# MA5204 Homework 4

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# 0.1 PROBLEM 1 2 3

**Problem 0.1.1** (problem 1)
Let $P_\bullet$ and $Q_\bullet$ be projective resolutions of $A$-modules $M$ and $N$, respectively. In class we proved that any $A$-module map $f : M \to N$ lifts to a chain complex map $\phi_\bullet : P_\bullet \to Q_\bullet$, but the lift depends on choices. Prove that any two lifts $\phi_\bullet, \psi_\bullet$ are homotopic.

*Proof.* Note that, the proof only requires $P_\bullet$ to be projective chain complex and $Q_\bullet$ to be exact. Recall the construction of lifts of $f$. Let $K_n = \ker(d : Q_n \to Q_{n-1}) = \operatorname{im}(d : Q_{n+1} \to Q_n)$, then there is a map $\alpha_n : P_{n+1} \to K_n$. Since $P_{n+1}$ is projective, $\alpha_n$ lifts into $\phi_{n+1} : P_{n+1} \to Q_{n+1}$

$$\begin{tikzcd}
P_{n+2} \arrow[rr, "d"] \arrow[dr, dashed, "\alpha_n"] & & P_{n+1} \arrow[rr, "d"] \arrow[dr, "\alpha_n"] \arrow[dd, dashed, "\phi_{n+1}"] & & P_n \arrow[dd, "\phi_n"] \\
& K_{n+1} \arrow[dr, hook] & & K_n \arrow[dr, hook] & \\
Q_{n+2} \arrow[rr, "d"] \arrow[ur, two heads] & & Q_{n+1} \arrow[rr, "d"] \arrow[ur, two heads] & & Q_n
\end{tikzcd}$$

Now, the composition $P_{n+2} \to P_{n+1} \to Q_{n+1} \to Q_n$ equals $P_{n+2} \to P_{n+1} \to P_n \to Q_n$ equals zero. Hence the map $P_{n+2} \to P_{n+1} \to Q_{n+1}$ factors through $K_{n+1}$ by a map $\alpha_{n+1} : P_{n+2} \to Q_{n+1}$.
Let $g_n = \phi_n - \psi_n$, then $g_\bullet$ is a chain map, we will show by induction that there exists a collection of maps $\{h_i : P_{i-1} \to Q_i\}_{i \in \mathbb{N}}$ so that

$$g_i = h_i d + d h_{i+1}$$

where $d$ are the appropriate maps on exact sequences $P_\bullet$ and $Q_\bullet$.
**Induction case**: Suppose we have maps $h_n : P_{n-1} \to Q_n$ and $h_{n-1} : P_{n-2} \to Q_{n-1}$ for some $n - 1 \geq 0$, so that

$$g_{n-1} = h_{n-1} d + d h_n$$

We construct $h_{n+1} : P_{n+2} \to Q_{n+1}$ as follows:

$$\begin{tikzcd}
P_n \arrow[r, "d"] \arrow[d, "g_n"'] & P_{n-1} \arrow[dl, "h_n"] \\
Q_n \arrow[r, "d"'] & Q_{n-1}
\end{tikzcd}$$

Consider the map $\beta_n = g_n - h_n d : P_n \to Q_n$ and the composition $d\beta_n : P_n \to Q_{n-1}$, we have

$$\begin{aligned}
d\beta_n &= d(g_n - h_n d) \\
&= dg_n - dh_n d && \text{($A$-module is preadditive)} \\
&= dg_n - (g_{n-1} - h_{n-1} d) d && \text{(induction hypothesis)} \\
&= dg_n - g_{n-1} d + h_{n-1} dd && \text{($A$-module is preadditive)} \\
&= dg_n - g_{n-1} d && \text{(top sequence is a chain complex)} \\
&= 0 && \text{($g_\bullet$ is a chain map)}
\end{aligned}$$

Then, $\beta_n$ factors through $K_n$ by a map $P_n \to K_n$, since $P_n$ is projective, it factors through $Q_{n+1}$ by a map $h_{n+1} : P_n \to Q_{n+1}$

$$\begin{tikzcd}
 & & P_n \arrow[dll, dashed, bend right, "h_{n+1}"'] \arrow[dl, dashed] \arrow[dd, "\beta_n"] & \\
 & K_n \arrow[dr] & & \\
Q_{n+1} \arrow[ur, two heads] \arrow[rr, "d"'] & & Q_n \arrow[r, "d"'] & Q_{n-1}
\end{tikzcd}$$

Precisely, we have $\beta_n = dh_{n+1}$. Hence, $g_n = h_n d + dh_{n+1}$

**Case case**: We construct $h_0 : 0 \to Q_0$ and $h_1 : P_0 \to Q_1$ as follows:

$$\begin{tikzcd}
 & P_0 \arrow[r] \arrow[dl, "h_1"'] \arrow[d, "g_0"] & 0 \arrow[dl, "h_0"] \\
Q_1 \arrow[r, two heads, "d"'] & Q_0 \arrow[r] & 0
\end{tikzcd}$$

$h_0 = 0$ is the unique zero map. $P_0$ is projective, so $g_0 = \phi_0 - \psi_0$ factors through $Q_1$ by a map $h_1 : P_0 \to Q_1$, then $g_0 = dh_1 = dh_1 + h_0 d$ □

**Problem 0.1.2** (problem 2)

Let $P_\bullet$ and $Q_\bullet$ be projective resolution of an $A$-module $M$. Prove that they are homotopy equivalent

*Proof.* The lifting from a map in $A$-modules into chain complexes of $A$-modules is a functor, that is if $f_\bullet, g_\bullet$ are lifts of $f, g$, then $g_\bullet \circ f_\bullet$ is a lift of $g \circ f$

$$\begin{tikzcd}
L \arrow[r, "f"'] \arrow[rr, bend left, "g \circ f"] & M \arrow[r, "g"'] & N
\end{tikzcd}$$

$$\begin{tikzcd}
P_\bullet \arrow[r, "f_\bullet"] \arrow[rr, bend right, "g_\bullet \circ f_\bullet"'] & Q_\bullet \arrow[r, "g_\bullet"] & R_\bullet
\end{tikzcd}$$

Hence, the identity map $1_M : M \to M$ lifts into 2 chain maps $f_\bullet : P_\bullet \to Q_\bullet$, $g_\bullet : Q_\bullet \to P_\bullet$ as in the diagram below

$$\begin{tikzcd}
M \arrow[r, "1_M"'] \arrow[rr, bend left, "1_M"] & M \arrow[r, "1_M"'] & M
\end{tikzcd}$$

$$\begin{tikzcd}
P_\bullet \arrow[r, "f_\bullet"] \arrow[rr, bend right, "g_\bullet \circ f_\bullet"'] & Q_\bullet \arrow[r, "g_\bullet"] & P_\bullet
\end{tikzcd}$$

Then, $g_\bullet \circ f_\bullet$ is also lift of $1_M : M \to M$. On the other hand, the identity chain map $1_{P_\bullet} : P_\bullet \to P_\bullet$ is also a lift of $1_M : M \to M$, by Problem 0.1.1, $g_\bullet \circ f_\bullet \sim 1_{P_\bullet}$. Using the same argument, $f_\bullet \circ g_\bullet \sim 1_{Q_\bullet}$, hence $P_\bullet$ and $Q_\bullet$ are homotopy equivalent.

□

**Problem 0.1.3** (problem 3)

If $0 \to L \to M \to N \to 0$ is a short exact sequence of $A$-modules, prove that we can find a compatible short exact sequence of projective resolutions $0 \to P_\bullet \to Q_\bullet \to R_\bullet \to 0$

*Proof.* Pick arbitrary projective resolutions $P_\bullet$ and $R_\bullet$ of $L$ and $N$ respectively. We will show by induction that there exists a projective resolution $Q_\bullet$ of $M$ so that $0 \to P_\bullet \to Q_\bullet \to R_\bullet \to 0$ is a short exact sequence.

Let $Q_n = P_n \oplus R_n$ and the canonical maps $f_n : P_n \hookrightarrow Q_n$ and $g_n : Q_n \twoheadrightarrow R_n$, we will construct maps $Q_0 \to M$ and $Q_{n+1} \to Q_n$ so that $Q_\bullet$ is a projective resolution of $M$. Note that, since both $P_n$ and $R_n$ are projective, $Q_n$ is also projective.

**Induction case**: For any $n \geq 0$, suppose we have a commutative diagram as follows (does not include dash arrows), all rows are exact, all columns split

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\ldots \arrow[r] & P_{n+1} \arrow[r, "d"] \arrow[d, hook, "f_{n+1}"] & P_n \arrow[r, "d"] \arrow[d, hook, "f_n"] & P_{n-1} \arrow[r] \arrow[d, hook, "f_{n-1}"] & \ldots \\
\ldots \arrow[r, dashed] & Q_{n+1} \arrow[r, dashed, "f_n d \oplus r_n d"] \arrow[d, two heads, "g_{n+1}"] & Q_n \arrow[r, "h"] \arrow[d, two heads, "g_n"] & Q_{n-1} \arrow[r] \arrow[d, two heads, "g_{n-1}"] & \ldots \\
\ldots \arrow[r] & R_{n+1} \arrow[r, "d"'] \arrow[d] \arrow[u, bend left, hook, "r_{n+1}"] & R_n \arrow[r, "d"] \arrow[d] \arrow[u, bend left, hook, "r_n"] & R_{n-1} \arrow[r] \arrow[d] \arrow[u, bend left, hook, "r_{n-1}"] & \ldots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Let the map $Q_{n+1} \to Q_n$ be $f_n d \oplus r_n d$. The sequence $Q_\bullet$ is exact at $Q_n$ since

$$
\begin{aligned}
&\operatorname{im}(f_n d \oplus r_n d) \\
&= \operatorname{im}(d : P_{n+1} \to P_n) \oplus \operatorname{im}(d : R_{n+1} \to R_n) \\
&= \ker(d : P_n \to P_{n-1}) \oplus \ker(d : R_n \to R_{n-1}) \\
&= \ker h
\end{aligned}
$$

**Base case**: Since $M \twoheadrightarrow N$ is surjective and $R_0$ is projective, there is a map $\phi : R_0 \to M$ lifted from $\eta : R_0 \to N$. Let the map $Q_0 \to M$ be $f\epsilon \oplus \phi$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\ldots \arrow[r] & P_0 \arrow[r, two heads, "\epsilon"] \arrow[d, hook] & L \arrow[r] \arrow[d, hook, "f"] & 0 \\
\ldots \arrow[r, dashed] & P_0 \oplus R_0 \arrow[r, dashed, two heads, "f\epsilon\oplus\phi"] \arrow[d, two heads] & M \arrow[r] \arrow[d, two heads] & 0 \\
\ldots \arrow[r] & R_0 \arrow[r, two heads, "\eta"'] \arrow[d] \arrow[ur, dashed, "\phi"] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}
$$

It remains to show that $f\epsilon \oplus \phi$ is surjective. By snake lemma,

$$0 = \operatorname{coker} \epsilon \to \operatorname{coker}(f\epsilon \oplus \phi) \to \operatorname{coker} \eta = 0$$

is exact. Hence, $\operatorname{coker}(f\epsilon \oplus \phi) = 0$, $f\epsilon \oplus \phi$ is surjective.

$\square$

## 0.2 PROBLEM 4

**Lemma 0.2.1**

Some basic facts about $\mathrm{Tor}$

1. If $L$ is a flat $A$-module, then $\mathrm{Tor}_n^A(-, L) = 0$ for $n \geq 1$
2. If $P$ is a projective $A$-module, then $\mathrm{Tor}_n^A(P, -) = 0$ for $n \geq 1$.
3. If $P$ is a projective $A$-module, then $P$ is flat. Hence, $\mathrm{Tor}_n^A(-, P) = 0$ for $n \geq 1$.

*Proof.*
($\mathrm{Tor}_n(-, L) = 0$ for $n \geq 1$) Let $... \to Q_2 \to Q_1 \to Q_0 \to 0$ be a projective resolution of $N$, since $L$ is flat, $... \to Q_2 \otimes L \to Q_1 \otimes L \to Q_0 \otimes L \to 0$ is exact at every $Q_n \otimes L$ for $n \geq 1$. Hence, $\mathrm{Tor}_n(-, L) = 0$ for $n \geq 1$
($\mathrm{Tor}_n(P, -) = 0$ for $n \geq 1$) A projective resolution for $P$ is $0 \to P \to P \to 0$
($P$ is flat) Let $Q \oplus P$ be a free module, hence also flat. Given any injection $M \hookrightarrow N$. In the diagram below, all columns split and the middle sequence is exact

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & M \otimes P \arrow[r] \arrow[d] & N \otimes P \arrow[d] \\
0 \arrow[r] & M \otimes (P \oplus Q) \arrow[r, hook] \arrow[d] & N \otimes (P \oplus Q) \arrow[d] \\
 & M \otimes Q \arrow[r] \arrow[d] & N \otimes Q \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

By snake lemma, $0 \to \ker(M \otimes P \to N \otimes P) \to 0$ is exact. So, $(- \otimes P)$ preserves injection

$\square$

**Lemma 0.2.2**

If $0 \to M \to N \to P \to 0$ is a short exact sequence of $A$-modules with $P$ being projective, then for any $A$-modules, the sequence

$$0 \to M \otimes J \to N \otimes J \to P \otimes J \to 0$$

is also exact

*Proof.* $P$ being projective, so $1_P : P \to P$ factors through $N$, that is, the sequence $0 \to M \to N \to P \to 0$ splits. $N = M \oplus P$. So

$$N \otimes J = (M \otimes J) \oplus (P \otimes J)$$

The induced maps $M \otimes J \to N \otimes J$ and $N \otimes J \to P \otimes J$ from $(- \otimes J)$ are precisely the canonical injection and canonical projection

$$
\begin{tikzcd}
M \arrow[r, "f"] & N \arrow[r, "g"] & P \\
m \arrow[r, mapsto, "f"] & (m,0) & \\
 & (m,p) \arrow[r, mapsto, "g"] & p \\
M \otimes J \arrow[r, "f\otimes 1"] & N \otimes J \arrow[r, "g\otimes 1"] & P \otimes J
\end{tikzcd}
$$

$$m \otimes j_1 \xmapsto{f\otimes 1} (m,0) \otimes j_1 = (m \otimes j_1, 0)$$

$$(m \otimes j_2, p \otimes j_2) = (m,p) \otimes j_2 \xmapsto{g\otimes 1} p \otimes j_2$$

Hence, $0 \to M \otimes J \to N \otimes J \to P \otimes J \to 0$ also splits □

**Lemma 0.2.3**

Tensor product preserves chain complex and chain homotopy, that is

1. If $C_\bullet$ is a chain complex then $C_\bullet \otimes J$

$$... \to C_{n+1} \otimes J \to C_n \otimes J \to C_{n-1} \otimes J \to ...$$

is also a chain complex for any $A$-module $J$.

2. If $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ are chain homotopic by a chain homotopy $h_\bullet$, then

$$
\begin{tikzcd}
 & C_n \arrow[r, "d"] \arrow[dl, "h_{n+1}"'] & C_{n-1} \arrow[dl, "h_n"] \\
D_{n+1} \arrow[r, "d"] & D_n &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C_n \otimes J \arrow[r, "Fd"] \arrow[dl, "Fh_{n+1}"'] & C_{n-1} \otimes J \arrow[dl, "Fh_n"] \\
D_{n+1} \otimes J \arrow[r, "Fd"] & D_n \otimes J &
\end{tikzcd}
$$

$Fh_\bullet$ is also a chain homotopy where $F(-)$ denotes the tensor product $(- \otimes J)$ functor

*Proof.* We write $F(-)$ for the functor $(- \otimes J)$

(Tensor product preserves chain complex)

$$(Fd)(Fd) = F(dd) = 0$$

(Tensor product preserves chain homotopy)

$$
\begin{aligned}
& (Fd)(Fh_{n+1}) + (Fh_n)(Fd) & \\
& = F(dh_{n+1}) + F(h_n d) & (F \text{ is a functor}) \\
& = F(dh_{n+1} + h_n d) & (F \text{ is additive}) \\
& = F(f_n - g_n) & (f_\bullet \sim g_\bullet \text{ by } h_\bullet) \\
& = F(f_n) - F(g_n) & (F \text{ is a functor})
\end{aligned}
$$

□

**Lemma 0.2.4**

Let $0 \to L \to M \to N \to 0$ be a short exact sequence of $A$-modules, then

1. (version 1) there exists a natural long exact sequence of $A$-modules

$$\begin{tikzcd}
 & \dots \arrow[r] & \operatorname{Tor}_2(J,N) \arrow[dll] \\
\operatorname{Tor}_1(J,L) \arrow[r] & \operatorname{Tor}_1(J,M) \arrow[r] & \operatorname{Tor}_1(J,N) \arrow[dll] \\
J \otimes L \arrow[r] & J \otimes M \arrow[r] & J \otimes N \arrow[r] & 0
\end{tikzcd}$$

2. (version 2) there exists a natural long exact sequence of $A$-modules

$$\begin{tikzcd}
 & \dots \arrow[r] & \operatorname{Tor}_2(N,J) \arrow[dll] \\
\operatorname{Tor}_1(L,J) \arrow[r] & \operatorname{Tor}_1(M,J) \arrow[r] & \operatorname{Tor}_1(N,J) \arrow[dll] \\
L \otimes J \arrow[r] & M \otimes J \arrow[r] & N \otimes J \arrow[r] & 0
\end{tikzcd}$$

*Proof.*

(version 1) Let $P_\bullet$ be a projective resolution of $J$, by Lemma 0.2.1 each $P_n$ is flat, hence

$$0 \to P_\bullet \otimes L \to P_\bullet \otimes M \to P_\bullet \otimes N \to 0$$

is a short exact sequence of chain complexes (rows are exact by Lemma 0.2.1, columns are chain complexes by Lemma 0.2.3). By fundamental lemma of homological algebra, there is a natual long exact sequence

$$\begin{tikzcd}
 & \dots \arrow[r] & H_2(P_\bullet \otimes N) \arrow[dll] \\
H_1(P_\bullet \otimes L) \arrow[r] & H_1(P_\bullet \otimes M) \arrow[r] & H_1(P_\bullet \otimes N) \arrow[dll] \\
H_0(P_\bullet \otimes L) \arrow[r] & H_0(P_\bullet \otimes M) \arrow[r] & H_0(P_\bullet \otimes N) \arrow[r] & 0
\end{tikzcd}$$

Since $(- \otimes L)$ and $(- \otimes M)$ are right exact, the rows in bottom diagram are exact.

$$\begin{tikzcd}
L \arrow[d, "f"] \\
M
\end{tikzcd}$$

$$\begin{tikzcd}
P_1 \otimes L \arrow[r] \arrow[d] & P_0 \otimes L \arrow[r] \arrow[d] & J \otimes L \arrow[r] \arrow[d, "1\otimes f"] & 0 \\
P_1 \otimes M \arrow[r] & P_0 \otimes M \arrow[r] & J \otimes M \arrow[r] & 0
\end{tikzcd}$$

Hence,

$$H_0(P_\bullet \otimes L) = \frac{\ker(P_0 \otimes L \to 0)}{\operatorname{im}(P_1 \otimes L \to P_0 \otimes L)} = \frac{P_0 \otimes L}{\operatorname{im}(d : P_1 \otimes L \to P_0 \otimes L)} = \operatorname{coker}(P_1 \otimes L \to P_0 \otimes L) = J \otimes L$$

and the map $1 \otimes f : J \otimes L \to J \otimes M$ is precisely the induced map from $P_0 \otimes L \to P_0 \otimes M$ into its map in homology.
(version 2) Let $P_\bullet, Q_\bullet, R_\bullet$ be projective resolutions of $M, N, L$ in Problem 0.1.3, then

$$0 \to P_\bullet \otimes J \to Q_\bullet \otimes J \to R_\bullet \otimes J \to 0$$

is also a short exact sequence of chain complexes (rows are exact by Lemma 0.2.2, columns are chain complexes by Lemma 0.2.3). By fundamental lemma of homological algebra, there is a natual long exact sequence

$$\begin{tikzcd}
 & \dots \arrow[r] & H_2(R_\bullet \otimes J) \arrow[dll] & \\
H_1(P_\bullet \otimes J) \arrow[r] & H_1(Q_\bullet \otimes J) \arrow[r] & H_1(R_\bullet \otimes J) \arrow[dll] & \\
H_0(P_\bullet \otimes J) \arrow[r] & H_0(Q_\bullet \otimes J) \arrow[r] & H_0(R_\bullet \otimes J) \arrow[r] & 0
\end{tikzcd}$$

Since $(- \otimes J)$ is right exact, the rows in bottom diagram are exact

$$\begin{tikzcd}
P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, "f"] & 0 \\
Q_1 \arrow[r] & Q_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
P_1 \otimes J \arrow[r] \arrow[d] & P_0 \otimes J \arrow[r] \arrow[d] & L \otimes J \arrow[r] \arrow[d, "f\otimes 1"] & 0 \\
Q_1 \otimes J \arrow[r] & Q_0 \otimes J \arrow[r] & M \otimes J \arrow[r] & 0
\end{tikzcd}$$

Hence,

$$H_0(P_\bullet \otimes J) = \frac{\ker(P_0 \otimes J \to 0)}{\operatorname{im}(P_1 \otimes J \to P_0 \otimes J)} = \frac{P_0 \otimes J}{\operatorname{im}(P_1 \otimes J \to P_0 \otimes J)} = \operatorname{coker}(P_1 \otimes J \to P_0 \otimes J) = L \otimes J$$

and the map $(f \otimes 1) : L \otimes J \to M \otimes J$ is precisely the induced map from $P_0 \otimes J \to Q_0 \otimes J$ into its map in homology. ☐

**Problem 0.2.5** (problem 4)
For $A$-modules $M$ and $N$, we define $\operatorname{Tor}_i^A(M, N) = H_i(P_\bullet \otimes_A N)$ where $P_\bullet$ is a projective resolution of $M$. Prove that $\operatorname{Tor}_i^A(M, N) \cong \operatorname{Tor}_i^A(N, M)$

*Proof.* Let $\dots \to P_2 \to P_1 \to P_0 \to M \to 0$ be a projective resolution of $M$, we have the following diagonal short exact sequences

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dr] & & 0 & & & \\
 & K_3 \arrow[dr, hook] & & & & K_1 \arrow[ur] \arrow[dr, hook] & & & & \\
\dots \arrow[rr] & & P_2 \arrow[rr, "d"] \arrow[dr, two heads] & & P_1 \arrow[rr, "d"] \arrow[ur, two heads] & & P_0 \arrow[rr, "\epsilon"] \arrow[dr, two heads] & & M \arrow[r] & 0 \\
 & & & K_2 \arrow[ur, hook] \arrow[dr] & & & & M \arrow[dr] & & \\
 & & 0 \arrow[ur] & & 0 & & & & 0 &
\end{tikzcd}$$

where $K_1 = \ker(\epsilon : P_0 \to M)$ and $K_n = \ker(d : P_{n-1} \to P_{n-2})$. For any $n \geq 1$, the map $P_n \to K_n$ is lifted from $d : P_n \to P_{n-1}$ since the composition $P_n \to P_{n-1} \to P_{n-2}$ is zero[1]. Since $K_n = \operatorname{im}(d : P_n \to P_{n-1})$, $P_n \to K_n$ is surjective, hence every diagonal sequence is exact.

From $0 \to K_1 \to P_0 \to M \to 0$, for any $n \geq 0$, by Lemma 0.2.4 we have two exact sequences

$$0 = \operatorname{Tor}_{n+1}(P_0, N) \longrightarrow \operatorname{Tor}_{n+1}(M, N) \longrightarrow \operatorname{Tor}_n(K_1, N) \longrightarrow \operatorname{Tor}_n(P_0, N) = 0$$

$$0 = \operatorname{Tor}_{n+1}(N, P_0) \longrightarrow \operatorname{Tor}_{n+1}(N, M) \longrightarrow \operatorname{Tor}_n(N, K_1) \longrightarrow \operatorname{Tor}_n(N, P_0) = 0$$

Hence, $\operatorname{Tor}_{n+1}(M, N) = \operatorname{Tor}_n(K_1, N)$ and $\operatorname{Tor}_{n+1}(N, M) = \operatorname{Tor}_n(N, K_1)$

From $0 \to K_2 \to P_1 \to K_1 \to 0$, for any $n \geq 0$, by Lemma 0.2.4 we have two exact sequences

$$0 = \operatorname{Tor}_{n+1}(P_1, N) \longrightarrow \operatorname{Tor}_{n+1}(K_1, N) \longrightarrow \operatorname{Tor}_n(K_2, N) \longrightarrow \operatorname{Tor}_n(P_1, N) = 0$$

$$0 = \operatorname{Tor}_{n+1}(N, P_1) \longrightarrow \operatorname{Tor}_{n+1}(N, K_1) \longrightarrow \operatorname{Tor}_n(N, K_2) \longrightarrow \operatorname{Tor}_n(N, P_1) = 0$$

Similarly, we have $\operatorname{Tor}_{n+1}(K_1, N) = \operatorname{Tor}_n(K_2, N)$ and $\operatorname{Tor}_{n+1}(N, K_1) = \operatorname{Tor}_n(K_2, N)$. Hence

$$\operatorname{Tor}_{n+1}(M, N) = \operatorname{Tor}_n(K_1, N) = \operatorname{Tor}_{n-1}(K_2, N) = \ldots = \operatorname{Tor}_1(K_n, N)$$
$$\operatorname{Tor}_{n+1}(N, M) = \operatorname{Tor}_n(N, K_1) = \operatorname{Tor}_{n-1}(N, K_2) = \ldots = \operatorname{Tor}_1(N, K_n)$$

From $0 \to K_{n+1} \to P_n \to K_n \to 0$, for any $n \geq 0$, by Lemma 0.2.1 and Lemma 0.2.4 we have two exact sequences

$$\begin{tikzcd}
0 = \operatorname{Tor}_1(P_n, N) \arrow[r] \arrow[d, "\sim"] & \operatorname{Tor}_1(K_n, N) \arrow[r] \arrow[d, dashed] & K_{n+1} \otimes N \arrow[r] \arrow[d, "\sim"] & P_n \otimes N \arrow[r] \arrow[d, "\sim"] & K_n \otimes N \arrow[r] & 0 \\
0 = \operatorname{Tor}_1(N, P_n) \arrow[r] & \operatorname{Tor}_1(N, K_n) \arrow[r] & N \otimes K_{n+1} \arrow[r] & N \otimes P_n \arrow[r] & N \otimes K_n \arrow[r] & 0
\end{tikzcd}$$

By five lemma, there is an isomorphism $\operatorname{Tor}_1(K_n, N) \xrightarrow{\sim} \operatorname{Tor}_1(N, K_n)$ completing the squares. Hence

$$\operatorname{Tor}_{n+1}(M, N) \cong \operatorname{Tor}_{n+1}(N, M)$$

for any $n \geq 0$. In Lemma 0.2.4, we showed that $\operatorname{Tor}_0(M, N) = M \otimes N$. So $\operatorname{Tor}_n(M, N) \cong \operatorname{Tor}_n(N, M)$ for any $n \geq 0$. $\square$

**Remark 0.2.6** (dimension shifting)

The technique is called *dimension shifting*, one can realize it in a different way. The exact sequence $\ldots \to P_{n+1} \to P_n \to P_{n-1} \to K_{n-1} \to 0$ is a projective resolution of $K_n$, tensoring with $N$ and taking homology gives

$$\operatorname{Tor}_1(K_{n-1}, N) = \operatorname{Tor}_n(M, N)$$

since $\ldots \to P_{n+1} \to P_n \to P_{n-1} \to 0$ is an subsequence of $\ldots \to P_2 \to P_1 \to P_0 \to 0$ but shifted by $n$ positions. More generally, let $L_n F$ be a left derived functor of a covariant functor, we have

$$(L_n F)(M) = (L_{n-1} F)(K_0) = \ldots = (L_1 F)(K_{n-1})$$

Similarly, we also have a version for right derived functor.

[1] $P_{-1} = M$

## 0.3 PROBLEM 5

**Problem 0.3.1** (problem 5)
Let $N$ be an $A$-module, then the following are equivalent

1. $\operatorname{Tor}_i^A(-,N)=0$ for any $i\geq 1$
2. $\operatorname{Tor}_1^A(-,N)=0$
3. $N$ is flat

*Proof.*
$(3\implies 1)$ Lemma 0.2.1
$(1\implies 2)$ clear
$(2\implies 3)$ Let $f:M\hookrightarrow L$ be an injective map, then the short exact sequence $0\to M\to L\to\operatorname{coker}f\to 0$ induces an exact sequence

$$0=\operatorname{Tor}_1(\operatorname{coker}f,N)\to M\otimes N\to L\otimes N$$

$\operatorname{Tor}_1(\operatorname{coker}f,N)=0$ implies $(-\otimes N)$ preserves injective map. ☐

## 0.4 PROBLEM 6

In this section, we will denote $\operatorname{Ext}_r$ for the version of $\operatorname{Ext}$ calculated using injective resolution and $\operatorname{Ext}_l$ for the version of $\operatorname{Ext}$ calculated using projective resolution

**Problem 0.4.1** (problem 6)
Prove that $\operatorname{Ext}_A^i(M,N)$ can be computed using either projective resolution of $M$ or an injective resolution of $N$

*Proof.* Let $0\to N\to I^0\to I^1\to I^2\to \ldots$ be a injective resolution of $N$, we have the following diagonal short exact sequences

$$\begin{tikzcd}
 & & & 0 \arrow[dr] & & 0 & & & & 0 \\
 & & & & C^1 \arrow[ur] \arrow[dr, hook] & & & & C^3 \arrow[ur] & \\
0 \arrow[r] & N \arrow[rr, hook, "\epsilon"] & & I^0 \arrow[ur, two heads] \arrow[rr, "d"] & & I^1 \arrow[rr, "d"] \arrow[dr, two heads] & & I^2 \arrow[rr, "d"] \arrow[ur, two heads] & & \ldots \\
 & & N \arrow[ur, hook] & & & & C^2 \arrow[ur, hook] \arrow[dr] & & & \\
 & 0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

where $C^1=\operatorname{coker}(\epsilon:N\to I^0)$ and $C^n=\operatorname{coker}(d:I^{n-2}\to I^{n-1})$. For any $n\geq 1$, the map $C^n\to I^n$ lifted from $d:I^{n-1}\to I^n$ since the composition $I^{n-2}\to I^{n-1}\to I^n$ is zero [2]. Since

$$C^n=\frac{I^{n-1}}{\operatorname{im}(d:I^{n-2}\to I^{n-1})}=\frac{I^{n-1}}{\ker(d:I^{n-1}\to I^n)}=\operatorname{im}(d:I^{n-1}\to I^n)=\ker(d:I^n\to I^{n+1})\hookrightarrow I^n$$

[2] $I^{-1}=N$

$C^n \to I^n$ is injective. Hence, every diagonal sequence is exact.
Dimension shifting 0.2.6 for right derived functor $\mathrm{Ext}_r(M, -)$

$$\mathrm{Ext}_r^{n+1}(M,N) = \mathrm{Ext}_r^n(M,C^1) = \ldots = \mathrm{Ext}_r^1(M,C^n)$$

From $0 \to N \to I^0 \to C^1 \to 0$ and $0 \to C^1 \to I^1 \to C^2$, we have

$$0 = \mathrm{Ext}_l^n(M,I^0) \longrightarrow \mathrm{Ext}_l^n(M,C^1) \longrightarrow \mathrm{Ext}_l^{n+1}(M,N) \longrightarrow \mathrm{Ext}_l^{n+1}(M,I^0) = 0$$

$$0 = \mathrm{Ext}_l^{n-1}(M,I^1) \longrightarrow \mathrm{Ext}_l^{n-1}(M,C^2) \longrightarrow \mathrm{Ext}_l^n(M,C^1) \longrightarrow \mathrm{Ext}_l^n(M,I^1) = 0$$

Hence, we have the same formula for $\mathrm{Ext}_l$

$$\mathrm{Ext}_l^{n+1}(M,N) = \mathrm{Ext}_l^n(M,C^1) = \ldots = \mathrm{Ext}_l^1(M,C^n)$$

From the exact sequence $0 \to C^n \to I^n \to C^{n+1}$, we have

$$\begin{tikzcd}
\mathrm{Hom}(M,I^n) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}(M,C^{n+1}) \arrow[r] \arrow[d, "\sim"] & \mathrm{Ext}_r^1(M,C^n) \arrow[r] \arrow[d, dashed] & \mathrm{Ext}_r^1(M,I^n) = 0 \arrow[d, "\sim"] \\
\mathrm{Hom}(M,C^{n+1}) \arrow[r] & \mathrm{Hom}(M,C^{n+1}) \arrow[r] & \mathrm{Ext}_l^1(K_n,N) \arrow[r] & \mathrm{Ext}_l^1(P_n,N) = 0
\end{tikzcd}$$

By five lemma, $\mathrm{Ext}_r^{n+1}(M,N) = \mathrm{Ext}_l^{n+1}(M,N)$. □

**Remark 0.4.2** (some notes on dimension shifting solution)
Given a short exact sequence $0 \to A \to B \to C \to 0$. In the above proof, we used the following results

1. Consider $\mathrm{Ext}_l^i(-,-)$ computed using projective resolution, we have
   (a) a variant of *dimension shifting* using in the proof
   (b) $\mathrm{Ext}_l^i(M,I) = 0$ for any injective module $I$
   (c) the two long exact sequences mentioned in class

$$0 \to \mathrm{Hom}(M,A) \to \mathrm{Hom}(M,B) \to \mathrm{Hom}(M,C) \to \mathrm{Ext}_l^1(M,A) \to \mathrm{Ext}_l^1(M,B) \to \mathrm{Ext}_l^1(M,C) \to \ldots$$
$$0 \to \mathrm{Hom}(C,N) \to \mathrm{Hom}(B,N) \to \mathrm{Hom}(A,N) \to \mathrm{Ext}_l^1(C,N) \to \mathrm{Ext}_l^1(B,N) \to \mathrm{Ext}_l^1(A,N) \to \ldots$$

2. Consider $R^iF = \mathrm{Ext}_r^i(M,-)$ computed using injective resolution as a right derived functor of the covariant functor $F = \mathrm{Hom}(M,-)$, we have
   (a) a mirror version of dimension shifting in 0.2.6 for $\mathrm{Ext}_r$ since if $0 \to N \to I^0 \to I^1 \to \ldots$ is an injective resolution for $N$ then $0 \to C^n \to I^n \to I^{n+1} \to \ldots$ is an shifted injective resolution for $C^n$, hence

$$\mathrm{Ext}_r^{n+1}(M,N) = \mathrm{Ext}_r^n(M,C^1) = \ldots = \mathrm{Ext}_r^1(M,C^n)$$

   (b) $\mathrm{Ext}_r^i(M,I) = 0$ for any injective module $I$ since $0 \to I \to I \to 0$ is an injective resolution for $I$
   (c) the long exact sequence for right derived functor

$$0 \to \mathrm{Hom}(M,A) \to \mathrm{Hom}(M,B) \to \mathrm{Hom}(M,C) \to \mathrm{Ext}_r^1(M,A) \to \mathrm{Ext}_r^1(M,B) \to \mathrm{Ext}_r^1(M,C) \to \ldots$$

*An alternative solution.* Since $\mathrm{Hom}(P, -)$ is exact for every projective module $P$ and $\mathrm{Hom}(-, I)$ is exact for every injective module $I$, we have the double complex $\mathrm{Hom}(P_\bullet, I^\bullet)$ [3] where every column except $\mathrm{Hom}(M, I^\bullet)$ is exact and every row except $\mathrm{Hom}(P_\bullet, N)$ is exact. We can construct a map

$$\phi : \ker(\mathrm{Hom}(M, I^n) \to \mathrm{Hom}(M, I^{n+1})) \to \ker(\mathrm{Hom}(P_n, N) \to \mathrm{Hom}(P_{n+1}, N))$$

as follows: for $i + j = n$,

$$\begin{aligned}
&\mathrm{Hom}(M, I^n) \to \mathrm{Hom}(P_0, I^n) \to \dots \\
&\dots \to \mathrm{Hom}(P_i, I^j) \to \mathrm{Hom}(P_{i+1}, I^j) \to \mathrm{Hom}(P_{i+1}, I^{j-1}) \to \dots \\
&\dots \to \mathrm{Hom}(P_n, I^0) \to \mathrm{Hom}(P_n, N)
\end{aligned}$$

Informally, the path zig-zags on the **top right** squares of the diagonal $i + j = n$

In the first square, let $x \in \ker(\mathrm{Hom}(M, I^n) \to \mathrm{Hom}(M, I^{n+1}))$, then $bax = 0$, then $cdx = 0$, then $dx \in \ker(\mathrm{Hom}(P_0, I^n) \to \mathrm{Hom}(P_0, I^{n+1}))$, then there is a lift $y$ of $dx$ in $\mathrm{Hom}(P_0, I^{n-1})$

$$\begin{tikzcd}
\mathrm{Hom}(M, I^{n+1}) \arrow[r, "b"] & \mathrm{Hom}(P_0, I^{n+1}) \\
\mathrm{Hom}(M, I^n) \arrow[u, "a"] \arrow[r, "d"] & \mathrm{Hom}(P_0, I^n) \arrow[u, "c"'] \\
& \mathrm{Hom}(P_0, I^{n-1}) \arrow[u, "e"']
\end{tikzcd}$$

In any intermediate square ($i + j = n$) and the last square ($i = n, j = 0$)

$$\begin{tikzcd}
\mathrm{Hom}(P_{i-1}, I^{j+1}) \arrow[r, "a"] & \mathrm{Hom}(P_i, I^{j+1}) \arrow[r, "b"] & \mathrm{Hom}(P_{i+1}, I^{j+1}) \\
& \mathrm{Hom}(P_i, I^j) \arrow[u, "c"] \arrow[r, "e"] & \mathrm{Hom}(P_{i+1}, I^j) \arrow[u, "d"] \\
& & \mathrm{Hom}(P_{i+1}, I^{j-1}) \arrow[u, "f"]
\end{tikzcd}$$

Let $y \in \mathrm{Hom}(P_i, I^j)$ be a lift of $x \in \mathrm{Hom}(P_{i-1}, I^{j+1})$ . Since $bax = 0$, then $bcy = 0$, then $dey = 0$, hence $ey \in \ker d$, so there exists a lift $z \in \mathrm{Hom}(P_{i+1}, I^{j-1})$ so that $fz = ey$

In the last square, let $y \in \mathrm{Hom}(P_n, N)$ be a lift of $x \in \mathrm{Hom}(P_{n-1}, I^0)$. Since $bax = 0$, then $bcy = 0$, then $dey = 0$. Since $d$ is injective, $ey = 0$, hence $y \in \ker(\mathrm{Hom}(P_n, N) \to \mathrm{Hom}(P_{n+1}, N))$. The map $\phi$ is well-defined.

$$\begin{tikzcd}
\mathrm{Hom}(P_{n-1}, I^0) \arrow[r, "a"] & \mathrm{Hom}(P_n, I^0) \arrow[r, "b"] & \mathrm{Hom}(P_{n+1}, I^0) \\
& \mathrm{Hom}(P_n, N) \arrow[u, hook, "c"] \arrow[r, "e"] & \mathrm{Hom}(P_{n+1}, N) \arrow[u, hook, "d"] \\
& 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Now we construct another map

$$\phi_0 : \mathrm{im}(\mathrm{Hom}(M, I^{n-1}) \to \mathrm{Hom}(M, I^n)) \to \mathrm{im}(\mathrm{Hom}(P_{n-1} \to N) \to \mathrm{Hom}(P_n \to N))$$

[3] $P_{-1} = M, I^{-1} = N$

as follows: for $i + j = n$

$$
\begin{aligned}
&\mathrm{Hom}(M, I^n) \to \mathrm{Hom}(P_0, I^n) \to ... \\
&... \to \mathrm{Hom}(P_i, I^j) \to \mathrm{Hom}(P_i, I^{j-1}) \to \mathrm{Hom}(P_{i+1}, I^{j-1}) \to ... \\
&... \to \mathrm{Hom}(P_n, I^0) \to \mathrm{Hom}(P_n, N)
\end{aligned}
$$

Informally, the path zig-zags on the **top right** and **bottom left** squares of the diagonal $i + j = n$

In the first square, let $x \in \mathrm{im}(\mathrm{Hom}(M, I^{n-1}) \to \mathrm{Hom}(M, I^n))$, let $z \in \mathrm{Hom}(M, I^{n-1})$ so that $az = x$. Let $y = bz$

$$
\begin{tikzcd}
 & \mathrm{Hom}(P_0, I^{n+1}) \\
\mathrm{Hom}(M, I^n) \arrow[r, "d"] & \mathrm{Hom}(P_0, I^n) \arrow[u] \\
\mathrm{Hom}(M, I^{n-1}) \arrow[u, "a"] \arrow[r, "b"] & \mathrm{Hom}(P_0, I^{n-1}) \arrow[u, "e"]
\end{tikzcd}
$$

In any intermediate square ($i + j = n$) and the last square ($i = n, j = 0$)

$$
\begin{tikzcd}
\mathrm{Hom}(P_{i-2}, I^{j+1}) \arrow[r, dashed] & \mathrm{Hom}(P_{i-1}, I^{j+1}) \arrow[r, "a"] & \mathrm{Hom}(P_i, I^{j+1}) \arrow[r, "b"] & \mathrm{Hom}(P_{i+1}, I^{j+1}) \\
 & \mathrm{Hom}(P_{i-1}, I^j) \arrow[u, dashed] \arrow[r, dashed] & \mathrm{Hom}(P_i, I^j) \arrow[u, "c"] \arrow[r, "e"] & \mathrm{Hom}(P_{i+1}, I^j) \arrow[u, "d"] \\
 & & \mathrm{Hom}(P_i, I^{j-1}) \arrow[u, dashed] \arrow[r, dashed] & \mathrm{Hom}(P_{i+1}, I^{j-1}) \arrow[u, "f"]
\end{tikzcd}
$$

Let $y \in \mathrm{Hom}(P_i, I^j)$ be a lift of $x \in \mathrm{Hom}(P_{i-1}, I^{j+1})$. Using exactly the same argument, we can construct $z \in \mathrm{Hom}(P_{i+1}, I^{j-1})$. However, this time, we also have $\tilde{x} \in \mathrm{Hom}(P_{i-2}, I^{j+1})$, $\tilde{y} \in \mathrm{Hom}(P_{i-1}, I^j)$ so that $\tilde{x}, \tilde{y}$ are mapped into $x, y$ respectively, using the same argument, we can construct $\tilde{z} \in \mathrm{Hom}(P_i, I^{j-1})$ so that $\tilde{z}$ is mapped into $z$. Hence, the map $\phi_0$ is well-defined.

Similarly, we can construct

$$\psi : \ker(\mathrm{Hom}(M, I^n) \to \mathrm{Hom}(M, I^{n+1})) \leftarrow \ker(\mathrm{Hom}(P_n, N) \to \mathrm{Hom}(P_{n+1}, N))$$

For any $x \in \ker(\mathrm{Hom}(M, I^n) \to \mathrm{Hom}(M, I^{n+1}))$, $x$ and $\psi\phi x$ differ by an element in $\mathrm{im}(\mathrm{Hom}(M, I^{n-1}) \to \mathrm{Hom}(M, I^n))$. Hence, $\psi\phi$ and $\phi\psi$ are identity maps in the level of cohomology. Hence

$$H^n(\mathrm{Hom}(M, I^\bullet)) = H^n(\mathrm{Hom}(P_\bullet, N))$$

$\square$

## 0.5 PROBLEM 7

**Problem 0.5.1** (problem 7)
If $M$ is flat $A$-module and $N$ is an injective $A$-module, prove that $\mathrm{Hom}_A(M, N)$ is an injective $A$-module

*Proof.* Let

$$0 \to X \to Y$$

be exact. Since $M$ is flat module, the covariant functor $(- \otimes M)$ is exact, so

$$0 \to X \otimes M \to Y \otimes M$$

is exact. Since $N$ is injective module, the contravariant functor $\mathrm{Hom}(-, N)$ is exact, so

$$\mathrm{Hom}(Y \otimes M, N) \to \mathrm{Hom}(X \otimes M, N) \to 0$$

is exact. By tensor-hom adjunction

$$\mathrm{Hom}(Y, \mathrm{Hom}(M, N)) \to \mathrm{Hom}(X, \mathrm{Hom}(M, N)) \to 0$$

is exact. Hence, the contravariant function $\mathrm{Hom}(-, \mathrm{Hom}(M, N))$ is exact. So, $\mathrm{Hom}(M, N)$ is injective. □

## 0.6 PROBLEM 8

**Lemma 0.6.1** (Baer's criterion)
An $A$-module $M$ is injective if and only if every map $I \to M$ from an ideal $I$ of $A$ can be extended into a map $R \to M$. In particular, if $A$ is a PID, $M$ is injective if and only if it is divisible, that is, for every nonzero $a \in A$ and every $m \in M$, there exists $n \in M$ so that $an = m$

*Proof.*
(divisible $\implies$ injective) If $M$ is divisible, for any ideal $(a) \in A$ and any map $\phi : (a) \to M$, let $m = \phi(a)$ and let $n \in M$ so that $an = m$. The extension is defined by

$$\begin{aligned} \tilde{\phi} : A &\to M \\ b &\mapsto bn \end{aligned}$$

(divisible $\impliedby$ injective) If $M$ is injective, then for any $a \in A$, $m \in M$, define

$$\begin{aligned} \phi : (a) &\to M \\ a &\mapsto m \end{aligned}$$

Then, the extension gives $n = \tilde{\phi}(1) \in M$ so that $an = m$ □

**Problem 0.6.2** (problem 8)
For a field $k$, consider the $k[x]$-module $M = k[x, x^{-1}]/xk[x]$. Prove that $M$ is an injective $k[x]$-module

*Proof.* We have

$$M = \frac{k[x, x^{-1}]}{xk[x]} = \left\{ \sum_{i \in \mathbb{Z}} a_i x^i : a_i \in k \text{ and all } a_i \text{ but finitely many of nonpositive indices are zeros} \right\}$$

Any element of $M$ can be written as

$$m(x) = a_0 + a_1 x^{-1} + ... + a_n x^{-n}$$

for some $a_0, ..., a_n \in k$ and $a_n \neq 0$. We will show that $M$ is divisible. For any nonzero polynomial $f(x) \in k[x]$

$$f(x) = b_0 + b_1 x + ... + b_m x^m$$

for some $b_0, ..., b_m \in k$ and $b_m \neq 0$. If $b_0 = 0$, then $f(x) = x^r g(x)$ with $g(x)$ having nonzero constant term. Hence, finding $n(x) \in M$ so that $m(x) = f(x)n(x) = x^r g(x)n(x)$ is equivalent to finding $n(x)$ so that $g(x)n(x) = x^{-r}m(x)$. Hence, we can assume that $b_0 \neq 0$. Let

$$n(x) = c_0 + c_1 x^{-1} + ... \in M$$

for some $c_0, c_1, ... \in k$. We have the following system of equations

$$\begin{aligned}
a_0 &= b_0 c_0 + b_1 c_1 + ... + b_m c_m \\
a_1 &= b_0 c_1 + b_1 c_2 + ... + b_m c_{m+1} \\
&... \\
a_n &= b_0 c_n + b_1 c_{n+1} + ... + b_m c_{m+n}
\end{aligned}$$

Pick $c_{n+1}, ..., c_{m+n}$ arbitrarily. Using the last equation, we can solve uniquely for $c_n$. And from bottom to top we can solve for unique $c_{n-1}, ..., c_0$ iteratively. Hence, $M$ is divisible, so injective.

□