MA4271 Tutorial Week 3

Nguyen Ngoc Khanh - A0275047B

August 2023

Problem 1 *Let $u, v, w \in \mathbb{R}^3$ be three linearly independent vectors. Prove that*

$$((u \times v) \cdot w)^2 = \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}$$

Let $z_1 = (u_1, v_1, v_1)$, $z_2 = (u_2, v_2, v_2)$, $z_3 = (u_3, v_3, v_3)$, we have $\det(z_1, z_2, z_3) = \det(u, v, w)$

$$\begin{aligned} A &= \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix} \\ &= \det(u_1 z_1 + u_2 z_2 + u_3 z_3, v_1 z_1 + v_2 z_2 + v_3 z_3, w_1 z_1 + w_2 z_2 + w_3 z_3) \end{aligned}$$

By multi-additivity of determinant and det is 0 if two columns/rows are linearly dependent,

$$A = $$

$$\det(u_1 z_1, v_2 z_2, w_3 z_3) + \det(u_1 z_1, v_3 z_3, w_2 z_2) +$$
$$\det(u_2 z_2, v_3 z_3, w_1 z_1) + \det(u_2 z_2, v_1 z_1, w_3 z_3) +$$
$$\det(u_3 z_3, v_1 z_1, w_2 z_2) + \det(u_3 z_3, v_2 z_2, w_1 z_1)$$

By multi-homogeneity of determinant,

$$A = $$

$$u_1 u_2 w_3 \det(z_1, z_2, z_3) + u_1 v_3 w_2 \det(z_1, z_3, z_2) +$$
$$u_2 v_3 w_1 \det(z_2, z_3, z_1) + u_2 v_1 w_3 \det(z_2, z_1, z_3) +$$
$$u_3 v_1 w_2 \det(z_3, z_1, z_2) + u_3 v_2 w_1 \det(z_3, z_2, z_1)$$

Perform swap on determinant.

$$\begin{aligned} A &= (u_1u_2w_3 - u_1v_3w_2 + u_2v_3w_1 - u_2v_1w_3 + u_3v_1w_2 - u_3v_2w_1)\det(z_1, z_2, z_3) \\ &= \det(u, v, w)^2 \\ &= ((u \times v) \cdot w)^2 \end{aligned}$$