

Künneth Theorem

subtitle

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1 UNIVERSAL COEFFICIENT THEOREM

Theorem 1 (universal coefficient theorem). *Let R be a PID, N be an R -module, C_\bullet be chain complexes of R -module, and C_\bullet is degree-wise free (each C_n is a free R -module). Then, there is a short exact sequence*

$$0 \longrightarrow H_n(C_\bullet) \otimes N \longrightarrow H_n(C_\bullet \otimes N) \longrightarrow \text{Tor}_1(H_{n-1}(C_\bullet), N) \longrightarrow 0$$

and this sequence splits (but not naturally)

2 KÜNNETH THEOREM FOR CHAIN COMPLEXES OF R -MODULES

Definition 1 (direct sum of chain complexes of R -module). *In the category of chain complexes of R -module ($\text{Ch}(R\text{-Mod})$), let $C_\bullet, D_\bullet \in \text{ob Ch}(R\text{-Mod})$, define the direct sum $C_\bullet \oplus D_\bullet \in \text{ob Ch}(R\text{-Mod})$ as follows:*

$$(C_\bullet \oplus D_\bullet)_n = C_n \oplus D_n$$

and the boundary map $\partial : (C_\bullet \oplus D_\bullet)_n \rightarrow (C_\bullet \oplus D_\bullet)_{n-1}$ is defined by

$$\begin{aligned} \partial : (C \oplus D)_n &\rightarrow (C \oplus D)_{n-1} \\ c \oplus d &\mapsto \partial c \oplus \partial d \end{aligned}$$

Definition 2 (tensor product of chain complexes of R -module). *In the category of chain complexes of R -module ($\text{Ch}(R\text{-Mod})$), let $C_\bullet, D_\bullet \in \text{ob Ch}(R\text{-Mod})$, define the tensor product $C_\bullet \otimes D_\bullet \in \text{ob Ch}(R\text{-Mod})$ as follows:*

$$(C_\bullet \otimes D_\bullet)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and the boundary map $\partial : (C_\bullet \otimes D_\bullet)_n \rightarrow (C_\bullet \otimes D_\bullet)_{n-1}$ is the linear extension of $\partial : C_p \otimes D_q \rightarrow (C_\bullet \otimes D_\bullet)_{n-1}$ where

$$\partial(c \otimes d) = \partial c \otimes d + (-1)^{|c|} c \otimes \partial d$$

where $c \otimes d \in C_p \otimes D_q$ and $|c| = p$

Proof. TODO - bilinear chain map factors through tensor product □

Definition 3 (the Tor functor). *TODO*

Theorem 2 (Künneth theorem for chain complexes). *Let R be a PID, C_\bullet, D_\bullet be chain complexes of R -module, and C_\bullet is degree-wise free (each C_n is a free R -module). Then, there is a short exact sequence*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(D_\bullet) \xrightarrow{\times} H_n(C_\bullet \otimes D_\bullet) \longrightarrow \bigoplus_{p+q=n-1} \text{Tor}_1^R(H_p(C_\bullet), H_q(D_\bullet)) \longrightarrow 0$$

and this sequence splits (but not naturally)

Proof. Consider the case where the boundary map in C_\bullet is zero, that is, for all $c \in C_n$, $\partial c = 0$. Then,

$$\begin{aligned} \partial : C_p \otimes D_q &\rightarrow C_p \otimes D_{q-1} \\ c \otimes d &\mapsto (-1)^{|c|} c \otimes \partial d \end{aligned}$$

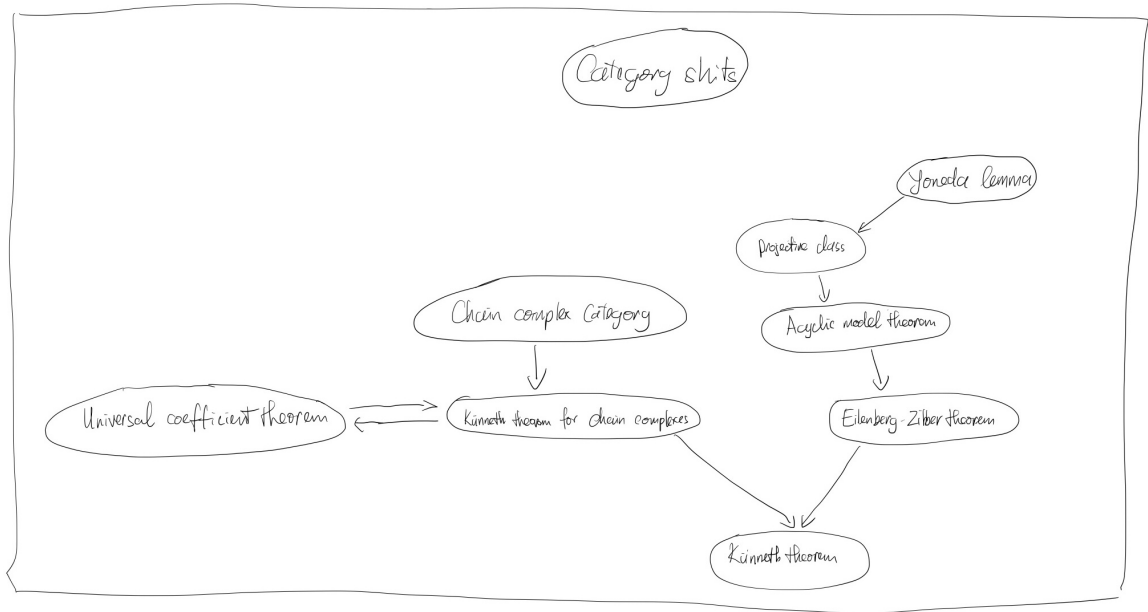


Figure 1: road map

Hence, $C_\bullet \otimes D_\bullet$ can be written as a direct sum of chain complexes $C_\bullet \otimes D_\bullet = \bigoplus_p C_p \otimes D_{\bullet-p}$. Therefore

$$\begin{aligned}
 H_n(C_\bullet \otimes D_\bullet) &= H_n\left(\bigoplus_p C_p \otimes D_{\bullet-p}\right) \\
 &= \bigoplus_p H_n(C_p \otimes D_{\bullet-p}) \\
 &= \bigoplus_p C_p \otimes H_n(D_{\bullet-p}) && (C_p \text{ is free, cons of UCT}) \\
 &= \bigoplus_{p+q=n} C_p \otimes H_q(D_\bullet) && (\text{shifted chain complex}) \\
 &= \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(D_\bullet) && (C_p = H_p(C_\bullet))
 \end{aligned}$$

Now let C_\bullet be an arbitrary chain complex, we have the short exact sequence of chain complexes

$$0 \longrightarrow Z_\bullet \longrightarrow C_\bullet \longrightarrow B_{\bullet-1} \longrightarrow 0$$

where $Z_n = \ker(\partial : C_n \rightarrow C_{n-1})$ and $B_n = \text{im}(\partial : C_{n+1} \rightarrow C_n)$ are n -cycle and n -boundary and consider $Z_\bullet, B_{\bullet-1}$ as chain complexes with zero boundary map. Each Z_n, B_n are free as they are submodules of free R -module C_n . As $B_{\bullet-1}$ is free, the sequence splits, hence, the sequence below is exact

$$0 \longrightarrow Z_\bullet \otimes D_\bullet \longrightarrow C_\bullet \otimes D_\bullet \longrightarrow B_{\bullet-1} \otimes D_\bullet \longrightarrow 0$$

That induces a long exact sequence in homology

$$\begin{array}{ccccccc}
 & & & & \dots & \longrightarrow & H_{n+1}(B_{\bullet-1} \otimes D_\bullet) \\
 & & & & \swarrow & & \uparrow (i_n)_* \\
 H_n(Z_\bullet \otimes D_\bullet) & \longrightarrow & H_n(C_\bullet \otimes D_\bullet) & \longrightarrow & H_n(B_{\bullet-1} \otimes D_\bullet) \\
 & & \swarrow & & \uparrow (i_{n-1})_* \\
 H_{n-1}(Z_\bullet \otimes D_\bullet) & \longrightarrow & \dots & &
 \end{array}$$

where the connecting homomorphisms $(i_n)_*, (i_{n-1})_*$ are induced by inclusion maps

$$\begin{array}{ccc}
 & & (B_{\bullet-1} \otimes D_\bullet)_{n+1} \\
 & \swarrow i_n & \nwarrow \\
 (Z_\bullet \otimes D_\bullet)_n & \longrightarrow & (C_\bullet \otimes D_\bullet)_n
 \end{array}$$

From the long exact sequence, we have the short exact sequence

$$0 \longrightarrow \text{coker}(i_n)_* \longrightarrow H_n(C_\bullet \otimes D_\bullet) \longrightarrow \ker(i_{n-1})_* \longrightarrow 0$$

Discussed in the previous argument, as Z_\bullet and $B_{\bullet-1}$ are free,

$$\begin{aligned} H_n(Z_\bullet \otimes D_\bullet) &= \bigoplus_{p+q=n} Z_p \otimes H_q(D_\bullet) \\ H_{n+1}(B_{\bullet-1} \otimes D_\bullet) &= \bigoplus_{p+q=n} B_p \otimes H_q(D_\bullet) \end{aligned}$$

Since tensor product is right-exact, exactness of the top sequence implies exactness of the bottom sequence

$$0 \longrightarrow B_p \xrightarrow{j} Z_p \longrightarrow H_p(C_\bullet) \longrightarrow 0$$

$$B_p \otimes H_q(D_\bullet) \xrightarrow{(i_*)_{p+q=j \otimes 1}} Z_p \otimes H_q(D_\bullet) \longrightarrow H_p(C_\bullet) \otimes H_q(D_\bullet) \longrightarrow 0$$

Hence, $\text{coker}(i_*)_n = \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(D_\bullet)$. On the other hand, the top sequence is the free resolution of $H_p(C_\bullet)$. Then, $\text{Tor}_1^R(H_p(C_\bullet), H_q(D_\bullet))$ is the first homology group of the bottom sequence

$$\begin{array}{ccccccc} & & 2 & & 1 & & 0 \\ 0 & \longrightarrow & B_p & \xrightarrow{j} & Z_p & \longrightarrow & 0 \\ & & & & & & \\ 0 & \longrightarrow & B_p \otimes H_q(D_\bullet) & \xrightarrow{(i_*)_{p+q=j \otimes 1}} & Z_p \otimes H_q(D_\bullet) & \longrightarrow & 0 \end{array}$$

That is, $\ker(i_*)_{n-1} = \bigoplus_{p+q=n-1} \text{Tor}_1^R(H_p(C_\bullet), H_q(D_\bullet))$

*TODO - here, there are to functors, one is $\otimes D$ composed with H and the other is $\otimes H(D)$ - need to prove
TODO - split*

□

3 KÜNNETH THEOREM FOR TOPOLOGICAL SPACES

3.1 FUNDAMENTAL THEOREM OF HOMOLOGICAL ALGEBRA

Definition 4 (initial object, terminal object, pointed category, zero map, kernel). *Given a category C , an object 0 is initial if for all $X \in \text{ob } C$, there is only one map in $\text{Hom}(0, X)$, an object $*$ is terminal if for all $X \in \text{ob } C$, there is only one map in $\text{Hom}(X, *)$. Category C is called pointed if it has initial and terminal objects and the unique map $0 \rightarrow *$ is an isomorphism.*

If C is a pointed category, we use the same symbol 0 for both initial object and terminal object. There exists a zero map between any two objects $M, N \in \text{ob } C$, defined by

$$\begin{array}{ccc} M & \longrightarrow & 0 \\ & \searrow 0 & \downarrow \\ & & N \end{array}$$

the composition of $M \rightarrow 0$ and $0 \rightarrow N$. Let $f : M \rightarrow N$ be a morphism in C , a kernel of f is a map $i : K \rightarrow M$ such that $fi = 0$ and such map is universal, that is, if $j : L \rightarrow M$ with $fj = 0$, then it factors through K

$$\begin{array}{ccccc} & & 0 & & \\ & \nearrow & \text{curved} & \searrow & \\ K & \xrightarrow{i} & M & \xrightarrow{f} & N \\ & \nearrow j & & & \\ L & \xrightarrow{\text{curved}} & & & 0 \end{array}$$

Category C has kernels if every morphism has a kernel.

Definition 5 (preadditive category, Ab-enriched category). A category C is called *preadditive category* (or *Ab-enriched category*) if for any two objects $M, N \in \text{ob } C$, $\text{Hom}(M, N)$ is an abelian group and composition is bilinear, that is, if f, g, h are morphisms in C

$$\begin{aligned} f(g + h) &= fg + fh \\ (f + g)h &= fh + gh \end{aligned}$$

Definition 6 (chain complex, acyclic chain complex, exact sequence). In a pointed category with kernels, a chain complex is a sequence such that given any subsequence $A \rightarrow B \rightarrow C$, $A \rightarrow B$ factors through $\ker(B \rightarrow C)$, that is, there exists a map $A \rightarrow \ker(B \rightarrow C)$ such that the diagram below commutes

$$\begin{array}{ccccccc} \dots & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow \dots \\ & & \downarrow & \nearrow & & & \\ & & \ker(B \rightarrow C) & & & & \end{array}$$

If there is a notion of epimorphism and the map $A \rightarrow \ker(B \rightarrow C)$ is an epimorphism, then the sequence is called *exact* at B . A sequence is called *exact sequence* or an *acyclic chain complex* if it is exact everywhere, possibly except the two ends.

Definition 7 (chain map, chain homotopy). Given two chain complexes C_\bullet, D_\bullet in a pointed category with kernels, for each $n \in \mathbb{Z}$, there is a map $f_n : C_n \rightarrow D_n$ such that the diagram below commutes, then f_\bullet is called a *chain map*

$$\begin{array}{ccccccc} \dots & \longleftarrow & C_{n-1} & \longleftarrow & C_n & \longleftarrow & C_{n+1} \longleftarrow \dots \\ & & \downarrow f_{n-1} & & \downarrow f_n & & \downarrow f_{n+1} \\ \dots & \longleftarrow & D_{n-1} & \longleftarrow & D_n & \longleftarrow & D_{n+1} \longleftarrow \dots \end{array}$$

Chain complexes and chain maps form a category and it is called the category of chain complexes.

Given two chain complexes C_\bullet, D_\bullet in a pointed **preadditive** category with kernels. Let $f_\bullet, g_\bullet : C_\bullet \rightarrow D_\bullet$ be two chain maps. A chain homotopy from f_\bullet to g_\bullet is a collection of maps $h_n : C_{n-1} \rightarrow D_n$ such that $\partial h_{n+1} + h_n \partial = f_n - g_n$

$$\begin{array}{ccccccc} \dots & \xleftarrow{\partial} & C_{n-1} & \xleftarrow{\partial} & C_n & \xleftarrow{\partial} & C_{n+1} \xleftarrow{\partial} \dots \\ & & \searrow h_n & & \searrow h_{n+1} & & \\ \dots & \xleftarrow{\partial} & D_{n-1} & \xleftarrow{\partial} & D_n & \xleftarrow{\partial} & D_{n+1} \xleftarrow{\partial} \dots \end{array}$$

Definition 8 (projective class). Let C be a pointed category with kernels. A projective class in C is a pair $(\mathcal{P}, \mathcal{E})$ where \mathcal{P} is a collection of objects (called **projectives**) and \mathcal{E} is a collection of morphisms (called **epimorphisms**) such that

1. An object P is **projective** if and only if P has the universal lifting property against every **epimorphism** $M \rightarrow N$, that is, given any **epimorphism** $M \rightarrow N$, if there is a map $P \rightarrow N$, then it factors through M

$$\begin{array}{ccc} M & \xrightarrow{\text{epi}} & N \\ & \nwarrow & \uparrow \\ & & P \end{array}$$

2. A morphism $f : M \rightarrow N$ is an **epimorphism** if and only if every **projective** has the universal lifting property against f , that is, given any **projective** P , if there is a map $P \rightarrow N$, then it factors through M

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & \nwarrow & \uparrow \\ & & P \end{array}$$

3. C has enough **projectives**, that is, given any object $M \in \text{ob } C$, for every **projective** P , there exists an **epimorphism** $P \rightarrow M$.

Theorem 3 (fundamental theorem of homological algebra). Let C be a pointed category with kernels and $(\mathcal{P}, \mathcal{C})$ be a projective class in C . Given $f : M \rightarrow M'$ in C and the diagram below

$$\begin{array}{ccccccc} 0 & \longleftarrow & M & \xleftarrow{\epsilon} & P_0 & \xleftarrow{d} & P_1 \xleftarrow{d} \dots \\ & & \downarrow f & & \downarrow f_0 & & \downarrow f_1 \\ 0 & \longleftarrow & M' & \xleftarrow{\epsilon'} & P'_0 & \xleftarrow{d'} & P'_1 \xleftarrow{d'} \dots \end{array}$$

where both chains are chain complexes, the top chain consists of projectives P_n and the bottom chain is acyclic. Then,

- There exists a chain map defined by $f_n : P_n \rightarrow P'_n$
- If C is preadditive, the lift is unique upto chain homotopy.

Proof.

1. The first statement is proved by induction

$$\begin{array}{ccccc}
 P_{n-2} & \longleftarrow & P_{n-1} & \longleftarrow & P_n \\
 \downarrow f_{n-2} & & \downarrow f_{n-1} & & \downarrow f_n \\
 P'_{n-2} & \longleftarrow & P'_{n-1} & \longleftarrow & P'_n
 \end{array}$$

K'_{n-1} is shown as a subobject of P'_{n-1} with maps $P_{n-1} \rightarrow K'_{n-1}$ and $P'_n \rightarrow K'_{n-1}$. Dashed arrows indicate the factorization of $P_n \rightarrow P'_{n-1}$ through K'_{n-1} .

Suppose there exist maps $f_{n-1} : P_{n-1} \rightarrow P'_{n-1}$ and $f_{n-2} : P_{n-2} \rightarrow P'_{n-2}$. Let $K'_{n-1} = \ker(P'_{n-1} \rightarrow P'_{n-2})$.

Since the bottom chain is acyclic, the map $P'_n \rightarrow P'_{n-1}$ factors through K'_{n-1} by an epimorphism.

Since the top chain is a chain complex, the composition $P_n \rightarrow P_{n-1} \rightarrow P'_{n-1} \rightarrow P'_{n-2}$ equals $P_n \rightarrow P_{n-1} \rightarrow P_{n-2} \rightarrow P'_{n-2}$ and equals 0 zero, so $P_n \rightarrow P_{n-1} \rightarrow P'_{n-1}$ factors through K'_{n-1} .

Since P_n is projective and $P'_n \rightarrow K'_{n-1}$ is an epimorphism, $P_n \rightarrow K'_{n-1}$ factors through P'_n by a map $f_n : P_n \rightarrow P'_n$.

Base case: $n = 0$, let $P_{n-1} = M, P'_{n-1} = M', P_{n-2} = 0, P'_{n-2} = 0$ and $f_{n-1} = f, f_{n-2} = 0$

2. Let $f_{\bullet}^{(1)}, f_{\bullet}^{(2)} : P_{\bullet} \rightarrow P'_{\bullet}$ be any two lifts from $f : M \rightarrow M'$

$$\begin{array}{ccc}
 M & \xleftarrow{\epsilon} & P_{\bullet} \\
 f \downarrow & & \downarrow f_{\bullet}^{(1)} \quad \downarrow f_{\bullet}^{(2)} \\
 M' & \xleftarrow{\epsilon'} & P'_{\bullet}
 \end{array}$$

We will prove that $g_{\bullet} = f_{\bullet}^{(1)} - f_{\bullet}^{(2)}$ is chain homotopic to zero, that is to find maps $h_{n+1} : P_n \rightarrow P'_{n+1}$ such that $d'h + hd = g$

$$\begin{array}{ccccccc}
 0 & \xleftarrow{d} & P_0 & \xleftarrow{d} & P_1 & \xleftarrow{d} & \dots \\
 0 \downarrow & & g_0 \downarrow & & g_1 \downarrow & & \\
 0 & \xleftarrow{d'} & P'_0 & \xleftarrow{d'} & P'_1 & \xleftarrow{d'} & \dots
 \end{array}$$

Suppose there exists map $h_{n-1} : P_{n-2} \rightarrow P'_{n-1}$ and $h_{n-2} : P_{n-3} \rightarrow P'_{n-2}$ such that

$$g_{n-2} - h_{n-2}d = d'h_{n-1}$$

$$\begin{array}{ccccc}
 P_{n-3} & \xleftarrow{d} & P_{n-2} & \xleftarrow{d} & P_{n-1} \\
 & \searrow h_{n-2} & & \searrow h_{n-1} & \\
 P'_{n-3} & \xleftarrow{d'} & P'_{n-2} & \xleftarrow{d'} & P'_{n-1}
 \end{array}$$

Consider the map $g_{n-1} - h_{n-1}d : P_{n-1} \rightarrow P'_{n-1}$,

$$\begin{aligned}
 d'(g_{n-1} - h_{n-1}d) &= d'g_{n-1} - d'h_{n-1}d && \text{(preadditive)} \\
 &= d'g_{n-1} - (g_{n-2} - h_{n-2}d)d && \text{(induction)} \\
 &= d'g_{n-1} - g_{n-2}d && \text{(preadditive, } dd = 0) \\
 &= 0 && (g_{\bullet} \text{ is a chain map})
 \end{aligned}$$

Let $K'_{n-1} = \ker(d' : P'_{n-1} \rightarrow P'_{n-2})$.

Since the bottom chain is acyclic, the map $d' : P'_n \rightarrow P'_{n-1}$ factors through K'_{n-1} by an epimorphism.

$$\begin{array}{ccccc}
P_{n-2} & \xleftarrow{d} & P_{n-1} & & \\
& \searrow h_{n-1} & \downarrow & \swarrow h_n & \\
& & K'_{n-1} & & \\
& \searrow & \downarrow & \swarrow & \\
P'_{n-2} & \xleftarrow{d'} & P'_{n-1} & \xleftarrow{d'} & P'_n
\end{array}$$

As $d'(g_{n-1} - h_{n-1}d) = 0$, $g_{n-1} - h_{n-1}d$ factors through K'_{n-1} , that is, $g_{n-1} - h_{n-1}d$ equals the composition $P_{n-1} \rightarrow K'_{n-1} \rightarrow P'_{n-1}$

Since P_{n-1} is projective and $P'_n \rightarrow K'_{n-1}$ is an epimorphism, $P_{n-1} \rightarrow K'_{n-1}$ factors through P'_n by a map $h_n : P_{n-1} \rightarrow P'_n$, that is, the $d'h_n$ equals the composition $P_{n-1} \rightarrow P'_n \rightarrow K'_{n-1} \rightarrow P'_{n-1}$ and equals the composition $P_{n-1} \rightarrow K'_{n-1} \rightarrow P'_{n-1}$, hence

$$d'h_n = g_{n-1} - h_{n-1}d$$

Base case: $n = 0$, let $P_{n-2} = 0, P'_{n-2} = 0, P_{n-1} = M, P'_{n-1} = M', h_{n-1} = 0$, then

$$d'(g_{n-1} - h_{n-1}d) = 0 \quad (d' : P'_{n-1} \rightarrow P'_{n-2} \text{ is the zero map } M' \rightarrow 0)$$

□

3.2 RESOLUTION AND TOR FUNCTOR

Definition 9 (resolution, projective resolution). *Let M be an object in a pointed category with kernels. A resolution of M is an exact sequence*

$$0 \longleftarrow M \xleftarrow{\epsilon} P_0 \xleftarrow{d} P_1 \xleftarrow{d} \dots$$

If P_n are projectives in a projective class $(\mathcal{P}, \mathcal{E})$, then the sequence is called \mathcal{P} -projective resolution.

Corollary 1. *Let M be an object in a pointed preadditive category with kernels. Any two projective resolutions of M are chain homotopy equivalent¹ or equivalently M defines a chain homotopy type.*

Definition 10 (Tor functor on R-Mod). *TODO*

3.3 EILENBERG-ZILBER THEOREM

Proposition 1 (projective module, projective class in R-Mod). *In the category of R -module (R-Mod), there is a projective class $(\mathcal{P}, \mathcal{E})$ defined by epimorphism being surjective homomorphism. Then, the following are equivalent*

1. $P \in \text{ob R-Mod}$ is projective
2. Every short exact sequence $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$ splits
3. P is a direct summand of a free R -module, that is, there exists $Q \in \text{ob R-Mod}$ such that $P \oplus Q$ is a free R -module.

Proof. *TODO*

□

Proposition 2 (models define projective class in $\text{Fun}(C, \text{R-Mod})$). *Given a category C , $\text{Fun}(C, \text{R-Mod})$ is a pointed preadditive category with kernels (more precisely, abelian category - will define in the future).*

Let \mathcal{M} be any set of objects in C (called models), then \mathcal{M} defines a projective class $(\mathcal{P}, \mathcal{E})$ in $\text{Fun}(C, \text{R-Mod})$ where a morphism $G \rightarrow F$ is an epimorphism (relative to \mathcal{M}) if for all $M \in \mathcal{M}$, $G(M) \rightarrow F(M)$ is surjective. Then, the following are equivalent

1. $P \in \text{ob Fun}(C, \text{R-Mod})$ is projective
2. P is a **retract of coproduct** of $R\text{Hom}(M, -)$ for some $M \in \mathcal{M}$ where $\text{Hom}(M, -)$ is a functor $\text{R-Mod} \rightarrow \text{Set}$, R is the free R -module functor $\text{Set} \rightarrow \text{R-Mod}$. In the case of R -module, **retract of coproduct** is the **direct summand** of a R -module

Proof. *TODO - prove using Yoneda lemma*

□

Theorem 4 (Eilenberg-Zilber theorem). *TODO*

¹two chain complexes C_\bullet, D_\bullet are chain homotopy equivalent if there are two chain maps $f_\bullet : C_\bullet \rightarrow D_\bullet, g_\bullet : D_\bullet \rightarrow C_\bullet$ such that gf and fg are chain homotopic to identity