Derivation of Equations Used in bfield_server.cpp

1 Introduction

Based FDTD update equations on a lecture series by Raymond C. Rumpf at CEM, UTEP, http://emlab.utep.edu/ee5390cem.htm, and *Understanding the Finite-Difference Time-Domain Method*, 2018, John Schneider, 2018. The source injection method comes from *Introduction to the Finite-Difference Time-Domain (FDTD) Method for Electromagnetics,* 2011, Stephen D. Gedney.

Using Rumpf's conventions:
— E on whole time step, H on half (see 15.7).
— Normalized H (H~ = H/Z0), in V/m.

Equation numbers from CEM refer to (Lecture.Slide). Those from Schneider have a 'U' prepended, and from Gedney, a 'G'.

2 gaussian()

Taken from the Gaussian function Wiki page.

$$f(x)=a\,e^{-(x-b)^2/(2c^2)}$$

where a is the height, b is the center position, and c is the RMS width.

LC's hard gaussian pulse source appears to be:

$$f(t)=s\,e^{-4(\ln 2)(t-t_R\sigma_R/\pi)^2/t_R^2}$$

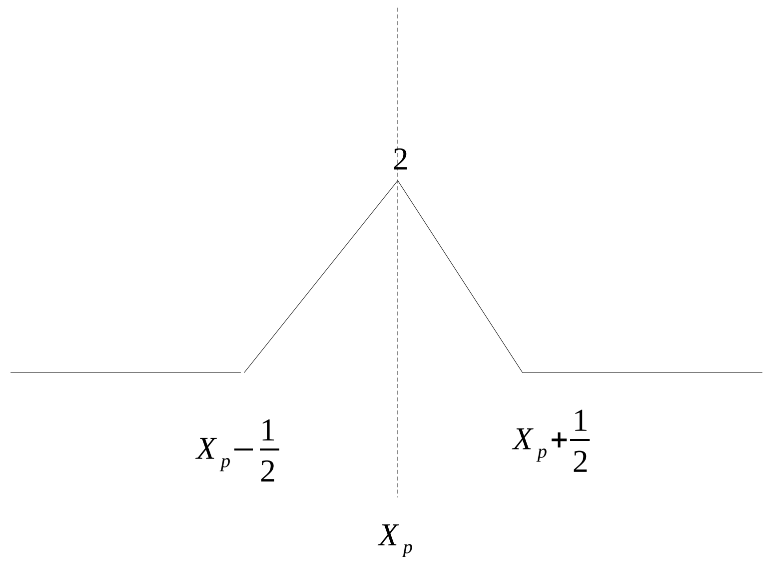
LC appears to have $t_R=FWTM$ (Full Width at Tenth Maximum), which is $2\sqrt{(2\ln 2)}c$.

3 MatBlock::place()

Material Block Placement. For placement of blocks on any boundary (and for proper placement in the 1/2-cell-displaced E field), consider on each axis the overlapping area between a function that is 1 from matStart.x to matEnd.x, and another function that is a triangle of height 2 and base 1, centered on the position in question:

$$f_{mat}(x)=\begin{cases}1 & \text{for} \quad x_s\le x<x_e\\ 0 & \text{otherwise}\end{cases} \tag{3.1}$$

$$f_{pos}(x)=\begin{cases}4\left(x+\frac{1}{2}-x_p\right) & \text{for} \quad -\frac{1}{2}\le(x-x_p)<0\\ -4\left(x-\frac{1}{2}-x_p\right) & \text{for} \quad 0\le(x-x_p)<\frac{1}{2}\\ 0 & \text{otherwise}\end{cases} \tag{3.2}$$



Integrating just (3.1):

$$c \;=\; x\Big|_{x_e}^{x_s} \;=\; x_s-x_e \quad ✔ \tag{3.3}$$

Integrating just (3.2):

$$c \;=\; \left(2x^2+(2-4x_p)x\right)\Big|_{x_p-\frac{1}{2}}^{x_p} \;+\; \left(-2x^2+(-2+4x_p)x\right)\Big|_{x_p}^{x_p+\frac{1}{2}}$$

$$c \;=\; \left(2x_p^2+(2-4x_p)x_p\right) \;-\; \left(2\left(x_p-\frac{1}{2}\right)^2+(2-4x_p)\left(x_p-\frac{1}{2}\right)\right)$$

$$+\; \left(-2\left(x_p+\frac{1}{2}\right)^2+(-2+4x_p)\left(x_p+\frac{1}{2}\right)\right) \;-\; \left(-2x_p^2+(-2+4x_p)x_p\right)$$

$$c \;=\; \left(2x_p^2+2x_p-4x_p^2-2x_p^2+2x_p-\frac{1}{2}-2x_p+4x_p^2+1-2x_p\right)$$

$$+\; \left(-2x_p^2-2x_p-\frac{1}{2}-2x_p+4x_p^2-1+2x_p+2x_p^2+2x_p-4x_p^2\right)$$

$$c \;=\; \left((0)x_p^2+(0)x_p+\frac{1}{2}\right) \;+\; \left((0)x_p^2+(0)x_p+\frac{1}{2}\right) \;=\; 1 \quad ✔ \tag{3.4}$$

The combined function is:

$$f(x)=\begin{cases} 4\left(x+\frac{1}{2}-x_s\right) & \mathit{for}\quad -\frac{1}{2}\le(x-x_s)<0 \\ -4\left(x-\frac{1}{2}-x_e\right) & \mathit{for}\quad 0\le(x-x_e)<\frac{1}{2} \\ 0 & \mathit{otherwise} \end{cases}$$

Then we want to integrate this function:

$$c = \int_{x_p - \frac{1}{2}}^{x_p + \frac{1}{2}} f(x)\, dx$$

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4 Source::injectAll()

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Source injecting current into E field:

$$\Delta E_z \;=\; -\frac{\Delta t}{\varepsilon}\left[\frac{I_z}{\Delta x \Delta y}\right]$$

For example: dE = -(4.16945e-13 s) / (8.8542e-12 F/m) * ((0.15V / 50 ohms) / (0.25mm * 0.25mm)) = -2260 V/m ???

From Gedney:

$$E^{n+1/2}_{z,i,j,k+1/2} \;=\; \frac{\dfrac{\varepsilon}{\Delta t} \;-\; \dfrac{\Delta z}{\Delta x \Delta y 2 R_{TH}}}{\dfrac{\varepsilon}{\Delta t} \;+\; \dfrac{\Delta z}{\Delta x \Delta y 2 R_{TH}}} E^{n-1/2}_{z,i,j,k+1/2} \;+\; \frac{1}{\dfrac{\varepsilon}{\Delta t} \;+\; \dfrac{\Delta z}{\Delta x \Delta y 2 R_{TH}}} \left[\left(\frac{H^n_{y,i+1/2,j,k+1/2} - H^n_{y,i-1/2,j,k+1/2}}{\Delta x}\right) \;-\; \left(\frac{H^n_{x,i,j+1/2,k+1/2} - H^n_{x,i,j-1/2,k+1/2}}{\Delta y}\right) \;-\; \frac{V^n_{oc,i,j,k+1/2}}{\Delta x \Delta y R_{TH}}\right] \tag{G4.28}$$

Converting this to E-on-the-whole-time-step and normalized H conventions:

$$H = \widetilde{H} / \eta_0 \tag{4.1}$$

$$E^{n}_{z,i,j,k+1/2} \;=\; \frac{\dfrac{\varepsilon}{\Delta t} \;-\; \dfrac{\Delta z}{\Delta x \Delta y 2 R_{TH}}}{\dfrac{\varepsilon}{\Delta t} \;+\; \dfrac{\Delta z}{\Delta x \Delta y 2 R_{TH}}} E^{n-1}_{z,i,j,k+1/2} \;+\; \frac{1}{\eta_0\left(\dfrac{\varepsilon}{\Delta t} \;+\; \dfrac{\Delta z}{\Delta x \Delta y 2 R_{TH}}\right)} \left[\left(\frac{\widetilde{H}^{n-1/2}_{y,i+1/2,j,k+1/2} - \widetilde{H}^{n-1/2}_{y,i-1/2,j,k+1/2}}{\Delta x}\right) \;-\; \left(\frac{\widetilde{H}^{n-1/2}_{x,i,j+1/2,k+1/2} - \widetilde{H}^{n-1/2}_{x,i,j-1/2,k+1/2}}{\Delta y}\right) \;-\; \frac{\eta_0 V^{n-1/2}_{oc,i,j,k+1/2}}{\Delta x \Delta y R_{TH}}\right] \tag{4.2}$$

Substituting sigma and J:

$$E^{n}_{z,i,j,k+1/2} \;=\; \frac{1-\sigma^E_{TH}}{1+\sigma^E_{TH}} E^{n-1}_{z,i,j,k+1/2} \;+\; \frac{\Delta t}{\eta_0 \varepsilon (1+\sigma^E_{TH})} \left[\left(\frac{\widetilde{H}^{n-1/2}_{y,i+1/2,j,k+1/2} - \widetilde{H}^{n-1/2}_{y,i-1/2,j,k+1/2}}{\Delta x}\right) \;-\; \left(\frac{\widetilde{H}^{n-1/2}_{x,i,j+1/2,k+1/2} - \widetilde{H}^{n-1/2}_{x,i,j-1/2,k+1/2}}{\Delta y}\right) \;-\; J'\right] \tag{4.3}$$

$$\text{where } \sigma^E_{TH} = \frac{\Delta t \Delta z}{\varepsilon \Delta x \Delta y 2 R_{TH}}, \quad J' = \frac{\eta_0 V^{n-1/2}_{oc,i,j,k+1/2}}{\Delta x \Delta y R_{TH}}$$

Forgoing non-cubic cells for now, so $\Delta x = \Delta y = \Delta z$ and we can remove the inner-loop divides:

$$E^{n}_{z,i,j,k+1/2} = m_{E1} E^{n-1}_{z,i,j,k+1/2} + \hat{m}_{E2}\left[\left(\widetilde{H}^{n-1/2}_{y,i+1/2,j,k+1/2} - \widetilde{H}^{n-1/2}_{y,i-1/2,j,k+1/2}\right) - \left(\widetilde{H}^{n-1/2}_{x,i,j+1/2,k+1/2} - \widetilde{H}^{n-1/2}_{x,i,j-1/2,k+1/2}\right) - \hat{J}\right] \tag{4.4}$$

where $\sigma^{E}_{TH} = \dfrac{\Delta t}{\varepsilon\, \Delta x\, 2 R_{TH}}$, $\hat{J} = \dfrac{\eta_0 V^{n-1/2}_{oc,i,j,k+1/2}}{\Delta x\, R_{TH}}$, $m_{E1} = \dfrac{1-\sigma^{E}_{TH}}{1+\sigma^{E}_{TH}}$, $\hat{m}_{E2} = \dfrac{\Delta t}{\Delta x\, \eta_0 \varepsilon \left(1+\sigma^{E}_{TH}\right)}$

ste: σ^{E}_{TH}, z0: η_0, ep0: ε_0, epr: ε_r, dx: Δx, val: V_{oc}

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// current density (*z0 due to mE2, ~H)
double epr = 1.;
double sige = 0.;
double sige += 1/(R*dx);
double ste = sige*dt / (ep0*epr*2);
double mE1 = (1. - ste) / (1. + ste);
double mE2 = dt / ((1. + ste) * dx*z0*ep0*epr);
double J = -z0 * val / (dx*R);
switch (src->axis) {
    case 0:
        sp->mE1.x[idx] = mE1;
        sp->mE2.x[idx] = mE2;
        sp->J.x[idx] = J / sn.i;
        break;
...
    E.z = mE1.z * E.z +
          mE2.z * ((H.y - (cp-i1)->H.y) -
                   (H.x - (cp-J1)->H.x) - J.z);
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4.1 Dimensional Analysis

For (4.3):

$$\sigma^{E}_{TH} \rightarrow \frac{(s)(m)}{\left(kg^{-1} m^{-3} s^{4} A^{2}\right)(m)(m)\left(kg^{1} m^{2} s^{-3} A^{-2}\right)} \rightarrow 1 \ ✔$$

$$J' \rightarrow \frac{(ohm)\left(kg^{1} m^{2} s^{-3} A^{-1}\right)}{(m)(m)(ohm)} \rightarrow \left(kg^{1} m^{0} s^{-3} A^{-1}\right)$$

For (4.4):

$$\sigma^{E}_{TH} \rightarrow \frac{(s)}{\left(kg^{-1} m^{-3} s^{4} A^{2}\right)(m)\left(kg^{1} m^{2} s^{-3} A^{-2}\right)} \rightarrow 1 \ ✔$$

$$\hat{J} \rightarrow \frac{(ohm)\left(kg^{1} m^{2} s^{-3} A^{-1}\right)}{(m)(ohm)} \rightarrow \left(kg^{1} m^{1} s^{-3} A^{-1}\right)$$

For true current density:

$$J = \frac{\hat{J}}{\eta_0 \Delta x} \rightarrow \frac{\left(kg^{1} m^{1} s^{-3} A^{-1}\right)}{\left(kg^{1} m^{2} s^{-3} A^{-2}\right)(m)} \rightarrow \frac{A}{m^{2}} \ ✔$$

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5 OuterSpace::initPMLarea()

Derivation of PML area Update Equations. Maxwell's equations for diagonal anisotropic media, incorporating loss

$$\begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = j\omega\varepsilon_0 \begin{bmatrix} \varepsilon_x + \sigma_x^E / j\omega & 0 & 0 \\ 0 & \varepsilon_y + \sigma_y^E / j\omega & 0 \\ 0 & 0 & \varepsilon_z + \sigma_z^E / j\omega \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$$\begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = -j\omega\mu_0 \begin{bmatrix} \mu_x + \sigma_x^H / j\omega & 0 & 0 \\ 0 & \mu_y + \sigma_y^H / j\omega & 0 \\ 0 & 0 & \mu_z + \sigma_z^H / j\omega \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} \tag{L13.S19}$$

PML parameters:

$$[s] = [\mu_r] = [\varepsilon_r] = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \quad a = b = \frac{1}{c}$$

$$[s_z] = \begin{bmatrix} s_z & 0 & 0 \\ 0 & s_z & 0 \\ 0 & 0 & s_z^{-1} \end{bmatrix} \quad s_z = \alpha - j\beta \tag{L13.S36}$$

PML along all the borders:

$$[s_i] = \begin{bmatrix} s_i^{-1} & 0 & 0 \\ 0 & s_i & 0 \\ 0 & 0 & d_i \end{bmatrix} \tag{L13.S37a}$$

Combined into a single tensor quantity:

$$[s] = [s_x] \cdot [s_y] \cdot [s_z] = \begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix} \tag{L13.S37b}$$

Maxwell's equations with uniaxial PML:

$$\nabla\times\vec{E}=k_0[\mu_r][s]\vec{\tilde{H}} \qquad \nabla\times\vec{\tilde{H}}=k_0[\varepsilon_r][s]\vec{E} \tag{L13.S40a}$$

Move PML tensor to the left side:

$$[s]^{-1}\nabla\times\vec{E}=k_0[\mu_r]\vec{\tilde{H}} \qquad [s]^{-1}\nabla\times\vec{\tilde{H}}=k_0[\varepsilon_r]\vec{E} \tag{L13.S41a}$$

Curl operator is now:

$$[s]^{-1}\nabla\times \;=\; \begin{bmatrix} s_z^{-1}s_y^{-1}s_x & 0 & 0 \\ 0 & s_z^{-1}s_y s_x^{-1} & 0 \\ 0 & 0 & s_z s_y^{-1}s_x^{-1} \end{bmatrix}\begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{s_x}{s_y}\left(\frac{1}{s_z}\frac{\partial}{\partial z}\right) & \frac{s_x}{s_z}\left(\frac{1}{s_y}\frac{\partial}{\partial y}\right) \\ \frac{s_y}{s_x}\left(\frac{1}{s_z}\frac{\partial}{\partial z}\right) & 0 & -\frac{s_y}{s_z}\left(\frac{1}{s_x}\frac{\partial}{\partial x}\right) \\ -\frac{s_z}{s_x}\left(\frac{1}{s_y}\frac{\partial}{\partial y}\right) & \frac{s_z}{s_y}\left(\frac{1}{s_x}\frac{\partial}{\partial x}\right) & 0 \end{bmatrix} \tag{L13.S41b}$$

Factors s_x, s_y, and s_z are effectively stretching the coordinates into a complex space.

Drop the non-stretching terms:

$$\nabla_s\times \;=\; \begin{bmatrix} 0 & -\frac{1}{s_z}\frac{\partial}{\partial z} & \frac{1}{s_y}\frac{\partial}{\partial y} \\ \frac{1}{s_z}\frac{\partial}{\partial z} & 0 & -\frac{1}{s_x}\frac{\partial}{\partial x} \\ -\frac{1}{s_y}\frac{\partial}{\partial y} & \frac{1}{s_x}\frac{\partial}{\partial x} & 0 \end{bmatrix} \tag{L13.S43a}$$

Introduce loss in the PML through fictitious conductivity terms, σ' (sigma prime), which are zero in the central problem space, but gradually increase going outward in the PML boundaries.

PML loss terms:

$$s_i(i) \;=\; 1+\frac{\sigma'_i(i)}{j\omega\varepsilon_0} \qquad \sigma'_i(i)=\frac{\varepsilon_0}{2\Delta t}\left(\frac{i}{L_i}\right)^3 \tag{L13.S46}$$

where $i=x,y,z$, and $L_i=$length of the PML in the i direction.

To add PML to Maxwell's equations, we do it in the frequency domain:

$$\nabla\times\vec{E}(\omega) = -j\omega\mu_0[\mu_r][s]\vec{H}(\omega)$$
$$\nabla\times\vec{H}(\omega) = \sigma\vec{E}(\omega)+j\omega[s]\vec{D}(\omega) \qquad \vec{D}(\omega) = \varepsilon_0[\varepsilon_r]\vec{E}(\omega) \tag{L13.S54}$$

Normalize the E field quantities according to:

$$\vec{\tilde{E}}=\sqrt{\frac{\varepsilon_0}{\mu_0}}\vec{E}=\frac{1}{\eta_0}\vec{E} \qquad \vec{\tilde{D}}=\frac{1}{\sqrt{\mu_0\varepsilon_0}}\vec{E}=c_0\vec{E} \tag{L13.S55a}$$

Maxwell's equations with the PML and normalized fields are:

$$\nabla\times\vec{\tilde{E}}(\omega) = -j\omega\frac{[\mu_r]}{c_0}[s]\vec{H}(\omega)$$
$$\nabla\times\vec{H}(\omega) = \eta_0\sigma\vec{\tilde{E}}(\omega)+\frac{j\omega}{c_0}[s]\vec{\tilde{D}}(\omega) \qquad \vec{\tilde{D}}(\omega) = [\varepsilon_r]\vec{\tilde{E}}(\omega) \tag{L13.S55b}$$

In matrix form, and with diagonal tensors substituted:

$$\begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}\begin{bmatrix} \tilde{E}_x(\omega) \\ \tilde{E}_y(\omega) \\ \tilde{E}_z(\omega) \end{bmatrix} = -\frac{j\omega}{c_0}\begin{bmatrix} \mu_{xx} & 0 & 0 \\ 0 & \mu_{yy} & 0 \\ 0 & 0 & \mu_{zz} \end{bmatrix}\begin{bmatrix} s_x^{-1}s_y s_z & 0 & 0 \\ 0 & s_x s_y^{-1} s_z & 0 \\ 0 & 0 & s_x s_y s_z^{-1} \end{bmatrix}\begin{bmatrix} H_x(\omega) \\ H_y(\omega) \\ H_z(\omega) \end{bmatrix}$$

$$\begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}\begin{bmatrix} H_x(\omega) \\ H_y(\omega) \\ H_z(\omega) \end{bmatrix} = \eta_0\begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}\begin{bmatrix} \tilde{E}_x(\omega) \\ \tilde{E}_y(\omega) \\ \tilde{E}_z(\omega) \end{bmatrix} + \frac{j\omega}{c_0}\begin{bmatrix} s_x^{-1}s_y s_z & 0 & 0 \\ 0 & s_x s_y^{-1} s_z & 0 \\ 0 & 0 & s_x s_y s_z^{-1} \end{bmatrix}\begin{bmatrix} \tilde{D}_x(\omega) \\ \tilde{D}_y(\omega) \\ \tilde{D}_z(\omega) \end{bmatrix}$$

$$\begin{bmatrix} \tilde{D}_x(\omega) \\ \tilde{D}_y(\omega) \\ \tilde{D}_z(\omega) \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix}\begin{bmatrix} \tilde{E}_x(\omega) \\ \tilde{E}_y(\omega) \\ \tilde{E}_z(\omega) \end{bmatrix}$$
(13.57)

Vector expansion:

$$\nabla\times\vec{\tilde{E}}(\omega)=-j\omega\frac{[\mu_r]}{c_0}[s]\vec{H}(\omega)\quad\rightarrow\quad\begin{aligned}\frac{\partial\tilde{E}_z(\omega)}{\partial y}-\frac{\partial\tilde{E}_y(\omega)}{\partial z}&=-j\omega\frac{\mu_{xx}}{c_0}\frac{s_y s_z}{s_x}H_x(\omega)\\\frac{\partial\tilde{E}_x(\omega)}{\partial z}-\frac{\partial\tilde{E}_z(\omega)}{\partial x}&=-j\omega\frac{\mu_{yy}}{c_0}\frac{s_x s_z}{s_y}H_y(\omega)\\\frac{\partial\tilde{E}_y(\omega)}{\partial x}-\frac{\partial\tilde{E}_x(\omega)}{\partial y}&=-j\omega\frac{\mu_{zz}}{c_0}\frac{s_x s_y}{s_z}H_z(\omega)\end{aligned}$$

$$\nabla\times\vec{H}(\omega)=\eta_0\sigma\vec{\tilde{E}}(\omega)+\frac{j\omega}{c_0}[s]\vec{\tilde{D}}(\omega)\quad\rightarrow\quad\begin{aligned}\frac{\partial H_z(\omega)}{\partial y}-\frac{\partial H_y(\omega)}{\partial z}&=\eta_0\sigma_{xx}\tilde{E}_x(\omega)+\frac{j\omega}{c_0}\frac{s_y s_z}{s_x}\tilde{D}_x(\omega)\\\frac{\partial H_x(\omega)}{\partial z}-\frac{\partial H_z(\omega)}{\partial x}&=\eta_0\sigma_{yy}\tilde{E}_y(\omega)+\frac{j\omega}{c_0}\frac{s_x s_z}{s_y}\tilde{D}_y(\omega)\\\frac{\partial H_y(\omega)}{\partial x}-\frac{\partial H_x(\omega)}{\partial y}&=\eta_0\sigma_{zz}\tilde{E}_z(\omega)+\frac{j\omega}{c_0}\frac{s_x s_y}{s_z}\tilde{D}_z(\omega)\end{aligned}$$

$$\vec{\tilde{D}}(\omega)=[\varepsilon_r]\vec{\tilde{E}}(\omega)\quad\rightarrow\quad\begin{aligned}\tilde{D}_x(\omega)&=\varepsilon_{xx}\tilde{E}_x(\omega)\\\tilde{D}_y(\omega)&=\varepsilon_{yy}\tilde{E}_y(\omega)\\\tilde{D}_z(\omega)&=\varepsilon_{zz}\tilde{E}_z(\omega)\end{aligned}$$

(L13.S58)

Substitute s_i from (13.46) to get final form of Maxwell's Equations with UPML:

$$\begin{aligned}j\omega\left(1+\frac{\sigma'_x}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma'_y}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\varepsilon_0}\right)H_x(\omega)&=-\frac{c_0}{\mu_{xx}}\left[\frac{\partial\tilde{E}_z(\omega)}{\partial y}-\frac{\partial\tilde{E}_y(\omega)}{\partial z}\right]\\j\omega\left(1+\frac{\sigma'_x}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_y}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma'_z}{j\omega\varepsilon_0}\right)H_y(\omega)&=-\frac{c_0}{\mu_{yy}}\left[\frac{\partial\tilde{E}_x(\omega)}{\partial z}-\frac{\partial\tilde{E}_z(\omega)}{\partial x}\right]\\j\omega\left(1+\frac{\sigma'_x}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_y}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\varepsilon_0}\right)^{-1}H_z(\omega)&=-\frac{c_0}{\mu_{zz}}\left[\frac{\partial\tilde{E}_y(\omega)}{\partial x}-\frac{\partial\tilde{E}_x(\omega)}{\partial y}\right]\end{aligned}\qquad\text{(L13.S59a)}$$

$$\begin{aligned}j\omega\left(1+\frac{\sigma'_x}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma'_y}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\varepsilon_0}\right)\tilde{D}_x(\omega)&=c_0\left[\frac{\partial H_z(\omega)}{\partial y}-\frac{\partial H_y(\omega)}{\partial z}\right]-\frac{\sigma_{xx}}{\varepsilon_0}\tilde{E}_x(\omega)\\j\omega\left(1+\frac{\sigma'_x}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_y}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma'_z}{j\omega\varepsilon_0}\right)\tilde{D}_y(\omega)&=c_0\left[\frac{\partial H_x(\omega)}{\partial z}-\frac{\partial H_z(\omega)}{\partial x}\right]-\frac{\sigma_{yy}}{\varepsilon_0}\tilde{E}_y(\omega)\\j\omega\left(1+\frac{\sigma'_x}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_y}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\varepsilon_0}\right)^{-1}\tilde{D}_z(\omega)&=c_0\left[\frac{\partial H_y(\omega)}{\partial x}-\frac{\partial H_x(\omega)}{\partial y}\right]-\frac{\sigma_{zz}}{\varepsilon_0}\tilde{E}_z(\omega)\end{aligned}\qquad\text{(L13.S59b)}$$

Assume no real loss (σ) in the PML areas.

Convert to time domain using:

$$F\{\frac{d^a}{dt^a}g(t)\}=(j\omega)^a G(\omega)\qquad\text{(L14.S11)}$$

To transform first equation in (13.59a), pull out Curl terms:

$$C_x^E(\omega) = \frac{\partial \tilde{E}_z(\omega)}{\partial y} - \frac{\partial \tilde{E}_y(\omega)}{\partial z} \qquad \text{(L14.S12b)}$$

Leaving:

$$j\omega \left(1+\frac{\sigma'_x}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma'_y}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\varepsilon_0}\right) H_x(\omega) = -\frac{c_0}{\mu_{xx}} C_x^E(\omega) \qquad \text{(L14.S12c)}$$

Move first (inverted) parenthesized term to right side and multiply everything out:

$$j\omega H_x(\omega) + \frac{\sigma'_y+\sigma'_z}{\varepsilon_0} H_x(\omega) + \frac{1}{j\omega}\frac{\sigma'_y\sigma'_z}{\varepsilon_0^2} H_x(\omega) = -\frac{c_0}{\mu_{xx}} C_x^E(\omega) - \frac{1}{j\omega}\frac{c_0\sigma'_x}{\varepsilon_0\mu_{xx}} C_x^E(\omega) \qquad \text{(L14.S13c)}$$

Apply (14.11) to get time-domain equation with PML:

$$\frac{\partial H_x(t)}{\partial t} + \frac{\sigma'_y+\sigma'_z}{\varepsilon_0} H_x(t) + \int_{-\infty}^{t} \frac{\sigma'_y\sigma'_z}{\varepsilon_0^2} H_x(\tau)\,d\tau = -\frac{c_0}{\mu_{xx}} C_x^E(t) - \int_{-\infty}^{t} \frac{c_0\sigma'_x}{\varepsilon_0\mu_{xx}} C_x^E(\tau)\,d\tau \qquad \text{(L14.S15)}$$

Pull constant terms out of integrals:

$$\frac{\partial}{\partial t} H_x(t) + \frac{\sigma'_y+\sigma'_z}{\varepsilon_0} H_x(t) + \frac{\sigma'_y\sigma'_z}{\varepsilon_0^2}\int_{-\infty}^{t} H_x(\tau)\,d\tau = -\frac{c_0}{\mu_{xx}} C_x^E(t) - \frac{c_0\sigma'_x}{\varepsilon_0\mu_{xx}}\int_{-\infty}^{t} C_x^E(\tau)\,d\tau$$

(L14.S16a)

Term 1. “This is the time derivative of the H field. We define the mag fields to exist at the half time steps, so this finite difference exists at the integer time steps, which is compatible with the electric field quantities.”:

$$\frac{\partial}{\partial t} H_x(t) \approx \frac{H_{x,t+\Delta t/2}^{i,j,k} - H_{x,t-\Delta t/2}^{i,j,k}}{\Delta t} \qquad \text{(L14.S19)}$$

Term 2. Need an H that exists at the integer time step. So we interpolate the half time steps:

$$\frac{\sigma'_y+\sigma'_z}{\varepsilon_0} H_x(t) \approx \frac{\sigma'^{i,j,k}_y+\sigma'^{i,j,k}_z}{\varepsilon_0}\,\frac{H_{x,t+\Delta t/2}^{i,j,k} - H_{x,t-\Delta t/2}^{i,j,k}}{2} \qquad \text{(L14.S20)}$$

Term 3. Need to integrate up to time t, not t plus half. Pull out the last term in the summation:

$$\int_{-\infty}^{t} \frac{\sigma'_y \sigma'_z}{\varepsilon_0^2} H_x(\tau)\, d\tau \quad \approx \quad \frac{\sigma'_y + \sigma'_z}{\varepsilon_0} \sum_{T=\Delta t/2}^{t+\Delta t/2} H_{x,T}^{i,j,k} \Delta t$$

$$\approx \quad \frac{\sigma'_y + \sigma'_z}{\varepsilon_0} \left(H_{x,t+\Delta t/2}^{i,j,k} \frac{\Delta t}{2} + \sum_{T=\Delta t/2}^{t-\Delta t/2} H_{x,T}^{i,j,k} \Delta t \right) \tag{L14.S22a}$$

Force the extracted term to only integrate over half of a time step:

$$\int_{-\infty}^{t} \frac{\sigma'_y \sigma'_z}{\varepsilon_0^2} H_x(\tau) d\tau \approx \frac{\sigma'_y + \sigma'_z}{\varepsilon_0} \left(\frac{\left(H_{x,t+\Delta t/2}^{i,j,k} + H_{x,t-\Delta t/2}^{i,j,k}\right)}{2} \frac{\Delta t}{2} + \sum_{T=\Delta t/2}^{t-\Delta t/2} H_{x,T}^{i,j,k} \Delta t \right) \tag{L14.S22b}$$

Term 4. Just a curl operation:

$$-\frac{c_0}{\mu_{xx}} C_x^E(t) \approx -\frac{c_0}{\mu_{xx}^{i,j,k}} C_{x,t}^{E,i,j,k} \tag{L14.S24a}$$

$$C_{x,t}^{E,i,j,k} = \frac{\tilde{E}_{z,t}^{i,j+1,k} - \tilde{E}_{z,t}^{i,j,k}}{\Delta y} - \frac{\tilde{E}_{y,t}^{i,j,k+1} - \tilde{E}_{y,t}^{i,j,k}}{\Delta z} \tag{L14.S24b}$$

Term 5. Integrating a term that exists at integer time steps.

$$-\frac{c_0 \sigma'_x}{\varepsilon_0 \mu_{xx}} \int_{-\infty}^{t} C_x^E(\tau) d\tau \approx \frac{c_0 \sigma'^{i,j,k}_x}{\varepsilon_0 \mu_{xx}^{i,j,k}} \sum_{T=0}^{t} C_{x,T}^{E,i,j,k} \Delta t \tag{L14.S25}$$

Since the sum goes up to t instead of t+t/2, we'll be updating the sum before updating the E field.

Same procedure is used to derive the y and z equations.

To derive the update equation, substitute the terms and solve for the future time value of H_x, giving:

$$H_{x,t+\Delta t/2}^{i,j,k} = \left(m_{Hx1}^{i,j,k}\right) H_{x,t-\Delta t/2}^{i,j,k} + \left(m_{Hx2}^{i,j,k}\right) C_{x,t}^{E,i,j,k} + \left(m_{Hx3}^{i,j,k}\right) I_{CEx,t}^{i,j,k} + \left(m_{Hx4}^{i,j,k}\right) I_{Hx,t}^{i,j,k} \tag{L14.S33c}$$

with PML update coefficients:

$$\begin{aligned}
m_{Hx0}^{i,j,k} &= \frac{1}{\Delta t} + \left(\frac{\sigma_y^{H,i,j,k} + \sigma_z^{H,i,j,k}}{2\varepsilon_0} \right) + \frac{\left(\sigma_y^{H,i,j,k}\right)\left(\sigma_z^{H,i,j,k}\right)\Delta t}{4\varepsilon_0^2} \\
m_{Hx1}^{i,j,k} &= \frac{1}{m_{Hx0}^{i,j,k}} \left[\frac{1}{\Delta t} - \left(\frac{\sigma_y^{H,i,j,k} + \sigma_z^{H,i,j,k}}{2\varepsilon_0} \right) - \frac{\left(\sigma_y^{H,i,j,k}\right)\left(\sigma_z^{H,i,j,k}\right)\Delta t}{4\varepsilon_0^2} \right] \\
m_{Hx2}^{i,j,k} &= -\frac{1}{m_{Hx0}^{i,j,k}} \frac{c_0}{\mu_x^{i,j,k}} \\
m_{Hx3}^{i,j,k} &= -\frac{1}{m_{Hx0}^{i,j,k}} \frac{c_0 \Delta t}{\varepsilon_0} \frac{\sigma_x^{H,i,j,k}}{\mu_{xx}^{i,j,k}} \\
m_{Hx4}^{i,j,k} &= -\frac{1}{m_{Hx0}^{i,j,k}} \frac{\Delta t}{\varepsilon_0^2} \left(\sigma_y^{H,i,j,k}\right)\left(\sigma_z^{H,i,j,k}\right)
\end{aligned} \tag{L14.S33a}$$

and integration terms:

$$\begin{aligned}
I_{CEx,t}^{i,j,k} &= \sum_{T=0}^{t} C_{x,T}^{E,i,j,k} \qquad I_{Hx,t-\frac{\Delta t}{2}}^{i,j,k} = \sum_{T=0}^{t-\frac{\Delta t}{2}} H_{x,T}^{i,j,k} \\
C_{x,t}^{E,i,j,k} &= \frac{\tilde{E}_{z,t}^{i,j+1,k} - \tilde{E}_{z,t}^{i,j,k}}{\Delta y} - \frac{\tilde{E}_{y,t}^{i,j,k+1} - \tilde{E}_{y,t}^{i,j,k}}{\Delta z}
\end{aligned} \tag{L14.S33b}$$

Similarly, the update equation for D_x is:

$$\widetilde{D}^{i,j,k}_{x,t+\Delta t} = \left(m^{i,j,k}_{Dx1}\right)\widetilde{D}^{i,j,k}_{x,t} + \left(m^{i,j,k}_{Dx2}\right)C^{H,i,j,k}_{x,t+\frac{\Delta t}{2}} + \left(m^{i,j,k}_{Dx3}\right)I^{i,j,k}_{CHx,t-\frac{\Delta t}{2}} + \left(m^{i,j,k}_{Dx4}\right)I^{i,j,k}_{Dx,t-\Delta t} \quad \text{(L14.S39c)}$$

with update coefficients:

$$\begin{aligned}
m^{i,j,k}_{Dx0} &= \frac{1}{\Delta t} + \left(\frac{\sigma^{D,i,j,k}_{y} + \sigma^{D,i,j,k}_{z}}{2\varepsilon_0}\right) + \frac{\left(\sigma^{D,i,j,k}_{y}\right)\left(\sigma^{D,i,j,k}_{z}\right)\Delta t}{4\varepsilon_0^2} \\
m^{i,j,k}_{Dx1} &= \frac{1}{m^{i,j,k}_{Dx0}}\left[\frac{1}{\Delta t} - \left(\frac{\sigma^{D,i,j,k}_{y} + \sigma^{D,i,j,k}_{z}}{2\varepsilon_0}\right) - \frac{\left(\sigma^{D,i,j,k}_{y}\right)\left(\sigma^{D,i,j,k}_{z}\right)\Delta t}{4\varepsilon_0^2}\right] \\
m^{i,j,k}_{Dx2} &= \frac{c_0}{m^{i,j,k}_{Dx0}} \\
m^{i,j,k}_{Dx3} &= \frac{1}{m^{i,j,k}_{Dx0}}\frac{c_0\ \Delta t\ \sigma^{D,i,j,k}_{x}}{\varepsilon_0} \\
m^{i,j,k}_{Dx4} &= -\frac{1}{m^{i,j,k}_{Dx0}}\frac{\Delta t}{\varepsilon_0^2}\left(\sigma^{D,i,j,k}_{y}\right)\left(\sigma^{D,i,j,k}_{z}\right)
\end{aligned} \quad \text{(L14.S39a)}$$

And integration terms:

$$\begin{aligned}
I^{i,j,k}_{CHx,t-\frac{\Delta t}{2}} &= \sum_{T=0}^{t-\frac{\Delta t}{2}} C^{H,i,j,k}_{x,T} \qquad I^{i,j,k}_{Dx,t-\Delta t} = \sum_{T=0}^{t-\Delta t} \widetilde{D}^{i,j,k}_{x,T} \\
C^{H,i,j,k}_{x,t-\frac{\Delta t}{2}} &= \frac{H^{i,j,k}_{z,t+\frac{\Delta t}{2}} - H^{i,j-1,k}_{z,t+\frac{\Delta t}{2}}}{\Delta y} - \frac{H^{i,j,k}_{y,t+\frac{\Delta t}{2}} - H^{i,j,k-1}_{y,t+\frac{\Delta t}{2}}}{\Delta z}
\end{aligned} \quad \text{(L14.S39b)}$$

Update coefficients for E_x, E_y, E_z (ε_{xx}, etc. are ε_r):

$$m^{i,j,k}_{Ex1} = \frac{1}{\varepsilon_{xx}^{\ i,j,k}} \qquad m^{i,j,k}_{Ey1} = \frac{1}{\varepsilon_{yy}^{\ i,j,k}} \qquad m^{i,j,k}_{Ez1} = \frac{1}{\varepsilon_{zz}^{\ i,j,k}} \quad \text{(L14.S42a)}$$

Update equations:

$$\begin{aligned}
\widetilde{E}^{i,j,k}_{x,t+\Delta t} &= \left(m^{i,j,k}_{Ex1}\right)\widetilde{D}^{i,j,k}_{x,t+\Delta t} \\
\widetilde{E}^{i,j,k}_{y,t+\Delta t} &= \left(m^{i,j,k}_{Ey1}\right)\widetilde{D}^{i,j,k}_{y,t+\Delta t} \\
\widetilde{E}^{i,j,k}_{z,t+\Delta t} &= \left(m^{i,j,k}_{Ez1}\right)\widetilde{D}^{i,j,k}_{z,t+\Delta t}
\end{aligned} \quad \text{(L14.S42b)}$$

5.1 My Transformations

Hatted variables ($\hat{\sigma}$) are my versions. From (13.46):

$$\sigma' = 2\varepsilon_0\hat{\sigma} \qquad L_i = n_b \qquad b = n_b + 1 \qquad \hat{\sigma}^H(i) = \frac{1}{2\varepsilon_0}\frac{\varepsilon_0}{2\Delta t}\left(\frac{i}{L_i}\right)^3 = \frac{1}{4\Delta t}\left(\frac{i}{n_b}\right)^3$$

$$\hat{\sigma}_x^{H,i,j,k} = \begin{cases} \hat{\sigma}^H(b-i+1) & for\, i<b \\ 0 & for\, b \le i < n_x - b \\ \hat{\sigma}^H(i-n_x+b) & for\, i \ge n_x - b \end{cases} \tag{5.1}$$

```
    // pre-calc PML gradients (sH are shifted over 0.5)
    double* sHg = new double[nb];
    double* sEg = new double[nb];
    for (int i = 0; i < nb; i++) {
        double r = double(i) / nb;
        sEg[i] = r*r*r / (4*dt);
        r += 0.5 / nb;
        sHg[i] = r*r*r / (4*dt);
    }
       double sHx1 = (i < b) ? sHg[b-i-1] : ((i >= nx-b) ? sHg[i-nx+b] : 0.);
```

Common terms from (L14.S33a):

$$m_{Hx0}^{i,j,k} = \frac{1}{\Delta t} + A + B \tag{5.2}$$

$$A = \frac{\sigma_y^{H,i,j,k} + \sigma_z^{H,i,j,k}}{2\varepsilon_0} = \frac{2\varepsilon_0\left(\hat{\sigma}_y^{H,i,j,k} + \hat{\sigma}_z^{H,i,j,k}\right)}{2\varepsilon_0} = \hat{\sigma}_y^{H,i,j,k} + \hat{\sigma}_z^{H,i,j,k}$$

$$B = \frac{\left(\sigma_y^{H,i,j,k}\right)\left(\sigma_z^{H,i,j,k}\right)\Delta t}{4\varepsilon_0^2} = \frac{\left(2\varepsilon_0\hat{\sigma}_y^{H,i,j,k}\right)\left(2\varepsilon_0\hat{\sigma}_z^{H,i,j,k}\right)\Delta t}{4\varepsilon_0^2} = \left(\hat{\sigma}_y^{H,i,j,k}\right)\left(\hat{\sigma}_z^{H,i,j,k}\right)\Delta t \tag{5.3}$$

```
// H update coefficients, mH2 & mH3 include "/dx"
double A = sHy + sHz;
double B = sHy * sHz * dt;
double R = 1 / (1/dt + A + B);
bp->mH1.x = R * (1/dt - A - B);
bp->mH2.x = -R * c0 / (dx * c.mur);
bp->mH3.x = -R * c0 * dt * sHx / (2 * dx);
bp->mH4.x = -R * 4*B;
```

Similar case for E common terms from (L14.S39a):

```
// E update coefficients (= D with 1/ep on H terms, 2 & 3)
A = sEy + sEz;
B = sEy * sEz * dt;
R = 1 / (1/dt + A + B);
bp->mE1.x = R * (1/dt - A - B);
bp->mE2.x = R * c0 / (dx * c.epr);
bp->mE3.x = R * c0 * dt * sEx / (2 * dx);
bp->mE4.x = -R * 4*B;
```

5.2 Dimensional Analysis

Using [kg,m,s,A] exponents.
Henry = [1,2,-2,-2], Farad = [-1,-2,4,2], Volt = [1,2,-3,-1]
Norm Mag Field = sqrt(H/F)*A/m = ([1,2,-2,-2]-[-1,-2,4,2])/2+[0,-1,0,1]

= [1,1,-3,-1] = V/m = Elec Field

Maxwell's equations

$$\nabla \times E = -\mu \frac{dH}{dt} \rightarrow (kg^1 m^0 s^{-3} A^{-1}) = (kg^1 m^1 s^{-2} A^{-2})(kg^0 m^{-1} s^{-1} A^1)$$

$$c_0^2 = \frac{1}{\mu_0 \varepsilon_0} \rightarrow (kg^0 m^2 s^{-2} A^0) = \frac{1}{(kg^1 m^1 s^{-2} A^{-2})(kg^{-1} m^{-3} s^4 A^2)} \quad ✔ \tag{5.4}$$

$$m_{Hx0}^{i,j,k} = \frac{1}{\Delta t} + \left(\frac{\sigma_y^{H,i,j,k} + \sigma_z^{H,i,j,k}}{2\varepsilon_0} \right) + \frac{\left(\sigma_y^{H,i,j,k}\right)\left(\sigma_z^{H,i,j,k}\right)\Delta t}{4\varepsilon_0^2} \rightarrow \frac{1}{(s^1)} + \frac{(s^{-1}\varepsilon_0)}{(\varepsilon_0)} + \frac{(s^{-1}\varepsilon_0)(s^{-1}\varepsilon_0)(s)}{(\varepsilon_0)^2} \quad ✔$$

(5.5)

$$m_{Hx1}^{i,j,k} = \frac{1}{m_{Hx0}^{i,j,k}} \left[\frac{\sigma_y^{H,i,j,k} + \sigma_z^{H,i,j,k}}{2\varepsilon_0} - \frac{\left(\sigma_y^{H,i,j,k}\right)\left(\sigma_z^{H,i,j,k}\right)\Delta t}{4\varepsilon_0^2} \right] \rightarrow \frac{1}{(s^{-1})}(s^{-1}) \rightarrow 1 \quad ✔ \tag{5.6}$$

$$\hat{m}_{Hx2}^{i,j,k} = -\frac{1}{m_{Hx0}^{i,j,k}} \frac{c_0}{\mu_r^{i,j,k} \Delta x} \rightarrow \frac{1}{(s^{-1})} \frac{(m/s)}{m} \rightarrow 1 \quad ✔ \tag{5.7}$$

$$\hat{m}_{Hx3}^{i,j,k} = -\frac{1}{m_{Hx0}^{i,j,k}} \frac{c_0 \Delta t}{\varepsilon_0} \frac{\sigma_x^{H,i,j,k}}{\mu_r^{i,j,k} \Delta x} \rightarrow \frac{1}{(s^{-1})} \frac{(kg^0 m^1 s^{-1} A^0)(s)}{(\varepsilon_0)} \frac{(s^{-1}\varepsilon_0)}{(m)} \rightarrow 1 \quad ✔ \tag{5.8}$$

$$m_{Hx4}^{i,j,k} = -\frac{1}{m_{Hx0}^{i,j,k}} \frac{\Delta t}{\varepsilon_0^2} \left(\sigma_y^{H,i,j,k}\right)\left(\sigma_z^{H,i,j,k}\right) \rightarrow \frac{1}{(s^{-1})} \frac{(s)}{(\varepsilon_0)^2} (s^{-1}\varepsilon_0)^2 \rightarrow 1 \quad ✔ \tag{5.9}$$

From http://personalpages.manchester.ac.uk/staff/fumie.costen/tmp/sourceexcitationpaperEMCblackwhite.pdf Eq (4):

$$\Delta E_z = \frac{-\Delta t}{\varepsilon} \left[\frac{I_z}{\Delta x \Delta y} \right] \rightarrow (kg^1 m^1 s^{-3} A^{-1}) = \frac{(s)(A)}{(kg^{-1} m^{-3} s^4 A^2)(m)(m)} \quad ✔ \tag{5.10}$$

---

6 Space::allocate() — Update Coefficients

Conduction form of Ez update equation, from Schneider:

$$E_z^{q+1}[m] = \frac{1 - \frac{\sigma \Delta t}{2\epsilon}}{1 + \frac{\sigma \Delta t}{2\epsilon}} E_z^q[m] + \frac{\frac{\Delta t}{\epsilon \Delta x}}{1 + \frac{\sigma \Delta t}{2\epsilon}} \left(H_y^{q+\frac{1}{2}}\left[m + \frac{1}{2}\right] - H_y^{q+\frac{1}{2}}\left[m - \frac{1}{2}\right] \right) \tag{U3.52}$$

(These are expanded to 3 dimensions in ch. 9, (U9.15) - (U9.32))

In our notation, with coefficients separated:

$$E^{i,j,k}_{z,t+\Delta t} = m^{i,j,k}_{Ez1} E^{i,j,k}_{z,t} + m^{i,j,k}_{Ez2}\left(H^{i,j,k}_{y,t+\Delta\frac{t}{2}} - H^{i,j-1,k}_{y,t+\Delta\frac{t}{2}}\right)$$

$$m^{i,j,k}_{Ez0} = 1 + \frac{\sigma^E \Delta t}{2\varepsilon}, \quad m^{i,j,k}_{Ez1} = \frac{1}{m^{i,j,k}_{Ez1}}\left[1 - \frac{\sigma^E \Delta t}{2\varepsilon}\right], \quad m^{i,j,k}_{Ez2} = \frac{1}{m^{i,j,k}_{Ez1}}\left[\frac{\Delta t}{\varepsilon \Delta x}\right], \tag{6.1}$$

Compare with PML coefficients:

$$m^{i,j,k}_{Ex0} = \frac{1}{\Delta t} + \left(\frac{\sigma^{D,i,j,k}_y + \sigma^{D,i,j,k}_z}{2\varepsilon_0}\right) + \frac{\left(\sigma^{D,i,j,k}_y\right)\left(\sigma^{D,i,j,k}_z\right)\Delta t}{4\varepsilon_0^2}$$

$$m^{i,j,k}_{Ex1} = \frac{1}{m^{i,j,k}_{Ex0}}\left[\frac{1}{\Delta t} - \left(\frac{\sigma^{D,i,j,k}_y + \sigma^{D,i,j,k}_z}{2\varepsilon_0}\right) - \frac{\left(\sigma^{D,i,j,k}_y\right)\left(\sigma^{D,i,j,k}_z\right)\Delta t}{4\varepsilon_0^2}\right] \tag{14.39a, 14.42}$$

$$\hat{m}^{i,j,k}_{Ex2} = \left(\frac{1}{\varepsilon_r}\right)\frac{c_0}{m^{i,j,k}_{Ex0}\Delta x}$$

If sigmas are isotropic:

$$m^{i,j,k}_{Ex0} = \frac{1}{\Delta t} + \left(\frac{\sigma^{D,i,j,k}}{\varepsilon_0}\right) + \frac{\left(\sigma^{D,i,j,k}\right)^2 \Delta t}{4\varepsilon_0^2}$$

$$m^{i,j,k}_{Ex1} = \frac{1}{m^{i,j,k}_{Ex0}}\left[\frac{1}{\Delta t} - \left(\frac{\sigma^{D,i,j,k}}{\varepsilon_0}\right) - \frac{\left(\sigma^{D,i,j,k}\right)^2 \Delta t}{4\varepsilon_0^2}\right] \tag{6.2}$$

$$\hat{m}^{i,j,k}_{Ex2} = \frac{c_0}{\varepsilon_r m^{i,j,k}_{Ex0}\Delta x}$$

Or:

$$m^{i,j,k}_{Ex1} = \frac{1-B}{1+B}, \quad B = 2A + A^2, \quad A = \frac{\sigma^{E,i,j,k}\Delta t}{2\varepsilon_0}$$

$$\hat{m}^{i,j,k}_{Ex2} = \frac{c_0 \Delta t}{\varepsilon_r \Delta x (1+B)} \tag{6.3}$$

So it appears to be equivalent, other than the PML version is a 2nd order approximation.

6.1 Dimensional Analysis

Dimensional analysis follows that for the PML in section 5.2. For sigma E specifically:

$$\frac{\sigma^E \Delta t}{\varepsilon_0} = k \rightarrow \frac{(kg^{-1}m^{-3}s^3A^2)(s)}{(kg^{-1}m^{-3}s^4A^2)} = 1 \ ✔ \tag{6.4}$$

$$\left(1 + \frac{\sigma \Delta t}{2\epsilon}\right) E^{q+1}_z[m] = \left(1 - \frac{\sigma \Delta t}{2\epsilon}\right) E^q_z[m] + \frac{\Delta t}{\epsilon \Delta x}\left(H^{q+\frac{1}{2}}_y\left[m + \frac{1}{2}\right] - H^{q+\frac{1}{2}}_y\left[m - \frac{1}{2}\right]\right) \tag{6.5}$$

7 Space::stepH(), E() — Update Equations

7.1 Derivation of My Update Coefficients

$$\mu \frac{\partial H_y}{\partial t}\bigg|_{(m+1/2)\Delta x, q\Delta t}$$

These are from Schneider

del(E) = -mu*d(H)/dt
c0^2 = 1/(mu0*ep0)

Normalizing mag field: (here Hu is unnormalized H)
Using H = sqrt(mu0/ep0)*Hu
So: del(E) = -mu*d(Hu)
del(E) = -mu*(1/sqrt(mu0/ep0))*d(H)
del(E) = -mu*(sqrt(mu0/ep0)/(mu0/ep0))*d(H)
del(E) = -mur*(sqrt(mu0/ep0)*ep0)*d(H)
del(E) = -mur*((sqrt(mu0)*ep0)/sqrt(ep0))*d(H)
del(E) = -mur*((sqrt(mu0)*ep0*sqrt(ep0))/ep0)*d(H)
del(E) = -mur*(1/c0))*d(H)
del(E) = -(mur/c0)*d(H)

Without conduction (sigma):
Kmu = dt*c0/(mur*dx) = 1 when dt/dx = c0
mu = mu0 * mur
c0^2 = (mur*epr)/(mu*ep)

With conduction:
stm = sig*dt/(2*mu)
Hx = (1-stm)/(1+stm)*Hx + (dt/((1+stm)*mu*dx))*(Eyp1-Ey)-(Ezp1-Ez))

From [Sch2018] 3.12 Lossy Material. Ampere's law with conduction term:
del(Hu) = sig*E + ep*d(E) [eq 3.48]

$$\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{H} \tag{U3.48}$$

In discretized form:
dHu/dx = sig*E + ep*dE/dt

$$\sigma E_x + \varepsilon \frac{\partial E_x}{\partial t} = \frac{\partial H_y}{\partial x} \tag{U3.49}$$

1D version, averaging in time the E node on either side of the H node:
(Huyp1 - Huy)/dx = sig*(Ezp1 + Ez)/2 + ep*(Ezp1 - Ez)/dt

$$\frac{H^{i,j,k+1}_{y,t+\frac{\Delta t}{2}} - H^{i,j,k}_{y,t-\frac{\Delta t}{2}}}{\Delta x} = \sigma \frac{\left(E^{i,j+1,k}_{z,t} - E^{i,j,k}_{z,t}\right)}{2} + \varepsilon \frac{\left(E^{i,j+1,k}_{z,t} - E^{i,j,k}_{z,t}\right)}{\Delta t} \tag{U3.51?}$$

Ezp1 = ((Huyp1 - Huy)/dx - (sig/2 - ep/dt)*Ez) / (sig/2 + ep/dt)

Ezp1 = (1/(dx*(sig/2 + ep/dt)))*(Hyup1 - Huy) -
((sig/2 - ep/dt)/(sig/2 + ep/dt))*Ez

Ezp1 = (dt/(dx*ep*(sig*dt/(2*ep) + 1))*(Huyp1 - Huy) -
((sig*dt/(2*ep) - 1)/(sig*dt/(2*ep) + 1))*Ez

Ezp1 = ((dt/(dx*ep))/(1 + sig*dt/(2*ep))*(Huyp1 - Huy) +
((1 - sig*dt/(2*ep))/(1 + sig*dt/(2*ep)))*Ez

$$E_z^{q+1}[m] = \frac{1-\frac{\sigma\Delta t}{2\epsilon}}{1+\frac{\sigma\Delta t}{2\epsilon}} E_z^q[m] + \frac{\frac{\Delta t}{\epsilon\Delta x}}{1+\frac{\sigma\Delta t}{2\epsilon}} \left(H_y^{q+\frac{1}{2}}\left[m+\frac{1}{2}\right] - H_y^{q+\frac{1}{2}}\left[m-\frac{1}{2}\right]\right) \tag{U3.52}$$

Ezp1 = ((dt/(dx*ep))/(1 + sig*dt/(2*ep))*(Hyp1 - Hy)/Z0 +
((1 - sig*dt/(2*ep))/(1 + sig*dt/(2*ep)))*Ez

Z0 = 1/(ep0*c0)

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$c_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \qquad \eta_0 = \frac{1}{\varepsilon_0 c_0} = \mu_0 c_0$$

non Kep = dt*c0/(dx*ep0*epr)

$$m_2^E = \frac{\Delta\ t\ c_0}{\Delta\ x\ \varepsilon_0\varepsilon_r}$$?

sig Kep = dt/(Z0*dx*ep0*epr) = dt*(ep0*c0)/(dx*ep0*epr)

Huyp1 = ((dt/(dx*mu))/(1 + sig*dt/(2*mu))*(Ezp1 - Ez) +
((1 - sig*dt/(2*mu))/(1 + sig*dt/(2*mu)))*Huy

Hyp1/Z0 = ((dt/(dx*mu))/(1 + sig*dt/(2*mu))*(Ezp1 - Ez) +
((1 - sig*dt/(2*mu))/(1 + sig*dt/(2*mu)))*Hy/Z0

Hyp1 = ((dt/(dx*mu))/(1 + sig*dt/(2*mu))*(Ezp1 - Ez)*Z0 +
((1 - sig*dt/(2*mu))/(1 + sig*dt/(2*mu)))*Hy

non Kmu = dt*c0/(dx*mu0*mur)

$$m_2^H = \frac{\Delta\ t\ c_0}{\Delta\ x\ \mu_0\mu_r}$$?

sig Kmu = dt*Z0/(dx*mu0*mur) = dt/(dx*ep0*c0*mu0*mur)
= dt*c0/(dx*ep0*c0^2*mu0*mur) = dt*c0/(dx*mur)

ste = sig*dt/(2*ep) = sig*dt/(2*ep0*epr)
Ksep = (1 - ste) / (1 + ste)

Kep = (dt/(dx*ep0*epr)) / (1 + ste)

8 OuterSpace::stepH(), E() — Conduction Border

?
Originally had (field_main2_844.cp, 180216):

```
        if (copyBorders)
        {
            // update E field borders
            for (size_t i = 1; i < nx-1; i++)
                for (size_t j = 1; j < ny-1; j++)
                {
                    Et(i,j,0) = Et(i,j,1);
                    Et(i,j,nz-1) = Et(i,j,nz-2);
                }
            for (size_t i = 1; i < nx-1; i++)
                for (size_t k = 1; k < nz-1; k++)
                {
                    Et(i,0,k) = Et(i,1,k);
                    Et(i,ny-1,k) = Et(i,ny-2,k);
                }
            for (size_t j = 1; j < ny-1; j++)
                for (size_t k = 1; k < nz-1; k++)
                {
                    Et(0,j,k) = Et(1,j,k);
                    Et(nx-1,j,k) = Et(nx-2,j,k);
                }
        }
```

Then (field_main2.cp@6777, 180216):

r6777 | scott | 2018-02-16 13:13:27 -0700 (Fri, 16 Feb 2018) | 2 lines
Changed copyBorders option to conductBorder, which matches small_lc_wall.out quite closely.

```
        // conductive border: tangential E and normal H are zero
        if (conductBorder)
        {
            for (size_t i = 1; i < nx-1; i++)
            {
                double3* Hp = &Ht(i,1,0);
                double3* Ep = &Et(i,1,0);
                double* Kmup = &Kmut(i,1,0);
                for (size_t j = 1; j < ny-1; j++)
                {
                    // step all H[i,j,0], but with z terms removed
                    double Km = *Kmup;
                    Hp->x += Km * ((Ep+1)->y - Ep->y);
                    Hp->y -= Km * ((Ep+1)->x - Ep->x);
                    // want H[i,j,nz-1], but it reaches beyond border
                    //double3* En = Ep + (nz-1)*k1;
                    //double3* Hn = Hp + (nz-1)*k1;
                    //Hn->x += Km * ((En+k1)->y - En->y);
                    //Hn->y -= Km * ((En+k1)->x - En->x);
                    Kmup += J1;
                    Ep += J1;
                    Ep += J1;
                }
            }
```

9 Notes

180311:

Normalizing H:

$$\nabla\times\vec{E}=-[\mu]\frac{\partial\vec{H}}{\partial t}\qquad\nabla\times\vec{H}=[\varepsilon]\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\tilde{H}}=\sqrt{\frac{\mu_0}{\varepsilon_0}}\vec{H}=\eta_0\vec{H}\quad\rightarrow\quad\begin{array}{c}\nabla\times\vec{E}=-\frac{[\mu_0\mu_r]}{\eta_0}\frac{\partial\vec{\tilde{H}}}{\partial t}\\ \nabla\times\vec{\tilde{H}}=\eta_0[\varepsilon_0\varepsilon_r]\frac{\partial\vec{E}}{\partial t}\end{array}\quad\rightarrow\quad\begin{array}{c}\nabla\times\vec{E}=-\frac{[\mu_r]}{c_0}\frac{\partial\vec{\tilde{H}}}{\partial t}\\ \nabla\times\vec{\tilde{H}}=\frac{[\varepsilon_r]}{c_0}\frac{\partial\vec{E}}{\partial t}\end{array}\tag{L5.S21}$$

Normalizing E (same equations, just different scales to the physical values):

$$\vec{\tilde{E}}=\sqrt{\frac{\varepsilon_0}{\mu_0}}\vec{E}=\frac{\vec{E}}{\eta_0}\quad\rightarrow\quad\begin{array}{c}\nabla\times\vec{\tilde{E}}=-\frac{[\mu_0\mu_r]}{\eta_0}\frac{\partial\vec{H}}{\partial t}\\ \nabla\times\vec{H}=\eta_0[\varepsilon_0\varepsilon_r]\frac{\partial\vec{\tilde{E}}}{\partial t}\end{array}\quad\rightarrow\quad\begin{array}{c}\nabla\times\vec{\tilde{E}}=-\frac{[\mu_r]}{c_0}\frac{\partial\vec{H}}{\partial t}\\ \nabla\times\vec{H}=\frac{[\varepsilon_r]}{c_0}\frac{\partial\vec{\tilde{E}}}{\partial t}\end{array}\tag{9.1}$$

Rearranging for Update Equation form:

$$\begin{aligned}\vec{H}&=-\frac{c_0}{[\mu_r]}\int\left(\nabla\times\vec{\tilde{E}}\right)dt\\ \vec{\tilde{E}}&=\frac{c_0}{[\varepsilon_r]}\int\left(\nabla\times\vec{H}\right)dt\end{aligned}\tag{9.2}$$

The existing non-PML coefficient calculations look correct, at least for the σ = 0 case (and they match LC).
How can the PML not match? It should just attenuate whatever levels of H & E it's given. Its coefficient values should match across the PML-center seam.

non-PML: mH2 = (dt/dx) * Z0 / (mu0*mur)
PML: mH2 = (dt/dx) * c0 / (mu0*mur) ???

There shouldn't be a mu0 (or ep0) in the PML case. See (13.58). muxx is relative, not absolute.

180315:

"It is possible for the charge in materials to move under the influence of an electric field such that currents flow. If the material has a non-zero conductivity σ, the current density is given by

$$\vec{J}=\sigma\vec{E}.\tag{U2.38}$$

"The current density has units of A/m2 and the conductivity has units of S/m." [Sch2018]

180504:

Checking how the sigmas of Space::allocate() and initPMLarea() correlate.

1/c0 = sqrt(mu0) * sqrt(ep0);
z0 = sqrt(mu0) / sqrt(ep0);
z0 = (1/(sqrt(ep0)*c0)) / sqrt(ep0);

```
z0 = 1/(ep0*c0);
c0 = 1/(z0*ep0);

----------
allocate():
mE1.z[i] = (1. - (sige[i]*dt / (2*ep0*epr[i]))) / (1. + (sige[i]*dt / (2*ep0*epr[i])));

mE2.z[i] = (dt/dx) / ((1. + (sige[i]*dt / (2*ep0*epr[i]))) * z0*ep0*epr[i]);



----------
initPMLarea():
mE1.z[i] = (1 / (1/dt + (r*r*r / (4*dt)))) * (1/dt - (r*r*r / (4*dt)));

mE2.z[i] = (1 / (1/dt + (r*r*r / (4*dt)))) * c0 / (dx * epr[i]);
mE2.z[i] = (1 / ((1/dt)(1 + r*r*r/4))) * c0 / (dx * epr[i]);
mE2.z[i] = (dt / (1 + r*r*r/4)) * c0 / (dx * epr[i]);
mE2.z[i] = (dt / (1 + r*r*r/4)) / (dx * z0*ep0*epr[i]);
mE2.z[i] = (dt/dx) / (1 + r*r*r/4) / (z0*ep0*epr[i]);

SVN revision $Id: $
```