adjunction

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July 2024

1 Adjunction

Definition 1 (adjunction). *Let C, D be categories, an adjunction between C and D is a pair of functors $L : C \to D$, $R : D \to C$ together with an isomorphism*

$$\phi_{X,Y} : D(LX, Y) \xrightarrow{\cong} C(X, RY)$$

for each object X in C and object Y in D that is natural in both components [1]*. L is called left adjoint and R is called right adjoint. We write*

$$L : C \rightleftarrows D : R$$

Example 1 (product-hom adjunction). *Let $C : \mathrm{Set} \to \mathrm{Set}$ and $D : \mathrm{Set} \to \mathrm{Set}$ be defined as follows:*

$$L := X \times - : \mathrm{Set} \to \mathrm{Set}$$
$$D := \mathrm{Set}(X, -) : \mathrm{Set} \to \mathrm{Set}$$

Then, L, R is an adjunction with isomorphism

$$\mathrm{Set}(LZ, Y) = \mathrm{Set}(X \times Z, Y) \cong \mathrm{Set}(Z, \mathrm{Set}(X, Y)) = \mathrm{Set}(Z, RY)$$

2 Unit and Counit of an Adjunction

Definition 2 (unit, counit). *Let $L : C \rightleftarrows D : R$ be an adjunction with the isomorphism*

$$\phi_{X,Y} : D(LX, Y) \xrightarrow{\cong} C(X, RY)$$

Let $Y = LX$, we have

$$\phi_{X,LX} : D(LX, LX) \xrightarrow{\cong} C(X, RLX)$$

Under this isomorphism, define $\eta_X : X \to RLX$ by the image of identity map $\mathrm{id}_{LX} : LX \to LX$ in $D(LX, LX)$ under $\phi_{X,LX}$, that is $\eta_X = \phi_{X,LX} \, \mathrm{id}_{LX}$. For each object X in C, there is map η_X and these maps assemble a natural isomorphism of functors $C \to C$

$$\eta : \mathrm{id}_C \to RL$$

η is called unit of the adjunction. Similarly, there is a natural isomorphism of functors $D \to D$

$$\epsilon : LR \to \mathrm{id}_D$$

ϵ is called counit of the adjunction.

Proof.
todo ∎

Example 2 (lifting property of unit). *Let $f : X \to RY$ where X is an object in C and Y is an object in D. By adjunction, $f : X \to RY$ is lifted into a map $g : RLX \to RY$*

$$\begin{tikzcd}
RLX \arrow[dr, dashed, "g=R\hat{f}"] & \\
X \arrow[u, "\eta_X"] \arrow[r, "f"'] & RY
\end{tikzcd}
\qquad
\begin{tikzcd}
LX \arrow[dr, dashed, "\hat{f}"] & \\
 & Y
\end{tikzcd}$$

[1] by being natural in X, for each Y, $\phi_{-,Y} : D(L-, Y) \to C(-, RY)$ is a natural transformation of functors $C^{op} \to \mathrm{Set}$, by being natural in Y, for each X, $\phi_{X,-} : D(LX, -) \to C(X, R-)$ is a natural transformation of functors $D \to \mathrm{Set}$

The map g can be explicitly defined by

$$g = R\hat{f}$$

where $\hat{f} : LX \to Y$ is the corresponding map to $f : X \to RY$ in the adjunction isomorphism

Proof.
The diagram below commutes because $\phi_{X,-} : D(LX,-) \to C(X,R-)$ is a natural transformation

$$\begin{tikzcd}
LX \arrow[d, "\hat{f}"'] & D(LX,LX) \arrow[rr, "\phi_{X,LX}"] \arrow[d, "D(LX,-)\hat{f}"'] & & C(X,RLX) \arrow[d, "C(X,R-)\hat{f}"] \\
Y & D(LX,Y) \arrow[rr, "\phi_{X,Y}"'] & & C(X,RY)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{id}_{LX} \arrow[r, mapsto] \arrow[d, mapsto] & \eta_X \arrow[d, mapsto] \\
\hat{f} \arrow[r, mapsto] & f
\end{tikzcd}$$

where the maps $D(LX,-)\hat{f} : D(LX,LX) \to D(LX,Y)$ and $C(X,R-)\hat{f} : C(X,RLX) \to C(X,RY)$ are defined by

$$(D(LX,-)\hat{f})(h) = \hat{f}h$$
$$(C(X,R-)\hat{f})(h) = (R\hat{f})h$$

By commutativity,

$$f = (R\hat{f})\eta_X$$

Then, define $g : RLX \to RY$ by

$$g = R\hat{f}$$

□

Example 3 (unit, counit of product-hom adjunction)**.** *Let $L : \mathrm{Set} \rightleftarrows \mathrm{Set} : R$ be the product-hom adjunction, that is*

$$L = X \times - : \mathrm{Set} \to \mathrm{Set} \;\; \textit{and} \;\; R = \mathrm{Set}(X,-) : \mathrm{Set} \to \mathrm{Set}$$

The counit of this adjunction is the evaluation map $\mathrm{eval} : X \times \mathrm{Set}(X,Y) \to Y$ defined by $\mathrm{eval}(x,f) = f(x)$. The unit of this adjunction is the map $Z \to \mathrm{Set}(X, X \times Z)$ defined by $z \mapsto (-,z)$ where $(-,z) : X \to X \times Z$ is the function $x \mapsto (x,z)$

Definition 3 (adjunction)**.** *An adjunction between categories C and D is a pair of functors $L : C \to D$ and $R : D \to C$ together with natural transformations $\eta : \mathrm{id}_C \to RL$ and $\epsilon : LR \to \mathrm{id}_D$ such that for all objects $X \in C$ and $Y \in D$ the following diagrams commute*

$$\begin{tikzcd}
& LRLX \arrow[dr, "\epsilon_{LX}"] & \\
LX \arrow[ur, "L\eta_X"] \arrow[rr, "\mathrm{id}_{LX}"'] & & LX
\end{tikzcd}
\qquad
\begin{tikzcd}
& RLRY \arrow[dr, "R\epsilon_Y"] & \\
RY \arrow[ur, "\eta_{RY}"] \arrow[rr, "\mathrm{id}_{RY}"'] & & RY
\end{tikzcd}$$

digest this

3 Free-Forgetful Adjunction in Algebra

Example 4 (free group)**.** *Let $U : \mathrm{Grp} \to \mathrm{Set}$ be the forgetful functor. A free group of a set S is a group FS and an injective map $\eta : S \to UFS$ satisfying the property that for any group G and map $f : S \to UG$, there exists a unique map $\hat{f} : FS \to G$ such that $f = (U\hat{f})\eta$*

$$\begin{tikzcd}
UFS \arrow[dr, dashed, "U\hat{f}"] & \\
S \arrow[u, "\eta"] \arrow[r, "f"'] & UG
\end{tikzcd}
\qquad
\begin{tikzcd}
FS \arrow[dr, dashed, "\hat{f}"] & \\
& G
\end{tikzcd}$$

The free functor and forgetful functor form an adjoint pair $F : \mathrm{Set} \rightleftarrows \mathrm{Grp} : U$ providing the isomorphism

$$\mathrm{Set}(S, UG) \cong \mathrm{Grp}(FS, G)$$

The unit of this adjunction is the inclusion $\eta : S \to UFS$

4 The Forgetful Functor $U : \mathrm{Top} \to \mathrm{Set}$ and Its Adjoint

Example 5 (left adjoint and right adjoint of the forgetful functor)**.** *Let $D : \mathrm{Set} \to \mathrm{Top}$ and $I : \mathrm{Set} \to \mathrm{Top}$ put the discrete topology and the indiscrete topology on any set. Then, D is the left adjoint and I is the right adjoint of the forgetful functor $U : \mathrm{Top} \to Set$.*

$$\mathrm{Top}(DX, Y) \cong \mathrm{Set}(X, UY)$$
$$\mathrm{Set}(UX, Y) \cong \mathrm{Top}(X, IY)$$

Theorem 1. *If $L : C \to D$ has a right adjoint, then L is cocontinuous. If $R : D \to C$ has a left adjoint, then R is continuous.*

Proof. todo □

Corollary 1. *Right adjoints preserve products.*

Remark 1. *That explains why the construction of products, coproducts, subspaces, quotients, equalizers, coequalizers, pullbacks, and pushouts in* Top *must have, as an underlying set, the corresponding construction in* Set*. That is, if a construction exists in* Top*, then the forgetful functor $U : \mathrm{Top} \to \mathrm{Set}$ preserves it.*

5 Adjoint Functor Theorem

Definition 4 (solution set condition)**.** *A functor $R : D \to C$ satisfies the solution set condition if for every object X in C, there exists a set of objects $\mathcal{Y} = \{Y_i\}$ in D and a set of morphisms*

$$\mathcal{S} = \{f_i : X \to RY_i : Y_i \in \mathcal{Y}\}$$

so that for any $f : X \to RY$, there exists Y_i and a morphism $g : Y_i \to Y$ in D such that the diagram below commutes

$$\begin{tikzcd} & RY_i \arrow[dr, dashed, "Rg"] & \\ X \arrow[ur, "f_i"] \arrow[rr, "f"'] & & RY \end{tikzcd} \qquad \begin{tikzcd} Y_i \arrow[dr, dashed, "g"] & \\ & Y \end{tikzcd}$$

Theorem 2 (adjoint functor theorem)**.** *Suppose D is complete and $R : D \to C$ is a continuous functor satisfying the solution set condition, then R has a left adjoint.*

6 Compactifications

Definition 5 (compactification)**.** *A compactification of a topological space is an embedding of that space as a dense subspace of a compact Hausdorff space.*

6.1 The One-Point Compactification

Definition 6 (one-point compactification)**.** *A compactification obtained by adding a single point is called one-point compactification*

Proposition 1. *A space X has a one-point compactification if and only if X is locally compact, Hausdorff, and X is not compact. If a space has one-point compactification, then the compactification is unique.*

Proof.
Let $X \hookrightarrow X^* = X \cup \{p\}$ be a compactification.
(one-point compactification implies Hausdorff)
Every subspace of a Hausdorff space is Hausdorff
(one-point compactification implies locally compact)
For any $x \in X$, as X^* is Hausdorff, x and p are separated by two open sets $U_x \ni x$ and $U_p \ni p$. $X^* \setminus U_p$ is a closed set in a compact space, then $X^* \setminus U_p$ is compact. Hence, X is locally compact.

(one-point compactification implies not compact)

If X is a compact subset of a Hausdorff space X^*, then X is closed, so that X cannot be dense in X^*. Therefore, X must not be compact.

(locally compact, Hausdorff, and not compact imply one-point compactification)

Given any locally compact, Hausdorff, and not compact space X, construct a new space by adding a point p and open neighbourhoods of p to be the complements of all compact subsets in X

(uniqueness of one-point compactification)

In X^*, the open neighbourhoods of p are precisely complements of compact subsets of X. Therefore, if there is another topology on X^* making it a compactification of X, the topology cannot be denser or coarser. Hence, uniqueness.

□

Theorem 3. *Suppose X is locally compact, Hausdorff, and not compact and let $i : X \to X^*$ be the one-point compactification of X. If $e : X \to Y$ is any other compactification of X, then there exists a quotient map $q : Y \to X^*$ such that the diagram below commutes*

$$\begin{tikzcd} & Y \arrow[d, dashed, "q"] \\ X \arrow[ur, "e"] \arrow[r, "i"'] & X^* \end{tikzcd}$$

Proof.

Let $X^* = X \cup \{*\}$ and $q : Y \to X^*$ as a set map is defined by

$$qy = \begin{cases} e^{-1}y & y \in eX \\ * & y \in Y \setminus eX \end{cases}$$

For any open set not containing p in X^*, its preimage under q is open due to homeomorphism. For any open set containing p, the preimage of its complement is closed due to homeomorphism. □

6.2 The Stone-Čech Compactification

Definition 7 (Stone-Čech compactification)**.** *Let CH be the category where objects are compact Hausdorff spaces and morphisms are continuous functions. Let $U : CH \to \mathrm{Top}$ be the inclusion functor. Then, U has a left adjoint $\beta : \mathrm{Top} \to CH$ called Stone-Čech compactification.*

Remark 2. *For every topological space X and compact Hausdorff space Y, we have*

$$CH(\beta X, Y) \cong \mathrm{Top}(X, UY) = \mathrm{Top}(X, Y)$$

That is equivalent to the universal lifting property as follows: Let βX be the Stone-Čech compactification of a topological space X. For every map $f : X \to Y$ where Y is a compact Hausdorff space, then there is a lift $\hat{f} : \beta X \to Y$ such that the diagram below commutes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta"'] & UY \\ U\beta X \arrow[ur, dashed, "U\hat{f}"'] & \end{tikzcd} \qquad \begin{tikzcd} & Y \\ \beta X \arrow[ur, dashed, "\hat{f}"] & \end{tikzcd}$$

where $\hat{f}$ is the adjoint of f and η is the unit of adjunction. In the case when X is locally compact, Hausdorff, the unit $\eta : X \to \beta X$ is a compactification of X

todo: some other remarks on ultrafilters, monad, etc

7 The Exponential Topology

Definition 8 (splitting, conjoining)**.** *Let X, Y be topological spaces. Given the product-hom adjunction on X, Y as sets.*

$$\mathrm{Set}(X \times Z, Y) \cong \mathrm{Set}(Z, \mathrm{Set}(X, Y))$$

A topology on $\mathrm{Top}(X, Y)$ is

- *splitting: if the continuity of $g : Z \times X \to Y$ implies the continuity of $\hat{g} : Z \to \mathrm{Top}(X, Y)$*
- *conjoining: if the continuity of $\hat{g} : Z \to \mathrm{Top}(X, Y)$ implies the continuity of $g : Z \times X \to Y$*

- *exponential: if it is both splitting and conjoining*

Lemma 1. *A topology on* $\mathrm{Top}(X,Y)$ *is conjoining if and only if the evaluation map* $\mathrm{eval} : X \times \mathrm{Top}(X,Y) \to Y$ *is continuous*

Proof.
(evaluation map is continuous implies conjoining)
Suppose $\mathrm{Top}(X,Y)$ has a topology such that the evaluation map is continuous, let $\hat{g} : Z \to \mathrm{Top}(X,Y)$ be a continuous map, the composition $\mathrm{eval}(\mathrm{id} \times \hat{g})$ is precisely $g : X \times Z \to Y$ the adjoint of $\hat{g}$

$$\begin{tikzcd} X \times Z \arrow[r, "\mathrm{id} \times \hat{g}"] \arrow[rr, bend right, "g"'] & X \times \mathrm{Top}(X,Y) \arrow[r, "\mathrm{eval}"] & Y \end{tikzcd}$$

The continuity of $\hat{g}$ implies the continuity of g
(conjoining implies evaluation map is continuous)
Suppose $\mathrm{Top}(X,Y)$ is equipped with a conjoining topology. Let $Z = \mathrm{Top}(X,Y)$, since the adjoint of evaluation map $\widehat{\mathrm{eval}} : \mathrm{Top}(X,Y) \to \mathrm{Top}(X,Y)$ is the identity which is continuous, conjoining implies eval is continuous. ☐

Lemma 2. *Every splitting topology on* $\mathrm{Top}(X,Y)$ *is coarser than every conjoining topology*

Proof. Let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on $\mathrm{Top}(X,Y)$ and $\mathcal{T}_1$ splitting, $\mathcal{T}_2$ conjoining. As $\mathcal{T}_2$ is conjoining, the evaluation map $\mathrm{eval} : X \times (\mathrm{Top}(X,Y), \mathcal{T}_2) \to Y$ is continuous. As $\mathcal{T}_1$ is splitting, the adjoint of $\mathrm{eval} : X \times (\mathrm{Top}(X,Y), \mathcal{T}_2) \to Y$ is continuous, that is, the identity map $\widehat{\mathrm{eval}} : (\mathrm{Top}(X,Y), \mathcal{T}_2) \to (\mathrm{Top}(X,Y), \mathcal{T}_1)$ is continuous. Then, $\mathcal{T}_1 \subseteq \mathcal{T}_2$ ☐

Theorem 4. *If there exists an exponential topology on* $\mathrm{Top}(X,Y)$*, it is unique.*

7.1 The Compact-Open Topology

Definition 9 (compact-open topology)**.** *Let* X, Y *be topological spaces. For each compact set* $K \subseteq X$ *and each open set* $U \subseteq Y$*, define*

$$S(K,U) = \{f \in \mathrm{Top}(X,Y) : fK \subseteq U\}$$

The collection $\{S(K,U)\}$ *forms a subbasis for a topology on* $\mathrm{Top}(X,Y)$ *called the compact-open topology.*

Definition 10 (finite-open topology)**.** *Let* X, Y *be topological spaces. For each finite set* $F \subseteq X$ *and each open set* $U \subseteq Y$*, define*

$$S(F,U) = \{f \in \mathrm{Top}(X,Y) : fF \subseteq U\}$$

The collection $\{S(F,U)\}$ *forms a subbasis for a topology on* $\mathrm{Top}(X,Y)$ *called the finite-open topology or product topology.*

Remark 3. *A sequence of functions* $\{f_n : [0,1] \to [0,1]\}_{n \in \mathbb{N}}$ *converges to a function* $f : [0,1] \to [0,1]$

- *in finite-open topology if and only if it converges pointwise.*
- *in compact-open topology if and only if it converges uniformly.*

Definition 11 (metric topology on $\mathrm{Top}(X,Y)$)**.** *Let* X *be compact, and* Y *be a metric space. Then,* $\mathrm{Top}(X,Y)$ *is a metric space via metric*

$$d(f,g) = \sup_{x \in X} d(fx, gx)$$

for $f, g \in \mathrm{Top}(X,Y)$*.*

Theorem 5. *Let* X *be compact and* Y *be a metric space. The compact-open topology on* $\mathrm{Top}(X,Y)$ *coincides with the metric topology.*

Proof.
(metric topology $\subseteq$ compact-open topology)
Given an open ball $\mathcal{B}(f,\epsilon)$, we will find an open set O in the compact-open topology such that $f \in O \subseteq \mathcal{B}(f,\epsilon)$. Since X is compact, fX is compact. The collection $\{\mathcal{B}(fx, \frac{\epsilon}{3})\}_{x \in X}$ is an open cover of $fX \subseteq Y$, it has a finite subcover

$$\left\{\mathcal{B}\Big(fx_1, \frac{\epsilon}{3}\Big), \mathcal{B}\Big(fx_2, \frac{\epsilon}{3}\Big), ..., \mathcal{B}\Big(fx_n, \frac{\epsilon}{3}\Big)\right\}$$

Define compact subsets $\{K_1, K_2, ..., K_n\}$ of X and open sets $\{U_1, U_2, ..., U_n\}$ of Y by

$$K_i = \overline{f^{-1}\mathcal{B}\Big(fx_i, \frac{\epsilon}{3}\Big)} \text{ and } U_i = \mathcal{B}\Big(fx_i, \frac{\epsilon}{2}\Big)$$

For any set A, $f\overline{A} \subseteq \overline{fA}$, then for each $i = 1, 2, .., n$

$$fK_i \subseteq \overline{\mathcal{B}\Big(fx_i, \frac{\epsilon}{3}\Big)} \subset U_i$$

Let $O = \bigcap_{i=1}^n S(K_i, U_i)$ be an open set in the compact-open topology, then $f \in O$. Moreover, let any $g \in O$, because $\{K_1, K_2, ..., K_n\}$ covers X, for any $x \in X$, there exists K_i such that $x \in K_i$, then $fx, gx \in U_i$, hence

$$d(fx, gx) \leq d(fx, fx_i) + d(fx_i, gx) \leq \epsilon$$

That is, $O \subseteq \mathcal{B}(f, \epsilon)$

(compact-open topology $\subseteq$ metric topology)

Given a subbasic open set $S(K, U)$ in compact-open topology where K is compact in X and U is open in Y, for every $f \in S(K, U)$, we will find an open ball $\mathcal{B}(f, \epsilon) \subseteq S(K, U)$. The open set U contains the compact set fK, then there exists $\epsilon > 0$ such that U contains every open ball centered in fK with radius ϵ. For every $g \in \mathcal{B}(f, \epsilon)$, for every $x \in X$, $d(fx, gx) < \epsilon$, that is, $gx \in \mathcal{B}(fx, \epsilon) \subseteq U$. Hence, $gK \subseteq gX \subseteq U$

$\square$

Lemma 3 (tube lemma)**.** *Given a product space $X \times Y$, let $A \subseteq X$, $B \subseteq Y$ be compact subsets. If $A \times B$ is contained in an open set $O \subseteq X \times Y$, then there exist open sets $U_A \subseteq X$, $U_B \subseteq Y$ such that*

$$A \times B \subseteq U_A \times U_B \subseteq O$$

Theorem 6. *For any spaces X, Y, the compact-open topology on $\mathrm{Top}(X, Y)$ is splitting.*

Proof. Let Z be any space, suppose $g : X \times Z \to Y$ is continuous. We will prove that the adjoint $\hat{g} : Z \to \mathrm{Top}(X, Y)$ is continuous where $\mathrm{Top}(X, Y)$ is equipped with the compact-open topology. Consider a subbasic open set $S(K, U)$ in compact-open topology where K is compact in X and U is open in Y. We will show that $\hat{g}^{-1}S(K, U) = \{z \in Z : g(K, z) \subseteq U\}$ is open in Z. For any $z \in \hat{g}^{-1}S(K, U)$, then we have $g(K, z) \subseteq U$. Since g is continuous by the premise, $g^{-1}U = \{(x, z) \in X \times Z : g(x, z) \subseteq U\}$ is open in $X \times Z$ and contains $K \times \{z\}$. By tube lemma, there exists open sets $U_X \subseteq X$, $U_Z \subseteq Z$ such that

$$K \times \{z\} \subseteq U_X \times U_Z \subseteq g^{-1}U$$

Then, $z \in U_Z \subseteq \hat{g}^{-1}S(K, U)$. $\square$

Remark 4. *Some notes about compact and locally compact*

- *closed subsets of a compact space are compact.*
- *compact subsets of a Hausdorff space are closed.*
- *a space X is locally compact if for every $x \in X$, there exists an open set U and a compact set K such that $x \in U \subseteq K$*
- *let X be locally compact and Hausdorff, S be an open set in X, and $x \in S$. then, there exists an open set U such that $x \in U \subseteq \overline{U} \subseteq S$ and $\overline{U}$ is compact.*

Proof. (proof of the last statement)

Let X be locally compact and Hausdorff, S be an open set in X and $x \in S$. As X is locally compact and Hausdorff, let $x \in T \subseteq \overline{T} \subseteq X$ such that T is open and $\overline{T}$ is compact. Let $U = S \cap T$

- If $U = \overline{U}$, $\overline{U}$ is closed subset of a compact set $\overline{T}$, hence compact. We have

$$x \in U \subseteq \overline{U} \subseteq S$$

- If $U \subset \overline{U}$. For each $y \in \overline{U} \setminus U$, by Hausdorff, let V_y, W_y be open sets separating y and x. As $\overline{U} \setminus U$ is compact, let $\{V_{y_i}\}_{i=1}^n$ be the finite open cover of $\overline{U} \setminus U$. Let $A = \bigcap_{i=1}^n W_{y_i}$, then $y \in A$, A open and does not intersect $\overline{U} \setminus U$. Let $B = \bigcap_{i=1}^n \overline{W_{y_i}}$, then $y \in A \subseteq B$, B closed and does not intersect $\overline{U} \setminus U$. We have, $B \cap \overline{U}$ is closed, contained in $U \subseteq \overline{T}$, then compact. Moreover, $W \cap \overline{U}$ is contained in $U \subseteq S$. We have,

$$x \in A \cap U \subseteq int(B) \cap U \subseteq B \cap U \subseteq B \cap \overline{U} \subseteq S$$

$\square$

Theorem 7. *If X is locally compact and Hausdorff and Y is any space, the compact-open topology on $\mathrm{Top}(X, Y)$ is conjoining.*

Proof. Let $\mathrm{Top}(X,Y)$ be equipped with the compact-open topology, we will show that the evaluation map $\mathrm{eval} : X \times \mathrm{Top}(X,Y) \to Y$ is continuous. Let $(x,f) \in X \times \mathrm{Top}(X,Y)$ and $U \subseteq Y$ be an open set containing $\mathrm{eval}(x,f) = fx$. As f is continuous, $f^{-1}U$ is an open set in X containing x. As X is locally compact and Hausdorff, there exists an open set $V \subseteq X$ such that $K := \overline{V}$ is compact and $x \in V \subseteq K \subseteq f^{-1}U$. Hence, $fx \in fK \subseteq U$. Then, $V \times S(K,U)$ is an open set in $X \times \mathrm{Top}(X,Y)$ with $(x,f) \in V \times S(K,U)$. Furthermore, for any $(x_1,f_1) \in V \times S(K,U)$, $f_1x_1 \in U$, that is, $\mathrm{eval}(V \times S(K,U)) \subseteq U$ □

Lemma 4. *If $f : X \to Y$ is a quotient map and Z is locally compact and Hausdorff then $f \times \mathrm{id}_Z : X \times Z \to Y \times Z$ is a quotient map.*

Proof. Let $f : X \to Y$ be a quotient map, we will show that the product $Y \to Z$ has the quotient topology inherited from the map $f \times \mathrm{id}_Z$. Let $(Y \times Z)_q$ denote the topological space $Y \times Z$ equipped with the quotient topology inherited from the map of sets $f \times \mathrm{id}_Z : X \times Y \to Z \times Y$ and let $\pi : X \times Y \to (Y \times Z)_q$ be the corresponding quotient map in Top. Let $Y \times Z$ denote the topological space $Y \times Z$ equipped with the product topology.

Since π is a quotient map, by characterization of quotient map, the continuity of $f \times \mathrm{id}_Z$ implies the continuity of $\mathrm{id} : (Y \to Z)_q \to Y \times Z$.

$$\begin{tikzcd}
X \times Z \arrow[d, "\pi"'] \arrow[dr, "f \times \mathrm{id}_Z"] & \\
(Y \times Z)_q \arrow[r, "\mathrm{id}"'] & Y \times Z
\end{tikzcd}$$

Now, we will show the continuity of $\mathrm{id} : Y \times Z \to (Y \times Z)_q$. As Z is locally compact Hausdorff space, the compact-open topology on $\mathrm{Top}(Z,(Y \times Z)_q)$ is conjoining, we will show the continuity of the adjoint $\widehat{\mathrm{id}} : Y \to \mathrm{Top}(Z,(Y \times Z)_q)$.

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[dr, dashed] & \\
Y \arrow[r, "\widehat{\mathrm{id}}"'] & \mathrm{Top}(Z,(Y \times Z)_q)
\end{tikzcd}$$

The composition $\widehat{\mathrm{id}}f$ is continuous as it is the adjoint of $\pi : X \times Y \to (Y \times Z)_q$. Hence, by characterization of quotient map, $\hat{\mathrm{id}}$ is continuous □

Theorem 8. *If $X_1 \to Y_1$ and $X_2 \to Y_2$ are quotient maps and X_2, Y_1 are locally compact and Hausdorff then $X_1 \times X_2 \to Y_1 \times Y_2$ is a quotient map*

Proof. The two maps below are quotient maps

$$f_1 \times \mathrm{id}_{X_2} : X_1 \times X_2 \to Y_1 \to X_2$$
$$\mathrm{id}_{Y_1} \times f_2 : Y_1 \times X_2 \to Y_1 \times Y_2$$

Hence, their composition. □

7.2 The Theorems of Ascoli and Arzela

I don't really understand this part

Theorem 9. *If X is any space and Y is Hausdorff then a subset $A \subseteq \mathrm{Top}(X,Y)$ has compact closure in the product topology if and only if for each $x \in X$, the set $A_x = \{fx \in Y : f \in A\}$ has compact closure in Y*

Proof. *todo* □

Definition 12 (equicontinuous)**.** *Let X be a topological space and (Y,d) be a metric space. A family $A \subseteq \mathrm{Top}(X,Y)$ is called equicontinuous at $x \in X$ if and only for every $\epsilon > 0$, there exists an open neighbourhood U of x so that for every $u \in U$ and for every $f \in A$, $d(fx, fu) < \epsilon$. If $\mathcal{F}$ is equicontinuous for every $x \in X$, the family A is simply called equicontinuous.*

Lemma 5. *Let X be a topological space and (Y,d) be a metric space. If $A \subseteq \mathrm{Top}(X,Y)$ is an equicontinuous family, then the subspace topology on A of $\mathrm{Top}(X,Y)$ with the compact-open topology is the same as the subspace topology on A of $\mathrm{Top}(X,Y)$ with the finite-open topology.*

Lemma 6. *If $A \subseteq \mathrm{Top}(X,Y)$ is equicontinuous then the closure of A in Top using the finite-open topology is also equicontinuous.*

Theorem 10 (Ascoli theorem)**.** *Let X be a locally compact Hausdorff and let (Y,d) be a metric space. A family $\mathcal{F} \subseteq \mathrm{Top}(X,Y)$ has compact closure if and only if $\mathcal{F}$ is equicontinuous and for every $x \in X$, the set $\mathcal{F}_x := \{fx : f \in \mathcal{F}\}$ has compact closure.*

Theorem 11 (Arzela theorem)**.** *Let X be compact, (Y,d) be a metric space and $\{f_n\}$ be a sequence of functions in $\mathrm{Top}(X,Y)$. If $\{f_n\}$ is equicontinuous and if for each $x \in X$ the set $\{f_nx\}$ is bounded then $\{f_n\}$ has a subsequence that converges uniformly.*

7.3 Enrich the Product-Hom Adjunction in Top

Definition 13. *Denote the set* $\mathrm{Top}(X, Y)$ *with exponential topology by* Y^X *provided it exists.*

Theorem 12. *If* X, Z *are locally compact Hausdorff then for any space* Y*, the isomorphism of sets* $\mathrm{Top}(Z \times X, Y) \to \mathrm{Top}(Z, \mathrm{Top}(X, Y))$ *is a homomorphism of spaces under compact-open topology.*

Proof.
(the map $(Y^Z)^X \to Y^{Z\times X}$ is continuous)

As X is locally compact Hausdorff, the compact-open topology on $\mathrm{Top}(X, Y^Z)$ is conjoining, then the evaluation map $X \times (Y^Z)^X \to Y^Z$ is continuous. As Z is locally compact Hausdorff, the compact-open topology on $\mathrm{Top}(Z, Y)$ is conjoining, then the evaluation map $Z \times Y^Z \to Y$ is continuous. Hence, the composition is continuous

$$(Z \times X) \times (Y^Z)^X \to Z \times Y^Z \to Y$$

As the compact-open topology on $\mathrm{Top}(Z \times X, Y)$ is splitting, the adjunct $(Y^Z)^X \to \mathrm{Top}(Z \times X, Y)$ is continuous.
the map $Y^{Z\times X} \to (Y^Z)^X$ is continuous)

As $Z \times X$ is locally compact Hausdorff, then the compact-open topology on $\mathrm{Top}(Z \times X, Y)$ is conjoining, the evaluation map is continuous

$$Z \times (X \times Y^{Z\times X}) \to Y$$

As the compact-open topology on $\mathrm{Top}(Z, Y)$ is splitting, then the adjoint $X \times Y^{Z\times X} \to Y^Z$ is continuous. As the compact-open topology on $\mathrm{Top}(X, Y^Z)$ is splitting, then the adjoint $Y^{Z\times X} \to (Y^Z)^X$ is continuous. $\square$

7.4 Compactly Generated Weakly Hausdorff Spaces

todo