MA5216 Test

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1 PROBLEM 1

Problem 1.1

Let $n \in \mathbb{N}$ and let $R > 0$ be a fixed positive number. Let

$$S^n(R) = \left\{ (x^1, ..., x^{n+1}) \in \mathbb{R}^n : \sum_{k=1}^{n+1} (x^k)^2 = R^2 \right\}$$

be the n-sphere in $\mathbb{R}^{n+1}$ centered at the origin and of radius R. Let $g_{S^n(R)}$ be the canonical Riemannian metric on $S^n(R)$ induced from the Euclidean metric $g_{\mathbb{R}^{n+1}}$ in $\mathbb{R}^{n+1}$, i.e. $g_{S^n(R)} = g_{\mathbb{R}^{n+1}}|_{S^n(R)}$. Similarly, consider the puncture Euclidean space $\mathbb{R}^{n+1} \setminus \{0\}$ and let $g_{\mathbb{R}^n \setminus \{0\}}$ be the Riemannian metric on $\mathbb{R}^n \setminus \{0\}$ induced from the Euclidean metric $g_{\mathbb{R}^{n+1}}$ on $\mathbb{R}^{n+1}$, i.e. $g_{\mathbb{R}^n \setminus \{0\}} = g_{\mathbb{R}^{n+1}}|_{\mathbb{R}^n \setminus \{0\}}$. Here with some abuse of notation, 0 also denotes the origin of $\mathbb{R}^{n+1}$

Consider the two maps $\mu : \mathbb{R}^{n+1} \setminus \{0\} \to (0, +\infty)$ and $\phi : \mathbb{R}^{n+1} \setminus \{0\} \to S^n(R)$ given as follows: For $x = (x^1, ..., x^{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}$

$$\mu(x) = \sqrt{(x^1)^2 + ... + (x^{n+1})^2}$$

$$\phi(x) = \left(\frac{Rx^1}{\mu(x)}, ..., \frac{Rx^{n+1}}{\mu(x)} \right)$$

(a) Show that the map $\phi : \mathbb{R}^{n+1} \setminus \{0\} \to S^n(R)$ is not a Riemannian submersion with respect to the metric $g_{\mathbb{R}^{n+1} \setminus \{0\}}$ on $\mathbb{R}^{n+1} \setminus \{0\}$ and the metric $g_{S^n(R)}$ on $S^n(R)$

(b) Does there exist a Riemannian metric g on $\mathbb{R}^{n+1} \setminus \{0\}$ so that the map $\phi : \mathbb{R}^{n+1} \setminus \{0\} \to S^n(R)$ is a Riemannian submersion with respect to the metric g on $\mathbb{R}^{n+1} \setminus \{0\}$ and the metric $g_{S^n(R)}$ on $S^n(R)$

1.1 a

Write $\phi(x) = (\phi_1(x), ..., \phi_n(x)) \in S^n(R)$ for every $x \in \mathbb{R}^{n+1} \setminus \{0\}$, each $\phi_i : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ is defined by

$$\phi_i(x) = \frac{Rx^i}{\|x\|} = Rx^i((x^1)^2 + ... + (x^{n+1})^2)^{-1/2}$$

For any $x \in \mathbb{R}^{n+1} \setminus \{0\}$, ϕ maps x into $y \in S^n(R)$. With respect to the canonical basis of $T_x\mathbb{R}^n$, $d\phi_x$ is a matrix in $\mathbb{R}^{n \times n}$ (index i for rows, index j for columns)

$$d\phi_x = \left[\tfrac{\partial \phi_i}{\partial x^j}|_x \right]_{1 \le i,j \le n} \in \mathbb{R}^{n \times n}$$

Let $x = (x^1, ..., x^n)$, then for every $1 \le i, j \le n$ and $i \neq j$

$$\frac{\partial \phi_i}{\partial x^j}|_x = Rx^i(-1/2)((x^1)^2 + ... + (x^{n+1})^2)^{-3/2} 2x^j = -\frac{R}{\|x\|^3} x^i x^j$$

$$\frac{\partial \phi_i}{\partial x^i}|_x = \frac{R}{\|x\|} - \frac{R}{\|x\|^3}(x^i)^2 = \frac{R}{\|x\|^3}(\|x\|^2 - (x^i)^2)$$

Then

$$d\phi_x = \frac{R}{\|x\|^3}\begin{bmatrix} \|x\|^2-(x^1)^2 & -x^1x^2 & ... & -x^1x^n \\ -x^2x^1 & \|x\|^2-(x^2)^2 & ... & -x^2x^n \\ ... & ... & ... & ... \\ -x^nx^1 & -x^nx^2 & ... & \|x\|^2-(x^n)^2 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

Let $x = (x^1, 0, ..., 0) \in \mathbb{R}^{n+1} \setminus \{0\}$ for some $x^1 > 0$, then $\|x\| = x^1$, consider the tangent vector $v = (0, 1, 0, ..., 0) \in T_p(\mathbb{R}^{n+1} \setminus \{0\}) \subseteq T_p\mathbb{R}^n$, then

$$g_{\mathbb{R}^{n+1}\setminus\{0\}}(v, v) = g_{\mathbb{R}^{n+1}}(v, v) = 1$$

Let $w = d\phi_x(v) \in T_yS^n(R) \subseteq T_y\mathbb{R}^n$, with respect to the canonical basis of $T_x\mathbb{R}^n$,

$$\begin{aligned} w &= \frac{R}{\|x\|^3}\begin{bmatrix} \|x\|^2-(x^1)^2 & -x^1x^2 & ... & -x^1x^n \\ -x^2x^1 & \|x\|^2-(x^2)^2 & ... & -x^2x^n \\ ... & ... & ... & ... \\ -x^nx^1 & -x^nx^2 & ... & \|x\|^2-(x^n)^2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ ... \\ 0 \end{bmatrix} \\ &= \frac{R}{\|x\|^3}\begin{bmatrix} -x^1x^2 \\ \|x\|^2-(x^2)^2 \\ ... \\ -x^nx^2 \end{bmatrix} \\ &= \frac{R}{\|x\|^3}\begin{bmatrix} 0 \\ \|x\|^2 \\ ... \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ R/\|x\| \\ ... \\ 0 \end{bmatrix} \in T_qS^n(R) \end{aligned}$$

Then

$$g_{S^n(R)}(w, w) = g_{\mathbb{R}^n}(w, w) = \left(\frac{R}{\|x\|}\right)^2$$

Pick $x^1 = R/2$, then $\|x\| = R/2$, then $g_{S^n(R)}(w, w) = 4 \neq g_{\mathbb{R}^{n+1}\setminus\{0\}}(v, v)$. Hence, ϕ is not a Riemannian submersion

1.2 b

For any $v_1, v_2 \in T_x(\mathbb{R}^{n+1} \setminus \{0\})$, let $d\phi_x$ maps v_1, v_2 into $w_1, w_2 \in T_y(S^n(R))$ respectively. We want to pick g so that

$$g(v_1, v_2) = g_{S^n(R)}(w_1, w_2)$$

Let $I = (0, +\infty) \subseteq \mathbb{R}$ with the usual metric dr^2 induced from Euclidean space, consider the product metric $dr^2 + g_{S^n(R)}$ in the product space $I \times S^n(R)$

$$\begin{tikzcd} \mathbb{R}^{n+1}\setminus\{0\} \arrow[r, "\psi"] \arrow[rr, bend left, "\phi"] & I \times S^n(R) \arrow[r, two heads] & S^n(R) \end{tikzcd}$$

There is a canonical Riemannian submersion $I \times S^n(R) \to S^n(R)$ mapping $(i, s) \mapsto s$. Define the diffeomorphism

$$\begin{aligned} \psi : \mathbb{R}^{n+1} \setminus \{0\} &\to I \times S^n(R) \\ x &\mapsto \left(\|x\|, R\frac{x}{\|x\|} \right) \end{aligned}$$

We write $\psi = (\chi, \phi)$ where $\chi : \mathbb{R}^{n+1} \setminus \{0\} \to I$ and $\phi : \mathbb{R}^{n+1} \setminus \{0\} \to S^n(R)$, then ϕ is precisely the function defined in the previous part.

Let g be the unique pullback metric on $\mathbb{R}^{n+1} \setminus \{0\}$ from $dr^2 + g_{S^n(R)}$ on $I \times S^n(R)$, then ϕ is a Riemannian submersion with respect to g and $g_{S^n(R)}$. In particular, in polar coordinate (r, s_n) of $\mathbb{R}^{n+1}$,

$$\begin{aligned} g &= dr^2 + r^2 ds_n \\ g_{S^n(R)} &= R^2 ds_n \end{aligned}$$

2 PROBLEM 2

Problem 2.1

Let (M, g) be a Riemannian manifold and let $f, h : M \to \mathbb{R}$ be smooth functions on M. Let X be a smooth vector field on M. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth function on R. Prove the following identities on M

(i) $\operatorname{div}(f \cdot X) = D_X f + f \cdot \operatorname{div} X$

(ii) $\Delta(f \cdot h) = h \cdot \Delta f + f \cdot \Delta h + 2g(\nabla f, \nabla h)$

(iii) $\operatorname{Hess}(f \cdot h) = h \cdot \operatorname{Hess} f + f \cdot \operatorname{Hess} h + df \otimes dh + dh \otimes df$

(iv) $\operatorname{Hess}(\phi \circ f) = (\phi'' \circ f) \cdot df \otimes df + (\phi' \circ f) \cdot \operatorname{Hess} f$

(v) $\Delta(\phi \circ f) = (\phi' \circ f) \cdot \Delta f + (\phi'' \circ f) \cdot |df|^2$

where $\cdot$ denotes the point-wise multiplication, $\circ$ denotes function composition, $|df|$ denotes the point-wise norm with respect to g

2.1 i

We use the definition $\operatorname{div} X = -\nabla^*\theta_X = \sum g(\nabla_{E_i} X, E_i)$ for some orthonormal frame E_i in proposition 2.2.7. We have

$$\begin{aligned}
&\operatorname{div}(f \cdot X) \\
&= \sum g(\nabla_{E_i}(f \cdot X), E_i) \\
&= \sum g((\nabla_{E_i} f) \cdot X + f \cdot \nabla_{E_i} X, E_i) && \text{(prop 2)} \\
&= \sum (\nabla_{E_i} f) \cdot g(X, E_i) + f \cdot g(\nabla_{E_i} X, E_i) && \text{(bilinearity of } g\text{)} \\
&= \Big(\sum (\nabla_{E_i} f) \cdot g(X, E_i)\Big) + f \cdot \operatorname{div} X && \text{(definition)} \\
&= \nabla_{\sum g(X,E_i)\cdot E_i} f + f \cdot \operatorname{div} X && \text{(prop 1)} \\
&= \nabla_X f + f \cdot \operatorname{div} X \\
&= D_X f + f \cdot \operatorname{div} X
\end{aligned}$$

2.2 ii

We use the definition $\Delta f = -\nabla^* \nabla f = \sum (\nabla_{E_i} \nabla f)(E_i)$ for some orthonormal frame E_i. Consider f and h as $(0,0)$-tensors, for any vector field X

$$\nabla_X (f \cdot h) = (\nabla_X f) \otimes h + f \otimes \nabla_X h = h \cdot \nabla_X f + f \cdot \nabla_X h$$

Then, as $(0,1)$-tensors, we have $\nabla(f \cdot h) = h \cdot \nabla f + f \cdot \nabla h$, then

$$\begin{aligned}
&\Delta(f \cdot h) \\
&= \sum \nabla_{E_i} \nabla(f \cdot h)(E_i) && \text{(definition)} \\
&= \sum \nabla_{E_i} (h \cdot \nabla f + f \cdot \nabla h)(E_i) \\
&= \sum \nabla_{E_i} (h \cdot \nabla f)(E_i) + \sum \nabla_{E_i} (f \cdot \nabla h)(E_i) && \text{(prop 2)} \\
&= 2 \sum \nabla_{E_i} h \cdot \nabla_{E_i} f + h \cdot \sum \nabla_{E_i} \nabla f(E_i) + f \cdot \sum \nabla_{E_i} \nabla h(E_i) && \text{(prop 2)} \\
&= 2\Big(\sum \nabla_{E_i} h \cdot \nabla_{E_i} f\Big) + h \cdot \Delta f + f \cdot \Delta h && \text{(definition)}
\end{aligned}$$

Finally, $\nabla_{E_i} h, \nabla_{E_i} f$ are directional derivative, then $\nabla_{E_i} h = g(E_i, \nabla f)$ and $\nabla_{E_i} f = g(E_i, \nabla h)$, hence

$$\sum \nabla_{E_i} h \cdot \nabla_{E_i} f = \sum g(E_i, \nabla f) g(E_i, \nabla h) = g(\nabla f, \nabla h)$$

We obtain

$$\Delta(f \cdot h) = 2g(\nabla f, \nabla h) + h \cdot \Delta f + f \cdot \Delta h$$

2.3 iii

We use the definition $\mathrm{Hess}\, f(X,Y) = \nabla^2_{X,Y} f = \nabla_X \nabla_Y f - \nabla_{\nabla_X Y} f$, then

$$\mathrm{Hess}(f \cdot h)(X,Y) = \nabla_X \nabla_Y (f \cdot h) - \nabla_{\nabla_X Y}(f \cdot h)$$

We have

$$\begin{aligned}
&\nabla_X \nabla_Y (f \cdot h) \\
&= \nabla_X (f \cdot \nabla_Y h + h \cdot \nabla_Y f) \\
&= \nabla_X (f \cdot \nabla_Y h) + \nabla_X (h \cdot \nabla_Y f) \\
&= f \cdot \nabla_X \nabla_Y h + \nabla_X f \cdot \nabla_Y h + h \cdot \nabla_X \nabla_Y f + \nabla_X h \cdot \nabla_Y f
\end{aligned}$$

We also have $\nabla_{\nabla_X Y}(f \cdot h) = f \cdot \nabla_{\nabla_X Y} h + h \cdot \nabla_{\nabla_X Y} f$, then

$$\mathrm{Hess}(f \cdot h)(X,Y) = f \cdot \mathrm{Hess}\, h(X,Y) + h \cdot \mathrm{Hess}(X,Y) + \nabla_X f \cdot \nabla_Y h + \nabla_X h \cdot \nabla_Y f$$

$\nabla_X f \cdot \nabla_Y h$ is precisely $(df \otimes dh)(X,Y)$ since by definition

$$(df \otimes dh)(X,Y) = df(X) \cdot dh(Y) = \nabla_X f \cdot \nabla_Y h$$

Similarly, $\nabla_X h \cdot \nabla_Y f = (dh \otimes df)(X,Y)$, we obtain

$$\mathrm{Hess}(f \cdot h) = h \cdot \mathrm{Hess}\, f + f \cdot \mathrm{Hess}\, h + df \otimes dh + dh \otimes df$$

2.4 iv

By proposition 2.2.6

$$\mathrm{Hess}\, f(X,Y) = g(\nabla_X \nabla f, Y) = (\nabla_X df)(Y)$$

Chain rule for $\phi \circ f$, for any $p \in M$, $d(\phi \circ f)_p = d\phi_{f(p)} df_p$, we can rewrite it as product of functions $TM \to \mathbb{R}$

$$d(\phi \circ f) = (\phi' \circ f) \cdot df$$

Similarly,

$$d(\phi' \circ f) = (\phi'' \circ f) \cdot df$$

Then

$$
\begin{aligned}
&\operatorname{Hess}(\phi \circ f)(X, Y) \\
&= (\nabla_X d(\phi \circ f))(Y) \\
&= (\nabla_X ((\phi' \circ f) \cdot df))(Y) \\
&= ((\phi' \circ f) \cdot \nabla_X df + (\nabla_X (\phi' \circ f)) \cdot df)(Y) \\
&= (\phi' \circ f) \cdot (\nabla_X df)(Y) + (\nabla_X (\phi' \circ f)) \cdot df(Y) \\
&= (\phi' \circ f) \cdot \operatorname{Hess} f(X, Y) + (\nabla_X (\phi' \circ f)) \cdot df(Y) \\
&= (\phi' \circ f) \cdot \operatorname{Hess} f(X, Y) + d(\phi' \circ f)(X) \cdot df(Y) && \text{(directional derivative)} \\
&= (\phi' \circ f) \cdot \operatorname{Hess} f(X, Y) + (\phi'' \circ f) \cdot df(X) \cdot df(Y) && \text{(chain rule)} \\
&= (\phi' \circ f) \cdot \operatorname{Hess} f(X, Y) + ((\phi'' \circ f) \cdot df \otimes df)(X, Y) && \text{(definition)}
\end{aligned}
$$

Hence,

$$\operatorname{Hess}(\phi \circ f) = (\phi' \circ f) \cdot \operatorname{Hess} f + (\phi'' \circ f) \cdot df \otimes df$$

2.5 v

We use the definition $\Delta f = -\nabla^* \nabla f = \sum_i \nabla^2_{E_i, E_i} f = \sum_i \operatorname{Hess} f(E_i, E_i)$ for some orthonormal frame E_i, we have

$$
\begin{aligned}
&\Delta f \\
&= \sum_i \operatorname{Hess}(\phi \circ f)(E_i, E_i) \\
&= \sum_i (\phi' \circ f) \cdot \operatorname{Hess} f(E_i, E_i) + (\phi'' \circ f) \cdot df(E_i)^2 \\
&= (\phi' \circ f) \Delta f + (\phi'' \circ f) \cdot \sum_i df(E_i)^2 \\
&= (\phi' \circ f) \Delta f + (\phi'' \circ f) \cdot |df|
\end{aligned}
$$

3 PROBLEM 3

Problem 3.1

Let (M, g) be a Riemannian manifold. Let S be a smooth (1, 1)-tensor field on M, and let X be a smooth vector field on M. Let ∇ denote the Riemannian connection on M

(a) Prove that

$$\operatorname{tr}(\nabla_X S) = \nabla_X \operatorname{tr} S$$

Here $\operatorname{tr} S$ denotes the trace of the $(1,1)$-tensor S.

(b) Consider the $(0,2)$-tensor T given by

$$T(Y, Z) = g(S(Y), Z)$$

for any smooth vector fields Y and Z on M. Prove that

$$(\nabla_X T)(Y, Z) = g((\nabla_X S)(Y), Z)$$

for any smooth vector fields Y and Z on M

3.1 a

For any $(1,1)$-tensor $S = S^i_j E_i \otimes \sigma^j$, $\operatorname{tr} S$ is defined by

$$\operatorname{tr} S = S^i_i$$

If $S = Y \otimes \omega$ for some vector field $Y = Y^i E_i \in TM$ and covector field $\omega = \omega_j \sigma^j \in T^*M$, then

$$\operatorname{tr} S = \operatorname{tr}(Y^i \sigma_j E_i \otimes \sigma^j) = Y^i \omega_i = \omega(Y)$$

Any $(1,1)$-tensor can be written as a sum of $\{Y \otimes \omega\}$, so we just need to prove for the case of basic tensor

$$\begin{aligned}
&\operatorname{tr} \nabla_X (Y \otimes \omega) \\
&= \operatorname{tr}(\nabla_X Y \otimes \omega + Y \otimes \nabla\omega) && \text{(general product rule)} \\
&= \operatorname{tr}(\nabla_X Y \otimes \omega) + \operatorname{tr}(Y \otimes \nabla_X \omega) \\
&= \omega(\nabla_X Y) + (\nabla_X \omega)(Y) \\
&= \nabla_X(\omega(Y)) && \text{(page 57, def of } \nabla_X T) \\
&= \nabla_X \operatorname{tr}(Y \otimes \omega)
\end{aligned}$$

3.2 b

$$
\begin{aligned}
&(\nabla_X T)(Y,Z) \\
&= \nabla_X (T(Y,Z)) - T(\nabla_X Y, Z) - T(Y, \nabla_X Z) && \text{(page 57, def of } \nabla_X T\text{)} \\
&= \nabla_X g(S(Y),Z) - g(S(\nabla_X Y),Z) - g(S(Y),\nabla_X Z) && \text{(premise)} \\
&= g(\nabla_X S(Y),Z) + g(S(Y),\nabla_X Z) - g(S(\nabla_X Y),Z) - g(S(Y),\nabla_X Z) && \text{(prop 4)} \\
&= g(\nabla_X S(Y),Z) - g(S(\nabla_X Y),Z) \\
&= g(\nabla_X S(Y) - S(\nabla_X Y),Z) \\
&= g((\nabla_X S)(Y),Z) && \text{(page 57, def of } \nabla_X T\text{)}
\end{aligned}
$$

4 PROBLEM 4

Definition 4.1 (totally geodesic, normal vector)
The hypersurface H is said to be *totally geodesic* if its second fundamental form $II : TH \times TH \to \mathbb{R}$ vanishes. For a point $p \in H$, a vector $v \in T_pM$ is said to be normal vector to H if $v \perp T_pH$

Problem 4.2

–

(a) Let V be a vector space and let $\langle,\rangle$ be an inner product on V. Let $B : V \times V \to \mathbb{R}$ be a symmetric bilinear form on V such that

$$B(v,w) = 0 \text{ for all } v,w \in V \text{ satisfying } \langle v,w\rangle = 0$$

Show that B is a multiple of the inner product, that is, there exists a constant c such that

$$B(v,w) = c \cdot \langle v,w\rangle$$

(b) Let (M,g) be a Riemmannian manifold, and fix a point $p \in M$. Show that there exists a constant k such that $\sec(\pi) = k$ for all planes $\pi \subseteq T_pM$ if and only if $R(v,w)z = 0$ for all mutually orthogonal $v,w,z \in T_pM$

(c) Let (M,g) be a Riemmannian manifold of dimension $n \geq 3$. Suppose that for any point $p \in M$ and any tangent vector $v \in T_pM$, v is a normal vector to a totally geodesic hypersurface H of M containing p. Show that (M,g) has constant sectional curvature

4.1 a

(assume V is a Hilbert space) For any nonzero $v \in V$, $B(-, v)$ is a linear, by Riesz representation theorem, there exists a nonzero $\bar{v} \in V$ so that

$$B(u, v) = \langle u, \bar{v} \rangle$$

for every $u \in V$. We can write $\bar{v} = \bar{v}^\perp + cv$ for some $\bar{v}^\perp \in (\operatorname{span} v)^\perp$ and $c \in \mathbb{R}$. For any nonzero $u \in V$, we have

$$\langle u, \bar{v} \rangle = \langle u, \bar{v}^\perp \rangle + c\langle u, v \rangle$$

Let $u = \bar{v}^\perp$, then

$$\langle u, v \rangle = 0 \implies \langle u, \bar{v} \rangle = 0 \implies \langle \bar{v}^\perp, \bar{v}^\perp \rangle = 0 \implies \bar{v}^\perp = 0$$

So

$$B(u, v) = c\langle u, v \rangle$$

4.2 b

Lemma 4.3 (proposition 3.1.3 - Riemann, 1954)
The following properties are equivalent

1. $\sec \pi = k$ for all 2-planes in T_pM
2. $R(v_1, v_2)v_3 = -k(v_1 \wedge v_2)(v_3)$

where k is a constant

($\implies$) If $\sec \pi = k$ for all planes $\pi \subseteq T_pM$, let v, w, z mutually orthogonal, then

$$\begin{aligned}
&R(v, w)z \\
&= -k(v \wedge w)(z) \\
&= -k(g(v, z)w - g(w, z)v) \\
&= 0
\end{aligned}$$

($\impliedby$) Consider curvature tensor as a symmetric bilinear map

$$\begin{aligned}
R : \wedge^2 TM \times \wedge^2 TM &\to \mathbb{R} \\
(x \wedge y, v \wedge w) &\mapsto R(x \wedge y, v \wedge w) = R(x, y, w, v) = g(R(x, y)w, v)
\end{aligned}$$

Sectional curvature is defined by

$$\sec(\operatorname{span}\{v, w\}) = \frac{g(R(w, v)v, w)}{g(v \wedge w, v \wedge w)} = \frac{R(w \wedge v, w \wedge v)}{g(w \wedge v, w \wedge v)}$$

From now, R and g are restricted to the tangent space T_pM. It suffices to show that if $g(\tilde{v}, \tilde{w}) = 0$ for some vectors $\tilde{v}, \tilde{w} \in \wedge^2 T_pM$, then $R(\tilde{v}, \tilde{w}) = 0$.

Suppose $\tilde{v}, \tilde{w} \in \wedge^2 T_pM$ so that $g(\tilde{v}, \tilde{w}) = 0$, each vector in $\wedge^2 T_pM$ corresponds to a 2-plane in T_pM, let $\tilde{v} = v_1 \wedge v_2$ for some orthogonal vectors $v_1, v_2 \in T_pM$. Similarly, let $\tilde{w} = w_1 \wedge w_2$ for some vectors $w_1, w_2 \in T_pM$ (not neccessarily orthogonal). $g(\tilde{v}, \tilde{w}) = 0$ implies

$$g(\tilde{v}, \tilde{w}) = g(v_1 \wedge v_2, w_1 \wedge w_2) = \det \begin{bmatrix} g(v_1, w_1) & g(v_1, w_2) \\ g(v_2, w_1) & g(v_2, w_2) \end{bmatrix} = 0$$

We want to pick a nonzero vector $w = aw_1 + bw_2 \in \operatorname{span}\{w_1, w_2\}$ so that $g(v_1, w) = g(v_2, w) = 0$. This is possible since a, b satisfies the following system of equations

$$\begin{bmatrix} g(v_1, w_1) & g(v_1, w_2) \\ g(v_2, w_1) & g(v_2, w_2) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} g(v_1, w) \\ g(v_2, w) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and the nullspace of $\begin{bmatrix} g(v_1, w_1) & g(v_1, w_2) \\ g(v_2, w_1) & g(v_2, w_2) \end{bmatrix}$ is non-trivial. That is, there exists a vector in $\operatorname{span}\{w_1, w_2\}$ that is orthogonal to both v_1 and v_2, let that vector be w_1' and pick another vector $w_2' \in \operatorname{span}\{w_1, w_2\}$ so that $w_1' \wedge w_2' = w_1 \wedge w_2 = \tilde{w}$. Now

$$R(\tilde{v}, \tilde{w}) = R(v_1 \wedge v_2, w_2' \wedge w_1') = g(R(v_1, v_2)w_1', w_2')$$

Since v_1, v_2, w_1' are mutually orthogonal, $R(\tilde{v}, \tilde{w}) = 0$. From part 1, we have

$$R = kg$$

for some constant k as symmetric bilinear forms $\wedge^2 T_pM \times \wedge^2 T_pM \to \mathbb{R}$. Hence

$$\sec(\operatorname{span}\{v, w\}) = \frac{g(R(w, v)v, w)}{g(v \wedge w, v \wedge w)} = \frac{R(w \wedge v, w \wedge v)}{g(w \wedge v, w \wedge v)} = k$$

4.3 c

Let nonzero $v, w, z \in T_pM$ be mutually orthogonal. Let z be normal to totally geodesic hypersurface H, its second fundamental form is identically zero, then by theorem 3.2.5

$$g(R(V, W)Y, Z) = 0$$

for some tangent vector field $V, W, Y \in TH$ and normal vector field Z. Pick V, W, Z so that $V(p) = v, W(p) = w, Z(p) = z$ and let $y = W(p)$, then

$$g(R(v, w)y, z) = 0$$

Note that, the choice of Y was arbitrary. We have

$$g(R(v, w)z, y) = -g(R(v, w)y, z) = 0$$

for every vector $y \in T_pH$, so $R(v, w)z$ must be parallel to z. However, $g(R(v, w)z, z) = 0$ due to symmetry of curvature tensor. So $R(v, w)z = 0$. From part 2, (M, g) has constant sectional curvature.