MA5204 Homework 2

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May 20, 2025

Problem 1 (chapter 2 problem 1)
Show that $\frac{\mathbb{Z}}{m\mathbb{Z}} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}}{n\mathbb{Z}} = 0$ if m, n are coprime

Proof. If m, n are coprime, $m\mathbb{Z} + n\mathbb{Z} = \mathbb{Z}$ as sum of ideals in $\mathbb{Z}$

$$\frac{\mathbb{Z}}{m\mathbb{Z}} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}}{n\mathbb{Z}} = \frac{\mathbb{Z}}{m\mathbb{Z} + n\mathbb{Z}} = \frac{\mathbb{Z}}{\mathbb{Z}} = 0$$

□

Problem 2 (chapter 2 problem 2)
Let A be a ring, $\mathfrak{a}$ be an ideal of A and M be an A-module. Show that $A/\mathfrak{a} \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$

Proof. The top sequence is exact with the canonical inclusion and projection. By right exactness of tensor product, the bottom sequence is also exact

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a} \arrow[r, hook, "i"] & A \arrow[r, two heads, "p"] & A/\mathfrak{a} \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} \mathfrak{a} \otimes M \arrow[r, "i\otimes 1"] & A \otimes M \arrow[r, two heads] & A/\mathfrak{a} \otimes M \arrow[r] & 0 \end{tikzcd}$$

Then, $A/\mathfrak{a} \otimes M \cong \operatorname{coker}(i \otimes 1)$. On the other hand, $A \otimes M \xrightarrow{\sim} M$ and the image of $\mathfrak{a} \otimes M$ in M under $i \otimes 1$ is $\mathfrak{a}M$

$$\begin{tikzcd} \mathfrak{a} \otimes M \arrow[r, "i\otimes 1"'] \arrow[rr, bend left, "i\otimes 1"] & A \otimes M \arrow[r, "\sim"] & M \end{tikzcd}$$

$$\textstyle\sum_i a_i \otimes m_i \longmapsto \sum_i a_i \otimes m_i \longmapsto \sum_i a_i m_i$$

Hence, $\operatorname{coker}(i \otimes 1) = \frac{A \otimes M}{(i\otimes 1)(\mathfrak{a} \otimes M)} \cong \frac{M}{\mathfrak{a}M}$ □

Problem 3 (chapter 2 problem 3)
Let A be a local ring, M and N be finitely generated A-modules. Prove that if $M \otimes_A N = 0$, then $M = 0$ or $N = 0$

Proof. Let $\mathfrak{m}$ be an ideal of A and $k = A/\mathfrak{m}$. If $M \otimes_A N = 0$, then

$$0 = (M \otimes_A N) \otimes_A k \otimes_A k \cong (k \otimes_A M) \otimes_A (k \otimes_A N)$$

By exercise 2, $k \otimes_A M \cong M/\mathfrak{m}M = M_k$ and $k \otimes_A N \cong N/\mathfrak{m}N = N_k$, then

$$M_k \otimes_A N_k = 0$$

Note that, given any ring A ideal $\mathfrak{a}$ and an $A/\mathfrak{a}$-module M, then M also carries A-module structure defined by

$$\begin{aligned} A \times M &\to M \\ (a, m) &\mapsto \bar{a}m \end{aligned}$$

$M_k \otimes_k N_K \cong M_k \otimes_A N_k$ by the following A-module isomorphism

$$M_k \otimes_k N_K \xrightarrow{\sim} M_k \otimes_A N_k$$
$$\bar{x} \otimes_k \bar{y} \mapsto \bar{x} \otimes_A \bar{y}$$

Hence, $M_k \otimes_k N_k = 0$. Let $\mathfrak{m}$ be the unique maximal ideal of the local ring A, then k is a field, hence $M_k = 0$ or $N_k = 0$. By Nakayama lemma version 1, $\mathfrak{m} \subseteq J(A)$, $\mathfrak{m}M = M$ or $\mathfrak{m}N = 0$ implies $M = 0$ or $N = 0$ $\square$

Problem 4 (chapter 2 problem 8)

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1. If M and N are flat A-modules, then so is $M \otimes_A N$
2. If B is a flat A-algebra and N is a flat B-module, then N is a flat as an A-module

Proof.

(1) Let $f : X \to Y$ be an injective A-module morphism and f_1 be the induced map by the functor $((- \otimes_A M) \otimes_A N)$. Let f_2 be the induced map by the functor $(- \otimes_A (M \otimes_A N))$. There is a natural isomorphism $g : ((- \otimes_A M) \otimes_A N) \to (- \otimes_A (M \otimes_A N))$ as follows:

$$\begin{tikzcd}
(X \otimes_A M) \otimes_A N \arrow[rrr, "f_1"] \arrow[ddd, "g_X"'] & & & (Y \otimes_A M) \otimes_A N \arrow[ddd, "g_Y"] \\
& (x \otimes m) \otimes n \arrow[r, mapsto] \arrow[d, mapsto] & (f(x) \otimes m) \otimes n \arrow[d, mapsto] & \\
& x \otimes (m \otimes n) \arrow[r, mapsto] & f(x) \otimes (m \otimes n) & \\
X \otimes_A (M \otimes_A N) \arrow[rrr, "f_2"] & & & Y \otimes_A (M \otimes_A N)
\end{tikzcd}$$

The square commutes since

$$g_Y f_1((x \otimes m) \otimes n) = g_Y((f(x) \otimes m) \otimes n) = f(x) \otimes (m \otimes n)$$
$$f_2 g_X((x \otimes m) \otimes n) = f_2(x \otimes (m \otimes n)) = f(x) \otimes (m \otimes n)$$

for all $x \in X, m \in M, n \in N$. f_1 being injective implies f_2 being injective. Hence $M \otimes_A N$ is flat.

(2) Note that B and N are (A, B)-bimodules, hence given any A-module M

$$M \otimes_A N \cong M \otimes_A (B \otimes_B N) \cong (M \otimes_A B) \otimes_B N$$

We will show the following: (2a) $(- \otimes_A B)$ is an exact functor from A-module into (A, B)-bimodule (2b) $((- \otimes_A B) \otimes_B N)$ is an exact functor from A-module into (A, B)-bimodule (2c) $((- \otimes_A B) \otimes_B N)$ is naturally isomorphic to $(- \otimes_A N)$

(2a) Let $f : X \to Y$ be an injective A-module morphism and f_1 be the induced A-module map by the functor $(- \otimes_A B)$

$$f : X \to Y$$
$$f_1 : X \otimes_A B \to Y \otimes_A B$$

$X \otimes_A B$ and $Y \otimes_A B$ canonically carry B-module structure (extension of scalars under the map $a \mapsto a1_B$) defined by

$$\begin{aligned} B \times (X \otimes_A B) &\to X \otimes_A B \\ (b_1, x \otimes_A b) &\mapsto x \otimes_A b_1 b \end{aligned}$$

Under that B-module structure, f_1 is also a B-module map because

$$f_1(b_1(x \otimes_A b)) = f_1(x \otimes_A b_1 b) = f(x) \otimes_A b_1 b = b_1(f(x) \otimes_A b) = b_1 f_1(x \otimes_A b)$$

Since B is flat as an A-module, f_1 as an A-module map is injective, so is f_1 as an (A, B)-bimodule map. Hence, $(- \otimes_A B)$ is an exact functor from A-module into (A, B)-bimodule
(2b) Let f_2 be the induced B-module map by the functor $((- \otimes_A B) \otimes_B N)$

$$\begin{aligned} f &: X \to Y \\ f_1 &: X \otimes_A B \to Y \otimes_A B \\ f_2 &: (X \otimes_A B) \otimes_B N \to (Y \otimes_A B) \otimes_B N \end{aligned}$$

Since N is flat as a B-module, $((- \otimes_A B) \otimes_B N)$ is exact as a functor from A-module to B-module, that is, f_2 is injective as a B-module map. $(X \otimes_A B) \otimes_B N$ and $(Y \otimes_A B) \otimes_B N$ carry an A-module structure defined by

$$\begin{aligned} A \times ((X \otimes_A B) \otimes_B N) &\to (X \otimes_A B) \otimes_B N \\ (a, (x \otimes_A b) \otimes_B n) &\mapsto (x \otimes_A ab) \otimes_B n \end{aligned}$$

Under that A-module structure, f_2 is also an A-module map because

$$f_2(a((x \otimes_A b) \otimes_B n)) = f_2((x \otimes_A ab) \otimes_B n) = (f(x) \otimes_A ab) \otimes_B n = a((f(x) \otimes_A b) \otimes_B n) = af_2((x \otimes_A b) \otimes_B n)$$

Hence, $((- \otimes_A B) \otimes_B N)$ is an exact functor from A-module into (A, B)-bimodule.
(2c) Note that, N carries a A-module structure defined by

$$\begin{aligned} A \times N &\to N \\ (a, n) &\mapsto (a1_B)n \end{aligned}$$

Let f_3 be the induced A-module map by the functor $(- \otimes_A N)$. The natural isomorphism $g : (- \otimes_A N) \to ((- \otimes_A B) \otimes_B N)$ is defined as follows:

$$\begin{tikzcd} X \otimes_A N \arrow[rrr, "f_3"] \arrow[ddd, "g_X"'] & & & Y \otimes_A N \arrow[ddd, "g_Y"] \\ & x \otimes_A n \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \otimes_A n \arrow[d, mapsto] & \\ & (x \otimes_A 1) \otimes_B n \arrow[r, mapsto] & (f(x) \otimes_A 1) \otimes_B n & \\ (X \otimes_A B) \otimes_B N \arrow[rrr, "f_2"] & & & (Y \otimes_A B) \otimes_B N \end{tikzcd}$$

The square commutes since

$$g_Y f_3(x \otimes_A n) = g_Y(f(x) \otimes_A N) = (f(x) \otimes_A 1) \otimes_B n$$
$$f_2 g_X(x \otimes_A n) = f_2((x \otimes_A 1) \otimes_B n) = (f(x) \otimes_A 1) \otimes_B n$$

for all $x \in X, n \in N$. f_2 being injective implies f_3 being injective. Hence, N is flat as an A-module by the above A-module structure. $\square$

Problem 5 (chapter 3 problem 4)
Let $f : A \to B$ be a homomorphism of rings and let S be a multiplicatively closed subset of A. Let $T = f(S)$. Show that $S^{-1}B$ and $T^{-1}B$ are isomorphic as $S^{-1}A$-modules.

Proof. B carries the A-module structure defined by

$$A \times B \to B$$
$$(a, b) \mapsto f(a)b$$

for $a \in A$ and $b \in B$. Hence, $S^{-1}B$ is a localization of A-module B on the multiplicatively closed subset S

$$S^{-1}B = B \times S/\sim$$

where $(b_1, s_1) \sim (b_2, s_2)$ for $b_1, b_2 \in B$ and $s_1, s_2 \in S$ if and only if there exists $s \in S$ so that $0 = s(s_2b1 - s_1b2) = f(s)(f(s_2)b_1 - f(s_1)b_2)$. The $S^{-1}A$-module structure on $S^{-1}B$ is defined by

$$S^{-1}A \otimes S^{-1}B \to S^{-1}B$$
$$\left(\frac{a}{s_1}, \frac{b}{s_2}\right) \mapsto \frac{ab}{s_1s_2} = \frac{f(a)b}{s_1s_2}$$

where $a \in A$, $b \in B$, and $s_1, s_2 \in S$. On the other hand, $T^{-1}B$ is a localization of ring B on the multiplicatively closed subset T

$$T^{-1}B = B \times T/\sim$$

where $(b_1, t_1) \sim (b_2, t_2)$ for $b_1, b_2 \in B$ and $t_1, t_2 \in T$ if and only if there exists $t \in T$ so that $0 = t(t_2b_1 - t_1b_2)$. We define the canonical $S^{-1}A$-module structure on $T^{-1}B$ by

$$S^{-1}A \times T^{-1}B \to T^{-1}B$$
$$\left(\frac{a}{s}, \frac{b}{t}\right) \mapsto \frac{ab}{st} = \frac{f(a)b}{f(s)t}$$

Now, we can define a pair of isomorphisms between $S^{-1}B$ and $T^{-1}B$ as $S^{-1}A$-modules as follows:

$$S^{-1}B \to T^{-1}B$$
$$\phi : \frac{b}{s} \mapsto \frac{b}{f(s)}$$
$$\psi : \frac{b}{s_t} \leftarrow\!\shortmid \frac{b}{t}$$

where $b \in B$, $s \in S$, $t \in T$, and $s_t = f^{-1}(t) \in S$ is any element in the preimage of t.

(ϕ is well-defined) Let $b_1/s_1 = b_2/s_2$ in $S^{-1}B$, then there exists $s \in S$, so that $0 = s(s_2 b_1 - s_1 b_2) = f(s)(f(s_2)b_1 - f(s_1)b_2)$, this is the condition for $\phi(b_1/s_1) = \phi(b_2/s_2)$
(ψ is well-defined) Let $s_1, s_2 \in S$ so that $b_1/f(s_1) = b_2/f(s_2)$ in $T^{-1}B$, then there exists $t \in T = f(S)$ so that $0 = t(f(s_2)b_1 - f(s_1)b_2)$. Let $s \in S$ so that $f(s) = t$, then we have $0 = f(s)(f(s_2)b_1 - f(s_1)b_2) = s(s_2 b_1 - s_1 b_2)$, this is the condition for $\psi(b_1/f(s_1)) = psi(b_2/f(s_2))$. Note that, this also show that the image of ψ is independent of the choice of s_t
($\phi\psi = 1$, $\psi\phi = 1$) this is clear from the definition
(ϕ and ψ are $S^{-1}A$-module maps)

$$\phi\left(\frac{a}{s_1}\frac{b_2}{s_2} + \frac{b_3}{s_3}\right) = \phi\left(\frac{f(s_3)f(a)b_2 + f(s_1)f(s_2)b_3}{s_1 s_2 s_3}\right) = \frac{f(s_3)f(a)b_2 + f(s_1)f(s_2)b_3}{f(s_1)f(s_2)f(s_3)} = \frac{a}{s_1}\phi\left(\frac{b_2}{s_2}\right) + \phi\left(\frac{b_3}{s_3}\right)$$

$$\psi\left(\frac{a}{s_1}\frac{b_2}{f(s_2)} + \frac{b_3}{f(s_3)}\right) = \psi\left(\frac{f(s_3)f(a)b_2 + f(s_1)f(s_2)b_3}{f(s_1)f(s_2)f(s_3)}\right) = \frac{f(s_3)f(a)b_2 + f(s_1)f(s_2)b_3}{s_1 s_2 s_3} = \frac{a}{s_1}\psi\left(\frac{b_2}{f(s_2)}\right) + \phi\left(\frac{b_3}{f(s_3)}\right)$$

□

Problem 6 (chapter 3 problem 12 - torsion submodule)
Let A be an integral domain and M an A-module. An element $x \in M$ is a torsion element of M if $\operatorname{ann}_A(x) = \{a \in A : ax = 0\} \neq 0$, that is if x is killed by some non-zero element of A. Show that the torsion elements of M form a submodule of M. This submodule is called the torsion submodule of M and denoted by $T(M)$. If $T(M) = 0$, the module M is said to be torsion-free. Show that

1. If M is any A-module, then $M/T(M)$ is torsion-free
2. If $f : M \to N$ is a module morphism then $f(T(M)) \subseteq T(N)$
3. If $0 \to M_l \to M \to M_r \to 0$ is an exact sequence, then the sequence $0 \to T(M_l) \to T(M) \to T(M_r)$ is exact, i.e. $T(-)$ is a left exact covariant functor
4. If M is any A-module, then $T(M)$ is the kernel of the mapping $x \mapsto 1 \otimes x$ of M into $K \otimes_A A$ where K is the field of fractions of A, i.e. $K = \mathrm{Frac}(A) = (A - 0)^{-1}A$

Proof. ($T(M)$ is a submodule of M) $0 \in T(M)$. If $x, y \in T(M)$, then $ax = 0$ and $by = 0$ for some $a, b \in A$. Hence $a(-x) = ax + a(-x) = a(x - x) = 0$ and $ab(x + y) = bax + aby = 0$. Moreover, for any $a_1 \in A$, then $a(a_1 x) = a_1 a x = 0$, hence $T(M)$ is a submodule of M
(1) Suppose $\bar{x} \in M/T(M)$ is nonzero and it is an element of the torsion submodule of $M/T(M)$, there exists $a \in A$ so that $0 = a\bar{x} = \overline{ax}$, hence $ax \in T(M)$, so there exists $b \in A$ so that $bax = 0$, that implies $x \in T(M)$ which contradicts the assumption of $\bar{x}$ being nonzero
(2) Let $x \in T(M)$, then there exists $a \in A$, so that $ax = 0$. Hence $0 = f(ax) = af(x)$, so $f(x) \in T(N)$
(3) $T(M_l) \to T(M)$ is injective since it is a restriction of the injective map $M_l \to M$. Moreover, by (2)

$$\ker(T(M) \to T(M_r)) = T(M) \cap \ker(M \to M_r) = T(M) \cap \operatorname{im}(M_l \to M) \supseteq \operatorname{im}(T(M_l) \to T(M))$$

Let denote the map $M_l \to M$ by $f : M_l \to M$ For any $x \in T(M) \cap \operatorname{im}(M_l \to M)$, there exists $y \in M_l$ so that $f(y) = x$. Since, $x \in T(M)$, there exists $a \in A$ so that $0 = ax = af(y) = f(ay)$. Since f is injective, $ay = 0$, i.e $y \in T(M_l)$, hence $T(M) \cap \operatorname{im}(M_l \to M) = \operatorname{im}(T(M_l) \to T(M))$
(4)

Let $S = A - 0$, then we have

$$S^{-1}A \otimes_A M \cong S^{-1}M$$

The composition $M \to S^{-1}M$ is defined by

$$\begin{aligned} M \to K \otimes M &\xrightarrow{\sim} S^{-1}M \\ x \mapsto 1 \otimes x &\mapsto \frac{x}{1} \end{aligned}$$

$$x \in \ker(M \to K \otimes M) \iff x/1 = 0 \text{ in } S^{-1}M \iff \exists a \in S, ax = 0 \iff x \in T(M)$$

Note, the suggestion in the book was really misleading and that costed me a whole night and not delivering the solution. My effort was as below

(K is a colimit of a diagram containing $A\mu$ for $\mu \in K - 0$)

Consider K as an A-module, for any $\mu \in K - 0$, $A\mu = \{\tilde{a}\mu : \tilde{a} \in A\}$ is a submodule of K, there is a canonical A-module map

$$\begin{aligned} f_{\mu\nu} : A\mu &\to A\nu \\ x &\mapsto (\nu\mu^{-1})x \end{aligned}$$

for any $\mu, \nu \in K - 0$ and $x \in A\mu$. Define

$$\begin{aligned} g_\mu : A\mu &\to K \\ x &\mapsto \mu^{-1}x \end{aligned}$$

Then, $g_\bullet$ is the colimit of the diagram consists of $f_{\bullet\bullet}$

$$\begin{tikzcd} A\mu \arrow[rr, "f_{\mu\nu}"] \arrow[dr, "g_\mu"] \arrow[ddr, "h_\mu"'] & & A\nu \arrow[dl, "g_\nu"'] \arrow[ddl, "h_\nu"] \\ & K \arrow[d, dashed, "k"] & \\ & L & \end{tikzcd}$$

Let h_μ and h_ν be defined so that the diagram commutes, then $k : K \to L$ is

$$\begin{aligned} k : K &\to L \\ x &\mapsto h_\mu(\mu x) \end{aligned}$$

This map is unique since if $k' : K \to L$ makes the diagram commutes, then

$$(k' - k)g_\mu = k'g_\mu - kg_\mu = 0$$

for all $\mu \in K - 0$. For any $x = a/b \in K$ for $a \in A$ and $b \in A - 0$, let $\mu = 1/b$, then $x \in A\mu$. In other words, the map $g : \coprod_{\mu \in K-0} A\mu \to K$ is surjective.

$$\begin{tikzcd} A\mu \arrow[r] \arrow[dr] & \coprod_{\mu \in K-0} A\mu = \bigoplus_{\mu \in K-0} A\mu \arrow[d, two heads] \\ & K \end{tikzcd}$$

Then, $0 = (k' - k)g_\mu(\mu x) = (k' - k)(x)$ for all $x \in K$. Hence, $k' - k$ is a zero function, so the factoring map k is unique. In particular, the diagram of $f_{\bullet\bullet}$ is a directed set, so K is the direct limit of the directed set

$$K = \operatorname{colim}_\mu A_\mu = \varinjlim_\mu A_\mu$$

(colimit are compatible with tensor product) The diagram consists of $A_\mu \otimes M$ and $g_{\mu\nu} \otimes 1 : A_\mu \otimes M \to A_\nu \otimes M$ is a directed set, A-module is a cocomplete category, hence the colimit exists

$$\operatorname{colim}_\mu(A_\mu \otimes M) = \varinjlim_\mu(A_\mu \otimes M)$$

We will show that $\operatorname{colim}_\mu(A_\mu \otimes M) \cong (\operatorname{colim}_\mu A_\mu) \otimes M$. For any A-module L, we have

$$\begin{aligned}
\operatorname{Hom}(\operatorname{colim}_\mu(A_\mu \otimes M), L) &\cong \lim_\mu \operatorname{Hom}(A_\mu \otimes M, L) && (\operatorname{Hom}(-, L) \text{ is contravariant}) \\
&\cong \lim_\mu \operatorname{Hom}(A_\mu, \operatorname{Hom}(M, L)) && (\text{tensor-hom adjunction}) \\
&\cong \operatorname{Hom}(\operatorname{colim}_\mu A_\mu, \operatorname{Hom}(M, L)) && (\operatorname{Hom}(-, \operatorname{Hom}(M, L)) \text{ is contravariant}) \\
&\cong \operatorname{Hom}((\operatorname{colim}_\mu A_\mu) \otimes M, L) && (\text{tensor-hom adjunction})
\end{aligned}$$

Hence, $\operatorname{colim}_\mu(A_\mu \otimes M) \cong (\operatorname{colim}_\mu A_\mu) \otimes M$ as a consequence of Yoneda lemma. In particular

$$K \otimes M \cong \operatorname{colim}_\mu(A_\mu \otimes M)$$

(main proof)
Now, the canonical isomorphism $M \mapsto A1 \otimes M$ defined by $x \mapsto 1 \otimes x$ and the canonical map $M \to K \otimes M$ defined by $x \mapsto 1 \otimes x$ make the diagram commutes.

$$\begin{tikzcd}
M \arrow[r, "\sim"] \arrow[drr] & A1 \otimes M \arrow[r, "f_{1\nu}\otimes 1"] \arrow[dr, "g_1 \otimes 1"] & A\nu \otimes M \arrow[d, "f_\nu \otimes 1"] \\
& & K \otimes M
\end{tikzcd}$$

Let $x \in \ker(M \to K \otimes M)$, for any $\nu \in K - 0$, the image on $A\nu \otimes M$ is

$$(f_{1\nu} \otimes 1)(1 \otimes x) = \nu \otimes x$$

Let $\nu = a/1$, then

$$(f_{1\nu} \otimes 1)(1 \otimes x) = \nu \otimes x = 1 \otimes ax$$

Pullback to M gives $ax \in \ker(M \to K \otimes M)$ $\square$

Problem 7 (chapter 3 problem 16 - faithfully flat)
Let B be a flat A-algebra. Then the following conditions are equivalent

1. $\mathfrak{a}^{ec} = \mathfrak{a}$ for all ideals $\mathfrak{a}$ of A
2. $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective
3. For every maximal ideal $\mathfrak{m}$ of A, we have $\mathfrak{m}^e \neq (1)$
4. If M is any non-zero A-module, then $M_B \neq 0$ for $M_B = M \otimes_A B$
5. For every A-module M, the mapping $x \to 1 \otimes x$ of M into M_B is injective

B is said to be faithfully flat over A

Lemma 8 (chapter 2 exercise 13)
Let $f : A \to B$ be a ring map and N be a B-module, then the map $g : N \to N \otimes_A B$ defined by $y \mapsto 1 \otimes y$ is injective.

Proof.
$(1 \implies 2)$ The map $\phi^* : \operatorname{Spec} B \to \operatorname{Spec} A$ is defined by

$$\phi^* : \operatorname{Spec} B \to \operatorname{Spec} A$$
$$\mathfrak{q} \mapsto \mathfrak{q}^c$$

For any prime ideal $\mathfrak{p}$ in $\operatorname{Spec} A$,

$$\mathfrak{p} = \mathfrak{p}^{ec} = \phi^*(\mathfrak{p}^e)$$

Hence, the map $\phi^* : \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective
$(2 \implies 3)$ Since $\phi^* : \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective, there exists a prime ideal $\mathfrak{n} \in \operatorname{Spec} B$ so that $\mathfrak{m} = \phi^*(\mathfrak{n}) = \mathfrak{n}^c$, hence

$$\mathfrak{m}^e = \mathfrak{n}^{ce} \subseteq \mathfrak{n} \subsetneq (1)$$

$(3 \implies 4)$ For any non-zero $x \in M$, let Ax be the submodule of M generated by x. Since B is flat, the top exact sequence induces the exactness of the bottom sequence

$$0 \longrightarrow Ax \longrightarrow M$$

$$0 \longrightarrow Ax \otimes_A B \longrightarrow M \otimes_A B$$

Since $Ax \otimes_A B \to M \otimes_A B$ is injective, in order to show $M \otimes_A B \neq 0$, it suffices to show that $Ax \otimes_A B \neq 0$. The module Ax generated by one element is isomorphic to $A/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of A and $\mathfrak{a} \neq A$ since Ax is nontrivial. Hence,

$$Ax \otimes_A B \cong \frac{A}{\mathfrak{a}} \otimes_A B \cong \frac{B}{\mathfrak{a}B} = \frac{B}{\mathfrak{a}^e B} = \frac{B}{\mathfrak{a}^e}$$

Since $\mathfrak{a}$ belongs to some maximal ideal $\mathfrak{m}$ in A and $\mathfrak{m}^e \neq (1)$, so $\mathfrak{a}^e \neq B$. Hence, $Ax \otimes_A B \neq 0$
$(4 \implies 5)$ Let $K = \ker(M \to M \otimes_A B)$, since B is a flat A-module, then top exact sequence induces the exactness of the bottom sequence

$$
\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & M \arrow[r] & M \otimes_A B
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & K \otimes_A B \arrow[r] & M \otimes_A B \arrow[r, "v"] & (M \otimes_A B) \otimes_A B
\end{tikzcd}
$$

Note that, from Lemma 8 with $N = M \otimes_A B$, the composition $t : M \otimes_A B \to (M \otimes_A B) \otimes_A B$ is injective due to the natural isomorphism $(x \otimes 1) \otimes b \mapsto (x \otimes b)$, hence the induced map v from $(- \otimes_A B)$ is injective.

$$
\begin{tikzcd}
M \arrow[r] & M \otimes_A B & M \otimes_A B \arrow[r, "v"'] \arrow[rr, bend left, "t"] & (M \otimes_A B) \otimes_A B \arrow[r, "\sim"] & (M \otimes_A B) \otimes_A B \\
x \arrow[r, mapsto] & x \otimes 1 & x \otimes b \arrow[r, mapsto] & (x \otimes 1) \otimes b \arrow[r, mapsto] & (x \otimes b) \otimes 1
\end{tikzcd}
$$

So, by exactness, $K \otimes_A B = 0$. From 4, $K = 0$

$(5 \implies 1)$ We always have $\mathfrak{a}^{ec} \supseteq \mathfrak{a}$ for all ideals $\mathfrak{a}$ in A, we will show the other direction $\mathfrak{a}^{ec} \subseteq \mathfrak{a}$. Let $f : A \to B$, for any $x \in \mathfrak{a}^{ec}$, then $f(x) \in \mathfrak{a}^e \subseteq B$. Let $M = \frac{A}{\mathfrak{a}}$, so the map below is is injective

$$
\begin{aligned}
\frac{A}{\mathfrak{a}} \to \frac{A}{\mathfrak{a}} \otimes_A B \xrightarrow{\sim} \frac{B}{\mathfrak{a}^e} \\
\bar{a} \mapsto \bar{a} \otimes 1 \mapsto \overline{f(a)}
\end{aligned}
$$

Since $f(x) \in \mathfrak{a}^e$, then $\overline{f(x)} = 0$ in $\frac{B}{\mathfrak{a}^e}$, by injectivity, $\bar{x} = 0$ in $\frac{A}{\mathfrak{a}}$, hence $x \in \mathfrak{a}$

□

Problem 9 (chapter 3 problem 18)
Let $f : A \to B$ be a flat homomorphism of rings (B is a flat A-module), let $\mathfrak{q}$ be a prime ideal of B and let $\mathfrak{p} = \mathfrak{q}^c$. Then $f^* : \operatorname{Spec} B_\mathfrak{q} \to \operatorname{Spec} A_\mathfrak{p}$ is surjective

Lemma 10
Let $f : A \to B$ be a ring map and $S \subseteq T$ be two multiplicative subsets of A, then

$$
T^{-1}A \cong \phi_S(T)^{-1}(S^{-1}A) \cong T^{-1}(S^{-1}A)
$$

as A-modules. Note, the result is a consequence of chapter 3 problem 4

Proof. The induced map $A_\mathfrak{p} \to B_\mathfrak{q}$ is

$$
\begin{aligned}
f_\mathfrak{q} : A_\mathfrak{p} \to B_\mathfrak{q} \\
\frac{a}{s} \mapsto \frac{f(a)}{f(s)}
\end{aligned}
$$

Let $S = A - \mathfrak{p}$ and $T = B - \mathfrak{q}$, then $f(S) \subseteq T$, from Lemma 10

$$
B_\mathfrak{q} = T^{-1}B \cong T^{-1}(f(S)^{-1}B) \cong T^{-1}(S^{-1}B) = (B_\mathfrak{p})_\mathfrak{q}
$$

The map is well-defined since $s \in A - \mathfrak{p} \iff f_\mathfrak{q}(s) \in B - \mathfrak{q}$. B is flat as an A-module, since flatness is a local property, $B_\mathfrak{p}$ is flat as an $A_\mathfrak{p}$-module, hence $B_\mathfrak{q}$ is also flat as an $A_\mathfrak{p}$-module because again $B_\mathfrak{q}$ is a localized module of $B_\mathfrak{p}$. Now, we will show that $B_\mathfrak{q}$ is faithfully flat over $A_\mathfrak{p}$. Let $\mathfrak{m} = \mathfrak{p}A_\mathfrak{p}$ be the unique maximal ideal of $A_\mathfrak{p}$, we have a one-to-one correspondence between prime ideals of A and $A_\mathfrak{p}$, of B and $B_\mathfrak{q}$

$$\begin{tikzcd}
A \arrow[rrr] \arrow[ddd] & & & B \arrow[ddd] \\
& \mathfrak{p} \arrow[r, mapsto] \arrow[d, mapsto] & \mathfrak{p}^e \arrow[d, mapsto] & \\
& \mathfrak{m} = \mathfrak{p}A_{\mathfrak{p}} \arrow[r, mapsto, dashed] & m^e = \mathfrak{p}^e B_{\mathfrak{q}} & \\
A_{\mathfrak{p}} \arrow[rrr, dashed] & & & B_{\mathfrak{q}}
\end{tikzcd}$$

Hence,

$$\mathfrak{m}^e = \mathfrak{p}^e B_{\mathfrak{q}} = \mathfrak{q}^{ce} B_{\mathfrak{q}} \subseteq \mathfrak{q}B_{\mathfrak{q}}$$

Since $\mathfrak{q}B_{\mathfrak{q}}$ is a maximal in $B_{\mathfrak{q}}$, $\mathfrak{m}^e \neq (1)$. Then $\operatorname{Spec} B_{\mathfrak{q}} \to \operatorname{Spec} A_{\mathfrak{p}}$ is surjective. □

Problem 11 (chapter 3 problem 19 - support of module)
Let A be a ring and M be an A-module. The support of M is defined to be the set $\operatorname{supp}(M)$ of prime ideal $\mathfrak{p}$ of A such that $M_{\mathfrak{p}} \neq 0$. Prove the following results:

1. $M \neq 0 \iff \operatorname{supp}(M) \neq \varnothing$
2. $V(\mathfrak{a}) = \operatorname{supp}(A/\mathfrak{a})$
3. If $0 \to M^l \to M \to M^r \to 0$ is an exact sequence, then $\operatorname{supp}(M) = \operatorname{supp}(M^l) \cup \operatorname{supp}(M^r)$
4. If $M = \sum_{i \in I} M_i$, then $\operatorname{supp}(M) = \bigcup_{i \in I} \operatorname{supp}(M_i)$
5. If M is finitely generated, then $\operatorname{supp}(M) = V(\operatorname{ann}_A(M))$ (and is therefore a closed subset of $\operatorname{Spec} A$)
6. If M, N are finitely generated, then $\operatorname{supp}(M \otimes_A N) = \operatorname{supp}(M) \cap \operatorname{supp}(N)$
7. If M is finitely generated and $\mathfrak{a}$ is an ideal of A, then $\operatorname{supp}(M/\mathfrak{a}M) = V(\mathfrak{a} + \operatorname{ann}_A(M))$
8. If $f : A \to B$ is a ring homomorphism and M is a finitely generated A-module, then $\operatorname{supp}(B \otimes_A M) = (f^*)^{-1}(\operatorname{supp}(M))$ where $f^* : \operatorname{Spec} B \to \operatorname{Spec} A$ is the induced map from f

Note, $V(\mathfrak{a})$ is the set of all prime ideals in A containing $\mathfrak{a}$

Lemma 12 (chapter 3 proposition 3.7)
Let M and N be A-modules and S be a multiplicatively closed subset of A, then there is an isomorphism

$$S^{-1}M \otimes_{S^{-1}A} S^{-1}N \xrightarrow{\sim} S^{-1}(M \otimes_A N)$$

In particular, if $S = A - \mathfrak{p}$ for some prime ideal $\mathfrak{p}$, then

$$M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \xrightarrow{\sim} (M \otimes_A N)_{\mathfrak{p}}$$

Lemma 13 (chapter 2 problem 3 extended)
Let A be a local ring with unique maximal ideal $\mathfrak{a}$, M and N be A-modules with $M_{\mathfrak{a}} \neq 0$ and N finitely generated. Prove that

$$M \otimes_A N \implies N = 0$$

Note, the proof is exactly in chapter 2 problem 3, except at the last step we only use Nakayama lemma version 1 for N and given $M_{\mathfrak{a}} \neq 0$, then N is zero.

Proof.
(1)

$$M = 0 \iff M_{\mathfrak{p}} = 0 \text{ for all prime ideal } \mathfrak{p} \subseteq A \iff \operatorname{supp}(M) = \varnothing$$

(2) For any prime ideal $\mathfrak{p}$ in A, the following are equivalent

(a) $(A/\mathfrak{a})_{\mathfrak{p}} \neq 0$

(b) $\exists x \in A, \forall t \in A - \mathfrak{p}, tx \notin \mathfrak{a}$

(c) $\mathfrak{a} \subseteq \mathfrak{p}$

$(b \implies c)$ suppose $(b \land \neg c)$, that is there exists $a \in \mathfrak{a} - \mathfrak{p} \subseteq A - \mathfrak{p}$, then $ax \in \mathfrak{a}$, that is a contradiction
$(b \impliedby c)$ suppose $(\neg b \land c)$, note that $(\neg b)$ is $\forall x \in A, \exists t \in A - \mathfrak{p}, tx \in \mathfrak{a}$. Since $\mathfrak{p}$ is prime, that is not the whole ring, choose $x \in A - \mathfrak{p}$, then there exists $t \in A - \mathfrak{p}$, but $tx \in \mathfrak{a} \subseteq \mathfrak{p}$, that is a contraction
Hence, $p \in \operatorname{supp}(A/\mathfrak{a}) \iff \mathfrak{p} \in V(\mathfrak{a})$, that is $\operatorname{supp}(A/\mathfrak{a}) = V(\mathfrak{a})$
(3) For any prime ideal $\mathfrak{p}$ in A, the functor $((A - \mathfrak{p})^{-1} -)$ is exact, hence both sequences are exact

$$0 \longrightarrow M^l \longrightarrow M \longrightarrow M^r \longrightarrow 0$$

$$0 \longrightarrow M^l_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow M^r_{\mathfrak{p}} \longrightarrow 0$$

Then

$$\mathfrak{p} \in \operatorname{supp}(M)^c \iff M_{\mathfrak{p}} = 0 \iff M^l_{\mathfrak{p}} = 0 \text{ and } M^r_{\mathfrak{p}} = 0 \iff \mathfrak{p} \in \operatorname{supp}(M^l)^c \cap \operatorname{supp}(M^r)^c$$

Hence, $\operatorname{supp}(M)^c = \operatorname{supp}(M^l)^c \cap \operatorname{supp}(M^r)^c$, that is equivalent to $\operatorname{supp}(M) = \operatorname{supp}(M^l) \cup \operatorname{supp}(M^r)$
(4) For any prime ideal $\mathfrak{p}$ in A, the functor $((A - \mathfrak{p})^{-1} -)$ is exact, hence both sequences are exact

$$0 \longrightarrow M_i \longrightarrow M$$

$$0 \longrightarrow (M_i)_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}}$$

Then

$$\mathfrak{p} \in \operatorname{supp}(M)^c \iff M_{\mathfrak{p}} = 0 \implies (M_i)_{\mathfrak{p}} = 0 \iff \mathfrak{p} \in \operatorname{supp}(M_i)^c$$

Hence, $\operatorname{supp}(M)^c \subseteq \bigcap_{i \in I} \operatorname{supp}(M_i)^c$, that is equivalent to $\operatorname{supp}(M) \supseteq \bigcup_{i \in I} \operatorname{supp}(M_i)$. To see the other direction, let $\mathfrak{p} \in \operatorname{supp}(M)$ but $(M_i)_{\mathfrak{p}} = 0$ for all $i \in I$. Let

$$x = \sum_{j \in J} x_j \in M$$

for some finite subset $J \subseteq I$ so that $\frac{x}{s} \neq 0$ in $M_{\mathfrak{p}}$ for some $s \in A - \mathfrak{p}$. Since $(M_j)_{\mathfrak{p}} = 0$, $\frac{x_j}{1} = 0$ in $(M_j)_{\mathfrak{p}}$, so there exists $t_j \in A - \mathfrak{p}$ so that $t_j x_j = 0$. Hence, let $t = \prod_{j \in J} t_j$, then $tx = 0$, so $\frac{x}{s} = 0$, that a contradiction

(5) Let $x_1, x_2, ..., x_n$ generates M, the each Ax_i is a submodule of M that is isomorphic to $A/\mathfrak{a}_i$ for some ideal $\mathfrak{a}_i$ in A. We will show that $\bigcap_{i=1}^n \mathfrak{a}_i = \mathrm{ann}_A(M)$. If $a \in A$ so that $aM = 0$, then $a(Ax_i) = 0$ for all i, hence $x \in \mathfrak{a}_i$ for all i. On the other hand, if $a \in \mathfrak{a}_i$, then a acts on any element of Ax_i resulting zero. Hence, $aM = 0$. We have

$$\mathrm{supp}(M) = \mathrm{supp}(\sum_{i=1}^n Ax_i) = \bigcup_{i=1}^n \mathrm{supp}(Ax_i) = \bigcup_{i=1}^n \mathrm{supp}(A/\mathfrak{a}_i) = \bigcup_{i=1}^n V(\mathfrak{a}_i) = V\left(\bigcap_{i=1}^n \mathfrak{a}_i\right) = V(\mathrm{ann}_A(M))$$

(6) For any prime ideal $\mathfrak{p}$ in A, since $A_\mathfrak{p}$ is a local ring, from chapter 2 problem 3, we have

$$\mathfrak{p} \in \mathrm{supp}(M \otimes_A N)^c \iff M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p} = (M \otimes N)_\mathfrak{p} = 0 \iff M_\mathfrak{p} = 0 \text{ or } N_\mathfrak{p} = 0 \iff \mathfrak{p} \in \mathrm{supp}(M)^c \cup \mathrm{supp}(N)^c$$

Hence, $\mathrm{supp}(M \otimes_A N) = \mathrm{supp}(M) \cap \mathrm{supp}(N)$

(7)

$$\mathrm{supp}(M/\mathfrak{a}M) = \mathrm{supp}(A/\mathfrak{a} \otimes_A M) = \mathrm{supp}(A/\mathfrak{a}) \cap \mathrm{supp}(M) = V(\mathfrak{a}) \cap V(\mathrm{ann}_A(M)) = V(\mathfrak{a} \cup \mathrm{ann}_A(M)) = V(\mathfrak{a} + \mathrm{ann}_A(M))$$

where the last equality is due to $\mathfrak{a} + \mathrm{ann}_A(M)$ being the smallest ideal containing $\mathfrak{a} \cup \mathrm{ann}_A(M)$

(8) The induced map $f^* : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is defined by

$$\begin{aligned} f^* : \mathrm{Spec}\, B &\to \mathrm{Spec}\, A \\ \mathfrak{q} &\mapsto \mathfrak{q}^c \end{aligned}$$

Let $\mathfrak{q} \in \mathrm{Spec}\, B$, and $\mathfrak{p} = f^*(\mathfrak{q}) = \mathfrak{q}^c$, then we have

$$\begin{aligned} (B \otimes_A M)_\mathfrak{q} &\cong B_\mathfrak{q} \otimes_B (B \otimes_A M) \\ &\cong (B_\mathfrak{q} \otimes_B B) \otimes_A M \\ &\cong B_\mathfrak{q} \otimes_A M \\ &\cong (B_\mathfrak{q} \otimes_{A_\mathfrak{p}} A_\mathfrak{p}) \otimes_A M && (B_\mathfrak{q} \text{ is an } A_\mathfrak{p}\text{-module, Lemma 10}) \\ &\cong B_\mathfrak{q} \otimes_{A_\mathfrak{p}} (A_\mathfrak{p} \otimes_A M) \\ &\cong B_\mathfrak{q} \otimes_{A_\mathfrak{p}} M_\mathfrak{p} \end{aligned}$$

The proof logic is as follows:

$$\begin{tikzcd} \mathfrak{q} \in (f^*)^{-1}(\mathrm{supp}(M))^c \arrow[r, leftrightarrow] & \mathfrak{p} \in \mathrm{supp}(M)^c \arrow[r, leftrightarrow] & M_\mathfrak{p} = 0 \arrow[d, bend right] \\ \mathfrak{q} \in \mathrm{supp}_B(B \otimes_A M)^c \arrow[r, leftrightarrow] & (B \otimes_A M)_\mathfrak{q} = 0 \arrow[r, leftrightarrow] & B_\mathfrak{q} \otimes_{A_\mathfrak{p}} M_\mathfrak{p} = 0 \arrow[u, bend right] \end{tikzcd}$$

where $B_\mathfrak{q} \otimes_{A_\mathfrak{p}} M_\mathfrak{p} = 0 \implies M_\mathfrak{p} = 0$ because $(B_\mathfrak{q})_\mathfrak{p} = B_\mathfrak{q} \neq 0$. $B_\mathfrak{q} \neq 0$ because $1/1 \neq 0$ in $B_\mathfrak{q}$ and $(B_\mathfrak{q})_\mathfrak{p} = B_\mathfrak{q}$ because $B - \mathfrak{q} \supseteq B - \mathfrak{p}^e$, localizing larger subset first then localizing smaller subset is equivalent to localizing only larger subset. □

Proof of Lemma 12.

$$\begin{aligned}
S^{-1}(M \otimes_A N) &\cong S^{-1}A \otimes_A (M \otimes_A N)\\
&\cong (S^{-1}A \otimes_A M) \otimes_A N\\
&\cong S^{-1}M \otimes_A N\\
&\cong (S^{-1}A \otimes_{S^{-1}A} S^{-1}M) \otimes_A N\\
&\cong (S^{-1}M \otimes_{S^{-1}A} S^{-1}A) \otimes_A N\\
&\cong S^{-1}M \otimes_{S^{-1}A} (S^{-1}A \otimes_A N)\\
&\cong S^{-1}M \otimes_{S^{-1}A} (S^{-1}A \otimes_A N)\\
&\cong S^{-1}M \otimes_{S^{-1}A} S^{-1}N
\end{aligned}$$

□

Problem 14 (chapter 5 problem 1)
Let $f : A \to B$ be an integral ring extension. Show that $f^* : \operatorname{Spec} B \to \operatorname{Spec} A$ is a closed mapping, that is, it maps closed sets into closed sets

Proof. Let $\mathfrak{b} \subseteq B$ be any ideal, then $V(\mathfrak{b})$ is a closed set in $\operatorname{Spec} B$ and

$$f^*V(\mathfrak{b}) = \{\mathfrak{q} \cap A : \mathfrak{q} \in V(\mathfrak{b})\}$$

is its image in $\operatorname{Spec} A$. We have

$$\mathfrak{p} \in f^*V(\mathfrak{b}) \iff \exists \mathfrak{q} \in V(\mathfrak{b}), \mathfrak{q} \cap A = \mathfrak{p} \implies \mathfrak{p} \in V(\mathfrak{b} \cap A)$$

That is, $f^*V(\mathfrak{b}) \subseteq V(\mathfrak{b} \cap A)$. On the other hand, for any $\mathfrak{p} \in V(\mathfrak{b} \cap A)$, the inclusion

$$\frac{A}{\mathfrak{b} \cap A} \hookrightarrow \frac{B}{\mathfrak{b}}$$

is an integral ring extension, $\bar{\mathfrak{p}}$ is a prime ideal in $\frac{A}{\mathfrak{b} \cap A}$, hence there exists a prime ideal $\mathfrak{q} \in V(\mathfrak{b})$ so that $\bar{\mathfrak{q}} \cap \frac{A}{\mathfrak{b} \cap A} = \bar{\mathfrak{p}}$. We have

$$\begin{aligned}
\bar{\mathfrak{q}} &= \{y + \mathfrak{b} : y \in \mathfrak{q}\}\\
\bar{\mathfrak{p}} = \bar{\mathfrak{q}} \cap \frac{A}{\mathfrak{b} \cap A} &= \{x + \mathfrak{b} \cap A : x \in A, f(x) = y\}
\end{aligned}$$

Since $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ are prime ideals of a quotient map $\mathfrak{p} = \{x \in A, f(x) = y\} = \mathfrak{q} \cap A$. Hence, $f^*V(\mathfrak{b}) = V(\mathfrak{b} \cap A)$, f^* is a closed map □

Problem 15 (chapter 5 problem 3)
Let $f : B \to B'$ be a A-algebra morphism and C be an A-algebra. If f is integral, show that $f \otimes 1 : B \otimes_A C \to B' \otimes_A C$ is integral

Proof. Since the integral closure of $B \otimes_A C$ in $B' \otimes_A C$ is a subring of $B' \otimes_A C$, it suffices to show that all every basic tensor $b' \otimes c$ is integral over $B \otimes_A C$. $f : B \to B'$ is integral, hence any $b' \in B$ satisfies a monic polynomial in B

$$(b')^n + f(b_1)(b')^{n-1} + ... + f(b_n) = 0$$

for some $b_1, ..., b_n \in B$. Note that, $(b' \otimes c)^k = (b')^n \otimes c^k$. Let c^n act on the monic polynomial, we have

$$(b' \otimes c)^n + (f(b_1) \otimes c)(b' \otimes c)^{n-1} + ... + (f(b_n) \otimes c^n) = 0$$

The coefficients $f(b_k) \otimes c^k = (f \otimes 1)(b_k \otimes c^k) \in \text{im}(f \otimes 1)$. Hence, $b' \otimes c$ is integral over $B \otimes_A C$ $\square$

Problem 16 (chapter 5 problem 5)
Let $A \hookrightarrow B$ be an integral ring extension

1. If $x \in A$ is a unit in B then it is a unit in A
2. The Jacobson radical of A is the contraction of the Jacobson radical of B

Proof.
(1)
Suppose x is not a unit in A, let $\mathfrak{m}_A$ be the maximal ideal containing x, then there exists a prime ideal $\mathfrak{m}_B$ in B so that $\mathfrak{m}_B \cap A = \mathfrak{m}_A$. Since $\mathfrak{m}_A$ is maximal, then $\mathfrak{m}_B$ is also maximal, but $x \in \mathfrak{m}_A \subseteq \mathfrak{m}_B$, then x is not a unit in B, contradiction.
(2)
Using the previous argument, any maximal ideal in A is the contraction of another maximal ideal in B, hence

$$J(A) \subseteq J(B) \cap A$$

Moreover, contraction of any maximal ideal in B is maximal in A, then

$$J(B) \cap A \subseteq J(A)$$

$\square$

Problem 17 (chapter 5 problem 12)
Let G be a finite group of automorphisms of a ring A and let A^G denote the subring of G-invariants, that is

$$A^G = \{x \in A : \sigma(x) = x \text{ for all } \sigma \in G\}$$

Prove that A is integral over A^G. Let S be a multiplicative closed subset of A such that $\sigma(S) \subseteq S$ for all $\sigma \in G$, let $S^G = S \cap A^G$. Show that the action of G on A extends to an action on $S^{-1}A$ and that $(S^G)^{-1}A^G \cong (S^{-1}A)^G$

Proof. (A is integral over A^G)
For any $x \in A$, since $1_A \in G \subseteq \text{Hom}(A, A)$ and $n = |G|$ is finite, x is a root of the polynomial

$$f(t) = \prod_{\sigma \in G} (t - \sigma(x)) \in A[t]$$

We will show that $f(t) \in A^G[t]$, that is

$$f(t) = a_0 + a_1 t^1 + ... + a_n t^n$$

with $a_0, ..., a_n \in A^G$. For any $\tau \in G \subseteq \mathrm{Hom}(A, A)$, it induces a $\tau \in \mathrm{Hom}(A[t], A[t])$, then

$$\tau(f(t)) = \tau\left(\prod_{\sigma \in G}(t - \sigma(x))\right) = \prod_{\sigma \in G}(t - (\tau\sigma)(x))$$

Since $\{\tau\sigma : \sigma \in G\} = G$, then $\tau(f(t)) = f(t)$, hence

$$\tau(a_k) = a_k$$

Hence, $a_k \in A^G$, thus $f(t) \in A^G[t]$

(the action of G on A extends to an action on $S^{-1}A$)

The action of G on A extends to an action of $S^1 A$ as follows:

$$G \times S^{-1}A \to S^{-1}A$$
$$\left(\sigma, \frac{a}{s}\right) \mapsto \frac{\sigma(a)}{\sigma(s)}$$

This is a well-defined group action since

$$1_A \frac{a}{s} = \frac{a}{s}$$
$$(\sigma\tau)\frac{a}{s} = \frac{\sigma\tau(a)}{\sigma\tau(s)} = \sigma\left(\tau\frac{a}{s}\right)$$

Moreover, it respects addition and multiplication on $S^{-1}A$, that is

$$\sigma\left(\frac{a_1}{s_1}\right) + \sigma\left(\frac{a_2}{s_2}\right) = \frac{\sigma a_1}{\sigma s_1} + \frac{\sigma a_2}{\sigma s_2} = \frac{\sigma(s_2 a_1 + s_1 a_2)}{\sigma(s_1 s_2)} = \sigma\left(\frac{a_1}{s_1} + \frac{a_2}{s_2}\right)$$
$$\sigma\left(\frac{a_1}{s_1}\right)\sigma\left(\frac{a_2}{s_2}\right) = \frac{\sigma a_1}{\sigma s_1}\frac{\sigma a_2}{\sigma s_2} = \frac{\sigma(a_1 a_2)}{\sigma(s_1 s_2)} = \sigma\left(\frac{a_1}{s_1}\frac{a_2}{s_2}\right)$$

($(S^G)^{-1}A^G \cong (S^{-1}A)^G$)

Note that

$$(S^{-1}A)^G = \left\{\frac{a}{s} \in S^{-1}A : \sigma\left(\frac{a}{s}\right) = \frac{a}{s} \text{ for all } \sigma \in G\right\}$$

We define a pair of isomorphism as follows:

$$(S^G)^{-1}A^G \to (S^{-1}A)^G$$
$$f : \frac{a}{s} \mapsto \frac{a}{s}$$
$$g : \frac{\mu(a)}{\mu(s)} \leftarrow\!\shortmid \frac{a}{s}$$

where $\mu(x) = \sum_{\sigma \in G} \sigma(x)$ for any $x \in A$

f is well-defined since if $a \in A^G$ and $s \in S^G$, then immediately $\sigma\left(\frac{a}{s}\right) = \frac{a}{s}$, hence $f\left(\frac{a}{s}\right) \in (S^{-1}A)^G \subseteq S^{-1}A$.

g is well-defined because $\mu(a) \in A^G$ for any $a \in A$ and $\mu(s) \in S^G$ for any $s \in S$. For any $\tau \in G$, since $\{\tau\sigma : \sigma \in G\} = G$

$$\tau\mu(x) = \tau\left(\sum_{\sigma \in G}\sigma(x)\right) = \sum_{\sigma \in G}\tau\sigma(x) = \sum_{\sigma \in G}\sigma(x) = \mu(x)$$

It is clear that $gf = 1_{(S^G)^{-1}A^G}$. On the other hand, For any $\frac{a}{s} \in (S^{-1}A)^G$, $fg\left(\frac{a}{s}\right) = \frac{\mu(a)}{\mu(s)}$. For each $\sigma \in G$,

$\frac{a}{s} \in (S^{-1}A)^G$ implies that there exists $t_\sigma \in S$ so that

$$t_\sigma s\sigma(a) = t_\sigma a\sigma(s)$$

Let $t = \prod_{\sigma \in G} t_\sigma \in S$, since t_σ is one of the factor of the product t, then $ts\sigma(a) = ta\sigma(s)$ for all $\sigma \in G$, summing over all $\sigma \in G$ gives $ts\mu(a) = ta\mu(s)$. Now, multiplying both sides by $\left(\prod_{\tau \in G - \{1_A\}} \tau(t)\right)$ gives

$$\nu(t)a\mu(s) = \nu(t)s\mu(a)$$

for $\nu(t) = \prod_{\tau \in G} \tau(t)$ using the same argument as above $\prod_{\tau \in G} \tau(t) \in A^G$, moreover since $\tau(S) \subseteq S$, then $\prod_{\tau \in G} \tau(t) \in A^G \cap S = S^G$. Theforefore, $\frac{\mu(a)}{\mu(s)} = \frac{a}{s}$, that is $fg = 1_{(S^{-1}A)^G}$ ☐

Problem 18 (chapter 5 problem 13)
Let $\mathfrak{p}$ be a prime ideal of A^G and let P be the set of prime ideals of A whose contraction is $\mathfrak{p}$. Show that G act transitively on P. In particular, P is finite.

Proof.
(on a fiber of $\operatorname{Spec} A \to \operatorname{Spec} A^G$, G maps sheets into sheets)
For any $\mathfrak{q} \in P$, then $\mathfrak{q} \cap A^G = \mathfrak{p}$, for any $\sigma \in G$, since σ is an isomorphism in $\operatorname{Hom}(A, A)$, then $\sigma(\mathfrak{q} \cap A^G) = \sigma(\mathfrak{q}) \cap \sigma(A^G)$, hence

$$\mathfrak{p} = \sigma(\mathfrak{p}) = \sigma(\mathfrak{q} \cap A^G) = \sigma(\mathfrak{q}) \cap \sigma(A^G) = \sigma(\mathfrak{q}) \cap A^G$$

Thus, $\sigma(\mathfrak{q})$ is another prime ideal whose contraction is $\mathfrak{p}$
(on a fiber of $\operatorname{Spec} A \to \operatorname{Spec} A^G$, given any two sheets, there is a $\sigma \in G$ maps from one to another)
Let $\mathfrak{q}_1$ and $\mathfrak{q}_2$ be prime ideals in A so that $\mathfrak{p} = \mathfrak{q}_1 \cap A^G = \mathfrak{q}_2 \cap A^G$ but $\mathfrak{q}_2$ is not on the G-orbit of $\mathfrak{q}_1$, that is there exists $x \in \mathfrak{q}_2$ so that $x \notin \sigma(\mathfrak{q}_1)$ for any $\sigma \in G$. From previous part, we have $\nu(x) = \prod_{\sigma \in G} \sigma(x) \in A^G$, moreover x is one of the factor of the product $\nu(x)$, hence

$$\nu(x) \in \mathfrak{q}_2 \cap A^G = \mathfrak{p} \subseteq \mathfrak{q}_1$$

Hence, there at least one $\sigma \in G$ so that $\sigma(x) \in \mathfrak{q}_1$. Thus, $\sigma^{-1} \in G$ map $\sigma(x) \in \mathfrak{q}_1$ into x which is a contradiction.
(P is finite)
In particular, since G is finite, P is also finite.

☐

Problem 19 (chapter 7 problem 4)
Which is the following rings Noetherian?

1. The ring of rational functions of z having no pole on the circle $|z| = 1$
2. The ring of power series in z with a positive radius of convergence
3. The ring of power series in z with an infinite radius of convergence
4. The ring of polynomials in z whose first k derivatives vanish at the origin (k being a fixed integer)
5. The ring of polynomial in z, w all of whose partial derivatives with respect to w vanish for $z = 0$

   In all cases the coefficients are complex numbers.

Proof.
(1) $\mathbb{C}$ is Noetherian, so $\mathbb{C}[z]$ is Noetherian. Define the multiplicative closed set $S \subseteq \mathbb{C}[x]$

$$S = \{q(z) \in \mathbb{C}[z] : q(x) \neq 0 \text{ for all } x \text{ on the circle } |x| = 1\}$$

Then the rational functions of z having no pole on the circle $|z| = 1$ is precisely $S^{-1}\mathbb{C}[x]$, hence Notherian
(2) The ring of power series in z with positive radius of convergence is

$$A = \left\{ \sum_{n=0}^{\infty} a_n z^n : R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}} > 0 \right\}$$

Note that we can write any $f(z) \in \mathbb{C}[[z]]$ as

$$f(z) = z^{\operatorname{ord} f(z)} g(z)$$

If $f(z) \in A$, so is $g(z)$. Hence, $f(z) \in (z^{\operatorname{ord} f(z)})$. Therefore, any ideal I in A is generated by z^n for

$$n = \min_{f(x)\in I} \operatorname{ord} f(x)$$

(3) The ring of power series in z with infinite radius of convergence is the ring of holomorphic function $\mathcal{O}$, let

$$I_n = \{f(x) \in \mathcal{O} : 0 = f(n) = f(n+1) = f(n+2) = ...\}$$

Then the chain of ideals $I_1 \subseteq I_2 \subseteq ...$ is strictly increasing. Hence, $\mathcal{O}$ is not Noetherian
(4) The ring of polynomials in z whose first k derivatives vanish at the origin (k being a fixed integer) is

$$B = \mathbb{C} + z^{k+1}\mathbb{C}[z] = \{a_0 + a_1 z + a_2 z^2 + ... + a_k z^k + ... + a_n z^n \in \mathbb{C}[z] : a_1 = a_2 = ... = a_k = 0\}$$

$\mathbb{C}[z^{k+1}]$ is Noetherian and a subring of M and M is a $\mathbb{C}[z^{k+1}]$-module generated by $1, z, z^2, ..., z^k$. Then, M is Noetherian
(5) Let

$$I_n = (z, zw, zw^2, ..., zw^n) \subseteq \mathbb{C}[z, w]$$

Then $I_1 \subseteq I_2 \subseteq ...$ is strictly increasing ($zw^{n+1} \in I_{n+1} - I_n$). Hence The ring of polynomial in z, w all of whose partial derivatives with respect to w vanish for $z = 0$ is not Noetherian ☐

Problem 20 (chapter 7 problem 5)
Let A be a Noetherian ring and B a finitely generated A-algebra, G is a finite group of A-automorphisms of B and B^G be the set of all elements of B which are left fixed by element element of G. Show that B^G is a finitely generated A-algebra.

Lemma 21 (chapter 7 proposition 7.8)
Let $A \subseteq B \subseteq C$ be rings. Suppose A is Noetherian, that C is finitely generated as an A-algebra and C is integral over B, then B is finitely generated as an A-algebra

Proof. We know that $B^G \to B$ is an integral ring extension, B being Noether follows from Lemma 21 and the chain

$$A \subseteq B^G \subseteq B$$

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Problem 22 (chapter 7 problem 8)
If $A[x]$ is Noetherian, is A necessarily Noetherian?

Proof. Since $A \cong A[x]/(x)$, then quotient ring A of a Noetherian ring $A[x]$ is Noetherian □

Problem 23 (chapter 7 problem 12)
Let A be a ring and B be a faithfully flat A-algebra. If B is Noetherian, show that A is also Noetherian

Proof. B is a failthfully flat A-algebra, hence under the map $A \to B$, for any ideal $\mathfrak{a} \subseteq A$, $\mathfrak{a}^{ec} = \mathfrak{a}$. Let

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq ...$$

be a chain of ideals in A. Then

$$\mathfrak{a}_1^e \subseteq \mathfrak{a}_2^e \subseteq ...$$

is a chain of ideals in B that must stablize at some point. Contracting back to A gives

$$\mathfrak{a}_1^{ec} \subseteq \mathfrak{a}_2^{ec} \subseteq ...$$

must stablize at some point. Hence, A is Noetherian □

Problem 24
If A is any Noetherian ring, then also the power series ring $A[[x]]$ is Noetherian

Proof. For any power series $f(x) = a_0 + a_1 x^1 + a^2 x^2 + ... + a^n x^n + ... \in A[[x]]$, define

$$\operatorname{ord} f(x) = \min\{n \geq 0 : a_n \neq 0\}$$

Suppose $A[[x]]$ is not Noetherian, let I be an ideal in $A[[x]]$ that is not finitely generated, we will inductively construct $f_0(x), f_1(x), f_2(x), ... \in A[x]$ and ideals $I_n = (f_0(x), f_1(x), ..., f_n(x))$ as follows:
Pick a nonzero $f_0(x) \in I$ of minimal order, set

$$I_0 = (f_0(x))$$

If we already pick $f_0(x), f_1(x), ..., f_{n-1}(x)$, the we pick a nonzero $f_n(x) \in I - I_{n-1}$ of minimal order and set

$$I_n = (f_0(x), f_1(x), ..., f_n(x))$$

By construction, we have

$$\operatorname{ord} f_0(x) \leq \operatorname{ord} f_1(x) \leq \operatorname{ord} f_2(x) \leq ...$$

Let a_n be the first nonzero coefficient of $f_n(x)$ and let $J \subseteq A$ be the ideal defined by

$$J = (a_0, a_1, a_2, ...)$$

Since A is Noetherian, J is finitely generated, that is

$$J = (a_0, a_1, a_2, ..., a_N)$$

for some $N \geq 0$. Let $f(x) \in I_n - I_N$ with $N < n$, by minimality of order of $f_i(x)$, we must have $\operatorname{ord} f(x) - \max\{\operatorname{ord} f_i(x)\} \geq 0$.

We will write $f(x)$ as a $A[[x]]$-linear combination of $f_1(x), f_2(x), ..., f_N(x)$, let $a \in A$ be the first nonzero coefficient of $f(x)$, then $a = \sum_{i=0}^N r_i a_i$ for some $r_0, r_1, ..., r_N \in A$. We can write

$$f(x) = f^{(1)}(x) - \sum_{i=0}^N r_i x^{\operatorname{ord} f(x) - \operatorname{ord} f_i(x)} f_i(x) = f^{(1)}(x) - \sum_{i=0}^N h_i^{(1)}(x) f_i(x)$$

for some $f^{(1)}(x) \in A[[x]]$ with $\operatorname{ord} f^{(1)}(x) \geq \operatorname{ord} f(x) + 1$ and $\operatorname{ord} h_i(x) \geq \operatorname{ord} f(x) - \max\{\operatorname{ord} f_i(x)\} \geq 0$.
Continue this process, we can write

$$f(x) = f^{(k)}(x) - \left(\sum_{i=0}^N h_i^{(k)}(x) f_i(x) + \sum_{i=0}^N h_i^{(k-1)}(x) f_i(x) + ... + \sum_{i=0}^N h_i^{(1)}(x) f_i(x) \right)$$

because at each step, order of $f^{(k)}(x)$ increases by at least 1, so $\operatorname{ord} f^{(k)} \geq \operatorname{ord} f(x) + k$, and

$$\operatorname{ord} h_i^{(k)}(x) \geq \operatorname{ord} f^{(k)}(x) - \max\{\operatorname{ord} f_i(x)\} \geq k + \operatorname{ord} f(x) - \max\{\operatorname{ord} f_i(x)\} \geq k$$

Then we can write

$$f(x) = \sum_{i=1}^N h_i(x) f_i(x)$$

for some $h_i(x) \in A[[x]]$. Because $\operatorname{ord} h_i^{(k)}(x) \geq k$ and $\operatorname{ord} f^{(k)} \geq k$, the process gives a construction of all coeffcients of degree $< k$ of $h_i(x)$ for any k. Hence, $I \subseteq A[[x]]$ is finitely generated.
Alternate proof: show $\mathbb{C}[[x]]$ is PID □

Problem 25

If A is any ring and $\mathfrak{p} \subset A$ is any prime ideal, then $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \cong \operatorname{Frac}(A/\mathfrak{p})$

Proof. Since localization is exact, let $S = A - \mathfrak{p}$, we have two short exact sequences

$$0 \longrightarrow \mathfrak{p} \longrightarrow A \longrightarrow A/\mathfrak{p} \longrightarrow 0$$

$$0 \longrightarrow S^{-1}\mathfrak{p} \longrightarrow A_\mathfrak{p} \longrightarrow S^{-1}(A/\mathfrak{p}) \longrightarrow 0$$

Note that $S^{-1}\mathfrak{p} = \left\{ \frac{p}{s} : p \in \mathfrak{p}, s \in S \right\} = \mathfrak{p}A_\mathfrak{p} \subseteq A_\mathfrak{p}$ is the maximal ideal of $A_\mathfrak{p}$. The surjection $A \to A/\mathfrak{p}$ sends S into $A/\mathfrak{p} - \{0\}$, there is an isomorphism of $S^{-1}A$-modules

$$S^{-1}(A/\mathfrak{p}) \cong \operatorname{Frac}(A/\mathfrak{p})$$

Hence, by exactness of the sequence sequence

$$\frac{A_\mathfrak{p}}{\mathfrak{p}A_\mathfrak{p}} \cong \operatorname{Frac}(A/\mathfrak{p})$$

as S^{-1}-modules □