

set

Khanh Nguyen

June 2023

some identities related to set

1 Zermelo–Fraenkel set theory

9 axioms in ZFC serve as the foundation of mathematics. All concepts in mathematics at the moment can be built from these 9 axioms using the language of first-order logic.¹

Axiom 1 (Axiom of extensionality) *Two sets are equal if they have the same elements*

$$\forall X \forall Y [\forall z (z \in X \iff z \in Y)] \implies X = Y$$

Axiom 2 (Axiom of regularity) *Every non-empty set of sets \mathcal{X} contains a element Y such that \mathcal{X} and Y are disjoint*

$$\forall \mathcal{X} [\mathcal{X} \neq \emptyset \implies \exists Y (Y \in \mathcal{X} \wedge Y \cap \mathcal{X} = \emptyset)]$$

Axiom 3 (Axiom schema of specification) *Given any predicate $\varphi(z)$ and a set X , the subset of elements of X obeying φ exists, namely Y*

$$\forall X \exists Y \forall z [z \in Y \iff (z \in X \wedge \varphi(z))]$$

we write $Y = \{z \in X : \varphi(z)\}$

Axiom 4 (Axiom of pairing) *If x and y are sets, there exists a set that contains x and y as elements, namely Z*

$$\forall x \forall y \exists Z [(x \in Z) \wedge (y \in Z)]$$

we use axiom ?? to construct the set of x and y , $\{x, y\}$

Axiom 5 (Axiom of union) *For any set of sets \mathcal{F} there exists a set containing every element that is a element x of some element Y of \mathcal{F} , namely A*

$$\forall \mathcal{F} \exists A \forall Y \forall x [Y \in \mathcal{F} \wedge x \in Y \implies x \in A]$$

we use axiom ?? to construct the union of \mathcal{F} , we write $\bigcup \mathcal{F} \subseteq A$

¹will give more insights once I master the topic

Axiom 6 (Axiom schema of replacement) Let $f : A \rightarrow B$, there exists a set containing the image of A , namely C

$$\forall A \forall B \forall (f : A \rightarrow B) \exists C \forall x (x \in A \implies f(x) \in C)$$

² we use axiom ?? to construct the image of A , we write $f(A) \subseteq C$. sometimes, we also write $f(A) = \{f(x) : x \in A\}$ and $f(\{x \in X : \varphi(x)\}) = \{f(x) : x \in X \wedge \varphi(x)\}$

Axiom 7 (Axiom of infinity) There exists a set X containing the empty set \emptyset and if y is a element of X then $y \cup \{y\}$ is also a element of X

$$\exists X [\emptyset \in X \wedge \forall y (y \in X \implies y \cup \{y\} \in X)]$$

this axiom asserts the existence of natural number (von Neumann ordinals)

Axiom 8 (Axiom of power set) For any set X , there is a set that contains every subsets Z of X , namely \mathcal{Y}

$$\forall X \exists \mathcal{Y} \forall Z (Z \subseteq X \implies Z \in \mathcal{Y})$$

we use axiom ?? to construct the power set of X , we write $\mathcal{P}(X) \subseteq \mathcal{Y}$

Axiom 9 (Axiom of choice) For any set \mathcal{X} of non-empty sets Y , there exists a choice function f that defined on \mathcal{X} and maps each set Y of \mathcal{X} to an element of that set.

$$\forall \mathcal{X} [\emptyset \notin \mathcal{X} \implies \exists (f : \mathcal{X} \rightarrow \bigcup \mathcal{X}) \forall A (A \in \mathcal{X} \wedge f(A) \in A)]$$

2 Some common identities

Theorem 1 ³ Given a set A , for all element $x \in A$ if $x \in U_x \subseteq A$ then

$$\bigcup_{x \in A} U_x = A$$

Proof

We immediately have $\bigcup_{x \in A} U_x \subseteq A$. On the other hand, for all $x \in A$, $\{x\} \subseteq U_x$. Then

$$A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} U_x$$

Theorem 2 ⁴ Let some property p be invariant over union of sets. Let J be an index set,

$$\forall j \in J, p(A_j) \implies p\left(\bigcup_{j \in J} A_j\right)$$

Given a set A , for all element $x \in A$, if $x \in A_j \subseteq A$ and $p(A_j)$ then $p(A)$

²sometimes, we write the formula $\forall x(x \in A \implies \varphi)$ as $\forall x \in A, \varphi$

³common techniques in set theory and general topology

⁴MIT 18.102 Intro to Functional Analysis - Dr. Casey Rodriguez

Proof

immediately from theorem ??

Theorem 3 ⁵ For any index set J and $A_j \cap B_j = \emptyset$ for all $j \in J$

$$\bigcap_{j \in J} A_j \cup B_j = \bigcup_{J_A \in \mathcal{P}(J)} \left[\left(\bigcap_{j \in J_A} A_j \right) \cap \left(\bigcap_{j \in J \setminus J_A} B_j \right) \right]$$

Proof

For all $x \in X = \bigcap_{j \in J} A_j \cup B_j$, for each $j \in J$, x must be either in A_j or B_j . Let $J_A(x) = \{j : j \in J \wedge x \in A_j\} \subseteq \mathcal{P}(J)$ be the set of indices where $x \in A_j$ and let $J_B(x) = J \setminus J_A(x)$. So that

$$x \in \left(\bigcap_{j \in J_A(x)} A_j \right) \cap \left(\bigcap_{j \in J \setminus J_A(x)} B_j \right)$$

On the other hand, $\bigcap_{j \in J_A} A_j \subseteq \bigcap_{j \in J_A} A_j \cup B_j$ and $\bigcap_{j \in J \setminus J_A} B_j \subseteq \bigcap_{j \in J \setminus J_A} A_j \cup B_j$, we have

$$\begin{aligned} \left(\bigcap_{j \in J_A(x)} A_j \right) \cap \left(\bigcap_{j \in J \setminus J_A(x)} B_j \right) &\subseteq \left(\bigcap_{j \in J_A(y)} A_j \cup B_j \right) \cap \left(\bigcap_{j \in J \setminus J_A(y)} A_j \cup B_j \right) \\ &= \bigcap_{j \in J} A_j \cup B_j \\ &= X \end{aligned}$$

Invoke theorem ??

Theorem 4 Let $f : X \rightarrow Y$ be a injective function. i.e $f(x_1) = f(x_2) \implies x_1 = x_2$. Let I be an index set and $A_i \in X$ for all $i \in I$

$$f\left(\bigcap_{i \in I} A_i\right) = \bigcap_{i \in I} f(A_i)$$

and

$$f\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f(A_i)$$

Proof

\bigcap
(\subseteq) Let $y \in f(\bigcap_{i \in I} A_i)$, there exists $x \in \bigcap_{i \in I} A_i$ such that $f(x) = y$. For any $i \in I$, $x \in A_i$ implies $y = f(x) \in f(A_i)$. Therefore, $y \in \bigcap_{i \in I} f(A_i)$

⁵Topology without tears - Sidney A. Morris

(\supseteq) Let $y \in \bigcap_{i \in I} f(A_i)$. For any $y \in A_i$, there exists $x_i \in A_i$ such that $f(x_i) = y$. Since f is injective ⁶, all x_i s equal, namely x . $f(x) = y$ and $x \in \bigcap_{i \in I} A_i$ implies $y \in f(\bigcap_{i \in I} A_i)$

(\subseteq) Let $y \in f(\bigcup_{i \in I} A_i)$, there exists $x \in \bigcup_{i \in I} A_i$ such that $f(x) = y$. Let $I_x = \{i \in I : x \in A_i\} \neq \emptyset$ be all indices i where $x \in A_i$ ⁷. So, for all $i \in I_x$, $y = f(x) \in f(A_i)$. Therefore, $y = f(x) \in \bigcup_{i \in I_x} f(A_i) \subseteq \bigcup_{i \in I} f(A_i)$

(\supseteq) Let $y \in \bigcup_{i \in I} f(A_i)$. Let $I_y = \{i \in I : y \in f(A_i)\} \neq \emptyset$ be all indices i where $y \in f(A_i)$ ⁸. So for all $i \in I_y$, $y \in f(A_i)$ there exists $x_i \in A_i$ such that $f(x_i) = y$. Since f is injective, all x_i s coincide, namely x . $f(x) = y$ and $x \in \bigcup_{i \in I_y} A_i$ implies $y \in f(\bigcup_{i \in I_y} A_i)$. Moreover, $\bigcup_{i \in I_y} A_i \subseteq \bigcup_{i \in I} A_i$, so $y \in f(\bigcup_{i \in I_y} A_i) \subseteq f(\bigcup_{i \in I} A_i)$

Corollary

$$f(A \setminus B) = f(A \cap B^C) = f(A) \cap f(B^C) = f(A) \cap f(B)^C = f(A) \setminus f(B)$$

⁶if f is not injective, consider $f(1 \mapsto 1, 2 \mapsto 1)$, then $f(\{1\} \cap \{2\}) \neq f(\{1\}) \cap f(\{2\})$

⁷here, we can choose an i such that $x \in A_i$ using AC to simplify the proof

⁸avoid AC