My notes for

"Ravi Vakil - The Rising Sea: Foundations of Algebraic Geometry"

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Contents

Chapter 1

PRELIMINARIES

1.1 JUST ENOUGH CATEGORY THEORY TO BE DANGEROUS

1.1.1 AN INTRODUCTION TO ABELIAN CATEGORIES

Definition 1.1.1 (additive category, additive functor)

A category $\mathcal{C}$ is said to be additive if it satisfies the following properties

1. for each $A, B \in \mathcal{C}$, $\mathrm{Hom}(A, B)$ is an abelian group and composition distributes over addition of morphisms

$$f \circ (g + h) = f \circ g + g \circ h$$
$$(g + h) \circ f = g \circ f + h \circ f$$

   for every morphisms f, g, h in $\mathcal{C}$

2. $\mathcal{C}$ has a zero object

3. $\mathcal{C}$ has biproduct of finitely many objects

A functor $F : \mathcal{C} \to \mathcal{D}$ from an additive category $\mathcal{C}$ to another additive category $\mathcal{D}$ is called additive if it preserves additive structure
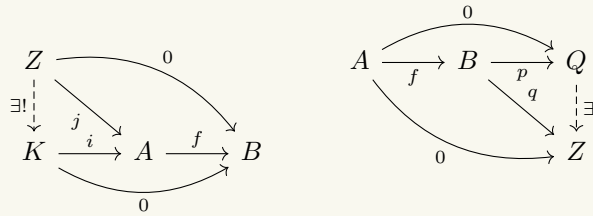
$$F(f \circ g) = F(f) \circ F(g)$$
$$F(f + g) = F(f) + F(g)$$

Definition 1.1.2 (kernel, cokernel, image, coimage)

Let $f : A \to B$ be a morphism in a category $\mathcal{C}$ with a zero object

1. a kernel for f is an object $K \in \mathcal{C}$ together with a universal map $i : K \to A$ so that $f \circ i = 0$

2. a cokernel for f is an object $Q \in \mathcal{C}$ together with a universal map $p : B \to Q$ so that $p \circ f = 0$. the object of cokernel of $f : A \to B$ is also written as A/B

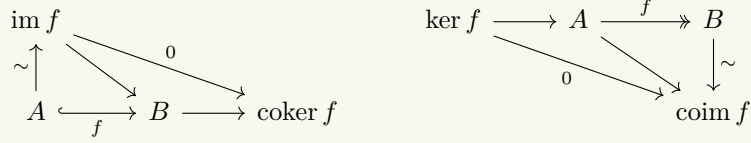3. image/coimage for f are defined by

$$\mathrm{im}\, f = \ker \mathrm{coker}\, f, \mathrm{coim}\, f = \mathrm{coker} \ker f$$

$$\operatorname{im} f \qquad\qquad \ker f \longrightarrow A \xrightarrow{\ f\ } B$$

(left diagram: $\operatorname{im} f$ with $\sim \uparrow$ from A, a 0 arrow to $\operatorname{coker} f$, $A \hookrightarrow_f B \longrightarrow \operatorname{coker} f$)

(right diagram: $\ker f \longrightarrow A \xrightarrow{f} B$, 0 arrow to $\operatorname{coim} f$, $B \downarrow \sim \operatorname{coim} f$)

By Frey-Mitchell Embedding Theorem, to prove something about diagram in some abelian category, we may assume that it is a diagram of modules over ring and use diagram chasing techniques
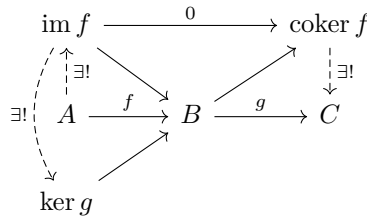
Proof. $\ker g / \operatorname{im} f$ is well-defined by the folllowing:

$$\operatorname{im} f \xrightarrow{\quad 0 \quad} \operatorname{coker} f$$

(diagram with $A \xrightarrow{f} B \xrightarrow{g} C$, maps $\exists!$ to $\operatorname{im} f$ and $\operatorname{coker} f$, and $\exists!$ dashed maps to $\ker g$)

$$\ker g$$

The composition $A \to B \to \operatorname{coker} f$ is zero, $\operatorname{im} f$ is the kernel of $B \to \operatorname{coker} f$, hence $f : A \to B$ factors through $\operatorname{im} f$ by a unique map $A \to \operatorname{im} f$

The composition $A \to B \to C$ is zero, $\operatorname{coker} f$ is the kernel of $f : A \to B$, hence $g : B \to C$ factors through $\operatorname{coker} f$ by a unique map $\operatorname{coker} f \to C$

The composition $\operatorname{im} f \to B \to coker f \to C$ is zero, hence the composition $\operatorname{im} f \to B \to C$ is zero, $\ker g$ is the kernel of $g : B \to C$, hence, $\operatorname{im} f \to B$ factors through $\ker g$ by a unique map $\operatorname{im} f \to \ker g$

Homology at B is defined by the cokernel of the map $\operatorname{im} f \to \ker g$ $\qquad\qquad\square$

Proposition 1.1.6 (factoring a long exact sequence into short exact sequences)

Given a long exact sequence $A^\bullet$

$$... \to A^{i-1} \xrightarrow{f^{i-1}} A^i \xrightarrow{f^i} A^{i+1} \to ...$$

Then it can be factored into short exact sequences

$$0 \to \ker f^i \to A^i \to \ker f^{i+1} \to 0$$

More generally, if $A^\bullet$ is only a complex then it can be factored into

$$0 \longrightarrow \ker f^i \longrightarrow A^i \longrightarrow \operatorname{im} f^i \longrightarrow 0$$

$$0 \longrightarrow \operatorname{im} f^{i-1} \longrightarrow \ker f^i \longrightarrow H^i(A^\bullet) \longrightarrow 0$$

$$0 \longrightarrow \operatorname{im} f^i \longrightarrow A^{i+1} \longrightarrow \operatorname{coker} f^i \longrightarrow 0$$

$$0 \longrightarrow H^i(A^\bullet) \longrightarrow \operatorname{coker} f^{i-1} \longrightarrow \operatorname{im} f^i \longrightarrow 0$$

Definition 1.1.7 (category of complexes)

Suppose $\mathcal{C}$ is an abelian category, then the category of complexes in $\mathcal{C}$ consists of objects being infinite complexes in $\mathcal{C}$ and morphisms $A^\bullet \to B^\bullet$ being commutative diagrams

$$
\begin{array}{ccccccc}
... \longrightarrow & A^{i-1} & \longrightarrow & A^i & \longrightarrow & A^{i+1} & \longrightarrow ... \\
& \downarrow & & \downarrow & & \downarrow & \\
... \longrightarrow & B^{i-1} & \longrightarrow & B^i & \longrightarrow & B^{i+1} & \longrightarrow ...
\end{array}
$$

The object and morphism are also called chain complex and chain map. The category of complexes in $\mathcal{C}$ is an abelian category.

Proposition 1.1.8 (chain map induces a map in homology)

A chain map $f : A^\bullet \to B^\bullet$ induces maps in homology $f^* : H^i(A^\bullet) \to H^i(B^\bullet)$

$$
\begin{array}{ccccc}
\ker f^i & \longrightarrow & A^i & \xrightarrow{f^i} & A^{i+1} \\
\downarrow & & \downarrow & & \downarrow \\
\ker g^i & \longrightarrow & B^i & \xrightarrow{g^i} & B^{i+1}
\end{array}
$$

Definition 1.1.9 (homotopic, homotopy, homotopy equivalence)

Two chain maps $f : A^\bullet \to B^\bullet$ and $g : A^\bullet \to B^\bullet$ are called homotopic if there exists a homotopy h from f into g, that is a sequence of maps $h : A^i \to B^{i-1}$ so that $f - g = dh + hd$

$$
\begin{array}{ccccccccc}
\ldots & \longrightarrow & A^{i-1} & \xrightarrow{\ d\ } & A^i & \xrightarrow{\ d\ } & A^{i+1} & \longleftarrow & \ldots \\
 & & & \searrow \ {}^h & \downarrow {}^h \swarrow & {}^h \searrow & & \swarrow & \\
\ldots & \longleftarrow & B^{i-1} & \xrightarrow{\ d\ } & B^i & \xrightarrow{\ d\ } & B^{i+1} & \longrightarrow & \ldots
\end{array}
$$

Being homotopic is an equivalence realtion. If f and g are homotopic, we write $f \sim g$.

If there is another chain map $f' : B^\bullet \to A^\bullet$ so that $f \circ f' \sim 1_{B^\bullet}$ and $f' \circ f \sim 1_{A^\bullet}$, then $A^\bullet$ and $B^\bullet$ are called homotopy equivalence which is also an equivalence relation.

Proposition 1.1.10 (homotopic maps induce the same map in homology)

If $f \sim g$ where $f : A^\bullet \to B^\bullet$ and $g : A^\bullet \to B^\bullet$ are two chain maps, then $f^* = g^*$ where $f^* : H^i(A^\bullet) \to H^i(B^\bullet)$ and $g^* : H^i(A^\bullet) \to H^i(B^\bullet)$ are the induced maps in homology

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Theorem 1.1.11 (short exact sequence of chain complexes induces a long exact sequence in homology)

A short exact sequence of chain complexes

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

induces a long exact sequence in homology

$$
\begin{array}{ccc}
H^{i+1}(A^\bullet) & \longrightarrow & \ldots \\
\ \ \ \nwarrow & & \\
H^i(A^\bullet) \longrightarrow H^i(B^\bullet) & \longrightarrow & H^i(C^\bullet) \\
& & \ \ \ \searrow \\
\ldots \longrightarrow & & H^{i-1}(C^\bullet)
\end{array}
$$

Definition 1.1.12 (exactness of functors)

Exactness of functor is characterized as follows:

1. If $F : \mathcal{A} \to \mathcal{B}$ is an additive covariant functor from abelian category to abelian category then

    (a) F is right-exact if the top sequence being exact imples the bottom sequence being exact

    $$A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$$

    $$F(A_1) \longrightarrow F(A_2) \longrightarrow F(A_3) \longrightarrow 0$$

    (b) F is left-exact if the top sequence being exact imples the bottom sequence being exact

    $$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3$$

    $$0 \longrightarrow F(A_1) \longrightarrow F(A_2) \longrightarrow F(A_3)$$

2. If $F : \mathcal{A} \to \mathcal{B}$ is an additive contravariant functor from abelian category into abelian category, then

    (a) F is right-exact if the top sequence being exact imples the bottom sequence being exact

    $$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3$$

    $$F(A_3) \longrightarrow F(A_2) \longrightarrow F(A_1) \longrightarrow 0$$

    (b) F is left-exact if the top sequence being exact imples the bottom sequence being exact

    $$A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$$

    $$0 \longrightarrow F(A_3) \longrightarrow F(A_2) \longrightarrow F(A_1)$$

A functor is exact if it is both left-exact and right-exact

Proposition 1.1.13

additive functor preserves chain complex, exact functor preserves exact sequence

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Remark 1.1.14

If A is a ring, $S \subseteq A$ is a multiplicative subset, and M is an A-module, then

1. Localization $(S^{-1}-)$ is an exact covariant functor $\mathrm{Mod}_A \to \mathrm{Mod}_{S^{-1}A}$

2. Tensor product $(- \otimes_A M)$ is a right-exact covariant functor $\mathrm{Mod}_A \to \mathrm{Mod}_A$

3. $\mathrm{Hom}(M, -)$ is a left-exact covariant functor $\mathrm{Mod}_A \to \mathrm{Mod}_A$

4. $\mathrm{Hom}(-, M)$ is a left-exact contravariant functor $\mathrm{Mod}_A \to \mathrm{Mod}_A$

Theorem 1.1.15 (FernbaHnHoF theorem, FHHF theorem)

Suppose $F : \mathcal{A} \to \mathcal{B}$ is a covariant functor from abelian category into abelian category and $C^\bullet$ is a chain complex in $\mathcal{A}$, then

1. If F is right-exact then there is a natural map

$$FH^\bullet \to H^\bullet F$$

   that is, for each i, there is a natural map $F(H^i(C^\bullet)) \to H^i(F(C^\bullet))$

2. If F is left-exact then there is a natural map

$$FH^\bullet \leftarrow H^\bullet F$$

   that is, for each i, there is a natural map $F(H^i(C^\bullet)) \leftarrow H^i(F(C^\bullet))$

3. If F is exact then there is a natural isomorphism

$$FH^\bullet \xrightarrow{\sim} H^\bullet F$$

   that is, exact functor preserves homology

Remark 1.1.16 (interaction of adjoint, limit, colimit, left-exact, right-exact)

interaction of adjoint, limit, colimit, left-exact, right-exact

1. limit commutes with limit and right adjoint functor. In particular, in abelian category, kernel is limit, hence both limit and right adjoint functor are left-exact

2. colimit commutes with colimit and left adjoint functor. In particular, in abelian category, cokernel is colimit, hence both colimit and left adjoint functor are right-exact

TODO - add some derived functor

1.2   SHEAVES

1.2.1   MOTIVATING EXAMPLE: THE SHEAF OF SMOOTH FUNCTIONS

TODO

1.2.2   DEFINITION OF SHEAVES AND PRESHEAVES

Definition 1.2.1 (presheaf, section)

A presheaf of sets $\mathscr{F}$ on a topological space X is the following data:

1. for each open $U \subseteq X$, there is a set $\mathscr{F}(U)$, the elements of $\mathscr{F}(U)$ are called sections of $\mathscr{F}$ over U

2. for each open set inclusion $U \hookrightarrow V$, there is a restriction map

$$\mathrm{res}_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$$

$$f \mapsto f|_U$$

The data is required to satisfy the following:

1. for each open set $U \subseteq X$, the restriction map $\mathrm{res}_{U,U}$ is identity $\mathrm{res}_{U,U} = \mathrm{id}_{\mathscr{F}(U)}$

2. if $U \hookrightarrow V \hookrightarrow W$ are inclusions of open sets, then the diagram below commutes

$$\mathscr{F}(W) \xrightarrow{\ \mathrm{res}\ } \mathscr{F}(V) \xrightarrow{\ \mathrm{res}\ } \mathscr{F}(U)$$
$$\underset{\mathrm{res}}{\underbrace{\qquad\qquad\qquad\qquad}}$$

In other words, presheaf is a functor from the category of open sets on X with restriction maps into the category of sets Set

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Definition 1.2.2 (germ, stalk)

For any $p \in X$, consider the collection of sections of $\mathscr{F}$ over all open sets containing p

$$\{(f, U) : p \subseteq U, f \in \mathscr{F}(U)\}$$

together with an equivalence relation $\sim$ as follows: $(f, U) \sim (g, V)$ if and only if there exists $W \subseteq U \cap V$ containing p and $f|_W = g|_W$. The equivalence class of $f \in \mathscr{F}(U)$ is called the germ of f at p. The collection of all germs at p is called stalk of $\mathscr{F}$ at p and denoted by $\mathscr{F}_p$

Remark 1.2.3 (filtered colimit, an equivalent definition of stalk)

(The Stacks project - 4.19 Filtered colimit) A diagram $M : \mathcal{I} \to \mathcal{C}$ is called directed (or filtered) if the following conditions hold:

1. the category $\mathcal{I}$ has at least one object

2. for every pair of objects x, y of $\mathcal{I}$, there exists an object z and morphisms $x \to z$ and $y \to z$

3. for every pair of morphisms $a, b : x \to y$ of $\mathcal{I}$, there exists a morphism $c : y \to z$ so that $M(c \circ a) = M(c \circ b)$ as morphisms in $\mathcal{C}$

after some gibberish - the colimit of M (filtered colimit) always exists. In particular, when $\mathcal{C}$ is Set, then

$$\mathrm{colim}_{\mathcal{I}} M = \left(\coprod_{i \in \mathcal{I}} M_i \right) / \sim$$

where $m_i \sim m_j$ if and only if there exists $k \in \mathcal{I}$, $\phi_{ik} : i \to k$, $\phi_{jk} : j \to k$, and $M(\phi_{ik})(m_i) = M(\phi_{jk})(m_j)$. In formally, two elements are equal if they eventually become equal. When $\mathcal{I}$ is the category of open sets on X with restriction maps containing p and M is $\mathscr{F}$, then stalk of $\mathscr{F}$ at p is precisely the colimit of the corresponding diagram.

Remark 1.2.4

On the condition 3 for directed diagram, the condition ensures that differences between parallel morphisms can eventually be "coequalized", or any two objects can be compared further along the line, and any two paths between the same objects eventually agree later on

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Remark 1.2.5

For any category $\mathcal{C}$, presheaf of $\mathcal{C}$ on X is also defined similarly

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Definition 1.2.6 (sheaf, separated presheaf)

A presheaf is called a sheaf if it satisfies two additional axioms: for any open set U with open cover $\{U_i\}_{i \in I}$

1. (identity axiom) if two sections $f, g \in \mathscr{F}(U)$ have $f|_{U_i} = g|_{U_i}$ for every $i \in I$, then $f = g$

2. (gluability axiom) if for each $i \in I$, there is a section $f_i \in \mathscr{F}(U_i)$ so that for every intersection $U = U_i \cap U_j$, $f_i|_U = f_j|_U$, then there exists a section $f \in \mathscr{F}(U)$ so that $f|_{U_i} = f_i$ for every $i \in I$

A presheaf satisfying only identity axiom is called separated presheaf

Remark 1.2.7

Some remarks on sheaf

1. Informally, identity axiom says there is at most one way to glue and gluability axiom says there is at least one way to glue. the identity axiom implies unique gluability

2. From the two axioms for sheaf, one can show that $\mathscr{F}(\varnothing)$ must be the final object in the target category. There is also a version of presheaf and sheaf where the empty set $\varnothing$ is not included.

3. germ and stalk on sheaf is germ and stalk on presheaf

4. The a presheaf being a sheaf is equivalent to say $\mathscr{F}\left(\bigcup_{i \in I} U_i\right)$ is the limit of the diagram consists of $\{\mathscr{F}(U_i)\}_{i \in I}$, all finite intersections of them, and restriction maps
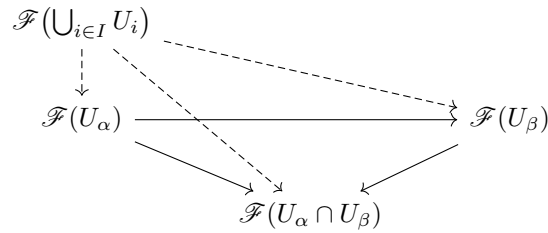
Definition 1.2.8 (restriction of sheaf)

Suppose $\mathscr{F}$ is a sheaf on topological X and $U \subseteq X$ is an open set, then the collection of $\{\mathscr{F}(V) : V \subseteq U\}$ forms a sheaf on the subspace U of topological space X

Remark 1.2.9 (skyscraper sheaf)

Let X be a topological space, $p \in X$, and a set S. Let $i_p : p \hookrightarrow X$ be the inclusion, then define $i_{p,*}S$ by

$$(i_{p,*}S)(U) = \begin{cases} S & \text{if } p \in U \\ \{\bullet\} & \text{if } p \notin U \end{cases}$$

together with evident restriction maps, they form a sheaf and called skyscraper sheaf supported at p

Remark 1.2.10 (constant presheaf, locally constant sheaf)

Let X be a topological space and S be any set, then $\underline{S}_{pre}(U) = S$ for all $U \subseteq S$ together with evident restriction maps, they form a presheaf and called constant presheaf.

Let X be a topological space and S be any set, let $\mathscr{F}(U)$ be the set of maps $U \to S$ that are locally constant, i.e. for any $p \in U$, there is an open neighbourhood of p where the function is constant, then together with evident restriction maps, they form a sheaf and called (locally) constant sheaf and denoted by $\underline{S}$

Definition 1.2.11 (pushforward sheaf)

Let $\pi : X \to Y$ be a continuous map and $\mathscr{F}$ be a presheaf on X. Define the presheaf $\pi_* \mathscr{F}$ on Y by

$$(\pi_* \mathscr{F})(V) = \mathscr{F}(\pi^{-1} V)$$

for every open set $V \subseteq Y$. Then, $\pi_* \mathscr{F}$ is a presheaf on Y and called pushforward of $\mathscr{F}$. If $\mathscr{F}$ is a sheaf on X then $\pi_* \mathscr{F}$ is a sheaf on Y.

Proposition 1.2.12 (pushforward induces maps of stalks)

Let $\pi : X \to Y$ be a continuous map and $\mathscr{F}$ be a presheaf on X. Let $\mathscr{G}$ be the pushforward of $\mathscr{F}$ on Y. If $p \in X$ and $q = \pi(p)$, then there is a natural map of stalks $\mathscr{G}_q \to \mathscr{F}_p$ induced from the map of representatives

---

Definition 1.2.13 (ringed space, $\mathscr{O}_X$-module)

Let $\mathscr{O}_X$ be a sheaf of rings on topological space X, then $(X, \mathscr{O}_X)$ is called a ringed space. $\mathscr{O}_X$ is called the structure sheaf of the ringed space. Sections of the structure sheaf $\mathscr{O}_X$ over an open set $U \subseteq X$ are called functions on U. A sheaf $\mathscr{F}$ of abelian groups on ring spaced $(X, \mathscr{O}_X)$ is called sheaf of $\mathscr{O}_X$-modules if each $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module and the structure compatible with restriction maps, that is, for every restriction of open sets $U \to V$, the diagram below commutes

$$
\begin{array}{ccc}
\mathscr{O}_X(U) \times \mathscr{F}(U) & \xrightarrow{\times} & \mathscr{F}(U) \\
{\scriptstyle (\text{res},\text{res})} \downarrow & & \downarrow {\scriptstyle \text{res}} \\
\mathscr{O}_X(V) \times \mathscr{F}(V) & \xrightarrow{\times} & \mathscr{F}(V)
\end{array}
$$

Remark 1.2.14 ($\mathbb{Z}$-modules and $\underline{\mathbb{Z}}$-modules)

a sheaf of $\underline{\mathbb{Z}}$-modules is precisely a sheaf of abelian groups where $\underline{\mathbb{Z}}$ denotes the locally constant sheaf to $\mathbb{Z}$.

Remark 1.2.15 (vector bundle)

a vector bundle is precisely a locally free sheaf of $\mathcal{E}$-modules where $\mathcal{E}$ is the sheaf of functions on the base space.

1.2.3 MORPHISMS OF PRESHEAVES AND SHEAVES

Definition 1.2.16 (morphism of presheaves, morphism of sheaves)

A morphism $\phi : \mathscr{F} \to \mathscr{G}$ of presheaves of sets on X is the data of maps $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ for every open set $U \subseteq X$ and compatible with restriction maps, that is, for every restriction of open sets $U \to V$, the diagram below commutes

$$
\begin{array}{ccc}
\mathscr{F}(U) & \xrightarrow{\phi(U)} & \mathscr{G}(U) \\
\downarrow{\scriptstyle\text{res}} & & \downarrow{\scriptstyle\text{res}} \\
\mathscr{F}(V) & \xrightarrow{\phi(V)} & \mathscr{G}(V)
\end{array}
$$

A morphism of sheaves of sets is a morphism of the underlying presheaves. Morphisms of (pre)sheaves make (pre)sheaves a category and called category of (pre)sheaves.

Let Set_X, Ab_X, etc denote the categories of sheaves of sets, abelian groups, etc.

Proposition 1.2.17 (morphism of (pre)sheaves induce moprhisms of stalks - stalkification functor)

If $\phi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves on X. If $p \in X$, then there is a natural map $\phi_p : \mathscr{F}_p \to \mathscr{G}_p$ induced from the map of representatives. On other words, there is a functor $\mathrm{Set}_X \to \mathrm{Set}$ and called stalkification functor.

Proof. Let $p \in V \subseteq U$

$\mathscr{G}_p$ is the colimit on the right side, by the property of $\mathscr{F}_p$ being a colimit, there is a map $\mathscr{F}_p \to \mathscr{G}_p$ that is compatible with the diagram.

The process of passing ϕ into $\bar{\phi}_p$ is a functor due to the uniqueness of the induced maps. $\qquad \square$

Proposition 1.2.18

Suppose $\pi : X \to Y$ is a continuous map of topological spaces, the pushforward gives a functor $\pi_* : \mathrm{Set}_X \to \mathrm{Set}_Y$

Proof. TODO $\qquad \square$

Definition 1.2.19 (sheaf Hom)

Suppose $\mathscr{F}$ and $\mathscr{G}$ are two sheaves of sets on X, let $Hom(\mathscr{F}, \mathscr{G})$ [a] be defined as follows: for every open set $U \subseteq X$, let

$$Hom(\mathscr{F}, \mathscr{G})(U) := \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$$

be the collection of morphisms of the restriction sheaves $\mathscr{F}|_U$, $\mathscr{G}|_U$. It is a sheaf and called "sheaf Hom"

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[a] *TODO - figure out how to write notation Hom correctly*

presheaves of abelian groups form an abelian category - kernel and cokernel in category of presheaves

Definition 1.2.20 (presheaf kernel, cokernel)

If $\phi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves. Define the presheaf kernel $\ker_{pre} \phi$ and the presheaf cokernel $\mathrm{coker}_{pre} \phi$ by

$$(\ker_{pre} \phi)(U) = \ker \phi(U)$$
$$(\mathrm{coker}_{pre} \phi)(U) = \mathrm{coker}\, \phi(U)$$

for every open set U. Presheaf kernel and presheaf cokernel are presheaves and kernel and cokernel in the category of presheaves.

Remark 1.2.21

presheaves of abelian groups form an abelian category and work "open set by open set". subpresheaf, image presheaf, quotient presheaf are then defined "open set by open set"

Proposition 1.2.22

Given a topological space X, if U is an open set on X, then $\mathscr{F} \mapsto \mathscr{F}(U)$ defines an exact functor from presheaves of abelian groups into abelian groups.

Proof. TODO $\square$

Proposition 1.2.23

A sequence of presheaves $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to ... \to \mathscr{F}_n \to 0$ is exact if and only if $0 \to \mathscr{F}_1(U) \to \mathscr{F}_2(U) \to ... \to \mathscr{F}_n(U) \to 0$ is exact for every open set U.

Proof. TODO $\square$

problem with sheaves

Proposition 1.2.24

If $\phi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then the presheaf kernel $\ker_{pre} \phi$ is a sheaf and the kernel in the category of sheaves.

Remark 1.2.25 (presheaf cokernel is *not cokernel* in the category of sheaves)

Let $\Omega \subseteq \mathbb{C}$ be an open set, a holomorphic function $f : \Omega \to \mathbb{C}$ is called admitting holomorphic logarithm if there exists a holomorphic function $g : \Omega \to \mathbb{C}$ so that $\exp g = f$.

Let $X = \mathbb{C}$ with the usual topology, let $\mathscr{O}_X$ be the sheaf of holomorphism functions and $\mathscr{F}$ be the presheaf of functions admitting a holomorphic logarithm, then there is an exact sequence of presheaves on X

$$0 \to \underline{\mathbb{Z}} \to \mathscr{O}_X \to \mathscr{F} \to 0$$

where $\underline{\mathbb{Z}}$ is the constant sheaf (to $\mathbb{Z}$) on X, for each open set $U \subseteq X$, $\underline{Z}(U) = \mathbb{Z}$. $\underline{\mathbb{Z}} \to \mathscr{O}_X$ is the natural inclusion, and $\mathscr{O}_X \to \mathscr{F}$ is given by $f \mapsto \exp(2\pi i f)$.

Then, $\mathscr{F}$ is not a sheaf. In other words, there is a holomorphic function that locally has holomorphic logarithm but does not have global logarithm

Proof. Let $A = \{z \in \mathbb{C} : 10 < |z| < 11\} \cap \{(x,y) \in \mathbb{C} : -1 < y\}$ and $B = \{z \in \mathbb{C} : 10 < |z| < 11\} \cap \{(x,y) \in \mathbb{C} : y < +1\}$, then the identity function $z \mapsto z$ has logarithm on A and B but not on $A \cup B$ $\qquad\square$

1.2.4 PROPERTIES DETERMINED AT THE LEVEL OF STALKS SHEAFIFICATION

properties determined at the level of stalks

Proposition 1.2.26 (section is determined by germs)

Given a seprated presheaf of sets $\mathscr{F}$ on X and any open set $U \subseteq X$, there exists an injective natural map

$$\mathscr{F}(U) \hookrightarrow \prod_{p \in U} \mathscr{F}_p$$

Proof. Let $U \subseteq X$ be an open set. For every $p \in U$, since $\mathscr{F}_p$ is the colimit of the diagram containing $\mathscr{F}(V)$ for all V containing p, hence there is a map $\mathscr{F}(U) \to \mathscr{F}_p$. The product of those maps is $\mathscr{F}(U) \to \prod_{p \in U} \mathscr{F}_p$.

We will show that this map is injective. Suppose two distinct functions $f, g \in \mathscr{F}(U)$, $f \neq g$ in $\mathscr{F}(U)$ but $f = g$ in $\prod_{p \in U} \mathscr{F}_p$. Then, $f = g$ in $\mathscr{F}_p$ for every $p \in U$. For each $p \in U$, since $f = g$ in $\mathscr{F}_p$, there exists an open set $V_p \subseteq U$ containing p so that $f = g$ in $\mathscr{F}(V_p)$. By **identity axiom**, $\bigcup_{p \in U} V_p = U$ and $f = g$ in $\mathscr{F}(V_p)$ for every p, then $f = g$ in $\mathscr{F}(U)$. Contradiction. $\qquad\square$

Remark 1.2.27

this feels like sheaf is a bunch of stalks bundled together, or at least some compressed version of stalks bundled together

Definition 1.2.28 (compatible germs)

An element $(s_p)_{p \in U} \in \prod_{p \in U} \mathscr{F}_p$ is called compatible germs if there is an open cover $\{U_i\}$ of U and each section $f_i \in \mathscr{F}(U_i)$ so that for every $p \in U_i$, the germ of f_i is precisely s_p

> **Proposition 1.2.29**
>
> The set of compatible germs on $\prod_{p \in U} \mathscr{F}_p$ is precisely the image of $\mathscr{F}(U) \hookrightarrow \prod_{p \in U} \mathscr{F}_p$

Proof.

(image of $\mathscr{F}(U) \hookrightarrow \prod_{p \in U} \mathscr{F}_p$ is a set of compatible germs) for any $s \in \mathscr{F}(U)$, the image $(s_p)_{p \in U} \in \prod_{p \in U} \mathscr{F}_p$ is compatible germs witnessed by s

(any compatible germs in $\prod_{p \in U} \mathscr{F}_p$ is the image of some section in $\mathscr{F}(U)$) let element $(s_p)_{p \in U} \in \prod_{p \in U} \mathscr{F}_p$ be compatible germs witnessed by $\{(\tilde{s}_i, U_i)\}$. For any two $(\tilde{s}_i, U_i)$ and $(\tilde{s}_j, U_j)$ with $V = U_i \cap U_j$. $\tilde{s}_i$ and $\tilde{s}_j$ agree on every $q \in V$. Since the map is injective, $\tilde{s}_i|_V = \tilde{s}_j|_V$. By gluability, there exists $s \in \mathscr{F}(U)$ that restricted to every $\tilde{s}_p$ which is the preimage of $(s_p)_{p \in U} \in \prod_{p \in U} \mathscr{F}_p$ in $\mathscr{F}(U)$ $\qquad\square$

> **Proposition 1.2.30** (isomorphism are determined by stalks)
>
> If $\phi_1, \phi_2 : \mathscr{F} \to \mathscr{G}$ are morphisms from presheaf of sets $\mathscr{F}$ to sheaf of sets $\mathscr{G}$ that induce the same map on each stalk, show that $\phi_1 = \phi_2$

Proof. For every open set $U \subseteq X$, the diagram below commutes

$$
\begin{array}{ccc}
\mathscr{F}(U) & \xrightarrow[\phi_1(U)]{\phi_2(U)} & \mathscr{G}(U) \\
\downarrow & & \downarrow \\
\prod_{p \in U} \mathscr{F}_p & \xrightarrow{\bar{\phi}_p} & \prod_{p \in U} \mathscr{G}_p
\end{array}
$$

$\mathscr{G}(U) \to \prod_{p \in U} \mathscr{G}_p$ being injective implies $\phi_2(U) = \phi_1(U)$ $\qquad\square$

> **Proposition 1.2.31** (isomorphisms are determined by stalks)
>
> A morphism of sheaves of sets is an isomorphism if and only if it induces an isomorphism of all stalks

Proof.

($\implies$) By uniqueness of the induced map, for each $p \in X$, $\bar{\phi}_p : \mathscr{F}_p \to \mathscr{G}_p$ is an isomorphism.

$$
\begin{array}{ccccc}
\mathscr{F}(U) & \xrightarrow{\phi(U)} & \mathscr{G}(U) & \xrightarrow{\psi(U)} & \mathscr{F}(U) \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{F}_p & \dashrightarrow{\bar{\phi}_p} & \mathscr{G}_p & \dashrightarrow{\bar{\psi}_p} & \mathscr{F}_p
\end{array}
$$

($\impliedby$) Since $\mathscr{G}(U) \to \prod_{p \in U} \mathscr{G}_p$ is injective, if $\phi(U)$ is not injective witnessed by distinct $a, b \in \mathscr{F}(U)$ being $\phi(U)(a) = \phi(U)(b)$, the diagram being commutative induces a contradiction from isomorphism $\prod_{p \in U} \mathscr{F}_p \to \prod_{p \in U} \mathscr{G}_p$ and composition $\mathscr{F}(U) \to \prod_{p \in U} \mathscr{F}_p \to \prod_{p \in U} \mathscr{G}_p$

$$
\begin{array}{ccc}
\mathscr{F}(U) & \xrightarrow{\phi(U)} & \mathscr{G}(U) \\
\downarrow & & \downarrow \\
\prod_{p \in U} \mathscr{F}_p & \xrightarrow{\sim} & \prod_{p \in U} \mathscr{G}_p
\end{array}
$$

TODO - show $\phi(U)$ being surjective $\qquad\square$

sheafification

Definition 1.2.32 (sheafification)

If $\mathscr{F}$ is a presheaf on X, then a morphism of presheaves $\mathrm{sh} : \mathscr{F} \to \mathscr{F}^{\mathrm{sh}}$ on X is a sheafification of $\mathscr{F}$ if $\mathscr{F}^{\mathrm{sh}}$ is a sheaf and for any sheaf $\mathscr{G}$ and any presheaf morphism $g : \mathscr{F} \to \mathscr{G}$, there exists a unique morphism of sheaves $f : \mathscr{F}^{\mathrm{sh}} \to \mathscr{G}$ making the diagram below commutes

$$
\begin{array}{ccc}
\mathscr{F} & \xrightarrow{\;\mathrm{sh}\;} & \mathscr{F}^{\mathrm{sh}} \\
& \searrow{\scriptstyle g} & \downarrow{\scriptstyle f} \\
& & \mathscr{G}
\end{array}
$$

Remark 1.2.33 (sheafification as a left adjoint functor)

Given a topological sapce, let $U : \mathrm{Sh} \to \mathrm{PSh}$ be the forgetful functor from the category of sheaves into the category of presheaves, then $\mathrm{sh} : \mathrm{PSh} \to \mathrm{Sh}$ is the unique functor that is left adjoint to U, that is

$$
\mathrm{Hom}_{\mathrm{Sh}}(\mathrm{sh}\, A, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{PSh}}(A, UB)
$$

for any presheaf A and any sheaf B

Proposition 1.2.34

Sheafification is a functor. Sheafification is unique up to isomorphism if exists. If $\mathscr{F}$ is a sheaf, then the sheafification is $\mathrm{id} : \mathscr{F} \to \mathscr{F}$

Proof. The proof is a direct consequence of universal property of sheafification. □

Remark 1.2.35 (construction of sheafification)

TODO