MA5216 Differential Geometry

this is a continuation of MA5210 Differential Manifolds - the course is about Riemannian Manifolds

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Contents

Chapter 0

PRELIMINARIES

0.1 DIFFERENTIABLE MANIFOLD

Definition 0.1.1 (manifold, coordinate chart, atlas)
A manifold M of dimension n is a Hausdorff, second-countable topological space such that every point $p \in M$ has an open neighbourhood U which is homeomorphic to an open subset $\Omega \subseteq \mathbb{R}^n$. Such an homeomorphism

$$x : U \to \Omega$$

is called (coordinate) chart. The component functions $x = (x_1, x_2, ..., x_n)$ are called local coordinates at the point p. An atlas $\mathcal{A}$ of M is collection of coordinate charts so that the collection of domains is an open cover of M.

Remark 0.1.2 (invariance of domain)
When M is connected, one can show that the dimension n is uniquely determined. This is known as invariance of domain

Definition 0.1.3 (transition function, differentiable manifold)
An atlas $\mathcal{A} = \{(U_\alpha, x_\alpha)\}_{\alpha \in A}$ on a manifold M is said to be differentiable if all the transition functions

$$x_{\alpha\beta} = x_\beta x_\alpha^{-1} : x_\alpha(U_\alpha \cap U_\beta)$$

are differentiable functions where $x_\alpha : U_\alpha \to \Omega_\alpha$, $x_\beta : U_\beta \to \Omega_\beta$ with $U_\alpha \cap U_\beta \neq \varnothing$. A differentiable manifold is a manifold with a differentiable atlas.

Remark 0.1.4 (equivalent differentiable manifold)
Two manifolds $(M, \mathcal{A}_1)$ and $(M, \mathcal{A}_2)$ are equivalent if and only if $(M, \mathcal{A}_1 \cup \mathcal{A}_2)$ is a differentiable manifold. This equivalence relation defines equivalence classes on the collection of differentiable manifolds. In each equivalence class $\{(M, \mathcal{A}_i)\}_{i \in I}$, there is a unique maximal atlas $\bigcup_{i \in I} \mathcal{A}_i$. In other words, two differentiable manifolds are said to be equivalent if they are contained in the same maximal atlas. A manifold is uniquely characterized by its maximal atlas.

Definition 0.1.5 (smooth map, differentiable structure)

Let M be a smooth manifold, a function $\phi : M \to \mathbb{R}$ is called smooth map on M if for each $x \in M$ and a chart $x : U_x \to \Omega_x$, the restriction of ϕ on U_x makes composition ϕx^{-1} smooth

$$\begin{tikzcd} U_x \arrow[r, "x"] \arrow[d, "\phi"'] & \Omega_x \arrow[dl, dashed, "\phi x^{-1}"] \\ \mathbb{R} & \end{tikzcd}$$

The set of all smooth maps on M is denoted by $\mathcal{E}(M)$. The set of all smooth maps on all open sets of M is called differentiable structure of M and denoted by S_M

Remark 0.1.6

There is a one-to-one correspondence between the differentiable structure on M and the maximal atlas on M

Remark 0.1.7

Smooth maps on open sets of M form a sheaf of rings denoted by $\mathcal{E}$. That is, for each open set U in M, $\mathcal{E}(U)$ is the ring of smooth functions defined on U.

Definition 0.1.8 (oriented differentiable manifold)

An atlas $\mathcal{A} = \{(U_\alpha, x_\alpha)\}_{\alpha \in A}$ is said to be oriented if all chart transition functions have positive Jacobian determinant. A manifold is said to be orientable if it admits an oriented atlas

Definition 0.1.9 (map of differentiable manifolds)

A map $f : M \to N$ from a differentiable manifold into a differentiable manifold is said to be a map of differentiable manifolds if the induced map $S_N \to S_M$ is well-defined, that is, if $\phi : U_N \to \mathbb{R}$ is in the differentiable structure of N, then $\psi = \phi f : U_M \to \mathbb{R}$ is in the differentiable structure of M. If h is a homeomorphism, then h is said to be an isomorphism of differentiable manifolds (or diffeomorphism).

$$\begin{tikzcd} U_M \arrow[r, "f"] \arrow[dr, dashed, "\psi"'] & U_N \arrow[d, "\phi"] \\ & \mathbb{R} \end{tikzcd}$$

Lemma 0.1.10 (partition of unity)

Let M be a differentiable manifold, and $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M. Then there exists a partition of unity $\{(U_\beta, \phi_\beta : M \to \mathbb{R})\}_{\beta \in B}$ subordinate to $\{U_\alpha\}_{\alpha \in A}$ such that

1. $\operatorname{supp} \phi_\beta$ is a compact subset of U_β for all $\beta \in B$
2. $0 \leq \phi_\beta \leq 1$ for for all $\beta \in B$
3. $\sum_{\beta \in B} \phi_\beta = 1$

Note that, since the open cover $\{U_\beta\}_{\beta \in B}$ is locally finite, the sum is well-defined at every point $x \in M$

0.2 VECTOR BUNDLE

Definition 0.2.1 (smooth vector bundle)
A smooth vector bundle of rank k is an surjective map of differentiable manifolds $\pi : E \to M$ so that for each $x \in M$, the fiber $E_x : \pi^{-1}(x)$ carries a structure of k-dimensional real vector space. Moreover, for each $x \in M$, there exists an open neighbourhood U and an isomorphism $\phi : \pi^{-1}U \to U \times \mathbb{R}^k$ so that the diagram below commutes

$$\begin{tikzcd}
\pi^{-1}U \arrow[r, "\phi"] \arrow[d, "\pi"] & U \times \mathbb{R}^k \arrow[dl] \\
U &
\end{tikzcd}$$

The map ϕ is called a local trivialization. The smooth vector bundle $M \times R^k \to M$ is called trivial bundle.

Proposition 0.2.2 (smooth bundle induces transition function)
Let $\phi : E \to M$ be a smooth bundle. Let $\phi_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times \mathbb{R}^k$ and $\phi_\beta : \pi^{-1}U_\beta \to U_\beta \times \mathbb{R}^k$ be two local trivializations with $U = U_\alpha \cap U_\beta \neq \varnothing$, then for each $x \in U$, the composition $\phi_\beta\phi_\alpha^{-1}$ define a map $\phi_{\alpha\beta} : U \to GL(k, \mathbb{R})$

$$\begin{tikzcd}
U \times \mathbb{R}^k \arrow[rr, bend left, dashed] \arrow[dr] & \pi^{-1}U \arrow[l, "\phi_\alpha"'] \arrow[r, "\phi_\beta"'] \arrow[d, "\pi"'] & U \times \mathbb{R}^k \arrow[dl] \\
& U &
\end{tikzcd}$$

with the following properties

1. $\phi_{\alpha\alpha} = 1_{\mathbb{R}^k}$
2. $\phi_{\alpha\beta}\phi_{\beta\alpha} = 1_{\mathbb{R}^k}$
3. $\phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} = 1_{\mathbb{R}^k}$

Proposition 0.2.3 (transition function induces smooth manifold)
Given a smooth manifold M and a open cover $\{U_\alpha\}_{\alpha\in A}$. If for each pair $\alpha, \beta \in A$ with $U = U_\alpha \cap U_\beta \neq \varnothing$, we have a transition function $\phi_{\alpha\beta} : U \to GL(k, \mathbb{R})$ with the properties

1. $\phi_{\alpha\alpha} = 1_{\mathbb{R}^k}$
2. $\phi_{\alpha\beta}\phi_{\beta\alpha} = 1_{\mathbb{R}^k}$
3. $\phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} = 1_{\mathbb{R}^k}$

Then, then the transition functions uniquely define a smooth vector bundle over M.

Definition 0.2.4 (section)
A section $s : M \to E$ of a smooth bundle $\pi : E \to M$ is a map so that the composition $\pi s = 1_M$. The space of sections of $\pi : E \to M$ is an $\mathcal{E}(M)$-module and denoted by $\mathcal{E}(M, E)$

Remark 0.2.5
The sections on open sets of M form a sheaf of $\mathcal{E}$-module denoted by $\mathcal{E}(-, E)$, that is, for each open set U in M, $\mathcal{E}(U, E)$ is the $\mathcal{E}(U)$-module of smooth sections defined on U

Definition 0.2.6 (frame)
A frame $\{f_1, f_2, ..., f_k\}$ of a smooth bundle $\pi : E \to M$ is a basis of the $\mathcal{E}(M)$-module $\mathcal{E}(M, E)$

Proposition 0.2.7 (dual space of smooth bundle)
Let $\pi : E \to M$ be a smooth bundle. There exists a smooth bundle $\pi^* : E^* \to M$ so that for each $x \in M$, the fiber E_x^* is the dual space of E_x. Moreover, the transition at x function is

$$\phi^*(x) = (\phi(x)^{-1})^t$$

Proposition 0.2.8 (tensor product, wedge product of smooth bundle)
Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be smooth bundles, then there exists a tensor product bundle denoted by $\pi_{E \otimes F} : E \otimes F \to M$ and wedge product bundles denoted by $\pi_{\wedge^n E} : \wedge^n E \to M$ so that for each $x \in M$, $(E \otimes F)_x \cong E_x \otimes F_x$, $(\wedge^n E)_x \cong \wedge^n E_x$. Moreover the transition functions at x are

$$\phi_{E \otimes F}(x) = \pi_E \otimes \pi_F$$
$$\phi_{\wedge^n E}(x) = \wedge^n \pi_E$$

Remark 0.2.9 (vector bundle and locally free sheaf)
There is a one-to-one correspondence between vector bundles and locally free sheaves. Hence, dual of vector bundles, tensor product and wedge product of vector bundles can be defined using sheaf.

0.3 TANGENT BUNDLE, COTANGENT BUNDLE, TENSOR

Definition 0.3.1 (germs of smooth functions)
On the sheaf of smooth functions of M, for each $x \in M$, the stalk of smooth functions at x is called the set of germs of smooth functions at x denoted by $\mathcal{E}_x$

$$\mathcal{E}_x = \varinjlim_{U \ni x} \mathcal{E}(U)$$

Note that, $\mathcal{E}_x$ is a real vector space

Definition 0.3.2 (tangent space)
On a smooth manifold M of dimension n, for each $x \in M$, the tangent space at x denoted by T_xM is the set of linear maps $D : E_x \to \mathbb{R}$ satisfy the product rule

$$D([f],[g]) = D([f])g(x) + f(x)D([g])$$

for all smooth maps $f : U_f \to \mathbb{R}$ and $g : U_g \to \mathbb{R}$ with $U_f, U_g \ni x$. Note that, T_xM is a real vector space of dimension n

Remark 0.3.3 (alternative definition for tangent space)
On $\mathbb{R}^n$, for each $x \in \mathbb{R}^n$, define the tangent space at x by

$$T_x\mathbb{R}^n = \operatorname{span}\left\{ \frac{\partial}{\partial x_1}\bigg|_x, \frac{\partial}{\partial x_2}\bigg|_x, ..., \frac{\partial}{\partial x_n}\bigg|_x \right\}$$

On a manifold M of dimension n, for each $x \in M$, let $x_1 : U_1 \to \Omega_1$ and $x_2 : U_2 \to \Omega_2$ be two charts. Let the Jacobian $d(x_2x_1^{-1})$ of $x_2x_1^{-1} : \Omega_1 \to \Omega_2$ maps the tangent space at $x_1(x)$ into the tange space at $x_2(x)$ so that $\frac{\partial}{\partial x_i}\Big|_x \mapsto \frac{\partial}{\partial y_i}\Big|_x$ for all $i = 1, 2, ..., n$. This defines an equivalence relation, for any $v \in T_{x_1(x)}\Omega_1$ and $w \in T_{x_2(x)}\Omega_2$, $v \sim w$ if and only if $d(x_2x_1^{-1})(v) = w$. The set of equivalence classes form the tangent space at x.

Definition 0.3.4 (differential)
A $f : M \to N$ be a smooth map naturally induces a linear map $df_x : T_xM \to T_{f(x)}N$ for each $x \in M$

Remark 0.3.5 (tangent bundle on $\mathbb{R}^n$)
On any open set U of $\mathbb{R}^n$, the tangent bundle is defined by the trivial bundle $\pi : TU \to U$ with $TU = U \times \mathbb{R}^n$ and

$$\pi^{-1}(x) = \{x\} \times \mathbb{R}^n \cong T_x\mathbb{R}^n = \operatorname{span}\left\{ \frac{\partial}{\partial x_1}\bigg|_x, \frac{\partial}{\partial x_2}\bigg|_x, ..., \frac{\partial}{\partial x_n}\bigg|_x \right\}$$

Definition 0.3.6 (tangent bundle, cotangent bundle)
On a manifold M of dimension n with atlas $\{U_\alpha\}_{\alpha \in A}$, the tangent bundle $TM \to M$ is the unique smooth bundle defined by transition functions

$$\begin{aligned} \phi_{\alpha\beta} : U_\alpha \cap U_\beta &\to \operatorname{Hom}(T_{x_\alpha(x)}\Omega_\alpha, T_{x_\beta(x)}\Omega_\beta) \xrightarrow{\sim} GL(k, \mathbb{R}) \\ x &\mapsto d(x_\beta x_\alpha^{-1})_x \end{aligned}$$

As a set, the tangent bundle TM can be identified with

$$TM = \coprod_{x \in M} T_xM$$

The cotangent bundle $T^*M \to M$ is the dual of the tangent bundle.

Remark 0.3.7 (local coordinates of tangent bundle and cotangent bundle)
Let M be a manifold of dimension n. Let
$$\{e_1, e_2, ..., e_n\}$$
be a frame of $TM \to M$. Let
$$\{e^1, e^2, ..., e^n\}$$
be a frame of $T^*M \to M$

Definition 0.3.8 (tensor)
Let $p, q \geq 0$, a p-contravariant q-covariant tensor (or (p,q)-tensor) on M is a section of the bundle
$$\left(\bigotimes^p TM\right) \otimes \left(\bigotimes^q T^*M\right)$$

Definition 0.3.9 (differential form)
Let $p \geq 0$, a p-form on M is a section of the bundle
$$\bigwedge^n T^*M$$

Remark 0.3.10
wedge product is a quotient space of tensor product, hence one can identify a p-form by its lift as a $(0,p)$-tensor

Remark 0.3.11 (local coordinates of tensor)
Let T be a (p,q)-tensor, then T can be written in local coordinates as follows:
$$\sum_{i_1,...,i_p,j_1,...,j_q} T^{j_1...j_q}_{i_1...i_p} e_{i_1} \otimes ... \otimes e_{i_p} \otimes e^{j_1} \otimes ... \otimes e^{j_q}$$
In Einstein notation, we also write
$$T^{j_1...j_q}_{i_1...i_p} e_{i_1} \otimes ... \otimes e_{i_p} \otimes e^{j_1} \otimes ... \otimes e^{j_q}$$

0.4 IMMERSION AND SUBMERSION

Definition 0.4.1 (immersion, differential embedding)
A map $f : M \to N$ is called an immersion for any $x \in M$, the differential
$$df_x : T_xM \to T_{f(x)}N$$
is injective. If f maps M homeomorphically into $f(M)$, then f is called a differential embedding and $f(M)$ is called a submanifold of M.

Lemma 0.4.2

Let $f : M \to N$ be an immersion from manifold of dimension m into a manifold of dimension n. For each $x \in M$, there exists a neighbourhood $U \ni x$ and a chart (V, y) on N so that $V \ni f(x)$ so that

1. $f|_U$ is a differential embedding
2. $y^{m+1} = y^{m+2} = ... = y^n = 0$ on $f(U) \cap N$

In other words, an immersion is locally differential embedding.

Lemma 0.4.3

Let $f : M \to N$ be map from manifold of dimension m into a manifold of dimension n with $m \geq n$. If $y \in N$ so that df_x have full rank for all $x \in f^{-1}(y) \subseteq M$, then $f^{-1}(y)$ is a differentiable submanifold of M of dimension $m - n$

Definition 0.4.4 (submersion)

A map $f : M \to N$ is called an immersion for any $x \in M$, the differential

$$df_x : T_x M \to T_{f(x)} N$$

is surjective

0.5 LIE BRACKET OF VECTOR FIELD

Remark 0.5.1 (vector field as an operator on smooth function)

Given a smooth manifold M and a vector field $X \in TM$, one can consider X as an operator on $\mathcal{E}(M)$ as follows:

$$\begin{aligned} X : \mathcal{E}(M) &\to \mathcal{E}(M) \\ f &\mapsto (x \mapsto df_x X(x)) \end{aligned}$$

Remark 0.5.2 (local representation of vector field as an operator on smooth function)

If we write $X = X^i \frac{\partial}{\partial x_i}$ where $\left\{ \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n} \right\}$ is the canonical frame, then

$$X(f) = X^i \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$$

Definition 0.5.3 (Lie bracket of vector field)

Let $X, Y \in TM$ be vector fields, define $[X, Y] \in TM$ by

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

for any smooth function $f \in \mathcal{E}(M)$

Remark 0.5.4 (product rule of Lie bracket)

$$[X, Y](f \cdot g) = [X, Y](f) \cdot g + f \cdot [X, Y](g)$$

Proposition 0.5.5 (Jacobi identity)

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

0.6 LOCAL FLOW GENERATED BY VECTOR FIELD

Proposition 0.6.1 (integral curve)
Given a vector field $X \in TM$ and $p \in M$, there exists a smooth curve $c : (-\epsilon, +\epsilon) \to M$ for some $\epsilon > 0$ with $c(0) = p$ and $c'(t) = X(c(t))$. Such smooth curve c is called an integral curve for X at p. Moreover, given any $\epsilon > 0$, if an integral curve exists then it is also unique.

Proposition 0.6.2 (local flow generated by a vector field)
Given a vector field $X \in TM$ and $p \in M$ and an open neighbourhood U of p , there exists an open neighbourhood $W \subseteq U$ of p and a smooth function $F : (-\epsilon, +\epsilon) \times W \to U$ so that $F(0, q) = q$ and

$$dF_{(t,q)}\left(\frac{\partial}{\partial t}, 0\right) = X(F(t, q))$$

for any $(t, q) \in (-\epsilon, +\epsilon) \times W$. Such F is called a local flow generated by the vector field X. In particular, $t \mapsto F(t, q)$ is an integral curve at $q \in W$. For any $t \in (-\epsilon, +\epsilon)$, define the smooth map

$$\begin{aligned} F_t : W &\to U \\ q &\mapsto F(t, q) \end{aligned}$$

F_t induces a group structure with $F_0 = 1_W$ and

$$F_{t+s} = F_t \circ F_s$$

0.7 EXERIOR DIFFERENTIATION AND INTERIOR MULTIPLICATI

0.7.1 EXTERIOR DIFFERENTIATION

Exterior derivative admits an axiomatic definition as follows:

Definition 0.7.1 (exterior derivative)
$d : \mathcal{E}^p(\mathbb{R}^n) \to \mathcal{E}^p(\mathbb{R}^n)$ is the unique $\mathbb{R}$-linear map satisfies the following:

1. for any $f \in \mathcal{E}^0(\mathbb{R}^n) = \mathcal{E}(\mathbb{R}^n)$
$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

2. for any $a \in \mathcal{E}^p(\mathbb{R}^n)$ and $b \in \mathcal{E}^q(\mathbb{R}^n)$
$$d(a \wedge b) = da \wedge b + (-1)^p a \wedge db$$

Proposition 0.7.2 (Poincaré lemma)
$d^2 = 0$

Proof. The proof can be done by induction, when $p = 0$,

$$d^2 f = \sum_{i=1}^{n} d\left(\frac{\partial f}{\partial x_i}\right) \wedge dx_i = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j \wedge dx_i = 0$$

The last equality is due to $dx_j \wedge dx_i + dx_i \wedge dx_j = 0$ for all $0 \le i, j \le n$. When $p > 0$, any p-form can be written as $f dx_I = f \wedge dx_I = a \wedge b$ for some subset $I \subseteq [n]$, then

$$d^2(a \wedge b) = d^2 a \wedge b + a \wedge d^2 b = 0$$

The last equality is due to a, b are of degree $< p$ □

0.7.2 INTERIOR MULTIPLICATION

Definition 0.7.3 (interior multiplication)
Let X be a vector field on manifold M, define

$$i_X : \mathcal{E}\left(\bigwedge^p T^* M\right) \to \mathcal{E}\left(\bigwedge^{p-1} T^* M\right)$$
$$\omega \mapsto (X_2, ..., X_k \mapsto \omega(X, X_2, ..., X_k))$$

where $\omega \in \mathcal{E}(\bigwedge^p T^* M)$ is identified as a map $\prod^p TM \to \mathbb{R}$

0.8 LIE GROUP AND LIE ALGEBRA

0.8.1 LIE GROUP

Definition 0.8.1 (Lie group, left multiplication, right multiplication)
A Lie group is a topological group G which is also a smooth manifold. For any $a \in G$, the map L_a is called left multiplication, the map R_a is called right multiplication

$$L_a : G \to G \qquad\qquad R_a : G \to G$$
$$x \mapsto ax \qquad\qquad x \mapsto xa$$

Left multiplication and right multiplication are diffeomorphism

Definition 0.8.2 (Lie group homomorphism)
A Lie group homomorphism is a smooth group homomorphism $f : G \to H$, that is

$$f(xy) = f(x)f(y)$$

or equivalently

$$fL_x = L_{f(x)}f$$

Remark 0.8.3
Some examples of Lie groups

1. general linear group
$$GL(n, \mathbb{R}) = \{A \in M(n, \mathbb{R}) : \det A \neq 0\}$$
2. orthogonal group
$$O(n) = \{A \in M(n, \mathbb{R}) : A^t A = 1\}$$

0.8.2 LIE ALGEBRA

Remark 0.8.4 (tangent space at identity)
Let G be a Lie group with the identity element e, for each $g \in G$, $L_g : G \to G$ is a diffeomorphism, so its differential at e

$$d(L_g)|_e : T_eG \to T_gG$$

is a vector space isomorphism. This defines an equivalent relation on the set of all tangent spaces of G and the equivalence classes are precisely T_eG.

Remark 0.8.5 (pushforward of vector field by left multiplication)
Given a vector field X and any $g, x \in G$, define the vector field $L_{g*}X$ by

$$(L_{g*}X)(x) = d(L_g)|_{g^{-1}x}X(g^{-1}x)$$

Definition 0.8.6 (left-invariant vector field)
A vector field X on Lie group G is called left-invariant if

$$L_{g*}X = X$$

Chapter 1

RIEMANNIAN MANIFOLD

1.1 RIEMANNIAN MANIFOLD

Definition 1.1.1 (Riemannian metric, Riemannian manifold)
A smooth manifold M equipped with a Riemannian metric g which is a positive definite and symmetric $(0,2)$-tensor. That is,

1. $g_x(v,v) \geq 0$ for any $v \in T_xM$ and $g_x(v,v) = 0$ if and only if $v = 0$
2. $g_x(v,w) = g_x(w,v)$ for any $v,w \in T_xM$

Definition 1.1.2 (Riemannian isometry)
A Riemannian isometry $f : (M, g_M) \to (N, g_N)$ is a diffeomorphism so that $f^*g_N = g_M$ where f^* is the induced metric on M from N. In other words, $g_M(v,w) = g_N(df_x(v), df_x(w))$ for all $v,w \in T_xM$

$$\begin{tikzcd}
T_xM \times T_xM \arrow[r, "df_x \times df_x"] \arrow[dr, "g_M = f^*g_N"'] & T_{f(x)}N \times T_{f(x)}N \arrow[d, "g_N"] \\
& \mathbb{R}
\end{tikzcd}$$

Two manifolds are said to be isometric if there exists a Riemannian isometry between them. Moreover, being isometric is an equivalent relation.

Definition 1.1.3 (Riemannian immersion)
Let $f : (M, g_M) \to (N, g_n)$ be an immersion, f is called a Riemannian immersion (or Riemannian embedding) if $f^*g_N = g_M$. Riemmannian immersion is also called isometric immersion.

Definition 1.1.4 (Riemannian submersion)
Let $f : (M, g_M) \to (N, g_N)$ be a submersion, f is called a Riemannian submersion if $f^*g_N = g_M$ on $(\ker df_x)^\perp$ for all $x \in M$.

Theorem 1.1.5
Every differentiable manifold M can be equipped with a Riemannian metric g

1.2 PSEUDO-RIEMANNIAN MANIFOLD

Definition 1.2.1 (pseudo-Riemannian metric, pseudo-Riemannian manifold)
A smooth manifold M equipped with a pseudo-Riemannian metric g which is a non-degenerate and symmetric $(0,2)$-tensor. That is,

1. for any nonzero $v \in T_xM$, there exists a $w \in T_xM$ so that $g_x(v,w) \neq 0$
2. $g_x(v,w) = g_x(w,v)$ for any $v, w \in T_xM$

Remark 1.2.2 (index)
Given a pseudo-Riemannian manifold (M,g), for any $x \in M$, T_xM admits a decomposition

$$T_xM = P \oplus N$$

such that g is positive definite on P and negative definite, that is

$$g(v,v) > 0 \text{ for all nonzero } v \in P$$
$$g(w,w) > 0 \text{ for all nonzero } w \in N$$

The decomposition is not unique but their dimensions are well-defined. For a connected pseudo-Riemannian manifold, the dimensions of decomposition is a constant and called index.

1.3 GROUP AND RIEMANNIAN MANIFOLD

1.3.1 ISOMETRY GROUP

Definition 1.3.1 (isometry group, isotropy subgroup, homogenenous Riemannian manifold)
Given a Riemannian manifold (M,g), let $\mathrm{Iso}(M,g)$ denote the group of Riemannian isometries. For any $x \in M$, the isotropy subgroup (or stabilizer subgroup) of $\mathrm{Iso}(M,g)$ is defined by

$$\mathrm{Iso}_x(M,g) = \{f \in \mathrm{Iso}(M,g) : f(x) = x\}$$

A Riemannian manifold is said to be homogenenous if $\mathrm{Iso}(M,g)$ acts transitively on M, that is given any pair of points $x, y \in M$, there exists a isometry $f \in \mathrm{Iso}(M,g)$ so that $f(x) = y$

Remark 1.3.2
The isometry group of $(\mathbb{R}^n, g_{\mathbb{R}^n})$ is

$$\mathrm{Iso}(\mathbb{R}^n, g_{\mathbb{R}^n}) = \mathbb{R}^n \rtimes O(n)$$

where $\mathbb{R}^n \rtimes O(n)$ is the semi-direct product of $\mathbb{R}^n$ with $O(n)$ the orthogonal subgroup of $GL(n,\mathbb{R})$. $\mathbb{R}^n$ and $O(n)$ are regarded as subgroup of $\mathrm{Iso}(\mathbb{R}^n, g_{\mathbb{R}^n})$ via the identification

$$\begin{aligned} \mathbb{R}^n &\hookrightarrow \mathrm{Iso}(\mathbb{R}^n, g_{\mathbb{R}^n}) \\ v &\mapsto (x \mapsto x + v) \end{aligned} \qquad \begin{aligned} O(n) &\hookrightarrow \mathrm{Iso}(\mathbb{R}^n, g_{\mathbb{R}^n}) \\ O &\mapsto (x \mapsto Ox) \end{aligned}$$

Under that identification

$$\mathbb{R}^n \cap O(n) = \{1_{\mathbb{R}^n}\}$$

Moreover, the isotropy subgroup at $x \in \mathbb{R}^n$ is isomorphic to $O(n)$

$$\mathrm{Iso}_p(\mathbb{R}^n, g_{\mathbb{R}^n}) \cong O(n) \text{ and } \mathbb{R}^n \cong \frac{\mathrm{Iso}(\mathbb{R}^n, g_{\mathbb{R}^n})}{\mathrm{Iso}_x(\mathbb{R}^n, g_{\mathbb{R}^n})}$$

Furthermore, any homogeneous Riemannian manifold (M, g) can be written as

$$(M, g) \cong \frac{\mathrm{Iso}(M, g)}{\mathrm{Iso}_x(M, g)}$$

for any $x \in M$

1.4 COVERING MAP

Definition 1.4.1 (covering map)
Let $\pi : M \to N$ be a smooth map between two smooth manifolds, then π is said to be a covering map if for each $q \in N$, there exists an open neighbourhood V of q in N so that $\pi^{-1}V$ is a disjoint union of open sets $\{U_i\}_{i\in I}$ in M such that $\pi|_{U_i} : U_i \to V$ is a diffeomorphism for every $i \in I$

Definition 1.4.2 (deck transformation, normal covering)
Let $\pi : M \to N$ be a covering map, the group of deck transformations is

$$\Gamma = \{\phi : M \to M : \phi \text{ is a diffeomorphism such that } \pi\phi = \pi\}$$

A covering $\pi : M \to N$ is said to be a normal covering if Γ acts transitively on the fibers, that is for each $q \in N$ and $p_1, p_2 \in \pi^{-1}q$, there exists $\phi \in \Gamma$ so that

$$\phi(q_1) = q_2$$

Definition 1.4.3 (local isometry)
A smooth map $f : (M, g_M) \to (N, g_N)$ is said to be a local isometry if for each $p \in M$, there exists a neighbourhood U of p so that $f|_U : U \to f(U)$ is a Riemannian isometry

Definition 1.4.4 (Riemannian covering)
A map $\pi : (M, g_M) \to (N, g_N)$ is said to be a Riemannian covering if it is a covering map and a local isometry. If π is a Riemannian covering, then each deck transformation $\phi \in \Gamma$ is a Riemannian isometry on (M, g)

1.5 LOCAL REPRESENTATION OF METRIC

the writing of Peterson's book was too bad, I couldn't summarize what he/she tried to say. This section just contains some examples without any proper mathematics

1.6 SOME TENSOR CONCEPTS

1.6.1 TYPE CHANGE BY RIEMANNIAN METRIC

Given a Riemannian manifold (M, g), a tensor of type (s, t) can be turned into a $(s-k, t+k)$ for any $k \in \mathbb{Z}$, the operation is called type change and is done as follows:
Let $\{E_1, E_2, ..., E_n\}$ be a frame of TM and $\{\sigma_1, \sigma_2, ..., \sigma_n\}$ be a frame of T^*M so that $\sigma^j(E_i) = \delta_i^j$. Then any $v \in TM$ and $\omega \in T^*M$ can be written as

$$v = v^i E_i = \sigma^i(v) E_i$$
$$\omega = \omega_j \sigma^j = \omega(E_j)\sigma^j$$

Any (s, t)-tensor T can be written as

$$T = T^{i_1 i_2 ... i_s}_{j_1 j_2 ... j_t} E_{i_1} \otimes ... \otimes E_{i_s} \otimes \sigma^{j_1} \otimes ... \otimes \sigma^{j_t}$$

where each $T^{i_1 i_2 ... i_s}_{j_1 j_2 ... j_t} \in \mathcal{E}(M)$ is a smooth function on M. In particular, for a Riemannian metric g

$$g = g_{ij}\sigma^i \otimes \sigma^j$$

One can use g to *change* a section in TM into a section in T^*M as follows: Let $v \in TM$, then the corresponding section in T^*M is $g(v, -)$

$$w \mapsto g(v, w)$$

Let $\omega \in T^*M$, then the corresponding section in TM is the unique section $v \in TM$ so that

$$g(v, w) = \omega(w)$$

for all $w \in TM$. Hence, one can *change* E_i into a section in T^*M and σ^j into a section in TM by a Riemannian metric g as follows:

$$E_i \mapsto g_{ij}\sigma^j$$
$$\sigma^j \mapsto g^{ij} E_i$$

Note that, (g_{ij}) is the matrix of g with respect to frame $\{\sigma_1, \sigma_2, ..., \sigma_n\}$, then $(g^{ij}) = (g_{ij})^{-1}$. If one choose an orthonormal frame of TM, then (g_{ij}) is the identity matrix.

THE RICCI TENSOR

A Ricci tensor ($(1,1)$-tensor) is of the form

$$\mathrm{Ric} = \mathrm{Ric}^i_j \, E_i \otimes \sigma^j$$

Its type change as a $(0,2)$-tensor is

$$\mathrm{Ric} = \mathrm{Ric}_{kj} \, \sigma^k \otimes \sigma^j = g_{ij} \, \mathrm{Ric}^i_j \, \sigma^k \otimes \sigma^j$$

Its type as a $(2,0)$-tensor is

$$\mathrm{Ric} = \mathrm{Ric}_{ik} \, E_i \otimes E_k = g^{kj} \, \mathrm{Ric}^i_j \, E_i \otimes E_k$$

THE CURVATURE TENSOR

A curvature tensor ($(1,3)$-tensor) is of the form

$$R = R^l_{ijk} E_l \otimes \sigma^i \otimes \sigma^j \otimes \sigma^k$$

As a $(0,4)$-tensor, we have

$$R = R_{mijk} \sigma^m \otimes \sigma^i \otimes \sigma^j \otimes \sigma^k = g_{lm} R^l_{ijk} \sigma^m \otimes \sigma^i \otimes \sigma^j \otimes \sigma^k$$

As a $(2,2)$-tensor we have

$$R = R^{lm}_{jk} E_l \otimes E_m \otimes \sigma^j \otimes \sigma^k = g^{mi} R^l_{ijk} E_l \otimes E_m \otimes \sigma^j \otimes \sigma^k$$

1.6.2 CONTRACTION

Given an (s,t)-tensor T for $s,t > 1$, contraction of T is a $(s-1, t-1)$ tensor. Let

$$T = T^{i_1 i_2 \ldots i_s}_{j_1 j_2 \ldots j_t} E_{i_1} \otimes \ldots \otimes E_{i_s} \otimes \sigma^{j_1} \otimes \ldots \otimes \sigma^{j_t}$$

Then, the contraction of T at index (j_m, i_n) is

$$C(T) = T^{i_1 \ldots i_{n-1} k i_{n+1} \ldots i_s}_{j_1 \ldots j_{m-1} k j_{m+1} \ldots j_t} E_{i_1} \otimes \ldots \otimes E_{i_{n-1}} \otimes E_{i_{n+1}} \otimes \ldots \otimes E_{i_s} \otimes \sigma^{j_1} \otimes \ldots \otimes \sigma^{j_{m-1}} \otimes \sigma^{j_{m+1}} \otimes \ldots \otimes \sigma^{j_t}$$

where each $T^{i_1 \ldots i_{n-1} k i_{n+1} \ldots i_s}_{j_1 \ldots j_{m-1} k j_{m+1} \ldots j_t} = \sum_k T^{i_1 \ldots i_{n-1} k i_{n+1} \ldots i_s}_{j_1 \ldots j_{m-1} k j_{m+1} \ldots j_t}$ in usual notation. Then the Ricci tensor ($(1,1)$-tensor) is a contraction of the curvature tensor ($(1,3)$-tensor)

1.6.3 INNER PRODUCT OF TENSOR

One can define an inner product of two tensors of the same type. *How to define it, the book didn't care enough to mention it rigorously*

Chapter 2

DERIVATIVE

2.1 LIE DERIVATIVE

2.1.1 DIRECTIONAL DERIVATIVE

Definition 2.1.1 (directional derivative)
Given a vector field $X \in TM$, directional derivative operator ∇_X, D_X, L_X sends a smooth function on M into a smooth function on M and defined by

$$\nabla_X f = D_X f = L_X f = X(f)$$

2.1.2 LIE DERIVATIVE

Given a vector field $X \in TM$ and $p \in M$, a local flow F^t at p for some small $t > 0$ sends p into $F^t(p)$. Lie derivative L_X is defined as the first order term of the Taylor expansion across the local flow.

LIE DERIVATIVE ON SMOOTH FUNCTION

Let $f : M \to \mathbb{R}$ be a smooth function, then Lie derivative $L_X f$ of f at p is defined by

$$f(F^t(p)) = f(p) + t(L_X f) + o(t)$$

This coincides with directional derivative

LIE DERIVATIVE ON VECTOR FIELD

Let $Y \in TM$ be a vector field, then consider the pullback of $Y(F^t(p))$ into T_pM. Note that, the map F^{-t} sends $F^t(p)$ into p, then Lie derivative $L_X Y$ of Y at p is defined by

$$d(F^{-t})Y(F^t(p)) = Y(p) + t(L_X Y) + o(t)$$

Proposition 2.1.2
Given two vector fields $X, Y \in TM$, then $L_X Y$ is a vector field and

$$L_X Y = [X, Y]$$

LIE DERIVATIVE ON TENSOR

. Given any tensor T of type $(0,k)$ Let T be a $(0,k)$-tensor, define the pullback $(F^t)^* : \bigotimes^k T^*M \to \bigotimes^k T^*M$ by

$$((F^t) * T)(Y_1, ..., Y_k) = T(d(F^t)(Y_1), ..., d(F^t)(Y_k))$$

Definition 2.1.3 (Lie derivative on tensor)
Lie derivative of a $(0,k)$-tensor T is the unique tensor $L_X T$ of type $(0,k)$ so that

$$(F^t)^* T = T + t(L_X T) + o(t)$$

Proposition 2.1.4
Given a $(0,k_1)$-tensor T_1 and $(0,k_2)$-tensor T_2, then

$$L_X(T_1 \cdot T_2) = (L_X T_1) \cdot T_2 + T_1 \cdot (L_X T_2)$$

Proposition 2.1.5
If T is a $(0,k)$-tensor and $f : M \to \mathbb{R}$ is a smooth function, then

$$L_{fX} T = f L_X T + \sum_{i=1}^{k} (L_{Y_i} f) T$$

Lemma 2.1.6
If a vector field X vanishes at p, then the Lie derivative $L_X T$ at p depends only on the value of T at p

Proposition 2.1.7 (the generalized Jacobi identity)
For all vector fields X, Y, Z and tensor T

$$(L_X L)_Y T = 0$$

THE FUCK?

2.1.3 LIE DERIVATIVE AND METRIC