adjunction

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1 Adjunction

Definition 1 (adjunction). Let C, D be categories, an adjunction between C and D is a pair of functors $L: C \to D$, $R: D \to C$ together with an isomorphism

$$\phi_{X,Y}: D(LX,Y) \xrightarrow{\cong} C(X,RY)$$

for each object X in C and object Y in D that is natural in both components 1 . L is called left adjoint and R is called right adjoint. We write

$$L: C \rightleftharpoons D: R$$

Example 1 (product-hom adjunction). Let $C : \text{Set} \to \text{Set}$ and $D : \text{Set} \to \text{Set}$ be defined as follows:

$$L := X \times - : \operatorname{Set} \to \operatorname{Set}$$

 $D := \operatorname{Set}(X, -) : \operatorname{Set} \to \operatorname{Set}$

Then, L, R is an adjunction with isomorphism

$$\operatorname{Set}(LZ,Y) = \operatorname{Set}(X \times Z,Y) \cong \operatorname{Set}(Z,\operatorname{Set}(X,Y)) = \operatorname{Set}(Z,RY)$$

2 Unit and Counit of an Adjunction

Definition 2 (unit, counit). Let $L: C \rightleftharpoons D: R$ be an adjunction with the isomorphism

$$\phi_{X,Y}: D(LX,Y) \xrightarrow{\cong} C(X,RY)$$

Let Y = LX, we have

$$\phi_{X,LX}: D(LX,LX) \xrightarrow{\cong} C(X,RLX)$$

Under this isomorphism, define $\eta_X: X \to RLX$ by the image of identity map $\mathrm{id}_{LX}: LX \to LX$ in D(LX, LX) under $\phi_{X,LX}$, that is $\eta_X = \phi_{X,LX} \mathrm{id}_{LX}$. For each object X in C, there is map η_X and these maps assemble a natural isomorphism of functors $C \to C$

$$\eta: \mathrm{id}_C \to RL$$

 η is called unit of the adjunction. Similarly, there is a natural isomorphism of functors $D \to D$

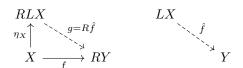
$$\epsilon: LR \to \mathrm{id}_D$$

 ϵ is called counit of the adjunction.

Proof.

todo

Example 2 (lifting property of unit). Let $f: X \to RY$ where X is an object in C and Y is an object in D. By adjunction, $f: X \to RY$ is lifted into a map $g: RLX \to RY$



¹by being natural in X, for each Y, $\phi_{-,Y}:D(L-,Y)\to C(-,RY)$ is a natural transformation of functors $C^{op}\to \mathrm{Set}$, by being natural in Y, for each X, $\phi_{X,-}:D(LX,-)\to C(X,R-)$ is a natural transformation of functors $D\to \mathrm{Set}$

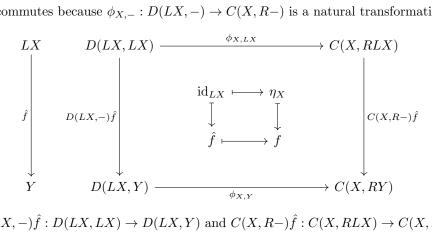
The map g can be explicitly defined by

$$q = R\hat{f}$$

where $\hat{f}: LX \to Y$ is the corresponding map to $f: X \to RY$ in the adjunction isomorphism

Proof.

The diagram below commutes because $\phi_{X,-}:D(LX,-)\to C(X,R-)$ is a natural transformation



where the maps $D(LX, -)\hat{f}: D(LX, LX) \to D(LX, Y)$ and $C(X, R-)\hat{f}: C(X, RLX) \to C(X, RY)$ are defined by

$$(D(LX, -)\hat{f})(h) = \hat{f}h$$
$$(C(X, R-)\hat{f})(h) = (R\hat{f})h$$

By commutativity,

$$f = (R\hat{f})\eta_X$$

Then, define $g: RLX \to RY$ by

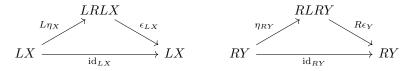
$$g = Rf$$

Example 3 (unit, counit of product-hom adjunction). Let $L : Set \rightleftharpoons Set : R$ be the product-hom adjunction, that is

$$L = X \times - : \mathsf{Set} \to \mathsf{Set} \ \ \mathit{and} \ R = \mathsf{Set}(X, -) : \mathsf{Set} \to \mathsf{Set}$$

The counit of this adjunction is the evaluation map eval: $X \times \text{Set}(X,Y) \to Y$ defined by eval(x,f) = f(x). The unit of this adjunction is the map $Z \to \text{Set}(X, X \times Z)$ defined by $z \mapsto (-, z)$ where $(-, z) : X \to X \times Z$ is the function $x \mapsto (x, z)$

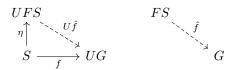
Definition 3 (adjunction). An adjunction between categories C and D is a pair of functors $L: C \to D$ and $R: D \to C$ together with natural transformations $\eta: id_C \to RL$ and $\epsilon: LR \to id_D$ such that for all objects $X \in C$ and $Y \in D$ the following diagrams commute



digest this

Free-Forgetful Adjunction in Algebra 3

Example 4 (free group). Let $U: \operatorname{Grp} \to \operatorname{Set}$ be the forgetful functor. A free group of a set S is a group FS and an injective map $\eta: S \to UFS$ satisfying the property that for any group G and map $f: S \to UG$, there exists a unique map $\hat{f}: FS \to G$ such that $f = (U\hat{f})\eta$



The free functor and forgetful functor form an adjoint pair $F: \operatorname{Set}
ightleftharpoons \operatorname{Grp} : U$ providing the isomorphism

$$Set(S, UG) \cong Grp(FS, G)$$

The unit of this adjunction is the inclusion $\eta: S \to UFS$

4 The Forgetful Functor $U: \text{Top} \to \text{Set}$ and Its Adjoint

Example 5 (left adjoint and right adjoint of the forgetful functor). Let $D : \text{Set} \to \text{Top}$ and $I : \text{Set} \to \text{Top}$ put the discrete topology and the indiscrete topology on any set. Then, D is the left adjoint and I is the right adjoint of the forgetful functor $U : \text{Top} \to Set$.

$$\operatorname{Top}(DX, Y) \cong \operatorname{Set}(X, UY)$$

 $\operatorname{Set}(UX, Y) \cong \operatorname{Top}(X, IY)$

Theorem 1. If $L: C \to D$ has a right adjoint, then L is cocontinuous. If $R: D \to C$ has a left adjoint, then R is continuous.

Proof. todo

Corollary 1. Right adjoints preserve products.

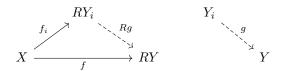
Remark 1. That explains why the construction of products, coproducts, subspaces, quotients, equalizers, coequalizers, pullbacks, and pushouts in Top must have, as an underlying set, the corresponding construction in Set. That is, if a construction exists in Top, then the forgetful functor $U: \text{Top} \to \text{Set}$ preserves it.

5 Adjoint Functor Theorem

Definition 4 (solution set condition). A functor $R: D \to C$ satisfies the solution set condition if for every object X in C, there exists a set of objects $\mathcal{Y} = \{Y_i\}$ in D and a set of morphisms

$$\mathcal{S} = \{ f_i : X \to RY_i : Y_i \in \mathcal{Y} \}$$

so that for any $f: X \to RY$, there exists Y_i and a morphism $g: Y_i \to Y$ in D such that the diagram below commutes



Theorem 2 (adjoint functor theorem). Suppose D is complete and $R: D \to C$ is a continuous functor satisfying the solution set condition, then R has a left adjoint.

6 Compactifications

Definition 5 (compactification). A compactification of a topological space is an embedding of that space as a dense subspace of a compact Hausdorff space.

6.1 The One-Point Compactification

Definition 6 (one-point compactification). A compactification obtained by adding a single point is called one-point compactification

Proposition 1. A space X has a one-point compactification if and only if X is locally compact, Hausdorff, and X is not compact. If a space has one-point compactification, then the compactification is unique.

Proof.

Let $X \hookrightarrow X^* = X \cup \{p\}$ be a compactification.

(one-point compactification implies Hausdorff)

Every subspace of a Hausdorff space is Hausdorff

(one-point compactification implies locally compact)

For any $x \in X$, as X^* is Hausdorff, x and p are separated by two open sets $U_x \ni x$ and $U_p \ni p$. $X^* \setminus U_p$ is a closed set in a compact space, then $X^* \setminus U_p$ is compact. Hence, X is locally compact.

(one-point compactification implies not compact)

If X is a compact subset of a Hausdorff space X^* , then X is closed, so that X cannot be dense in X^* . Therefore, X must not be compact.

(locally compact, Hausdorff, and not compact imply one-point compactification)

Given any locally compact, Hausdorff, and not compact space X, construct a new space by adding a point p and open neighbourhoods of p to be the complements of all compact subsets in X

(uniqueness of one-point compactification)

In X^* , the open neighbourhoods of p are precisely complements of compact subsets of X. Therefore, if there is another topology on X^* making it a compactification of X, the topology cannot be denser or coarser. Hence, uniqueness.

Theorem 3. Suppose X is locally compact, Hausdorff, and not compact and let $i: X \to X^*$ be the one-point compactification of X. If $e: X \to Y$ is any other compactification of X, then there exists a quotient map $q: Y \to X^*$ such that the diagram below commutes

$$X \xrightarrow{e} \stackrel{|_{q}}{\downarrow_{q}} X^{*}$$

Proof.

Let $X^* = X \cup \{*\}$ and $q: Y \to X^*$ as a set map is defined by

$$qy = \begin{cases} e^{-1}y & y \in eX \\ * & y \in Y \setminus eX \end{cases}$$

For any open set not containing p in X^* , its preimage under q is open due to homeomorphism. For any open set containing p, the preimage of its complement is closed due to homeomorphism.

6.2 The Stone-Čech Compactification

Definition 7 (Stone-Čech compactification). Let CH be the category where objects are compact Hausdorff spaces and morphisms are continuous functions. Let $U: CH \to \text{Top}$ be the inclusion functor. Then, U has a left adjoint $\beta: \text{Top} \to CH$ called Stone-Čech compactification.

Remark 2. For every topological space X and compact Hausdorff space Y, we have

$$CH(\beta X, Y) \cong \text{Top}(X, UY) = \text{Top}(X, Y)$$

That is equivalent to the universal lifting property as follows: Let βX be the Stone-Čech compactification of a topological space X. For every map $f: X \to Y$ where Y is a compact Hausdorff space, then there is a lift $\hat{f}: \beta X \to Y$ such that the diagram below commutes

$$\begin{array}{ccc}
X & \xrightarrow{f} UY \\
\uparrow \\
U\beta X & \beta X
\end{array}$$

where \hat{f} is the adjoint of f and η is the unit of adjunction. In the case when X is locally compact, Hausdorff, the unit $\eta: X \to \beta X$ is a compactification of X

todo: some other remarks on ultrafilters, monad, etc

7 The Exponential Topology

Definition 8 (splitting, conjoining). Let X, Y be topological spaces. Given the product-hom adjunction on X, Y as sets.

$$Set(X \times Z, Y) \cong Set(Z, Set(X, Y))$$

A topology on Top(X, Y) is

- splitting: if the continuity of $g: Z \times X \to Y$ implies the continuity of $\hat{g}: Z \to \text{Top}(X,Y)$
- conjoining: if the continuity of $\hat{g}: Z \to \text{Top}(X,Y)$ implies the continuity of $g: Z \times X \to Y$

• exponential: if it is both splitting and conjoining

Lemma 1. A topology on Top(X,Y) is conjoining if and only if the evaluation map eval : $X \times \text{Top}(X,Y) \to Y$ is continuous *Proof.*

(evaluation map is continuous implies conjoining)

Suppose Top(X,Y) has a topology such that the evaluation map is continuous, let $\hat{g}: Z \to \text{Top}(X,Y)$ be a continuous map, the composition eval(id $\times \hat{g}$) is precisely $g: X \times Z \to Y$ the adjoint of \hat{g}

$$X \times Z \xrightarrow{\operatorname{id} \times \hat{g}} X \times \operatorname{Top}(X, Y) \xrightarrow{\operatorname{eval}} Y$$

The continuity of \hat{g} implies the continuity of g

(conjoining implies evaluation map is continuous)

Suppose Top(X,Y) is equipped with a conjoining topology. Let Z = Top(X,Y), since the adjoint of evaluation map $\widehat{\text{eval}} : \text{Top}(X,Y) \to \text{Top}(X,Y)$ is the identity which is continuous, conjoining implies eval is continuous.

Lemma 2. Every splitting topology on Top(X,Y) is coarser than every conjoining topology

Proof. Let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on Top(X,Y) and \mathcal{T}_1 splitting, \mathcal{T}_2 conjoining. As \mathcal{T}_2 is conjoining, the evaluation map eval: $X \times (\text{Top}(X,Y), \mathcal{T}_2) \to Y$ is continuous. As \mathcal{T}_1 is splitting, the adjoint of eval: $X \times (\text{Top}(X,Y), \mathcal{T}_2) \to Y$ is continuous, that is, the identity map $\overrightarrow{\text{eval}} : (\text{Top}(X,Y), \mathcal{T}_2) \to (\text{Top}(X,Y), \mathcal{T}_1)$ is continuous. Then, $\mathcal{T}_1 \subseteq \mathcal{T}_2$

Theorem 4. If there exists an exponential topology on Top(X,Y), it is unique.

7.1 The Compact-Open Topology

Definition 9 (compact-open topology). Let X, Y be topological spaces. For each compact set $K \subseteq X$ and each open set $U \subseteq Y$, define

$$S(K, U) = \{ f \in \text{Top}(X, Y) : fK \subseteq U \}$$

The collection $\{S(K,U)\}\$ forms a subbasis for a topology on Top(X,Y) called the compact-open topology.

Definition 10 (finite-open topology). Let X,Y be topological spaces. For each finite set $F \subseteq X$ and each open set $U \subseteq Y$, define

$$S(F, U) = \{ f \in \text{Top}(X, Y) : fF \subseteq U \}$$

The collection $\{S(F,U)\}\$ forms a subbasis for a topology on Top(X,Y) called the finite-open topology or product topology.

Remark 3. A sequence of functions $\{f_n:[0,1]\to[0,1]\}_{n\in\mathbb{N}}$ converges to a function $f:[0,1]\to[0,1]$

- in finite-open topology if and only if it converges pointwise.
- in compact-open topology if and only if it converges uniformly.

Definition 11 (metric topology on Top(X,Y)). Let X be compact, and Y be a metric space. Then, Top(X,Y) is a metric space via metric

$$d(f,g) = \sup_{x \in X} d(fx, gx)$$

for $f, g \in \text{Top}(X, Y)$.

Theorem 5. Let X be compact and Y be a metric space. The compact-open topology on Top(X,Y) coincides with the metric topology.

Proof.

 $(metric\ topology \subseteq compact-open\ topology)$

Given an open ball $\mathcal{B}(f,\epsilon)$, we will find an open set O in the compact-open topology such that $f \in O \subseteq \mathcal{B}(f,\epsilon)$. Since X is compact, fX is compact. The collection $\left\{\mathcal{B}(fx,\frac{\epsilon}{3})\right\}_{x\in X}$ is an open cover of $fX\subseteq Y$, it has a finite subcover

$$\left\{\mathcal{B}\left(fx_1,\frac{\epsilon}{3}\right),\mathcal{B}\left(fx_2,\frac{\epsilon}{3}\right),...,\mathcal{B}\left(fx_n,\frac{\epsilon}{3}\right)\right\}$$

Define compact subsets $\{K_1, K_2, ..., K_n\}$ of X and open sets $\{U_1, U_2, ..., U_n\}$ of Y by

$$K_i = \overline{f^{-1}\mathcal{B}\Big(fx_i, \frac{\epsilon}{3}\Big)}$$
 and $U_i = \mathcal{B}\Big(fx_i, \frac{\epsilon}{2}\Big)$

For any set A, $f\overline{A} \subseteq \overline{fA}$, then for each i = 1, 2, ..., n

$$fK_i \subseteq \overline{\mathcal{B}\left(fx_i, \frac{\epsilon}{3}\right)} \subset U_i$$

Let $O = \bigcap_{i=1}^n S(K_i, U_i)$ be an open set in the compact-open topology, then $f \in O$. Moreover, let any $g \in O$, because $\{K_1, K_2, ..., K_n\}$ covers X, for any $x \in X$, there exists K_i such that $x \in K_i$, then $fx, gx \in U_i$, hence

$$d(fx, gx) \le d(fx, fx_i) + d(fx_i, gx) \le \epsilon$$

That is, $O \subseteq \mathcal{B}(f, \epsilon)$

(compact-open topology \subseteq metric topology)

Given a subbasic open set S(K,U) in compact-open topology where K is compact in X and U is open in Y, for every $f \in S(K,U)$, we will find an open ball $\mathcal{B}(f,\epsilon) \subseteq S(K,U)$. The open set U contains the compact set fK, then there exists $\epsilon > 0$ such that U contains every open ball centered in fK with radius ϵ . For every $g \in \mathcal{B}(f,\epsilon)$, for every $x \in X$, $d(fx,gx) < \epsilon$, that is, $gx \in \mathcal{B}(fx,\epsilon) \subseteq U$. Hence, $gK \subseteq gX \subseteq U$

Lemma 3 (tube lemma). Given a product space $X \times Y$, let $A \subseteq X$, $B \subseteq Y$ be compact subsets. If $A \times B$ is contained in an open set $O \subseteq X \times Y$, then there exist open sets $U_A \subseteq X$, $U_B \subseteq Y$ such that

$$A \times B \subseteq U_A \times U_B \subseteq O$$

Theorem 6. For any spaces X, Y, the compact-open topology on Top(X, Y) is splitting.

Proof. Let Z be any space, suppose $g: X \times Z \to Y$ is continuous. We will prove that the adjoint $\hat{g}: Z \to \operatorname{Top}(X,Y)$ is continuous where $\operatorname{Top}(X,Y)$ is equipped with the compact-open topology. Consider a subbasic open set S(K,U) in compact-open topology where K is compact in X and U is open in Y. We will show that $\hat{g}^{-1}S(K,U) = \{z \in Z : g(K,z) \subseteq U\}$ is open in Z. For any $z \in \hat{g}^{-1}S(K,U)$, then we have $g(K,z) \subseteq U$. Since g is continuous by the premise, $g^{-1}U = \{(x,z) \in X \times Z : g(x,z) \subseteq U\}$ is open in $X \times Z$ and contains $X \times \{z\}$. By tube lemma, there exists open sets $X \subseteq X$ 0 such that

$$K \times \{z\} \subseteq U_X \times U_Z \subseteq g^{-1}U$$

Then, $z \in U_Z \subseteq \hat{q}^{-1}S(K, U)$.

Remark 4. Some notes about compact and locally compact

- closed subsets of a compact space are compact.
- compact subsets of a Hausdorff space are closed.
- a space X is locally compact if for every $x \in X$, there exists an open set U and a compact set K such that $x \in U \subseteq K$
- let X be locally compact and Hausdorff, S be an open set in X, and $x \in S$. then, there exists an open set U such that $x \in U \subseteq \overline{U} \subseteq S$ and \overline{U} is compact.

Proof. (proof of the last statement)

Let X be locally compact and Hausdorff, S be an open set in X and $x \in S$. As X is locally compact and Hausdorff, let $x \in T \subseteq \overline{T} \subseteq X$ such that T is open and \overline{T} is compact. Let $U = S \cap T$

• If $U = \overline{U}$, \overline{U} is closed subset of a compact set \overline{T} , hence compact. We have

$$x \in U \subseteq \overline{U} \subseteq S$$

• If $U \subset \overline{U}$. For each $y \in \overline{U} \setminus U$, by Hausdorff, let V_y, W_y be open sets separating y and x. As $\overline{U} \setminus U$ is compact, let $\{V_{y_i}\}_{i=1}^n$ be the finite open cover of $\overline{U} \setminus U$. Let $A = \bigcap_{i=1}^n W_{y_i}$, then $y \in A$, A open and does not intersect $\overline{U} \setminus U$. Let $B = \bigcap_{i=1}^n \overline{W_{y_i}}$, then $y \in A \subseteq B$, B closed and does not intersect $\overline{U} \setminus U$. We have, $B \cap \overline{U}$ is closed, contained in $U \subseteq \overline{T}$, then compact. Moreover, $W \cap \overline{U}$ is contained in $U \subseteq S$. We have,

$$x \in A \cap U \subset int(B) \cap U \subset B \cap U \subset B \cap \overline{U} \subset S$$

Theorem 7. If X is locally compact and Hausdorff and Y is any space, the compact-open topology on Top(X,Y) is conjoining.

Proof. Let $\operatorname{Top}(X,Y)$ be equipped with the compact-open topology, we will show that the evaluation map eval : $X \times \operatorname{Top}(X,Y) \to Y$ is continuous. Let $(x,f) \in X \times \operatorname{Top}(X,Y)$ and $U \subseteq Y$ be an open set containing $\operatorname{eval}(x,f) = fx$. As f is continuous, $f^{-1}U$ is an open set in X containing x. As X is locally compact and Hausdorff, there exists an open set $V \subseteq X$ such that $K := \overline{V}$ is compact and $X \in V \subseteq K \subseteq f^{-1}U$. Hence, $fx \in fK \subseteq U$. Then, $V \times S(K,U)$ is an open set in $X \times \operatorname{Top}(X,Y)$ with $(x,f) \in V \times S(K,U)$. Furthermore, for any $(x_1,f_1) \in V \times S(K,U)$, $f_1x_1 \in U$, that is, $\operatorname{eval}(V \times S(K,U)) \subseteq U$

Lemma 4. If $f: X \to Y$ is a quotient map and Z is locally compact and Hausdorff then $f \times id_Z: X \times Z \to Y \times Z$ is a quotient map.

Proof. Let $f: X \to Y$ be a quotient map, we will show that the product $Y \to Z$ has the quotient topology inherited from the map $f \times \mathrm{id}_Z$. Let $(Y \times Z)_q$ denote the topological space $Y \times Z$ equipped with the quotient topology inherited from the map of sets $f \times \mathrm{id}_Z: X \times Y \to Z \times Y$ and let $\pi: X \times Y \to (Y \times Z)_q$ be the corresponding quotient map in Top. Let $Y \times Z$ denote the topological space $Y \times Z$ equipped with the product topology.

Since π is a quotient map, by characterization of quotient map, the continuity of $f \times \mathrm{id}_Z$ implies the continuity of $\mathrm{id}: (Y \to Z)_q \to Y \times Z$.

$$X \times Z$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

Now, we will show the continuity of id: $Y \times Z \to (Y \times Z)_q$. As Z is locally compact Hausdorff space, the compact-open topology on $\text{Top}(Z, (Y \times Z)_q)$ is conjoining, we will show the continuity of the adjoint $\widehat{\text{id}}: Y \to \text{Top}(Z, (Y \times Z)_q)$.

$$X$$

$$f \downarrow \qquad \qquad \qquad Y \xrightarrow{\widehat{\text{id}}} \text{Top}(Z, (Y \times Z)_q)$$

The composition $\widehat{\operatorname{id}} f$ is continuous as it is the adjoint of $\pi: X \times Y \to (Y \times Z)_q$. Hence, by characterization of quotient map, $\widehat{\operatorname{id}}$ is continuous

Theorem 8. If $X_1 \to Y_1$ and $X_2 \to Y_2$ are quotient maps and X_2, Y_1 are locally compact and Hausdorff then $X_1 \times X_2 \to Y_1 \times Y_2$ is a quotient map

Proof. The two maps below are quotient maps

$$f_1 \times \mathrm{id}_{X_2} : X_1 \times X_2 \to Y_1 \to X_2$$

 $\mathrm{id}_{Y_1} \times f_2 : Y_1 \times X_2 \to Y_1 \times Y_2$

Hence, their composition.

7.2 The Theorems of Ascoli and Arzela

I don't really understand this part

Theorem 9. If X is any space and Y is Hausdorff then a subset $A \subseteq \text{Top}(X,Y)$ has compact closure in the product topology if and only if for each $x \in X$, the set $A_x = \{fx \in Y : f \in A\}$ has compact closure in Y

Definition 12 (equicontinuous). Let X be a topological space and (Y,d) be a metric space. A family $A \subseteq \text{Top}(X,Y)$ is called equicontinuous at $x \in X$ if and only for every $\epsilon > 0$, there exists an open neighbourhood U of x so that for every $u \in U$ and for every $f \in A$, $d(fx, fu) < \epsilon$. If \mathcal{F} is equicontinuous for every $x \in X$, the family A is simply called equicontinuous.

Lemma 5. Let X be a topological space and (Y, d) be a metric space. If $A \subseteq \text{Top}(X, Y)$ is an equicontinuous family, then the subspace topology on A of Top(X, Y) with the compact-open topology is the same as the subspace topology on A of Top(X, Y) with the finite-open topology.

Lemma 6. If $A \subseteq \text{Top}(X,Y)$ is equicontinuous then the closure of A in Top using the finite-open topology is also equicontinuous.

Theorem 10 (Ascoli theorem). Let X be a locally compact Hausdorff and let (Y,d) be a metric space. A family $\mathcal{F} \subseteq \operatorname{Top}(X,Y)$ has compact closure if and only if \mathcal{F} is equicontinuous and for every $x \in X$, the set $\mathcal{F}_x := \{fx : f \in \mathcal{F}\}$ has compact closure.

Theorem 11 (Arzela theorem). Let X be compact, (Y, d) be a metric space and $\{f_n\}$ be a sequence of functions in Top(X, Y). If $\{f_n\}$ is equicontinuous and if for each $x \in X$ the set $\{f_nx\}$ is bounded then $\{f_n\}$ has a subsequence that converges uniformly.

7.3 Enrich the Product-Hom Adjunction in Top

Definition 13. Denote the set Top(X,Y) with exponential topology by Y^X provided it exists.

Theorem 12. If X, Z are locally compact Hausdorff then for any space Y, the isomorphism of sets $\text{Top}(Z \times X, Y) \to \text{Top}(Z, \text{Top}(X, Y))$ is a homomorphism of spaces under compact-open topology.

Proof.

(the map $(Y^Z)^X \to Y^{Z \times X}$ is continuous)

As X is locally compact Hausdorff, the compact-open topology on $\text{Top}(X,Y^Z)$ is conjoining, then the evaluation map $X \times (Y^Z)^X \to Y^Z$ is continuous. As Z is locally compact Hausdorff, the compact-open topology on Top(Z,Y) is conjoining, then the evaluation map $Z \times Y^Z \to Y$ is continuous. Hence, the composition is continuous

$$(Z\times X)\times (Y^Z)^X\to Z\times Y^Z\to Y$$

As the compact-open topology on $\text{Top}(Z \times X, Y)$ is splitting, the adjunct $(Y^Z)^X \to \text{Top}(Z \times X, Y)$ is continuous. the map $Y^{Z \times X} \to (Y^Z)^X$ is continuous)

As $Z \times X$ is locally compact Hausdorff, then the compact-open topology on $\text{Top}(Z \times X, Y)$ is conjoining, the evaluation map is continuous

$$Z \times (X \times Y^{Z \times X}) \to Y$$

As the compact-open topology on Top(Z,Y) is splitting, then the adjoint $X\times Y^{Z\times X}\to Y^Z$ is continuous. As the compact-open topology on $\text{Top}(X,Y^Z)$ is splitting, then the adjoint $Y^{Z\times X}\to (Y^Z)^X$ is continuous.

7.4 Compactly Generated Weakly Hausdorff Spaces

todo