Complex Analysis

mot2ba4nam6

Real Numbers:

An ordered field $\mathbb{R}$ satisfies least upper bound property

every non-empty set with upper bound has a least upper bound

Complex Numbers

$\mathbb{C} = \mathbb{R} \times \mathbb{R}$ with additional structure

- Complex Addition: $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$
- Complex Multiplication: $(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1)$

$(\mathbb{C}, +, \cdot)$ forms the field of complex numbers

Theorem: $R = \{(a,0) : a \in \mathbb{R}\} \subset \mathbb{C}$

R with complex addition and complex multiplication is isomorphic to $\mathbb{R}$

$(R, +, \cdot) \cong (\mathbb{R}, +, \cdot)$

We write $(a, 0)$ as a

Standard form of Complex Number

Any complex number can be written as $a + bi$ for $a, b \in R$ and $i = (0,1) \in \mathbb{C}$

$$a + bi = (a,0) + (b,0)(0,1) = (a,b)$$

Polar form of Complex Number

Polar form for a complex number $a + bi$, polar form

$$(r, \theta) \in [0, \infty) \times \mathbb{R}$$

where $r = \sqrt{a^2 + b^2}$

$\tan\theta = \frac{b}{a}$

Two polar forms are equivalent if they represent the same complex number

Common Complex functions

Modulus $|a + bi| = \sqrt{a^2 + b^2}$ Real part $\mathrm{Re}(a + bi) = a$

Argument $\arg(a + bi) = \arctan\frac{b}{a}$ Imaginary part $\mathrm{Im}(a + bi) = b$

Conjugate $\overline{a + bi} = a - bi$ // $z\bar{z} = |z|^2$

Additive inverse $-(a + bi) = -a - bi$

Multiplicative inverse $z^{-1} = \frac{\bar{z}}{|z|^2}$

Exponential $e^{a + bi} = e^a(\cos b + i \sin b)$ // coincide with polar form $re^{i\theta} = (r, \theta)$

Log $\log : \mathbb{C} \to \mathcal{P}(\mathbb{C})$

$\log(z) = \{x : x \in \mathbb{C} \wedge e^x = z\}$

$\log(re^{i\theta}) = \{\log r + i(\theta + 2\pi k) : k \in \mathbb{Z}\}$

principle branch: choose k such that $\theta + 2\pi k \in (-\pi, +\pi)$

Power $z^a = e^{a \log z}$

Properties of Exponential and Power

$z^{a+b} = z^a z^b$

$z^{-a} = (z^a)^{-1}$

$z^{ab} = (z^a)^b$

Sequence of Complex Numbers

Given $\{c_n\}_{n=1}^{\infty} \subseteq \mathbb{C}$ converges to c

$$\lim_{n \to \infty} c_n = c$$

if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t

$n \geq N \Rightarrow |c - c_n| < \varepsilon$

From Real Analysis

- Bounded Sequence: $\{a_n\}_{n=1}^{\infty} \in \mathbb{C}$ is bounded if $\exists M \in \mathbb{R}, \forall n \in \mathbb{N}, |a_n| < M$
- ~~Monotone Sequence~~: not exist in $\mathbb{C}$
- Cauchy Sequence: $\{a_n\}_{n=1}^{\infty} \in \mathbb{C}$ is Cauchy if $\forall \varepsilon > 0, \exists N > \mathbb{N}$ s.t $m > n \geq N \Rightarrow |a_m - a_n| < \varepsilon$

Theorem:

- Convergent Sequence is bounded
- A sequence converges $\Leftrightarrow$ it is Cauchy

A sequence of Complex Number converges if and only if the sequences of real parts and imaginary parts converge

Limit of Complex function

Given $f : \mathbb{C} \to \mathbb{C}$

$$\lim_{z \to z_0} f = L$$

if $\forall \varepsilon > 0, \exists \delta > 0$ s.t

$|z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon$

f is continuous at z_0 if $\lim_{z \to z_0} f = f(z_0)$

Re, Im, $z \mapsto |z|$ are continuous everywhere

$$\lim_{z \to z_0}(f + g) = \lim_{z \to z_0} f + \lim_{z \to z_0} g$$

$$\lim_{z \to z_0} fg = \left(\lim_{z \to z_0} f\right)\left(\lim_{z \to z_0} g\right)$$

$$\lim_{z \to z_0} \frac{f}{g} = \frac{\lim_{z \to z_0} f}{\lim_{z \to z_0} g}$$

Differentiable function

Given $f : \mathbb{C} \to \mathbb{C}$ is differentiable if

$$f'(z_0) = \lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}$$ exists and finite

$z \mapsto z^n$ is differentiable everywhere

$z \mapsto \bar{z}$ is nowhere differentiable

Analytic function

A function f is analytic on an open set $U \subseteq \mathbb{C}$ if it is complex differentiable at every $z \in U$ and $f'(z)$ is continuous

path : $[0,1] \to \mathbb{C}$ continuous

- If f is analytic on path-connected open set $U \subseteq \mathbb{C}$
  define $g = \overline{f(\bar{z})}$
  g is analytic on $D^* = \{\bar{z} : z \in D\}$

Cauchy Rieman Equation

Let $f(z) = u(z) + iv(z)$ where $u, v : \mathbb{C} \to \mathbb{R}$

View $u, v : \mathbb{R}^2 \to \mathbb{R}$ $u = u(x,y)$, $v = v(x,y)$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

// Cauchy Rieman Equation

Cauchy Rieman Equation $\Leftrightarrow$ Analytic

Corollary

If f is analytic on path connected $D \subseteq \mathbb{C}$

$f'(z) = 0 \quad \forall z \in D \Rightarrow f$ is constant

$f'(z) \in \mathbb{R} \quad \forall z \in D \Rightarrow f$ is constant

Jacobian Matrix

if $f = u + iv$ $\quad J_f = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ where $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $v_x = \frac{\partial v}{\partial x}$, $v_y = \frac{\partial v}{\partial y}$

Laplacian

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2}$$

$g : \mathbb{R}^n \to \mathbb{R}$, $\Delta g = 0$ g is harmonic

- if f is analytic, $\det J_f = |f'(z)|^2$

Let f be analytic on D, $z_0 \in D$ and $f'(z_0) \neq 0$, Then

$\exists \varepsilon > 0$ s.t $U = z_0 + \varepsilon\mathbb{B} \subseteq D$, f is one-to-one on U

open unit ball

$f(U)$ is open, $f^{-1} : f(U) \to U$ is analytic with

$$(f^{-1})'(f(z)) = \frac{1}{f'(z)} \quad \text{for all } z \in U$$

If $f = u + iv$ is analytic and u, v have continuous 2nd partial derivatives then u, v are harmonic

Harmonic Conjugate

Given u is harmonic, v is a harmonic conjugate of u if v is harmonic and $f = u + iv$ is analytic

Harmonic conjugate are unique upto a constant

Conformal Mapping

Suppose $U \subseteq \mathbb{C}$, $f : U \to V$ is one-to-one and onto

angle between tangent

f is conformal at $z_0 \in U$ if it preserves angle between directed intersecting curves at z_0

- f is conformal at z_0 and $f'(z_0) \neq 0$ $\Rightarrow$ f is analytic at z_0

Möbius Transformation

$\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$

$$f : \mathbb{C}^* \to \mathbb{C}^* \qquad f(z) = \frac{az + b}{cz + d} \quad \text{with } ad - bc \neq 0$$

take $f(\infty) = \frac{a}{c}$, $f(-\frac{c}{d}) = \infty$

Given distinct $z_0, z_1, z_2 \in \mathbb{C}^*$ and $w_0, w_1, w_2 \in \mathbb{C}^*$
There is a unique Möbius transformation f that maps $z_0 \mapsto w_0$, $z_1 \mapsto w_1$, $z_2 \to w_2$

Möbius Transformations map circles to circles

Green's Theorem

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

different from harmonic in multivariable calculus

$h : D \to \mathbb{C}$ where $D \subseteq \mathbb{C}$ is harmonic if $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$

Gauss' Mean Value Property

$u : D \to \mathbb{C}$ is harmonic, $\forall z_0 \in D, \forall r > 0 \wedge z_0 + r\mathbb{B} \subseteq D$, we have

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$$

Strict Maximum Principle

Let $h : D \to \mathbb{C}$ be a bounded harmonic function, D is open
If $\forall z \in D$, $|h(z)| \leq M$ and $\exists z_0 \in D$, $|h(z_0)| = M$ then h is a constant

Maximum Principle

Let h be a harmonic function on bounded open D that extend continuous to ∂D
if $|h(z)| \leq M$ $\forall z \in \partial D$ then $|h(z)| \leq M$ $\forall z \in D$

Complex Integration (line integral)

Fundamental Theorem of Calculus for Analytic Function

If f is continuous and analytic on D such that $F'(z) = f(z)$
called a primitive of f
For any path γ from α to β

$$\int_\alpha^\beta f(z)\,dz = \int_\gamma f(z)\,dz = F(\beta) - F(\alpha)$$

- If γ is piecewise smooth and f is continuous on γ

$$\left|\int_\gamma f(z)\,dz\right| \leq \int_\gamma |f(z)|\,|dz| \leq ML$$

where $|f(z)| \leq M$ $\forall z \in \gamma$ and $L = \text{length}(L)$

- An analytic function f has a primitive on D if and only if

$$\int_\gamma f(z)\,dz = 0 \quad \text{for every closed } \gamma \subseteq D$$

Cauchy's Theorem

Let D be a bounded domain and f be analytic on D that extends smoothly to ∂D then

$$\int_{\partial D} f(z)\,dz = 0$$

Cauchy's Integral Formula

$$f(z) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(w)}{w - z}\,dw \quad \text{for } z \in D$$

Corollary

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\partial D} \frac{f(w)}{(w-z)^{n+1}}\,dw \quad \text{for } z \in D$$

Cauchy's Estimates

Suppose f is analytic on $z_0 + R\mathbb{B}$, If $|f(z)| \leq M$ on the boundary then $\left|f^{(n)}(z_0)\right| \leq \frac{n!}{R^n} M$

Liouville's Theorem

Every bounded entire function is constant

analytic on $\mathbb{C}$

Corollary: Every non-constant polynomial has a complex root

Morera's Theorem

Let f be continuous on D, if $\oint_{\partial R} f(z)\,dz$ for every rectangle R parallel to x-y coordinates then f is analytic on D

Corollary: Suppose for all $t \in [a,b]$, $g(t,z)$ is a continuous function on D. If for each $t \in [a,b]$, $g(t,z)$ is analytic then

$$G(z) = \int_a^b g(t,z)\,dt \quad \text{is analytic on } D$$

Corollary: Suppose f is continuous on D and analytic on $D \setminus \mathbb{R}$ then f is analytic on D

Goursat's Theorem

If $f : D \to \mathbb{C}$ has $f'(z)$ $\forall z \in D$ then f is analytic on D

no need continuous condition

New form of Cauchy Rieman Equation

$$f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

Define $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ $\Rightarrow$ $\frac{\partial f}{\partial \bar{z}} = 0 \iff f$ is analytic

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Complex Green's Theorem

closure $D \cup \partial D$

If D is a bounded with piecewise smooth boundary and $g(z)$ is smooth on $\bar{D}$ then

$$\oint_{\partial D} g(z)\,dz = 2i \iint_D \frac{\partial g}{\partial \bar{z}}\,dz$$

Pompeiu's Formula

$$g(w) = \frac{1}{2\pi i} \oint_{\partial D} \frac{g(z)}{z - w}\,dz - \frac{1}{\pi} \iint_D \frac{\partial g}{\partial \bar{z}} \frac{1}{z - w}\,dz$$

Complex Series

Given $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{C}$, $\sum_{n=1}^{\infty} a_n = \lim_{N \to \infty}\left(\sum_{n=1}^{N} a_n\right)$

$\sum_{n=1}^{\infty} a_n$ converges absolutely $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ converges

For $n \in \mathbb{N}$, $f_n : A \to \mathbb{C}$ where $A \subseteq \mathbb{C}$
$f_n \to f$ point wise on A if forall $z \in A$

$$\lim_{n\to\infty} f_n(z) = f(z)$$

$f_n \to f$ uniformly on A if forall $\varepsilon > 0$, exists $N \in \mathbb{N}$ s.t

$$|f_n(z) - f(z)| < \varepsilon \quad \text{forall } z \in A$$

- $f_n \to f$ uniformly on A, f_n continuous $\Rightarrow f$ continuous
- $C \subseteq \mathbb{C}$ be a piecewise smooth curve, $f_n \to f$ uniformly

$$\lim_{n\to\infty} \int_C f_n(z)dz = \int_C f(z)dz$$

- $f_n \to f$ uniformly on $D \subseteq \mathbb{C}$ and f_n is analytic for all $n \in \mathbb{N}$, so is f
- f_n is analytic on $z_0 + R\mathbb{B}$ and $f_n \to f$ uniformly. Then forall $r < R$ and $m \geq 1$, $f_n^{(m)} \to f^{(m)}$ uniformly of $z_0 + r\mathbb{B}$

Power Series

A power series centered at $z_0 \in \mathbb{C}$ is

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for } a_n \in \mathbb{C}$$

For any power series $\sum_{n=0}^{\infty} a_n z^n$, there is $R \geq 0$ such that
possibly infinite
for all $|z| < R$ the series absolutely converges and forall $|z| > R$ it diverges // R: radius of convergence

Suppse $f(z) = \sum_{n=0} a_n z^n$ has a radius of convergence R then f is analytic on $R\mathbb{B}$

$$a_m = \frac{f^{(m)}(z_0)}{m!} = \frac{1}{2\pi i} \oint_{|w-z|=r} \frac{f(w)}{(w-z)^{m+1}} dw, \quad r < R$$

$f(z)$ is analytic at ∞ if $f\left(\frac{1}{z}\right)$ is analytic at $z = 0$

So $f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^n$, $|z| < R$

Thus

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}, \quad |z| > R$$

where $a_n = \frac{1}{2\pi i} \oint_{|z|=r} z^{n-1} f(z)dz$, $r > R$

Suppose f is analytic at z_0, we say f has a zero of order n at z_0 if

$$0 = f(z_0) = f'(z_0) = \dots = f^{(n-1)}(z_0)$$

$$\Leftrightarrow f(z) = a_n(z-z_0)^n + a_{n+1}(z-z_0)^{n+1} + \dots$$
$$= (z-z_0)^n g(z), \quad g(z_0) \text{ is analytic at } z_0$$

We say f has zero of order n at ∞ if $f\left(\frac{1}{z}\right)$ has zero of order n at 0

$$f(z) = \frac{a_n}{z^n} + \frac{a_{n+1}}{z^{n+1}} + \dots$$

If $f: D \to \mathbb{C}$ is analytic where D is a domain, then all of the zeroes of $f \neq 0$ are of finite order

$\exists \varepsilon > 0,\ z + \varepsilon\mathbb{B} = \{z\}$

If $f: D \to \mathbb{C}$ where $f \neq 0$ and $D \subseteq \mathbb{C}$ is a domain then all zeroes of f are isolated points.

Suppose $0 \leq r < R \leq +\infty$ and f is analytic on the annulus $r < |z - z_0| < R$, Then

$$f(z) = f_0(z) + f_1(z)$$

where f_0 is analytic for $|z - z_0| < R$ and f_1 is analytic for $r < |z - z_0|$

Laurent Decomposition

$$f_0(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad f_1(z) = \sum_{n=-\infty}^{0} \frac{b_n}{(z-z_0)^n}$$

Hence, $f(z) = \sum_{n\in\mathbb{Z}} a_n (z-z_0)^n$

where $a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}} dz$ where γ is a curve enclosing z_0 on the annulus (r, R)

Singularity

z_0 is an isolated singularity of $f(z)$ if $f(z)$ is analytic on $z_0 - R\mathbb{B} \setminus \{z_0\}$ for some $R > 0$
$f(z)$ has an isolated singularity at ∞ if $f\left(\frac{1}{z}\right)$ has an isolated singularity at 0

- z_0 is a removable singularity if $a_n = 0\ \forall\ n \leq -1$
- z_0 is a pole of order N if $a_{-N} \neq 0$, $a_n = 0\ \forall\ n < -N$
- z_0 is an essential singularity if $a_n \neq 0$ for infinitely many $n \leq -1$

where $f(z) = \sum_{n=\mathbb{Z}} a_n (z-z_0)^n$, $0 < |z - z_0| < R$

Suppso z_0 is an isolated singularity of f, If f is bounded near z_0, it is a removable singularity

The following are equivalent
① z_0 is a pole of $f(z)$ of order N
② $g(z) = (z-z_0)^N f(z)$ is analytic at z_0 with $g(z_0) \neq 0$
③ $\frac{1}{f(z)}$ is analytic at z_0 with zero of order N

An isolated singularity z_0 of $f(z)$ is a pole if and only if $\lim_{z\to z_0} |f(z_0)| = \infty$

Ksarati Virustras Theorem // Casorati - Weierstrass Theorem

If z_0 is an essential singularity of $f(z)$ then for all $w \in \mathbb{C}$ exists $\{z_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} z_n = z_0 \quad \text{and} \quad \lim_{n\to\infty} f(z_n) = w$$

Meromorphic

$f: D \to \mathbb{C}$ is meromorphic in D if it is analytic on D except a set of finitely many poles

Residue

Suppse z_0 is an isolated point of f and $f(z) = \sum_{n\in\mathbb{Z}} a_n (z-z_0)^n$ for $0 < |z - z_0| < R$
The residue of f at z_0 is

$$\text{Res}(f, z_0) = a_{-1} = \frac{1}{2\pi i} \oint_{|z-z_0|=r} f(z)dz$$

// generally, $f(z) = \frac{\text{Res}(f,z_0)}{z-z_0} + f_1(z)$ analytic at z_0

If f has a simple pole (order 1) at z_0 then $\text{Res}(f, z_0) = \lim_{z\to z_0} (z-z_0)f(z)$

If $f(z) = \frac{p(z)}{q(z)}$, $p(z_0) \neq 0$, $q(z)$ has a zero of order 1 at z_0 then $\text{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}$

If f has a pole of order N at z_0 then $\text{Res}(f, z_0) = \frac{1}{(N-1)!} \lim_{z\to z_0} \left(\frac{d}{dz}\right)^{N-1} \left[(z-z_0)^N f(z)\right]$

Cauchy Residue

Suppse f is analytic on a domain D and its boundary ∂D except for a finite set of isolated singularities $z_1, z_2, \dots, z_n \in D$ then

$$\int_{\partial D} f(z)dz = 2\pi i \sum_{j=1}^{n} \text{Res}(f, z_j)$$