set

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some identities related to set

1 Zermelo–Fraenkel set theory

9 axioms in ZFC serve as the foundation of mathematics. All concepts in mathematics at the moment can be built from these 9 axioms using the language of first-order logic. [1]

Axiom 1 (Axiom of extensionality) *Two sets are equal if they have the same elements*

$$\forall X \forall Y [\forall z (z \in X \iff z \in Y)] \implies X = Y$$

Axiom 2 (Axiom of regularity) *Every non-empty set of sets $\mathcal{X}$ contains a element Y such that $\mathcal{X}$ and Y are disjoint*

$$\forall \mathcal{X} [\mathcal{X} \neq \varnothing \implies \exists Y (Y \in \mathcal{X} \wedge Y \cap \mathcal{X} = \varnothing)]$$

Axiom 3 (Axiom schema of specification) *Given any predicate $\varphi(z)$ and a set X, the subset of elements of X obeying φ exists, namely Y*

$$\forall X \exists Y \forall z [z \in Y \iff (z \in X \wedge \varphi(z))]$$

we write $Y = \{z \in X : \varphi(z)\}$

Axiom 4 (Axiom of pairing) *If x and y are sets, there exists a set that contains x and y as elements, namely Z*

$$\forall x \forall y \exists Z [(x \in Z) \wedge (y \in Z)]$$

we use axiom 3 to construct the set of x and y, $\{x, y\}$

Axiom 5 (Axiom of union) *For any set of sets $\mathcal{F}$ there exists a set containing every element that is a element x of some element Y of $\mathcal{F}$, namely A*

$$\forall \mathcal{F} \exists A \forall Y \forall x [Y \in \mathcal{F} \wedge x \in Y \implies x \in A]$$

we use axiom 3 to construct the union of $\mathcal{F}$, we write $\bigcup \mathcal{F} \subseteq A$

[1] will give more insights once I master the topic

Axiom 6 (Axiom schema of replacement) *Let $f : A \to B$, there exists a set containing the image of A, namely C*

$$\forall A \forall B \forall (f : A \to B) \exists C \forall x (x \in A \implies f(x) \in C)$$

[2] *we use axiom 3 to construct the image of A, we write $f(A) \subseteq C$. sometimes, we also write $f(A) = \{f(x) : x \in A\}$ and $f(\{x \in X : \varphi(x)\}) = \{f(x) : x \in X \wedge \varphi(x)\}$*

Axiom 7 (Axiom of infinity) *There exists a set X containing the empty set $\varnothing$ and if y is a element of X then $y \cup \{y\}$ is also a element of X*

$$\exists X [\varnothing \in X \wedge \forall y (y \in X \implies y \cup \{y\} \in X)]$$

this axiom asserts the existence of natural number (von Neumann ordinals)

Axiom 8 (Axiom of power set) *For any set X, there is a set that contains every subsets Z of X, namely $\mathcal{Y}$*

$$\forall X \exists \mathcal{Y} \forall Z (Z \subseteq X \implies Z \in \mathcal{Y})$$

we use axiom 3 to construct the power set of X, we write $\mathcal{P}(X) \subseteq \mathcal{Y}$

Axiom 9 (Axiom of choice) *For any set $\mathcal{X}$ of non-empty sets Y, there exists a choice function f that defined on $\mathcal{X}$ and maps each set Y of $\mathcal{X}$ to an element of that set.*

$$\forall \mathcal{X} [\varnothing \notin \mathcal{X} \implies \exists (f : \mathcal{X} \to \bigcup \mathcal{X}) \forall A (A \in \mathcal{X} \wedge f(A) \in A)]$$

2 Some common identities

Theorem 1 [3] *Given a set A, for all element $x \in A$ if $x \in U_x \subseteq A$ then*

$$\bigcup_{x \in A} U_x = A$$

Proof

We immediately have $\bigcup_{x \in A} U_x \subseteq A$. On the other hand, for all $x \in A$, $\{x\} \subseteq U_x$. Then

$$A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} U_x$$

Theorem 2 [4] *Let some property p be invariant over union of sets. Let J be an index set,*

$$\forall j \in J, p(A_j) \implies p\left(\bigcup_{j \in J} A_j\right)$$

Given a set A, for all element $x \in A$, if $x \in A_j \subseteq A$ and $p(A_j)$ then $p(A)$

[2] sometimes, we write the formula $\forall x (x \in A \implies \varphi$ as $\forall x \in A, \varphi$

[3] common techniques in set theory and general topology

[4] MIT 18.102 Intro to Functional Analysis - Dr. Casey Rodriguez

Proof
immediately from theorem 1

Theorem 3 [5] *For any index set J and $A_j \cap B_j = \varnothing$ for all $j \in J$*

$$\bigcap_{j \in J} A_j \cup B_j = \bigcup_{J_A \in \mathcal{P}(J)} \left[\left(\bigcap_{j \in J_A} A_j \right) \cap \left(\bigcap_{j \in J \setminus J_A} B_j \right) \right]$$

Proof
For all $x \in X = \bigcap_{j \in J} A_j \cup B_j$, for each $j \in J$, x must be either in A_j or B_j. Let $J_A(x) = \{j : j \in J \wedge x \in A_j\} \subseteq \mathcal{P}(J)$ be the set of indices where $x \in A_j$ and let $J_B(x) = J \setminus J_A(x)$. So that

$$x \in \left(\bigcap_{j \in J_A(x)} A_j \right) \cap \left(\bigcap_{j \in J \setminus J_A(x)} B_j \right)$$

On the other hand, $\bigcap_{j \in J_A} A_j \subseteq \bigcap_{j \in J_A} A_j \cup B_j$ and $\bigcap_{j \in J \setminus J_A} B_j \subseteq \bigcap_{j \in J \setminus J_A} A_J \cup B_j$, we have

$$\begin{aligned} \left(\bigcap_{j \in J_A(x)} A_j \right) \cap \left(\bigcap_{j \in J \setminus J_A(x)} B_j \right) &\subseteq \left(\bigcap_{j \in J_A(y)} A_j \cup B_j \right) \cap \left(\bigcap_{j \in J \setminus J_A(y)} A_j \cup B_j \right) \\ &= \bigcap_{j \in J} A_j \cup B_j \\ &= X \end{aligned}$$

Invoke theorem 1

Theorem 4 *Let $f : X \to Y$ be a injective function. i.e $f(x_1) = f(x_2) \implies x_1 = x_2$. Let I be an index set and $A_i \in X$ for all $i \in I$*

$$f(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} f(A_i)$$

and

$$f(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f(A_i)$$

Proof
$\bigcap$
($\subseteq$) Let $y \in f(\bigcap_{i \in I} A_i)$, there exists $x \in \bigcap_{i \in I} A_i$ such that $f(x) = y$. For any $i \in I$, $x \in A_i$ implies $y = f(x) \in f(A_i)$. Therefore, $y \in \bigcap_{i \in I} f(A_i)$

[5] Topology without tears - Sidney A. Morris

($\supseteq$) Let $y \in \bigcap_{i \in I} f(A_i)$. For any $y \in A_i$, there exists $x_i \in A_i$ such that $f(x_i) = y$. Since f is injective [6], all x_is equal, namely x. $f(x) = y$ and $x \in \bigcap_{i \in I} A_i$ implies $y \in f(\bigcap_{i \in I} A_i)$

$\bigcup$

($\subseteq$) Let $y \in f(\bigcup_{i \in I} A_i)$, there exists $x \in \bigcup_{i \in I} A_i$ such that $f(x) = y$. Let $I_x = \{i \in I : x = A_i\} \neq \varnothing$ be all indices i where $x \in A_i$ [7]. So, for all $i \in I_x$, $y = f(x) \in f(A_i)$. Therefore, $y = f(x) \in \bigcup_{i \in I_x} f(A_i) \subseteq \bigcup_{i \in I} f(A_i)$

($\supseteq$) Let $y \in \bigcup_{i \in I} f(A_i)$. Let $I_y = \{i \in I : y \in f(A_i)\} \neq \varnothing$ be all indices i where $y \in f(A_i)$ [8]. So for all $i \in I_y$, $y \in f(A_i)$ there exists $x_i \in A_i$ such that $f(x_i) = y$. Since f is injective, all x_is coincide, namely x. $f(x) = y$ and $x \in \bigcup_{i \in I_y} A_i$ implies $y \in f(\bigcup_{i \in I_y} A_i)$. Moreover, $\bigcup_{i \in I_y} A_i \subseteq \bigcup_{i \in I} A_i$, so $y \in f(\bigcup_{i \in I_y} A_i) \subseteq f(\bigcup_{i \in I} A_i)$

Corollary

$f(A \setminus B) = f(A \cap B^C) = f(A) \cap f(B^C) = f(A) \cap f(B)^C = f(A) \setminus f(B)$

[6] if f is not injective, consider $f(1 \mapsto 1, 2 \mapsto 1)$, then $f(\{1\} \cap \{2\}) \neq f(\{1\}) \cap f(\{2\})$

[7] here, we can choose an i such that $x \in A_i$ using AC to simplify the proof

[8] avoid AC