le p \le +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$||f||_p = \sup_{||g||_q \le 1} ||fg||_1$$

where $f : X \to [-\infty, +\infty]$ is a μ-measurable function. Moreover,

$$||f||_p = \sup_{||g||_q \le 1} ||fg||_1 = \sup_{||g||_q \le 1} \int_X fg d\mu$$

From Holder Inequality,

$$||f||_p \ge ||f||_p ||g||_q \ge ||fg||_1$$

Hence,

$$||f||_p \ge \sup_{||g||_q \le 1} ||fg||_1$$

Now, we will prove the converse: $||f||_p \le \sup_{||g||_q \le 1} ||fg||_1$

Case 1: $||f||_p = 0$, then $f = 0$ almost everywhere

not interesting

Case 2: $||f||_p$ finite

Assume $||f||_p = 1$ (if not, divide both sides by $||f||_p$). Pick $g = |f|^{p/q}$, then $||q||_q^q = \int_X |f|^p = ||f||_p^p = 1$. Furthermore,

$$||fg||_1 = \int_X |f|^{p/q+1} d\mu = \int_X |f|^p d\mu = ||f||_p^p = 1 = ||f||_p$$

Hence,

$$||f||_p \le \sup_{||g||_q \le 1} ||fg||_1$$

Case 3: $||f||_p = +\infty$

Let $f_k, k \in \mathbb{N}$ be defined as

$$f_k(x) = \begin{cases} 0 & \text{if } ||x|| \ge k \\ \min\{|f(x)|, k\} & \text{if } ||x|| < k \end{cases}$$

Each f_k has $||f_k||_p$ finite, then there exists $g_k, ||g_k||_q = 1$ such that $||f_k g_k||_1 = ||f_k||_p$. Since $f_k \le |f|$, we have $||f_k||_p \le ||f_k g_k||_1 \le ||f g_k||$. As $f_k \nearrow |f|$, then $||f_k||_p \nearrow ||f||_p$, $||f g_k||$ is arbitrary large. Then,

$$+\infty = ||f||_p \le \sup_{||g||_q \le 1} ||fg||_1 = +\infty$$

Theorem 8 (Minkowski Inequality) *Given a measurable space (X, M, μ) and μ-measurable functions $f, g : X \to [-\infty, +\infty]$. If $1 \le p \le +\infty$, then*

$$||f + g||_p \le ||f||_p + ||g||_p$$

Case 1: $p = +\infty$
not interesting
Case 2: $1 \leq p < +\infty$ and $||f||_p = +\infty$ (or $||g||_p = +\infty$)
not interesting
Case 3: $1 \leq p < +\infty$ and both $||f||_p$ and $||g||_p$ finite

$$\begin{aligned}
||f+g||_p^p &= \int_X |f+g|^p d\mu \\
&= \int_X |f+g|^{p-1}|f+g| d\mu \\
&\leq \int_X |f+g|^{p-1}(|f|+|g|) d\mu \\
&= \int_X |f+g|^{p-1}|f| d\mu + \int_X |f+g|^{p-1}|g| d\mu \\
&= ||(|f+g|^{p-1}|f|)||_1 + ||(|f+g|^{p-1}|g|)||_1 \\
&\leq ||(f+g)^{p-1}||_q ||f||_p + ||(f+g)^{p-1}||_q ||g||_p && \text{(where } q > 0 \text{ and } \frac{1}{p}+\frac{1}{q}=1) \\
&= ||f+g||_p^{p-1}(||f||_p + ||g||_p)
\end{aligned}$$

Hence,

$$||f+g||_p \leq ||f||_p + ||g||_p$$

6.3 Banach Space, Metric Space

Definition 25 (Normed Space) *Given a vector space V, a function $||\cdot|| : V \to [0, +\infty)$ is said to be a norm if it satisfies the following*

- *Positive definiteness: $||x|| = 0$ if and only if $x = 0$*
- *Absolute homogeneity: $||\lambda x|| = |\lambda| ||x||$ for all $\lambda \in F$ and $x \in V$*
- *Triangle inequality: $||x+y|| \leq ||x|| + ||y||$ for all $x, y \in V$*

If $||\cdot||$ satisfies only Absolute homogeneity and Triangle inequality, it is said to be a semi-norm. A vector space equipped with a norm is called a normed space.

Definition 26 (Complete Metric Space) *A set M equipped with a metric $d : M \times M \to [0, +\infty)$ is said to be a metric space if it satisfies the following*

- *Positive definiteness: $d(x, y) = 0$ if and only if $x = y$*
- *Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in M$*
- *Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$*

A sequence $x_1, x_2, \ldots$ ($x : \mathbb{N} \to X$) is said to be Cauchy if given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n > N$, then $d(x_m, x_n) < \epsilon$. A metric space (X, d) is said to be complete if every Cauchy sequence converges.

Definition 27 (Banach Space - Complete Normed Space) *A space X is said to be a Banach space over the field F if it satisfies the following*

- *X is a normed space over F*
- *X is a complete metric space induced by its norm, i.e. $d(x, y) = ||x - y||$*

Definition 28 (Separable Topological Space) *A set X equipped with a topology $\mathcal{T} \subseteq \mathcal{P}(X)$ is said to be a topological space if it satisfies the following*

- *$\varnothing, X \in \mathcal{T}$*
- *$\bigcup_{i \in I} U_i \in \mathcal{T}$ if $U_i \in \mathcal{T}$ for all $i \in I$ where I is any index set.*
- *$\bigcap_{n=1}^{N} U_i \in \mathcal{T}$ if $U_i \in \mathcal{T}$ for all $i = 1, 2, \ldots, N$ where $N \in \mathbb{N}$*

A set in $\mathcal{T}$ is said to be an open set. Given a subset $E \subseteq X$, a point $x \in X$ is said to be a limit point of E if any open neighbour of x (open set containing x) contains a point in E other than x. The union of E and all of its limit points is said to be the closure of E, written as $\overline{E}$. A subset $E \subseteq X$ is said to be dense if $\overline{E} = X$. A topological space is said to be separable if it has a countable dense subset.

Definition 29 (Metric Space as a Topological Space) *Given a metric space (X, d), define a topology $\mathcal{T}$ consists of all open balls $\mathcal{B}_r(x)$ for all $x \in X$ and $r > 0$*

Lemma 9 *Given a measurable space (X, M, μ) and μ-measurable functions $f, g : X \to [-\infty, +\infty]$. Then*

- $||f + g||_p \leq ||f||_p + ||g||_p$ *(if $1 \leq p \leq +\infty$)*
- $||f + g||_p \leq K(||f||_p + ||g||_p)$ *for $K > 0$ (if $0 < p < 1$)*

Theorem 9 (Classification of L^p) *Some classifications of L^p*

1. *if $0 < p < 1$, L^p is a separable metric space with metric $d(f, g) = ||f - g||_p^p$*
2. *if $1 \leq p < +\infty$, L^p is a separable Banach space with norm $||f|| = ||f||_p$*
3. *if $p = +\infty$, L^p is a Banach space with norm $||f|| = ||f||_p$*

Lemma 10 $(a + b)^p \leq a^p + b^p$ *if $a, b \geq 0$ and $0 < p < 1$*

$$(a+b)^p = (a+b)^p \left(\frac{a}{a+b} + \frac{b}{a+b} \right) \leq (a+b)^p \left[\left(\frac{a}{a+b} \right)^p + \left(\frac{b}{a+b} \right)^p \right] = a^p + b^p$$

Proposition 14 *If $0 < p < 1$, $d(f, g) = ||f - g||_p^p$ is a metric*

$$||f + g||_p^p = \int_X |f(x) + g(x)|^p d\mu \leq \int_X |f(x)|^q d\mu + \int_X |g(x)|^p d\mu \leq ||f||_p^p + ||g||_p^p$$

Proposition 15 *If $1 \leq p < +\infty$, L^p is complete.*

Let $\{f_k\}_{k \in \mathbb{N}}$ be a Cauchy sequence in L^p, that is given any $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that for all $m, n > N$, $||f_m - f_n||_p < \epsilon$. Hence, we can construct a subsequence $\{f_{k_i}\}$ such that

$$\int_X |f_{k_i} - f_{k_{i-1}}| d\mu \leq \int_X |f_{k_i} - f_{k_{i-1}}|^p d\mu = ||f_{k_i} - f_{k_{i-1}}||_p^p < 2^i$$

Put $g_i = \sum_{i=1}^n |f_{k_i} - f_{k_{i-1}}|$ and $g = \sum_{i=1}^\infty |f_{k_i} - f_{k_{i+1}}| = \lim_{i \to \infty} g_i$. By Fatou lemma, $\int_X g d\mu \leq \liminf_{i \to \infty} \int_X g_i d\mu \leq 1$. Hence, g is finite almost everywhere.

Therefore, the following infinite series absolute converges for almost every $x \in X$

$$f(x) = f_{k_0}(x) + \sum_{i=1}^\infty (f_{k_i}(x) - f_{k_{i-1}}(x)) = \lim_{i \to \infty} f_{k_i}$$

Now, we prove that f is also a limit of $\{f_k\}$ in L^p. As $|\cdot|^p$ is continous, $|f_{k_i} - f_k|^p \to |f - f_k|^p$, then by Fatou lemma

$$||f - f_k||_p^p = \int_X |f - f_k|^p d\mu \leq \liminf_{i \to \infty} \int_X |f_{k_i} - f_k|^p d\mu$$

Given any $\epsilon > 0$, pick K large enough such that for all $k > K$, $\liminf_{i \to \infty} \int_X |f_{k_i} - f_k|^p d\mu < \epsilon^{1/p}$. We have the convergence in L^p. Furthermore, by Minkowski inequality

$$||f||_p \leq ||f - f_k||_p + ||f_k||_p < +\infty$$

Proposition 16 *L^∞ is complete.*

Let $\{f_k\}_{k \in \mathbb{N}}$ be a Cauchy sequence in L^∞. Let $Z_{kmn} = \{x \in X : |f_k(x)| > ||f_k||_\infty$ and $|f_m(x) - f_n(x)| > ||f_m - f_n||_\infty\}$ with $\mu(Z_{kmn}) = 0$ and $E = X \setminus \bigcup_{(k,m,n) \in \mathbb{N}^3} Z_{kmn}$ with $\mu(E) = \mu(X)$. For each $x \in E$, $f_k(x)$ is a Cauchy sequence, hence $f_k(x) \to y < M$. Let $f : X \to \mathbb{R}$ be the bounded function where $f(x) = 0$ on $X \setminus E$ and $f_k(x) \to f(x)$ uniformly on E. Hence, f is also a limit of $\{f_k\}$ in L^∞.

Proposition 17 *If $0 < p < +\infty$, L^p is separable in $\mathbb{R}^n$* [1]

not interesting

Theorem 10 (Continuity in $L^p(\mathbb{R}^n)$) *If $f \in L^p(\mathbb{R}^n)$ given $0 < p < +\infty$, then*

$$\lim_{|h|\to 0} d(f(x+h), f(x)) = 0$$

where $d(a,b) = ||a-b||_p$ for $1 \leq p < +\infty$ and $d(a,b) = ||a-b||_p^p$ for $0 < p < 1$

Outline of the proof

1. Let C^p be the set of functions satisfying the continuity conditions
2. C^p is a vector space
3. $f \in C^p$ if $f_k \to f$ in L^p ($||f_k - f|| \to 0$ as $k \to \infty$) and $f_k \in C^p, k \in \mathbb{N}$
4. There is a dense set D of L^p belongs to C^p (set of linear combinations of indicator functions of cubes)
5. $C^p = L^p = \overline{D}$

[1] probably true given X σ-finite

Chapter 7

Hilbert Space

Definition 30 (Inner Product Space) *Given a vector space V over field F, a function $\langle \cdot, \cdot \rangle : V \times V \to F$ is said to be an inner product if it satisfies the following*

- *Positive definiteness: $\langle x, x \rangle > 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$*
- *Conjugate Symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in V$*
- *Additivity in the first argument: $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in V$*
- *Homogeneity in the first argument: $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for all $x, y \in V$ and $\alpha \in F$*

A vector space equipped with an inner product is called an inner product space. Consequently, an inner product space is also a norm space where the norm is defined as $||x|| = |\langle x, x \rangle|$

Definition 31 (Hilbert Space) *Hilbert Space is a Banach Space where the norm is induced from an inner product.*

7.1 Orthogonality

Definition 32 (Linear Independence, Orthogonal, Orthogonal System, Span, Basis) *Some definitions related to Hilbert Space:*

- *A collection of vectors $\{x_i : i \in I\}$ is said to be linearly independent if for any finite subcollection $J \subseteq I$, $\sum_{j \in J} a_j x_j = 0 \iff a_j = 0 \, \forall j \in J$*
- *Non-zero $x, y \in V$ are said to be orthogonal if $\langle x, y \rangle = 0$*
- *A set of non-zero pairwise orthogonal vectors is said to be an orthogonal system.*
- *An orthogonal system is said to be maximal (or complete) if no other vector is orthogonal to every vector in the system.*
- *Span of a collection of vectors is the set of all finite linear combinations.*
- *A collection of vectors is said to be a basis if its closure of span is the whole space.*

Theorem 11 *Cardinality of any orthogonal system in a Hilbert Space is at most its density. In particular, any orthogonal system in a separable Hilbert Space is countable.*

Assume $\{x_i : i \in I\}$ is orthonormal where I is any index set. If $a \neq b$, then

$$||x_a - x_b||^2 = |\langle x_a - x_b, x_a - x_b \rangle| = ||x_a||^2 + ||x_b||^2 = 2$$

Hence, $||x_a - x_b|| < \sqrt{2}$. Let $B = \{\mathcal{B}_{\sqrt{2}/2}(x_i) : i \in I\}$ be a collection of disjoint balls centered at x_i. Let D be a dense set, then for each $B_i \in B$, there is at least one element from D. By Axiom of Choice, we can construct a injection from B to D. Hence, $\text{card}(I) \leq \text{card}(D)$

Theorem 12 *Any orthogonal system in a Hilbert Space is linearly independent.*

Let $\{x_j : j \in J\}$ be finite orthonormal collection of vectors. Then

$$0 = \left\langle x_k, \sum_{j \in J} a_j x_j \right\rangle = a_k \langle x_k, x_k \rangle = a_k$$

Theorem 13 (Gram-Schmidt Process) *Given a countable collection of vectors $\{x_i\}, i \in \mathbb{N}$. There exists a countable orthogonal system $\{y_i\}$ defined by*

$$
\begin{aligned}
y_1 &= x_1 \\
y_2 &= a_{21}x_1 + x_2 \\
&... \\
y_k &= a_{k1}x_1 + ... + a_{kk-1}x_{k-1} + x_k \\
&...
\end{aligned}
$$

Theorem 14 *An orthogonal system is maximal if and only if it is a basis.*

Let $\{x_i : i \in I\}$ be an orthonormal basis, we will prove that it is maximal. Suppose there exists y such that $\langle y, x_i \rangle$ for all $i \in I$ and $||y|| = 1$, then

$$1 = \langle y, y \rangle = \left\langle y, y - \sum_{j \in J} a_j x_j \right\rangle \leq ||y|| \cdot \left|\left| y - \sum_{j \in J} a_j x_j \right|\right|$$

for any finite subcollection $J \subseteq I$ and choice of a_j. Since $\{x_i : i \in I\}$ is a basis, RHS is arbitrary small. Hence, $\{x_i : i \in I\}$ is maximal.

On the other hand, let $\{x_i : i \in I\}$ be an maximal orthonormal system. Suppose, it is not a basis, that is, there exists y and $\epsilon > 0$ such that

$$\left|\left| y - \sum_{j \in J} a_j x_j \right|\right| > \epsilon$$

for all finite subcollection $J \subseteq I$, and choice of a_j. Let $a_j = \langle y, x_i \rangle$ and $y_k = y - \sum_{j=1}^{k} a_j x_j$. We will prove that y_k is a Cauchy sequence then converges. By Pythagorean theorem

$$||y||^2 = ||y_k||^2 + ||\sum_{j=1}^{k} a_j x_j||^2 \geq ||\sum_{j=1}^{k} a_j x_j||^2 = \sum_{j=1}^{k} |a_j|^2$$

for all $k \in \mathbb{N}$. The monotone increasing sequence $(\sum_{j=1}^{k} a_j^2)$ is bounded above, hence converges. Given any $m < n$, we have

$$||y_m - y_n|| = ||\sum_{j=m+1}^{n} a_j x_j|| = \sqrt{\sum_{j=m+1}^{n} |a_j|^2}$$

So, we can pick N large enough such that $N < m < n$ and $||y_m - y_n||$ is arbitrary small. Let $y_k \to y_\infty$, by the assumption, $||y_\infty|| \geq ||y_k|| - ||y_k - y_\infty||$, then $||y_\infty|| \geq \epsilon$. We now prove that y_∞ is orthogonal to all x_i which leads to a contradiction. For any x_i, choose $k > i$ so that y_k orthogonal to x_i

$$\langle y_\infty, x_i \rangle = \langle y_k + (y_\infty - y_k), x_i \rangle = \langle y_k, x_i \rangle + \langle y_\infty - y_k, x_i \rangle = \langle y_\infty - y_k, x_i \rangle \leq ||y_\infty - y_k|| ||x_i||$$

By choosing k large enough, we can make RHS arbitrary small. Hence, $\langle y_\infty, x_i \rangle = 0$

Theorem 15 *$L^P(\mathbb{R}^n)$ is a separable Hilbert Space and every basis is countably infinite.*

7.2 Fourier Series

Definition 33 (Fourier Series) *Given an orthonormal system $\{x_i : i \in \mathbb{N}\}$ in a Hilbert space H and $y \in H$, then the number*

$$c_i = \langle y, x_i \rangle$$

is said to be the Fourier coefficient. The series

$$S[y] = \sum_{i=1}^{\infty} c_i x_i$$

is said to be the Fourier series

Theorem 16 *Given an orthonormal system $\{x_i : i \in \mathbb{N}\}$ in a Hilbert space H and $y \in H$. Given $N \in \mathbb{N}$, then*

$$\min_{z\in \text{span}\{x_1,\dots,x_N\}} ||y - z||^2 = ||y - S[y]||^2 = ||y|| - \sum_{i=1}^{N} |c_i|^2$$

where $\{c_i : i \in \mathbb{N}\}$ are Fourier coefficients of y. Moreover,

$$\sum_{i=1}^{\infty} |c_i|^2 \le ||y||^2$$

Let $z = \sum_{i=1}^{N} a_i x_i$, then

$$\begin{aligned}
||y - z||^2 &= \langle y - z, y - z\rangle \\
&= ||y||^2 - 2\,\text{Re}(\langle y, z\rangle) + ||z||^2 \\
&= ||y||^2 - 2\,\text{Re}(\langle y, \sum_{i=1}^{N} a_i x_i\rangle) + ||z||^2 \\
&= ||y||^2 + \sum_{i=1}^{N} \left[-2\,\text{Re}(\overline{a_i} c_i) + |a_i|^2\right] \\
&= ||y||^2 - \sum_{i=1}^{N} |c_i|^2 + \sum_{i=1}^{N} |a_i - c_i|^2 \\
&\ge ||y||^2 - \sum_{i=1}^{N} |c_i|^2
\end{aligned}$$

The equality holds when $a_i = c_i$. Let $y_k = \sum_{i=1}^{k} c_i x_i$, by Pythagorean theorem

$$||y||^2 = ||y_k||^2 + ||y - y_k||^2$$

Then $\sum_{i=1}^{k} |c_i|^2 = ||y_k||^2 \le ||y||^2$ for all $k \in \mathbb{N}$. Hence, $\sum_{i=1}^{\infty} |c_i|^2 \le ||y||^2$

Proposition 18 (Parseval formula) *Let $\{c_i : i \in \mathbb{N}\}$ be Fourier coefficients of y, then $\sum_{i=1}^{\infty} |c_i|^2 = ||y||^2$ if and only if $y_k \to y$ where $y_k = \sum_{i=1}^{k} c_i x_i$*

$$\lim_{k\to\infty} ||y - y_k||^2 = ||y||^2 - \lim_{k\to\infty} ||y_k||^2 = ||y||^2 - \sum_{i=1}^{\infty} |c_i|^2$$

Theorem 17 (Riesz-Fischer Theorem) *Given an orthonormal system $\{x_i : i \in \mathbb{N}\}$ in a Hilbert space H and $(a_1, a_2, \dots) \in l^2$ ($\sum_{i=1}^{\infty} |a_i|^2 < +\infty$), then there exists $y \in H$ such that $\langle y, x_i\rangle = a_i$ and $\sum_{i=1}^{\infty} |a_i|^2 = ||y||^2$*

Let $y_k = \sum_{i=1}^{k} a_i x_i$. We will prove that $\{y_k : k \in \mathbb{N}\}$ is a Cauchy sequence. Given any $m < n$,

$$||y_n - y_m||^2 = \sum_{i=m+1}^{n} |a_i|^2$$

As $\sum_{i=1}^{\infty} |a_i|^2 < +\infty$, we can choose N large enough such that the RHS is arbitrary small for all $m, n > N$. Hence, $\{y_k : k \in \mathbb{N}\}$ is a Cauchy sequence. Let $y_k \to y$, for any x_i, choose $k > i$

$$\langle y, x_i\rangle = \langle y_k + (y - y_k), x_i\rangle = \langle y_k, x_i\rangle + \langle y - y_k, x_i\rangle = a_i + \langle y - y_k, x_i\rangle$$

we can choose k large enough such that $|\langle y, x_i\rangle - a_i|$ arbitrary small. Hence, $\langle y, x_i\rangle = a_i$. By Pythagorean theorem [1],

$$||y||^2 = ||y_k + (y - y_k)||^2 = ||y_k||^2 + ||y - y_k||^2 = \sum_{i=1}^{k} |a_i|^2 + ||y - y_k||^2$$

Then

$$\sum_{i=1}^{\infty} |a_i|^2 = ||y||^2 - \lim_{k\to\infty} ||y - y_k||^2 = ||y||^2$$

[1] $\langle y - y_k, y_k\rangle = \langle y, y_k\rangle - \langle y_k, y_k\rangle = \langle y, \sum_{i=1}^{k} a_i x_i\rangle - \sum_{i=1}^{k} |a_i|^2 = 0$

Theorem 18 *Given an orthonormal system $\{x_i : i \in \mathbb{N}\}$ in a Hilbert space H. $\{x_i : i \in \mathbb{N}\}$ is a basis if and only if for every $y \in H$, $y_k \to y$ where $y_k = \sum_{i=1}^{k} c_i x_i$ and $c_i = \langle y, x_i \rangle$*

For every $y \in H$, $y_k \to y$, then $\{x_i : i \in \mathbb{N}\}$ is dense. On the other hand, let $\{x_i : i \in \mathbb{N}\}$ be a basis but $y_k \to y_\infty$ and $||y - y_\infty|| > 0$, then $y - y_\infty$ is orthogonal to all x_i. Choose $k > i$

$$\langle y - y_\infty, x_i \rangle = \langle (y - y_k) + (y_k - y_\infty), x_i \rangle = \langle y - y_k, x_i \rangle + \langle y_k - y_\infty, x_i \rangle \leq ||y_k - y_\infty|| ||x_i||$$

Choose k large enough make RHS arbitrary small. Contradiction

Definition 34 (Isometry - Isomorphism of Metric Space) *Two metric spaces A, B are isometric if there exists a bijection $T : A \to B$ such that it preserves distance, i.e. $d_B(T(x), T(y)) = d_A(x, y)$ for all $x, y \in A$*

Lemma 11 *All vector spaces have a basis*

Prove using Zorn lemma

Theorem 19 *All separable Hilbert spaces are linearly isometric*

Given a separable Hilbert space H, we will prove that it is isometric to l^2. Given a basis of H, using Gram-Schmidt Process, we can construct an orthonormal basis $\{x_i : i \in \mathbb{N}\}$

Let $(a_1, a_2, ...) \in l^2$ and the map $T(a_1, a_2, ...) = y$ where $y_k \to y$ with $y_k = \sum_{i=1}^{k} a_i x_i$.

T is one-to-one. Suppose $T(a_1, a_2, ...) = T(b_1, b_2, ...) = y$ with $a_i \neq b_i$. By Riesz-Fischer Theorem, $a_i = \langle y, x_i \rangle = b_i$. Contradiction

T is onto. By theorem 18, for any $y \in H$ the sequence of Fourier coefficients $(c_1, c_2, ...)$ satisfies $y_k \to y$ with $y_k = \sum_{i=1}^{k} c_i x_i$.

T is an isometry. Let $y = T(a_1, a_2, ...)$, $z = T(b_1, b_2, ...)$

$$||y - z||^2 = ||\sum_{i=1}^{\infty} (a_i - b_i) x_i|| = \sum_{i=1}^{\infty} (a_i - b_i)^2 = ||(a_1, a_2, ...) - (b_1, b_2, ...)||^2$$

T is linear.

not interesting

Chapter 8

Approximation of Identity

Definition 35 (Convolution) *Given f, g measurable on X, define convolution*

$$(f * g)(x) = \int_X f(t)g(x-t)dt$$

provided the integral exists.

Theorem 20 (Young Convolution Theorem) *Let $1 \leq p, q \leq +\infty$ and $\frac{1}{p} + \frac{1}{q} \geq 1$ and let $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. If $f \in L^p$ and $g \in L^q$, then $f * g \in L^r$ and*

$$||f * g||_r \leq ||f||_p ||g||_q$$

Moreover, theorem still holds if we relax either $f \in L^p$ or $g \in L^q$ to weak-type.

Definition 36 (Partial derivative) *C^m denotes the class of functions whose partial derivatives of order m exist and continuous. C_0^m denotes a subset of C^m with compact support. Similar notations C^∞, C_0^∞ for infinitely differentiable function.*

Let $\alpha = (\alpha_1, ..., \alpha_n), \sum_{i=1}^n \alpha \leq m$, denote the α partial derivative of $f \in C^m$ as

$$(D^\alpha f)(x) = \left(\frac{\partial^\alpha f}{\partial x^\alpha}\right)(x) = \left(\frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial {x_n}^{\alpha_n}}\right)(x)$$

Theorem 21 *If $1 \leq p \leq +\infty$, $f \in L^p$ and $K \in C_0^m$, then $f * K \in C^m$ with bounded partial derivatives of all orders at most m and*

$$D^\alpha(f * K)(x) = (f * D^\alpha K)(x)$$

where $\alpha = (\alpha_1, ..., \alpha_n), \sum_{i=1}^n \alpha \leq m$

todo

Definition 37 (Approximation of Identity) *The ring of measurable functions on X with usual addition and convolution as the multiplication is a ring without unity, i.e. there is no $e : X \to [-\infty, +\infty]$ such that $f * e = e * f = f$ for all $f : X \to [-\infty, +\infty]$. We instead approximate the identity by a kernel K. Given $K(x)$ and $\epsilon > 0$, let*

$$K_\epsilon(x) = \epsilon^{-n} K\left(\frac{x}{\epsilon}\right)$$

Lemma 12 *If $K \in L^1(\mathbb{R}^n)$ and $\epsilon > 0$, then*

1. *$\int K_\epsilon = \int K$*
2. *$\lim_{\epsilon \to 0^+} \int_{|x|>\delta} |K_\epsilon| = 0$ for any fixed δ*

By change of variables $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $y = T(x) = \frac{x}{\epsilon}$

$$\int_{\mathbb{R}^n} K_\epsilon(y)dy = \int_{T^{-1}(\mathbb{R}^n)} K_\epsilon(T(x))|(\det T)(x)|dx = \int_{\mathbb{R}^n} K(x)dx$$

Similarly,

$$\int_{|x|>\delta} |K_\epsilon(y)|dy = \int_{|x|>\delta/\epsilon} |K(x)|dx$$

As $\int_{\mathbb{R}^n} |K(x)|dx < +\infty$, by monotonicity, $lim_{\epsilon \to 0^+} \int_{|x|>\delta} |K_\epsilon| = 0$

Theorem 22 *Let $f_\epsilon = f * K_\epsilon$ where $K \in L^1(\mathbb{R}^n)$ and $\int K = 1$. If $f \in L^p(\mathbb{R}^n), 1 \le p < +\infty$, then*

$$\lim_{\epsilon \to 0} ||f_\epsilon - f||_p = 0$$

If $1 < p < +\infty$, then

$$\begin{aligned}
|f_\epsilon(x) - f(x)| &= \left| \int f(x-t)K_\epsilon(t)dt - f(x) \right| \\
&= \left| \int [f(x-t) - f(x)]K_\epsilon(t)dt \right| \\
&\le \int |f(x-t) - f(x)||K_\epsilon(t)|dt && \text{(analogous to triangle inequality)} \\
&= \int |f(x-t) - f(x)||K_\epsilon(t)|^{1/p}|K_\epsilon(t)|^{1/q}dt && \left(\frac{1}{p} + \frac{1}{q} = 1\right) \\
&= \left|\left| \left(|f(x-t) - f(x)||K_\epsilon(t)|^{1/p}\right) \left(|K_\epsilon(t)|^{1/q}\right) \right|\right|_1 && \text{(norm as a function of } t\text{)} \\
&\le \left|\left| |f(x-t) - f(x)||K_\epsilon(t)|^{1/p} \right|\right|_p \left|\left| |K_\epsilon(t)|^{1/q} \right|\right|_q && \text{(Holder inequality for } p, q\text{, norm as a function of } t\text{)} \\
&= \left(\int |f(x-t) - f(x)|^p |K_\epsilon(t)|dt \right)^{1/p} \left(\int |K_\epsilon(t)|dt \right)^{1/q} \\
&= \left(\int |f(x-t) - f(x)|^p |K_\epsilon(t)|dt \right)^{1/p}
\end{aligned}$$

We also have the same inequality if $p = 1$ (not interested). Therefore,

$$\begin{aligned}
||f_\epsilon - f||_p^p &= \int |f_\epsilon(x) - f(x)|^p dx \\
&\le \int \left(\int |f(x-t) - f(x)|^p |K_\epsilon(t)|dt \right) dx \\
&= \int \left(\int |f(x-t) - f(x)|^p |K_\epsilon(t)|dx \right) dt && \text{(Tonelli theorem)} \\
&= \int |K_\epsilon(t)| \left(\int |f(x-t) - f(x)|^p dx \right) dt
\end{aligned}$$

Let $\phi(t) = \int |f(x-t) - f(x)|^p dx = ||f(x-t) - f(x)||_p^p$ (norm as a function of x), by continuity of L^p function, $f \in L^p(\mathbb{R}^n), 1 \le p \le +\infty$, for any $\eta > 0$, there exists $\delta > 0$ such that $|t| < \delta$ then $\phi(t) < \eta$. We have

$$\begin{aligned}
||f_\epsilon - f||_p^p &\le \int |K_\epsilon(t)|\phi(t)dt \\
&= \int_{|t|<\delta} |K_\epsilon(t)|\phi(t)dt + \int_{|t|\ge\delta} |K_\epsilon(t)|\phi(t)dt \\
&\le \eta \int_{|t|<\delta} |K_\epsilon(t)|dt + \int_{|t|\ge\delta} |K_\epsilon(t)|\phi(t)dt && \text{(continuity of } L^p\text{)} \\
&\le \eta M + \int_{|t|\ge\delta} |K_\epsilon(t)|\phi(t)dt && \left(\int_{|t|<\delta} |K_\epsilon(t)|dt \le \int |K_\epsilon(t)|dt = M < +\infty \right) \\
&\le \eta M + 2^p ||f||_p^p \int_{|t|\ge\delta} |K_\epsilon(t)|dt && (||f(x-t) - f(x)||_p^p \le (||f(x-t)||_p + ||f(x)||_p)^p = 2^p ||f||_p^p)
\end{aligned}$$

As $\lim_{\epsilon \to 0} \int_{|t|\ge\delta} |K_\epsilon(t)|dt = 0$, we can make $||f_\epsilon - f||_p^p$ arbitrarily small

8.1 Marcinkiewicz Interpolation Theorem

Definition 38 (Weak-L^p) *A measurable function on X is said to belong to weak $L^p(X), 0 < p < \infty$ if there is a constant $A \ge 0$ such that $\mu\{x \in X : f(x) > \alpha\} \le \frac{A}{\alpha^p}$. Some properties of weak L^p*

- *if $f \in L^p$, then $f \in$ weak-L^p*
- *if $f \in$ weak L^1 and $f \in$ weak L^r, then $f \in L^p$ with $1 < r < p < \infty$*

Theorem 23 (Marcinkiewicz Interpolation Theorem) *Given a operator T that is sublinear [1], that is*

$$|T(f+g)(x)| \leq |Tf(x)| + |Tg(x)|$$

almost everywhere. Then, for any $p < r < q$, if T maps L^p into weak-L^p and T maps a L^q into a weak-L^q. Then, T maps L^r into a L^r.

Given any function f and any $\alpha > 0$, define the following

- $f^\alpha = \chi_{\{|f|>\alpha\}} f$
- $f_\alpha = \chi_{\{|f|\leq\alpha\}} f$

Then, $f = f^\alpha + f_\alpha$. By sublinearity of T,

$$|Tf(x)| \leq |Tf^\alpha(x)| + |Tf_\alpha(x)|$$

almost everywhere. Then

$$\{|Tf| > \alpha\} \subseteq \{|Tf^\alpha| > \alpha/2\} \cup \{|Tf_\alpha| > \alpha/2\}$$

By sublinearity of measure,

$$\mu\{|Tf| > \alpha\} \leq \mu\{|Tf^\alpha| > \alpha/2\} + \mu\{|Tf_\alpha| > \alpha/2\}$$

Now, we write the L_r norm of Tf

$$\begin{aligned}
||Tf||_r^r &= \int |Tf(x)|^r dx \\
&= r\int_0^{+\infty} \alpha^{r-1}\mu\{|Tf| > \alpha\} d\alpha \\
&\leq r\int_0^{+\infty} \alpha^{r-1}\mu\{|Tf^\alpha| > \alpha/2\} d\alpha + r\int_0^{+\infty} \alpha^{r-1}\mu\{|Tf_\alpha| > \alpha/2\} d\alpha
\end{aligned}$$

We will prove that each term is bounded given $||f||_r$ finite. For the second term, pick α small enough such that for all $x \in [0, \alpha]$, $x^q < x^r$. Then $f \in L^r \implies f_\alpha \in L^r \implies f_\alpha \in L^q$. As T maps L^q to weak-L^q, we have

$$\mu\{|Tf_\alpha| > \alpha/2\} \leq c\alpha^{-q}||f_\alpha||_q^q$$

for some constant c. Then

$$\begin{aligned}
\int_0^{+\infty} \alpha^{r-1}\mu\{|Tf_\alpha| > \alpha/2\} d\alpha &\leq c\int_0^{+\infty} \alpha^{r-1}\alpha^{-q}\left(\int \chi_{\{|f|\leq\alpha\}}|f(x)|^q dx\right) d\alpha \\
&= c\int |f(x)|^q \left(\int_0^{+\infty} \alpha^{-(q-r)-1}\chi_{\{|f|\leq\alpha\}} d\alpha\right) dx \qquad \text{(Tonelli theorem)} \\
&= c\int |f(x)|^q \left(\int_{|f(x)|}^{+\infty} \alpha^{-(q-r)-1} d\alpha\right) dx \\
&= c\int |f(x)|^q \frac{1}{(q-r)|f(x)|^{q-r}} dx \\
&= \frac{c}{q-r}\int |f(x)|^r dx
\end{aligned}$$

For the first term, $f \in L^r \implies f^\alpha \in L^r \implies f^\alpha \in L^p$. As T maps L^p to weak-L^p, we have

$$\mu\{|Tf^\alpha| > \alpha/2\} \leq c\alpha^{-p}||f_\alpha||_p^p$$

for some constant c. Then

[1] can be generalized to $|T(f+g)(x)| \leq c(|Tf(x)| + |Tg(x)|)$

$$\begin{aligned}
\int_0^{+\infty} \alpha^{r-1} \mu\{|Tf^\alpha| > \alpha/2\} d\alpha &\leq c \int_0^{+\infty} \alpha^{r-1} \alpha^{-p} \left(\int \chi_{\{|f|>\alpha\}} |f(x)|^p dx \right) d\alpha \\
&= c \int |f(x)|^p \left(\int_0^{+\infty} \alpha^{(r-p)-1} \chi_{\{|f|>\alpha\}} d\alpha \right) dx \\
&= c \int |f(x)|^p \left(\int_0^{|f(x)|} \alpha^{(r-p)-1} d\alpha \right) dx \\
&= c \int |f(x)|^p \frac{|f(x)|^{r-p}}{r-p} dx \\
&= \frac{c}{r-p} \int |f(x)|^r dx
\end{aligned}$$