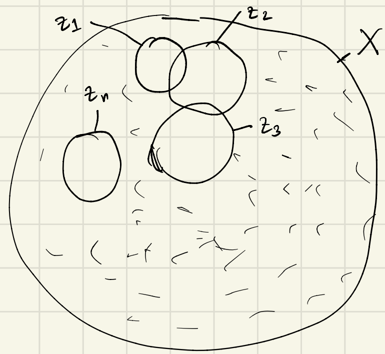
2023/04/01, I found an interesting proof on sum of measurable functions being measurable while watching MIT 18.102 Functional Analysis, Lecture 9: Lebesgue Measurable Functions, Spring 2021 by Dr Casey Rodriguez.

The proof can be summarized as follows:

We want to prove some property p on set X given the premise
If p is true on a countable collection of sets Z, p is true on the union

$$\bigwedge_{n=1}^{\infty} p(Z_n) \implies p\left(\bigcup_{n=1}^{\infty} Z_n\right)$$



for every element $x \in X$
if we can find a patch Z on X that covers x : $\exists Z \subseteq X, x \in Z$
Then we have

$$\bigcup Z = X$$

In the proof for sum of measurable functions being measurable, the author associates each set Z to a rational number which makes $\bigcup Z$ being a countable union.
The full proof is as below:

<u>Dfn</u>: $f : E \to [-\infty, +\infty]$ where $E \subseteq \mathbb{R}$ is Lebesgue measurable if $\forall \alpha \in \mathbb{R}$, $f^{-1}((\alpha, +\infty])$ is a measurable set. $(\in \mathcal{M})$

<u>Col</u>: $\bigwedge_{n=1}^{\infty} Z_n \in \mathcal{M} \Rightarrow \left(\bigcup_{n=1}^{\infty} Z_n\right) \in \mathcal{M}$

Let $f, g \in \mathcal{M}$
Let $\alpha \in \mathbb{R}$, consider x such that

$f(x) + g(x) > \alpha$

$\Rightarrow f(x) > \alpha - g(x)$

Let r be a rational between LHS and RHS

$f(x) > r > \alpha - g(x)$

$$\Rightarrow \begin{cases} f(x) > r \\ g(x) > \alpha - r \end{cases}$$

So,

$$x \in Z_r = \left(\overbrace{f^{-1}((r, +\infty])}^{\text{measurable}} \cap \overbrace{g^{-1}((\alpha - r, +\infty])}^{\text{measurable}}\right)$$

where $Z_r \subseteq (f+g)^{-1}((\alpha, +\infty])$

Hence

$$\bigcup_{r \in R} Z_r = (f+g)^{-1}((\alpha, +\infty])$$

for some subset $R \subseteq \mathbb{Q}$ rationals

$\square$