ma5259_hw2

Khanh Nguyen

October 2024

1 Q1

Let $\xi_1, \xi_2, ...$ be a sequence of i.i.d coin tosses with bias $P(\xi_1 = H) = p$. (1) Find the probability that we will see HH before seeing TT. (2) Conditioned on $\xi_1\xi_2 = HT$, find the probability that we will see HT again before seeing either HH or TT

1.1 seeing HH before seeing TT

Let $X_1, X_2, ...$ be a stochastic process with state space $S = (HH, HT, TH, TT)$ defined by $X_i = (\xi_i, \xi_{i+1})$. Then $X_1, X_2, ...$ is a Markov chain with transition matrix Π defined by

$$\begin{aligned}
\Pi : S \times S &\to \mathbb{R} \\
(HH, HH) &\mapsto p \\
(HH, HT) &\mapsto 1 - p \\
(HT, TH) &\mapsto p \\
(HT, TT) &\mapsto 1 - p \\
(TH, HH) &\mapsto p \\
(TH, HT) &\mapsto 1 - p \\
(TT, TH) &\mapsto p \\
(TT, TT) &\mapsto 1 - p
\end{aligned}$$

For $x \in S$, let $T_x = \min\{n \geq 0 : X_n = x\}$ be the first time seeing x. We want to find $P(T_{HH} < T_{TT})$
We have

$$\begin{aligned}
P_{HH}(T_{HH} < T_{TT}) = 1 \\
P_{TT}(T_{HH} < T_{TT}) = 0
\end{aligned}$$

For any $x \in S \setminus \{HH, TT\}$, we have

$$\begin{aligned}
P_x(T_{HH} < T_{TT}) &= \sum_{y \in S} P_x(X_1 = y, T_{HH} < T_{TT}) \\
&= \sum_{y \in S} P_x(X_1 = y) P_x(T_{HH} < T_{TT} | X_1 = y) \\
&= \sum_{y \in S} \Pi(x, y) P_y(T_{HH} < T_{TT})
\end{aligned}$$

Then

$$\begin{aligned}
P_{HT}(T_{HH} < T_{TT}) &= \Pi(HT, TH) P_{TH}(T_{HH} < T_{TT}) + \Pi(HT, TT) P_{TT}(T_{HH} < T_{TT}) \\
&= p P_{TH}(T_{HH} < T_{TT}) \\
P_{TH}(T_{HH} < T_{TT}) &= \Pi(TH, HT) P_{HT}(T_{HH} < T_{TT}) + \Pi(TH, HH) P_{HH}(T_{HH} < T_{TT}) \\
&= (1 - p) P_{HT}(T_{HH} < T_{TT}) + p
\end{aligned}$$

Solve the system of two equations, we have

$$P_{HT}(T_{HH} < T_{TT}) = \frac{p^2}{1-(1-p)p}$$
$$P_{TH}(T_{HH} < T_{TT}) = \frac{p}{1-(1-p)p}$$

Hence

$$\begin{aligned}
P(T_{HH} < T_{TT}) &= \sum_{x\in S} P(X_0 = x, T_{HH} < T_{TT}) && (1)\\
&= \sum_{x\in S} P(X_0 = x) P_x(T_{HH} < T_{TT}) && (2)\\
&= P(X_0 = HH) + P(X_0 = HT)\frac{p^2}{1-(1-p)p} + P(X_0 = TH)\frac{p}{1-(1-p)p} && (3)\\
&= p^2 + p(1-p)\frac{p^2}{1-(1-p)p} + p(1-p)\frac{p}{1-(1-p)p} && (4)\\
&= \frac{2p^2 - p^3}{1-p+p^2} && (5)
\end{aligned}$$

When $p = 1/2$, $P(T_{HH} < T_{TT}) = 1/2$

1.2 conditioned on $\xi_1\xi_2 = HT$, seeing HT again before seeing either HH or TT

For $x \in S$, let $T_x = \min\{n > 0 : X_n = x\}$ be the first time seeing x in positive time. Let $A = \{T_{HT} < T_{HH}, T_{TT}\}$, we want to find $P_{HT}(A)$

We have

$$\begin{aligned}
P_{HT}(A) &= \sum_{x\in S} P_{HT}(X_1 = x, A)\\
&= \sum_{x\in S} P_{HT}(X_1 = x) P_{HT}(A|X_1 = x)\\
&= \sum_{x\in S} \Pi(HT, x) P_{HT}(A|X_1 = x)
\end{aligned}$$

As $P_{HT}(A|X_1 = TT) = 0$, then

$$\begin{aligned}
P_{HT}(A) &= \Pi(HT, TH) P_{HT}(A|X_1 = TH)\\
&= pP_{TH}(A)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
P_{TH}(A) &= \sum_{x\in S} P_{TH}(X_1 = x, A)\\
&= \sum_{x\in S} P_{TH}(X_1 = x) P_{TH}(A|X_1 = x)\\
&= \sum_{x\in S} \Pi(TH, x) P_{TH}(A|X_1 = x)
\end{aligned}$$

As $P_{TH}(A|X_1 = HH) = 0$ and $P_{TH}(A|X_1 = HT) = 1$, then

$$\begin{aligned}
P_{TH}(A) &= \Pi(TH, HT) P_{TH}(A|X_1 = HT)\\
&= 1 - p
\end{aligned}$$

Hence,

$$P_{HT}(A) = pP_{TH}(A) = p(1-p)$$

Remark 1. *This question is simple enough that we can enumerate all possible sequences length 4 of coin flips and analyze the probability* $\{HTHH, HTHT, HTTH, HTTT\}$

2 Q2

Consider a 3-state Markov chain with transition $\Pi(1,1) = 1 - \Pi(1,2) = p$, $\Pi(2,1) = 1 - \Pi(2,3) = q$ and $\Pi(3,3) = 1$. Let

$$G(x,y) = \mathbb{E}\left[\sum_{n=0}^{\infty} 1_{\{X_n=y\}} \middle| X_0 = x\right] = \sum_{n=0}^{\infty} \Pi^n(x,y)$$

denote the expected number of visits to state y given that the Markov chain starts at x. Find $G(1,2)$ and $G(2,1)$
Transition matrix

$$\Pi = \begin{bmatrix} p & 1-p & 0 \\ q & 0 & 1-q \\ 0 & 0 & 1 \end{bmatrix}$$

We have

$$G(x,y) = \Pi^0(x,y) + \sum_{n=1}^{\infty} \Pi^n(x,y) = \Pi^0(x,y) + \Pi(x,y) + \sum_{n=2}^{\infty} \Pi^n(x,y)$$

We have

$$\begin{aligned}
&G(x,y) - \Pi^0(x,y) - \Pi(x,y) \\
&= \sum_{n=2}^{\infty} \Pi^n(x,y) \\
&= \sum_{n=2}^{\infty} P(X_n = y | X_0 = x) \\
&= \sum_{n=2}^{\infty} \sum_{z \in S} P(X_n = y, X_1 = z | X_0 = x) && \text{(marginalize)} \\
&= \sum_{n=2}^{\infty} \sum_{z \in S} P(X_1 = z | X_0 = x) P(X_n = y | X_1 = z, X_0 = x) && \text{(conditional probability)} \\
&= \sum_{n=2}^{\infty} \sum_{z \in S} \Pi(x,z) P(X_n = y | X_1 = z) && \text{(Markov)} \\
&= \sum_{z \in S} \Pi(x,z) \sum_{n=2}^{\infty} P(X_n = y | X_1 = z) && \text{(Tonelli)} \\
&= \sum_{z \in S} \Pi(x,z) \sum_{m=1}^{\infty} \Pi^m(z,y) && (m = n-1) \\
&= \sum_{z \in S} \Pi(x,z)(G(z,y) - \Pi^0(z,y))
\end{aligned}$$

If $x = y$, we have

$$(1 - \Pi(x,x))G(x,x) = 1 + \sum_{z \in S - \{x\}} \Pi(x,z) G(z,x)$$

If $x \neq y$, we have

$$(1 - \Pi(x,x))G(x,y) = \Pi(x,y)G(y,y) + \sum_{z \in S - \{x,y\}} \Pi(x,z) G(z,y)$$

Given that $G(3,1) = G(3,2) = 0$, we have the following equations

$$\begin{aligned}
(1-p)G(1,1) &= 1 + (1-p)G(2,1) \\
(1-p)G(1,2) &= (1-p)G(2,2) \\
G(2,1) &= qG(1,1) \\
G(2,2) &= 1 + qG(1,2)
\end{aligned}$$

Solve the system of equations, we have

$$\begin{aligned}
G(2,1) &= \frac{q}{(1-p)(1-q)} \\
G(1,2) &= \frac{1}{1-q}
\end{aligned}$$

3 Q3

Consider a random walk on $\{0,1,...,L\}$ with transition matrix $\Pi(i,i+1) = \Pi(i,i-1) = \frac{1}{2}$ for each $i \in \{1,...,L-1\}$ and $\Pi(0,0) = \Pi(L,L) = 1$. For each $i \in \{0,1,...,L\}$, find the return probability $f_{ii} = P(X_n = i \text{ for some } n \in \mathbb{N} \,|X_0 = i)$

Let $A_i^m = \{X_n = i \text{ for some } n \in \mathbb{N} : n \geq m\}$. It is clear that $f_{00} = f_{LL} = 1$, for any $i \in \{1,...,L-1\}$, note that $\{A_i^1, X_1 = i \pm 1\} = \{A_i^2, X_1 = i \pm 1\}$, we have

$$\begin{aligned}
f_{ii} &= P(A_i^1|X_0 = i) \\
&= P(A_i^1, X_1 = i-1|X_0 = i) + P(A_i^1, X_1 = i+1|X_0 = i) \\
&= P(A_i^2, X_1 = i-1|X_0 = i) + P(A_i^2, X_1 = i+1|X_0 = i) \\
&= P(A_i^2|X_1 = i-1, X_0 = i)P(X_1 = i-1|X_0 = i) \\
&+ P(A_i^2|X_1 = i+1, X_0 = i)P(X_1 = i+1|X_0 = i) \\
&= P(A_i^2|X_1 = i-1)\Pi(i,i-1) + P(A_i^2|X_1 = i+1)\Pi(i,i+1) \qquad \text{(Markov property)} \\
&= P(A_i^1|X_0 = i-1)\Pi(i,i-1) + P(A_i^1|X_0 = i+1)\Pi(i,i+1)
\end{aligned}$$

For any $a,b,c \in \{0,1,...,L\}$ such that $a < b < c$. Starting from a, probability of visiting a before visiting c is $\frac{c-b}{c-a}$, probability of visiting c before visiting a is $\frac{b-a}{c-a}$, therefore

$$\begin{aligned}
P(A_i^1|X_0 = i-1) &= \frac{i-1}{i} \\
P(A_i^1|X_0 = i+1) &= \frac{L-i-1}{L-i}
\end{aligned}$$

Hence, for $i \in \{1,...,L\}$

$$f_{ii} = \frac{1}{2}\left(\frac{i-1}{i} + \frac{L-i-1}{L-i}\right) = 1 - \frac{L}{2i(L-i)}$$

4 Q4

Let $X = (X_n)_{n\in\mathbb{N}_0}$ be a symmetric random walk on $\mathbb{Z}$. For $x \in \mathbb{Z}$, let $P_x(\cdot)$ respectively $\mathbb{E}_x[\cdot]$ denote the probability and expectation for X with initial condition $X_0 = x$. Let $T_y = T_y(X) = \min\{n \geq 0 : X_n = y\}$ be the first time X visits the point $y \in \mathbb{Z}$. Let $L \in \mathbb{N}$ and define $T = \min\{T_0, T_L\}$ the first time hitting either 0 or L. Show that for $0 \leq x \leq L$, $P_x(T < \infty) = 1$ and furthermore, $\mathbb{E}_x[T] < \infty$.

When X starts at either 0 or L, $T = 0$. Suppose $F = \mathbb{Z} \cap [1, L-1]$ is non-empty and X starts at $x \in F$. Let $T_{F^c} = \min\{n \geq 0 : X_n \notin F\}$. Since the only transition from a state in F to a state in F^c is transitioning to 0 or L. Hence, $T = T_{F^c}$. Therefore, there exists constant $C > 0$ and $\rho \in (0,1)$ such that

$$P_x(T \geq n) \leq C\rho^n$$

Then,

$$P_x(T < \infty) = 1 - P_x(T = \infty) = 1 - \lim_{n\to\infty} P_x(T \geq n) = 1$$

Moreover, we have

$$\begin{aligned}
\mathbb{E}_x[T] &= \sum_{n=0}^{\infty} nP_x(T = n) \\
&= \sum_{n=1}^{\infty} P_x(T \geq n) \\
&\leq \sum_{n=1}^{\infty} C\rho^n \\
&= \frac{C\rho}{1-\rho} < \infty
\end{aligned}$$

5 Q5

Let X be a simple symmetric random walk on $\mathbb{Z}$ with $X_0 = 0$. Let $a \in \mathbb{N}$

(a) By conditioning on the Markov chain's first visit to a and using the symmetry of the random walk, show that

$$P_0\left(\max_{0 \le i \le n} X_i \ge a\right) = P_0(X_n = a) + 2P_0(X_n \ge a+1) = P_0(X_n \notin [-a, a-1])$$

(b) Deduce from the above identity that for any $a \in \mathbb{N}$

$$P_a(T_0 > n) = P_0\left(\max_{0 \le i \le n} X_i \le a-1\right) = P_0(X_n \in [-a, a-1])$$

(c) Deduce from (b) that $E_a[T_0] = \infty$ for all $a \in \mathbb{N}$

(d) Deduce from (b) for all $n \in \mathbb{N}$

$$P_0(X_1, ..., X_{2n} \neq 0) = P_0(X_{2n} = 0)$$

5.1 a

Let $T_a = \min\{n \ge 0 : X_n = a\}$ be the first time X visits a. Then,

$$\left\{\max_{0 \le i \le n} X_i \ge a\right\} = \{T_a \le n\}$$

Moreover, we can write $\{T_a \le n\}$ as a disjoint union of events

$$\begin{aligned}
&\{T_a \le n\} \\
&= \{T_a \le n, X_n = a\} \amalg \{T_a \le n, X_n > a\} \amalg \{T_a \le n, X_n < a\} \\
&= \{X_n = a\} \amalg \{X_n > a\} \amalg \{T_a \le n, X_n < a\}
\end{aligned}$$

We can write $\{T_a \le n, X_n < a\}$ as a disjoint union of events

$$\{T_a \le n, X_n < a\} = \coprod_{i=0}^{n} \{T_a = i, X_n < a\}$$

We have

$$\begin{aligned}
P_0(T_a \le n, X_n < a) &= \sum_{i=0}^{n} P_0(T_a = i, X_n < a) \\
&= \sum_{i=0}^{n} P_0(X_n < a | T_a = i) P_0(T_a = i) && \text{(conditional probability)} \\
&= \sum_{i=0}^{n} P_0(X_n < a | X_i = a) P_0(T_a = i) && \text{(Markov property)} \\
&= \sum_{i=0}^{n} P_0(X_n > a | X_i = a) P_0(T_a = i) && \text{(symmetric random walk)} \\
&= \sum_{i=0}^{n} P_0(X_n > a | T_a = i) P_0(T_a = i) && \text{(Markov property)} \\
&= \sum_{i=0}^{n} P_0(X_n > a, T_a = i) && \text{(conditional probability)} \\
&= P_0(X_n > a)
\end{aligned}$$

Therefore,

$$P_0\left(\max_{0 \le i \le n} X_i \ge a\right) = P_0(T_a \le n) = P_0(X_n = a) + 2P_0(X_n > a)$$

5.2 b

In previous part, $P_0(T_a \leq n)$ is the probability of visiting a within n steps starting from 0. By symmetry, let $P_a(T_0 \leq n)$ be the probability of visiting 0 within n steps starting from 0, then $P_a(T_0 \leq n) = P_0(T_a \leq n)$. We have

$$P_a(T_0 > n) = 1 - P_a(T_0 \leq n) = 1 - P_0(T_a \leq n) = 1 - P_0\left(\max_{0\leq i\leq n} X_i \geq a\right)$$

5.3 c

We have

$$\begin{aligned}
\mathbb{E}_a[T_0] &= \sum_{n=0}^{\infty} nP_a(T_0 = n) \\
&= \sum_{n=1}^{\infty} P_a(T_0 \geq n) \\
&= \sum_{n=0}^{\infty} P_a(T_0 > n) \\
&= \sum_{n=0}^{\infty} P_0(X_n \in [-a, a-1]) \\
&= \mathbb{E}_0\left[\sum_{n=0}^{\infty} 1_{\{X_n \in [-a,a-1]\}}\right] \\
&\geq \mathbb{E}_0\left[\sum_{n=0}^{\infty} 1_{\{X_n = 0\}}\right]
\end{aligned}$$

Since symmetric random walk is recurrent, starting from 0, X visits 0 infinitely many times. Therefore, $\mathbb{E}_a[T_0] \geq \mathbb{E}_0\left[\sum_{n=0}^{\infty} 1_{\{X_n=0\}}\right] = \infty$

5.4 d

Starting at 0, at time 1, $X_1 = 1$ or $X_1 = -1$. By symmetry,

$$P_0(X_1, ..., X_{2n} \neq 0) = P_1(X_2, ..., X_{2n} \neq 0) = P_1(T_0 > 2n) = P_0(X_{2n} \in [-1, 0])$$

Since, $2n$ is even X_{2n} cannot be -1, $P_0(X_{2n} \in [-1, 0]) = P_0(X_{2n} = 0)$. Hence

$$P_0(X_1, ..., X_{2n} \neq 0) = P_0(X_{2n} = 0)$$

6 Q6

A general random walk X on $\mathbb{Z}$ is a Markov chain on $\mathbb{Z}$ with transition probability $\Pi(x, y) = \Pi(0, y-x) = \mu(y-x)$ for all $x, y \in \mathbb{Z}$ where μ is a probability measure on $\mathbb{Z}$. In particular, the increment $X_1 - X_0$, $X_2 - X_1$, ... are i.i.d with distribution μ. Find a stationary measure for X. Can X be positive recurrent?

6.1 a stationary measure for X

Let $\nu : \mathbb{Z} \to [0, \infty)$ be a stationary measure on $\mathbb{Z}$ satisfying $\sum_{x\in\mathbb{Z}} \nu(x) > 0$. By stationary, we must have $\nu = \nu\Pi$, that is, for every $y \in \mathbb{Z}$,

$$\nu(y) = \sum_{x\in\mathbb{Z}} \nu(x)\Pi(x, y)$$

We have

$$\sum_{x\in\mathbb{Z}} \nu(0)\mu(-x) = \nu(0) = \sum_{x\in\mathbb{Z}} \nu(x)\Pi(x, 0) = \sum_{x\in\mathbb{Z}} \nu(x)\mu(-x)$$

Therefore,

$$\sum_{z\in\mathbb{Z}} (\nu(x) - \nu(0))\mu(-x) = 0$$

A stationary measure for x is one so that $x \mapsto \nu(x) - \nu(0)$ is orthogonal to $x \mapsto \mu(-x)$. In particular, one choice of ν is the uniform measure, that is $\nu(x) = 1$ for all $x \in \mathbb{Z}$

6.2 can X be positive recurrent?

No in general. Because suppose X is positive recurrent, then there exists a unique stationary distribution. Let $p : S \to \mathbb{R}$ be a non-zero vector that is orthogonal to $x \mapsto \mu(-x)$, then we can construct two stationary measures. Let $x \in S$ such that $p(x) > 0$, and take $\nu_1(0) = \nu_2(0) = 1$

$$\nu_1(x) - \nu_1(0) = p(x)$$
$$\nu_2(x) - \nu_2(0) = 2p(x)$$

then,

$$\nu_2(x) = 1 + 2(\nu_1(x) - 1)$$

which is not linear.

7 Q7

Let X be a random walk on $\{0, 1, ..., L\}$ with transition matrix Π such that $\Pi(0, 1) = \Pi(L, L-1) = 1$, $\Pi(i, i+1) = 1 - \Pi(i, i-1) = p$ for all $1 \leq i \leq L-1$. Find the stationary distribution of this random walk. If $X_0 = 0$, find the expected number of visits to L before returning 0 as well as the expected time of returning.

We can assume that $p \in (0, 1)$ as if $p = 0$ or $p = 1$, the answers are trivial. We can also assume that $L \geq 3$ as if $L = 1$ or $L = 2$, the answers are trivial.

It is clear that X is irreducible since for any pair of states, there exists a path with positive probability connecting them. As the number of states are finite, therefore, X is recurrent. Therefore, there exists a unique stationary distribution for X

7.1 stationary distribution of X

Let $\mu : \{0, 1, ..., L\} \to \mathbb{R}$ be the stationary measure of X. Then, we have $\mu = \mu\Pi$, that is, for every $y \in \{0, 1, ..., L\}$,

$$\mu_y = \sum_{x \in \{0,1,...,L\}} \mu_x \Pi(x, y)$$

We have the following

$$\begin{aligned}
\mu_0 &= \mu_1 \Pi(1, 0) = \mu_1(1-p) \\
\mu_1 &= \mu_0 \Pi(0, 1) + \mu_2 \Pi(2, 1) = \mu_0 + \mu_2(1-p) \\
\mu_L &= \mu_{L-1} \Pi(L-1, L) = \mu_{L-1} p \\
\mu_{L-1} &= \mu(L-2)\Pi(L-2, L-1) + \mu_L \Pi(L, L-1) = \mu_{L-2} p + \mu_L \\
\mu_i &= \mu_{i-1} \Pi(i-1, i) + \mu_{i+1} \Pi(i+1, i) = \mu_{i-1} p + \mu_{i+1}(1-p) && \text{(if } i \in \{2, ..., L-2\}\text{)}
\end{aligned}$$

For convenient, let $\nu : \{0, 1, ..., L\} \to \mathbb{R}$ be defined by

$$\nu_i = \begin{cases} \frac{\mu_0}{p} & \text{if } y = 0 \\ \mu_i & \text{if } y \in \{1, ..., L-1\} \\ \frac{\mu_L}{1-p} & \text{if } y = L \end{cases}$$

We have the relation, for all $i \in \{1, ..., L-1\}$,

$$\nu_i = \nu_{i-1} p + \nu_{i+1}(1-p)$$

Let r^i be a basic solution for ν_i, then

$$1 = \frac{1}{r} p + r(1-p)$$

Solve for r we have

$$r_1 = 1 \text{ or } r_2 = \frac{p}{1-p}$$

Hence, general solution for ν_i is

$$\nu_i = a r_1^i + b r_2^i = a + b\left(\frac{p}{1-p}\right)^i$$

Since, $\mu_0 = \mu_1(1-p)$ and $\mu_L = \mu_{L-1} p$, then

$$p\nu_0 = \nu_1(1-p) \text{ and } (1-p)\nu_L = \nu_{L-1} p$$

We must have $a = 0$, then $\nu_i = br_2^i$. As $\mu : \{0, 1, ..., L\} \to \mathbb{R}$ is a distribution, we must have

$$\begin{aligned}
1 &= \mu_0 + \mu_L + \sum_{i=1}^{L-1} \mu_i \\
&= p\nu_0 + (1-p)\nu_L + \sum_{i=1}^{L-1} \nu_i \\
&= pb + (1-p)br_2^L + \sum_{i=1}^{L-1} br_2^i \\
&= b\left(p + (1-p)r_2^L + \sum_{i=1}^{L-1} r_2^i\right) \\
&= b\left(p + (1-p)r_2^L + \frac{r_2 - r_2^L}{1 - r_2}\right) \\
&= b\frac{2p(1-p)(1-r_2^L)}{1-2p}
\end{aligned}$$

Therefore,

$$b = \frac{1-2p}{2p(1-p)(1-r_2^L)} \text{ and } \nu_i = br_2^i$$

$$\mu_i = \begin{cases} pb & \text{if } i = 0 \\ br_2^i & \text{if } i \in \{1, ..., L-1\} \\ (1-p)br_2^L & \text{if } i = L \end{cases}$$

7.2 find the expected number of visits to L before returning to 0, as well as the expected time of returning

Let $T_x = \min\{n > 0 : X_n = x\}$ be the first time X visits $x \in S$ in positive time, then the cycle trick construction states that if x is recurrent then, $\mu_x : S \to [0, \infty)$ defined by

$$\mu_x(y) = \mathbb{E}_x\left[\sum_{n=0}^{T_x - 1} 1_{\{X_n = y\}}\right] = \sum_{n=0}^{\infty} P_x(X_n = y, n < T_x)$$

is a stationary measure and if $\mathbb{E}_x[T_x] < \infty$, then $\frac{\mu_x(y)}{\mathbb{E}_x[T_x]}$ is a stationary distribution. Since X is recurrent, then stationary distribution exists and unique. That is, $\frac{\mu_x(y)}{\mathbb{E}_x[T_x]}$ coincides with the stationary distribution calculated in the previous part.

Hence, we need to calculate $\mu_0(L)$ and $\mathbb{E}_0[T_0]$. We have

$$\begin{aligned}
\mathbb{E}_0[T_0] &= \sum_{n=1}^{\infty} nP_0(T_0 = n) \\
&= \sum_{n=1}^{\infty} P_0(T_0 \geq n)
\end{aligned}$$

if $n = 1$, $P_0(T_0 \geq 1) = 1$ and if $n = 2$

$$P_0(T_0 \geq 2) = P_0(X_1 \neq 0) = 1$$

if $n \geq 3$, without loss of generality, let X_i distributed according to the stationary distribution

$$\begin{aligned}
P_0(T_0 \geq n) &= P_0(X_1 \neq 0, ..., X_{n-2} \neq 0, X_{n-1} \neq 0) \\
&= P(X_{n-1} \neq 0 | X_{n-2} \neq 0) P_0(X_1 \neq 0, ..., X_{n-2} \neq 0) && \text{(Markov property)} \\
&= P(X_1 \neq 0 | X_0 \neq 0) P_0(T_0 \geq n-1) && \text{(stationary)}
\end{aligned}$$

We will calculate the constant $P(X_1 \neq 0 | X_0 \neq 0)$

$$
\begin{aligned}
P(X_1 \neq 0 | X_0 \neq 0) &= \frac{P(X_1 \neq 0, X_0 \neq 0)}{P(X_0 \neq 0)} \\
&= \frac{P(X_1 \neq 0) - P(X_1 \neq 0, X_0 = 0)}{P(X_0 \neq 0)} \\
&= \frac{P(X_1 \neq 0) - (P(X_0 = 0) - P(X_1 = 0, X_0 = 0))}{P(X_0 \neq 0)} \\
&= \frac{P(X_1 \neq 0) - P(X_0 = 0)}{P(X_0 \neq 0)} \\
&= \frac{1 - 2pb}{1 - pb}
\end{aligned}
$$

Therefore, for $n \geq 3$,

$$P_0(T_0 \geq n) = \left(\frac{1 - 2pb}{1 - pb}\right)^{n-2}$$

Then,

$$\mathbb{E}_0[T_0] = 1 + 1 + \sum_{n=3}^{\infty} \left(\frac{1 - 2pb}{1 - pb}\right)^{n-2} = \frac{1}{pb} = \frac{2(1-p)(1-r_2^L)}{1-2p}$$

Then,

$$\mu_0(L) = \mathbb{E}_0[T_0](1-p)br_2^L = r_2^{L-1} = \left(\frac{p}{1-p}\right)^{L-1}$$

8 Q8

At each time $n \in \mathbb{N}$, we light up ξ_n many candles where $(\xi_n)_{n \in \mathbb{N}}$ are i.i.d Poisson random variables with mean 1. The lifetimes of candles are assume to be i.i.d integer-valued with a common probability mass function $f = (f_k)_{k \in \mathbb{N}}$. Assume that the mean lifetime $\lambda = \sum_{k=1}^{\infty} kf(k) < \infty$. Let X_n denote the number of candles burning at time n. Is X a Markov chain? Is it irreducible? Is it aperiodic? Find the stationary distribution if it exists.

8.1 is X a Markov chain?

Let A_t be the number of new candles at time $t \in \mathbb{N}_0$. At time n, let z be the lifetime of a candle ($z = 0$ means that a candle does not last until next time step), then the number of candles burning at time t is

$$
\begin{aligned}
X_t &= \sum_{i=1}^{A_0} 1_{\{z_i^0 \geq t\}} + ... + \sum_{i=1}^{A_n} 1_{\{z_i^n \geq t-n\}} + ... + \sum_{i=1}^{A_t} 1_{\{z_i^t \geq 0\}} \\
X_{t+1} &= \sum_{i=1}^{A_0} 1_{\{z_i^0 \geq t+1\}} + ... + \sum_{i=1}^{A_n} 1_{\{z_i^n \geq t-n+1\}} + ... + \sum_{i=1}^{A_t} 1_{\{z_i^t \geq 1\}} + \sum_{i=1}^{A_{t+1}} 1_{\{z_i^{t+1} \geq 0\}}
\end{aligned}
$$

Then,

$$X_{t+1} = X_t - \left(\sum_{i=1}^{A_0} 1_{\{z_i^0 = t\}} + ... + \sum_{i=1}^{A_n} 1_{\{z_i^n = t-n\}} + ... + \sum_{i=1}^{A_t} 1_{\{z_i^t = 0\}}\right) + \sum_{i=1}^{A_{t+1}} 1_{\{z_i^{t+1} \geq 0\}}$$

That is, the number of burning candles at time $t+1$ is the number of candles at time t that have positive remaining times plus a random variable. It is generally not a Markov chain unless we can infer the number of candles at time t that have positive remaining times from the total number of candles at time t.

Let's construct a counterexample. Suppose distribution of candle lifetime is

$$f(k) = \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{if } k = 1 \end{cases}$$

Then,

$$
\begin{aligned}
P(X_2 = \cdot | X_1 = 1, X_0 = 0) &= 1 + Pois(1) \\
P(X_2 = \cdot | X_1 = 1, X_0 = 1) &= Pois(1)
\end{aligned}
$$

8.2 is it irreducible?

Yes, it is possible to go from a state to any other state. Let $X_n = N$, there is a positive probability such that $X_{n+1} = N+k$ for $k \in \mathbb{N}$ equals to probability of burning k plus the number of candles at time n with 0 remaining time. To go to 0, there is a finite number of steps so that all burning candles turn off, that's a positive probability transition ($(e^{-1})^{\text{time}}$)

8.3 is it aperiodic?

Yes, it is aperiodic since there is a positive probability of staying at the same state by burning exactly the number of candles with 0 remaining time in the next time step.

8.4 find the stationary distribution if it exists

We have

$$X_t = \sum_{i=1}^{A_0} 1_{\{z_i^0 \geq t\}} + ... + \sum_{i=1}^{A_n} 1_{\{z_i^n \geq t-n\}} + ... + \sum_{i=1}^{A_t} 1_{\{z_i^t \geq 0\}}$$

Since $\infty > \lambda = \sum_{k=1}^{\infty} kf(k) = \sum_{k=1}^{\infty} P(z \geq k)$. Therefore, $P(z \geq k) \to 0$ as $k \to \infty$.
TODO

9 Q9

A birth-death chain is a Markov chain with state space $\mathbb{N}_0$ and transition probabilities $\Pi(0,1) = p_0$, $\Pi(0,0) = r_0 = 1 - p_0$ and for each $k \in \mathbb{N}$, we have $\Pi(k, k+1) = p_k$, $\Pi(k, k-1) = q_k$, $\Pi(k,k) = r_k$ with $p_k + q_k + r_k = 1$. Find a necessary and sufficient condition for this Markov chain to be irreducible. Show that when X is irreducible, it is in fact reversible and identify the reversible measure. Find a necessary and sufficient condition for this Markov chain to be positive recurrent.

9.1 Find a necessary and sufficient condition for this Markov chain to be irreducible

Let $0 \leq m < n$, since the Markov chain can only transition to the adjacent natural numbers, a valid transition from m to n must go through $n-1$, therefore, $\Pi(n-1, n) = p_{n-1} > 0$ for all $n > 0$. Similarly, let $0 \leq n < m$, all valid transition from m to n must go through $n+1$, therefore, $\Pi(n+1, n) = q_{n+1} > 0$ for all $n \geq 0$. Hence, there necessary condition is

$$p_k > 0 \text{ for all } k \geq 0$$
$$q_k > 0 \text{ for all } k \geq 1$$

Moreover, it is also the sufficient condition. If $0 \leq m < n$, there exists a path with positive probability

$$\Pi(m, m+1)\Pi(m+1, m+2)...\Pi(n-1, n) = p_m p_{m+1} ... p_{n-1} > 0$$

If $0 \leq n < m$, there exists a path with positive probability

$$\Pi(n, n-1)\Pi(n-1, n-2)...\Pi(m+1, m) = q_n q_{n-1} ... q_{m+1} > 0$$

9.2 Show that when X is irreducible, it is in fact reversible and identify the reversible measure

When X is irreducible, it satisfies the loop condition since probability of every loop is a product of terms $\Pi(k, k+1)\Pi(k, k-1)$. If we reverse direction, the probability stays the same.
Detailed balance condition for reversible measure $\nu : \mathbb{N}_0 \to \mathbb{R}$

$$\nu(k)\Pi(k, k+1) = \nu(k+1)\Pi(k+1, k)$$

therefore $\frac{\nu(k+1)}{\nu(k)} = \frac{\Pi(k,k+1)}{\Pi(k+1,k)} = \frac{p_k}{q_{k+1}}$. Let $\nu(0) = 1$, then

$$\nu(n) = \prod_{k=0}^{n} \frac{p_k}{q_{k+1}}$$

9.3 Find a necessary and sufficient condition for this Markov chain to be positive recurrent

In order to avoid confusion, we define the following definitions:

1. **irreducible**: there exists a positive probability path from any state to any state
2. **recurrent**: starting from any state, the probability of returning in finite time is 1
3. **positive recurrent**: for all $x \in S$, $\mathbb{E}_x[T_x] < \infty$

It easy to see that **positive recurrent** implies **recurrent**, since if there exists a probability of escaping, $\mathbb{E}_x[T_x] = \infty$. Moreover, **recurrent** does not imply **irreducible** as we can make probability of staying in state being 1, so that $\mathbb{E}_x[T_x] = 1$

The question is to find the necessary and sufficient condition so that X is all of **irreducible** and **positive recurrent**.

9.3.1 necessary part

If X is **irreducible** and **recurrent**, then it has a stationary distribution, since the reversible measure is stationary, then it can be normalized. i.e. $\sum_{n\in\mathbb{N}_0} \nu(v) < \infty$. Let $\mu : \mathbb{N}_0 \to \mathbb{R}$ be the stationary distribution

$$\mu(v) = \frac{\nu(v)}{\sum_{n\in\mathbb{N}_0} \nu(v)}$$

We have

$$\mathbb{E}_x[T_x] = \sum_{n=1}^{\infty} nP_x(T_x = n) = \sum_{n=1}^{\infty} P_x(T_x \geq n)$$

Note that $P_x(T_x \geq 2m-1) = P_x(T_x \geq 2m)$, then

$$\mathbb{E}_x[T_x] = 2\sum_{m=1} P_x(T_x \geq 2m)$$

We will calculate $P_x(T_x \geq 2m)$,

$$P_x(T_x \geq 2) = 1 - P_x(T_x < 2) = P_x(T_x = 0) = r_0$$

If $m \geq 2$, suppose X is distributed according to the stationary distribution, we have

$$\begin{aligned}
P_x(T_x \geq 2m) &= P_x(T_x \geq 2m-1) \\
&= P_x(X_1 \neq 0, ..., X_{2m-2} \neq 0) \\
&= P(X_{2m-2} \neq 0 | X_{2m-3} \neq 0) P_x(X_1 \neq 0, ..., X_{2m-3} \neq 0) \\
&= P(X_{2m-2} \neq 0 | X_{2m-3} \neq 0) P_x(T_x \geq 2m-2) \\
&= P(X_{2m-2} \neq 0) P_x(T_x \geq 2m-2) && (\{X_{2m-3} \neq 0\} \text{ is a sure event}) \\
&= (1 - \mu(0)) P_x(T_x \geq 2m-2)
\end{aligned}$$

Therefore,

$$\mathbb{E}_x[T_x] = 2\sum_{m=1} P_x(T_x \geq 2m) = 2\sum_{m=1} r_0(1-\mu(0))^{m-1}$$

As long as $\mu(0) < 1$, the series converges. Therefore, the necessary condition is all of (1) X is **irreducible** (2) $\sum_{n\in\mathbb{N}_0} \nu(v) < \infty$ and (3) $\mu(0) < 1$.

9.3.2 sufficient part

On the other hand, if X is (1) **irreducible** and (2) $\sum_{n\in\mathbb{N}_0} \nu(v) < \infty$, the stationary distribution is defined. Moreover, if (3) $\mu(0) < 1$ is true, for all $x \in \mathbb{N}_0$, $\mathbb{E}_x[T_x] < 0$, then X is positive recurrent. Therefore, (1) (2) (3) is the sufficient condition.

10 Question 10

Consider a knight jumping randomly on a 5×5 square. At each step, it picks one of the admissible moves with equal probability. Find the stationary distribution for this Markov chain by identifying Markov chain as a random walk on a graph.

The Markov chain is finite state and irreducible, therefore, there exists a stationary distribution. Construct the graph and set all conductances to 1. So that the reversible measure of a node is the number edges on that node.

We denote the symmetry by the matrix below:

$$\begin{bmatrix} d & e & f & e & d \\ e & b & c & b & e \\ f & c & a & c & f \\ e & b & c & b & e \\ d & e & f & e & d \end{bmatrix}$$

The reversible measure $\nu : \{a, b, c, d, e, f\} \to \mathbb{R}$ is as follows

$$\begin{aligned} \nu(a) &= 8 \\ \nu(b) &= 4 \\ \nu(c) &= 6 \\ \nu(d) &= 2 \\ \nu(e) &= 3 \\ \nu(f) &= 4 \end{aligned}$$

The stationary distribution $\mu : \{a, b, c, d, e, f\} \to \mathbb{R}$ is as follows with $s = \nu(a)+4\nu(b)+4\nu(c)+4\nu(d)+8\nu(e)+4\nu(f) = 96$

$$\begin{aligned} \mu(a) &= \frac{\nu(a)}{s} = \frac{1}{12} \\ \mu(b) &= \frac{\nu(b)}{s} = \frac{1}{24} \\ \mu(c) &= \frac{\nu(c)}{s} = \frac{1}{16} \\ \mu(d) &= \frac{\nu(d)}{s} = \frac{1}{48} \\ \mu(e) &= \frac{\nu(e)}{s} = \frac{1}{32} \\ \mu(f) &= \frac{\nu(f)}{s} = \frac{1}{24} \end{aligned}$$

11 Question 11

Suppose that the knight starts at the center

1. Find the probability that the knight will reach one of the corners before returning to the center
2. Find the expected time it takes for the knight to reach one of the corners

11.1 the knight will reach one of the corners before returning to the center

Let D be the set of corners, a the center, and $A = \{a\}$. Let $T_A = \min\{n \geq 0 : X_n \in A\}$ be the first time X visits A, $T_D = \min\{n \geq 0 : X_n \in D\}$ be the first time X visits D. Since starting from a, it must go to one of the 8 squares labelled by e, then the probability of hitting D before A is

$$\phi(e) = P_e(T_D < T_A)$$

We have, for any $x \notin A \cup D$,

$$\begin{aligned} \phi(x) = P_x(T_D < T_A) &= \sum_{y \in S} P_x(X_1 = y, T_D < T_A) \\ &= \sum_{y \in S} P_x(T_D < T_A | X_1 = y) P(X_1 = y | X_0 = x) \\ &= \sum_{y \in S} \phi(y) \Pi(x, y) \end{aligned}$$

We have the following system of equations

$$
\begin{aligned}
&\phi(a) = 0 \\
&\phi(b) = 2\phi(e)\Pi(b,e) + 2\phi(c)\Pi(b,c) = \frac{1}{2}\phi(e) + \frac{1}{2}\phi(c) \\
&\phi(c) = 2\phi(d)\Pi(c,d) + 2\phi(f)\Pi(c,f) + 2\phi(b)\Pi(c,b) = \frac{1}{3}\phi(d) + \frac{1}{3}\phi(f) + \frac{1}{3}\phi(b) \\
&\phi(d) = 1 \\
&\phi(e) = \phi(b)\Pi(e,b) + \phi(a)\Pi(e,a) + \phi(f)\Pi(e,f) = \frac{1}{3}\phi(b) + \frac{1}{3}\phi(a) + \frac{1}{3}\phi(f) \\
&\phi(f) = 2\phi(e)\Pi(f,e) + 2\phi(c)\Pi(f,c) = \frac{1}{2}\phi(e) + \frac{1}{2}\phi(c)
\end{aligned}
$$

Solving the system of equations, we have $\phi(e) = \frac{1}{3}$

11.2 the expected time it takes for the knight to reach one of the corners

Let D be the set of corners, $T_D = \min\{n \geq 0 : X_n \in D\}$ be the first time X visits D. We want to find

$$\mathbb{E}_a[T_D]$$

We have, for any $x \notin D$,

$$
\begin{aligned}
\psi(x) &= \mathbb{E}_x[T_D] \\
&= \sum_{y \in S} \mathbb{E}_x[T_D 1_{\{X_1 = y\}}] \\
&= 1 + \sum_{y \in S} \mathbb{E}_x[(T_D - 1)1_{\{X_1 = y\}}] \\
&= 1 + \sum_{y \in S} P(X_1 = y | X_0 = x)\mathbb{E}_x[\mathbb{E}_x[T_D - 1 | X_1 = y]] \\
&= 1 + \sum_{y \in S} P(X_1 = y | X_0 = x)\mathbb{E}_y[T_D] \\
&= 1 + \sum_{y \in S} \Pi(x,y)\psi(y)
\end{aligned}
$$

We have the following system of equations

$$
\begin{aligned}
&\psi(a) = 1 + 8\psi(e)\Pi(a,e) = 1 + \psi(e) \\
&\psi(b) = 1 + 2\psi(e)\Pi(b,e) + 2\psi(c)\Pi(b,c) = 1 + \frac{1}{2}\psi(e) + \frac{1}{2}\psi(c) \\
&\psi(c) = 1 + 2\psi(d)\Pi(c,d) + 2\psi(f)\Pi(c,f) + 2\psi(b)\Pi(c,b) = 1 + \frac{1}{3}\psi(d) + \frac{1}{3}\psi(f) + \frac{1}{3}\psi(b) \\
&\psi(d) = 0 \\
&\psi(e) = 1 + \psi(b)\Pi(e,b) + \psi(a)\Pi(e,a) + \psi(f)\Pi(e,f) = 1 + \frac{1}{3}\psi(b) + \frac{1}{3}\psi(a) + \frac{1}{3}\psi(f) \\
&\psi(f) = 1 + 2\psi(e)\Pi(f,e) + 2\psi(c)\Pi(f,c) = 1 + \frac{1}{2}\psi(e) + \frac{1}{2}\psi(c)
\end{aligned}
$$

Solving the system of equations, we have $\psi(a) = 18$