Supplementary material

October 7, 2025

1 Gauss-Hermite quadrature

The univariate Gauss-Hermite quadrature approximates any integral of the form

$$\int_{-\infty}^{+\infty} \exp(-z^2)\mathrm{g}(z)\,\mathrm{d}z,$$

by a weighted sum,

$$\int_{-\infty}^{+\infty} \exp(-z^2)\mathrm{g}(z)\,\mathrm{d}z \approx \sum_{k=1}^{Q} w_k \mathrm{g}(z_k), \tag{1}$$

where $\mathrm{g}(z) : \Re \to \Re$, Q denotes the order of the approximation, the z_k are the zeros of the Qth order Hermite polynomial,

$$H_Q(z) = (-1)^Q e^{z^2/2} \frac{d^Q e^{-z^2/2}}{dz^Q},$$

and the w_k are the corresponding weights.

The multivariate Gauss-Hermite quadrature considers the m-dimensional vector $\boldsymbol{z} = (z_1, \dots, z_m)$ and the function $g(\boldsymbol{z}) : \Re^m \to \Re$. The expression (1) may be applied to approximate the m-dimensional integral

$$\int_{\Re^m} w(\boldsymbol{z})\mathrm{g}(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} = \int_{\Re} \cdots \int_{\Re} w(z_1, \dots, z_m)\mathrm{g}(z_1, \dots, z_m)\,\mathrm{d}z_1 \cdots \mathrm{d}z_m, \tag{2}$$

where $w(z_1, \dots, z_m) = \exp(-z_1^2) \cdot \ldots \cdot \exp(-z_m^2)$.

Then, the univariate Gauss-Hermite quadrature applies to each of the components of z in turn. Using Q_r nodes in the rth dimension, $r = 1, \dots, m$, yields the approximation

$$\int_{\Re^m} w(\boldsymbol{z})\mathrm{g}(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} \approx \sum_{k_1=1}^{Q_1} w_{k_1}^{(1)} \cdots \sum_{k_m=1}^{Q_m} w_{k_m}^{(m)} \mathrm{g}(z_{k_1}^{(1)}, \dots, z_{k_m}^{(m)}), \tag{3}$$

where $z_{k_r}^{(r)}$ is the k_rth zero of the Hermite polynomial with degree Q_r and $w_{k_r}^{(r)}$ is the corresponding weight. The $z_{k_r}^{(r)}$ are also denoted as quadrature nodes or quadrature points.

In equation (6), we need to evaluate the following integral:

$$\begin{aligned} I_i &= \int\int_{\Re^2} \prod_{j=1}^{n_i} f\left(y_{ij} \mid \gamma_{i1}, \gamma_{i2}; \boldsymbol{\beta}_1, \boldsymbol{\beta}_2\right) \cdot f(\gamma_{i1}; \lambda_1) f(\gamma_{i2}; \lambda_2)\, d\gamma_{i1} d\gamma_{i2} \qquad (4) \\ &= \int\int_{\Re^2} \prod_{j=1}^{n_i} f\left(y_{ij} \mid \gamma_{i1}, \gamma_{i2}; \boldsymbol{\beta}_1, \boldsymbol{\beta}_2\right) \frac{e^{-\frac{\gamma_{i1}^2}{2\lambda_1^2}}}{\sqrt{2\pi}\lambda_1} \frac{e^{-\frac{\gamma_{i2}^2}{2\lambda_2^2}}}{\sqrt{2\pi}\lambda_2}\, d\gamma_{i1} d\gamma_{i2}. \end{aligned}$$

Performing the following change of variables:

$$b_{i1} = \frac{\gamma_{i1}}{\sqrt{2}\lambda_1}, \quad b_{i2} = \frac{\gamma_{i2}}{\sqrt{2}\lambda_2},$$

we obtain the integral I_i as follows:

$$I_i = \int\int_{\Re^2} \prod_{j=1}^{n_i} f\left(y_{ij} \mid \sqrt{2}\lambda_1 b_{i1}, \sqrt{2}\lambda_2 b_{i2}; \boldsymbol{\beta}_1, \boldsymbol{\beta}_2\right) \frac{\exp(-b_{i1}^2)\exp(-b_{i2}^2)}{\pi}\, db_{i1} db_{i2}, \quad (5)$$

which has the form (2).Using the multivariate Gauss-Hermite quadrature, the integral I_i is aproximated as

$$I_i \cong \sum_{k_1=1}^{Q_1} \sum_{k_2=1}^{Q_2} \prod_{j=1}^{n_i} f(y_{ij} \mid \sqrt{2}\lambda_1 z_{k_1}, \sqrt{2}\lambda_2 z_{k_2}; \boldsymbol{\beta}_1, \boldsymbol{\beta}_2) \frac{w_{k_1} w_{k_2}}{\pi},$$

where z_{k_1} and z_{k_2} are the quadrature nodes and w_{k_1} and w_{k_2} are the corresponding weights.

2 Score function

The score function is given by $\mathbf{U}(\boldsymbol{\theta}) = (\mathbf{U}_{\boldsymbol{\beta}_1}^{\mathrm{T}}(\boldsymbol{\theta}), \mathbf{U}_{\boldsymbol{\beta}_2}^{\mathrm{T}}(\boldsymbol{\theta}), U_{\lambda_1}(\boldsymbol{\theta}), U_{\lambda_2}(\boldsymbol{\theta}))^{\mathrm{T}}$, with

$$\mathbf{U}_{\boldsymbol{\beta}_1}(\boldsymbol{\theta}) = \sum_{i=1}^{N} \frac{\mathbf{X}_{i1}^{\mathrm{T}}\mathbf{D}_{i1}}{\mathrm{B}_i},$$

$$\mathbf{U}_{\boldsymbol{\beta}_2}(\boldsymbol{\theta}) = \sum_{i=1}^{N} \frac{\mathbf{X}_{i2}^{\mathrm{T}}\mathbf{D}_{i2}}{\mathrm{B}_i},$$

$$U_{\lambda_1}(\boldsymbol{\theta}) = \sum_{i=1}^{N} \frac{D_{i3}}{\mathrm{B}_i},$$

$$U_{\lambda_2}(\boldsymbol{\theta}) = \sum_{i=1}^{N} \frac{D_{i4}}{\mathrm{B}_i},$$

where $\mathbf{X}_{i1}$ and $\mathbf{X}_{i2}$ are the $(n_i \times p_1)$ and $(n_i \times p_2)$ matrices containing covariates related to the models for μ and σ, respectively. The n_i-dimensional vectors $\mathbf{D}_{i1}$ and $\mathbf{D}_{i2}$ and the quantities D_{i3}, D_{i4}, and B_i are defined as follows.

Let $\mathrm{B}_i = \left(\sum_{k_1=1}^{\mathrm{Q}_1} \sum_{k_2=1}^{\mathrm{Q}_2} \mathrm{A}_i w_{k_1} w_{k_2}/\pi \right)$ and

$$\mathrm{A}_i = \exp\left(\sum_{j=1}^{n_i} \log\Gamma\left(\frac{1-\sigma_{ij}^2}{\sigma_{ij}^2}\right) - \log\Gamma\left(\mu_{ij}\frac{1-\sigma_{ij}^2}{\sigma_{ij}^2}\right) - \log\Gamma\left((1-\mu_{ij})\frac{1-\sigma_{ij}^2}{\sigma_{ij}^2}\right) \right.$$
$$\left. + \left(\mu_{ij}\frac{1-\sigma_{ij}^2}{\sigma_{ij}^2} - 1\right)\log y_{ij} + \left((1-\mu_{ij})\frac{1-\sigma_{ij}^2}{\sigma_{ij}^2} - 1\right)\log\left(1-y_{ij}\right) \right),$$

with $\mathrm{g}_1(\mu_{ij}) = \eta_{ij1} = \mathbf{x}_{ij1}^{\mathrm{T}}\boldsymbol{\beta}_1 + \sqrt{2}\lambda_1 z_{k_1}$ and $\mathrm{g}_2(\sigma_{ij}) = \eta_{ij2} = \mathbf{x}_{ij2}^{\mathrm{T}}\boldsymbol{\beta}_2 + \sqrt{2}\lambda_2 z_{k_2}$.

The n_i-dimensional vector $\mathbf{D}_{i1}$ defined in $\mathbf{U}_{\boldsymbol{\beta}_1}(\boldsymbol{\theta})$ is expressed as

$$\mathbf{D}_{i1} = \sum_{k_1=1}^{\mathrm{Q}_1} \sum_{k_2=1}^{\mathrm{Q}_2} \mathrm{A}_i \mathbf{Q}_{i1} \mathbf{P}_i \frac{w_{k_1} w_{k_2}}{\pi},$$

with $\mathbf{Q}_{i1} = \mathrm{diag}\{\mathrm{d}\mu_{i1}/\mathrm{d}\eta_{i11}, \ldots, \mathrm{d}\mu_{in_i}/d\eta_{in_i1}\}$ and $\mathbf{P}_i = (p_{i1}, \ldots, p_{in_i})^{\mathrm{T}}$, where $p_{ij} = ((1-\sigma_{ij}^2)/\sigma_{ij}^2)(y_{ij}^* - \mu_{ij}^*)$, $y_{ij}^* = \log\{y_{ij}/(1-y_{ij})\}$, and $\mu_{ij}^* = \psi(\mu_{ij}((1-\sigma_{ij}^2)/\sigma_{ij}^2)) - \psi((1-\mu_{ij})((1-\sigma_{ij}^2)/\sigma_{ij}^2))$ and $\psi(\cdot)$ is the digamma function, i.e., $\psi(z) = \mathrm{d}\log\Gamma(z)/\mathrm{d}z$, for $z > 0$.

The n_i-dimensional vector $\mathbf{D}_{i2}$ defined in $\mathbf{U}_{\boldsymbol{\beta}_2}(\boldsymbol{\theta})$ and is given by

$$\mathbf{D}_{i2} = \sum_{k_1=1}^{\mathrm{Q}_1} \sum_{k_2=1}^{\mathrm{Q}_2} \mathrm{A}_i \mathbf{Q}_{i2} \mathbf{R}_i \frac{w_{k_1} w_{k_2}}{\pi},$$

with $\mathbf{Q}_{i2} = \text{diag}\{\text{d}\sigma_{i1}/\text{d}\eta_{i12}, \ldots, \text{d}\sigma_{in_i}/\text{d}\eta_{in_i2}\}$ and $\mathbf{R}_i = (r_{i1}, \ldots, r_{in_i})^{\mathrm{T}}$, where $r_{ij} = -2a_{ij}/\sigma_{ij}^3$ and

$$a_{ij} = \mu_{ij}(y_{ij}^* - \mu_{ij}^*) + \log(1 - y_{ij}) + \psi\left(\frac{1-\sigma_{ij}^2}{\sigma_{ij}^2}\right) - \psi\left((1-\mu_{ij})\frac{1-\sigma_{ij}^2}{\sigma_{ij}^2}\right).$$

The quantity D_{i3} defined in $U_{\lambda_1}(\boldsymbol{\theta})$ is expressed as:

$$D_{i3} = \sum_{k_1=1}^{\mathrm{Q}_1}\sum_{k_2=1}^{\mathrm{Q}_2} \mathrm{A}_i \mathbf{Q}_{i1}^* \mathbf{P}_i \frac{\sqrt{2} z_{k_1} w_{k_1} w_{k_2}}{\pi},$$

with $\mathbf{Q}_{i1}^* = (\text{d}\mu_{i1}/\text{d}\eta_{i11}, \ldots, \text{d}\mu_{in_i}/d\eta_{in_i1})$.

The quantity D_{i3} defined in $U_{\lambda_2}(\boldsymbol{\theta})$ is given by

$$D_{i4} = \sum_{k_1=1}^{\mathrm{Q}_1}\sum_{k_2=1}^{\mathrm{Q}_2} \mathrm{A}_i \mathbf{Q}_{i2}^* \mathbf{R}_i \frac{\sqrt{2} z_{k_2} w_{k_1} w_{k_2}}{\pi},$$

with $\mathbf{Q}_{i2}^* = (\text{d}\sigma_{i1}/\text{d}\eta_{i12}, \ldots, \text{d}\sigma_{in_i}/\text{d}\eta_{in_i2})$.

3 Observed information matrix

The elements of the $(p \times p)$ observed information matrix are defined as follows:

$$\mathbf{J}_{\boldsymbol{\beta}_1\boldsymbol{\beta}_1} = -\sum_{i=1}^{N} \mathbf{X}_{i1}^{\mathrm{T}} \left(\frac{\mathbf{D}_{i1}\mathbf{D}_{i1}^{\mathrm{T}}}{\mathrm{B}_i^2} - \frac{\mathbf{D}_{i5}}{\mathrm{B}_i} \right) \mathbf{X}_{i1},$$

$$\mathbf{J}_{\boldsymbol{\beta}_1\boldsymbol{\beta}_2} = -\sum_{i=1}^{N} \mathbf{X}_{i1}^{\mathrm{T}} \left(\frac{\mathbf{D}_{i1}\mathbf{D}_{i2}^{\mathrm{T}}}{\mathrm{B}_i^2} - \frac{\mathbf{D}_{i6}}{\mathrm{B}_i} \right) \mathbf{X}_{i2},$$

$$\mathbf{J}_{\boldsymbol{\beta}_1\lambda_1} = -\sum_{i=1}^{N} \mathbf{X}_{i1}^{\mathrm{T}} \left(\frac{\mathbf{D}_{i1}D_{i3}}{\mathrm{B}_i^2} - \frac{\mathbf{D}_{i7}}{\mathrm{B}_i} \right),$$

$$\mathbf{J}_{\boldsymbol{\beta}_1\lambda_2} = -\sum_{i=1}^{N} \mathbf{X}_{i1}^{\mathrm{T}} \left(\frac{\mathbf{D}_{i1}D_{i4}}{\mathrm{B}_i^2} - \frac{\mathbf{D}_{i8}}{\mathrm{B}_i} \right),$$

$$\mathbf{J}_{\boldsymbol{\beta}_2\boldsymbol{\beta}_2} = -\sum_{i=1}^{N} \mathbf{X}_{i2}^{\mathrm{T}} \left(\frac{\mathbf{D}_{i2}\mathbf{D}_{i2}^{\mathrm{T}}}{\mathrm{B}_i^2} - \frac{\mathbf{D}_{i9}}{\mathrm{B}_i} \right) \mathbf{X}_{i2},$$

$$\mathbf{J}_{\boldsymbol{\beta}_2\lambda_1} = -\sum_{i=1}^{N} \mathbf{X}_{i2}^{\mathrm{T}} \left(\frac{\mathbf{D}_{i2}D_{i3}}{\mathrm{B}_i^2} - \frac{\mathbf{D}_{i10}}{\mathrm{B}_i} \right),$$

$$\mathbf{J}_{\boldsymbol{\beta}_2\lambda_2} = -\sum_{i=1}^{N} \mathbf{X}_{i2}^{\mathrm{T}} \left(\frac{\mathbf{D}_{i2}D_{i4}}{\mathrm{B}_i^2} - \frac{\mathbf{D}_{i11}}{\mathrm{B}_i} \right),$$

$$\mathrm{J}_{\lambda_1\lambda_1} = -\sum_{i=1}^{N} \left(\frac{D_{i3}^2}{\mathrm{B}_i^2} - \frac{D_{i12}}{\mathrm{B}_i} \right),$$

$$\mathrm{J}_{\lambda_1\lambda_2} = -\sum_{i=1}^{N} \left(\frac{D_{i3}D_{i4}}{\mathrm{B}_i^2} - \frac{D_{i13}}{\mathrm{B}_i} \right),$$

$$\mathrm{J}_{\lambda_2\lambda_2} = -\sum_{i=1}^{N} \left(\frac{D_{i4}^2}{\mathrm{B}_i^2} - \frac{D_{i14}}{\mathrm{B}_i} \right).$$

The matrix $\mathbf{D}_{i5}$ of order $(n_i \times n_i)$ defined in $\mathbf{J}_{\boldsymbol{\beta}_1\boldsymbol{\beta}_1}$ is given by

$$\mathbf{D}_{i5} = \sum_{k_1=1}^{\mathrm{Q}_1} \sum_{k_2=1}^{\mathrm{Q}_2} \mathrm{A}_i \left(\mathbf{Q}_{i1}\mathbf{P}_i(\mathbf{Q}_{i1}\mathbf{P}_i)^{\mathrm{T}} + \mathbf{T}_i \right) \frac{w_{k_1}w_{k_2}}{\pi},$$

with $\mathbf{T}_i = \mathrm{diag}\{t_{i1}, \ldots, t_{in_i}\}$ and

$$t_{ij} = \frac{1-\sigma_{ij}^2}{\sigma_{ij}^2} \left((y_{ij}^* - \mu_{ij}^*) \frac{\mathrm{d}^2\mu_{ij}}{\mathrm{d}\eta_{ij1}^2} - b_{ij} \frac{1-\sigma_{ij}^2}{\sigma_{ij}^2} \left(\frac{\mathrm{d}\mu_{ij}}{\mathrm{d}\eta_{ij1}} \right)^2 \right),$$

where $b_{ij} = \psi^{'}(\mu_{ij}((1-\sigma_{ij}^2)/\sigma_{ij}^2)) + \psi^{'}((1-\mu_{ij})((1-\sigma_{ij}^2)/\sigma_{ij}^2))$ and $\psi^{'}(\cdot)$ is the trigamma function.

The matrix $\mathbf{D}_{i6}$ of order $(n_i \times n_i)$ defined in $\mathbf{J}_{\boldsymbol{\beta}_1\boldsymbol{\beta}_2}$ is expressed as

$$\mathbf{D}_{i6} = \sum_{k_1=1}^{\mathrm{Q}_1} \sum_{k_2=1}^{\mathrm{Q}_2} \mathrm{A}_i \left(\mathbf{Q}_{i1} \mathbf{P}_i (\mathbf{Q}_{i2} \mathbf{R}_i)^{\mathrm{T}} + \mathbf{V}_i \mathbf{Q}_{i1} \mathbf{Q}_{i2} \right) \frac{w_{k_1} w_{k_2}}{\pi},$$

with $\mathbf{V}_i = \mathrm{diag}\{v_{i1}, \ldots, v_{in_i}\}$ and

$$v_{ij} = -\frac{2}{\sigma_{ij}^3} \left((y_{ij}^* - \mu_{ij}^*) - \frac{1-\sigma_{ij}^2}{\sigma_{ij}^2} \left(\mu_{ij} b_{ij} - \psi^{'} \left((1-\mu_{ij}) \frac{1-\sigma_{ij}^2}{\sigma_{ij}^2} \right) \right) \right).$$

The n_i-dimensional vector $\mathbf{D}_{i7}$ defined in $\mathbf{J}_{\boldsymbol{\beta}_1 \lambda_1}$ is given by

$$\mathbf{D}_{i7} = \sum_{k_1=1}^{\mathrm{Q}_1} \sum_{k_2=1}^{\mathrm{Q}_2} \mathrm{A}_i \left(\mathbf{Q}_{i1} \mathbf{P}_i \mathbf{Q}_{i1}^* \mathbf{P}_i + \mathbf{T}_i^* \right) \frac{\sqrt{2} z_{k_1} w_{k_1} w_{k_2}}{\pi},$$

with $\mathbf{T}_i^* = (t_{i1}, \ldots, t_{in_i})^{\mathrm{T}}$.

The n_i-dimensional vector $\mathbf{D}_{i8}$ defined in $\mathbf{J}_{\boldsymbol{\beta}_1 \lambda_2}$ is expressed as

$$\mathbf{D}_{i8} = \sum_{k_1=1}^{\mathrm{Q}_1} \sum_{k_2=1}^{\mathrm{Q}_2} \mathrm{A}_i \left(\mathbf{Q}_{i1} \mathbf{P}_i \mathbf{Q}_{i2}^* \mathbf{R}_i + \mathbf{Q}_{i1} \mathbf{Q}_{i2} \mathbf{V}_i^* \right) \frac{\sqrt{2} z_{k_2} w_{k_1} w_{k_2}}{\pi},$$

with $\mathbf{V}_i^* = (v_{i1}, \ldots, v_{in_i})^{\mathrm{T}}$.

The matrix $\mathbf{D}_{i9}$ of order $(n_i \times n_i)$ defined in $\mathbf{J}_{\boldsymbol{\beta}_2 \boldsymbol{\beta}_2}$ is given by

$$\mathbf{D}_{i9} = \sum_{k_1=1}^{\mathrm{Q}_1} \sum_{k_2=1}^{\mathrm{Q}_2} \mathrm{A}_i \left(\mathbf{Q}_{i2} \mathbf{R}_i (\mathbf{Q}_{i2} \mathbf{R}_i)^{\mathrm{T}} + \mathbf{Z}_i \right) \frac{w_{k_1} w_{k_2}}{\pi},$$

with $\mathbf{Z}_i = \mathrm{diag}\{z_{i1}, \ldots, z_{in_i}\}$ and

$$z_{ij} = \frac{-2}{\sigma_{ij}^3} \left(a_{ij} \frac{\mathrm{d}^2 \sigma_{ij}}{\mathrm{d}\eta_{ij2}^2} - \frac{1}{\sigma_{ij}} \left(3a_{ij} - \frac{2}{\sigma_{ij}^2} c_{ij} \right) \left(\frac{\mathrm{d}\sigma_{ij}}{\mathrm{d}\eta_{ij2}} \right)^2 \right),$$

where

$$c_{ij} = \mu_{ij}^2 b_{ij} - 2\mu_{ij} \psi^{'} \left((1-\mu_{ij}) \frac{1-\sigma_{ij}^2}{\sigma_{ij}^2} \right) - \psi^{'} \left(\frac{1-\sigma_{ij}^2}{\sigma_{ij}^2} \right) + \psi^{'} \left((1-\mu_{ij}) \frac{1-\sigma_{ij}^2}{\sigma_{ij}^2} \right).$$

The n_i-dimensional vector $\mathbf{D}_{i10}$ defined in $\mathbf{J}_{\boldsymbol{\beta}_2 \lambda_1}$ is expressed as

$$\mathbf{D}_{i10} = \sum_{k_1=1}^{\mathrm{Q}_1} \sum_{k_2=1}^{\mathrm{Q}_2} \mathrm{A}_i \left(\mathbf{Q}_{i2} \mathbf{R}_i \mathbf{Q}_{i1}^* \mathbf{P}_i + \mathbf{Q}_{i1} \mathbf{Q}_{i2} \mathbf{V}_i^* \right) \frac{\sqrt{2} z_{k_1} w_{k_1} w_{k_2}}{\pi}.$$

The n_i-dimensional vector $\mathbf{D}_{i11}$ defined in $\mathbf{J}_{\boldsymbol{\beta}_2 \lambda_2}$ is given by

$$\mathbf{D}_{i11} = \sum_{k_1=1}^{\mathrm{Q}_1} \sum_{k_2=1}^{\mathrm{Q}_2} \mathrm{A}_i \left(\mathbf{Q}_{i2} \mathbf{R}^*_{\mathbf{Q}_{i2}} \mathbf{R}_i + \mathbf{Z}^*_i \right) \frac{\sqrt{2} z_{k_1} w_{k_1} w_{k_2}}{\pi},$$

where $\mathbf{Z}^*_i = (z_{i1}, \ldots, z_{in_i})^{\mathrm{T}}$.

The quantity D_{i12} defined in $\mathrm{J}_{\lambda_1 \lambda_1}$ is given by

$$D_{i12} = \sum_{k_1=1}^{\mathrm{Q}_1} \sum_{k_2=1}^{\mathrm{Q}_2} \mathrm{A}_i \left(\mathbf{Q}^*_{i1} \mathbf{P}_i \mathbf{Q}^*_{i1} \mathbf{P}_i + \mathrm{trace}(\mathbf{T}^*_i) \right) \frac{2 z^2_{k_1} w_{k_1} w_{k_2}}{\pi}.$$

The quantity D_{i13} defined in $\mathrm{J}_{\lambda_1 \lambda_2}$ is expressed as

$$D_{i13} = \sum_{k_1=1}^{\mathrm{Q}_1} \sum_{k_2=1}^{\mathrm{Q}_2} \mathrm{A}_i \left(\mathbf{Q}^*_{i1} \mathbf{P}_i \mathbf{Q}^*_{i2} \mathbf{R}_i + \mathrm{trace}(\mathbf{V}^*_i) \right) \frac{2 z_{k_1} z_{k_2} w_{k_1} w_{k_2}}{\pi}.$$

The quantity D_{i14} defined in $\mathrm{J}_{\lambda_2 \lambda_2}$ is given by

$$D_{i14} = \sum_{k_1=1}^{\mathrm{Q}_1} \sum_{k_2=1}^{\mathrm{Q}_2} \mathrm{A}_i \left(\mathbf{Q}^*_{i2} \mathbf{R}_i \mathbf{Q}^*_{i2} \mathbf{R}_i + \mathrm{trace}(\mathbf{Z}^*_i) \right) \frac{2 z^2_{k_2} w_{k_1} w_{k_2}}{\pi}.$$