### ma5209 - algebraic topology

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### Chapter 1

# CATEGORY AND HOMOLOGICAL ALGEBRA

#### 1.1 BASIC HOMOLOGICAL ALGEBRA

**Definition 1** (chain complex). A chain complex  $A_{\bullet}$  is a sequence of abelian groups and homomorphisms, namely boundary operator

$$\ldots \xleftarrow{\partial} A_{-2} \xleftarrow{\partial} A_{-1} \xleftarrow{\partial} A_0 \xleftarrow{\partial} A_1 \xleftarrow{\partial} A_2 \xleftarrow{\partial} \ldots$$

such that  $\partial^2 = 0$ , that is,  $\operatorname{im}(\partial : A_{n+1} \to A_n) \subseteq \ker(\partial : A_n \to A_{n-1})$ 

**Definition 2** (cycle, boundary, homology group). Given a chain complex  $A_{\bullet}$  with boundary operator  $\partial$ .

• An n-cycle is an element of  $Z_n(A)$  defined by

$$Z_n(A_{\bullet}) = \ker(\partial : A_n \to A_{n-1})$$

• An n-boundary is an element of  $B_n(A)$  defined by

$$B_n(A_{\bullet}) = \operatorname{im}(\partial : A_{n+1} \to A_n)$$

• The nth homology group of A is the quotient

$$H_n(A_{\bullet}) = \frac{Z_n(A_{\bullet})}{B_n(A_{\bullet})}$$

**Definition 3** (chain map). Let  $A_{\bullet}$ ,  $B_{\bullet}$  be chain complexes. A chain map  $g: A_{\bullet} \to B_{\bullet}$  is a sequence of homomorphisms such that  $\partial g = g\partial$ . The following square commutes

$$\begin{array}{ccc} A_n & \stackrel{\partial}{\longrightarrow} & A_{n-1} \\ \downarrow^g & & \downarrow^g \\ B_n & \stackrel{\partial}{\longrightarrow} & B_{n-1} \end{array}$$

**Proposition 1.** Let  $g: A \to B$  be a chain map. Then, g determines a homomorphism  $H_n(g) = g_*: H_n(A) \to H_n(B)$  for all n

$$A_{n} \xrightarrow{g} B_{n}$$

$$\downarrow \subseteq \qquad \qquad \downarrow \subseteq$$

$$Z_{n}(A) \xrightarrow{g} Z_{n}(B)$$

$$\downarrow \qquad \qquad \downarrow$$

$$H_{n}(A) \xrightarrow{g_{*}} H_{n}(B)$$

Proof.

 $(g: Z_n(A) \to Z_n(B))$  well-defined - g maps cycles to cycles) Let  $\sigma \in Z_n(A)$ , then  $\partial \sigma = 0$  and  $\partial g \sigma = g \partial \sigma = g \partial \sigma = 0$   $(g: B_n(A) \to B_n(B))$  well-defined - g maps boundaries to boundaries) Let  $\sigma \in B_n(A)$ , then  $\sigma = \partial b$  where  $b \in C_{n+1}(A)$ , and  $g\sigma = g \partial b = \partial g b \in B_n(B)$ 

 $(g_*: H_n(A) \to H_n(B))$  well-defined) Any element of  $Z_n(A)$  in equivalence class  $[a] \in H_n(A)$  can be written as  $a + \partial c$ , then  $g(a + \partial c) = ga + g\partial c = ga$ . That is, g maps elements from the same equivalence class to the same value.

 $(g_*: H_n(A) \to H_n(B))$  is a homomorphism) Let  $a + \partial c, b + \partial d \in Z_n(A)$  be in the equivalence classes  $[a], [b] \in H_n$ . Then  $g(a + \partial c + b + \partial d) = g(a + \partial c) + g(b + \partial d)$ . Therefore,  $g_*([a] + [b]) = g_*[a] + g_*[b]$ 

**Proposition 2.** Let  $1: A \to A, f: B \to C, g: A \to B$  be chain maps.

1. 
$$H_n(1) = 1$$

2. 
$$H_n(fg) = H_n(f)H_n(g)$$

Proof.

(1) obvious

(2) 
$$H_n(fg)[a] = [fga] = H_n(f)[ga] = H_n(f)H_n(g)[a]$$

**Definition 4** (chain homotopy). Let  $g_0, g_1 : A_{\bullet} \to B_{\bullet}$  be chain maps. A chain homotopy from  $g_0$  to  $g_1$  is a sequence of homomorphisms  $h : A_n \to B_{n+1}$  such that  $\partial h + h\partial = g_0 - g_1$ 

$$\dots \stackrel{\partial}{\longleftarrow} A_0 \stackrel{\partial}{\longleftarrow} A_1 \stackrel{\partial}{\longleftarrow} A_2 \stackrel{\partial}{\longleftarrow} \dots$$

$$\dots \stackrel{\partial}{\longleftarrow} B_0 \stackrel{\partial}{\longleftarrow} B_1 \stackrel{\partial}{\longleftarrow} B_2 \stackrel{\partial}{\longleftarrow} \dots$$

If there is a chain homotopy from  $g_0$  to  $g_1$ , we said  $g_0$  and  $g_1$  are chain homotopic or of the same chain homotopy type.

**Proposition 3.** Two chain homotopic maps induce the same map in homology. That is, if  $g_0, g_1$  are chain homotopic, then  $g_{0*} = g_{1*}$ 

*Proof.* Let 
$$a \in Z_n(A)$$
, then  $\partial a = 0$  and  $g_0(a) = g_1(a) + \partial ha + h\partial a = g_1(a) + \partial ha$ . So,  $[g_0(a)] = [g_1(a)]$ 

**Definition 5** (chain homotopy equivalence). A map  $f: X \to Y$  is a chain homotopy equivalence if there exists  $g: Y \to X$  such that  $fg: Y \to Y$  is chain homotopic to  $1_Y$  and  $gf: X \to X$  is chain homotopic to  $1_X$ 

**Definition 6** (short exact sequence). A sequence of groups and homomorphisms

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

is a short exact sequence if one of the following equivalent conditions satisfies

- (change needed) p is surjective and i = inclusion of ker p
- (change needed) i is injective and p = projection onto coker i
- i is injective, p is surjective, and im  $i = \ker p$

**Definition 7** (exact). Given sequences of groups

- $A \xrightarrow{i} B \xrightarrow{p} C$  is exact at B if im  $i = \ker p$
- ullet ...  $\longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow \dots$  is exact if it is exact at every group

**Theorem 1.** Short exact sequence of chain complexes induces long exact sequence in homology. That is, Let  $0 \longrightarrow A \stackrel{i}{\longrightarrow} B \stackrel{p}{\longrightarrow} C \longrightarrow 0$  be a short exact sequence of chain complexes and i, p be chain maps.

$$0 \longrightarrow A_{n+1} \xrightarrow{i} B_{n+1} \xrightarrow{p} C_{n+1} \longrightarrow 0$$

$$\downarrow \partial \qquad \qquad \downarrow \partial \qquad \qquad \downarrow \partial \qquad \qquad \downarrow \partial$$

$$0 \longrightarrow A_{n} \xrightarrow{i} B_{n} \xrightarrow{p} C_{n} \longrightarrow 0$$

$$\downarrow \partial \qquad \qquad \downarrow \partial \qquad \qquad \downarrow \partial$$

$$0 \longrightarrow A_{n-1} \xrightarrow{i} B_{n-1} \xrightarrow{p} C_{n-1} \longrightarrow 0$$

$$\downarrow \partial \qquad \qquad \downarrow \partial \qquad \qquad \downarrow \partial$$

$$\vdots \qquad \qquad \downarrow \partial \qquad \qquad \downarrow \partial$$

$$\vdots \qquad \qquad \downarrow \partial \qquad \qquad \downarrow \partial$$

$$\vdots \qquad \qquad \downarrow \partial \qquad \qquad \downarrow \partial$$

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$$\vdots \qquad \qquad \vdots \qquad \qquad \vdots \qquad \qquad \downarrow \partial$$

$$\vdots \qquad \qquad \vdots \qquad \qquad \vdots \qquad \qquad \downarrow \partial$$

$$\vdots \qquad \qquad \vdots \qquad \qquad \vdots \qquad \qquad \vdots \qquad \qquad \vdots$$

Then there exists connecting homomorphisms  $\partial$  such that the sequence

$$H_{n+1}(A) \xrightarrow{i_*} H_{n+1}(B) \xrightarrow{p_*} H_{n+1}(C)$$

$$H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{p_*} H_n(C)$$

$$H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{p_*} H_{n-1}(C)$$

is exact.

Proof.

(1 defining of  $\partial: H_{n+1}(C) \to H_n(A)$ )

Given  $[c] \in H_{n+1}(C)$ , (1) take any representative  $c \in Z_{n+1}(C)$ . As  $p: B_{n+1} \to C_{n+1}$  is surjective, (2) take any  $b \in B_{n+1}$  such that pb = c. As  $p\partial b = \partial pb = \partial c = 0$  and  $\ker(p: B_n \to C_n) = \operatorname{im}(i: A_n \to B_n)$ , take  $a \in A_n$  such that  $ia = \partial b$ , this choice is unique as i is injective.  $i\partial a = \partial ia = \partial^2 b = 0$ , as i is an injective homomorphism,  $\partial a = 0$ , then  $a \in Z_n(A)$ . The construction is done by  $[c] \mapsto [a]$ 

We will verify that the homology class of a does not depend on (1) the representative c and (2) the value of b.

(1.1 [a]does not depend on the choice of b)

$$n+1: \qquad \qquad d \stackrel{i}{\longmapsto} b-b_1 \stackrel{p}{\longmapsto} 0$$
 
$$\downarrow \partial \qquad \qquad \downarrow \partial \qquad \qquad \downarrow \partial$$
 
$$n: \qquad \partial d=a-a_1 \stackrel{i}{\longrightarrow} \partial b-\partial b_1$$

Let  $b_1 \in B_{n+1}$  such that  $pb_1 = c$  and  $a_1 \in A_n$  such that  $ia_1 = \partial b_1$ . We want to prove that a and  $a_1$  belong to the same homology class, that is  $a - a_1$  is a boundary. As  $p(b - b_1) = c - c = 0$ , there exists a unique  $d \in A_{n+1}$  such that  $id = b - b_1$ . Then,  $i\partial d = \partial id = \partial (b - b_1) = \partial b - \partial b_1 = ia - ia_1 = i(a - a_1)$ . As i is injective,  $a - a_1 = \partial d$ , that is,  $a - a_1$  is a boundary. (1.1 [a] does not depend on the choice of c)

$$n+2: \qquad \qquad f \xrightarrow{p} e \\ \downarrow \partial \qquad \downarrow \partial \\ n+1: \qquad d \xrightarrow{i} b-b_1-\partial f \xrightarrow{p} 0 \qquad \partial f \xrightarrow{p} c-c_1 = \partial e \\ \downarrow \partial \qquad \qquad a-a_1 = \partial d$$

Let  $c_1 = c - \partial e$  for  $e \in C_{n+2}$  be in the same homology class with  $c, b_1 \in B_{n+1}$  such that  $pb_1 = c_1$ , and  $a_1 \in A_n$  such that  $ia_1 = \partial b_1$ . We want to prove that a and  $a_1$  belong to the same homology class, that is  $a - a_1$  is a boundary. Let  $f \in B_{n+2}$  such that pf = e. Then,  $p\partial f = \partial pf = \partial e$ . Hence,  $p(b - b_1 - \partial f) = c - c_1 - \partial e = 0$ . There exists unique  $d \in A_{n+1}$  such that  $id = b - b_1 - \partial f$ , then  $i\partial d = \partial id = \partial (b - b_1 - \partial f) = \partial b - \partial b_1 - \partial^2 f = ia - ia_1 = i(a - a_1)$ . As i is injective,  $a - a_1 = \partial d$ , that is,  $a - a_1$  is a boundary.

 $(1.3 \partial \text{ is a homomorphism})$ 

Let  $c_1 \in Z_{n+1}(C)$ ,  $b_1 \in B_{n+1}$  such that  $pb_1 = c_1$ , and  $a_1 \in A_n$  such that  $ia_1 = \partial b_1$ . We want to prove that  $\partial(c + c_1) = a + a_1$ . Indeed, this is done verifying  $p(b + b_1) = c + c_1$  and  $i(a + a_1) = \partial(b + b_1)$ .

(2 exactness at  $H_n(A), H_n(B), H_n(C)$ ) It is equivalent to  $\operatorname{im} \partial = \ker i_*, \operatorname{im} i_* = \ker p_*, \operatorname{im} p_* = \ker \partial$ 

 $(2.1 \text{ im } \partial \subseteq \ker i_* \iff \text{ for all } [c] \in H_{n+1}(C), \ i_* \partial [c] = 0) \ i_* \partial [c] = i_* [a] = [ia] = [\partial b] = 0$ 

 $(2.2 \text{ im } i_* \subseteq \ker p_* \iff \text{ for all } [a] \in H_n(A), \ p_*i_*[a] = 0) \ p_*i_*[a] = p_*[ia] = [pia] = 0$ 

(2.3 im  $p_* \subseteq \ker \partial \iff$  for all  $[b] \in H_n(B)$ ,  $\partial p_*[b] = 0$ )  $\partial p_*[b] = \partial [pb] = i^{-1}(\partial^*b)$  where  $\partial^* : B_n \to B_{n-1}$  denotes the boundary operator. As  $b \in Z_n(B)$ ,  $\partial^*b = 0$  and i is an injective homomorphism, then  $\partial p_*[b] = i^{-1}(\partial^*b) = i^{-1}(0) = 0$ 

 $(2.4 \text{ im } \partial \supseteq \ker i_* \iff \text{for all } [\overline{a}] \in H_n(A), \text{ if } i_*[\overline{a}] = 0, \text{ then there exists } [c] \in H_{n+1}(C) \text{ such that } \partial[c] = [\overline{a}])$   $i_*[\overline{a}] = 0 \implies [i\overline{a}] = 0, \text{ that is } i\overline{a} \text{ is a boundary, then there exists } b \in B_n \text{ such that } \partial^*b = i\overline{a}. \text{ Take } c = pb.$   $(2.5 \text{ im } i_* \supseteq \ker p_*) \text{ TODO} - \text{look at the notes from prof}$   $(2.6 \text{ im } p_* \supseteq \ker \partial \iff \text{for all } [c] \in H_n(C), \text{ if } \partial[c] = 0, \text{ then there exists } [\overline{b}] \in H_n(B) \text{ such that } p_*[\overline{b}] = [c])$ 

As p is surjective, let  $b \in B_n$  such that pb = c,  $a \in A_{n-1}$  such that  $ia = \partial^* b$ , then  $[a] = \partial[c] = 0$ , that is a is a boundary. Then, there exists  $d \in A_n$  such that  $a = \partial^* d$ . Take  $\bar{b} = b - id \in B_n$ , then  $p_*[\bar{b}] = [p(b - id)] = [pb - pid] = [pb] = [c]$ , and  $\bar{b} \in Z_n(B)$  because  $\partial^* \bar{b} = \partial^* (b - id) = \partial^* b - \partial^* id = \partial^* b - i\partial^* d = \partial^* b - ia = 0$ 

**Definition 8** (short exact sequence splits). A short exact sequence splits if TODO

#### 1.2 BASIC CATEGORY THEORY

#### 1.2.1 CATEGORY, FUNCTOR, NATURAL TRANSFORMATION

**Definition 9** (category). A category C is a class ob C of objects such that

- for all  $X, Y \in \text{ob } C$ , a class C(X, Y) (or  $\text{hom}_C(X, Y)$ , hom(X, Y)) of morphisms
- for all  $X \in ob C$ ,  $1_X \in C(X,X)$ : the identity morphism
- for all  $X, Y, Z \in ob C$ , there exists a function

$$\circ: C(Y,Z) \times C(X,Y) \to C(X,Z)$$
$$(q,f) \mapsto qf$$

 $f \in C(X,Y)$  is also written as  $f: X \to Y$ 

and

1. Let 
$$f: X \to Y$$
,  $1_Y f = f$ ,  $f1_X = f$ 

$$X \xrightarrow{f} Y \qquad X$$

$$\downarrow_{1_Y f = f} \downarrow_{1_Y} \qquad \downarrow_{1_X} \downarrow_{f \downarrow_{X} = f}$$

$$Y \qquad X \xrightarrow{f} Y$$

2. Let 
$$f: X \to Y, g: Y \to Z, h: Z \to W, h(gf) = (hg)f$$

#### Remark 1. :

Some remarks on categories

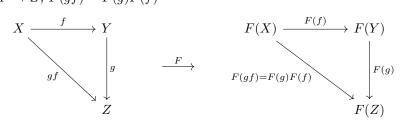
- Set: the category of sets and functions
- Top: the category of topological spaces and continuous maps
- Grp: the category of groups and homomorphisms
- Ab: the category of abelian groups and homomorphisms

Some remarks on morphisms: given a morphism  $f: A \to B$ 

- f is said to be a monomorphism if for all  $g_1, g_2: X \to A$ ,  $fg_1 = fg_2 \implies g_1 = g_2$
- f is said to be an epimorphism if for all  $g_1, g_2 : B \to X$ ,  $g_1 f = g_2 f \implies g_1 = g_2$
- f is said to be an isomorphism if there exists  $g: B \to A$  such that  $fg = 1_B$  and  $gf = 1_A$

**Definition 10** (functor). Let C, D be categories. A functor from C to  $D, F: C \to D$  is

- $F : \operatorname{ob} C \to \operatorname{ob} D$
- for all  $X, Y \in \text{ob } C$ , there is a function  $F: C(X,Y) \to C(F(X),F(Y))$  such that
  - for all  $X \in ob C$ ,  $F(1_X) = 1_{F(X)}$
  - for all  $f: X \to Y, g: Y \to Z, F(gf) = F(g)F(f)$



Remark 2. Some remarks on functors

- identity functor:  $1: C \to C$
- forgetful functors:  $U: \text{Grp} \to \text{Set}, \ U: \text{Top} \to Set$
- $singular\ homology:\ H_n: Top \to Ab$
- free abelian group functor:  $\mathbb{Z} : \operatorname{Set} \to \operatorname{Ab}$

**Definition 11** (natural transformation). Let C, D be categories and  $F, G: C \to D$  be functors. A natural transformation from F to G is an assignment of each  $X \in \text{ob } C$  to a morphism  $\theta_X: F(X) \to G(X)$  in D such that for all  $f: X \to Y$  in C. The diagram below commutes

$$\begin{array}{ccc}
X & F(X) & \xrightarrow{\theta_X} G(X) \\
\downarrow^f & F(f) \downarrow & \downarrow^{G(f)} \\
Y & F(Y) & \xrightarrow{\theta_Y} G(Y)
\end{array}$$

Notation:

- ob Fun(C, D): functors  $C \to D$
- Fun(C, D)(F, G): natural transformations  $F \to G$

Remark 3. Some remarks on natural transformations

• The boundary operator  $\partial$  is a natural transformation from  $C_n$  to  $C_{n-1}$  where  $C_n$  is the functor from the Top to the category of free abelian group of n-chains

$$\begin{array}{ccc}
X & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) \\
\downarrow^f & C_n(f) = f_{\#} & & \downarrow^{C_{n-1}(f) = f_{\#}} \\
Y & C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y)
\end{array}$$

- The connecting homomorphism  $\partial$  is a natural transformation from  $H_n(C)$  to  $H_{n-1}(A)$  where  $H_n(C)$  and  $H_{n-1}(A)$  are two functors from the category of short exact sequence of chain complexes to Ab and defined by
  - $-H_n(C)(upper\ chain) = H_n(C_{\bullet})$
  - $-H_n(C)(lower\ chain) = H_n(C'_{\bullet})$
  - $H_n(C)(f) = H_n(f) = f_*$
  - $-H_{n-1}(A)(upper\ chain) = H_{n-1}(A_{\bullet})$
  - $-H_{n-1}(A)(lower\ chain) = H_{n-1}(A'_{\bullet})$
  - $H_{n-1}(A)(f) = H_n(f) = f_*$

$$0 \longrightarrow A_{\bullet} \longrightarrow B_{\bullet} \longrightarrow C_{\bullet} \longrightarrow 0 \qquad H_{n}(C_{\bullet}) \stackrel{\partial}{\longrightarrow} H_{n-1}(A_{\bullet})$$

$$\downarrow f \qquad \downarrow f \qquad \downarrow f \qquad \downarrow H_{n}(C)(f) = f_{*} \downarrow \qquad \downarrow H_{n-1}(A)(f) = f_{*}$$

$$0 \longrightarrow A'_{\bullet} \longrightarrow B'_{\bullet} \longrightarrow C'_{\bullet} \longrightarrow 0 \qquad H_{n}(C'_{\bullet}) \stackrel{\partial}{\longrightarrow} H_{n-1}(A'_{\bullet})$$

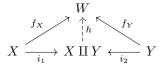
#### 1.2.2 UNIVERSAL MAPPING PROPERTY

**Definition 12** (product). Let  $X, Y \in \text{ob } C$ , the product an object Z together with  $p_1: Z \to X$ ,  $p_2: Z \to Y$ . Moreover, the triplet  $(Z, p_1, p_2)$  is universal, that is, for each diagram  $X \xleftarrow{f_X} W \xrightarrow{f_Y} Y$ , there exists a unique  $h: W \to Z$  such that the diagram commutes

$$X \stackrel{f_X}{\longleftarrow} X \times Y \stackrel{f_Y}{\longrightarrow} Y$$

The coproduct is written by  $Z = X \times Y$ 

**Definition 13** (coproduct). Let  $X,Y \in \text{ob } C$ , the coproduct an object Z together with  $i_1: X \to Z$ ,  $i_2: Y \to Z$ . Moreover, the triplet  $(Z,i_1,i_2)$  is universal, that is, for each diagram  $X \xrightarrow{f_X} W \xleftarrow{f_Y} Y$ , there exists a unique  $h: Z \to W$  such that the diagram commutes



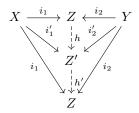
The coproduct is written by  $Z = X \coprod Y$ 

Remark 4. Some remarks on coproduct

- direct sum on abelian groups is a coproduct
- disjoint union on sets is a coproduct
- free product on groups is a coproduct
- coproduct is unique upto isomorphism

Proof.

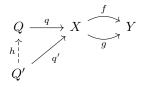
Let  $(Z, i_1, i_2), (Z', i'_1, i'_2)$  be both coproduct of X, Y. Using universal property, we can construct unique  $h: Z \to Z'$  and  $h': Z' \to Z$  such that the diagram below commutes



By uniqueness of h, h', we must have  $h'h = 1_Z$ . Similarly,  $hh' = 1_{Z'}$ 

• Construction of coproduct from pushout: coproduct of X, Y is the pushout of  $X \longleftarrow 0 \longrightarrow Y$  where 0 is the initial object.

**Definition 14** (equalizer). Let  $f, g: X \to Y$ , the equalizer is an object Q and a map  $Q \to X$  such that fq = gq. Moreover, the pair (Q, q) is universal, that is, if (Q', q') satisfies the same condition, there exists a unique map  $h: Q' \to Q$  such that the diagram below commutes



**Definition 15** (coequalizer). Let  $f, g: X \to Y$ , the coequalizer is an object Q and a map  $q: Y \to Q$  such that qf = qg. Moreover, the pair (Q, q) is universal, that is, if (Q', q') satisfies the same conditions, there exists a unique map  $h: Q \to Q'$  such that the diagram below commutes

$$X \xrightarrow{g} Y \xrightarrow{q} Q$$

$$\downarrow h$$

$$Q'$$

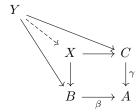
**Definition 16** (pullback). The pullback of  $C \xrightarrow{\gamma} A \xleftarrow{\beta} B$  is X and  $X \to B, X \to C$  such that the diagram below commutes

$$\begin{array}{ccc} X & \longrightarrow & C \\ \downarrow & & & \downarrow^{\gamma} \\ B & \stackrel{\beta}{\longrightarrow} & A \end{array}$$

Moreover, X is universal, that is, if there exists Y and the commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & C \\ \downarrow & & \downarrow^{\gamma} \\ B & \stackrel{\beta}{\longrightarrow} & A \end{array}$$

then there exists a unique map  $Y \to X$  such that the diagram below commutes



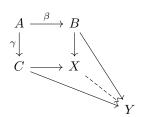
**Definition 17** (pushout). The pushout of  $C \xleftarrow{\gamma} A \xrightarrow{\beta} B$  is X and  $A \to X$ ,  $B \to X$  such that the diagram below commutes

$$\begin{array}{ccc}
A & \xrightarrow{\beta} & B \\
\uparrow \downarrow & & \downarrow \\
C & \longrightarrow & X
\end{array}$$

Moreover, X is universal, that is, if there exists Y and the commutative diagram

$$\begin{array}{ccc} A & \stackrel{\beta}{\longrightarrow} & B \\ \uparrow \downarrow & & \downarrow \\ C & \longrightarrow & Y \end{array}$$

then there exists a unique map  $X \to Y$  such that the diagram below commutes



#### Remark 5. Some remarks on pushout

ullet Construction of pushout from coproduct and coequalizer: the pushout of  $C \xleftarrow{\gamma} A \xrightarrow{\beta} B$  , is the coequalizer of

$$A \underbrace{\bigcap_{i_2 \gamma}^{i_1 \beta}}_{i_2 \gamma} B \coprod C$$

- Consider the category of sets, then the pushout  $X = B \coprod C / \sim$  where the equivalence relation is defined by  $\beta(a) \sim \gamma(a)$  for all  $a \in A$
- $\mathbb{R}P^{n-1}$ : real projective space space of lines through the origin in  $\mathbb{R}^n$ ,  $\mathbb{R}P^n$  is the pushout of  $\mathbb{R}P^{n-1} \longleftarrow S^{n-1} \longleftarrow e^n$ , the diagram below commutes

$$S^{n-1} \longleftrightarrow D^n$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathbb{R}P^{n-1} \longrightarrow \mathbb{R}P^n$$

where  $S^{n-1}$  denotes the equator,  $D^n$  denotes the closed upper hemisphere.

•  $\mathbb{C}P^{n-1}$ : complex projective space - space of lines through the origin in  $\mathbb{C}^n$ , similarly,  $\mathbb{C}P^n$  is the pushout of the diagram below

$$S^{2n-1} \longleftrightarrow D^{2n}$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathbb{C}P^{n-1} \longleftrightarrow \mathbb{C}P^n$$

**Definition 18** (base change, cobase change). :

- 1. if X is the pullback of  $C \xrightarrow{\gamma} A \xleftarrow{\beta} B$  then the transform from  $B \to A$  to  $X \to C$  is called base change.
- 2. if X is the pushout of  $C \xleftarrow{\gamma} A \xrightarrow{\beta} B$ , then the transform from  $A \to B$  to  $C \to X$  is called cobase change.

**Definition 19** (direct limit). Given a category C with

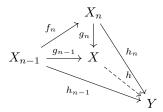
$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \dots$$

the direct limit of this sequence is an object X with maps  $g_n: X_n \to X$  for all n such that the diagram below commutes for all n

$$X_n \xrightarrow{f_n} X_n$$

$$X_{n-1} \xrightarrow{g_{n-1}} X$$

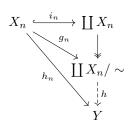
Moreover,  $(X, g_{\bullet})$  is universal, that is, if  $(Y, h_{\bullet})$  is also a direct limit, then there exists a unique map  $h: X \to Y$  such that the diagram below commutes for all n



The direct limit is written by  $X = \lim_{n \to \infty} X_n$ 

Remark 6. Some remarks on direct limit

- Construction of direct limit from pushout: the direct limit of  $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \dots$  is X such that X is the pushout of  $X_{n-1} \xleftarrow{1} X_{n-1} \xrightarrow{f_n} X_n$  for all n
- Consider the category of sets, then the direct limit of  $(X_n)$  is  $\coprod X_n / \sim$  where the equivalence relation is defined by  $(x \in X_{n-1}) \sim (f_n(x) \in X_n)$



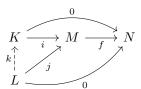
- if each  $f_n$  is mono,  $\lim X_n = \bigcup X_n$
- if each  $f_n$  is inclusion of a subspace, then  $\varinjlim X_n$  describe a topology on  $\bigcup X_n$ , called weak topology, that is,  $U \subseteq \bigcup X_n$  is said to be open if  $U \cap X_n$  open for all n

# 1.2.3 PREADDITIVE CATEGORY, ADDITIVE FUNCTOR, PROJECTIVE CLASS, DERIVE FUNCTOR

**Definition 20** (initial object, terminal object, zero, pointed category, zero map). Let C be a category. 0 is initial if #C(0,X) = 1 for all X, \* is terminal if #C(X,\*) = 1 for all X. C is called pointed if it has initial and terminal objects and the unique map  $0 \to *$  is an isomorphism. We write 0 for both initial and terminal objects. Consequently, given any two objects M, N there exists a zero map

$$M \longrightarrow 0 \longrightarrow N$$

**Definition 21** (kernel). Given a pointed category C, let  $f: M \to N$ , a kernel of f is a map  $i: K \to M$  such that fi = 0. Moreover, i is universal, that is, if  $j: L \to M$  such that fj = 0, then there exists a unique  $k: L \to K$  such that the diagram below commutes

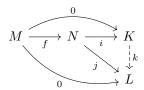


The category C has kernels if every morphism has a kernel.

Remark 7. Some remarks on kernels

• Construction of kernel from equalizer: the kernel of  $f: X \to Y$  is the equalizer of  $f: X \to Y$  and  $0: X \to Y$ 

**Definition 22** (cokernel). Given a pointed category C, let  $f: M \to N$ , a cokernel of f is a map  $i: N \to K$  such that if = 0. Moreover, i is universal, that is, if  $j: N \to L$  such that jf = 0, then there exists a unique  $k: K \to L$  such that the diagram below commutes



**Definition 23** (preadditive category, Ab-enriched category). A preadditive category is a category C together with an abelian group structure on Hom(M, N) for all  $M, N \in \text{ob } C$  such that composition map are bilinear. That is,

$$fh + gh = (f + g)h$$
 and  $hf + hg = h(f + g)$ 

Preadditive category is also called Ab-enriched category, or enriched over a monoidal category.

**Definition 24** (additive functor). A functor F between preadditive categories is additive if

$$\operatorname{Hom}(M,N) \to \operatorname{Hom}(F(M),F(N))$$

is a homomorphism between two abelian groups.

**Definition 25** (P-element, P-exact, P-epic - https://arxiv.org/pdf/math/9912157.pdf). For any two objects P, A, a map  $P \to A$  is called P-element of A. If P is a collection of objects, then P-element is a P-element for some  $P \in P$ . Let C be a (locally small) pointed category. Let  $P \in ob C$ , a sequence

$$A \longrightarrow B \longrightarrow C$$

is said to be P-exact if the composite  $A \rightarrow B \rightarrow C$  is the zero map and

$$C(P,A) \longrightarrow C(P,B) \longrightarrow C(P,C)$$

is an exact sequence of pointed sets (based point is the zero map). If  $\mathcal{P}$  is a class of objects in C, a  $\mathcal{P}$ -exact sequence in one which is P-exact for all  $P \in \mathcal{P}$ . A map  $A \to B$  is called P-epic if it induces a surjection of P-elements.

**Definition 26** (projective class - https://arxiv.org/pdf/math/9912157.pdf). A projective class on a pointed category C is  $(\mathcal{P}, \mathcal{E})$  where  $\mathcal{P}$  is a class of objects (projectives) and  $\mathcal{E}$  is a class of morphisms (epimorphisms) such that

- 1.  $\mathcal{E}$  is precisely the collection of all  $\mathcal{P}$ -exact sequences
- 2. P is precisely the collection of all object P such that each sequence in  $\mathcal{E}$  is P-exact.
- 3. any map  $A \to B$  can be extended to a sequence  $P \to A \to A$  in  $\mathcal{E}$  with  $P \in \mathcal{P}$

**Definition 27** (projective class). Let C be a pointed category with kernels. A projective class in C is  $(\mathcal{P}, \mathcal{E})$  where  $\mathcal{P}$  is a class of objects (projectives) and  $\mathcal{E}$  is a class of morphisms (epimorphisms) such that

1.  $P \in \text{ob } C \text{ lies in } \mathcal{P} \text{ if and only if for all } f: M \to N \text{ in } \mathcal{E}, \text{ the map } C(P,M) \to C(P,N) \text{ defined by } g \mapsto fg \text{ is onto.}$ 

$$P \xrightarrow{g} M$$

$$fg \xrightarrow{f} N$$

2. A morphism  $f: M \to N$  lies in  $\mathcal{E}$  if and only if for all  $P \in \mathcal{P}$ , the map  $C(P, M) \to C(P, N)$  defined by  $g \mapsto fg$  is onto.

$$P \xrightarrow{g} M$$

$$\downarrow f$$

$$\downarrow f$$

$$N$$

3. For any object  $M \in \text{ob } C$ , there exists  $P \to M$  in  $\mathcal{E}$  such that  $P \in P$ . This condition is also called C has enough projectives

**Definition 28** (left derived functor). Given a pointed preadditive category C with kernels and projective class (P, E) in C. Let  $F: C \to Ab$  be an additive functor (it maps chain complexes to chain complexes). The left derived functors of F with respect to P are  $L_nF: C \to Ab$  for  $n \ge 0$  defined by

$$L_n F(X) = H_n(F(P_{\bullet}))$$

where  $X \leftarrow P_{\bullet}$  is a  $\mathcal{P}$ -projective resolution. (analogous to Tor functor)

Remark 8. Some remarks on left derived functor

• Let C = R-Mod,  $\mathcal{P}$  be the categorical projectives (= direct summands of free R-modules),  $F : R\text{-Mod} \to Ab$  defined by  $A \mapsto A \oplus_R B$  where  $B \in R\text{-Mod}$ , then

$$L_n F(A) = \operatorname{Tor}_n^R(A, B)$$

•  $F(X) \leftarrow L_0F(X)$  might not be an isomorphism. It is an isomorphism if and only if F is right exact

**Definition 29** (retract). P is a retract of F is there exists a commutative diagram



The map  $P \to F$  is called section and the map  $F \to P$ 

**Proposition 4.** Class of projectives  $\mathcal{P}$  is closed under coproduct and retract

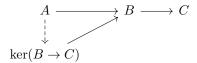
#### 1.2.4 AXIOMATISE FUNDAMENTAL THEOREM OF HOMOLOGICAL ALGEBRA

**Theorem 2** (fundamental theorem of homological algebra). Let C be a pointed category with kernels. Let  $(P, \mathcal{E})$  be a projective class in C.

1. For any  $M \in \text{ob } \mathcal{C}$ , there is an exact sequence w.r.t  $(\mathcal{P}, \mathcal{E})$  ( $\mathcal{P}$ -projective resolution)

$$0 \longleftarrow M \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \dots$$

where  $P_n \in \mathcal{P}$  and exactness  $A \to B \to C$  is defined by the composition being the zero map and  $A \to B$  is factored through  $\ker(B \to C)$  by an epimorphism, that is, there exists an epimorphism  $A \to \ker(B \to C)$  such that the diagram below commutes



2. Any map  $f: M \to M'$  lifts to maps of projectives

$$0 \longleftarrow M \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \dots$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longleftarrow M' \longleftarrow P'_0 \longleftarrow P'_1 \longleftarrow \dots$$

Furthermore, if C is preadditive, the lifts are unique upto chain homotopy.

Proof. TODO

#### 1.2.5 YONEDA LEMMA

**Definition 30** (functor category, evaluation functor). Let A, B be categories, the functor category  $\operatorname{Fun}(A, B)$  is defined by functors  $A \to B$  as objects and natural transformations as morphisms. Let  $X \in \operatorname{ob} A$ , define the evaluation functor  $\operatorname{Ev}_X : \operatorname{Fun}(A, B) \to B$  by

$$\operatorname{Ev}_X(F) = F(X)$$

where  $F \in \text{ob Fun}(A, B)$ . define application of morphism

**Definition 31** (hom functor). Let C be a locally small category, for any  $A \in \text{ob } C$ , the hom functor  $h_A : C \to \text{Set is defined } by$ 

$$X \xrightarrow{h_A} C(A, X)$$

$$f \downarrow \qquad \qquad \downarrow h_A(f)$$

$$Y \xrightarrow{h_A} C(A, Y)$$

where  $h_A(f)(g) = fg$  for any  $g \in C(A, X)$ .  $h_A$  is also written as C(A, -)

Remark 9. Some remarks on hom functor

• The singular chain complex functor  $C_n : \text{Top} \to \text{Ab}$  can be defined by

$$C_n = \mathbb{Z}\operatorname{Top}(\Delta^n, -)$$

**Theorem 3** (Yoneda lemma). Given a locally small category C and  $A \in \text{ob } C$ . Let  $F : C \to \text{Set } be$  a functor, then both  $h_A$  and F are objects in Fun(C, Set) and the set of morphisms  $h_A \to F$  in Fun(C, Set) is isomorphic to F(A),

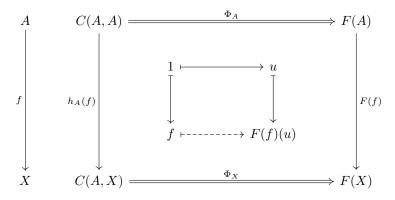
$$\operatorname{Nat}(h_A, F) \cong F(A)$$

where  $Nat(h_A, F)$  denotes the set of natural transformation from  $h_A$  to F. Since,  $F(A) = Ev_A(F)$ , then

$$\Phi: \operatorname{Hom}(A,-) \to \operatorname{Ev}_A$$

is a natural isomorphism. We say that  $\operatorname{Ev}_A$  is corepresentable.  $(\Phi, A)$  is called corepresentation of  $\operatorname{Ev}_A$  Proof.

Let  $\Phi: h_A \to F$  be a natural transformation



By commutativity,  $\Phi_X: C(A,X) \to F(X)$  is defined by  $f \mapsto F(f)(u)$ . That is,

1. for each  $u \in F(A)$ , there is a unique natural transformation  $\Phi$  defined by

$$\Phi_X(f) = F(f)(u)$$

2. for each natural transformation  $\Phi$ , there is unique  $u \in F(A)$  defined by

$$u = \Phi_A(1)$$

Remark 10. Some remarks on Yoneda lemma

- Take C be a category such objects are sets, take F to be the forgetful functor and  $A \in ob C$ . Then, the set of natural transformations  $h_A \to F$  is exactly A
- in C = Set, each functor  $F : C \to \text{Set}$  corresponds to a natural transformation  $1 \to F$ . From Yoneda lemma, for each natural transformation  $1 \to F$ , the map  $X \to F(X)$  is factored through  $X \to C(X,X)$  defined by  $x \mapsto 1$

**Lemma 1.** Let C be any category, then Fun(C, Ab) is preadditive. That is, given any two functors  $G, F : C \to Ab$ , the class of natural transformations of  $G \to F$  is an abelian group.

**Proposition 5** (Yoneda lemma for Ab-enriched functor). Given C is preadditive, for any  $A \in \text{ob } C$  and functor  $F: C \to \text{Ab}$ , then we have an isomorphism between abelian groups

$$Nat(h_A, F) \cong F(A)$$

Corollary 1. Given  $X, Y \in \text{ob } C$ ,  $X \cong Y$  if and only if  $C(X, -) \cong C(Y, -)$ 

#### 1.2.6 ACYCLIC MODEL

**Remark 11** (kernel of natural transformation between Ab-enriched functors). Given Fun(C, Ab) is pointed, has kernels, and preadditive. Let  $F, G \in \text{Fun}(C, \text{Ab})$ , and  $v : G \to F$  be a natural transformation, then

$$\ker(v:G\to F)(X) = \ker(v_X:G(X)\to F(X))$$

where  $X \in \text{ob } C$ 

**Definition 32** (acyclic models). Let  $\mathcal{M}$  be any set of objects in C ("models"). It defines a projective class in  $\operatorname{Fun}(C,\operatorname{Ab})$  as follows: an object is projective (relative to  $\mathcal{M}$ ) if it is a retract of a coproduct of  $\mathbb{Z}C(M,-)s$ . A morphism  $Y:G\to F$  is an epimorphism (relative to  $\mathcal{M}$ ) if  $G(M) \twoheadrightarrow F(M)$  is surjective for all  $M\in \mathcal{M}$ 

Remark 12. Some remarks on acyclic model

- $\mathbb{Z}C(M,-)$  is analogous to free  $\mathbb{Z}$ -module (free abelian group), then the coproduct is direct product and retract is direct summand (A is a summand of B if  $B = A \oplus C$ )
- C = Top,  $\mathcal{M} = \{\Delta^n : n \geq 0\}$ , this models  $\mathcal{M}$  recovers projectives being singular n-chains and left derive functor of evaluation map at X of  $H_0$  is  $H_n$  (lecture 16)

*Proof.* We will show the that the construction is indeed a projective class in Fun(C, Ab) by the following

1. If H is a projective and  $G \to F$  epimorphism, then there is a lift from  $H \to F$  to  $H \to G$  (forward direction of (1) and (2))

It suffices to show for the case if H = C(M, -).

$$C(M,-) \longrightarrow F \qquad F(M)$$

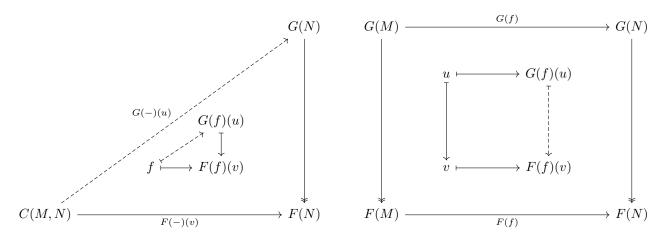
By Yoneda lemma, let natural transformation  $C(M,-) \to F$  determined by  $v \in F(M)$ 

$$C(M,N) \to F(N)$$
  
 $f \mapsto F(f)(v)$ 

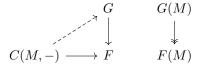
Since  $G(M) \to F(M)$  is surjective, let  $u \in G(M)$  such that  $u \mapsto v$  by  $G(M) \to F(M)$ . Construct the natural transformation  $C(M, -) \to G$  by

$$C(M,N) \to G(N)$$
  
 $f \mapsto G(f)(u)$ 

The lift make diagram commutes is due to naturality of  $G \to F$  as in the diagram below



2. If for every projective H there is a lift from  $H \to F$  to  $H \to G$ , then  $G \to F$  epimorphism (backward direction of (2)) Let H = C(M, -),



FOR LATER - TOO LONG

#### 1.3 EILENBERG-STEENROD-MILNOR AXIOMS

**Definition 33** (category of pairs of topological spaces). The category of pairs of topological spaces Top<sub>2</sub> is defined as follows

- objects: pair (X, A) where  $A \subseteq X$
- morphisms: continuous map  $f:(X,A)\to (Y,B)$  that is a continuous map  $f:X\to Y$  such that  $f(A)\subseteq B$
- composition law for morphisms: compose a set maps

**Axiom 1** (Eilenberg-Steenrod-Milnor axioms). Let C be a category of pairs of topological spaces (X, A) such that if  $(X, A) \in O$  ob C, then  $(A, \varnothing) \to (X, \varnothing) \to (X, A)$  are also in C. A homology theory on C is a functor  $H_n : OD C \to AD$ ,  $n \in \mathbb{Z}$  and a natural transformation (connecting homomorphism)  $\partial : H_n(X, A) \to H_{n-1}(A, \varnothing)$  such that

- 1. **homotopy axiom**: if  $f \simeq g$ , then  $H_n(f) = H_n(g)$  where  $f, g : (X, A) \to (Y, B)$  are homotopic if they are homotopic as both a function  $X \to Y$  and  $A \to B$ .
- 2. exactness axiom: for all  $(X, A) \in ob C$ , the sequence below is exact

$$H_n(A) \xrightarrow{\dots} H_{n+1}(X, A)$$

$$H_n(X) \xrightarrow{\dots} H_n(X, A)$$

$$H_{n-1}(A) \xrightarrow{\dots} \dots$$

where  $H_n(X)$  denotes  $H_n(X,\emptyset)$ 

- 3. excision axiom: if  $(X, A) \in \text{ob } C$  and  $U \subseteq \overline{U} \subseteq \text{int } A$  and  $(X U, A U) \hookrightarrow (X, A)$  in C, then  $H_n(X U, A U) \cong H_n(X, A)$
- 4. additivity axiom (Milnor): homology commutes with coproduct. That is, let  $\{X_{\alpha} : \alpha \in \mathcal{A}\}$  be a set of topological spaces. If  $(X_{\alpha}, \varnothing) \in \text{ob } C$  for all  $\alpha \in \mathcal{A}$  and  $(\coprod X_{\alpha}, \varnothing) \in \text{ob } C$ , then

$$\bigoplus_{i_{\alpha}} H_n(X_{\alpha}, \varnothing) \xrightarrow{\cong} H_n(\coprod X_{\alpha}, \varnothing)$$

$$H_n(X_{\alpha}, \varnothing)$$

5. dimension axiom:  $H_n(*,\varnothing) = \begin{cases} \mathbb{Z}, & n=0 \\ 0, & n \neq 0 \end{cases}$ 

Remark 13. Some remarks on Eilenberg-Steenrod axioms

- we omit dimension axiom: extraordinary cohomology theory.
- varying of dimension axiom: homology with coefficients. Let A be any abelian group,  $H_n(*,\varnothing;A) = \begin{cases} A, & n=0\\ 0, & n\neq 0 \end{cases}$

### Chapter 2

### ALGEBRAIC TOPOLOGY

#### 2.1 SINGULAR HOMOLOGY

#### 2.1.1 DEFINITION OF SINGULAR HOMOLOGY

**Definition 34** (n-simplex, barycentric coordinate, barycentre). The standard n-simplex is the convex hull of  $e_0, e_1, ..., e_n$  in  $\mathbb{R}^{n+1}$  where  $e_0, e_1, ..., e_n$  are the standard basis vectors, denoted by  $\Delta^n$ ,

$$\Delta^n = \{ x \in \mathbb{R}^{n+1} : \sum_{i=0}^n x_i = 1, x_i \ge 0 \}$$

We also write  $\Delta^n = [e_0, ..., e_n] = [0, ..., n]$ .  $\{x_0, ..., x_n\}$  is called the barycentric coordinate. The point  $b = \left(\frac{1}{n+1}, ..., \frac{1}{n+1}\right)$  is called the barycentre.

**Definition 35** (singular n-simplex). A singular n-simplex in X is a continuous function  $\sigma: \Delta^n \to X$ . The set of singular n-simplicies in X is denoted by  $S_n(X)$ 

**Definition 36** (singular n-chain). A singular n-chain is an element of the free abelian group  $C_n(X) = \mathbb{Z}[S_n(X)]$  generated by  $S_n$ . Alternatively, n-chain is a finite formal linear combination of singular n-simplicies with integer coefficients. If  $n \leq 0$ ,  $C_n(X) = 0$  is defined to be the trivial group.

**Definition 37** (face map). The face map  $d^i: \Delta^{n-1} \to \Delta^n$  is the affine map sending vertices to vertices, in order, missing i. That is,

$$d^{i}(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \ge i \end{cases}$$

**Proposition 6.** If i < j, then  $d^j \circ d^i = d^i \circ d^{j-1}$ . Equivalently,  $d_i d_j = d_{j-1} d_i$ 

**Definition 38** (boundary operator). Define the boundary operator  $\partial: C_n(X) \to C_{n-1}(X)$  as a linear extension of  $\partial: S_n(X) \to C_{n-1}(X)$ 

$$\partial \sigma = \sum_{i=0}^{n} (-1)^{i} \sigma \circ d^{i} = \sum_{i=0}^{n} (-1)^{i} d_{i} \sigma \in C_{n-1}(X)$$

where  $\sigma \in S_n(X)$ .

**Proposition 7.**  $\partial^2 = 0$ 

$$C_{n-1}(X) \stackrel{\partial}{\longleftarrow} C_n(X) \stackrel{\partial}{\longleftarrow} C_{n+1}(X)$$

**Proposition 8** (singular homology). The singular n-chains  $C_n(X)$  together with the boundary operator  $\partial$  is a chain complex, written as  $C_{\bullet}(X)$ . The corresponding n-homology group is denoted by  $H_n(X)$ .

#### 2.1.2 PATH-COMPONENT DECOMPOSITION

Define  $x \sim y$  if and only if there exists a path in X from x to y, then  $\sim$  is an equivalent relation. Each equivalence class of  $\sim$  is called a path component of X. The set of path components of X is denoted by  $\pi_0(X)$ . Since each singular n-simplex  $\sigma: \Delta^n \to X$  land entirely in a single path component, we have

$$S_n(X) = \coprod_{\alpha \in \pi_0(X)} S_n(X_\alpha)$$

where  $X_{\alpha}$  denotes the subspace induced by  $\alpha$ , and

$$C_n(X) = \bigoplus_{\alpha \in \pi_0(X)} C_n(X_\alpha)$$

**Proposition 9.**  $H_n(X) \cong \bigoplus_{\alpha \in \pi_0(X)} H_n(X_\alpha)$ 

#### **2.1.3** SOME SPECIAL CASES: $H_0, H_n(*)$

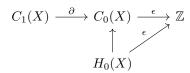
Assume X is path connected, then  $Z_0(X) = C_0(X)$ , and  $H_0(X) = \frac{Z_0(X)}{B_0(X)} = \frac{C_0(X)}{\operatorname{im}(\partial:C_1(X) \to C_0(X))} = \operatorname{coker} \partial C_0(X)$ 

**Definition 39** (augmentation map). The augmentation map  $\epsilon: C_n(X) \to \mathbb{Z}$  is defined by

$$\epsilon \left( \sum_{x \in S_n(X)} a_x x \right) = \sum_{x \in S_n(X)} a_x$$

**Lemma 2.** Let  $\epsilon: C_0(X) \to \mathbb{Z}$ , then  $\epsilon \partial = 0$ 

*Proof.* For each  $\sigma \in S_1(X)$ , then  $\epsilon \partial \sigma = \epsilon (d_0 \sigma - d_1 \sigma) = \epsilon d_0 \sigma - \epsilon d_1 \sigma = 1 - 1 = 0$ .



**Lemma 3.** If X is path connected, then the augmentation map induces an isomorphism  $\epsilon: H_0(X) \to \mathbb{Z}$ 

Proof.

(well-defined) Let a and  $a + \partial b$  be two elements from the same cosets of  $B_0(X)$ , then

$$\epsilon[a + \partial b] = \epsilon(a + \partial b) = \epsilon a = \epsilon[a]$$

(homomorphism)

$$\epsilon([a] + [b]) = \epsilon[a + b] = \epsilon(a + b) = \epsilon a + \epsilon b = \epsilon[a] + \epsilon[b]$$

(surjective) Let any  $x \in S_0(X), n \in \mathbb{Z}$ , then

$$\epsilon[nx] = \epsilon(nx) = n\epsilon x = n$$

(injective) As  $\epsilon$  is a homomorphism, we just need to prove that the kernel of  $\epsilon$  is 0, that is, for all  $c \in C_0(X)$ , if  $\epsilon c = 0$ , then [c] = 0, that is,  $c \in B_0(X)$ . Let  $c = \sum_{x \in S_0(X)} a_x x$  and  $\epsilon c = 0$ , then  $\sum_{x \in S_0(X)} a_x = 0$ . We can rewrite

$$c = \sum_{x \in S_0(X)} a_x x = (x_1 + \dots + x_n) - (y_1 + \dots + y_n)$$

where  $x_1, ..., x_n, y_1, ..., y_n \in S_0(X)$ . For each i = 1, ..., n, pick  $\sigma_i \in S_1(X)$  such that  $d_0\sigma_i = x_i$  and  $d_1\sigma_i = y_i$  (pick  $\sigma_i : \Delta^1 \to X$  mapping the end points of  $\Delta^1$  to  $x_i$  and  $y_i$ ), then  $\sum_{i=1}^n \sigma_i \in C_1(X)$ , and

$$\partial \left(\sum_{i=1}^{n} \sigma_{i}\right) = \sum_{i=1}^{n} (d_{0}\sigma_{i} - d_{1}\sigma_{i}) = \sum_{i=1}^{n} (x_{i} - y_{i}) = c$$

Corollary 2.  $H_0(X) = \bigoplus_{a \in \pi_0(X)} \mathbb{Z}$  for any space X

**Proposition 10.**  $H_n(*) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}$  where \* denotes the space of one point.

*Proof.* For any  $n = 0, 1, ..., S_n(*) = *$  then  $C_n(*) = \mathbb{Z}$ . For n > 0, and  $* \in S_n(X)$ ,

$$\partial * = \sum_{i=0}^{n} (-1)^i d_i * = \sum_{i=0}^{n} (-1)^i * = \begin{cases} 0, & n \text{ odd} \\ *, & n \text{ even} \end{cases}$$

We have the diagram

$$0 \stackrel{\partial=0}{\longleftarrow} C_0(*) \stackrel{\partial=0}{\longleftarrow} C_1(*) \stackrel{\partial=1}{\longleftarrow} C_2(*) \stackrel{\partial=0}{\longleftarrow} C_3(*) \stackrel{\partial=1}{\longleftarrow} \dots$$

 $H_0(*) = \mathbb{Z}$  by the previous explanation. If n > 0 (n odd or even),  $Z_n(*) = B_n(*) = C_n(*) = \mathbb{Z}$ , then  $H_n(*) = 0$ .

#### 2.1.4 INDUCED FUNCTION BETWEEN TWO TOPOLOGICAL SPACES

**Proposition 11** (naturality of induced map). A function  $f: X \to Y$  induces a function  $f_\#: C_n(X) \to C_n(Y)$  defined by a linear extension of  $f_\#: S_n(X) \to S_n(Y)$  with  $f_\# \sigma = f \sigma$ 

$$\begin{array}{ccc}
\Delta^n & \xrightarrow{\sigma} & X \\
\downarrow^f \\
f_\#\sigma = f\sigma & \downarrow^f \\
V
\end{array}$$

Then,  $f_{\#}\partial = \partial f_{\#}$ , that is, the following square commutes

$$C_n(X) \xrightarrow{\partial} C_{n-1}(X)$$

$$\downarrow f_{\#} \qquad \qquad \downarrow f_{\#}$$

$$C_n(Y) \xrightarrow{\partial} C_{n-1}(Y)$$

*Proof.* Let  $\sigma \in S_n(X)$ , then

$$f_{\#}\partial\sigma = f_{\#} \sum_{i=0}^{n} (-1)^{i} \sigma d^{i} = f \sum_{i=0}^{n} (-1)^{i} \sigma d^{i} = \sum_{i=0}^{n} (-1)^{i} f \sigma d^{i} = \partial f_{\#}\sigma$$

**Remark 14** (chain map). The function  $f_{\#}: C_n(X) \to C_n(Y)$  induced from  $f: X \to Y$  is a chain map of from chain complex  $C_{\bullet}(X)$  to chain complex  $C_{\bullet}(Y)$ . The induced map in homology is written as  $H_n(f) = f_*: H_n(A) \to H_n(B)$ 

#### **2.1.5 HOMOTOPY**

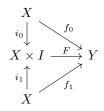
**Definition 40** (homotopy). Let  $f_0, f_1 : X \to Y$  be continuous maps. A homotopy from  $f_0$  to  $f_1$  is a continuous map  $F : X \times I \to Y$  where I = [0,1] such that  $F(x,0) = f_0(x)$  and  $F(x,1) = f_1(x)$ . Being homotopic is an equivalence relation on maps:  $(X,Y) \to [X,Y]$  where (X,Y) denotes the set of all maps from X to Y and [X,Y] denotes the equivalence classes of homotopy of maps from X to Y.  $f_0$  and  $f_1$  being homotopic is denoted by  $f_0 \simeq f_1$ .

**Theorem 4** (homotopy invariance). Two homotopic maps induce the same map in homology. That is, if  $f_0 \simeq f_1$ , then  $H_n(f_0) = H_n(f_1) : H_n(X) \to H_n(Y)$  for all n

Proof. Consequence of Proposition 3 and Proposition 13

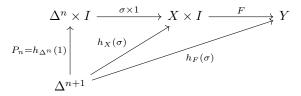
#### prism operator

Let  $f_0, f_1: X \to Y$  be homotopic, the diagram below commutes



where  $i_0: x \mapsto (x,0)$ , and  $i_1: x \mapsto (x,1)$ 

**Definition 41** (prism operator). Given homotopy  $F: X \times I \to Y$ , the prism operator  $h_F: C_n(X) \to C_{n+1}(Y)$  is defined by a linear extension of  $h_F: S_n(X) \to C_{n+1}(Y)$  as follows. Let  $\sigma \in S_n(X)$ ,

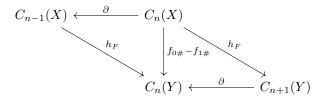


Let  $[v_0,...,v_n],[w_1,...,w_n]$  be the bases of the prism  $\Delta^n \times I$ . Define  $\sigma_i:\Delta^{n+1} \to \Delta^n \times I$  be the affine map from  $\Delta^{n+1}$  to the simplex  $[v_0,v_1,...,v_i,w_i,w_{i+1},...,w_n]$ . Define  $P_n=\sum_{i=0}^n (-1)^i\sigma_i$ , and  $h_F(\sigma)=F(\sigma\times 1)P_n$  is then defined accordingly.

**Proposition 12.**  $\partial P_n = -P_{n-1}\partial + i_1 - i_0$  and  $\partial h_F = -h_F\partial + f_1 - f_0$  where  $i_1 : \Delta^n \mapsto (\Delta^n, 1), i_0 : \Delta^n \mapsto (\Delta^n, 0)$ 

Proof. TODO

**Proposition 13.** Two homotopic maps induce chain homotopic maps. That is, if  $f_0 \simeq f_1 : X \to Y$ , then  $f_{0\#}, f_{1\#} : C_n(X) \to C_n(Y)$  are chain homotopic



Proof. Proposition 12

**Definition 42** (homotopy equivalence, homotopy type). A map  $f: X \to Y$  is a homotopy equivalence if there exists  $g: Y \to X$  such that  $1_Y \simeq fg: Y \to Y$  and  $1_X \simeq gf: X \to X$ . If there exists a homotopy equivalence between X and Y, X and Y are said to be of the same homotopy type.

**Definition 43** (contractible). X is contractible if  $X \to *$  is a homotopy equivalence

Corollary 3. Any homotopy equivalence induces an isomorphism in homology

*Proof.* As  $fg \simeq 1_Y$ , then  $H_n(f)H_n(g) = 1_{H_n(Y)}$ . Similarly,  $H_n(g)H_n(f) = 1_{H_n(X)}$ . Therefore,  $H_n(f)$  is an isomorphism  $\square$ 

#### 2.1.6 LOCALITY OF $H_*(-)$

Let  $\mathcal{U}$  be an open cover of a topological space X.

**Definition 44** (*U*-small). A singular simplex  $\sigma: \Delta^n \to X$  is *U*-small if its image lies entirely in a single open set  $U \in \mathcal{U}$ . Let  $C_n^{\mathcal{U}}(X)$  denote the subgroup of  $C_n(X)$  generated by *U*-small singular simplicies, namely *U*-small n-chains

$$C_n^{\mathcal{U}}(X) \longrightarrow C_n(X)$$

$$\downarrow_{\partial} \qquad \qquad \downarrow_{\partial}$$

$$C_{n-1}^{\mathcal{U}}(X) \longrightarrow C_{n-1}(X)$$

The boundary map, homology on U-small chains is defined accordingly.

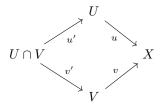
**Theorem 5** (locality principle). The inclusion map  $C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$  is a chain homotopy equivalence

Corollary 4.  $H_n(C_n^{\mathcal{U}}(X)) = H_n(C_n(X)) = H_n(X)$ 

*Proof.* chain homotopy equivalence induces isomorphism in homology

#### 2.1.7 LONG EXACT SEQUENCE IN HOMOLOGY

Special case:  $\mathcal{U} = \{U, V\}, X = U \cup V, U, V \text{ open. Define } u, v, u', v' \text{ being the inclusion maps.}$ 



**Proposition 14.** The diagram below is a short exact sequence

$$C_n(U \cap V) \xrightarrow{i = \begin{bmatrix} u'_{\#} \\ -v'_{\#} \end{bmatrix}} C_n(U) \oplus C_n(V) \xrightarrow{p = \begin{bmatrix} u_{\#}, v_{\#} \end{bmatrix}} C_n^{\mathcal{U}}(X)$$

$$\downarrow \subseteq$$

$$C_n(X)$$

where  $i = c \mapsto (c, -c)$  and  $p = (c_1, c_2) \mapsto c_1 + c_2$ 

**Theorem 6** (Mayer-Vietoris sequence). Special case:  $U = \{U, V\}$ ,  $X = U \cup V$ , U, V open. There exists a long exact sequence

$$H_{n}(U \cap V) \stackrel{i_{*}}{\longleftarrow} H_{n}(U) \oplus H_{n}(V) \stackrel{p_{*}}{\longrightarrow} H_{n}(C_{n}^{\mathcal{U}}(X))$$

$$H_{n-1}(U \cap V) \stackrel{i_{*}}{\longleftarrow} H_{n-1}(U) \oplus H_{n-1}(V) \stackrel{p_{*}}{\longrightarrow} H_{n-1}(C_{n-1}^{\mathcal{U}}(X))$$

$$\downarrow 0$$

#### 2.1.8 PROOF OF LOCALITY PRINCIPLE

#### barycentric subdivision

**Definition 45** (cone operator). Given a star-shaped subset D of  $\mathbb{R}^n$  (there exists a point p such that the line segment from p to any point  $x \in D$  belongs to D). The cone operator  $c_p : C_n(D) \to C_{n+1}(D)$  is defined by a linear extension of  $c_p : S_n(D) \to C_{n+1}(D)$ . Let  $\sigma \in S_n(D)$ , then  $c_p(\sigma)$  is defined by

$$c_p(\sigma): (x_0, x_1, ..., x_{n+1}) \mapsto x_0 p + (1 - x_0) \sigma(\overline{x_1}, ..., \overline{x_{n+1}})$$

where  $\overline{x_i} = \frac{x_i}{x_1 + \dots + x_{n+1}}$ 

**Definition 46** (subdivision operator). The subdivision operator  $\operatorname{Sd}_n: C_n(\Delta^n) \to C_n(\Delta^n)$  is defined by

$$Sd_0 = 1$$

$$Sd_n = c_{b(n)} Sd_{n-1} \partial$$

where  $c_{b(n)}$  is the cone operator with respective to the barycentre of  $\Delta^n$ 

**Lemma 4.** Given a chain  $s \in C_n(\Delta^p)$ , we can define a unique operator, a linear transformation

$$S^{\bullet}: C_n(\bullet) \to C_n(\bullet)$$

such that

- 1.  $S^{\Delta^p}(1) = s$  where  $1: \Delta^p \mapsto \Delta^p$  is the identity simplex
- 2. for any map  $f: X \to Y$ , the diagram below commutes

$$\begin{array}{ccc} C_p(X) & \xrightarrow{S^X} & C_n(X) \\ f_\# \downarrow & & f_\# \downarrow \\ C_p(Y) & \xrightarrow{S^Y} & C_n(Y) \end{array}$$

*Proof.* (Uniqueness of  $S^X$ )

$$C_{p}(\Delta^{p}) \xrightarrow{S^{\Delta^{p}}} C_{n}(\Delta^{p}) \qquad 1 \xrightarrow{S^{\Delta^{p}}} s$$

$$\sigma_{\#} \downarrow \qquad \sigma_{\#} \downarrow \qquad \int_{\sigma_{\#}} \sigma_{\#} \qquad \int_{\sigma_{\#}} f_{\#}$$

$$C_{p}(X) \xrightarrow{S^{X}} C_{n}(X) \qquad \sigma \xrightarrow{S^{X}} S^{X} \sigma_{\#} 1 = \sigma_{\#} S^{\Delta^{p}} 1$$

Let  $\sigma: \Delta^n \to X$ , the commutativity implies the unique definition of  $S^X$  by  $S^X \sigma = S^X \sigma_\# 1 = \sigma_\# S^{\Delta^p} 1 = \sigma_\# s$ 

**Definition 47** (subdivision operator). Given  $\operatorname{Sd}_n: C_n(\Delta^n) \to C_n(\Delta^n)$ , by Lemma 4, naturality of  $S^X$  induces a unique map  $\operatorname{Sd}_n: C_n(X) \to C_n(X)$  such that for all  $\sigma: \Delta^n \to X$ ,  $\operatorname{Sd}_n \sigma = \operatorname{Sd}_n \sigma_\# 1 = \sigma_\# \operatorname{Sd}^n 1$  where  $1: \Delta^n \to \Delta^n$  is the identity simplex.

**Proposition 15** (boundary of cone operator). Given the cone operator  $c_p: C_n(D) \to C_{n+1}(D)$  then its boundary is

$$\partial c_p = \begin{cases} 1 - c_p \partial, & n > 0 \\ 1 - p\epsilon, & n = 0 \end{cases}$$

where  $\epsilon: C_0(X) \to \mathbb{Z}$  is the augmentation map

**Proposition 16.**  $Sd_n: C_n(X) \to C_n(X)$  is a chain map, that is  $Sd_n \partial = \partial Sd_{n-1}$  apply formula

Proposition 17. Sd is chain homotopic to 1

Proof.

Let defined  $T_n: C_n(X) \to C_{n+1}(X)$  as follows:

$$T_0 = 0$$
  

$$T_n = c_{b(n)}(1 - \operatorname{Sd}_n - T_{n-1}\partial)$$

We will prove that  $\partial T_n = 1 - \operatorname{Sd}_n - T_{n-1}\partial$  by induction. Base case: n = 1, then

$$\partial T_1 = \partial c_{b(1)} (1 - \operatorname{Sd}_1 - T_0 \partial) 
= (1 - c_{b(1)} \partial) (1 - \operatorname{Sd}_1) 
= 1 - \operatorname{Sd}_1 - c_{b(1)} \partial + c_{b(1)} \partial \operatorname{Sd}_1 
= 1 - \operatorname{Sd}_1 - c_{b(1)} \partial + c_{b(1)} \operatorname{Sd}_0 \partial 
= 1 - \operatorname{Sd}_1 - c_{b(1)} \partial + c_{b(1)} \partial 
= 1 - \operatorname{Sd}_1 
= 1 - \operatorname{Sd}_1 
= 1 - \operatorname{Sd}_1 - T_0$$

Induction case, suppose  $\partial T_{n-1} = 1 - \operatorname{Sd}_{n-1} - T_{n-2}\partial$ , then

$$\begin{split} \partial T_n &= (1 - c_p \partial) (1 - \operatorname{Sd}_n - T_{n-1} \partial) \\ &= (1 - \operatorname{Sd}_n - T_{n-1} \partial) - c_p (\partial - \partial \operatorname{Sd}_n - \partial T_{n-1} \partial) \\ &= (1 - \operatorname{Sd}_n - T_{n-1} \partial) - c_p (\partial - \partial \operatorname{Sd}_n - (1 - \operatorname{Sd}_{n-1} - T_{n-2} \partial) \partial) \\ &= 1 - \operatorname{Sd}_n - T_{n-1} \partial \end{split}$$

**Proposition 18.**  $\operatorname{Sd}^k: C_{\bullet}(X) \to C_{\bullet}(X)$  is a chain map

*Proof.* Let g be a chain map, then  $\partial gg = g\partial g = gg\partial$ 

Corollary 5. 1 and  $Sd^k$  are chain homotopic

*Proof.* Let  $G = 1 + \text{Sd} + \text{Sd}^2 + ... + \text{Sd}^{k-1}$ , then

$$1 - \mathrm{Sd}^k = G(1 - \mathrm{Sd}) = G(\partial T + T\partial) = \partial(GT) + (GT)\partial$$

Theorem 7 (locality principle).

Proof.

TODO - assignment 1

#### 2.1.9 RELATIVE HOMOLOGY

**Definition 48** (pair, quotient chain complex, relative homology). A pair (X, A) is a space X together with a subspace  $A \subseteq X$ 

$$0 \longrightarrow C_n(A) \hookrightarrow C_n(X) \longrightarrow C_n(X)/C_n(A) \longrightarrow 0$$

$$\downarrow \partial \qquad \qquad \downarrow \partial \qquad \qquad \downarrow \partial$$

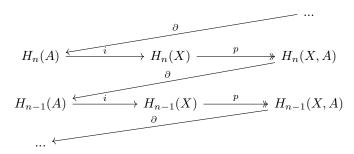
$$0 \longrightarrow C_{n-1}(A) \hookrightarrow C_{n-1}(X) \longrightarrow C_{n-1}(X)/C_{n-1}(A) \longrightarrow 0$$

$$\dots \hookrightarrow c + a \longrightarrow [c] \longrightarrow 0$$

$$\downarrow \partial \qquad \qquad \downarrow \partial$$

$$\dots \hookrightarrow \partial c + \partial a \longrightarrow [\partial c] \longrightarrow 0$$

The boundary map on quotient chain complex  $C_n(X)/C_n(A)$  is induced from the boundary map on chain complex  $C_n(X)$  such that the diagram remains commutative. The quotient chain complex is also denoted by  $C_n(X,A) = C_n(X)/C_n(A)$ . The relative homology is defined by the homology on  $C_n(X,A)$  and denoted by  $H_n(X,A) = H_n(C_n(X,A))$ . The short exact sequence  $C_n(A) \stackrel{\iota}{\smile} C_n(X) \stackrel{p}{\longrightarrow} C_n(X)/C_n(A)$  induces a long exact sequence



**Remark 15.**  $C_n(X,\varnothing) = C_n(X)$ , then  $H_n(X,\varnothing) = H_n(X)$ 

**Definition 49** (map of pairs).  $f:(X,A)\to (Y,B)$  is a map of pairs of  $f:X\to Y$  and  $f(A)\subseteq B$ 

**Definition 50** (naturality of induced map in pair). Let  $f:(X,A)\to (Y,B)$ , the induced map  $f_\#:C_n(X,A)\to C_n(Y,B)$  is well-defined since the map  $f_\#:C_n(X)\to C_n(Y)$  takes  $C_n(A)$  to  $C_n(B)$ . The naturality is followed by naturality in absolute chains

$$C_n(X,A) \xrightarrow{\partial} C_{n-1}(X,A)$$

$$\downarrow f_{\#} \qquad \qquad \downarrow f_{\#}$$

$$C_n(Y,B) \xrightarrow{\partial} C_{n-1}(Y,B)$$

**Theorem 8** (excision). Let (X, A) be a pair and  $U \subseteq X$  such that  $\overline{U} \subseteq \text{int } A$ . Then,

$$H_n(X-U,A-U) \cong H_n(X,A)$$

Proof.

TODO

Proof.

TODO

**Lemma 5** (five lemma). Given  $A \to B \to C \to D \to E$  and  $A' \to B' \to C' \to D' \to E'$  exact. Then

 $C \to C'$  is a monomorphism if

- ullet  $D \to D'$  is a monomorphism
- ullet  $B \to B'$  is a monomorphism
- $A \rightarrow A'$  is a epimorphism

 $C \to C'$  is a epimorphism if

- $B \to B'$  is a epimorphism
- $D \rightarrow D'$  is a epimorphism
- $E \to E'$  is a monomorphism

Proof.

TODO

**Definition 51** (pointed space, reduced homology). A pointed space is a pair (X, \*) such that \* is a single point. The reduced homology is a pointed space is denoted by  $\overline{H}_n(X) = H_n(X, *)$ 

**Proposition 19.** Some propositions on reduced homology

- $H_0(X) = \overline{H}_0(X) \oplus \mathbb{Z}$
- $H_n(X) = \overline{H}_n(X)$  for  $n \ge 1$

#### 2.1.10 EXPRESSING RELATIVE HOMOLOGY IN TERMS OF REDUCE HOMOLOGY

**Definition 52** (cone). Let A be a topological space, the cone on A is defined by

$$CA = A \times [0,1]/A \times \{0\}$$

Note:  $A/\emptyset = A \coprod *$ , then  $C\emptyset = *$ 

**Lemma 6.** The cone on any space A is contractible, that is  $CA \simeq *$ . In other words, any space can be embedded into a contractible space.

Proof.

TODO

**Definition 53** (cone of the pair). The cone of the pair (X, A) is

$$X \cup_A CA = X \coprod CA/a \sim (a,1)$$

where  $a \sim (a, 1)$  denotes the equivalence relation between point  $a \in A$  and point  $(a, 1) \in CA$ 

**Proposition 20.**  $H_n(X,A) = \overline{H}_n(X \cup_A CA)$ 

Proof.

TODO

**Definition 54** (suspension). The cone of pair (CA, A), denoted by  $\Sigma A$ 

#### 2.2 CELLULAR HOMOLOGY

closure-finite weak topology

**Definition 55** (cell complex). A cell complex is a space X with a filtration  $X_0 \subseteq X_1 \subseteq ... \subseteq X$  by subspaces such that

- 1.  $X_0$  is discrete
- 2. for each n,  $X_n$  is the pushout of the diagram below

$$\coprod_{\alpha \in \mathcal{A}_n} \partial D^n \longleftrightarrow \coprod_{\alpha \in \mathcal{A}_n} D^n$$

$$\downarrow \qquad \qquad \downarrow$$

$$X_{n-1} \longleftrightarrow X_n$$

where  $D^n$  is a space homeomorphic to an n-dimensional disk,  $\partial D^n$  is the boundary of  $D^n$   $(S^{n-1})$ .  $X_n$  is called the n-skeleton of X,  $\coprod \partial D^n \to X_{n-1}$  is called the attaching map,  $\coprod D^n$  is called the characteristic map.

3. 
$$X = \lim_{n \to \infty} X_n$$

Remark 16. Some remarks on cell complex

- $\mathbb{R}P^{n-1}$  is a cell complex with  $(\mathbb{R}P^{n-1})_k = \mathbb{R}P^k \subseteq \mathbb{R}P^n$ .  $\mathbb{R}P^\infty = \bigcup \mathbb{R}P^n = \lim \mathbb{R}P^n$
- $S^n$  is a cell complex
- Grassmannians,  $Gr_k(\mathbb{R}^n)$ , the set of k-dimensional subspace of  $\mathbb{R}^n$ , is a cell complex

**Definition 56** (finite type, finite dimensional). A cell complex  $X_0 \subseteq X_1 \subseteq ... \subseteq X$  is of finite type if  $X_0$  is finite and each  $A_n$  is finite  $(|X_0|)$  is the number of 0-cells,  $|A_n|$  is the number of n-cells). X is of finite dimensional if  $A_n = \emptyset$  for n >> 0. X is of finite if it is both finite type and finite dimensional.

#### 2.2.1 THE CONSTRUCTION OF CELLULAR HOMOLOGY

Let  $X_0$  be the pushout of the diagram below

$$\coprod_{\alpha \in \mathcal{A}_0} \partial D^0 = \varnothing \hookrightarrow \longrightarrow \coprod_{\alpha \in \mathcal{A}_0} D^0$$

$$\downarrow \qquad \qquad \downarrow$$

$$X_{-1} = \varnothing \hookrightarrow X_0$$

Then, from the definition of cell complex,

$$X_n/X_{n-1} = \prod_{\alpha \in \mathcal{A}_n} D^n / \prod_{\alpha \in \mathcal{A}_n} \partial D^n = \bigvee_{\alpha \in \mathcal{A}_n} S^n$$

where  $S^n = D^n/\partial D^n$  (why do the two quotient topologies coincide? think about it.  $X_n/X_{n-1}$  makes  $X_{n-1}$  to be one point). The wedge sum  $\vee$  is defined by follows: Let  $(A, *_A), (B, *_B)$  be pointed spaces, then

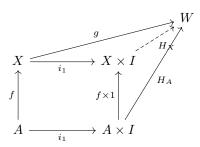
$$A \vee B = A \coprod B/*_A \sim *_B$$

Remark 17. Wedge sum is the coproduct in the category of pointed spaces

Let H be a homology functor satisfying Eilenberg-Steenrod axioms. Given  $A \subseteq X$ , as CA contractible,  $X \cup_A CA \simeq X/A$ , then  $\overline{H}_q(X \cup_A CA) \cong \overline{H}_q(X/A)$ . Therefore,  $H_q(X,A) = \overline{H}_q(X/A)$ 

$$H_a(X,A) \xrightarrow{\cong} \overline{H}_a(X \cup_A CA) \xrightarrow{\cong} \overline{H}_a(X/A)$$

**Definition 57** (cofibration). A map  $f: A \to X$  is a cofibration if the homotopy  $H_A: A \times I \to W$  can be extended to  $H_X: X \times I \to W$ , namely, homotopy extension property. That is, given the diagram below, there exists a map  $H_X: X \times I \to W$  such that the diagram below commutes



where  $i_1$  denotes the map  $x \mapsto (x,1)$ 

Proof. Let  $Z = X \cup_A (A \times I)$  be the pushout of  $X \leftarrow_f A \xrightarrow{i_1} A \times I$  where  $X \cup_A (A \times I) = X \coprod (A \times I) / \sim$  with  $f(a) \sim (a,1)$  for all  $a \in A$ . There exists a retraction  $r: X \times I \to Z$ . Define H = hr where  $h: Z \to W$  is the unique map from pushout Z to W

Lemma 7. Some lemmas on cofibration

- 1. the composite of cofibrations is a cofibration
- 2. the coproduct of (closed) cofibrations is a (closed) cofibration
- 3. (closed) cofibration is closed under cobase change, that is, if  $f:A\to X$  is a (closed) cofibration and  $Y=X\cup_A B$  is the pushout of  $B\xleftarrow{g}A\xrightarrow{f}X$ , then  $\overline{f}:B\to Y$  is a (closed) cofibration

$$\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow g & & \downarrow \\
B & \xrightarrow{\overline{f}} & Y
\end{array}$$

where  $X \cup_A B = X \coprod B / \sim$  with  $f(a) \sim g(a)$  for all  $a \in A$ 

Corollary 6.  $X_q \hookrightarrow X_n$  for  $q \leq n$  is a closed cofibration

*Proof.* the composite of cofibrations is a cofibration (inclusion map is a cofibration)

Corollary 7.  $H_q(X_n, X_{n-1}) = \overline{H}_q(X_n/X_{n-1}) = \overline{H}_q(\bigvee_{\alpha \in \mathcal{A}_n} S^n) = \bigoplus_{\alpha \in \mathcal{A}_n} \overline{H}_q(S^n) = \begin{cases} 0, & q \neq n \\ \mathbb{Z}\mathcal{A}_n, & q = n \end{cases}$  where  $\mathbb{Z}$  denotes the free abelian group functor, that is,  $\mathbb{Z}\mathcal{A}_n$  is the free abelian group generated by  $\mathcal{A}_n$ 

*Proof.* In the category of pointed space, morphisms are the continuous maps that fix the basepoint and wedge sum is the coproduct. As  $H_q$  is a functor on that category. Let X,Y be pointed spaces, we must have the coproduct of two abelian groups  $H_q(X)$  and  $H_q(Y)$  being  $H_q(X \vee Y)$ , that is,  $H_q(X) \oplus H_q(Y) = H_q(X \vee Y)$ 

The long exact sequence for  $(X_n, X_{n-1})$ 

$$H_{q+1}(X_n, X_{n-1})$$

$$H_q(X_{n-1}) \xrightarrow{i_*} H_q(X_n) \longrightarrow H_q(X_n, X_{n-1})$$

the map  $i_*: H_q(X_{n-1}) \to H_q(X_n)$  is an isomorphism unless q=n,n-1. This is due to  $H_q(X_n,X_{n-1})=0$  and  $H_{q+1}(X_n,X_{n-1})$  otherwise. Therefore,

**Corollary 8.**  $H_q(X_n) \cong H_q(X_{n-1}) \cong ... \cong H_q(X_0) = 0$  for  $q > n \geq 0$ . That is, homology groups do not see the skeleton of dimension less than its.

**Corollary 9.**  $H_q(X_n) \cong H_q(X_{n+1}) \cong ... \cong H_q(X)$  for q < n. That is, homology groups see the same thing on skeleton of dimension more than its.

When q = n, the long exact sequence

$$0 = H_n(X_{n-1}) \xrightarrow[i_*]{} H_q(X_n) \xrightarrow[\partial]{} H_q(X_n, X_{n-1}) = \mathbb{Z}A_n$$

Then

**Corollary 10.**  $H_n(X_n)$  is free abelian (a subgroup of  $H_n(X_n, X_{n-1}) = \mathbb{Z}A_n$ ). Moreover, the number of generators of  $H_n(X_n)$  is bounded above by number of n-cells.

Now using the lemmas, we put the long exact sequence of the pairs  $(X_{n+1}, X_n), (X_n, X_{n-1})$  together,

**Definition 58** (cellular chain complex, cellular homology). The abelian groups  $H_n(X_n, X_{n-1})$  and homomorphisms  $d_n: H_{n+1}(X_{n+1}, X_n) \to H_n(X_n, X_{n-1})$  form a chain complex and called cellular chain complex of the cell complex  $X_0 \subseteq X_1 \subseteq ... \subseteq X$  with respect to homology H. Denote  $H_n(X_n, X_{n-1})$  by  $C_n^{CW}(X)$ . We have

$$C_n^{CW}(X) = \mathbb{Z}\mathcal{A}_n$$

That is, the number of generators of  $C_n^{CW}(X)$  is exactly the number of n-cells. Cellular homology is defined from the cellular chain complex and denoted by  $H_n(C_n^{CW}(X)) = H_n^{CW}(X)$ .

**Theorem 9.**  $H_n(X) = H_n^{CW}(X)$  and the isomorphism is also natural in the cell complex. *Proof.* 

$$H_n(X) = H_n(X_{n+1})$$

$$= \operatorname{im}(H_n(X_n) \to H_n(X_{n+1}))$$

$$= \frac{H_n(X_n)}{\ker(H_n(X_n) \to H_n(X_{n+1}))}$$

$$= \frac{H_n(X_n)}{\operatorname{im} \partial_{n+1}}$$

$$= \frac{\operatorname{im} i_n}{\operatorname{im} d_n}$$

$$= \frac{\ker d_{n-1}}{\operatorname{im} d_n}$$

$$= \frac{\operatorname{Her} d_{n-1}}{\operatorname{im} d_n}$$

$$= H_n^{CW}(X)$$

$$(first isomorphism theorem)$$

$$(exactness)$$

$$(i_n is injective)$$

$$(exactness)$$

Theorem 10 (cellular boundary formula). TODO

#### 2.2.2 COEFFICIENTS IN AN ABELIAN GROUP

**Definition 59.** Let A be an abelian group, cellular chain complex of X is

$$C_n^{CW}(X;A) = H_n(X_n, X_{n-1}; A) = H_n(X_n, X_{n-1}) \otimes A = \bigoplus_{\alpha \in \mathcal{A}_n} A = \left(\bigoplus_{\alpha \in \mathcal{A}_n} \mathbb{Z}\right) \otimes A$$
$$H_n(X;A) = H_n^{CW}(X;A)$$

#### 2.3 HOMOLOGY WITH COEFFICIENTS

#### 2.3.1 TENSOR PRODUCT

**Definition 60** (tensor product). Let A, B, C be abelian groups. A bilinear map  $A \times B \to C$  is a function  $f: A \times B \to C$  such that

$$f(a + a', b) = f(a, b) + f(a', b)$$
  
$$f(a, b + b') = f(a, b) + f(a, b')$$

The set of bilinear maps  $A \times B \to C$  is denoted by Bil(A, B; C). There exists an abelian group  $A \otimes B$  and bilinear map  $A \times B \to A \otimes B$  such that for any C and any bilinear map  $g: A \times B \to C$ , there exists a unique homomorphism  $A \otimes B \to C$  such that the following diagram commutes

Moreover,  $A \otimes B$  is universal. universal mapping property

#### Remark 18. :

- 1. If  $A \times B \to D$  is bilinear and  $D \to C$  is a homomorphism, then the composite  $A \times B \to D \to C$  is bilinear
- 2. Let B be an abelian group, then  $\otimes B$ , the right tensoring of B, is a functor on the category of abelian groups Ab
- 3.  $\mathbb{Z} \otimes B = B$

Proof.

1. trivial

- 2. consequence of  $A \times B \to A \otimes B$  being a functor
- 3. define the map  $f: Z \times B \to B$  by f(n, b) = nb and prove that B satisfies the tensor product factoring property. Hence,  $\mathbb{Z} \otimes B = B$

**Proposition 21.**  $\operatorname{Hom}(A \otimes B, C) \cong \operatorname{Bil}(A, B; C) \cong \operatorname{Hom}(A, \operatorname{Hom}(B, C))$ 

*Proof.* Hom $(A \otimes B, C) \cong \text{Bil}(A, B; C)$  is done by the definition of tensor product. Bil $(A, B; C) \cong \text{Hom}(A, \text{Hom}(B, C))$  is done by natural evaluation map.

Corollary 11.  $\operatorname{Hom}((\bigoplus_{\alpha} A_{\alpha}) \otimes B, C) = \operatorname{Hom}(\bigoplus_{\alpha} A_{\alpha} \otimes B, C)$ 

Proof.

$$\operatorname{Hom}\left(\left(\bigoplus_{\alpha} A_{\alpha}\right) \otimes B, C\right) = \operatorname{Hom}\left(\bigoplus_{\alpha} A_{\alpha}, \operatorname{Hom}(B, C)\right)$$

$$= \prod_{\alpha} \operatorname{Hom}(A_{\alpha}, \operatorname{Hom}(B, C))$$

$$= \prod_{\alpha} \operatorname{Hom}(A_{\alpha} \otimes B, C)$$

$$= \operatorname{Hom}\left(\bigoplus_{\alpha} A_{\alpha} \otimes B, C\right)$$

**Lemma 8.** Let C be a category, then  $f \in C(X,Y)$  is an isomorphism in C if and only if for all  $W \in ob C$ ,  $C(Y,W) \to C(X,W)$  defined by  $g \mapsto gf$  is bijective

Proof. By surjectivity,  $1 \in C(X, X)$  has a preimage  $g \in C(Y, X)$  such that  $g \mapsto gf = 1$ . Now,  $fg \in C(Y, Y)$  is mapped to  $g = gfg \in C(X, Y)$ . Moreover,  $1 \in C(Y, Y)$  is also mapped to  $g \in C(X, Y)$ . By injectivity, fg = 1.

**Proposition 22.**  $\beta: \bigoplus (A_{\alpha} \otimes B) \to (\bigoplus A_{\alpha}) \otimes B$  is an isomorphism

*Proof.* From the corollary above

**Lemma 9.** Suppose  $A' \to A \to A'' \to 0$  in Ab. Then  $A' \to A \to A'' \to 0$  is exact if and only if for all C,  $\operatorname{Hom}(A',C) \leftarrow \operatorname{Hom}(A,C) \leftarrow \operatorname{Hom}(A'',C) \leftarrow 0$  is exact.

Proof. no proof provided

Corollary 12 (right exactness of tensoring abelian group). If  $A' \to A \to A'' \to 0$  is exact, then  $A' \otimes B \to A \otimes B \to A'' \otimes B \to 0$  is also exact

Proof.

A sequence of exact sequences

- 1.  $A' \rightarrow A \rightarrow A'' \rightarrow 0$
- 2.  $\operatorname{Hom}(A', \operatorname{Hom}(B, C)) \to \operatorname{Hom}(A, \operatorname{Hom}(B, C)) \to \operatorname{Hom}(A'', \operatorname{Hom}(B, C)) \to 0$
- 3.  $\operatorname{Hom}(A' \otimes B, C) \to \operatorname{Hom}(A \otimes B, C) \to \operatorname{Hom}(A'' \otimes B, C) \to 0$
- 4.  $A' \otimes B \to A \otimes B \to A'' \otimes B \to 0$

**Proposition 23** (an example of free resolution of abelian group). Let A be an arbitrary abelian group, if  $S \subseteq A$  generates A, then the natural projection  $\eta: \mathbb{Z}S \to A$  is a surjection where  $\mathbb{Z}$  is the free abelian group functor. Let  $F_0 = \mathbb{Z}S$ ,  $F_1 = \ker \eta$  ( $F_1$  is free abelian as it is a subgroup of  $F_0$ ) and  $M: F_1 \to F_0$  be the inclusion map, then the sequence below is short exact

$$0 \longrightarrow F_1 \stackrel{M}{\longrightarrow} F_0 \stackrel{\eta}{\longrightarrow} A \longrightarrow 0$$

Let  $F = \bigoplus_{\alpha} \mathbb{Z}$  be a free abelian group, then  $F \otimes B = (\bigoplus_{\alpha} \mathbb{Z}) \otimes B = \bigoplus_{\alpha} (\mathbb{Z} \otimes B) = \bigoplus_{\alpha} B$ . Therefore, for any abelian group B, the sequence below is also exact

$$\bigoplus_{S_1} B \xrightarrow{M} \bigoplus_{S_0} B \xrightarrow{\eta} A \otimes B \longrightarrow 0$$

**Remark 19.** Let  $A = \mathbb{Z}/m\mathbb{Z}$ ,  $S = \{1\}$ , then  $m\mathbb{Z} \cong \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow A \longrightarrow 0$  is exact. And

is exact. Therefore,  $\mathbb{Z}/m\mathbb{Z}\otimes B=B/mB$ 

**Lemma 10.** There is a natural isomorphism  $A \otimes B \cong B \otimes A$ 

*Proof.* short  $\Box$ 

**Lemma 11** (right exactness of tensoring free abelian group). If F is a free abelian group and  $A' \longrightarrow A \longrightarrow A''$  is exact at A, then

$$A' \otimes F \longrightarrow A \otimes F \longrightarrow A'' \otimes F$$

is exact at  $A \otimes F$ 

*Proof.* As F is free, write 
$$F = \bigoplus_{\alpha} \mathbb{Z}$$
, then  $A \otimes F = A \otimes \bigoplus_{\alpha} \mathbb{Z} = \bigoplus_{\alpha} A$ 

#### Proposition 24. :

1. In the category  $Ch \times Ab$  where Ch is the category of chain complexes and Ab is the category of abelian groups, then there is a natural transformation  $\alpha$ 

$$H_n(C_{\bullet}) \otimes A \xrightarrow{\alpha_{C_{\bullet},A}} H_n(C_{\bullet} \otimes A)$$

$$\downarrow \qquad \qquad \downarrow$$

$$H_n(C'_{\bullet}) \otimes A' \xrightarrow{\alpha_{C'_{\bullet},A'}} H_n(C'_{\bullet} \otimes A')$$

where  $\alpha: H_n(C_{\bullet}) \times A \to H_n(C_{\bullet} \otimes A)$  is defined by a bilinear map as follows

$$[z], a \mapsto [z \otimes a]$$

for  $z \in C_{\bullet}$ ,  $a \in A$ .

2.  $\alpha$  is compatible with direct sum

3. if A is free abelian then  $\alpha$  is an isomorphism. An example when  $\alpha$  is not an isomorphism:

$$n-1$$
  $n$   $n+1$   $n+1$ 
 $C_{ullet}$   $0 \longleftarrow \mathbb{Z} \stackrel{2}{\longleftarrow} \mathbb{Z} \longleftarrow 0$ 
 $H(C_{ullet}) \otimes \mathbb{Z}/2$   $\mathbb{Z}/2$   $0$ 
 $C_{ullet} \otimes \mathbb{Z}/2$   $0 \longleftarrow \mathbb{Z}/2 \stackrel{0}{\longleftarrow} \mathbb{Z}/2 \longleftarrow 0$ 
 $H(C_{ullet}) \otimes \mathbb{Z}/2$   $0 \longleftarrow \mathbb{Z}/2 \stackrel{0}{\longleftarrow} \mathbb{Z}/2 \longleftarrow 0$ 

*Proof.*  $(\alpha: H_n(C_{\bullet}) \otimes A \to H_n(C_{\bullet} \otimes A)$  is mono)

Let  $C_{\bullet}$  be a chain complex and A be an abelian group, the natural projection  $F_0 = \mathbb{Z}S \to A$  where  $S \subseteq A$  generates A,  $F_1 = \ker(F_0 \to A)$ . Suppose  $C_{\bullet}$  is a free abelian group, we have two short exact sequences

$$0 \longrightarrow F_1 \stackrel{d}{\longrightarrow} F_0 \longrightarrow A \longrightarrow 0$$

$$0 \longrightarrow C_{\bullet} \otimes F_1 \xrightarrow{1 \otimes d} C_{\bullet} \otimes F_0 \longrightarrow C_{\bullet} \otimes A \longrightarrow 0$$

Define the natural boundary maps to make  $C_{\bullet} \otimes F_1, C_{\bullet} \otimes F_0, C_{\bullet} \otimes A$  chain complexes. The short exact sequence of chain complexes induces a long exact sequence in homology

$$H_n(C_{\bullet} \otimes F_1) \xrightarrow[H_n(1 \otimes d)]{} H_n(C_{\bullet} \otimes F_0) \xrightarrow{} H_n(C_{\bullet} \otimes A)$$

$$H_{n-1}(C_{\bullet} \otimes F_1) \xrightarrow[H_{n-1}(1 \otimes d)]{} H_{n-1}(C_{\bullet} \otimes F_0) \xrightarrow{} H_{n-1}(C_{\bullet} \otimes A)$$

Hence, coker  $H_n(1 \otimes d) \to H_n(C_{\bullet} \otimes A)$  is a monomorphism and  $H_n(C_{\bullet} \otimes A) \to \ker H_{n-1}(1 \otimes d)$  is a epimorphism. Hence, the sequence below is exact

$$0 \longrightarrow \operatorname{coker} H_n(1 \otimes d) \longrightarrow H_n(C_{\bullet} \otimes A) \longrightarrow \ker H_{n-1}(1 \otimes d) \longrightarrow 0$$

Since  $H_n(C_{\bullet})$  is abelian, we have another exact sequence

$$H_n(C_{\bullet}) \otimes F_1 \xrightarrow{1 \otimes d} H_n(C_{\bullet}) \otimes F_0 \longrightarrow H_n(C_{\bullet}) \otimes A \longrightarrow 0$$

The diagram below commutes

$$H_{n}(C_{\bullet}) \otimes F_{1} \xrightarrow{1 \otimes d} H_{n}(C_{\bullet}) \otimes F_{0} \longrightarrow H_{n}(C_{\bullet}) \otimes A \longrightarrow 0$$

$$\downarrow^{\alpha(\cong)} \qquad \downarrow^{\alpha(\cong)} \qquad \downarrow^{\alpha} \qquad \downarrow^$$

We have  $H_n(C_{\bullet}) \otimes A = \operatorname{coker}(1 \otimes d) = \operatorname{coker} H_n(1 \otimes d)$ . Since the sequence  $0 \to \operatorname{coker} H_n(1 \otimes d) \to H_n(C_{\bullet} \otimes A) \to \ker H_{n-1}(1 \otimes d) \to 0$  is exact, then  $\operatorname{coker} H_n(1 \otimes d) \to H_n(C_{\bullet} \otimes A)$  is mono. Hence  $\alpha : H_n(C_{\bullet}) \otimes A \to H_n(C_{\bullet} \otimes A)$  is mono.

#### 2.3.2 *R*-MODULE

**Definition 61** (R-module). Given a commutative ring R, an R-module is an abelian group M with bilinear map  $R \times M \to M$  such that

- 1m = m
- r(sm) = (rs)m

for all  $r, s \in R, m \in M$ 

Remark 20. Some remarks on R-module

- R-module is similar to group acting on a set
- R-module is a generalization of a vector space where the field of scalars ( $\mathbb{R}$  or  $\mathbb{C}$ ) is taken as a commutative ring R
- R-module is a generalization of abelian group that is every abelian group is a Z-module.

**Definition 62** (free R-module). An R-module M is free if there is a subset B, namely basis, such that

- B generates M: every element of M can be written as a linear combination of elements in B
- elements of B are linearly independent: for every  $\{e_1, e_2, ..., e_n\} \subseteq B$ , if  $r_1e_1 + r_2e_2 + ... + r_ne_n = 0$ , then  $r_1 = r_2 = ... = r_n = 0$

**Definition 63** (tensor product of R-modules). Let M, N, L be R-modules and a bilinear map  $f: M \times N \to L$  such that

$$f(rm, n) = rf(m, n) = f(m, rn)$$

Define the tensor product  $M \otimes_R N$  with universal property as follows

$$Bil_R(M, N; L) = Hom_R(M \otimes_R N, L)$$

The story above works for chain complexes over R-module, that is, the sequence below is exact

$$0 \longrightarrow F_1 \longrightarrow F_0 \stackrel{\epsilon}{\longrightarrow} A \longrightarrow 0$$

if  $F_1$  is free. In particular, if R is a principal ideal domain (PID) .

#### 2.3.3 FUNDAMENTAL THEOREM OF HOMOLOGICAL ALGEBRA

**Definition 64** (resolution, free resolution). Let  $\mathcal{M}$  be the category of R-modules (more generally, abelian category). A resolution of  $M \in \text{ob } \mathcal{M}$  is a chain complex of R-modules  $(F_{\bullet}, d)$  together with a map  $\epsilon : F_0 \to M$  such that the sequence below is exact

$$0 \longleftarrow M \stackrel{\epsilon}{\longleftarrow} F_0 \stackrel{d}{\longleftarrow} F_1 \stackrel{d}{\longleftarrow} F_2 \stackrel{d}{\longleftarrow} \dots$$

d is called boundary map,  $\epsilon$  is called augmentation map. If  $F_{\bullet}$  are free, the sequence is called free resolution of M

Remark 21. Some remarks on resolution

• If we write free resolution as a chain map

$$0 \longleftarrow F_0 \xleftarrow{d} F_1 \xleftarrow{d} F_2 \xleftarrow{d} \dots$$

$$\downarrow \qquad \qquad \downarrow^{\epsilon} \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longleftarrow M \longleftarrow 0 \longleftarrow 0 \longleftarrow \dots$$

then  $\epsilon$  is an isomorphism in homology.

• If A is a  $\mathbb{Z}$ -module (abelian group), then  $0 \longleftarrow A \xleftarrow{\epsilon} \mathbb{Z}S \xleftarrow{d} \ker \epsilon \longleftarrow 0$  is the free resolution of A where  $S \subseteq A$  generates A

Proposition 25. Free resolution always exists

*Proof.* Denote M by  $F_{-1}$ , 0 by  $F_{-2}$ . Let  $S_n = \ker(F_{n-1} \to F_{n-2})$ , construct  $F_n = RS_n$  free R-module generated by  $S_n$ 

$$F_{n-2} \longleftarrow F_{n-1} \longleftarrow F_n = RS_n$$

$$S_n = \ker(F_{n-1} \to F_{n-2})$$

ask prof why in her proof, M has a generator

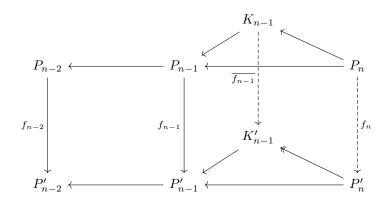
**Theorem 11** (fundamental theorem of homological algebra, FTHA). Let  $f: M \to M'$  be a map of R-modules and let

be such that  $P_n$  is free, the top chain is a chain complex, the bottom chain is exact. Then

- 1. for each n, there exists a chain map (lift)  $f_n: P_n \to P'_n$  lifting f
- 2. any two lifts are chain homotopic

Proof.

#### 1. We will prove the first statement by induction.



Suppose  $f_{n-1}: P_{n-1} \to P'_{n-1}$  and  $f_{n-2}: P_{n-2} \to P'_{n-2}$  were constructed. Let  $K_{n-1} = \ker(P_{n-1} \to P_{n-2})$  and  $K'_{n-1} = \ker(P'_{n-1} \to P'_{n-2})$ . Since the both chains are chain complexes, there are maps  $P_n \to K_{n-1}$  and  $P'_n \to K'_{n-1}$ . Moreover, since the bottom chain is exact, the map  $P'_n \to K'_{n-1}$  is surjective. We also have the restriction of  $f_{n-1}$  on  $K_{n-1}$  by  $\overline{f_{n-1}}: K_{n-1} \to K'_{n-1}$ . This map is well-defined because if  $x \in K_{n-1} \subseteq P_{n-1}$ , x is mapped to 0 through  $P_{n-1} \to P'_{n-2} \to P'_{n-2}$ , by commutativity, x is also mapped to 0 through  $P_{n-1} \to P'_{n-1} \to P'_{n-2}$ , that is, x is mapped to  $K'_{n-1}$  through  $f_{n-1}$ .

Now, since  $P'_n \to K'_{n-1}$  is surjective, there exists a map  $f_n: P_n \to P'_n$  such that the diagram remains commutative. For the base case when n=0, let  $P_{n-1}=M, P'_{n-1}=M', P_{n-2}=P'_{n-2}=0, f_{n-1}=f, f_{n-2}=0$ 

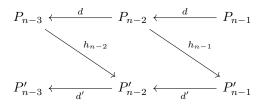
#### 2. Let $f_{\bullet}^{(1)}, f_{\bullet}^{(2)}: P_{\bullet} \to P_{\bullet}'$ are two lifts.

$$M \leftarrow_{\epsilon} P_{\bullet}$$

$$f \downarrow \qquad f^{(1)} (\ ) \downarrow f^{(2)} \downarrow f^{(2)} \downarrow \qquad M' \leftarrow_{\epsilon'} P'_{\bullet}$$

We will prove that  $g_{\bullet} = f_{\bullet}^{(1)} - f_{\bullet}^{(2)}$  is chain homotopic to zero, that is, find map  $h_{n+1}: P_n \to P'_{n+1}$  such that d'h + hd = g

Suppose  $h_{n-1}: P_{n-2} \to P'_{n-1}$  and  $h_{n-2}: P_{n-3} \to P'_{n-2}$  were constructed



Consider the term below

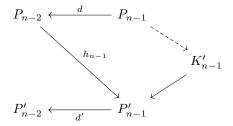
$$d'(g_{n-1} - h_{n-1}d) = d'g_{n-1} - d'h_{n-1}d$$

$$= d'g_{n-1} - (g_{n-2} - h_{n-2}d)d$$

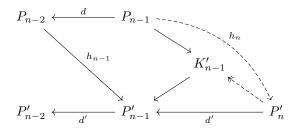
$$= d'g_{n-1} - g_{n-2}d$$

$$= 0$$
(top chain is a chain complex)
$$= 0$$

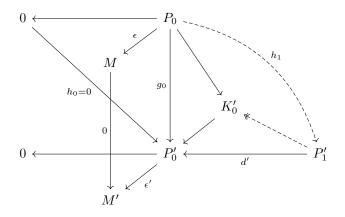
Hence, there exists a map  $P_{n-1} \to K'_{n-1}$  where  $K'_{n-1} = \ker(P'_{n-1} \to P'_{n-2})$  such that the diagram below commutes



Using the same argument as above, there exists a surjective map  $P'_n \to K'_{n-1}$  such that the diagram below commutes. Then,  $h_n: P_{n-1} \to P'_n$  is constructed from  $P_{n-1} \to K'_{n-1}$  and surjectivity of  $P'_n \to K'_{n-1}$ .

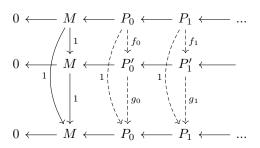


For the base case, let  $K_0' = \ker \epsilon'$ , the map  $M \to M'$  is zero, then  $g_0 : P_0 \to P_0'$  is factored through  $K_0'$ . The map  $P_1 \to K_0'$  is surjective since  $M' \leftarrow P_0' \leftarrow P_1'$  is exact. Hence,  $h_1 : P_0 \to P_1'$  can be constructed.



**Corollary 13.** A free resolution of M is well-defined as a chain homotopy type.

*Proof.* Suppose we have two free resolutions  $0 \leftarrow M \leftarrow P_{\bullet}$  and  $0 \leftarrow M \leftarrow P'_{\bullet}$ 



By FTHA, there are three lifts  $1_{\bullet}: P_{\bullet} \to P_{\bullet}, f_{\bullet}: P_{\bullet} \to P'_{\bullet}, g_{\bullet}: P'_{\bullet} \to P_{\bullet}$ . As both  $1_{\bullet}$  and  $g_{\bullet}f_{\bullet}$  are both lifts  $P_{\bullet} \to P_{\bullet}$ . By FTHA,  $g_{\bullet}f_{\bullet} \simeq 1$ . Similarly,  $f_{\bullet}g_{\bullet} \simeq 1$ . Hence, the two free resolutions are of the same chain homotopy type.

TODO - whether free resolution of two R-modules can be chain homotopic?

Corollary 14. There exists a functor R-Mod  $\rightarrow$  Ho(Ch(R-Mod)) where Ho denotes the category of homology

*Proof.* Given M being an R-module, the functor is defined by the homology of the chain complex from free resolution of M. The functor is well-defined because a free resolution defines a chain homotopy type and chain homotopic chain complexes induces the same homology. Given any map between R-modules M and M', the functor is defined by the chain map between free resolutions of M and M'. The functor is well-defined because two chain homotopic maps induces the same map in homology.

#### 2.3.4 ADDITIVE FUNCTOR, TOR FUNCTOR R-Mod $\rightarrow$ Ab

**Remark 22.** Let  $F: R\text{-Mod} \to Ab$  be a functor from R-module to abelian group. Let  $P_{\bullet}$  be a chain complex of R-module

$$0 \longleftarrow P_0 \longleftarrow^d P_1 \longleftarrow^d \dots$$

$$0 \longleftarrow FP_0 \xleftarrow{Fd} FP_1 \xleftarrow{Fd} \dots$$

then,

$$(Fd)(Fd) = F(dd) = F(0)$$

If F(0) = 0, then F sends a chain complex to a chain complex.

**Definition 65** (additive functor R-Mod  $\rightarrow$  Ab).  $F: \text{R-Mod} \rightarrow \text{Ab}$  is a additive functor if for any  $M, N \in \text{ob R-Mod}$ ,

$$\operatorname{Hom}_R(M,N) \to \operatorname{Hom}(F(M),F(N))$$

is a homomorphism of abelian groups.

Remark 23. Some remarks on additive functor of R-Mod  $\rightarrow$  Ab

- in the categories R-Mod or Ab, Hom is an abelian groups. More generally, they are preadditive categories or Abenriched categories where Hom is equipped with abelian group structure and composition is bilinear.
- if  $F : R\text{-Mod} \to Ab$  is additive, then F sends a chain complex to a chain complex, that is, there is an extension  $Ch(R\text{-Mod}) \to Ch(Ab)$
- tensor product is an additive functor.

**Lemma 12.** If F is additive and h is a chain homotopy  $f_0 \simeq f_1 : C_{\bullet} \to C'_{\bullet}$  in R-Mod, then F(h) is a chain homotopy  $F(f_0) \simeq F(f_1) : F(C_{\bullet}) \to F(C'_{\bullet})$ 

*Proof.* Suppose  $d'h + hd = f_0 - f_1$  is a chain homotopy. Then

$$F(d')F(h) + F(h)F(d) = F(d'h) + F(hd)$$
 (F is a functor)  
 $= F(d'h + hd)$  (F is an additive functor)  
 $= F(f_0 - f_1)$   
 $= F(f_0) - F(f_1)$  (F is an additive functor)

Corollary 15. Chain homotopy type of  $F(P_{\bullet})$  is well-defined (where  $M \leftarrow P_{\bullet}$  is a free resolution)

Remark 24. TODO

**Definition 66** (Tor functor). Let R be a ring and M be R-modules, define the Tor functor as the homology of chain complex  $P_{\bullet} \otimes_R N$ 

$$\operatorname{Tor}_n^R(M,N) = H_n(P_{\bullet} \otimes_R N)$$

for  $M \leftarrow P_{\bullet}$  is any free resolution.

Remark 25. Some remarks on Tor

- $\operatorname{Tor}_0^R(M,N) = M \otimes_R N$
- Some example of Tor TODO

**Proposition 26.** Tor is well-defined and functorial in both variables, that is,  $\operatorname{Tor}_n^R(M,-)$  and  $\operatorname{Tor}_n^R(-,N)$  are functors  $R\operatorname{-Mod} \to \operatorname{Ab}$ .

#### 2.3.5 UNIVERSAL COEFFICIENT THEOREM

**Theorem 12** (universal coefficient theorem). Let R be a PID and N be an R-module, then for any chain complex  $C_{\bullet}$ , then there is a short exact sequence

$$0 \longrightarrow H_n(C_{\bullet}) \otimes_R N \stackrel{\alpha}{\longrightarrow} H_n(C_{\bullet} \otimes_R N) \longrightarrow \operatorname{Tor}_1^R(H_{n-1}(C_{\bullet}), N) \longrightarrow 0$$

*Proof.* Since R is a PID, N admits a free resolution

$$0 \longleftarrow N \longleftarrow P_0 \xleftarrow{d} P_1 \longleftarrow 0$$

By exactness of tensor product, the sequence below is exact

$$0 \longleftarrow C_{\bullet} \otimes_{R} N \longleftarrow C_{\bullet} \otimes_{R} P_{0} \stackrel{1 \otimes d}{\longleftarrow} C_{\bullet} \otimes_{R} P_{1} \longleftarrow 0$$

The short exact sequence induces a long exact sequence

$$H_n(C_{\bullet} \otimes_R P_1) \xrightarrow{H_n(1 \otimes_R d)} H_n(C_{\bullet} \otimes_R P_0) \longrightarrow H_n(C_{\bullet} \otimes_R N) \longrightarrow H_{n-1}(C_{\bullet} \otimes_R P_1) \xrightarrow{H_{n-1}(1 \otimes_R d)} H_{n-1}(C_{\bullet} \otimes_R P_0)$$

That induces a short exact sequence

$$0 \longrightarrow \operatorname{coker} H_n(1 \otimes_R d) \longrightarrow H_n(C_{\bullet} \otimes_R N) \longrightarrow \ker H_{n-1}(1 \otimes_R d) \longrightarrow 0$$

As discussed earlier, coker  $H_n(1 \otimes_R d) = H_n(C_{\bullet}) \otimes_R N$ . Furthermore, we can identify  $\ker H_n(1 \otimes_R d)$  by  $\ker(1 \otimes_R d)$ 

$$H_{n-1}(C_{\bullet}) \otimes_{R} P_{1} \xrightarrow{1 \otimes_{R} d} H_{n-1}(C_{\bullet}) \otimes_{R} P_{0}$$

$$\downarrow^{\alpha(\cong)} \qquad \qquad \downarrow^{\alpha(\cong)}$$

$$H_{n-1}(C_{\bullet} \otimes_{R} P_{1}) \xrightarrow{H_{n-1}(1 \otimes_{R} d)} H_{n-1}(C_{\bullet} \otimes_{R} P_{0})$$

And  $\ker(1 \otimes_R d) = \operatorname{Tor}_1^R(H_{n-1}(C_{\bullet}), N)$  by exactness of tensor product, the sequence below is exact

$$0 \longleftarrow H_{n-1}(C_{\bullet}) \otimes_R N \longleftarrow H_{n-1}(C_{\bullet}) \otimes_R P_0 \stackrel{1 \otimes d}{\longleftarrow} H_{n-1}(C_{\bullet}) \otimes_R P_1 \longleftarrow 0$$

**Remark 26.** Let  $C_{\bullet} = C_{\bullet}(X; R) = C_{\bullet}(X) \otimes_{\mathbb{Z}} R$ , then

$$0 \longrightarrow H_n(X;R) \otimes_R N \stackrel{\alpha}{\longrightarrow} H_n(X;N) \longrightarrow \operatorname{Tor}_1^R(H_{n-1}(X;R),N) \longrightarrow 0$$

#### 2.3.6 KÜNNETH THEOREM

**Definition 67** (tensor product of chain complexes). Let  $C_{\bullet}$ ,  $D_{\bullet}$  be chain complexes of R-module. Define the tensor product  $C_{\bullet} \otimes D_{\bullet}$  by

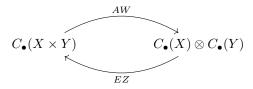
$$(C_{\bullet} \otimes D_{\bullet})_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and define the boundary map  $\partial: (C_{\bullet} \otimes D_{\bullet})_n \to (C_{\bullet} \otimes D_{\bullet})_{n-1}$  by a extension of  $\partial: C_p \otimes D_q \to (C_{\bullet} \otimes D_{\bullet})_{n-1}$ 

$$\partial(x \otimes y) = \partial x \otimes y + (-1)^p x \otimes \partial y$$

where  $x \in C_p, y \in D_q$ 

**Theorem 13** (Eilenberg-Zilber theorem). The Alexander-Whitney (AW) map and Eilenberg-Zilber (EZ) map are natural transformation of functors  $\text{Top}^2 \to \text{Ch}(\text{Ab})$  which are naturally chain homotopy inverses



TODO: method of proof

Remark 27. Two chain complexes of functors  $\mathrm{Top}^2 \to \mathrm{Ab}$ 

$$(X,Y) \mapsto C_n(X \times Y)$$
  
 $(X,Y) \mapsto (C_{\bullet}(X) \otimes C_{\bullet}(Y))_n$ 

Theorem 14. There is a natural short exact sequence

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C_{\bullet}) \otimes_R H_q(D_{\bullet}) \longrightarrow H_n(C_{\bullet} \otimes D_{\bullet}) \longrightarrow \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R(H_p(C_{\bullet}), H_q(D_{\bullet})) \longrightarrow 0$$

if R is a PID,  $C_{ullet}$  degree-wise free chain complex and this sequence splits

Corollary 16. Let R be a PID and  $C'_{\bullet}, C_{\bullet}$  degree-wise R-free. If  $C'_{\bullet} \to C_{\bullet}$  and  $D'_{\bullet} \to D_{\bullet}$  are homology isomorphisms, so is  $C'_{\bullet} \otimes D'_{\bullet} \to C_{\bullet} \otimes D_{\bullet}$ 

# Chapter 3

# OUT OF PLACE DISCUSSION

3.1 (BROUWER) DEGREE OF  $f: S^n \to S^n$  ()