On the set function in $\mathbb{R}^n$

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In $\mathbb{R}^n$, let u be a set function on $\mathbb{R}^n$ which is a non-negative function on a subcollection of the collection of all subsets of $\mathbb{R}^n$, i.e. $u : \mathcal{U} \to [0, +\infty], \mathcal{U} \subseteq \mathcal{P}(\mathbb{R}^n)$.

We will construct the smallest set of assumptions on u such that the following limit converges for almost every x in a bounded outer measure X, i.e. $|X| < +\infty$

$$Du(x) = \lim_{B \searrow x} \frac{u(B)}{|B|}$$

where $B \searrow x$ denotes the contraction of B into a singleton x, and $|B|$ denotes the Lebesgue measure of B.

Assumption 1 (Closed balls) *$\overline{\mathcal{B}_r(x)} \in \mathcal{U}$ for all $x \in \mathbb{R}^n$ and $r > 0$*

Consider a simpler version where B is a closed ball of radius $r > 0$ centered at x, i.e. $B = \overline{\mathcal{B}_r(x)}$. Define lim sup, lim inf as follows

- $\overline{D}u(x) = \limsup_{B \searrow x} \frac{u(B)}{|B|} = \lim_{r \to 0^+} \left(\sup_{t \in (0,r)} \frac{u(B_t(x))}{|B_t(x)|} \right)$
- $\underline{D}u(x) = \liminf_{B \searrow x} \frac{u(B)}{|B|} = \lim_{r \to 0^+} \left(\inf_{t \in (0,r)} \frac{u(B_t(x))}{|B_t(x)|} \right)$

The necessary and sufficient condition for the existence of $Du(x)$ is $\overline{D}u(x) = \underline{D}u(x)$. If $\overline{D}u(x) > \underline{D}u(x)$, let $\alpha, \beta \in \mathbb{Q}$ such that $\overline{D}u(x) > \alpha > \beta > \underline{D}u(x) \geq 0$. Then $Du(x)$ exists for almost every $x \in X$ if

$$A_{\alpha\beta} = \{x \in X : \overline{D}u(x) > \alpha > \beta > \underline{D}u(x)\}$$

has measure zero for all $\alpha, \beta \in \mathbb{Q}$. If $x \in A_{\alpha\beta}$, then given any $\delta > 0$, there exist balls $B^{(\alpha)}, B^{(\beta)}$ centered at x with radius less than δ such that $\frac{u(B^{(\alpha)})}{|B^{(\alpha)}|} > \alpha > \beta > \frac{u(B^{(\beta)})}{|B^{(\beta)}|}$. That is, the collections $\{B^{(\alpha)}\}$ and $\{B^{(\beta)}\}$ each cover $A_{\alpha\beta}$ finely.

Lemma 1 (Vitali Covering Theorem for Lebesgue measure) *A collection $\mathcal{V}$ of closed balls is said to cover a set E finely if for every $x \in E$ and $\eta > 0$, there is a ball B containing x whose radius is less than η. Suppose E is covered finely by a collection $\mathcal{V}$ of balls such that $0 < \mu_e(E) < +\infty$* [1]*. Then, given any $\epsilon > 0$, there is a countable collection M of disjoint balls such that*

$$\mu\left(E \setminus \bigcup_{B_k \in M} B_k\right) = 0 \text{ and } \sum_{B_k \in M} \mu(B_k) < \mu_e(E) + \epsilon$$

Claim 1 *Given any open ball U contains a subset $E \subseteq A_{\alpha\beta}$, the $u(\overline{U}) \geq \alpha|E|_e$*

Given any open ball U contains a subset $E \subseteq A_{\alpha\beta}$, since U is open, for each $x \in E$, we can find a subcollection of balls $\{B_U^{(\alpha)}\} \subseteq \{B^{(\alpha)}\}$ that finely covers E and is contained in U. By Vitali Covering Theorem for Lebesgue measure on finite outer measure E with $\{B_U^{(\alpha)}\}$, given any $\eta > 0$, there exists a finite collection of pairwise disjoint balls $\{B_i\}_{i=1}^N$ contained in U such that $\left|E \setminus \bigcup_{i=1}^N B_i\right|_e < \eta/\alpha$ and $u(B_i) > \alpha|B_i|$

Assumption 2 (Additivity) *$A, B \in \mathcal{U}$ and $A \cap B = \varnothing \implies A \cup B \in \mathcal{U}$ and $u(A \cup B) = u(A) + u(B)$*

Assumption 3 (Subset) *$A, B \in \mathcal{U}$ and $A \subseteq B \implies u(A) \leq u(B)$*

[1] if $\mu_e(E) = 0$, the result is immediate

Then,

$$
\begin{aligned}
u(\overline{U}) &\geq u\left(\bigcup_{i=1}^{N} B_i\right) && \text{(subset)} \\
&= \sum_{i=1}^{N} u(B_i) && \text{(additivity)} \\
&> \alpha \sum_{i=1}^{N} |B_i| && (B_i \in \{B_U^{(\alpha)}\} \subseteq \{B^{(\alpha)}\}) \\
&= \alpha \left|\bigcup_{i=1}^{N} B_i\right| && \text{(additivity of Lebesgue measure)} \\
&= \alpha \left|E \cap \bigcup_{i=1}^{N} B_i\right|_e && \text{(subset of Lebesgue measure)} \\
&= \alpha \left(|E|_e - \left|E \setminus \bigcup_{i=1}^{N} B_i\right|_e\right) && \text{(Caratheodory condition)} \\
&= \alpha|E|_e - \eta && \text{(Vitali Covering Theorem for Lebesgue measure)}
\end{aligned}
$$

As the inequality is true for all $\eta > 0$, $u(\overline{U}) \geq \alpha|E|_e$

By Vitali Covering Theorem for Lebesgue measure on finite outer measure $A_{\alpha\beta}$ ($|A_{\alpha\beta}|_e \leq |X|_e < +\infty$) with $\{B^{(\beta)}\}$, given any $\epsilon > 0$, there exists a finite collection of pairwise disjoint balls $\{B_i\}_{i=1}^{N}$ such that $\left|A_{\alpha\beta} \setminus \bigcup_{i=1}^{N} B_i\right|_e < \epsilon$, $\sum_{i=1}^{N} |B_i| < |A_{\alpha\beta}|_e + \epsilon$, and $u(B_i) < \beta|B_i|$

Then

$$
\begin{aligned}
|A_{\alpha\beta}|_e + \epsilon &> \sum_{i=1}^{N} |B_i| && \text{(Vitali Covering Theorem for Lebesgue measure)} \\
&> \frac{1}{\beta} \sum_{i=1}^{N} \mu(B_i) && (B_i \in \{B_U^{(\beta)}\} \subseteq \{B^{(\beta)}\}) \\
&\geq \frac{\alpha}{\beta} \sum_{i=1}^{N} |B_i^o \cap A_{\alpha\beta}|_e && \text{(by claim 1 as open ball } B_i^o \text{ contains } B_i^o \cap A_{\alpha\beta} \subseteq A_{\alpha\beta}) \\
&\geq \frac{\alpha}{\beta} \left|A_{\alpha\beta} \cap \bigcup_{i=1}^{N} B_i^o\right|_e && \text{(subadditivity of Lebesgue outer measure)} \\
&\geq \frac{\alpha}{\beta} \left|A_{\alpha\beta} \cap \bigcup_{i=1}^{N} B_i\right|_e && (|A \cap B|_e \leq |A \cap B^o|_e + |A \cap B^b|_e\ {}^2) \\
&\geq \frac{\alpha}{\beta} \left(|A_{\alpha\beta}|_e - \left|A_{\alpha\beta} \setminus \bigcup_{i=1}^{N} B_i\right|_e\right) && \text{(Caratheodory condition)} \\
&> \frac{\alpha}{\beta}|A_{\alpha\beta}|_e - \frac{\alpha}{\beta}\epsilon && \text{(Vitali Covering Theorem for Lebesgue measure)}
\end{aligned}
$$

Send ϵ to 0^+ implies $|A_{\alpha\beta}| \geq \frac{\alpha}{\beta}|A_{\alpha\beta}|$. Hence $|A_{\alpha\beta}| = 0$

Theorem 1 *Given a set function $u : \mathcal{U} \to [0, +\infty]$ defined for all closed balls $\overline{\mathcal{B}_r(x)}, r > 0$ that is equipped with (1) additivity and (2) subset, Then*

$$Du(x) = \lim_{B \searrow x} \frac{u(B)}{|B|}$$

is defined for almost every $x \in X$ where $X \subseteq \mathbb{R}^n$ is any finite outer measure set and $B \searrow x$ denotes the contraction of closed balls $\overline{\mathcal{B}_r(x)}, r > 0$ into a singleton x

Theorem 1 is true for every $X_r = \mathcal{B}_r(0), r \in \mathbb{N}$, then it is also true for $\mathbb{R}^n$.

Corollary 1 *Given a set function $u : \mathcal{U} \to [0, +\infty]$ defined for all closed balls $\overline{\mathcal{B}_r(x)}, r > 0$ that is equipped with (1) additivity and (2) subset, Then*

$$Du(x) = \lim_{B \searrow x} \frac{u(B)}{|B|}$$

is defined for almost every $x \in \mathbb{R}^n$ set where $B \searrow x$ denotes the contraction of closed balls $\overline{\mathcal{B}_r(x)}, r > 0$ into a singleton x

Remark 1 *Some remarks*

- *If u is the Lebesgue measure, then $Du(x) = 1$ everywhere.*
- *If u is any positive measure on $\mathbb{R}^n$, $Du(x)$ is defined everywhere.*
- *If $f : \mathbb{R}^n \to [0, +\infty]$ is Lebesgue measurable, let $u(E) = \int_E f$, then we have the Lebesgue Differentiation Theorem*

Definition 1 (Absolute Continuous Set Function) *A set function $u : \mathcal{U} \to [0, +\infty]$ is defined to be absolute continous if given any $\epsilon > 0$, there exists a $\delta > 0$ such that if $E \in \mathcal{U}$, $|E| < \delta$, then $u(E) < \epsilon$*

Lemma 2 (Theorem 7.28) *If $u : \mathcal{U} \to [0, +\infty]$ is an absolute continuous set function, $E \in \mathcal{U}$ with $|E| < +\infty$ and $\lim_{B \searrow x} \frac{u(B)}{|B|} = 0$ for every $x \in E$, then $u(E) = 0$*

$u : \mathcal{U} \to [0, +\infty]$ is absolute continous, that is, given any $\epsilon > 0$, there exists a $\delta > 0$ such that $E \in \mathcal{U}, |E| < \delta$ then $u(E) < \epsilon$

For each $x \in E$, $\lim_{B \searrow x} \frac{u(B)}{|B|} = 0$, that is, given any $\epsilon > 0$, then there exists a $\delta > 0$ such that for all ball B with radius less than δ, i.e. $r(B) < \delta$, then $u(B) < \epsilon|B|$. Fix $\epsilon > 0$, there is a collection of balls $\{B^{(\epsilon)}\}$ that finely covers E and $u(B) < \epsilon|B|$ for all $B \in \{B^{(\epsilon)}\}$. By Vitali Covering Theorem for Lebesgue measure on finite measure E with $\{B^{(\epsilon)}\}$, given any $\epsilon, \delta > 0$, there exists a finite collection of pairwise disjoint balls $\{B_i\}_{i=1}^N$ such that $|E \setminus \bigcup_{i=1}^N B_i| < \delta$ and $\sum_{i=1}^N |B_i| < |E| + \epsilon$. Hence,

$$\begin{aligned}
u(E) &= u\left(E \setminus \bigcup_{i=1}^N B_i\right) + u\left(E \cap \bigcup_{i=1}^N B_i\right) && \left(\text{additivity, suppose } E \setminus \bigcup_{i=1}^N B_i, E \cap \bigcup_{i=1}^N B_i \in \mathcal{U}\right) \\
&\le \epsilon + u\left(E \cap \bigcup_{i=1}^N B_i\right) && (\text{Vitali Covering Theorem for Lebesgue measure, absolute continuous}) \\
&\le \epsilon + u\left(\bigcup_{i=1}^N B_i\right) && (\text{subset}) \\
&= \epsilon + \sum_{i=1}^N u(B_i) && (\text{additivity}) \\
&\le \epsilon + \epsilon \sum_{i=1}^N |B_i| && (B_i \in \{B^{(\epsilon)}\}) \\
&\le \epsilon + \epsilon(|E| + \epsilon) && (\text{Vitali Covering Theorem for Lebesgue measure})
\end{aligned}$$

As $|E| < +\infty$, send ϵ to 0^+ implies $u(E) = 0$

Remark 2 *Put $u([a, b]) = |f(b) - f(a)|$, $\lim_{B \searrow x} \frac{u(B)}{|B|} = 0$ implies f being singular, recover **Theorem 7.28***

Corollary 2 *If $u : \mathcal{U} \to [0, +\infty]$ is an absolute continuous set function, $E \in \mathcal{U}$ with $|E| < +\infty$ and $\lim_{B \searrow x} \frac{u(B)}{|B|} = 0$ for **almost** every $x \in E$, then $u(E) = 0$*

Let $F \subseteq E$ with $|F| = 0$ and $\lim_{B \searrow x} \frac{u(B)}{|B|} = 0$ in $E \setminus F$. $u(E) = u(E \setminus F) + u(F) = u(F)$. By absolute continuity of u, $u(E) = u(F) = 0$

Theorem 2 (Theorem 7.29) *If $u : \mathcal{U} \to [0, +\infty]$ is an absolute continuous set function. Given $E \in \mathcal{U}$ with $|E|_e < +\infty$, given necessary conditions (additivity, subset) then $u'(x) = \lim_{B \searrow x} \frac{u(B)}{|B|}$ is defined almost every $x \in E$. Furthermore,*

$$u(E) = \int_E u'$$

for all $E \in \mathcal{U} \cap \mathcal{M}(\mathbb{R}^n)$.

u' is defined almost every x in a finite outer measure E is immediate from theorem 1. For the second part, let $v(E) = u(E) - \int_E u'$ for all $E \in \mathcal{U} \cap \mathcal{M}(\mathbb{R}^n)$, then

$$\lim_{B \searrow x} \frac{v(E)}{|E|} = \lim_{B \searrow x} \frac{u(E) - \int_E u'}{|E|} = \lim_{B \searrow x} \frac{u(E)}{|E|} - \lim_{B \searrow x} \frac{\int_E u'}{|E|} = 0 \text{ for almost every } x \in E$$

By corollary 2, $v(E) = 0$ almost every where.

Remark 3 *Put* $u([a,b]) = |f(b) - f(a)|$, $f' = u' = \lim_{B \searrow x} \frac{u(B)}{|B|}$, *recover* ***Theorem 7.29***