

adjunction

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1 Adjunction

Definition 1 (adjunction). Let C, D be categories, an adjunction between C and D is a pair of functors $L : C \rightarrow D$, $R : D \rightarrow C$ together with an isomorphism

$$\phi_{X,Y} : D(LX, Y) \xrightarrow{\cong} C(X, RY)$$

for each object X in C and object Y in D that is natural in both components¹. L is called left adjoint and R is called right adjoint. We write

$$L : C \rightleftarrows D : R$$

Example 1 (product-hom adjunction). Let $C : \text{Set} \rightarrow \text{Set}$ and $D : \text{Set} \rightarrow \text{Set}$ be defined as follows:

$$\begin{aligned} L &:= X \times - : \text{Set} \rightarrow \text{Set} \\ D &:= \text{Set}(X, -) : \text{Set} \rightarrow \text{Set} \end{aligned}$$

Then, L, R is an adjunction with isomorphism

$$\text{Set}(LZ, Y) = \text{Set}(X \times Z, Y) \cong \text{Set}(Z, \text{Set}(X, Y)) = \text{Set}(Z, RY)$$

2 Unit and Counit of an Adjunction

Definition 2 (unit, counit). Let $L : C \rightleftarrows D : R$ be an adjunction with the isomorphism

$$\phi_{X,Y} : D(LX, Y) \xrightarrow{\cong} C(X, RY)$$

Let $Y = LX$, we have

$$\phi_{X,LX} : D(LX, LX) \xrightarrow{\cong} C(X, RLX)$$

Under this isomorphism, define $\eta_X : X \rightarrow RLX$ by the image of identity map $\text{id}_{LX} : LX \rightarrow LX$ in $D(LX, LX)$ under $\phi_{X,LX}$, that is $\eta_X = \phi_{X,LX} \text{id}_{LX}$. For each object X in C , there is map η_X and these maps assemble a natural isomorphism of functors $C \rightarrow C$

$$\eta : \text{id}_C \rightarrow RL$$

η is called unit of the adjunction. Similarly, there is a natural isomorphism of functors $D \rightarrow D$

$$\epsilon : LR \rightarrow \text{id}_D$$

ϵ is called counit of the adjunction.

Proof.

todo

□

Example 2 (lifting property of unit). Let $f : X \rightarrow RY$ where X is an object in C and Y is an object in D . By adjunction, $f : X \rightarrow RY$ is lifted into a map $g : RLX \rightarrow RY$

$$\begin{array}{ccc} RLX & & LX \\ \eta_X \uparrow & \searrow g=Rf & \searrow \hat{f} \\ X & \xrightarrow{f} & RY & & Y \end{array}$$

¹by being natural in X , for each Y , $\phi_{-,Y} : D(L-, Y) \rightarrow C(-, RY)$ is a natural transformation of functors $C^{op} \rightarrow \text{Set}$, by being natural in Y , for each X , $\phi_{X,-} : D(LX, -) \rightarrow C(X, R-)$ is a natural transformation of functors $D \rightarrow \text{Set}$

The map g can be explicitly defined by

$$g = R\hat{f}$$

where $\hat{f} : LX \rightarrow Y$ is the corresponding map to $f : X \rightarrow RY$ in the adjunction isomorphism

Proof.

The diagram below commutes because $\phi_{X,-} : D(LX, -) \rightarrow C(X, R-)$ is a natural transformation

$$\begin{array}{ccccc}
 LX & D(LX, LX) & \xrightarrow{\phi_{X, LX}} & C(X, RLX) & \\
 \downarrow \hat{f} & \downarrow D(LX, -)\hat{f} & & \downarrow C(X, R-)\hat{f} & \\
 Y & D(LX, Y) & \xrightarrow{\phi_{X, Y}} & C(X, RY) & \\
 & & & & \downarrow \\
 & & & & C(X, RY)
 \end{array}$$

$\begin{array}{ccc} \text{id}_{LX} & \xrightarrow{\quad} & \eta_X \\ \downarrow & & \downarrow \\ \hat{f} & \xrightarrow{\quad} & f \end{array}$

where the maps $D(LX, -)\hat{f} : D(LX, LX) \rightarrow D(LX, Y)$ and $C(X, R-)\hat{f} : C(X, RLX) \rightarrow C(X, RY)$ are defined by

$$\begin{aligned}
 (D(LX, -)\hat{f})(h) &= \hat{f}h \\
 (C(X, R-)\hat{f})(h) &= (R\hat{f})h
 \end{aligned}$$

By commutativity,

$$f = (R\hat{f})\eta_X$$

Then, define $g : RLX \rightarrow RY$ by

$$g = R\hat{f}$$

□

Example 3 (unit, counit of product-hom adjunction). Let $L : \text{Set} \rightleftarrows \text{Set} : R$ be the product-hom adjunction, that is

$$L = X \times - : \text{Set} \rightarrow \text{Set} \text{ and } R = \text{Set}(X, -) : \text{Set} \rightarrow \text{Set}$$

The counit of this adjunction is the evaluation map $\text{eval} : X \times \text{Set}(X, Y) \rightarrow Y$ defined by $\text{eval}(x, f) = f(x)$. The unit of this adjunction is the map $Z \rightarrow \text{Set}(X, X \times Z)$ defined by $z \mapsto (-, z)$ where $(-, z) : X \rightarrow X \times Z$ is the function $x \mapsto (x, z)$

Definition 3 (adjunction). An adjunction between categories C and D is a pair of functors $L : C \rightarrow D$ and $R : D \rightarrow C$ together with natural transformations $\eta : \text{id}_C \rightarrow RL$ and $\epsilon : LR \rightarrow \text{id}_D$ such that for all objects $X \in C$ and $Y \in D$ the following diagrams commute

$$\begin{array}{ccc}
 & LRLX & \\
 L\eta_X \nearrow & & \searrow \epsilon_{LX} \\
 LX & \xrightarrow{\text{id}_{LX}} & LX
 \end{array}
 \qquad
 \begin{array}{ccc}
 & RLRY & \\
 \eta_{RY} \nearrow & & \searrow R\epsilon_Y \\
 RY & \xrightarrow{\text{id}_{RY}} & RY
 \end{array}$$

digest this

3 Free-Forgetful Adjunction in Algebra

Example 4 (free group). Let $U : \text{Grp} \rightarrow \text{Set}$ be the forgetful functor. A free group of a set S is a group FS and an injective map $\eta : S \rightarrow UFS$ satisfying the property that for any group G and map $f : S \rightarrow UG$, there exists a unique map $\hat{f} : FS \rightarrow G$ such that $f = (U\hat{f})\eta$

$$\begin{array}{ccc}
 UFS & & FS \\
 \eta \uparrow & \searrow U\hat{f} & \searrow \hat{f} \\
 S & \xrightarrow{f} & UG & & G
 \end{array}$$

The free functor and forgetful functor form an adjoint pair $F : \text{Set} \rightleftarrows \text{Grp} : U$ providing the isomorphism

$$\text{Set}(S, UG) \cong \text{Grp}(FS, G)$$

The unit of this adjunction is the inclusion $\eta : S \rightarrow UFS$

4 The Forgetful Functor $U : \text{Top} \rightarrow \text{Set}$ and Its Adjoint

Example 5 (left adjoint and right adjoint of the forgetful functor). Let $D : \text{Set} \rightarrow \text{Top}$ and $I : \text{Set} \rightarrow \text{Top}$ put the discrete topology and the indiscrete topology on any set. Then, D is the left adjoint and I is the right adjoint of the forgetful functor $U : \text{Top} \rightarrow \text{Set}$.

$$\text{Top}(DX, Y) \cong \text{Set}(X, UY)$$

$$\text{Set}(UX, Y) \cong \text{Top}(X, IY)$$

Theorem 1. If $L : C \rightarrow D$ has a right adjoint, then L is cocontinuous. If $R : D \rightarrow C$ has a left adjoint, then R is continuous.

Proof. *todo* □

Corollary 1. Right adjoints preserve products.

Remark 1. That explains why the construction of products, coproducts, subspaces, quotients, equalizers, coequalizers, pullbacks, and pushouts in Top must have, as an underlying set, the corresponding construction in Set . That is, if a construction exists in Top , then the forgetful functor $U : \text{Top} \rightarrow \text{Set}$ preserves it.

5 Adjoint Functor Theorem

Definition 4 (solution set condition). A functor $R : D \rightarrow C$ satisfies the solution set condition if for every object X in C , there exists a set of objects $\mathcal{Y} = \{Y_i\}$ in D and a set of morphisms

$$\mathcal{S} = \{f_i : X \rightarrow RY_i : Y_i \in \mathcal{Y}\}$$

so that for any $f : X \rightarrow RY$, there exists Y_i and a morphism $g : Y_i \rightarrow Y$ in D such that the diagram below commutes

$$\begin{array}{ccc} & RY_i & \\ f_i \nearrow & & \searrow Rg \\ X & \xrightarrow{f} & RY \end{array} \qquad \begin{array}{ccc} Y_i & & \\ & \searrow g & \\ & & Y \end{array}$$

Theorem 2 (adjoint functor theorem). Suppose D is complete and $R : D \rightarrow C$ is a continuous functor satisfying the solution set condition, then R has a left adjoint.

6 Compactifications

Definition 5 (compactification). A compactification of a topological space is an embedding of that space as a dense subspace of a compact Hausdorff space.

6.1 The One-Point Compactification

Definition 6 (one-point compactification). A compactification obtained by adding a single point is called one-point compactification.

Proposition 1. A space X has a one-point compactification if and only if X is locally compact, Hausdorff, and X is not compact. If a space has one-point compactification, then the compactification is unique.

Proof.

Let $X \hookrightarrow X^* = X \cup \{p\}$ be a compactification.

(one-point compactification implies Hausdorff)

Every subspace of a Hausdorff space is Hausdorff

(one-point compactification implies locally compact)

For any $x \in X$, as X^* is Hausdorff, x and p are separated by two open sets $U_x \ni x$ and $U_p \ni p$. $X^* \setminus U_p$ is a closed set in a compact space, then $X^* \setminus U_p$ is compact. Hence, X is locally compact.

(one-point compactification implies not compact)

If X is a compact subset of a Hausdorff space X^* , then X is closed, so that X cannot be dense in X^* . Therefore, X must not be compact.

(locally compact, Hausdorff, and not compact imply one-point compactification)

Given any locally compact, Hausdorff, and not compact space X , construct a new space by adding a point p and open neighbourhoods of p to be the complements of all compact subsets in X

(uniqueness of one-point compactification)

In X^* , the open neighbourhoods of p are precisely complements of compact subsets of X . Therefore, if there is another topology on X^* making it a compactification of X , the topology cannot be denser or coarser. Hence, uniqueness. \square

Theorem 3. Suppose X is locally compact, Hausdorff, and not compact and let $i : X \rightarrow X^*$ be the one-point compactification of X . If $e : X \rightarrow Y$ is any other compactification of X , then there exists a quotient map $q : Y \rightarrow X^*$ such that the diagram below commutes

$$\begin{array}{ccc} & & Y \\ & \nearrow e & \downarrow q \\ X & \xrightarrow{i} & X^* \end{array}$$

Proof.

Let $X^* = X \cup \{*\}$ and $q : Y \rightarrow X^*$ as a set map is defined by

$$qy = \begin{cases} e^{-1}y & y \in eX \\ * & y \in Y \setminus eX \end{cases}$$

For any open set not containing p in X^* , its preimage under q is open due to homeomorphism. For any open set containing p , the preimage of its complement is closed due to homeomorphism. \square

6.2 The Stone-Čech Compactification

Definition 7 (Stone-Čech compactification). Let CH be the category where objects are compact Hausdorff spaces and morphisms are continuous functions. Let $U : CH \rightarrow \text{Top}$ be the inclusion functor. Then, U has a left adjoint $\beta : \text{Top} \rightarrow CH$ called Stone-Čech compactification.

Remark 2. For every topological space X and compact Hausdorff space Y , we have

$$CH(\beta X, Y) \cong \text{Top}(X, UY) = \text{Top}(X, Y)$$

That is equivalent to the universal lifting property as follows: Let βX be the Stone-Čech compactification of a topological space X . For every map $f : X \rightarrow Y$ where Y is a compact Hausdorff space, then there is a lift $\hat{f} : \beta X \rightarrow Y$ such that the diagram below commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \eta \downarrow & \nearrow U\hat{f} & \\ U\beta X & & \beta X \end{array}$$

where \hat{f} is the adjoint of f and η is the unit of adjunction. In the case when X is locally compact, Hausdorff, the unit $\eta : X \rightarrow \beta X$ is a compactification of X

todo: some other remarks on ultrafilters, monad, etc

7 The Exponential Topology

Definition 8 (splitting, conjoining). Let X, Y be topological spaces. Given the product-hom adjunction on X, Y as sets.

$$\text{Set}(X \times Z, Y) \cong \text{Set}(Z, \text{Set}(X, Y))$$

A topology on $\text{Top}(X, Y)$ is

- *splitting*: if the continuity of $g : Z \times X \rightarrow Y$ implies the continuity of $\hat{g} : Z \rightarrow \text{Top}(X, Y)$
- *conjoining*: if the continuity of $\hat{g} : Z \rightarrow \text{Top}(X, Y)$ implies the continuity of $g : Z \times X \rightarrow Y$

- *exponential: if it is both splitting and conjoining*

Lemma 1. *A topology on $\text{Top}(X, Y)$ is conjoining if and only if the evaluation map $\text{eval} : X \times \text{Top}(X, Y) \rightarrow Y$ is continuous*

Proof.

(evaluation map is continuous implies conjoining)

Suppose $\text{Top}(X, Y)$ has a topology such that the evaluation map is continuous, let $\hat{g} : Z \rightarrow \text{Top}(X, Y)$ be a continuous map, the composition $\text{eval}(\text{id} \times \hat{g})$ is precisely $g : X \times Z \rightarrow Y$ the adjoint of \hat{g}

$$\begin{array}{ccc} X \times Z & \xrightarrow{\text{id} \times \hat{g}} & X \times \text{Top}(X, Y) \xrightarrow{\text{eval}} Y \\ & \searrow g & \nearrow \end{array}$$

The continuity of \hat{g} implies the continuity of g

(conjoining implies evaluation map is continuous)

Suppose $\text{Top}(X, Y)$ is equipped with a conjoining topology. Let $Z = \text{Top}(X, Y)$, since the adjoint of evaluation map $\widehat{\text{eval}} : \text{Top}(X, Y) \rightarrow \text{Top}(X, Y)$ is the identity which is continuous, conjoining implies eval is continuous. \square

Lemma 2. *Every splitting topology on $\text{Top}(X, Y)$ is coarser than every conjoining topology*

Proof. Let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on $\text{Top}(X, Y)$ and \mathcal{T}_1 splitting, \mathcal{T}_2 conjoining. As \mathcal{T}_2 is conjoining, the evaluation map $\text{eval} : X \times (\text{Top}(X, Y), \mathcal{T}_2) \rightarrow Y$ is continuous. As \mathcal{T}_1 is splitting, the adjoint of $\text{eval} : X \times (\text{Top}(X, Y), \mathcal{T}_2) \rightarrow Y$ is continuous, that is, the identity map $\widehat{\text{eval}} : (\text{Top}(X, Y), \mathcal{T}_2) \rightarrow (\text{Top}(X, Y), \mathcal{T}_1)$ is continuous. Then, $\mathcal{T}_1 \subseteq \mathcal{T}_2$ \square

Theorem 4. *If there exists an exponential topology on $\text{Top}(X, Y)$, it is unique.*

7.1 The Compact-Open Topology

Definition 9 (compact-open topology). *Let X, Y be topological spaces. For each compact set $K \subseteq X$ and each open set $U \subseteq Y$, define*

$$S(K, U) = \{f \in \text{Top}(X, Y) : fK \subseteq U\}$$

The collection $\{S(K, U)\}$ forms a subbasis for a topology on $\text{Top}(X, Y)$ called the compact-open topology.

Definition 10 (finite-open topology). *Let X, Y be topological spaces. For each finite set $F \subseteq X$ and each open set $U \subseteq Y$, define*

$$S(F, U) = \{f \in \text{Top}(X, Y) : fF \subseteq U\}$$

The collection $\{S(F, U)\}$ forms a subbasis for a topology on $\text{Top}(X, Y)$ called the finite-open topology or product topology.

Remark 3. *A sequence of functions $\{f_n : [0, 1] \rightarrow [0, 1]\}_{n \in \mathbb{N}}$ converges to a function $f : [0, 1] \rightarrow [0, 1]$*

- *in finite-open topology if and only if it converges pointwise.*
- *in compact-open topology if and only if it converges uniformly.*

Definition 11 (metric topology on $\text{Top}(X, Y)$). *Let X be compact, and Y be a metric space. Then, $\text{Top}(X, Y)$ is a metric space via metric*

$$d(f, g) = \sup_{x \in X} d(fx, gx)$$

for $f, g \in \text{Top}(X, Y)$.

Theorem 5. *Let X be compact and Y be a metric space. The compact-open topology on $\text{Top}(X, Y)$ coincides with the metric topology.*

Proof.

(metric topology \subseteq compact-open topology)

Given an open ball $\mathcal{B}(f, \epsilon)$, we will find an open set O in the compact-open topology such that $f \in O \subseteq \mathcal{B}(f, \epsilon)$. Since X is compact, fX is compact. The collection $\{\mathcal{B}(fx, \frac{\epsilon}{3})\}_{x \in X}$ is an open cover of $fX \subseteq Y$, it has a finite subcover

$$\left\{ \mathcal{B}\left(fx_1, \frac{\epsilon}{3}\right), \mathcal{B}\left(fx_2, \frac{\epsilon}{3}\right), \dots, \mathcal{B}\left(fx_n, \frac{\epsilon}{3}\right) \right\}$$

Define compact subsets $\{K_1, K_2, \dots, K_n\}$ of X and open sets $\{U_1, U_2, \dots, U_n\}$ of Y by

$$K_i = \overline{f^{-1}\mathcal{B}\left(fx_i, \frac{\epsilon}{3}\right)} \text{ and } U_i = \mathcal{B}\left(fx_i, \frac{\epsilon}{2}\right)$$

For any set A , $f\overline{A} \subseteq \overline{fA}$, then for each $i = 1, 2, \dots, n$

$$fK_i \subseteq \overline{\mathcal{B}\left(fx_i, \frac{\epsilon}{3}\right)} \subset U_i$$

Let $O = \bigcap_{i=1}^n S(K_i, U_i)$ be an open set in the compact-open topology, then $f \in O$. Moreover, let any $g \in O$, because $\{K_1, K_2, \dots, K_n\}$ covers X , for any $x \in X$, there exists K_i such that $x \in K_i$, then $fx, gx \in U_i$, hence

$$d(fx, gx) \leq d(fx, fx_i) + d(fx_i, gx) \leq \epsilon$$

That is, $O \subseteq \mathcal{B}(f, \epsilon)$

(compact-open topology \subseteq metric topology)

Given a subbasic open set $S(K, U)$ in compact-open topology where K is compact in X and U is open in Y , for every $f \in S(K, U)$, we will find an open ball $\mathcal{B}(f, \epsilon) \subseteq S(K, U)$. The open set U contains the compact set fK , then there exists $\epsilon > 0$ such that U contains every open ball centered in fK with radius ϵ . For every $g \in \mathcal{B}(f, \epsilon)$, for every $x \in X$, $d(fx, gx) < \epsilon$, that is, $gx \in \mathcal{B}(fx, \epsilon) \subseteq U$. Hence, $gK \subseteq gX \subseteq U$

□

Lemma 3 (tube lemma). *Given a product space $X \times Y$, let $A \subseteq X$, $B \subseteq Y$ be compact subsets. If $A \times B$ is contained in an open set $O \subseteq X \times Y$, then there exist open sets $U_A \subseteq X$, $U_B \subseteq Y$ such that*

$$A \times B \subseteq U_A \times U_B \subseteq O$$

Theorem 6. *For any spaces X, Y , the compact-open topology on $\text{Top}(X, Y)$ is splitting.*

Proof. Let Z be any space, suppose $g : X \times Z \rightarrow Y$ is continuous. We will prove that the adjoint $\hat{g} : Z \rightarrow \text{Top}(X, Y)$ is continuous where $\text{Top}(X, Y)$ is equipped with the compact-open topology. Consider a subbasic open set $S(K, U)$ in compact-open topology where K is compact in X and U is open in Y . We will show that $\hat{g}^{-1}S(K, U) = \{z \in Z : g(K, z) \subseteq U\}$ is open in Z . For any $z \in \hat{g}^{-1}S(K, U)$, then we have $g(K, z) \subseteq U$. Since g is continuous by the premise, $g^{-1}U = \{(x, z) \in X \times Z : g(x, z) \subseteq U\}$ is open in $X \times Z$ and contains $K \times \{z\}$. By tube lemma, there exists open sets $U_X \subseteq X$, $U_Z \subseteq Z$ such that

$$K \times \{z\} \subseteq U_X \times U_Z \subseteq g^{-1}U$$

Then, $z \in U_Z \subseteq \hat{g}^{-1}S(K, U)$.

□

Remark 4. *Some notes about compact and locally compact*

- closed subsets of a compact space are compact.
- compact subsets of a Hausdorff space are closed.
- a space X is locally compact if for every $x \in X$, there exists an open set U and a compact set K such that $x \in U \subseteq K$
- let X be locally compact and Hausdorff, S be an open set in X , and $x \in S$. then, there exists an open set U such that $x \in U \subseteq \overline{U} \subseteq S$ and \overline{U} is compact.

Proof. (proof of the last statement)

Let X be locally compact and Hausdorff, S be an open set in X and $x \in S$. As X is locally compact and Hausdorff, let $x \in T \subseteq \overline{T} \subseteq X$ such that T is open and \overline{T} is compact. Let $U = S \cap T$

- If $U = \overline{U}$, \overline{U} is closed subset of a compact set \overline{T} , hence compact. We have

$$x \in U \subseteq \overline{U} \subseteq S$$

- If $U \subset \overline{U}$. For each $y \in \overline{U} \setminus U$, by Hausdorff, let V_y, W_y be open sets separating y and x . As $\overline{U} \setminus U$ is compact, let $\{V_{y_i}\}_{i=1}^n$ be the finite open cover of $\overline{U} \setminus U$. Let $A = \bigcap_{i=1}^n W_{y_i}$, then $y \in A$, A open and does not intersect $\overline{U} \setminus U$. Let $B = \bigcap_{i=1}^n \overline{W_{y_i}}$, then $y \in A \subseteq B$, B closed and does not intersect $\overline{U} \setminus U$. We have, $B \cap \overline{U}$ is closed, contained in $U \subseteq \overline{T}$, then compact. Moreover, $W \cap \overline{U}$ is contained in $U \subseteq S$. We have,

$$x \in A \cap U \subseteq \text{int}(B) \cap U \subseteq B \cap U \subseteq B \cap \overline{U} \subseteq S$$

□

Theorem 7. *If X is locally compact and Hausdorff and Y is any space, the compact-open topology on $\text{Top}(X, Y)$ is conjoining.*

Proof. Let $\text{Top}(X, Y)$ be equipped with the compact-open topology, we will show that the evaluation map $\text{eval} : X \times \text{Top}(X, Y) \rightarrow Y$ is continuous. Let $(x, f) \in X \times \text{Top}(X, Y)$ and $U \subseteq Y$ be an open set containing $\text{eval}(x, f) = fx$. As f is continuous, $f^{-1}U$ is an open set in X containing x . As X is locally compact and Hausdorff, there exists an open set $V \subseteq X$ such that $K := \overline{V}$ is compact and $x \in V \subseteq K \subseteq f^{-1}U$. Hence, $fx \in fK \subseteq U$. Then, $V \times S(K, U)$ is an open set in $X \times \text{Top}(X, Y)$ with $(x, f) \in V \times S(K, U)$. Furthermore, for any $(x_1, f_1) \in V \times S(K, U)$, $f_1x_1 \in U$, that is, $\text{eval}(V \times S(K, U)) \subseteq U$ \square

Lemma 4. *If $f : X \rightarrow Y$ is a quotient map and Z is locally compact and Hausdorff then $f \times \text{id}_Z : X \times Z \rightarrow Y \times Z$ is a quotient map.*

Proof. Let $f : X \rightarrow Y$ be a quotient map, we will show that the product $Y \rightarrow Z$ has the quotient topology inherited from the map $f \times \text{id}_Z$. Let $(Y \times Z)_q$ denote the topological space $Y \times Z$ equipped with the quotient topology inherited from the map of sets $f \times \text{id}_Z : X \times Y \rightarrow Z \times Y$ and let $\pi : X \times Y \rightarrow (Y \times Z)_q$ be the corresponding quotient map in Top . Let $Y \times Z$ denote the topological space $Y \times Z$ equipped with the product topology.

Since π is a quotient map, by characterization of quotient map, the continuity of $f \times \text{id}_Z$ implies the continuity of $\text{id} : (Y \times Z)_q \rightarrow Y \times Z$.

$$\begin{array}{ccc} X \times Z & & \\ \pi \downarrow & \searrow f \times \text{id}_Z & \\ (Y \times Z)_q & \xrightarrow{\text{id}} & Y \times Z \end{array}$$

Now, we will show the continuity of $\text{id} : Y \times Z \rightarrow (Y \times Z)_q$. As Z is locally compact Hausdorff space, the compact-open topology on $\text{Top}(Z, (Y \times Z)_q)$ is conjoining, we will show the continuity of the adjoint $\hat{\text{id}} : Y \rightarrow \text{Top}(Z, (Y \times Z)_q)$.

$$\begin{array}{ccc} X & & \\ f \downarrow & \searrow \hat{\text{id}} & \\ Y & \xrightarrow{\text{id}} & \text{Top}(Z, (Y \times Z)_q) \end{array}$$

The composition $\hat{\text{id}}f$ is continuous as it is the adjoint of $\pi : X \times Y \rightarrow (Y \times Z)_q$. Hence, by characterization of quotient map, $\hat{\text{id}}$ is continuous \square

Theorem 8. *If $X_1 \rightarrow Y_1$ and $X_2 \rightarrow Y_2$ are quotient maps and X_2, Y_1 are locally compact and Hausdorff then $X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is a quotient map*

Proof. The two maps below are quotient maps

$$\begin{array}{l} f_1 \times \text{id}_{X_2} : X_1 \times X_2 \rightarrow Y_1 \times X_2 \\ \text{id}_{Y_1} \times f_2 : Y_1 \times X_2 \rightarrow Y_1 \times Y_2 \end{array}$$

Hence, their composition. \square

7.2 The Theorems of Ascoli and Arzela

I don't really understand this part

Theorem 9. *If X is any space and Y is Hausdorff then a subset $A \subseteq \text{Top}(X, Y)$ has compact closure in the product topology if and only if for each $x \in X$, the set $A_x = \{fx \in Y : f \in A\}$ has compact closure in Y*

Proof. *todo* \square

Definition 12 (equicontinuous). *Let X be a topological space and (Y, d) be a metric space. A family $A \subseteq \text{Top}(X, Y)$ is called equicontinuous at $x \in X$ if and only for every $\epsilon > 0$, there exists an open neighbourhood U of x so that for every $u \in U$ and for every $f \in A$, $d(fx, fu) < \epsilon$. If \mathcal{F} is equicontinuous for every $x \in X$, the family A is simply called equicontinuous.*

Lemma 5. *Let X be a topological space and (Y, d) be a metric space. If $A \subseteq \text{Top}(X, Y)$ is an equicontinuous family, then the subspace topology on A of $\text{Top}(X, Y)$ with the compact-open topology is the same as the subspace topology on A of $\text{Top}(X, Y)$ with the finite-open topology.*

Lemma 6. *If $A \subseteq \text{Top}(X, Y)$ is equicontinuous then the closure of A in Top using the finite-open topology is also equicontinuous.*

Theorem 10 (Ascoli theorem). *Let X be a locally compact Hausdorff and let (Y, d) be a metric space. A family $\mathcal{F} \subseteq \text{Top}(X, Y)$ has compact closure if and only if \mathcal{F} is equicontinuous and for every $x \in X$, the set $\mathcal{F}_x := \{fx : f \in \mathcal{F}\}$ has compact closure.*

Theorem 11 (Arzela theorem). *Let X be compact, (Y, d) be a metric space and $\{f_n\}$ be a sequence of functions in $\text{Top}(X, Y)$. If $\{f_n\}$ is equicontinuous and if for each $x \in X$ the set $\{f_nx\}$ is bounded then $\{f_n\}$ has a subsequence that converges uniformly.*

7.3 Enrich the Product-Hom Adjunction in Top

Definition 13. Denote the set $\text{Top}(X, Y)$ with exponential topology by Y^X provided it exists.

Theorem 12. If X, Z are locally compact Hausdorff then for any space Y , the isomorphism of sets $\text{Top}(Z \times X, Y) \rightarrow \text{Top}(Z, \text{Top}(X, Y))$ is a homomorphism of spaces under compact-open topology.

Proof.

(the map $(Y^Z)^X \rightarrow Y^{Z \times X}$ is continuous)

As X is locally compact Hausdorff, the compact-open topology on $\text{Top}(X, Y^Z)$ is conjoining, then the evaluation map $X \times (Y^Z)^X \rightarrow Y^Z$ is continuous. As Z is locally compact Hausdorff, the compact-open topology on $\text{Top}(Z, Y)$ is conjoining, then the evaluation map $Z \times Y^Z \rightarrow Y$ is continuous. Hence, the composition is continuous

$$(Z \times X) \times (Y^Z)^X \rightarrow Z \times Y^Z \rightarrow Y$$

As the compact-open topology on $\text{Top}(Z \times X, Y)$ is splitting, the adjunct $(Y^Z)^X \rightarrow \text{Top}(Z \times X, Y)$ is continuous.
the map $Y^{Z \times X} \rightarrow (Y^Z)^X$ is continuous)

As $Z \times X$ is locally compact Hausdorff, then the compact-open topology on $\text{Top}(Z \times X, Y)$ is conjoining, the evaluation map is continuous

$$Z \times (X \times Y^{Z \times X}) \rightarrow Y$$

As the compact-open topology on $\text{Top}(Z, Y)$ is splitting, then the adjunct $X \times Y^{Z \times X} \rightarrow Y^Z$ is continuous. As the compact-open topology on $\text{Top}(X, Y^Z)$ is splitting, then the adjunct $Y^{Z \times X} \rightarrow (Y^Z)^X$ is continuous. \square

7.4 Compactly Generated Weakly Hausdorff Spaces

todo