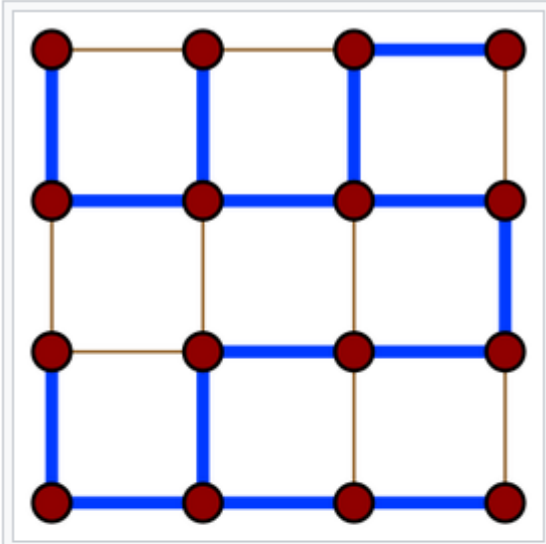


Zorn's Lemma

Nguyen Ngoc Khanh

October 2023



Zorn's lemma can be used to show that every connected graph has a spanning tree. The set of all sub-graphs that are trees is ordered by inclusion, and the union of a chain is an upper bound. Zorn's lemma says that a maximal tree must exist, which is a spanning tree since the graph is connected.^[1] Zorn's lemma is not needed for finite graphs, such as the one pictured here.

Definition 1 (Partially Order Set) A partial order on a set X is a binary relation, denoted by \leq such that

1. reflexivity: $x \leq x$ for all $x \in X$
2. anti-symmetry: $x = y$ if $x \leq y$ and $y \leq x$ for all $x, y \in X$
3. transitivity: $x \leq z$ if $x \leq y$ and $y \leq z$ for all $x, y, z \in X$

A set X equipped with a partial order \leq is said to be a partially ordered set, denoted by (X, \leq)

Definition 2 (Linearly Ordered Set) A partially ordered set (X, \leq) is said to be a linearly ordered set if every two elements are comparable, that is, $x \leq y$ or $y \leq x$ for all $x, y \in X$. The order \leq is said to be a linear order.

Definition 3 (Upper Bound) Given a partially ordered set (X, \leq) , an element $t \in X$ is said to be an upper bound of a subset $Y \subseteq X$ if $y \leq t$ for all $y \in Y$

Definition 4 (Maximal) Given a partially ordered set (X, \leq) , an element $w \in X$ is said to be maximal if $x \in X$ and $w \leq x$, then $x = w$

Axiom 1 (Zorn's Lemma) Let (X, \leq) be a non-empty partially ordered set in which every linearly ordered subset has an upper bound. Then, (X, \leq) has a maximal element.