Differential Forms and Stoke's Theorem

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this is my notes on Differential Forms and Stoke's Theorem from the book: all the mathematics you missed by Thomas A. Garrity

1 Volumes of Parallelepipeds

Theorem 1 *In R^n, the volume of the parallelepiped spanned by the columns of matrix $A \in \mathbb{R}^{k\times n}$ is*

$$\sqrt{\det(AA^T)}$$

2 Differential Forms and the Exterior Derivative

2.1 Elementary k-forms

In $\mathbb{R}^n$, let $I = \{i_1, i_2, ..., i_k\} \subseteq \{1, 2, ..., n\}$ be an index sequence. The elementary k-forms dx_I is defined as the operator measuring the signed volume of the projection of a a parallelepiped into the subspace formed by $e_I = \{e_{i_1}, e_{i_2}, ..., e_{i_k}\}$. [1]

$$dx_I(A) = \det\left[\pi_{e_I}(A)\right]$$

2.2 The Vector Space of k-forms

Definition 1 (Multilinear map) *A function $f : V_1 \times V_2 \times ... \times V_n \to W$ is a multilinear map if it is a linear map w.r.t each variable, i.e. for each $k \in \{1, 2, ..., n\}$,*

- $f(v_1, ..., a_k + b_k, ..., v_n) = f(v_1, ..., a_k, ..., v_n) + f(v_1, ..., b_k, ..., v_n)$
- $f(v_1, ..., \lambda v_k, ..., v_n) = \lambda f(v_1, ..., v_k, ..., v_n)$

Definition 2 (Determinant) *Determinant of an $n \times n$ matrix A is defined as the unique real-valued multilinear map w.r.t each column of A*

$$det : \mathbb{R}^{n\times n} \to \mathbb{R}$$

[1] We also write $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}$

with $det(I) = 1$

Definition 3 (k-forms) *A k-form ω is a real-valued multilinear map w.r.t each column of a $n \times k$ matrix*

$$\omega : \mathbb{R}^{n \times k} \to \mathbb{R}$$

By the properties of determinant, each elementary k-form is a k-form defined in definition 3. Furthermore,

Theorem 2 *In $\mathbb{R}^n$, the set of elementary k-forms with increasing indices is precisely the basis of the vector space of k-forms over the field $\mathbb{R}$ denoted by $\bigwedge^k(\mathbb{R}^n)$. The dimensional of this vector space is $\binom{n}{k}$.*

2.3 Rules for Manipulating k-forms

Definition 4 *In the symmetric group S_m [2], let $k + l = m$ and $\sigma \in S_m$ be the (k, l)-shuffle which has the property that*

$$\sigma(1) < \sigma(2) < .. < \sigma(k)$$

and

$$\sigma(k + 1) < \sigma(k_2) < .. < \sigma(k + l)$$

The set of all (k, l)-shuffles is denoted by $S(k, l)$

Definition 5 (Wedge Product) *Let $A = (A_1, A_2, ..., A_{k+l})$ be an $n \times (k + l)$ matrix. Let τ be a k-form and ω be an l-form, we define the wedge product*

$$(\tau \wedge \omega)(A) = \sum_{\sigma \in S(k,l)} (-1)^{\text{sign}(\sigma)} \tau(A_{\sigma(1)}, ..., A_{\sigma(k)}) \omega(A_{\sigma(k+1)}, ..., A_{\sigma(k+l)}))$$

2.4 Differential k-forms and the Exterior Derivative

Definition 6 *In the symmetric group S_n, let $I = \{i_1, i_2, ..., i_k\} \in S_n$ be the (k)-shuffle which has the property that*

$$i_1 < i_2 < .. < i_k$$

The set of all (k)-shuffles is denoted by $S(k)$

Definition 7 (Differential k-forms) *A differential k-form is defined as*

$$\omega = \sum_{I \in S(k)} f_I dx_I$$

where each $f_I \in C^1(\mathbb{R}^n, \mathbb{R})$: a differentiable function

[2] permutation of m elements

Definition 8 (Exterior derivative) *Given a differential k-form* $\omega = \sum_{I \in S(k)} f_I dx_I$, *the exterior derivaive* $d\omega$ *is*

$$d\omega = \sum_{I \in S(k)} df_I \wedge dx_I$$

where $df_I = \sum_{i \in I} \frac{\partial f_I}{\partial x_i} dx_i$

Proposition 1 *For any differential k-form* ω*, we have*

$$d(d\omega) = 0$$

3 Differential Forms and Vector Fields

Definition 9 (T_0, T_1, T_2, T_3) *In* $\mathbb{R}^3$ *with standard coordinates* x, y, z

Let T_0 *be the identity map on the space of* 0*-form* [3]

$$T_0(f) = f$$

Let T_1 *be the map from the space of* 1*-form into* $\mathbb{R}^3$

$$T_1(f_1 dx + f_2 dy + f_3 dz) = (f_1, f_2, f_3)$$

Let T_2 *be the map from the space of* 2*-form into* $\mathbb{R}^3$

$$T_2(f_1 dx \wedge dy + f_2 dy \wedge dz + f_3 dz \wedge dx) = (f_2, f_3, f_1)$$

Let T_3 *be the map from the space of* 3*-form* [4] *into* $C^1(\mathbb{R}^n, \mathbb{R})$

$$T_3(f dx \wedge dy \wedge dz) = f$$

Theorem 3 *In* $\mathbb{R}^3$*, let* ω_k *denote a differential k-form. Then*

$$\begin{aligned} T_1(d\omega_0) &= \mathrm{grad}(T_0(\omega_0)) \\ T_2(d\omega_1) &= \mathrm{curl}(T_1(\omega_1)) \\ T_3(d\omega_2) &= \mathrm{div}(T_2(\omega_2)) \end{aligned}$$

In $\mathbb{R}^2$ with coordinates $x_1, x_2, ..., x_n$. There is a single elementary n-form, namely $dx_1 \wedge dx_2 \wedge ... \wedge dx_n$. Define the map from space of n-forms into $\mathbb{R}$

$$T : \bigwedge^n(\mathbb{R}^n) \to \mathbb{R}$$

by $T(\alpha dx_1 \wedge dx_2 \wedge ... \wedge dx_n) = \alpha$. For k-forms, the dual space of $\bigwedge^k(\mathbb{R}^n)$ is isomorphic to $\bigwedge^{n-k}(\mathbb{R}^n)$. Let ω_{n-k} be a $n-k$-form, the associated linear map in the dual space of $\bigwedge^k(\mathbb{R}^n)$ is

[3] $C^1(R^n, R)$
[4] isomorphic to $C^1(\mathbb{R}^n, \mathbb{R})$

$$T_{\omega_{n-k}} : \bigwedge^k(\mathbb{R}^n) \to \mathbb{R}$$

defined by $T_{\omega_{n-k}}(\omega_k) = T(\omega_{n-k} \wedge \omega_k)$. Moreover, dimension of the dual space equals dimension of the original space, i.e. $\dim \bigwedge^k(\mathbb{R}^n) = \dim \bigwedge^{n-k}(\mathbb{R}^n)$

4 Manifolds