Scheme Theory

this is my note for scheme theory. this is loosely based on (1) Görtz - Wedhorn AG 1 (2) Borcherds online lecture (3) Vakil FOAG

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Chapter 1

AFFINE SCHEME

1.1 DEFINITION OF AFFINE SCHEME

Definition 1.1.1 (ringed space, locally ringed space)
A ringed space $(X, \mathscr{O})$ is a topological space X together with a sheaf of rings $\mathscr{O}$. A ringed space is a locally ringed space if for every point $x \in X$, the stalk $\mathscr{O}_x$ is a local ring.

Definition 1.1.2 (affine scheme)
An affine scheme is a locally ringed space that is isomorphic to the spectrum of some ring A

1.2 CONSTRUCT AFFINE SCHEME FROM RING

Let A be a ring, let $X = \operatorname{Spec} A$ be the set of prime ideals in A

$$\operatorname{Spec} A = \{\text{prime } \mathfrak{p} \subseteq A\}$$

We put a topology on $\operatorname{Spec} A$ generated by the basis of open sets

$$D(f) = \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}\}$$

for every $f \in A$. The constructed topology is called Zariski's topology and the open subsets of the form $D(f)$ for some $f \in A$ are called **principal open subsets**. We put a sheaf of rings $\mathscr{O}$ on $\operatorname{Spec} A$ generated by

$$\mathscr{O}(D(f)) = A_f$$

If $D(g) \subseteq D(f)$, that is $g \in \sqrt{(f)}$, $g^n = fh$ for some $h \in A$ and $n \geq 1$. Since f is a unit in A_g, the restriction map is well-defined and unique

$$\begin{aligned} A_f &\to A_g \\ \frac{x}{f^m} &\mapsto x\left(\frac{h}{g^n}\right)^m \end{aligned}$$

Under this construction, $(X, \mathscr{O})$ is an affine scheme. An element $f \in A$ is called **function**, a element $x \in X$ is called **point**, when refering x as a prime ideal in A, we write $\mathfrak{p}_x \subseteq A$, and function evaluation is equivalent to sending f to the residue field of stalk $\mathscr{O}_X = A_{\mathfrak{p}_x}$.
When A is a polynomial ring of n variables over ring R, we write $\mathbb{A}^n_R = \operatorname{Spec} A$

1.3 SOME EXAMPLES OF AFFINE SCHEME

example: (integers) Let $A = \mathbb{Z}$, then

$$X = \operatorname{Spec} \mathbb{Z} = \{(2), (3), (5), ..., (0)\}$$

for each prime number $p \in \mathbb{Z}$, (p) is a closed point. (0) is a generic point, and closure of (0) is the whole space X.
example: (field) Let k be a field. Let $A = k$, then

$$X = \operatorname{Spec} k = \{(0)\}$$

is a singleton set.
example: (polynomial ring of two variables over an algebraically closed field) Let A be an arbitrary ring, maximal ideals are closed points and other non-maximal primes are generic points. Moreover, the closure of a point $\mathfrak{p}$ is the set of prime ideals containing $\mathfrak{p}$

$$V(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} A : \mathfrak{p} \subseteq \mathfrak{q}\}$$

In particular, let k be an algebraically closed field, let $A = k[x, y]$, by Nullstellensatz, the set of closed points are

$$\{(x - a, y - b) : a \in k, b \in k\}$$

X admits other generic points (0) and (f) for every irreducible $f \in k[x, y]$. By dimensionality argument, the prime (f) is of height 1 and the closure of (f) consists of (f) and $(x - a, y - b)$ for $(a, b) \in k^2$ in the vanishing set of f. Similarly, closure of (0) is the whole space.

1.4 MORPHISM OF AFFINE SCHEMES

Remark 1.4.1 (inverse image, direct image)
Let $\psi : X \to Y$ be a continuous map. The inverse image functor ψ^{-1} and direct image functor ψ_* is an adjoint pair between the category of sheaves on X and the category of sheaves on Y. Let $\mathscr{F}$ and $\mathscr{G}$ be a sheaf on X and a sheaf on Y respectively, then

$$\operatorname{Hom}_{\mathrm{Sh}(X)}(\psi^{-1}\mathscr{G}, \mathscr{F}) \cong \operatorname{Hom}_{\mathrm{Sh}(Y)}(\mathscr{G}, \psi_*\mathscr{F})$$

The direct image functor ψ_* is defined as follows: for every open subset $V \subseteq Y$, then

$$(\psi_*\mathscr{F})(V) = \mathscr{F}(\psi^{-1}(U))$$

The inverse image functor ψ^{-1} is defined as follows: for every open subset $U \subseteq X$, then

$$(\psi^{-1}\mathscr{G})(U) = \operatorname{colim}_{V \subseteq Y : \psi(U) \subseteq V} \mathscr{G}(V)$$

Definition 1.4.2 (morphism of ringed spaces)
A morphism of ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is defined by a continuous map $\psi : X \to Y$ and a morphism of sheaves of rings $\psi^\flat : \mathscr{O}_Y \to \psi_* \mathscr{O}_X$. By adjunction between ψ^{-1} and ψ_*, this is equivalent to a morphism of sheaves of rings $\psi^\# : \psi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$. Moreover, $\psi^\#$ induces a map on stalks

$$\psi^\#_x : (\psi^{-1} O_Y)_x = \mathscr{O}_{Y,\psi(x)} \to \mathscr{O}_{X,x}$$

Remark 1.4.3 (compatible germs)
Given a sheaf of sets $\mathscr{F}$ on X, for every open subset $U \subseteq X$, the natural map from sections into product of stalks

$$\mathscr{F}(U) \hookrightarrow \prod_{x \in U} \mathscr{F}_x$$

is injective. An element $(s_x)_{x \in U} \subseteq \prod_{x \in U} \mathscr{F}_x$ in the image if this map is called compatible germs.

From the equivalence between set of sections and set of compatible germs, one can identify a morphism of ringed spaces by its topological space map $\psi : X \to Y$ and its map on stalks $\psi^\#_x : (\psi^{-1} O_Y)_x = \mathscr{O}_{Y,\psi(x)} \to \mathscr{O}_{X,x}$.

Definition 1.4.4 (morphism of locally ringed spaces, morphism of affine schemes)
A morphism of ringed spaces $(\psi, \psi^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of locally ringed spaces if for every $x \in X$, the induced map on stalks $\psi^\#_x : (\psi^{-1} O_Y)_x = \mathscr{O}_{Y,\psi(x)} \to \mathscr{O}_{X,x}$ is a local homomorphism. A morphism of affine schemes is a morphism of locally ringed spaces.

We obtain the category of affine schemes, denoted by AffSch. The local homomorphism condition of the map between stalks ensures that any zero function $g \in \mathscr{O}_{Y,\psi(x)}$ at $\psi(x) \in Y$ will be sent to a zero function $\psi^\#_x(g) \in \mathscr{O}_{X,x}$ at $x \in X$

Proposition 1.4.5 (equivalence between commutative rings and affine schemes)
The functor Spec from the opposite category of commutative rings into the category of affine schemes is fully faithful and essentially surjective.

$$\mathrm{Spec} : \mathrm{CRing}^{\mathrm{op}} \xrightarrow{\sim} \mathrm{AffSch}$$

1.5 CONSTRUCT MORPHISM OF AFFINE SCHEMES FROM MAP OF RINGS

Let $\phi : A \to B$ be a map of rings, let $X = \mathrm{Spec}\, B$ and $Y = \mathrm{Spec}\, A$ be the corresponding affine schemes. Then, the corresponding morphism of locally ringed space $(\psi, \psi^\flat) : X \to Y$ is defined as follows: for every $x \in X$

$$\begin{aligned} \psi : X &\to Y \\ x &\mapsto \phi^{-1}(x) \end{aligned}$$

The local homomorphism $\psi^\#_x : \mathscr{O}_{Y,\psi(x)} = A_{\psi(x)} \to B_x$ is induced from $\phi : A \to B$

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \\ A_{\psi(x)} \arrow[r, "\psi^\#_x"] & B_x \end{tikzcd}$$

1.6 SOME EXAMPLES OF MORPHISM OF AFFINE SCHEMES

example: (quotient) Let A be a ring and $\mathfrak{a} \subseteq A$ be an ideal. The map $A \twoheadrightarrow A/\mathfrak{a}$ induces an injective map of sets

$$\operatorname{Spec} A/\mathfrak{a} \hookrightarrow \operatorname{Spec} A$$

As sets, we have $\operatorname{Spec} A/\sqrt{\mathfrak{a}} = \operatorname{Spec} A/\mathfrak{a} = \operatorname{Spec} A/\mathfrak{a}^n$ for every $n \geq 1$. However, as affine schemes, in general, they are different, $A/\sqrt{\mathfrak{a}}$ is reduced, i.e. has no nilpotent while $\operatorname{Spec} A/\mathfrak{a}$ and $\operatorname{Spec} A/\mathfrak{a}^n$ might have nilpotents.
Furthermore, when $\mathfrak{a} = \mathfrak{p}_x = x$ is prime, quotient by $\mathfrak{p}_x$ is the action of taking closed subscheme [1] of X

> *keep all primes containing $\mathfrak{p}_x$. equivalently, keep all points contained in x*

example: (localization) Let A be a ring and $S \subseteq A$ be a multiplicatively closed subset. Localization at S induces an injective map of sets

$$\operatorname{Spec} S^{-1}A \hookrightarrow \operatorname{Spec} A$$

Furthermore, when $A - S = \mathfrak{p}_x = x$ is prime, localization at $\mathfrak{p}_x$ is the action of taking open subscheme [2] of X

> *keep all primes contained in $\mathfrak{p}_x$. equivalently, keep all points containing x*

example: (disjoint union) Let A, B be rings, then the projection $A \times B \to A$ corresponds to the monomorphism of affine schemes

$$\operatorname{Spec} A \hookrightarrow \operatorname{Spec} A \amalg \operatorname{Spec} B \cong \operatorname{Spec} A \times B$$

Categorically, $A \times B$ is product in rings and $\operatorname{Spec} A \amalg \operatorname{Spec} B$ is coproduct in affine schemes
example: (non-trivial fiber) Consider the ring map $\phi : k[u] \to k[x, y]$ defined by $u \mapsto y^2 - x$. It induces a morphism of affine schemes

$$\begin{aligned} \psi : \operatorname{Spec} k[x,y] &\to \operatorname{Spec} k[u] \\ (0) &\mapsto (0) \\ (x-a, y-b) &\mapsto (u - (b^2 - a)) \\ (y^2 - x) &\mapsto (u) \\ (f) &\mapsto (0) \end{aligned}$$

for every $a, b \in k$ and f is irreducible other than multiple of $y^2 - x$. *(check this carefully)*.

[1] meaning will be revealed later
[2] meaning will be revealed later

Chapter 2

SCHEME

2.1 DEFINITION OF SCHEME

Definition 2.1.1 (scheme, morphism of schemes)
A scheme is a locally ringed space $(X, \mathscr{O})$ such that for every point $x \in X$, there is an open neighbourhood U_x containing x so that $(U_x, \mathscr{O}|_{U_x})$ is an affine scheme. Morphism of schemes is morphism of locally ringed spaces.

We obtain the category of schemes, denoted by Sch

Definition 2.1.2 (scheme over S)
Let S be a scheme, the category (Sch/S) of schemes over S is the category where an object is a morphism of schemes $X \to S$ and a morphism is a morphism of schemes $X \to Y$ so that $X \to S$ factors through Y by the map $X \to Y$

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

The collection of morphisms from $X \to S$ into $Y \to S$ is denoted by $\mathrm{Hom}_S(X, Y)$. If $S = \operatorname{Spec} A$, we also call scheme over S as scheme over A

Since $\operatorname{Spec} \mathbb{Z}$ is a terminal object in the category of schemes, the category of schemes is canonically equivalent the category of schemes over $\mathbb{Z}$ *(proof for $\operatorname{Spec} \mathbb{Z}$ is terminal in next section)*

2.2 MORPHISM INTO AFFINE SCHEMES, GLUING OF MORPHISMS

Definition 2.2.1 (open subscheme)
Let $(X, \mathscr{O}_X)$ be a scheme and $U \subseteq X$ be an open subset. Then, the locally ringed space $(U, \mathscr{O}_X|_U)$ is a scheme and we call it an open subscheme of X. If the

Lemma 2.2.2

Let X be a scheme and U, V be affine open subschemes of X. Then for every $x \in U \cap V$, there exists an open subscheme $W \subseteq U \cap V$ containing x such that W is principal open in U as well as in V.

Proposition 2.2.3 (gluing of morphisms)

Let X, Y be locally ringed spaces. Then $U \mapsto \mathrm{Hom}(U, Y)$ sending open subset $U \subseteq X$ into the set of morphisms $(U, \mathscr{O}_X|_U) \to (Y, \mathscr{O}_Y)$ is a sheaf of sets on X.

In other words, if $X = \bigcup_i U_i$ is an open covering, then a family of morphisms $U_i \to Y$ glues into a morphism $X \to Y$ if and only if the morphisms coincide on intersections $U_i \cap U_j$ and the resulting morphism $X \to Y$ is uniquely determined. *TODO - proof idea - one can identify the morphism by map between stalks, it feels local hence gluable*

Proposition 2.2.4

Let X be a locally ringed space and $Y = \operatorname{Spec} A$ an affine scheme. Then, the natural map

$$\begin{aligned} \mathrm{Hom}(X, Y) &\to \mathrm{Hom}(A, \Gamma(X)) \\ (f, f^\flat) &\mapsto f^\flat_Y \end{aligned}$$

is a bijection.

Proof. *(TODO - use lemma above to prove for the case X is a scheme, full proof in EGAInew 1.6.3)* □

$\operatorname{Spec} \mathbb{Z}$ being terminal object in the category of schemes follows this proposition. Moreover, when $A = \Gamma(X)$, there corresponds to $\mathrm{id}_{\Gamma(X)}$ a morphism

$$c_X : X \to \operatorname{Spec} \Gamma(X)$$

which we call canonical.

2.3 MORPHISM FROM $\operatorname{Spec} k$ INTO SCHEMES